ECE 30200 - Probabilistic Methods in Electrical and Computer Engineering

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Background

The following formulas will be instrumental and may be familar.

Series

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{2}$$

$$\sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2} \tag{3}$$

Combinatorics

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{4}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{5}$$

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \tag{6}$$

$$P(n,k) = \frac{n!}{(n-k)!} \tag{7}$$

where $P(n,k)$ is the number of ways to arrange k objects out of n (permutations).

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{8}$$

where $C(n,k)$ is the number of ways to choose k objects out of n (combinations).

Approximations

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \tag{9}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{10}$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{11}$$

$$= e^x \tag{12}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{13}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{14}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \tag{15}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{16}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \tag{17}$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \tag{18}$$

Calculus

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x) \tag{19}$$

$$\int_a^b f'(x)\, dx = f(b) - f(a) \tag{20}$$

$$\int f(g(x)) g'(x)\, dx = \int f(u)\, du \tag{21}$$

$$\int u\, dv = uv - \int v\, du \tag{22}$$

$$\int \frac{1}{(x-a)(x-b)}\, dx = \frac{1}{b-a} \ln \left| \frac{x-a}{x-b} \right| + C \tag{23}$$

Linear Algebra

$$\vec{y} = \beta_1 \vec{x}_1 + \beta_2 \vec{x}_2 + \cdots + \beta_N \vec{x}_N \tag{24}$$

$$\langle \vec{a}, \vec{b} \rangle = \vec{a}\vec{b}^T \tag{25}$$
$$= \sum_{i=1}^{n} a_i b_i \tag{26}$$

where $\langle \vec{a}, \vec{b} \rangle$ denotes the inner product of vectors $\vec{a}$ and $\vec{b}$.

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{27}$$

where $\|\vec{x}\|_p$ is the p-norm (or ℓ_p-norm) of vector $\vec{x}$.

$$\cos(\theta) = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|_2 \|\vec{b}\|_2} \tag{28}$$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

$$\hat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\vec{y} \tag{29}$$

where $\hat{\beta}$ is the vector of least squares coefficients, $\mathbf{X}$ is the data matrix, and $\vec{y}$ is the target vector

Probability Laws

A probability law must satisfy three axioms:

1. Non-negativity: $P(A) \geq 0 \forall A \in F$
2. Normalization: $P(\Omega) = 1$
3. Additivity: For any disjoint subsets $\{A_1, A_2, \dots\}$, it holds that

$$P\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} P\left[A_n\right]$$

Probability Properties

$$P\left[A \cup B\right] = P\left[A\right] + P\left[B\right] - P\left[A \cap B\right] \tag{30}$$

$$P\left[A \cup B\right] \leq P\left[A\right] + P\left[B\right] \tag{31}$$

$$A \subseteq B \implies P\left[A\right] \leq P\left[B\right] \tag{32}$$

Formal Definitions

Outcomes

An *outcome* is the result of some *experiment*. If that experiment is flipping a coin, the outcome is either heads or tails. We could express the outcome of heads as H, and the outcome of tails as T. The set of all possible outcomes for an experiment is known as a sample space and is denoted by Ω. In this case $\Omega = \{H, T\}$.

Events

An *event* F is a subset of the sample space Ω. The formal definitions of probability are expressed with set notation. So the event where we have neither heads nor tails is written as $\{\}$. The event of heads could be expressed as $\{H\}$, and the event of tails could be expressed as $\{T\}$. The event of either heads or tails is $\{H, T\}$.

Probability Laws

A *probability law* is a function P that maps an event A to a real number in $[0, 1]$. For the coin example, the probability law might be $P(\{\}) = 0$, $P(\{H\}) = 0.5$, $P(\{T\}) = 0.5$, and $P(\{\Omega\}) = 1$. A probability law must satisfy three axioms:

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$$P\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} P\left[A_n\right]$$

Probability Space

A probability space is a triplet Ω, F, P.

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$$A \subseteq B \implies P\left[A\right] \leq P\left[B\right] \tag{35}$$

$$P\left[A|B\right] = \frac{P[A \cap B]}{P[B]} \tag{36}$$

Outcomes are statistically *independent* if $P(A|B) = P(A)$ (assuming P(B) > 0), or equivalently $P(A \cap B) = P(A)P(B)$.

A *random variable* X is a function $X : \Omega \implies \Re$ that maps an outcome $\epsilon \in \Omega$ to a number $X(\epsilon)$ on the real line. We call it a variable because it has multiple states. The mapping X

Bayes Theorem states that for any two events A and B such that $P[A] > 0$ and $P[B] > 0$,

$$P[A|B] = \frac{P[B|A]P[A]}{P[B]} \tag{37}$$

The *Law of Total Probability* states that if $\{A_1, A_2, \ldots, A_n\}$ is a partition of Ω, then for any $B \subseteq \Omega$,

$$P[B] = \sum_{i=1}^{n} P[B|A_i]P[A_i] \tag{38}$$

PMFs and CDFs

The *probability mass function* (PMF) $p_X(a)$ of a random variable X specifies the probability of obtaining a number $X(\epsilon) = a$. We denote a PMF as

$$p_X(a) = P[X = a] \tag{39}$$

PMFs are represented with histograms. A PMF should satisfy

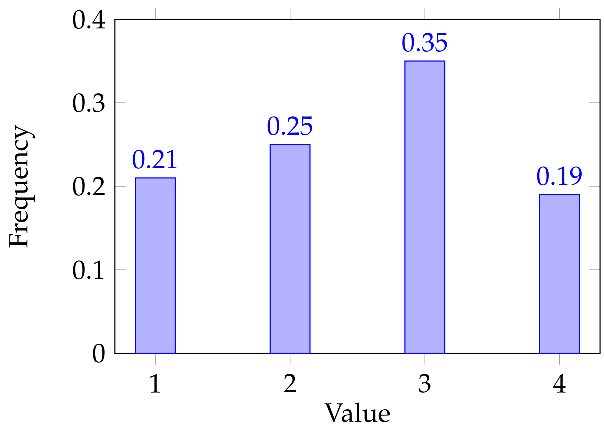

Figure 1: PMF

$$\sum_{x \in X(\Omega)} p_X(x) = 1 \tag{40}$$

The *cumulative distribution function* is given by

$$F_X(x) = P[X \leq x] \tag{41}$$

$$= \sum_{u \leq x} p_X(u) \tag{42}$$

and represents the sum of every impulse of the PMF up to x.

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