

## Sum of Squares

$$\sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

## Sum of Geometric Series

$$\sum_{k=0}^n ar^k = a \frac{1-r^{n+1}}{1-r}$$

## Sum of Weighted Geometric Series

$$\sum_{k=0}^n kr^k = r \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$$

## Memoryless

A *memoryless* system's output depends only on the input at time  $t$  (or  $n$  for discrete-time systems), and not on past or future states. For example,  $y(t) = 2x(t)$  is memoryless, while  $y[n] = x[n-1]$  has memory.

## Linearity

A system is *linear* if it satisfies superposition and homogeneity. Formally, for any inputs  $x_1(t) \rightarrow y_1(t)$  and  $x_2(t) \rightarrow y_2(t)$ , and constants  $a, b$ , the system satisfies:

$$S\{ax_1(t) + bx_2(t)\} = aS\{x_1(t)\} + bS\{x_2(t)\} = ay_1(t) + by_2(t).$$

An example is  $y(t) = kx(t)$ , where  $k$  is a constant.

## Time Invariance

A system is *time invariant* if a time shift in the input results in an identical shift in the output. For a discrete-time system  $S$ , if:

$$x[n] \rightarrow y[n],$$

then for any integer  $n_0$ :

$$x[n - n_0] \rightarrow y[n - n_0].$$

## Linear Time Invariance

An LTI system satisfies both linearity and time invariance. LTI systems are fully characterized by their impulse response  $h(t)$  or  $h[n]$ , enabling analysis via convolution:  $y(t) = x(t) * h(t)$ . The impulse response is found by plugging in  $\delta(t)$  or  $\delta[n]$  to the system. The result is the impulse response. Mathematically,

$$\begin{aligned} h(t) &= S(\delta(t)) \\ h[n] &= S(\delta[n]). \end{aligned}$$

## Causality

A system is *causal* if its output at time  $t$  depends only on present and past inputs. For causal systems, the impulse response must be 0 for values of  $n$  ( $t$ ) less than 0.

## Stability

A system is *stable* if bounded inputs produce bounded outputs. Formally, there exist constants  $B, M > 0$  such that:

$$|x(t)| < B \quad \forall t \implies |y(t)| < M \quad \forall t.$$

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |h[k]| &< \infty. \\ \int_{-\infty}^{\infty} |h(\tau)| d\tau &< \infty. \end{aligned}$$

## HRCE

$$x_k(t) = e^{jk\omega_0 t}, k \in \mathbb{Z}.$$

$$x_k[n] = e^{jk\omega_0 n}, k \in \{0, 1, \dots, N\}.$$

## Energy

$$\begin{aligned} E &= \int_{t_1}^{t_2} |x(t)|^2 dt \\ &= \int_{t_1}^{t_2} (x_{Re}(t)^2 + x_{Im}(t)^2) dt. \end{aligned}$$

$$\begin{aligned} E &= \sum_{n=n_1}^{n_2} |x[n]|^2 \\ &= \sum_{n=n_1}^{n_2} (x_{Re}[n]^2 + x_{Im}[n]^2). \end{aligned}$$

## Power

$$P_{avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt.$$

$$P_{avg} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2.$$

$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$P_{\infty} = \lim_{N \rightarrow \infty} \frac{1}{2N + 1} \sum_{n=-N}^N |x[n]|^2$$

## Poisson Summation Formula

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

## Analysis Equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

## Synthesis Equation

$$a_k = \frac{1}{T} \int_{<T>} x(t) e^{-jk \frac{2\pi}{T} t} dt$$

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk \frac{2\pi}{T} t} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \end{aligned}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk \frac{2\pi}{N} n}$$

$$\begin{aligned} x[n] &= \sum_{k=<N>} a_k e^{jk \frac{2\pi}{N} n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}$$

## Symmetry Properties

If  $x(t)$  is real, then  $|X(j\omega)|$  is even and  $\angle X(j\omega)$  is odd. Moreover, if  $x(t)$  is real and even, then  $X(j\omega)$  must be purely real and even, and if  $x(t)$  is real and odd,  $X(j\omega)$  is purely imaginary and odd.

## Duality Property

$$\mathcal{F}\{X(t)\} = 2\pi x(-\omega).$$

## Modulation Property

$$x(t) \cos(\omega_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2} (X(j(\omega - \omega_0)) + X(j(\omega + \omega_0)))$$
$$x[n] \cos(\omega_0 n) \xrightarrow{\mathcal{F}} \frac{1}{2} (X(e^{j(\omega - \omega_0)}) + X(e^{j(\omega + \omega_0)}))$$

## LTI System Invertibility Criterion

An LTI system with frequency response  $H(j\omega)$  (CT) or  $H(e^{j\omega})$  (DT) is invertible if and only if  $H(j\omega) \neq 0$  for all  $\omega$  (CT) or  $H(e^{j\omega}) \neq 0$  for all  $\omega \in [-\pi, \pi]$  (DT). If the system is invertible, then the inverse is  $\frac{1}{H(j\omega)}$  (CT) or  $\frac{1}{H(e^{j\omega})}$  (DT).

## LTI System Input/Output

$$Y(j\omega) = H(j\omega)X(j\omega)$$
$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

## LTI with Periodic Complex Exponential Input

Let the input to an LTI system be  $e^{j\omega_0 t}$ , and let the impulse response be  $h(t)$ . Then

$$y(t) = H(j\omega_0)e^{j\omega_0 t}.$$

## Differential Equations

For a system described by:

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

The frequency response is:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^M b_k (j\omega)^k}{\sum_{k=0}^N a_k (j\omega)^k}$$

## Difference Equations

For a system described by:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

The frequency response is:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^M b_k e^{-j\omega k}}{\sum_{k=0}^N a_k e^{-j\omega k}}$$

## Sampling Theorem

Let  $x(t)$  be a band-limited signal with  $X(j\omega) = 0$  for  $|\omega| > \omega_M$ . Then  $x(t)$  is uniquely determined by its samples  $x(nT)$ ,  $n = 0, \pm 1, \pm 2, \dots$  if

$$\omega_s > 2\omega_M,$$

where

$$\omega_s = \frac{2\pi}{T}.$$

Given these samples, we can reconstruct  $x(t)$  by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain  $T$  and cutoff frequency greater than  $\omega_M$  and less than  $\omega_s - \omega_M$ . The resulting output signal will exactly equal  $x(t)$ .

Table 1: Properties of Fourier Transforms

Property	CT Time Domain	CT Frequency Domain	DT Time Domain	DT Frequency Domain
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$	$Ax_1[n] + Bx_2[n]$	$AX_1(e^{j\omega}) + BX_2(e^{j\omega})$
Time Shifting	$x(t - t_0)$	$X(j\omega)e^{-j\omega t_0}$	$x[n - n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$
Frequency Shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega - \omega_0)})$
Time Scaling	$x(at), a \neq 0$	$\frac{1}{ a } X\left(\frac{j\omega}{a}\right)$	—	—
Time Reversal	$x(-t)$	$X(-j\omega)$	$x[-n]$	$X(e^{-j\omega})$
Conjugation	$x^*(t)$	$X^*(-j\omega)$	$x^*[n]$	$X^*(e^{-j\omega})$
Differentiation	$\frac{d^n}{dt^n} x(t)$	$(j\omega)^n X(j\omega)$	$x[n] - x[n - 1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Integration	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$	$\sum_{k=-\infty}^n x[k]$	$\frac{X(e^{j\omega})}{1 - e^{-j\omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
Convolution	$(x_1 * x_2)(t)$	$X_1(j\omega)X_2(j\omega)$	$(x_1 * x_2)[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi} (X_1 * X_2)(j\omega)$	$x_1[n]x_2[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega - \theta)}) d\theta$
Parseval's Theorem	$\int_{-\infty}^{\infty}  x(t) ^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty}  X(j\omega) ^2 d\omega$	$\sum_{n=-\infty}^{\infty}  x[n] ^2$	$\frac{1}{2\pi} \int_{-\pi}^{\pi}  X(e^{j\omega}) ^2 d\omega$

Table 2: Fourier Transform Pairs

CT Time Domain $x(t)$	CT Fourier Transform $X(j\omega)$	DT Time Domain $x[n]$	DT Fourier Transform $X(e^{j\omega})$
$\delta(t)$	1	$\delta[n]$	1
1	$2\pi \delta(\omega)$	1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$u(t)$	$\pi \delta(\omega) + \frac{1}{j\omega}$	$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$e^{-at}u(t), \Re(a) > 0$	$\frac{1}{a + j\omega}$	$a^n u[n],  a  < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$e^{-a t }, a > 0$	$\frac{2a}{a^2 + \omega^2}$	$a^{ n },  a  < 1$	$\frac{1 - a^2}{1 - 2a \cos \omega + a^2}$
$te^{-at}u(t), \Re(a) < 0$	$\frac{1}{(a + j\omega)^2}$	$na^n u[n],  a  < 1$	$\frac{ae^{-j\omega}}{(1 - ae^{-j\omega})^2}$
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t), \Re(a) < 0$	$\frac{1}{(a + j\omega)^n}$	$n a^n u[n],  a  < 1$	$\frac{a e^{-j\omega}}{(1 - ae^{-j\omega})^n}$
$\cos(\omega_0 t)$	$\pi \left[ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$	$\cos(\omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty} \left[ \delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k) \right]$
$\sin(\omega_0 t)$	$\frac{\pi}{j} \left[ \delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right]$	$\sin(\omega_0 n)$	$\frac{\pi}{j} \sum_{k=-\infty}^{\infty} \left[ \delta(\omega - \omega_0 - 2\pi k) - \delta(\omega + \omega_0 - 2\pi k) \right]$
$\begin{cases} 1, &  t  \leq T/2 \\ 0, &  t  > T/2 \end{cases}$	$2 \frac{\sin(\frac{\omega T}{2})}{\omega}$	$\begin{cases} 1, & 0 \leq n \leq N \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)} e^{-j\omega N/2}$
$\frac{\sin(Wt)}{\pi t}$	$\begin{cases} 1, &  \omega  \leq W \\ 0, &  \omega  > W \end{cases}$	$\frac{\sin(Wn)}{\pi n}$	$\begin{cases} 1, & 0 \leq  \omega  \leq W \\ 0, & W <  \omega  \leq \pi \end{cases}, X(e^{j\omega}) \text{ periodic with period } 2\pi.$

Table 3: Properties of Fourier Series

Property	CT Time Domain	CT Frequency Domain ( $a_k$ )	DT Time Domain	DT Frequency Domain ( $a_k$ )
Linearity	$Ax_1(t) + Bx_2(t)$	$Aa_k + Ba_k$	$Ax_1[n] + Bx_2[n]$	$Aa_k + Ba_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$	$x[n - n_0]$	$a_k e^{-j\frac{2\pi}{N}kn_0}$
Frequency Shifting	$x(t)e^{jM\omega_0 t}$	$a_{k-M}$	$x[n]e^{j\frac{2\pi}{N}Mn}$	$a_{(k-M) \bmod N}$
Time Reversal	$x(-t)$	$a_{-k}$	$x[-n]$	$a_{-k} \text{ (indices mod } N)$
Conjugation	$x^*(t)$	$a_{-k}^*$	$x^*[n]$	$a_{-k}^* \text{ (indices mod } N)$
Periodic Convolution	$(x * y)(t)$	$Ta_k b_k$	$(x \circledast y)[n]$	$Na_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$	$x[n]y[n]$	$\sum_{m=0}^{N-1} a_m b_{(k-m) \bmod N}$
Differentiation	$\frac{d}{dt} x(t)$	$jk\omega_0 a_k$	$x[n] - x[n - 1]$	$a_k \left(1 - e^{-j\frac{2\pi}{N}k}\right)$
Integration	$\int x(t) dt$	$\frac{a_k}{jk\omega_0} \text{ (} a_0 = 0 \text{)}$	—	—
Parseval's Theorem	$\frac{1}{T} \int_T  x(t) ^2 dt$	$\sum_{k=-\infty}^{\infty}  a_k ^2$	$\frac{1}{N} \sum_{n=0}^{N-1}  x[n] ^2$	$\sum_{k=0}^{N-1}  a_k ^2$
Running Sum	$y(t) = \int_{-\infty}^t x(\tau) d\tau$	$\frac{a_k}{jk\omega_0} \text{ (} k \neq 0, a_0 = 0 \text{)}$	$y[n] = \sum_{m=-\infty}^n x[m]$	$\frac{a_k}{1 - e^{-j\frac{2\pi}{N}k}} \text{ (} k \neq 0, a_0 = 0 \text{)}$
Symmetry (Real)	$x(t)$ real	$a_k = a_{-k}^*$	$x[n]$ real	$a_k = a_{-k}^*$
Symmetry (Real+Even)	$x(t)$ real and even	$a_k$ real and even	$x[n]$ real and even	$a_k$ real and even
Symmetry (Real+Odd)	$x(t)$ real and odd	$a_k$ purely imaginary and odd	$x[n]$ real and odd	$a_k$ purely imaginary and odd