Notes for ECE 30100 - Signals and Systems

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Contents

Course Description

Classification, analysis and design of systems in both the time- and frequency-domains. Continuous-time linear systems: Fourier Series, Fourier Transform, bilateral Laplace Transform. Discrete-time linear systems: difference equations, Discrete-Time Fourier Transform, bilateral z-Transform. Sampling, quantization, and discrete-time processing of continuous-time signals. Discrete-time nonlinear systems: median-type filters, threshold decomposition. System design examples such as the compact disc player and AM radio.

Introduction

As this course studies signals and systems, it behooves us to understand what signals and systems are. A signal is a quantity that varies over time. Examples include voltage waveform on a circuit, height as a function of age, or pulses of light through fiber optic.

We distinguish between continuous time (CT) and discrete time (DT) signals. CT signals have a continuous independent variable, such as time. DT signals have a discrete independent variable, such as the date. The indices are a set of integers.

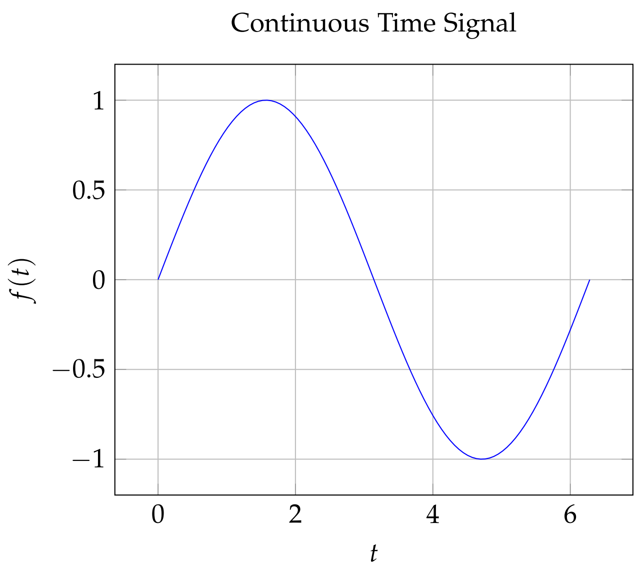


Figure 1: Continuous Time Signal

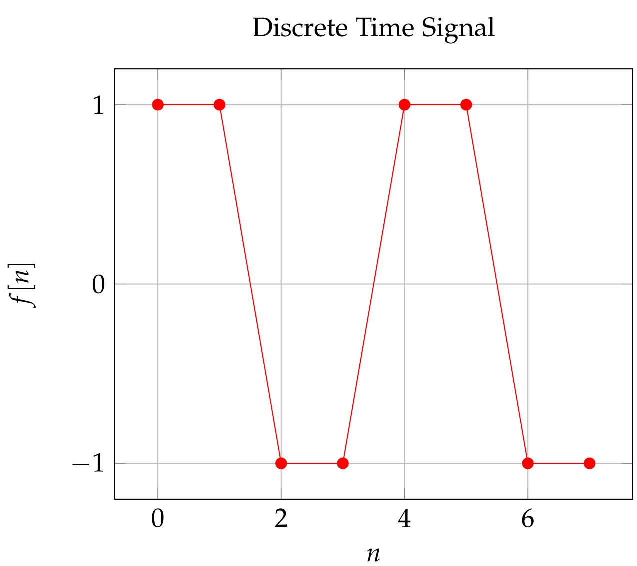


Figure 2: Discrete Time Signal

In the most general terms, a systems transform inputs to outputs. They're interconnections of subsystems. Examples of systems include, topically, circuits.

Similarly to signals, there are continuous time systems and discrete time systems. In a CT system, the input and output are continuous. Conversely, DT systems have discrete inputs and outputs.

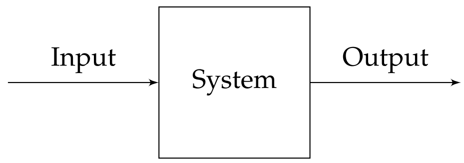


Figure 3: System Diagram

The astute reader will notice systems operate much like functions. We use function notation to describe systems. For a CT system, we write $y(t) = S(x(t))$ with parentheses to show it's CT. For DT, we use brackets like $y[t] = S(x[t])$.

It's easy to imagine a system with continuous input and discrete output, or vice versa. These are called samplers and reconstructors respectively. We'll mostly be looking at linear time-invariant discrete systems, since they have the greatest application to ECE.

Linearity

Readers are familiar with the concept of linearity, which mathematically may be expressed as

$$f(a+b) = f(a) + f(b) \tag{1}$$

Linear systems possess the property of superposition, so given an input as a sum of weighted inputs the output is a sum of weighted outputs.

The necessary and sufficient conditions for linearity in a CT system are if the input is $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ the output is $S(\alpha_1 x_1(t)) + S(\alpha_2 x_2(t))$. Formally,

$$S(\alpha_1 x_1(t) + \alpha_2 x_2(t)) = \alpha_1 S(x_1(t)) + \alpha_2 S(x_2(t)). \tag{2}$$

Likewise for DT systems,

$$S[\alpha_1 x_1[t] + \alpha_2 x_2[t]] = \alpha_1 S[x_1[t]] + \alpha_2 S[x_2[t]]. \tag{3}$$

This equality for hold for any real valued α_1 and α_2.

Consider the CT system S given by $y(t) = tx(t)$. We are interested in determining if the system is linear. We test it with the definition of linearity,

$$\begin{aligned} y(\alpha_1 x_1(t) + \alpha_2 x_2(t)) &= t(\alpha_1 x_1(t) + \alpha_2 x_2(t)) && (4)\\ &= t\alpha_1 x_1(t) + t\alpha_2 x_2(t) && (5)\\ &= \alpha_1 y(x_1(t)) + \alpha_2 y(x_2(t)) && (6) \end{aligned}$$

Since this is the definition of linearity, the system is linear.

Why do we care? We care because linearity gives us many useful properties and makes solving systems much easier. If we know the output for any set of inputs, we can find the output for any linear combination of those inputs.

Classifying Signal Types

Before we proceed we must be able to classify signal types. There are five ways to divide signal types.

- DT vs. CT
- Periodic vs. aperiodic
- Finite energy vs. finite power
- Even and odd
- Complex exponential

DT vs. CT

- DT: $x[n]$ is a sequence of complex values, including purely real values. numbers. Example: $x[n] = \frac{n}{2}$.
- CT: $x(t)$ is complex (including purely real) and continuous for all real values of t. Example: $x(t) = \frac{t}{2} - jt$.
- Complex:

For DT, complex $x[n]$ can be represented in Cartesian or polar form. For Cartesian,

$$x[n] = x_{Re}[n] + jx_{Im}[n]. \tag{7}$$

For polar,

$$x[n] = A[n]e^{j\Theta[n]}. \tag{8}$$

We can swap between the two with Euler's formula.

$$A[n]e^{j\Theta[n]} = A[n]\cos(\Theta[n]) + jA[n]\sin(\Theta[n]) \tag{9}$$

$$x_{Re}[n] + jx_{Im}[n] = \sqrt{x_{Re}[n]^2 + x_{Im}[n]^2} \times e^{j\arctan(\frac{x_{Im}[n]}{x_{Re}[n]})} \tag{10}$$

Energy vs. Power

DT vs. CT is one option to classify signals. Another possibility is Energy vs. Power. For this class, energy in a continuous time system is the area under the squared magnitude of the signal. Mathematically energy over (t_1, t_2) is equal to

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt \tag{11}$$

$$= \int_{t_1}^{t_2} (x_{Re}(t)^2 + x_{Im}(t)^2) dt. \tag{12}$$

For DT systems, the formula for energy is

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2 \tag{13}$$

$$= \sum_{n=n_1}^{n_2} (x_{Re}[n]^2 + x_{Im}[n]^2). \tag{14}$$

The total energy E_∞ is the energy from $t = -\infty$ to $t = \infty$.

Power is energy per unit time, or in terms of calculus $P(t) = \frac{d}{dt}E(t)$. For CT, average power is

$$P_{avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt. \tag{15}$$

For DT,

$$P_{avg} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2. \tag{16}$$

The overall average power is

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{17}$$

for CT and

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{18}$$

for DT time.

Transformations

Just as with functions, signals can be transformed in time. Here are the different transformations that can be applied to signals.

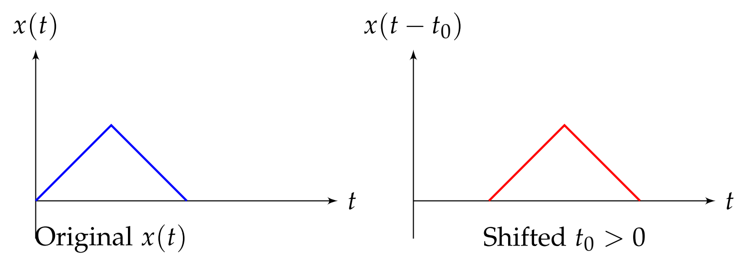
Time Shift

A CT time shift is given by $x(t) \to x(t - t_0)$, where t_0 is real.

- $t_0 > 0$: shifted to the right or delayed by t_0
- $t_0 < 0$: shifted to the left or advanced by t_0

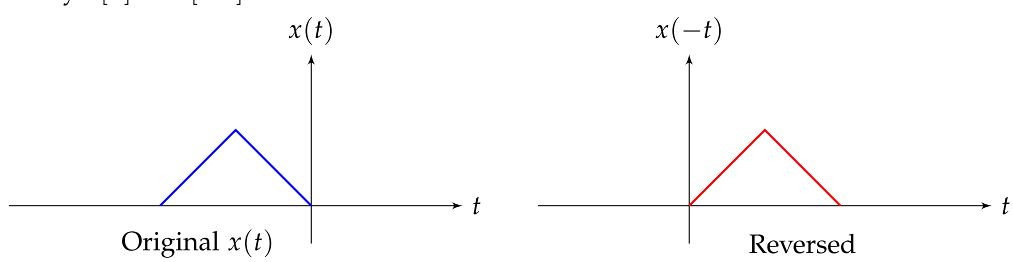
A DT time shift is given by $x[n] \to x[n - n_0]$, where n_0 is an integer.

- $n_0 > 0$: shifted to the right or delayed by n_0
- $n_0 < 0$: shifted to the left or advanced by n_0



Time Reversal

A CT time reversal is given by $x(t) \to x(-t)$. A DT time reversal is given by $x[n] \to x[-n]$.



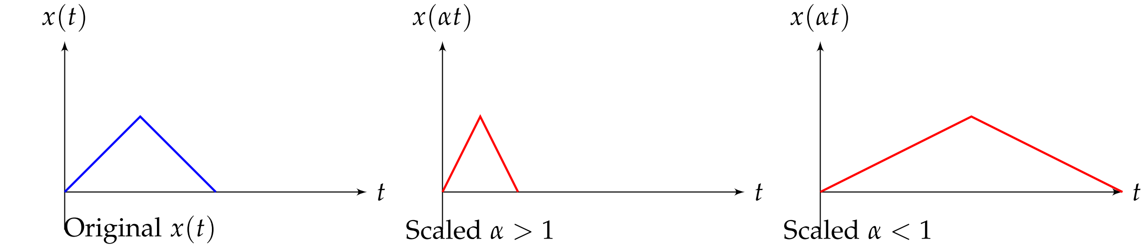
Time Scaling

A CT time scaling is given by $x(t) \to x(\alpha t)$, where $\alpha > 0$ is the time scaling factor.

If $\alpha < 0$, that's viewed as a combination of reversal and scaling.

- $\alpha > 1$: shorter timescale, or sped up
- $\alpha < 1$: longer timescale, or slowed down

A DT time scaling is given by $x[n] \to x[\alpha n]$.



A signal is even if it's symmetric with respect to the dependent axis. Mathematically, if $x(t) = x(-t)$.

A signal is odd if it's symmetric with respect to the origin. Mathematically, if $x(-t) = -x(t)$.

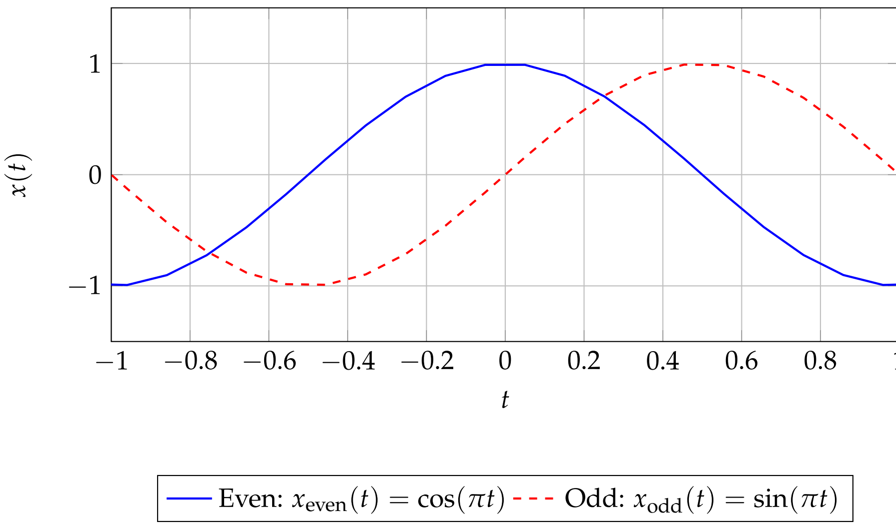


Figure 4: Even and odd signals

Signals can be odd, even, both, or neither. $x(t) = 0$, for instance, is both even and odd. $x + 1$ is neither.

The product of two odd signals is even (e.g. $x \times x^3$). The product of two evens is even ($x^2 \times 2$). The product of an odd and an even is odd ($x \times x^2$).

Any signal can be written as the sum of an even and an odd signal using these formulas:

$$x(t) = x_{even}(t) + x_{odd}(t) \tag{19}$$

$$= \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} \tag{20}$$

$$\tag{21}$$

$$\begin{aligned} x[n] &= x_{even}[n] + x_{odd}[n] && (22) \\ &= \frac{x[n] + x[-n]}{2} + \frac{x[n] - x[-n]}{2} && (23) \\ & && (24) \end{aligned}$$

Periodicity

A system is periodic if $x(t) = x(t+T)$, or in the case of discrete time, if $x[n] = x[n+N]$.

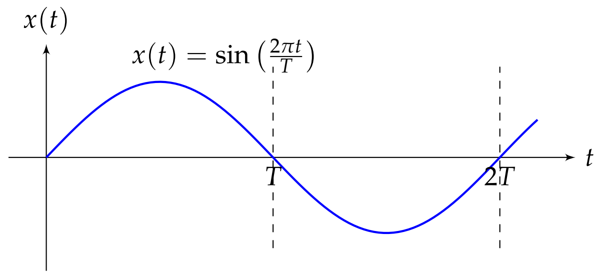


Figure 5: Periodic CT Signal

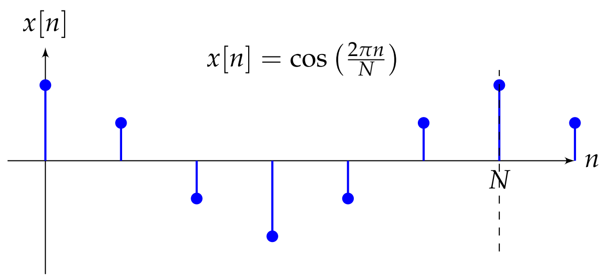


Figure 6: Periodic DT Signal

The fundemental period is the smallest T_0 (or N_0) such that $x(t) = x(t+T_0)$ (or $x[n] = x[n+N_0]$). If $x(t)$ is periodic, $x_{Re}(t) + jx_{Im}(t)$ is also periodic. However, if $x_1(t)$ and $x_2(t)$ are periodic then it is not necessarily the case that $x_1(t) + x_2(t)$ is periodic. Consider $x_1(t) = \sin(t)$ and $x_2(t) = \sin(\sqrt{2}t)$. $x_1(t) + x_2(t) = \sin(t) + \sin(\sqrt{2}t)$. $x_1(t)$ has period 2π. $x_2(t)$ has period $\frac{2\pi}{\sqrt{2}}$. However, their sum is not periodic and in fact the sum of any $x_1(t)$, $x_2(t)$ will be aperiodic when the ratio of their periods is irrational. To get the average power of a periodic signal, we can just calculate the power over one period.

Periodicty in Discrete Time

Periods in discrete time differ in some unintuitive ways from the continuous time case. For instance, in CT, two functions with different ω will never represent the same function. However, in DT, they can if $\omega_2 = \omega_1 + 2\pi n$. Functions like $x[n] = \cos(n)$ are aperiodic in discrete time since the period as given by $\frac{2\pi}{\omega}$ must be a rational number, and in this case it's 2π. When $\omega = k\pi$, $k = 1, 3, 5, \ldots$ and $e^{jkn} = (-1)^n$. In general, if

$$x_k[n] = e^{jk\frac{2\pi}{N}n} \tag{25}$$

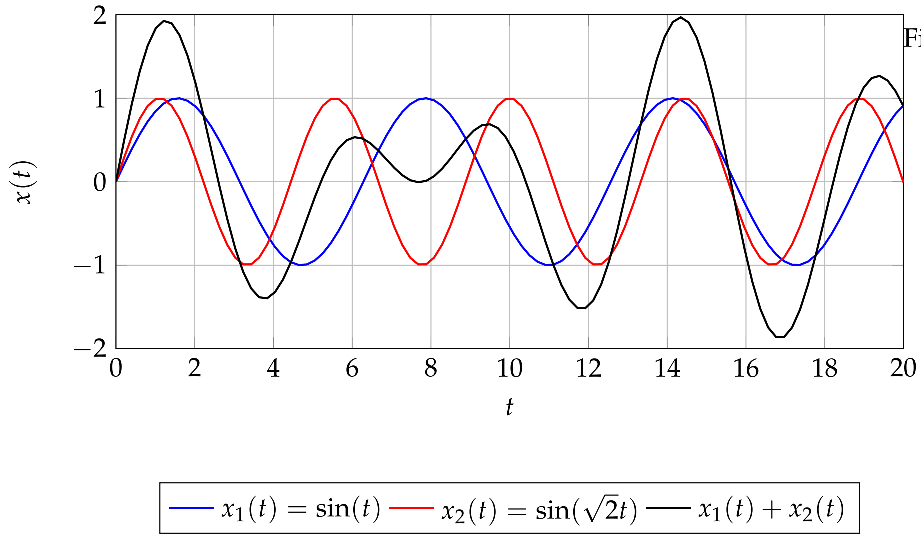


Figure 7: Periodic Signals Sum

then $x_k[n]$ is unique only for $k = 0, 1, \ldots, N-1$. Also in general,

$$N_k = LCM(\frac{N}{k}, 1) \tag{26}$$

Unit Step Signal

The unit step signal, also known as the Heaviside step function, is 0 for values less than 0 and 1 otherwise.

Discrete-Time Unit Step Signal

The discrete-time unit step signal is defined as:

$$u[n] = \begin{cases} 1 & n \geq 0, \\ 0 & n < 0. \end{cases}$$

Or alternatively,

$$u[n] = \sum_{i=-\infty}^{n} \delta[i] \tag{27}$$

In either definition,

$$u[n] = 1 \text{ for } n \geq 0, \text{ and } u[n] = 0 \text{ for } n < 0.$$

Figure 8 is the plot for the discrete-time unit step signal.

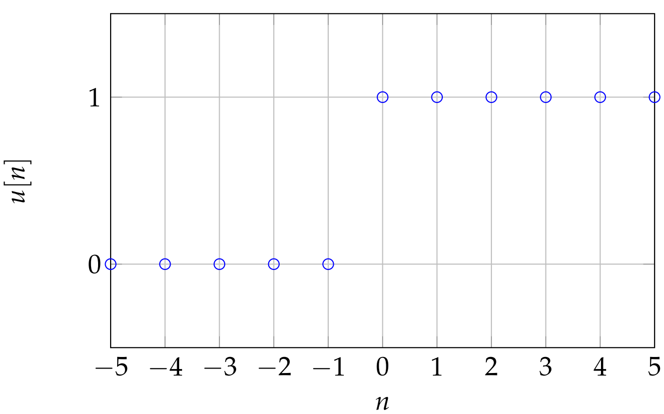


Figure 8: Discrete-Time Unit Step Signal

A useful property for signal sampling is that, if we just want to consider a section of the signal between 0 and n_0, we can just multiply $x[n]$ by $u[n] - u[n - n_0]$.

Continuous-Time Unit Step Signal

The continuous-time unit step signal is defined as:

$$u(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Or alternatively,

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \tag{28}$$

We can also make the substitution $\sigma = t - \tau$, which gives

$$u(t) = \int_0^{\infty} \delta(t-\sigma)d\sigma \tag{29}$$

Figure 9 is the plot for the continuous-time unit step signal:

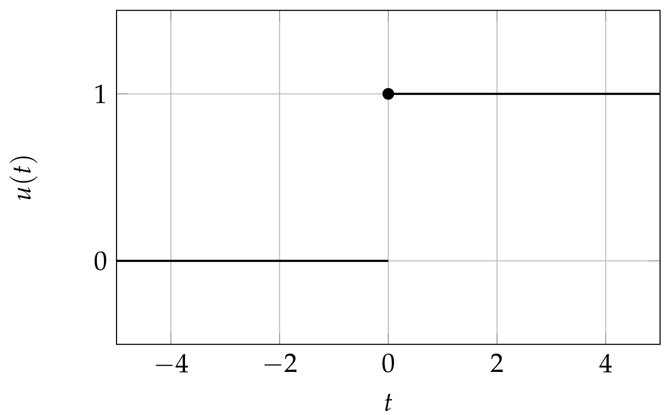


Figure 9: Continuous-Time Unit Step Signal

Discrete-Time Delta Function

The discrete-time delta function, known as the Kronecker delta function, is defined as:

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Figure 10 is the plot for the discrete-time delta function:

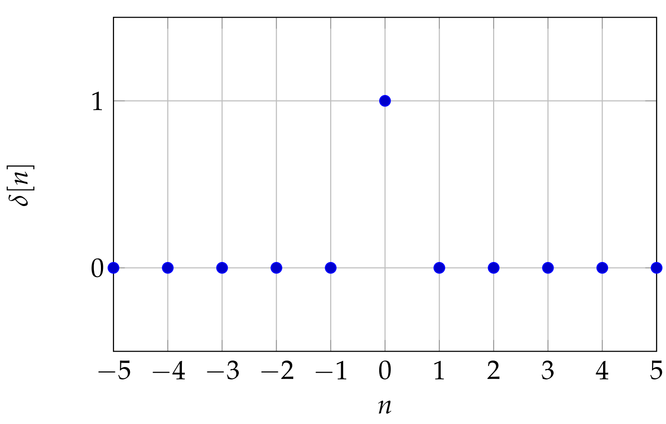


Figure 10: Discrete-Time Delta Function

A useful property of the delta function in DT is that, since it is just equal to one when its argument is 0, then

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{30}$$

Continuous-Time Delta Function

The continuous-time delta function, known as the Dirac delta function or unit impulse, is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0, \\ 0 & t \neq 0, \end{cases}$$

with the property that:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

An alternative definition for $\delta(t)$ is

$$\delta(t) = \frac{d}{dt} u(t) \tag{31}$$

Figure 11 is the plot for the continuous-time delta function. The height of the arrow indicates not the value of the function but its area, since the value is technically undefined.

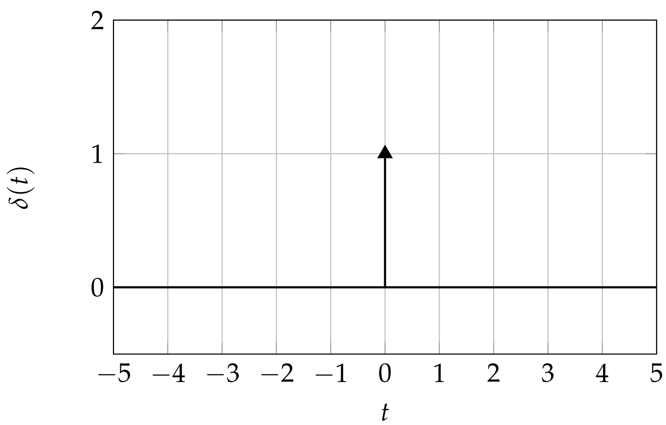


Figure 11: Continuous-Time Delta Function

The *sampling property* is the property that

$$x(t)\delta(t-\sigma) = x(\sigma)\delta(t-\sigma). \tag{32}$$

We can decompose $x(t)$

$$\begin{aligned} x(t) &= x(t) \times 1 && (33) \\ &= x(t) \int_{-\infty}^{\infty} \delta(t-\sigma) d\sigma && (34) \\ &= \int_{-\infty}^{\infty} x(t)\delta(t-\sigma) d\sigma && (35) \\ &= \int_{-\infty}^{\infty} x(\sigma)\delta(t-\sigma) d\sigma. && (36) \end{aligned}$$

In DT, this becomes

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{37}$$

Complex Exponential

A CT complex exponential signal is of the form $x(t) = Ce^{\alpha t}$, where C and α are in general complex. Alternatively,

$$x(t) = |C|e^{\sigma t}e^{j(\omega t+\phi)} \tag{38}$$

where ϕ is the angle between the real axis and C when plotted on the complex plane and $\alpha = \sigma + j\omega$.

The σ term determines whether the signal has exponential growth, decay, or neither. If $\sigma = 0$ then we are left with the periodic complex exponential $e^{j(\omega t+\phi)}$ with period $\frac{2\pi}{\omega}$.

ω is called the fundemental frequency, ϕ is called the phase.

The signal $x(t) = |C|e^{\sigma t}e^{j(\omega t+\phi)}$ forms a family of signals called harmonically related complex exponentials (HRCEs), each of the form

$$x_k(t) = e^{jk\omega_0 t}, k \in \mathbb{Z}. \tag{39}$$

These signals will serve as our building blocks when we construct Fourier series of complex exponential signals later on.

Let's now look at the discrete time complex exponential,

$$x[n] = C\alpha^n. \tag{40}$$

In general, C and α can be complex. This can be rewritten

$$x[n] = |C|e^{\sigma n}e^{j(\omega n+\phi)}. \tag{41}$$

Where the continuous and discrete begin to diverge is when we consider the $e^{j(\omega n+\phi)}$ term. This term is not always periodic, unlike the case of continuous time. For the CT case, the fundemental frequency is ω and larger values of ω produce higher rates of oscillation.

Now say we want to compute the fundemental period of $x[n] = \cos(3\pi n)$. In the continuous case it's easy, $\frac{2}{3}$. In the discrete case, we need to find the least common multiple of the fundemental period of the CT signal (in this case, $\frac{2}{3}$) and the sampling period. This is why the discrete case may not be periodic. Imagine the sampling period is 1 and the fundemental period of the CT signal is 2π. Then the least common multiple does not exist.

Let's do an example problem. Consider the signal

$$x(t) = e^{j2t} + e^{j5t}. \tag{42}$$

We can rewrite this, using the identity

$$\cos(\omega t) = \frac{e^{-j\omega t} + e^{j\omega t}}{2} \tag{43}$$

to the expression

$$e^{j3.5t}(e^{j1.5t} + e^{-j1.5t}) \tag{44}$$

$$= 2e^{j3.5t}\cos(1.5t) \tag{45}$$

$$|x(t)| = 2|\cos(1.5t)| \tag{46}$$

$$x[n] = C\alpha^n \tag{47}$$

$$= Ce^{\beta n} \tag{48}$$

is the general form of a discrete complex exponential and β is in general complex. If α is real and less than 0, then β must be complex.

System Connections

Often we analyze a complex system as a set of subsystems connected to one another. These connections can take many forms, but some of the more common ones are in series, parallel, and feedback.

Series

$$y(t) = S_2(S_1(x(t))) \tag{49}$$

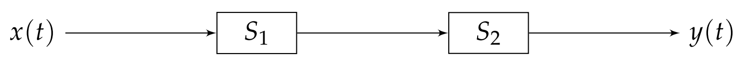


Figure 12: Series connection block diagram.

Parallel

$$y(t) = S_1(x(t)) + S_2(x(t)) \tag{50}$$

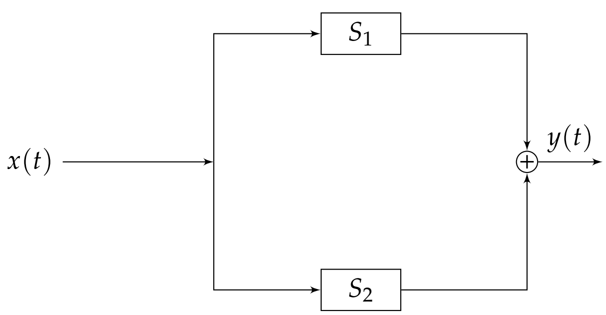


Figure 13: connection block diagram.

Feedback

$$y(t) = S_1\left(x(t) - S_2(y(t))\right) \tag{51}$$

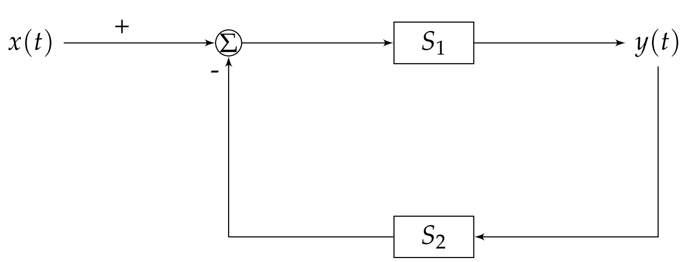


Figure 14: connection block diagram.

System Properties

Memoryless

A *memoryless* system output is dependent only on the input at time t, and not previous states.

Invertable

A system is *invertable* is distinct inputs lead to distinct outputs. A system is not invertable if there exists any $x_1(t) \neq x_2(t)$ such that $y_1(t) = y_2(t)$.

Three practical examples are modulation, sampling, and encoding. I do not understand modulation and cannot explain it. Encoding isn't always invertable, e.g. lossy image compression.

For an invertable system S_1, there exist an inverse system S_2 such that $S_2(S_1(x(t))) = S_1(S_2(x(t))) = x(t)$.

Linear

Causality

A system is *causal* when the output at any time is dependent only on past or present inputs. All memoryless systems are trivially causal. An example of a non-causal system is the moving average of a signal.

Stability

A system is stable if small changes in the input lead only to small changes in the output. For instance, if a pendulum is hanging down, it is stable to displacements. But if it's a vertical pendulum, it is not stable.

Mathematically, bounded inputs produce bounded outputs. For finite B, M,

$$|x(t)| < B \rightarrow |y(t)| < M. \tag{52}$$

Convolution

Recall the sifting property of the discrete unit impulse,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{53}$$

This property is useful in that it allows us to represent $x[n]$ as a series of scaled very simple functions. For a linear system, the response will