Sum of Squares

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum of Geometric Series

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^n}{1-r}$$

Sum of Weighted Geometric Series

$$\sum_{k=0}^{n} kr^k = r\frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$$

Memoryless

A *memoryless* system's output depends only on the input at time t (or n for discrete-time systems), and not on past or future states. For example, $y(t) = 2x(t)$ is memoryless, while $y[n] = x[n-1]$ has memory.

Linearity

A system is *linear* if it satisfies superposition and homogeneity. Formally, for any inputs $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, and constants a, b, the system satisfies:

$$S\{ax_1(t) + bx_2(t)\} = aS\{x_1(t)\} + bS\{x_2(t)\} = ay_1(t) + by_2(t).$$

An example is $y(t) = kx(t)$, where k is a constant.

Time Invariance

A system is *time invariant* if a time shift in the input results in an identical shift in the output. For a discrete-time system S, if:

$$x[n] \to y[n],$$

then for any integer n_0:

$$x[n-n_0] \to y[n-n_0].$$

Linear Time Invariance

An LTI system satisfies both linearity and time invariance. LTI systems are fully characterized by their impulse response $h(t)$ or $h[n]$, enabling analysis via convolution: $y(t) = x(t) * h(t)$. The impulse response is found by plugging in $\delta(t)$ or $\delta[n]$ to the system. The result is the impulse response. Mathematically,

$$\begin{aligned} h(t) &= S(\delta(t)) \\ h[n] &= S(\delta[n]). \end{aligned}$$

Causality

A system is *causal* if its output at time t depends only on present and past inputs. For causal systems, the impulse response must be 0 for values of n (t) less than 0.

Stability

A system is *stable* if bounded inputs produce bounded outputs. Formally, there exist constants $B, M > 0$ such that:

$$|x(t)| < B \ \forall t \implies |y(t)| < M \ \forall t.$$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.$$

HRCE

$$\begin{aligned} x_k(t) &= e^{jk\omega_0 t}, k \in \mathbb{Z}. \\ x_k[n] &= e^{jk\omega_0 n}, k \in \{0, 1, ..., N\}. \end{aligned}$$

Energy

$$\begin{aligned} E &= \int_{t_1}^{t_2} |x(t)|^2 dt \\ &= \int_{t_1}^{t_2} (x_{Re}(t)^2 + x_{Im}(t)^2) dt. \end{aligned}$$

$$\begin{aligned} E &= \sum_{n=n_1}^{n_2} |x[n]|^2 \\ &= \sum_{n=n_1}^{n_2} (x_{Re}[n]^2 + x_{Im}[n]^2). \end{aligned}$$

Power

$$P_{avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt.$$

$$P_{avg} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2.$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Poisson Summation Formula

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Analysis Equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Synthesis Equation

$$\begin{aligned} a_k &= \frac{1}{T} \int_{<T>} x(t) e^{-jk\frac{2\pi}{T}t} dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} \\ x[n] &= \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

Symmetry Properties

If $x(t)$ is real, then $|X(j\omega)|$ is even and $\angle X(j\omega)$ is odd. Moreover, if $x(t)$ is real and even, then $X(j\omega)$ must be purely real and even, and if $x(t)$ is real and odd, $X(j\omega)$ is purely imaginary and odd.

Duality Property

$$\mathcal{F}\{X(t)\} = 2\pi x(-\omega).$$

Modulation Property

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}\left(X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right)$$
$$x[n]\cos(\omega_0 n) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}\left(X(e^{j(\omega-\omega_0)}) + X(e^{j(\omega+\omega_0)})\right)$$

Amplitude Modulation

Modulate $x(t)$ with carrier $e^{j\omega_c t}$ or $\cos(\omega_c t)$. For complex exponential modulation,

$$x_{\mathrm{AM}}(t) = x(t)e^{j\omega_c t}$$

In frequency domain:

$$X_{\mathrm{AM}}(j\omega) = X(j(\omega-\omega_c))$$

To demodulate, multiply by $e^{-j\omega_c t}$ and lowpass filter. For sinusoidal modulation, modulate $x(t)$ with $\cos(\omega_c t)$:

$$x_{\mathrm{AM}}(t) = x(t)\cos(\omega_c t)$$

In frequency domain:

$$X_{\mathrm{AM}}(j\omega) = \frac{1}{2}\left[X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))\right]$$

To demodulate, multiply received AM signal by $\cos(\omega_c t)$ and lowpass filter:

$$y(t) = x_{\mathrm{AM}}(t)\cdot\cos(\omega_c t)$$

This yields terms at baseband and $2\omega_c$; lowpass filtering recovers $x(t)$. Alternatively, for $x(t)$ slowly varying and $x_{\mathrm{AM}}(t) = [A + x(t)]\cos(\omega_c t)$, the envelope (magnitude) of $x_{\mathrm{AM}}(t)$ approximates $A + x(t)$.

Frequency Division Multiplexing

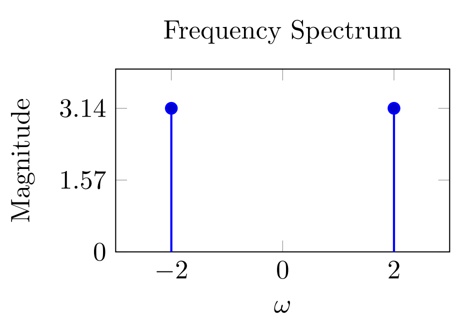
Multiple signals $x_k(t)$ modulated onto different carrier frequencies ω_k:

$$s(t) = \sum_k x_k(t)\cos(\omega_k t)$$

Each $x_k(t)$ occupies a unique frequency band; signals are separated at the receiver by bandpass filtering and demodulation.

Frequency Spectrum

The plot of the magnitude of a signal's Fourier transform is its frequency spectrum. For example, the Fourier transform of $\cos(2t)$ is $\pi\left[\delta(\omega-2) + \delta(\omega+2)\right]$ so its frequency spectrum is



LTI System Invertibility Criterion

An LTI system with frequency response $H(j\omega)$ (CT) or $H(e^{j\omega})$ (DT) is invertible if and only if $H(j\omega) \neq 0$ for all ω (CT) or $H(e^{j\omega}) \neq 0$ for all $\omega \in [-\pi, \pi]$ (DT). If the system is invertible, then the inverse is $\frac{1}{H(j\omega)}$ (CT) or $\frac{1}{H(e^{j\omega})}$ (DT).

LTI System Input/Output

$$Y(j\omega) = H(j\omega)X(j\omega)$$
$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

LTI with Periodic Complex Exponential Input

Let the input to an LTI system be $e^{j\omega_0 t}$, and let the impulse response be $h(t)$. Then

$$y(t) = H(j\omega_0)e^{j\omega_0 t}.$$

Differential Equations

For a system described by:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

The frequency response is:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Difference Equations

For a system described by:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The frequency response is:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

Sampling Theorem

Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm1, \pm2, \ldots$ if

$$\omega_s > 2\omega_M,$$

where

$$\omega_s = \frac{2\pi}{T}.$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which succesive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$. If $\omega_s < 2\omega_M$, spectral replicas overlap (aliasing), and $x(t)$ cannot be recovered exactly, though samples coincide: $x_r(nT) = x(nT)$.

Interpolation and Holds

Reconstruction via LTI filters or "holds":

- Zero-Order Hold (ZOH): impulse response $h_0(t) = u(t) - u(t-T)$, output holds each sample constant.
- First-Order Hold (Linear Interpolation): triangular impulse $h_1(t) = \frac{1}{T}(1 - \frac{|t|}{T})$ for $|t| < T$, cascaded ZOHs.
- Ideal Band-Limited Interpolation (Sinc): $h_{\mathrm{ideal}}(t) = \frac{\omega_s}{\pi}\,\mathrm{sinc}(\omega_s t/2)$ gives perfect reconstruction when conditions met.

Table 1: Properties of Fourier Transforms

Property	**CT Time Domain**	**CT Frequency Domain**	**DT Time Domain**	**DT Frequency Domain**
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$	$Ax_1[n] + Bx_2[n]$	$AX_1(e^{j\omega}) + BX_2(e^{j\omega})$
Time Shifting	$x(t - t_0)$	$X(j\omega)e^{-j\omega t_0}$	$x[n - n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$
Frequency Shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$
Time Scaling	$x(at)$, $a \neq 0$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\omega}{a}\right)$	—	—
Time Reversal	$x(-t)$	$X(-j\omega)$	$x[-n]$	$X(e^{-j\omega})$
Conjugation	$x^*(t)$	$X^*(-j\omega)$	$x^*[n]$	$X^*(e^{-j\omega})$
Differentiation	$\frac{d^n}{dt^n} x(t)$	$(j\omega)^n X(j\omega)$	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$	$\sum_{k=-\infty}^{n} x[k]$	$\frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
Convolution	$(x_1 * x_2)(t)$	$X_1(j\omega)X_2(j\omega)$	$(x_1 * x_2)[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}(X_1 * X_2)(j\omega)$	$x_1[n]x_2[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

Table 2: Fourier Transform Pairs

CT Time Domain $x(t)$	**CT Fourier Transform** $X(j\omega)$	**DT Time Domain** $x[n]$	**DT Fourier Transform** $X(e^{j\omega})$
$\delta(t)$	1	$\delta[n]$	1
1	$2\pi\,\delta(\omega)$	1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$	$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$e^{-at}u(t), \quad \Re(a) > 0$	$\dfrac{1}{a + j\omega}$	$a^n u[n], \quad \lvert a\rvert < 1$	$\dfrac{1}{1 - ae^{-j\omega}}$
$e^{-a\lvert t\rvert}$, $a > 0$	$\dfrac{2a}{a^2 + \omega^2}$	$a^{\lvert n\rvert}, \quad \lvert a\rvert < 1$	$\dfrac{1 - a^2}{1 - 2a\cos\omega + a^2}$
$te^{-at}u(t), \quad \Re(a) < 0$	$\dfrac{1}{(a + j\omega)^2}$	$na^n u[n], \quad \lvert a\rvert < 1$	$\dfrac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re(a) < 0$	$\frac{1}{(a+j\omega)^n}$	$n\,a^n\,u[n]$, $\lvert a\rvert < 1$	$\frac{a\,e^{-j\omega}}{\left(1-a\,e^{-j\omega}\right)^2}$
$\cos(\omega_0 t)$	$\pi\Big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\Big]$	$\cos(\omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty}\Big[\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k)\Big]$
$\sin(\omega_0 t)$	$\dfrac{\pi}{j}\Big[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\Big]$	$\sin(\omega_0 n)$	$\dfrac{\pi}{j} \sum_{k=-\infty}^{\infty}\Big[\delta(\omega - \omega_0 - 2\pi k) - \delta(\omega + \omega_0 - 2\pi k)\Big]$
$\begin{cases} 1, & \lvert t\rvert \leq T/2 \\ 0, & \lvert t\rvert > T/2 \end{cases}$	$2\frac{\sin\left(\frac{\omega T}{2}\right)}{\omega}$	$\begin{cases} 1, & 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin\Big(\omega(N+1)/2\Big)}{\sin\Big(\omega/2\Big)}\,e^{-j\omega N/2}$
$\frac{\sin(Wt)}{\pi t}$	$\begin{cases} 1, & \lvert\omega\rvert \leq W \\ 0, & \lvert\omega\rvert > W \end{cases}$	$\frac{\sin(Wn)}{\pi n}$	$\begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$, $X(e^{j\omega})$ periodic with period 2π.

Table 3: Properties of Fourier Series

Property	**CT Time Domain**	**CT Frequency Domain** (a_k)	**DT Time Domain**	**DT Frequency Domain** (a_k)
Linearity	$Ax_1(t) + Bx_2(t)$	$Aa_k + Bb_k$	$A\,x_1[n] + B\,x_2[n]$	$A\,a_k + B\,b_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$	$x[n - n_0]$	$a_k\,e^{-j\,\frac{2\pi}{N}kn_0}$
Frequency Shifting	$x(t)e^{jM\omega_0 t}$	a_{k-M}	$x[n]\,e^{j\,\frac{2\pi}{N}Mn}$	$a_{(k-M)\bmod N}$
Time Reversal	$x(-t)$	a_{-k}	$x[-n]$	a_{-k} (indices mod N)
Conjugation	$x^*(t)$	a^*_{-k}	$x^*[n]$	a^*_{-k} (indices mod N)
Periodic Convolution	$(x * y)(t)$	$Ta_k b_k$	$(x \circledast y)[n]$	$Na_k\,b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$	$x[n]\,y[n]$	$\sum_{m=0}^{N-1} a_m\,b_{(k-m)\bmod N}$
Differentiation	$\frac{d}{dt}x(t)$	$jk\omega_0 a_k$	$x[n] - x[n-1]$	$a_k\Big(1 - e^{-j\,\frac{2\pi}{N}k}\Big)$
Integration	$\int x(t)dt$	$\frac{a_k}{jk\omega_0}$ $(a_0 = 0)$	—	—
Parseval's Theorem	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt$	$\sum_{k=-\infty}^{\infty} \lvert a_k\rvert^2$	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$	$\sum_{k=0}^{N-1} \lvert a_k\rvert^2$
Running Sum	$y(t) = \displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau$	$\dfrac{a_k}{j\,k\,\omega_0} \quad (k \neq 0,\ a_0 = 0)$	$y[n] = \displaystyle\sum_{m=-\infty}^{n} x[m]$	$\dfrac{a_k}{1 - e^{-j\,\frac{2\pi}{N}k}} \quad (k \neq 0,\ a_0 = 0)$
Symmetry (Real)	$x(t)$ real	$a_k = a^*_{-k}$	$x[n]$ real	$a_k = a^*_{-k}$
Symmetry (Real+Even)	$x(t)$ real and even	a_k real and even	$x[n]$ real and even	a_k real and even
Symmetry (Real+Odd)	$x(t)$ real and odd	a_k purely imaginary and odd	$x[n]$ real and odd	a_k purely imaginary and odd