Notes for ECE 30100 - Signals and Systems

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Contents

Course Description

Classification, analysis and design of systems in both the time- and frequency-domains. Continuous-time linear systems: Fourier Series, Fourier Transform, bilateral Laplace Transform. Discrete-time linear systems: difference equations, Discrete-Time Fourier Transform, bilateral z-Transform. Sampling, quantization, and discrete-time processing of continuous-time signals. Discrete-time nonlinear systems: median-type filters, threshold decomposition. System design examples such as the compact disc player and AM radio.

Introduction

As this course studies signals and systems, it behooves us to understand what signals and systems are. A signal is a quantity that varies over time. Examples include voltage waveform on a circuit, height as a function of age, or pulses of light through fiber optic.

We distinguish between continuous time (CT) and discrete time (DT) signals. CT signals have a continuous independent variable, such as time. DT signals have a discrete independent variable, such as the date. The indices are a set of integers.

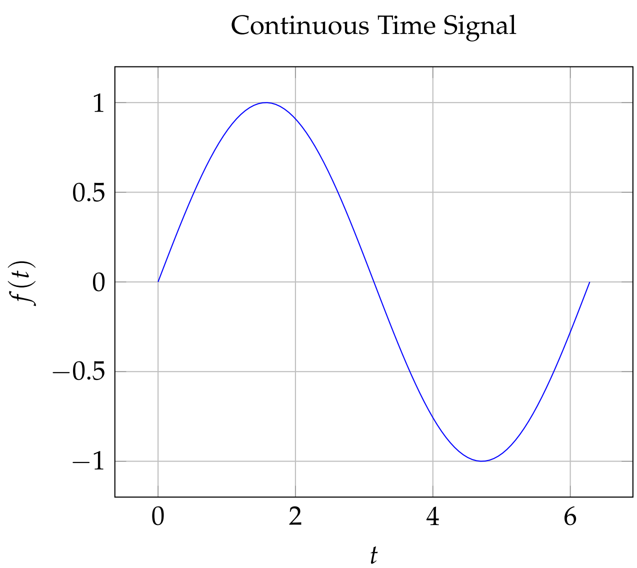


Figure 1: Continuous Time Signal

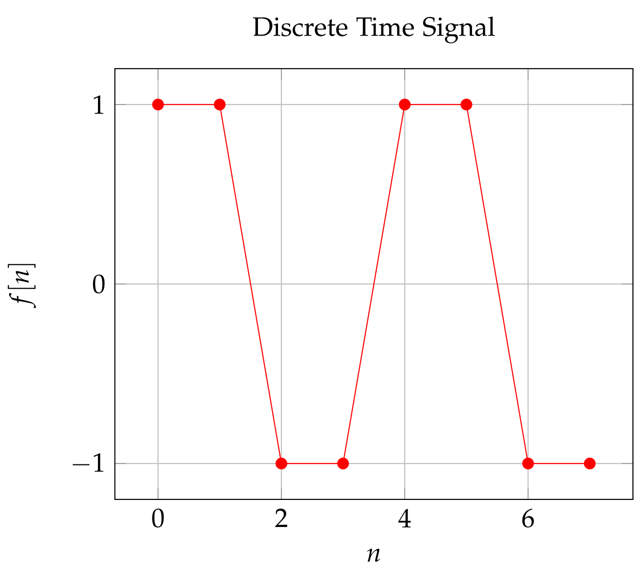


Figure 2: Discrete Time Signal

In the most general terms, a systems transform inputs to outputs. They're interconnections of subsystems. Examples of systems include, topically, circuits.

Similarly to signals, there are continuous time systems and discrete time systems. In a CT system, the input and output are continuous. Conversely, DT systems have discrete inputs and outputs.

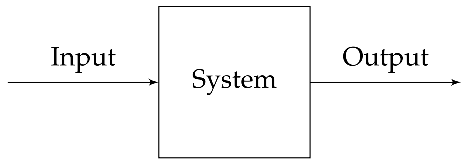


Figure 3: System Diagram

The astute reader will notice systems operate much like functions. We use function notation to describe systems. For a CT system, we write $y(t) = S(x(t))$ with parentheses to show it's CT. For DT, we use brackets like $y[t] = S(x[t])$.

It's easy to imagine a system with continuous input and discrete output, or vice versa. These are called samplers and reconstructors respectively. We'll mostly be looking at linear time-invariant discrete systems, since they have the greatest application to ECE.

Linearity

Readers are familiar with the concept of linearity, which mathematically may be expressed as

$$f(a+b) = f(a) + f(b) \tag{1}$$

Linear systems possess the property of superposition, so given an input as a sum of weighted inputs the output is a sum of weighted outputs.

The necessary and sufficient conditions for linearity in a CT system are if the input is $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ the output is $S(\alpha_1 x_1(t)) + S(\alpha_2 x_2(t))$. Formally,

$$S(\alpha_1 x_1(t) + \alpha_2 x_2(t)) = \alpha_1 S(x_1(t)) + \alpha_2 S(x_2(t)). \tag{2}$$

Likewise for DT systems,

$$S[\alpha_1 x_1[t] + \alpha_2 x_2[t]] = \alpha_1 S[x_1[t]] + \alpha_2 S[x_2[t]]. \tag{3}$$

This equality for hold for any real valued α_1 and α_2.

Consider the CT system S given by $y(t) = tx(t)$. We are interested in determining if the system is linear. We test it with the definition of linearity,

$$\begin{aligned} y(\alpha_1 x_1(t) + \alpha_2 x_2(t)) &= t(\alpha_1 x_1(t) + \alpha_2 x_2(t)) && (4)\\ &= t\alpha_1 x_1(t) + t\alpha_2 x_2(t) && (5)\\ &= \alpha_1 y(x_1(t)) + \alpha_2 y(x_2(t)) && (6) \end{aligned}$$

Since this is the definition of linearity, the system is linear.

Why do we care? We care because linearity gives us many useful properties and makes solving systems much easier. If we know the output for any set of inputs, we can find the output for any linear combination of those inputs.

Classifying Signal Types

Before we proceed we must be able to classify signal types. There are five ways to divide signal types.

- DT vs. CT
- Periodic vs. aperiodic
- Finite energy vs. finite power
- Even and odd
- Complex exponential

DT vs. CT

- DT: $x[n]$ is a sequence of complex values, including purely real values. numbers. Example: $x[n] = \frac{n}{2}$.
- CT: $x(t)$ is complex (including purely real) and continuous for all real values of t. Example: $x(t) = \frac{t}{2} - jt$.
- Complex:

For DT, complex $x[n]$ can be represented in Cartesian or polar form. For Cartesian,

$$x[n] = x_{Re}[n] + jx_{Im}[n]. \tag{7}$$

For polar,

$$x[n] = A[n]e^{j\Theta[n]}. \tag{8}$$

We can swap between the two with Euler's formula.

$$A[n]e^{j\Theta[n]} = A[n]\cos(\Theta[n]) + jA[n]\sin(\Theta[n]) \tag{9}$$

$$x_{Re}[n] + jx_{Im}[n] = \sqrt{x_{Re}[n]^2 + x_{Im}[n]^2} \times e^{j\arctan(\frac{x_{Im}[n]}{x_{Re}[n]})} \tag{10}$$

Energy vs. Power

DT vs. CT is one option to classify signals. Another possibility is Energy vs. Power. For this class, energy in a continuous time system is the area under the squared magnitude of the signal. Mathematically energy over (t_1, t_2) is equal to

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt \tag{11}$$

$$= \int_{t_1}^{t_2} (x_{Re}(t)^2 + x_{Im}(t)^2) dt. \tag{12}$$

For DT systems, the formula for energy is

$$E = \sum_{n=n_1}^{n_2} |x[n]|^2 \tag{13}$$

$$= \sum_{n=n_1}^{n_2} (x_{Re}[n]^2 + x_{Im}[n]^2). \tag{14}$$

The total energy E_∞ is the energy from $t = -\infty$ to $t = \infty$.

Power is energy per unit time, or in terms of calculus $P(t) = \frac{d}{dt}E(t)$. For CT, average power is

$$P_{avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt. \tag{15}$$

For DT,

$$P_{avg} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2. \tag{16}$$

The overall average power is

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{17}$$

for CT and

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{18}$$

for DT time.

Transformations

Just as with functions, signals can be transformed in time. Here are the different transformations that can be applied to signals.

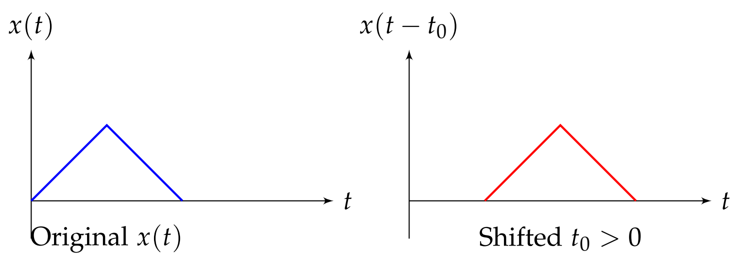
Time Shift

A CT time shift is given by $x(t) \to x(t - t_0)$, where t_0 is real.

- $t_0 > 0$: shifted to the right or delayed by t_0
- $t_0 < 0$: shifted to the left or advanced by t_0

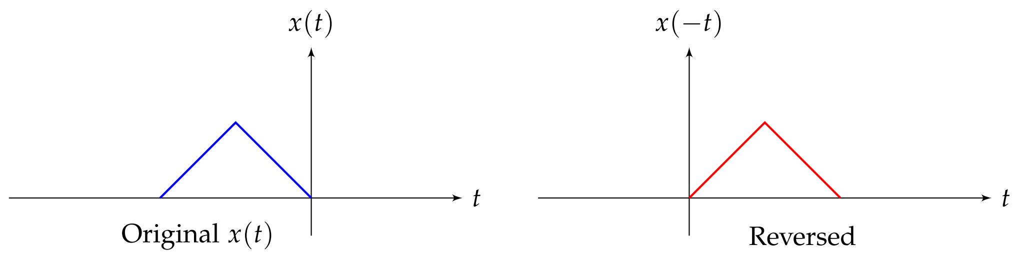
A DT time shift is given by $x[n] \to x[n - n_0]$, where n_0 is an integer.

- $n_0 > 0$: shifted to the right or delayed by n_0
- $n_0 < 0$: shifted to the left or advanced by n_0



Time Reversal

A CT time reversal is given by $x(t) \to x(-t)$. A DT time reversal is given by $x[n] \to x[-n]$.



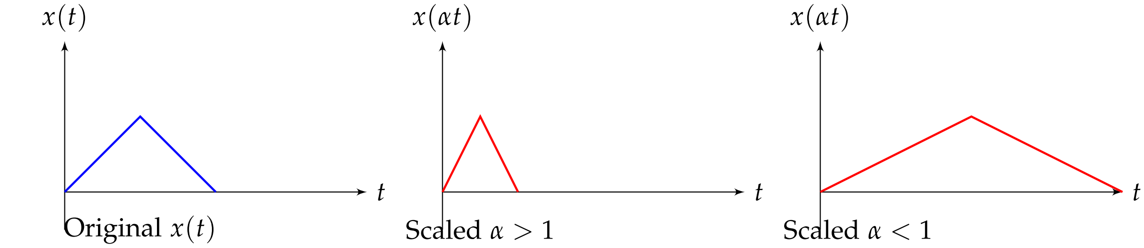
Time Scaling

A CT time scaling is given by $x(t) \to x(\alpha t)$, where $\alpha > 0$ is the time scaling factor.

If $\alpha < 0$, that's viewed as a combination of reversal and scaling.

- $\alpha > 1$: shorter timescale, or sped up
- $\alpha < 1$: longer timescale, or slowed down

A DT time scaling is given by $x[n] \to x[\alpha n]$.



A signal is even if it's symmetric with respect to the dependent axis. Mathematically, if $x(t) = x(-t)$.

A signal is odd if it's symmetric with respect to the origin. Mathematically, if $x(-t) = -x(t)$.

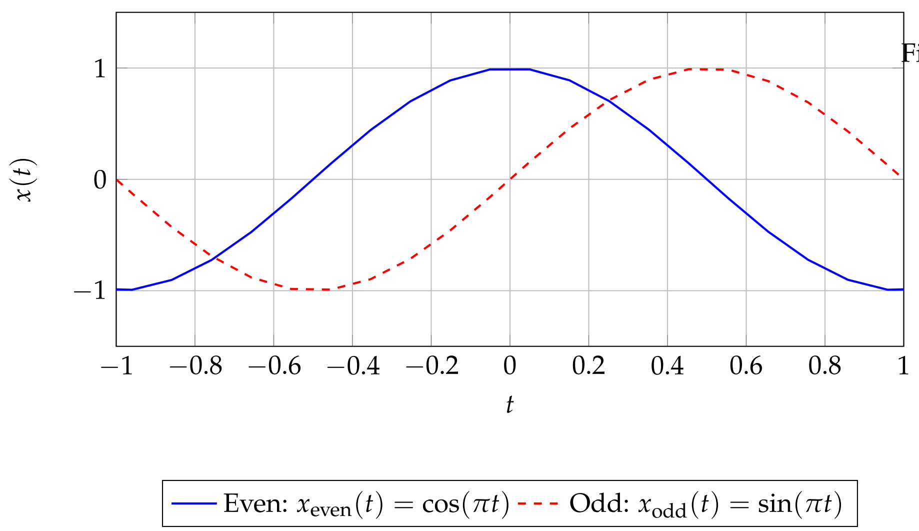


Figure 4: Even and odd signals

Signals can be odd, even, both, or neither. $x(t) = 0$, for instance, is both even and odd. $x + 1$ is neither.

The product of two odd signals is even (e.g. $x \times x^3$). The product of two evens is even ($x^2 \times 2$). The product of an odd and an even is odd ($x \times x^2$).

Any signal can be written as the sum of an even and an odd signal using these formulas:

$$
\begin{aligned}
x(t) &= x_{even}(t) + x_{odd}(t) && (19) \\
&= \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} && (20) \\
& && (21)
\end{aligned}
$$

$$
\begin{aligned}
x[n] &= x_{even}[n] + x_{odd}[n] \quad (22) \\
&= \frac{x[n] + x[-n]}{2} + \frac{x[n] - x[-n]}{2} \quad (23) \\
& \quad (24)
\end{aligned}
$$

Periodicity

A system is periodic if $x(t) = x(t+T)$, or in the case of discrete time, if $x[n] = x[n+N]$.

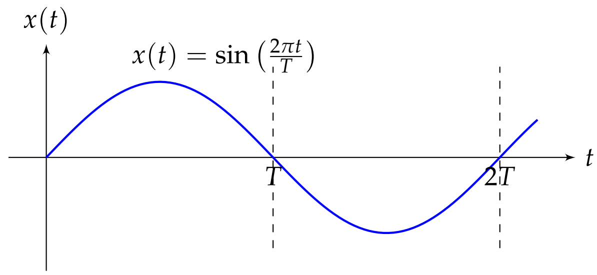


Figure 5: Periodic CT Signal

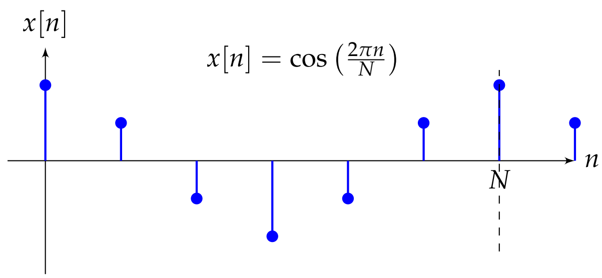


Figure 6: Periodic DT Signal

The fundemental period is the smallest T_0 (or N_0) such that $x(t) = x(t+T_0)$ (or $x[n] = x[n+N_0]$). If $x(t)$ is periodic, $x_{Re}(t) + jx_{Im}(t)$ is also periodic. However, if $x_1(t)$ and $x_2(t)$ are periodic then it is not necessarily the case that $x_1(t) + x_2(t)$ is periodic. Consider $x_1(t) = \sin(t)$ and $x_2(t) = \sin(\sqrt{2}t)$. $x_1(t) + x_2(t) = \sin(t) + \sin(\sqrt{2}t)$. $x_1(t)$ has period 2π. $x_2(t)$ has period $\frac{2\pi}{\sqrt{2}}$. However, their sum is not periodic and in fact the sum of any $x_1(t)$, $x_2(t)$ will be aperiodic when the ratio of their periods is irrational. To get the average power of a periodic signal, we can just calculate the power over one period.

Periodicty in Discrete Time

Periods in discrete time differ in some unintuitive ways from the continuous time case. For instance, in CT, two functions with different ω will never represent the same function. However, in DT, they can if $\omega_2 = \omega_1 + 2\pi n$. Functions like $x[n] = \cos(n)$ are aperiodic in discrete time since the period as given by $\frac{2\pi}{\omega}$ must be a rational number, and in this case it's 2π. When $\omega = k\pi$, $k = 1, 3, 5, \ldots$ and $e^{jkn} = (-1)^n$. In general, if

$$x_k[n] = e^{jk\frac{2\pi}{N}n} \tag{25}$$

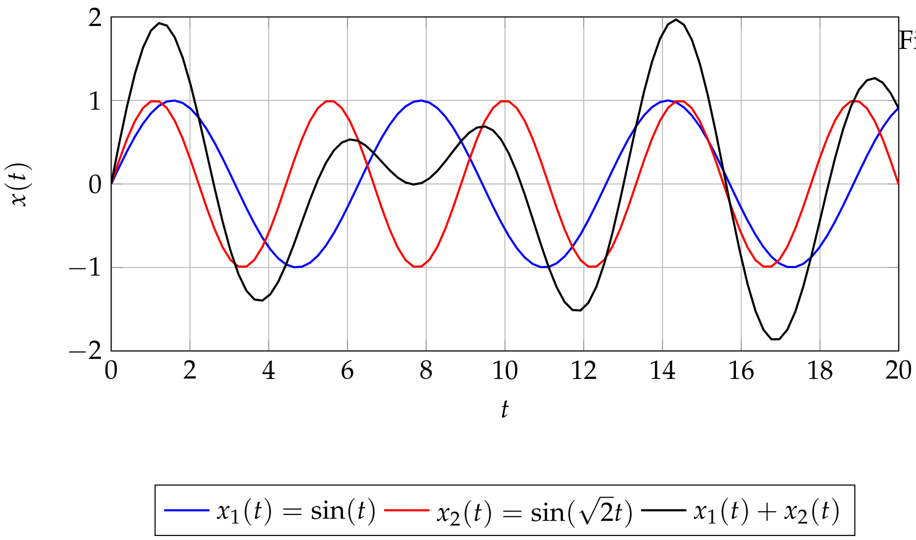


Figure 7: Periodic Signals Sum

then $x_k[n]$ is unique only for $k = 0, 1, \ldots, N - 1$. Also in general,

$$N_k = LCM(\frac{N}{k}, 1) \tag{26}$$

Unit Step Signal

The unit step signal, also known as the Heaviside step function, is 0 for values less than 0 and 1 otherwise.

Discrete-Time Unit Step Signal

The discrete-time unit step signal is defined as:

$$u[n] = \begin{cases} 1 & n \geq 0, \\ 0 & n < 0. \end{cases}$$

Or alternatively,

$$u[n] = \sum_{i=-\infty}^{n} \delta[i] \tag{27}$$

In either definition,

$$u[n] = 1 \text{ for } n \geq 0, \text{ and } u[n] = 0 \text{ for } n < 0.$$

Figure 8 is the plot for the discrete-time unit step signal.

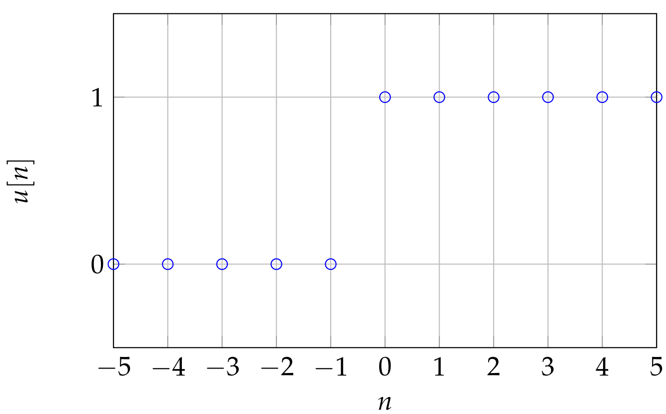


Figure 8: Discrete-Time Unit Step Signal

A useful property for signal sampling is that, if we just want to consider a section of the signal between 0 and n_0, we can just multiply $x[n]$ by $u[n] - u[n - n_0]$.

Continuous-Time Unit Step Signal

The continuous-time unit step signal is defined as:

$$u(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t < 0. \end{cases}$$

Or alternatively,

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \tag{28}$$

We can also make the substitution $\sigma = t - \tau$, which gives

$$u(t) = \int_0^{\infty} \delta(t - \sigma) d\sigma \tag{29}$$

Figure 9 is the plot for the continuous-time unit step signal:

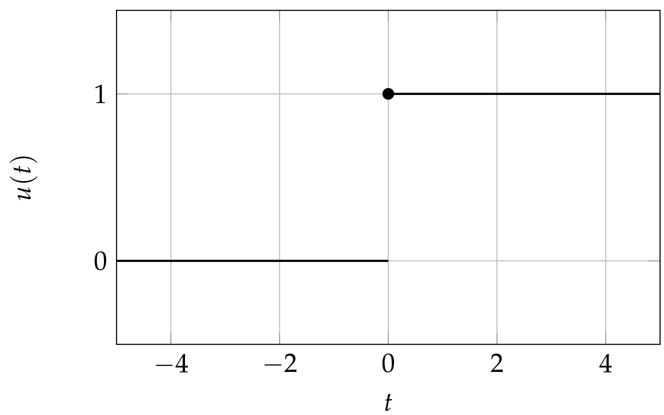


Figure 9: Continuous-Time Unit Step Signal

Discrete-Time Delta Function

The discrete-time delta function, known as the Kronecker delta function, is defined as:

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Figure 10 is the plot for the discrete-time delta function:

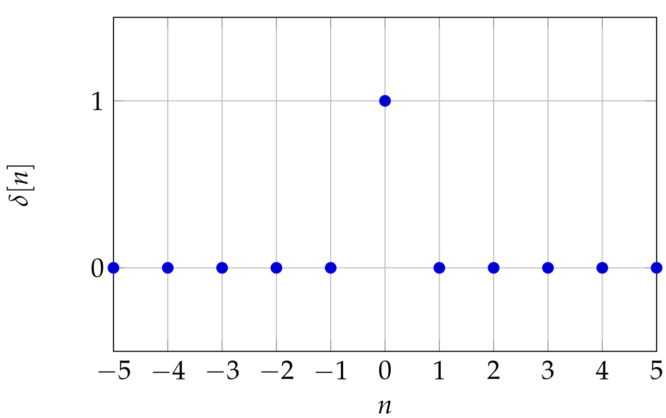


Figure 10: Discrete-Time Delta Function

A useful property of the delta function in DT is that, since it is just equal to one when its argument is 0, then

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{30}$$

Continuous-Time Delta Function

The continuous-time delta function, known as the Dirac delta function or unit impulse, is defined as:

$$\delta(t) = \begin{cases} \infty & t = 0, \\ 0 & t \neq 0, \end{cases}$$

with the property that:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1.$$

An alternative definition for $\delta(t)$ is

$$\delta(t) = \frac{d}{dt} u(t) \tag{31}$$

Figure 11 is the plot for the continuous-time delta function. The height of the arrow indicates not the value of the function but its area, since the value is technically undefined.

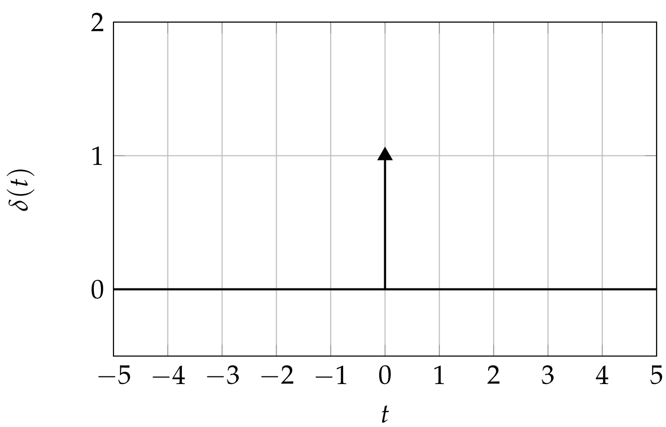


Figure 11: Continuous-Time Delta Function

The *sampling property* is the property that

$$x(t)\delta(t - \sigma) = x(\sigma)\delta(t - \sigma). \tag{32}$$

We can decompose $x(t)$

$$\begin{aligned} x(t) &= x(t) \times 1 && (33) \\ &= x(t) \int_{-\infty}^{\infty} \delta(t - \sigma) d\sigma && (34) \\ &= \int_{-\infty}^{\infty} x(t)\delta(t - \sigma) d\sigma && (35) \\ &= \int_{-\infty}^{\infty} x(\sigma)\delta(t - \sigma) d\sigma. && (36) \end{aligned}$$

In DT, this becomes

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k] \tag{37}$$

Complex Exponential

A CT complex exponential signal is of the form $x(t) = Ce^{\alpha t}$, where C and α are in general complex. Alternatively,

$$x(t) = |C|e^{\sigma t}e^{j(\omega t+\phi)} \tag{38}$$

where ϕ is the angle between the real axis and C when plotted on the complex plane and $\alpha = \sigma + j\omega$.

The σ term determines whether the signal has exponential growth, decay, or neither. If $\sigma = 0$ then we are left with the periodic complex exponential $e^{j(\omega t+\phi)}$ with period $\frac{2\pi}{\omega}$.

ω is called the fundemental frequency, ϕ is called the phase.

The signal $x(t) = |C|e^{\sigma t}e^{j(\omega t+\phi)}$ forms a family of signals called harmonically related complex exponentials (HRCEs), each of the form

$$x_k(t) = e^{jk\omega_0 t}, k \in \mathbb{Z}. \tag{39}$$

These signals will serve as our building blocks when we construct Fourier series of complex exponential signals later on.

Let's now look at the discrete time complex exponential,

$$x[n] = C\alpha^n. \tag{40}$$

In general, C and α can be complex. This can be rewritten

$$x[n] = |C|e^{\sigma n}e^{j(\omega n+\phi)}. \tag{41}$$

Where the continuous and discrete begin to diverge is when we consider the $e^{j(\omega n+\phi)}$ term. This term is not always periodic, unlike the case of continuous time. For the CT case, the fundemental frequency is ω and larger values of ω produce higher rates of oscillation.

Now say we want to compute the fundemental period of $x[n] = \cos(3\pi n)$. In the continuous case it's easy, $\frac{2}{3}$. In the discrete case, we need to find the least common multiple of the fundemental period of the CT signal (in this case, $\frac{2}{3}$) and the sampling period. This is why the discrete case may not be periodic. Imagine the sampling period is 1 and the fundemental period of the CT signal is 2π. Then the least common multiple does not exist.

Let's do an example problem. Consider the signal

$$x(t) = e^{j2t} + e^{j5t}. \tag{42}$$

We can rewrite this, using the identity

$$\cos(\omega t) = \frac{e^{-j\omega t} + e^{j\omega t}}{2} \tag{43}$$

to the expression

$$e^{j3.5t}(e^{j1.5t} + e^{-j1.5t}) \tag{44}$$

$$= 2e^{j3.5t}\cos(1.5t) \tag{45}$$

$$|x(t)| = 2|\cos(1.5t)| \tag{46}$$

$$x[n] = C\alpha^n \tag{47}$$

$$= Ce^{\beta n} \tag{48}$$

is the general form of a discrete complex exponential and β is in general complex. If α is real and less than 0, then β must be complex.

System Connections

Often we analyze a complex system as a set of subsystems connected to one another. These connections can take many forms, but some of the more common ones are in series, parallel, and feedback.

Series

$$y(t) = S_2(S_1(x(t))) \tag{49}$$

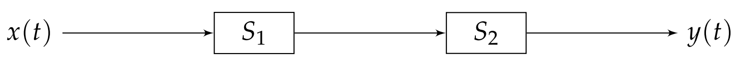


Figure 12: Series connection block diagram.

Parallel

$$y(t) = S_1(x(t)) + S_2(x(t)) \tag{50}$$

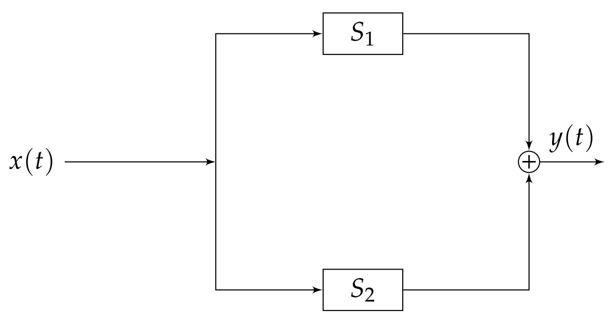


Figure 13: connection block diagram.

Feedback

$$y(t) = S_1\left(x(t) - S_2(y(t))\right) \tag{51}$$

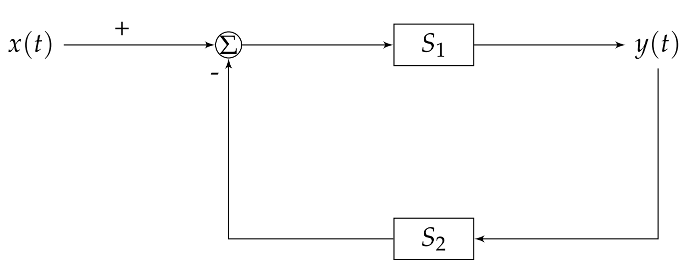


Figure 14: connection block diagram.

System Properties

Memoryless

A *memoryless* system's output depends only on the input at time t (or n for discrete-time systems), and not on past or future states. For example, $y(t) = 2x(t)$ is memoryless, while $y[n] = x[n-1]$ has memory.

Recall that for any system,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{52}$$

For a memoryless system, $y[n]$ must be dependent only on $x[n]$ and not some past or future inputs. That forces $h[n]$ to be 0 for all values of n except 0. Thus, for a memoryless system, the impulse response is the delta function times some scalar.

Linear

A system is *linear* if it satisfies superposition and homogeneity. Formally, for any inputs $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, and constants a, b, the system satisfies:

$$S\{ax_1(t) + bx_2(t)\} = aS\{x_1(t)\} + bS\{x_2(t)\} = ay_1(t) + by_2(t).$$

An example is $y(t) = kx(t)$, where k is a constant.

Time Invariance

A system is *time invariant* if a time shift in the input results in an identical shift in the output. For a discrete-time system S, if:

$$x[n] \to y[n],$$

then for any integer n_0:

$$x[n-n_0] \to y[n-n_0].$$

Informally, a shift in the input signal must result in the output signal being shifted the same amount.

For continuous-time systems, this becomes $x(t-\tau) \to y(t-\tau)$. Time-dependent operations (e.g., $y(t) = x(2t)$) or explicit time-varying components make a system *time variant*.

Linear Time Invariant (LTI)

An LTI system satisfies both linearity and time invariance. LTI systems are fully characterized by their impulse response $h(t)$ or $h[n]$, enabling analysis via convolution: $y(t) = x(t) * h(t)$. The impulse response is found by plugging in $\delta(t)$ or $\delta[n]$ to the system. The result is the impulse response. Mathematically,

$$h(t) = S(\delta(t)) \tag{53}$$
$$h[n] = S(\delta[n]). \tag{54}$$

Invertible

A system is *invertible* if distinct inputs produce distinct outputs. If there exist $x_1(t) \neq x_2(t)$ such that $y_1(t) = y_2(t)$, the system is not invertible. Examples include:

- **Modulation**: $y(t) = x(t)\cos(\omega t)$ (invertible with synchronous demodulation).
- **Sampling**: Invertible if the Nyquist criterion is satisfied.
- **Encoding**: Lossy compression (e.g., JPEG) is non-invertible.

For an invertible system S_1, there exists an inverse S_2 such that:

$$S_2(S_1(x(t))) = S_1(S_2(x(t))) = x(t).$$

It would be useful to have a general method of determining the inverse, so that we can algorithmically find the inverse or prove a system is noninvertable. For LTI systems we know that $y[n] = x[n] * h[n]$. When this is fed into the inverse system we get $x[n] = x[n] * h[n] * h_I[n]$. So the impulse response of the inverse system convolved with the impulse response of the original system must be δ. We will find a perfectly systematic way of determining $h_I[n]$ when we examine Fourier transforms in the section on **??**.

Causal

A system is *causal* if its output at time t depends only on present and past inputs. All memoryless systems are causal (e.g., $y(t) = x(t)^2$). A non-causal system (e.g., moving average $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$) requires future inputs.

For causal systems, the impulse response must be 0 for values of n (t) less than 0.

Stable

A system is *stable* if bounded inputs produce bounded outputs. Formally, there exist constants $B, M > 0$ such that:

$$|x(t)| < B \ \forall t \implies |y(t)| < M \ \forall t.$$

Examples:

- Stable: $y(t) = e^{-t}x(t)$.
- Unstable: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ (unbounded integral for a constant input).

For a stable LTI system, the impulse response must be absolutely summable. That is,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{55}$$

Or for continuous time, absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty. \tag{56}$$

Convolution

The *convolution* of a signal x with a signal h is given by

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{57}$$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{58}$$

Graphically, we can envision this as flipping h and sliding it along the horizontal axis, finding and summing the area of h times x as we slide h along. x remains stationary, so the bounds of the integral (or summation in the discrete case) are determined by the window of overlap between x and the flipped h (see figure 15).

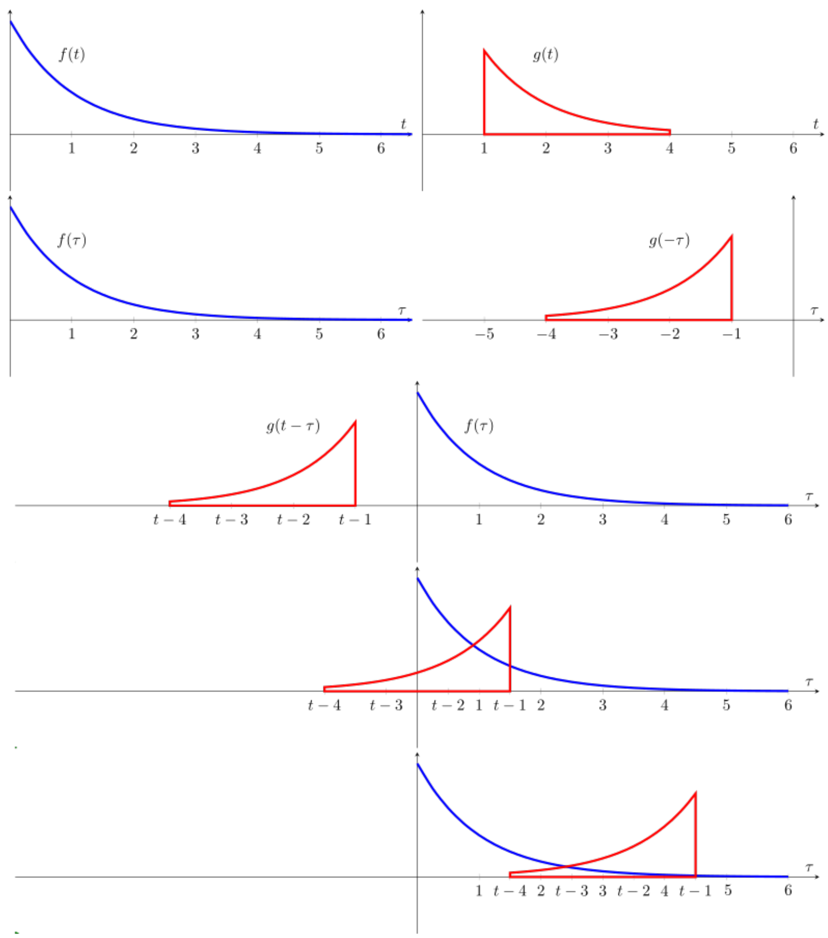


Figure 15: Convolution

This definition may seem arbitrary at first, but convolution has incredibly useful properties in signal analysis. For instance, an LTI system is completely characterized by its impulse response h, the value obtained by plugging in δ to the system, with $y = x * h$.

Recall the sifting property of the unit impulse,

$$x(t) = \int_{-\infty}^{\infty} x[\tau]\delta[\tau - t]d\tau \tag{59}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{60}$$

This property is useful in that it allows us to represent $x(t)$ or $x[n]$ as a series of scaled very simple functions. It also tells us that x convolved with δ yields x.

Convolution has the following properties:

- Commutativity: $x_1(t) * x_2(t) = x_2(t) * x_1(t)$
- Distributivity over addition: $x_1(t) * (x_2(t) + x_3(t)) = x_1(t) * x_2(t) + x_1(t) * x_3(t)$
- Associative: $x_1(t) * (x_2(t) * x_3(t)) = (x_1(t) * x_2(t)) * x_3(t)$

Fourier Series

Continuous Time Fourier Series

Consider an arbitrary periodic CT signal $x(t)$ with fundemental period T and fundemental frequency ω_0. The Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{61}$$

$$= a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{2j\omega_0 t} + \dots \tag{62}$$

where a_k is the kth Fourier coefficient and can be found with the formula

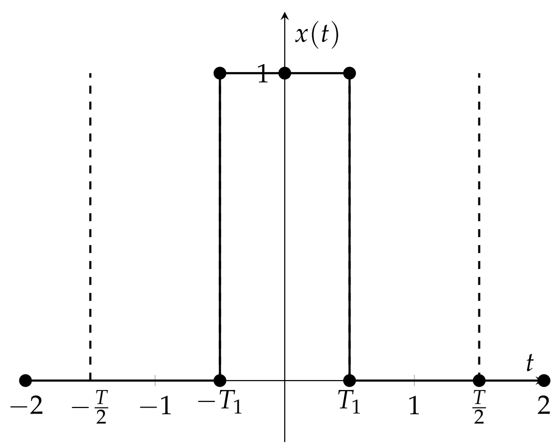
$$a_k = \frac{1}{T} \int_{<T>} x(t) e^{-jk\omega_0 t} dt \tag{63}$$

with $< T >$: $[0, T], [-\frac{T}{2}, \frac{T}{2}, ...]$. Equation 61 is known as the synthesis equation. The process of of finding a_k is Fourier analysis.

Notably, a_0 gives the DC component of the signal. In general the $a_{\pm k}$ are the kth harmonic components.

Let's see an example. Let $x(t)$ be a square wave given by

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases} \tag{64}$$



We calculate a_k as

$$a_k = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt \tag{65}$$
$$= -\frac{1}{jk\omega_0 T}\left[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right] \tag{66}$$
$$= \frac{1}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{j}\right] \tag{67}$$
$$= \frac{2}{\omega_0 T}\sin(k\omega_0 T_1) \tag{68}$$
$$= \frac{1}{\pi k}\sin(k\omega_0 T_1) \tag{69}$$

This expression for a_0 is fine, however, there is a problem. When $k = 0$ we have a discontinuity. In general we may have to find a_0 separately. It's not a big deal: $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} x(t)dt$ in general''.

Discrete Time Fourier Series

In discrete time, an arbitrary periodic signal $x[n]$ with fundemental period N and fundemental frequency ω_0. The Fourier series representation of $x[n]$ is

$$x[n] = \sum_{k=<N>} a_k x_k[n] \tag{70}$$
$$= \sum_{k=<N>} a_k e^{jk\omega_0 n} \tag{71}$$

This expression yields a system of n linear equations with n unknowns, namely $x[0], x[1], \ldots, x[n]$. In theory we could solve this, perhaps using computers or linear algebra, but in practice there is a simpler way. Multiply equation 70 by $e^{-jr\frac{2\pi}{N}n}$ for any r and sum over N terms. Then

$$\sum_{n=<N>} x_k[n]e^{-jr\frac{2\pi}{N}n} = \sum_{n=<N>} a_k \sum_{n=<N>} e^{j(k-r)(\frac{2\pi}{N})n}. \tag{72}$$

Solving for a_k,

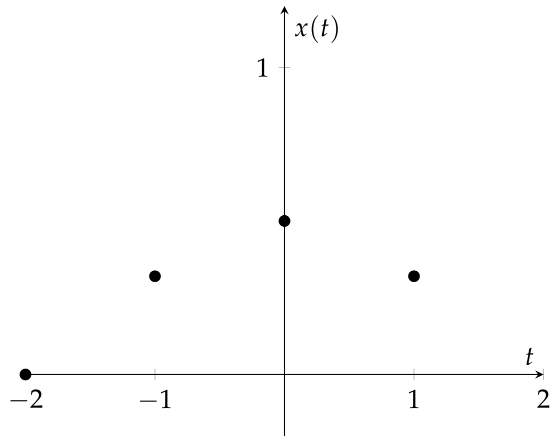
$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}, \tag{73}$$

which is the synthesis equation in discrete time.

A fun property is that $a_{k+N} = a_k$, so a_k are periodic in N.

Spectrum Analysis

a_k as a function of k given the *spectrum* of $x(t)$. Let's see an example: let $T_1 = \frac{T}{4}$. Then $a_k = \frac{\sin(\pi\frac{k}{2})}{k\pi}$.



Fourier Series Properties

Some useful properties follow from the definition of the Fourier series.

Property	Time Domain		Frequency Domain
Linearity	$Ax_1(t) + Bx_2(t)$	$\overset{FS}{\leftrightarrow}$	$Aa_k + Bb_k$
Even Symmetry	$x(t)$ even	$\overset{FS}{\leftrightarrow}$	a_k even
Odd Symmetry	$x(t)$ odd	$\overset{FS}{\leftrightarrow}$	a_k odd
Time Shifting	$x(t - t_0)$	$\overset{FS}{\leftrightarrow}$	$a_k e^{-jk\omega_0 t_0}$
Frequency Shifting	$x(t)e^{jn\omega_0 t}$	$\overset{FS}{\leftrightarrow}$	a_{k-n}
Time Reversal	$x(-t)$	$\overset{FS}{\leftrightarrow}$	a_{-k}
Conjugation	$x^*(t)$	$\overset{FS}{\leftrightarrow}$	a^*_{-k}
Periodic Convolution	$(x * y)(t)$	$\overset{FS}{\leftrightarrow}$	$a_k b_k$
Multiplication	$x(t)y(t)$	$\overset{FS}{\leftrightarrow}$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$
Differentiation	$\frac{d}{dt}x(t)$	$\overset{FS}{\leftrightarrow}$	$jk\omega_0 a_k$
Integration	$\int x(t)dt$	$\overset{FS}{\leftrightarrow}$	$\frac{a_k}{jk\omega_0}$ if $a_0 = 0)$
Parseval's Theorem	$\frac{1}{T}\int_T \|x(t)\|^2 dt$	$\overset{FS}{\leftrightarrow}$	$\sum_{k=-\infty}^{\infty} \|a_k\|^2$

6

Property	**Time Domain**		**Frequency Domain**
Linearity	$A\,x_1[n] + B\,x_2[n]$	$\overset{DTFS}{\leftrightarrow}$	$A\,a_k + B\,b_k$
Even Symmetry	$x[n]$ even (i.e., $x[n] = x[-n]$)	$\overset{DTFS}{\leftrightarrow}$	a_k even
Odd Symmetry	$x[n]$ odd (i.e., $x[n] = -x[-n]$)	$\overset{DTFS}{\leftrightarrow}$	a_k odd
Time Shifting	$x[n - n_0]$	$\overset{DTFS}{\leftrightarrow}$	$a_k\, e^{-j\frac{2\pi}{N}kn_0}$
Frequency Shifting	$x[n]\, e^{j\frac{2\pi}{N}n_0 n}$	$\overset{DTFS}{\leftrightarrow}$	$a_{(k-n_0) \bmod N}$
Time Reversal	$x[-n]$	$\overset{DTFS}{\leftrightarrow}$	a_{-k} (indices mod N)
Conjugation	$x^*[n]$	$\overset{DTFS}{\leftrightarrow}$	a^*_{-k}
Circular Convolution	$(x \circledast y)[n]$	$\overset{DTFS}{\leftrightarrow}$	$a_k\, b_k$
Multiplication	$x[n]\, y[n]$	$\overset{DTFS}{\leftrightarrow}$	$\frac{1}{N}\sum_{m=0}^{N-1} a_m\, b_{(k-m) \bmod N}$
Difference Operator	$x[n] - x[n-1]$	$\overset{DTFS}{\leftrightarrow}$	$a_k\left(1 - e^{-j\frac{2\pi}{N}k}\right)$
Parseval's Theorem	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$	$\overset{DTFS}{\leftrightarrow}$	$\sum_{k=0}^{N-1} \lvert a_k\rvert^2$

Fourier Transforms

Throughout this class we will sometimes see the expression $X(j\omega)$ or $X(e^{j\omega})$. Note that these are defined as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{74}$$

for CT and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{75}$$

for DT. This transformation is known as the *Fourier Transform* and works for well-behaved signals. A signal $x(t)$ is well-behaved iff

- $x(t)$ is absolutely integrable over its domain.
- $x(t)$ has finite maximum and minimum.
- $x(t)$ has finite discontinuities.

The original signal can be recovered with

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{76}$$

in CT and

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \tag{77}$$

in DT.

If the Fourier series of

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{78}$$

then the Fourier transform is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0). \tag{79}$$

This property quickly tells us that the Fourier transform of $e^{jk\omega_0 t}$ is $2\pi\delta(\omega - k\omega_0)$. From this the Fourier transforms of sinusoidals quickly follow.

Another important periodic signal is the impulse train,

$$\sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{80}$$

The Fourier series coefficients for the impulse train are simply $a_k = \frac{1}{T}$, so the Fourier transform is $X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$

Note that as T increases, the impulse train gets wider and wider while its Fourier transform gets narrower and narrower. This is because, as we will see often, time and frequency are inversely related. In general, we will find that

$$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{j\omega}{a}). \tag{81}$$

If $x(t)$ is real, then $|X(j\omega)|$ is even and $\angle X(j\omega)$ is odd. Moreover, if $x(t)$ is real and even, then $X(j\omega)$ must be purely real and even, and if $x(t)$ is real and odd, $X(j\omega)$ is purely imaginary and odd.

Arguably the most important property we will see is duality. As an example, a sinc function in time is a rectangular pulse in frequency, while a rectangular pulse in time is a sinc function in frequency. Formally,

$$F\{X(t)\} = 2\pi x(-\omega). \tag{82}$$

Duality

The Fourier transform displays an interesting symmetry known as duality. Figure 16 shows this graphically. Mathematically, if

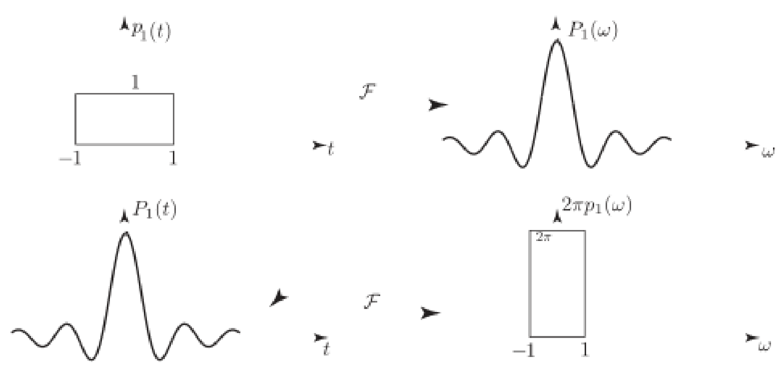


Figure 16: Duality of the Fourier Transform

$$x(t) \overset{F}{\leftrightarrow} X(j\omega) \tag{83}$$

then

$$X(t) \overset{F}{\leftrightarrow} 2\pi x(-\omega). \tag{84}$$

It doesn't *quite* give you the original function, but up to scaling and a reversal you get the original back.

Some interesting properties derive from duality. Notable ones are:

- Differentiation in frequency: If $x(t) \overset{F}{\leftrightarrow} X(j\omega)$, then

$$tx(t) \overset{F}{\leftrightarrow} j\frac{dX(j\omega)}{d\omega}. \tag{85}$$

This is the dual property of differentiation in time,

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\leftrightarrow} j\omega X(j\omega). \tag{86}$$

- Frequency shifting: this is the dual of shifting in time.

$$x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\leftrightarrow} X(j(\omega-\omega_0)). \tag{87}$$

- Parseval's relation:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{88}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \tag{89}$$

Fourier Transform Properties

Table 1 tabulates useful properties of Fourier transforms.

Property	Time Domain		Frequency Domain
Linearity	$Ax_1(t) + Bx_2(t)$	$\overset{\mathcal{F}}{\leftrightarrow}$	$AX_1(j\omega) + BX_2(j\omega)$
Time Shifting	$x(t-t_0)$	$\overset{\mathcal{F}}{\leftrightarrow}$	$X(j\omega)e^{-j\omega t_0}$
Frequency Shifting	$x(t)e^{j\omega_0 t}$	$\overset{\mathcal{F}}{\leftrightarrow}$	$X(j(\omega-\omega_0))$
Time Scaling	$x(at)$, $a > 0$	$\overset{\mathcal{F}}{\leftrightarrow}$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Time Reversal	$x(-t)$	$\overset{\mathcal{F}}{\leftrightarrow}$	$X(-j\omega)$
Conjugation	$x^*(t)$	$\overset{\mathcal{F}}{\leftrightarrow}$	$X^*(-j\omega)$
Differentiation	$\frac{d^n}{dt^n}x(t)$	$\overset{\mathcal{F}}{\leftrightarrow}$	$(j\omega)^n X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\overset{\mathcal{F}}{\leftrightarrow}$	$\frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Convolution	$(x_1 * x_2)(t)$	$\overset{\mathcal{F}}{\leftrightarrow}$	$X_1(j\omega)X_2(j\omega)$
Multiplication	$x_1(t)x_2(t)$	$\overset{\mathcal{F}}{\leftrightarrow}$	$\frac{1}{2\pi}(X_1 * X_2)(j\omega)$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$\overset{\mathcal{F}}{\leftrightarrow}$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$

Table 1: Fourier Transforms

Table 2 tabulates the Fourier transform of common continuous time functions.

Table 3 tabulates the Fourier transform of common discrete time functions.

The most important part of this entire course is the *convolution property*, which states that for an LTI system with impulse response $h(t)$, if

$$y(t) = x(t) * h(t), \tag{90}$$

then

$$Y(j\omega) = X(j\omega)H(j\omega). \tag{91}$$

Table 2: CT Fourier Transforms

Time Domain Function	**Fourier Transform**
$\delta(t)$	1
1	$2\pi\,\delta(\omega)$
$u(t)$	$\pi\,\delta(\omega) + \frac{1}{j\omega}$
$e^{-at}u(t), \quad \Re(a) > 0$	$\dfrac{1}{a + j\omega}$
$\cos(\omega_0 t)$	$\pi\Big[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\Big]$
$\sin(\omega_0 t)$	$\dfrac{\pi}{j}\Big[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\Big]$
$\text{rect}(t/T)$	$T\,\text{sinc}\Big(\dfrac{\omega T}{2}\Big)$
e^{-t^2}	$\sqrt{\pi}\,e^{-\omega^2/4}$

Table 3: DT Fourier Transforms

Time Domain Function	**Fourier Transform**
$\delta[n]$	1
1	$2\pi\,\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k)$
$a^n u[n], \quad \lvert a\rvert < 1$	$\dfrac{1}{1 - ae^{-j\omega}}$
$\cos(\omega_0 n)$	$\pi\,\sum_{k=-\infty}^{\infty}\Big[\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k)\Big]$
$\sin(\omega_0 n)$	$\dfrac{\pi}{j}\,\sum_{k=-\infty}^{\infty}\Big[\delta(\omega - \omega_0 - 2\pi k) - \delta(\omega + \omega_0 - 2\pi k)\Big]$
$\text{rect}\Big(\frac{n}{N}\Big)$	$\dfrac{\sin\Big(\frac{\omega N}{2}\Big)}{\sin\Big(\frac{\omega}{2}\Big)}\,e^{-j\omega\frac{N-1}{2}}$

Essentially, convolution in the time domain is equivalent to multiplication in the frequency domain.

We can use this property to easily calculate the impulse response to LTI systems. Consider

$$y(t) = x(t - t_0). \tag{92}$$

Say we want to find $H(j\omega)$. One way would could do this is to find $h(t)$ and calculate the Fourier transform. However, with the convolution property we could also just do this:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \tag{93}$$

$$= \frac{X(j\omega)e^{-j\omega t_0}}{X(j\omega)} \tag{94}$$

$$= e^{-j\omega t_0} \tag{95}$$

The caveat to this is that $X(j\omega)$ must be nonzero for all ω. Popular choices for $x(t)$ include the rectangular pulse, scaled exponentials, and δ function.

Suppose we have an input to an LTI system with $x(t) = e^{-bt}u(t)$, and $h(t) = e^{-at}u(t)$, with $b > 0$ and $a > 0$. Then

$$Y(j\omega) = X(j\omega)H(j\omega) \tag{96}$$
$$= \frac{1}{b+j\omega}\frac{1}{a+j\omega} \tag{97}$$
$$= \frac{1}{(b+j\omega)(a+j\omega)} \tag{98}$$

Use partial fraction decomposition to perform a reverse Fourier transform on $Y(j\omega)$ and you'll get

$$y(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t) \tag{99}$$

provided $a \neq b$. If $a = b$, then

$$Y(j\omega) = \frac{1}{(a+j\omega)^2} \tag{100}$$
$$y(t) = te^{-at}u(t) \tag{101}$$

This property finally lets us determine if a system is invertible. Previously, we said that a system is invertible if $h(t) * h^{-1}(t) = \delta(t)$ for some inverse $h^{-1}(t)$. Essentially, saying a system is invertible if an inverse exists. Really begs the question of how to determine if an inverse exists. Taking the Fourier transform of both sides gives us

$$H(j\omega)H^{-1}(j\omega) = 1. \tag{102}$$

Rearranging,

$$H^{-1}(j\omega) = \frac{1}{H(j\omega)}. \tag{103}$$

This only works if $H(j\omega) \neq 0$ for all ω. Therefore, an LTI system is only invertible if $H(j\omega) \neq 0 \forall \omega$.

Modulation

Signal modulation is the process of varying one or more properties of a periodic waveform in electronics and telecommunication for the purpose of transmitting information on a shared medium where signals might otherwise interfere. You can change the amplitude, the frequency, the phase, all with different pros and cons.

The modulation property is as follows:

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}\left(X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right) \tag{104}$$

for CT and

$$x[n]\cos(\omega_0 n) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2}\left(X(e^{j(\omega-\omega_0)}) + X(e^{j(\omega+\omega_0)})\right) \tag{105}$$

for DT.

Consider a generic signal $s(t)$ and its spectrum, $S(j\omega)$. Say the spectrum is band-limited. That is, its is only nonzero between $-\omega_1$ and ω_1, with $s(0) = A$ being the max amplitude.

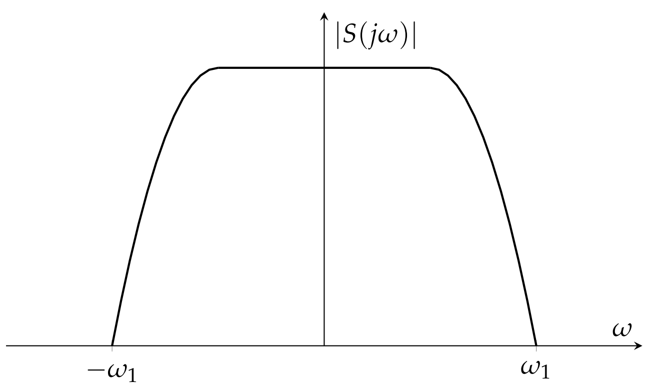


Figure 17: Spectrum of $s(t)$

Consider a modulating signal $x(t)$ and carrier signal $c(t)$. The modulated signal is $y(t) = x(t)c(t)$. We can have $c(t) = e^{j(\omega_c t+\phi_c)}$ or even just $c(t) = \cos(\omega_c t + \phi_c)$. In either case ω_c is called the carrier frequency.

Sampling

Under certain conditions, a CT signal can be completely represented by and recovered from a set of sampled values. This fact shocked me when I first heard it. In general, if you consider n points, you would expect that infinitely many possible CT signals could pass through the given points, as in Figure 18. The conditions under which a set of

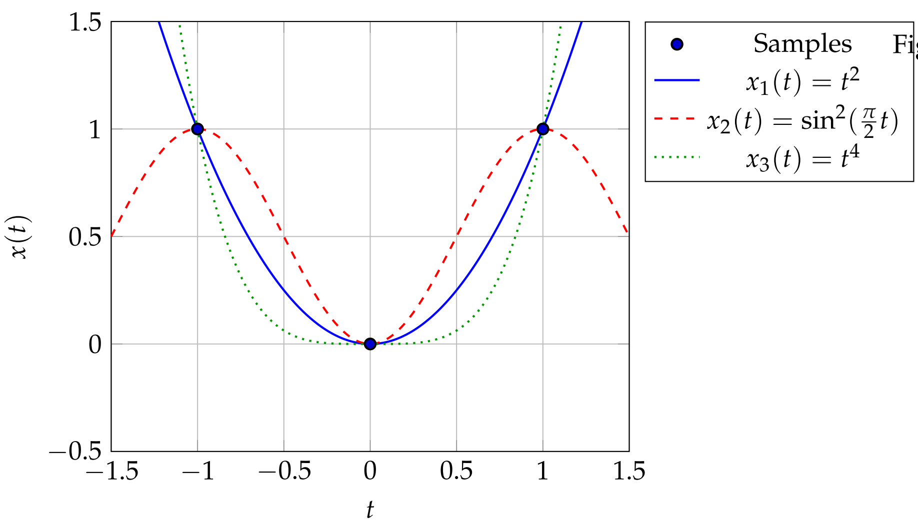


Figure 18: Sampling Ambiguity

samples uniquely determine a CT signal is if its Fourier transform is zero outside a finite band of frequencies, and if the samples are taken sufficiently close together in relation to the highest frequency present in the signal. The result is known as the *sampling theorem*, and it is both surprising and extremely useful.

Sampling Theorem

Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \dots$ if

$$\omega_s > 2\omega_M, \tag{106}$$

where

$$\omega_s = \frac{2\pi}{T}. \tag{107}$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which succesive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain T and cutoff frequency greater than ω_M and less than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.