Memoryless

A memoryless system's output depends only on the input at time t (or n for discrete-time systems), and not on past or future states. For example, y(t) = 2x(t) is memoryless, while y[n] = x[n-1] has memory.

Linearity

A system is *linear* if it satisfies superposition and homogeneity. Formally, for any inputs $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, and constants a, b, the system satisfies:

$$S\{ax_1(t) + bx_2(t)\} = aS\{x_1(t)\} + bS\{x_2(t)\} = ay_1(t) + by_2(t).$$

An example is y(t) = kx(t), where k is a constant.

Time Invariance

A system is *time invariant* if a time shift in the input results in an identical shift in the output. For a discrete-time system S, if:

$$x[n] \to y[n],$$

then for any integer n_0 :

$$x[n-n_0] \to y[n-n_0].$$

Linear Time Invariance

An LTI system satisfies both linearity and time invariance. LTI systems are fully characterized by their impulse response h(t) or h[n], enabling analysis via convolution: y(t) = x(t) * h(t). The impulse response is found by plugging in $\delta(t)$ or $\delta[n]$ to the system. The result is the impulse response. Mathematically,

$$h(t) = S(\delta(t)) \tag{1}$$

$$h[n] = S(\delta[n]). \tag{2}$$

Causality

A system is *causal* if its output at time t depends only on present and past inputs. For causal systems, the impulse response must be 0 for values of n (t) less than 0.

Stability

A system is *stable* if bounded inputs produce bounded outputs. Formally, there exist constants B, M > 0 such that:

$$|x(t)| < B \ \forall t \implies |y(t)| < M \ \forall t.$$
 (3)

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{4}$$

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty. \tag{5}$$

HRCE

$$x_k(t) = e^{jk\omega_0 t}, k \in \mathbb{Z}. \tag{6}$$

$$x_k[n] = e^{jk\omega_0 n}, k \in \{0, 1, ..., N\}.$$
 (7)

Energy

$$E = \int_{t_{-}}^{t_{2}} |x(t)|^{2} dt \tag{8}$$

$$= \int_{t_1}^{t_2} (x_{Re}(t)^2 + x_{Im}(t)^2) dt.$$
 (9)

$$E = \sum_{n=-n}^{n_2} |x[n]|^2 \tag{10}$$

$$= \sum_{n=n_1}^{n_2} (x_{Re}[n]^2 + x_{Im}[n]^2). \tag{11}$$

Power

$$P_{avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt.$$
 (12)

$$P_{avg} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2.$$
 (13)

$$P_{\infty} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \tag{14}$$

$$P_{\infty} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$
 (15)

Poisson Summation Formula

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{T}\right)$$
 (16)

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$
 (17)

Analysis Equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
 (18)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
(19)

Synthesis Equation

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\frac{2\pi}{T}t} dt \tag{20}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$
 (21)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{22}$$

$$a_k = \frac{1}{N} \sum_{n = \langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$$
 (23)

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}$$
 (24)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{25}$$

Symmetry Properties

If x(t) is real, then $|X(j\omega)|$ is even and $\angle X(j\omega)$ is odd. Moreover, if x(t) is real and even, then $X(j\omega)$ must be purely real and even, and if x(t) is real and odd, $X(j\omega)$ is purely imaginary and odd.

Duality Property

$$\mathcal{F}\left\{X(t)\right\} = 2\pi x(-\omega). \tag{26}$$

Modulation Property

$$x(t)\cos(\omega_0 t) \stackrel{\mathcal{F}}{\leftrightarrow} \frac{1}{2} \left(X(j(\omega - \omega_0)) + X(j(\omega + \omega_0)) \right)$$
 (27)

$$x[n]\cos(\omega_0 n) \stackrel{\mathcal{F}}{\leftrightarrow} \frac{1}{2} \left(X(e^{j(\omega - \omega_0)}) + X(e^{j(\omega + \omega_0)}) \right)$$
 (28)

LTI System Invertibility Criterion

An LTI system with frequency response $H(j\omega)$ (CT) or $H(e^{j\omega})$ (DT) is invertible if and only if $H(j\omega) \neq 0$ for all ω (CT) or $H(e^{j\omega}) \neq 0$ for all $\omega \in [-\pi, \pi]$ (DT).

LTI System Input/Output

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Differential Equations

For a system described by:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$
 (31)

The frequency response is:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\sum_{k=0}^{N} a_k(j\omega)^k}$$
(32)

Difference Equations

For a system described by:

(29)

(30)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$
 (33)

The frequency response is:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$
(34)

Table 1: Properties of Fourier Transforms

Property	CT Time Domain	CT Frequency Domain	DT Time Domain	DT Frequency Domain
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$	$Ax_1[n] + Bx_2[n]$	$AX_1(e^{j\omega}) + BX_2(e^{j\omega})$
Time Shifting	$x(t-t_0)$	$X(j\omega)e^{-j\omega t_0}$	$x[n-n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$
Frequency Shifting	$x(t)e^{j\omega_0t}$	$X(j(\omega-\omega_0))$	$x[n]e^{j\omega_0n}$	$X(e^{j(\omega-\omega_0)})$
Time Scaling	$x(at), a \neq 0$	$\frac{1}{ a }X\left(\frac{j\omega}{a}\right)$	_	_
Time Reversal	x(-t)	$X(-j\omega)$	x[-n]	$X(e^{-j\omega})$
Conjugation	$x^*(t)$	$X^*(-j\omega)$	$x^*[n]$	$X^*(e^{-j\omega})$
Differentiation	$\frac{d^n}{dt^n}x(t)$	$(j\omega)^n X(j\omega)$	x[n] - x[n-1]	$(1 - e^{-j\omega})X(e^{j\omega})$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$	$\sum_{k=-\infty}^{n} x[k]$	$\frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
Convolution	$(x_1 * x_2)(t)$	$X_1(j\omega)X_2(j\omega)$	$(x_1 * x_2)[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}(X_1*X_2)(j\omega)$	$x_1[n]x_2[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$
Parseval's Theorem	$\int_{-\infty}^{\infty} x(t) ^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) ^2 d\omega$	$\sum_{n=-\infty}^{\infty} x[n] ^2$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) ^2 d\omega$

Table 2: Fourier Transform Pairs

CT Time Domain $x(t)$	CT Fourier Transform $X(j\omega)$	$\mathbf{DT} \ \mathbf{Time} \ \mathbf{Domain} \ x[n]$	DT Fourier Transform $X(e^{j\omega})$
$\delta(t)$	1	$\delta[n]$	1
1	$2\pi \delta(\omega)$	1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
u(t)	$\pi \delta(\omega) + rac{1}{j\omega}$	u[n]	$\frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$
$e^{-at}u(t), \Re(a) > 0$	$\frac{1}{a+j\omega}$	$a^n u[n], a < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$e^{-a t },a>0$	$\frac{2a}{a^2 + \omega^2}$	$a^{ n }, a < 1$	$\frac{1-a^2}{1-2a\cos\omega+a^2}$ $\frac{1-a^2}{ae^{-j\omega}}$
$te^{at}u(t), \Re(a) < 0$	$\frac{1}{(a+j\omega)^2}$	$na^nu[n], a < 1$	$\overline{(1-ae^{-j\omega})^2}$
$\frac{t^{n-1}}{(n-1)!}e^{at}u(t), \Re(a)<0$	$\frac{1}{(a+j\omega)^n}$	$n a^n u[n], a < 1$	$\frac{a e^{-j\omega}}{\left(1 - a e^{-j\omega}\right)^2}$
$\cos(\omega_0 t)$	$\pi \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$	$\cos(\omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty} \left[\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k) \right]$
$\sin(\omega_0 t)$	$\pi \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$ $\frac{\pi}{j} \left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right]$	$\sin(\omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty} \left[\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k) \right]$ $\frac{\pi}{j} \sum_{k=-\infty}^{\infty} \left[\delta(\omega - \omega_0 - 2\pi k) - \delta(\omega + \omega_0 - 2\pi k) \right]$
$\begin{cases} 1, & t \le T/2 \\ 0, & t > T/2 \end{cases}$	$2\frac{\sin\left(\frac{\omega T}{2}\right)}{2}$	$\begin{cases} 1, & 0 \le n \le N \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)} e^{-j\omega N/2}$
(0, t > T/2)	ω	(0, otherwise)	$\sin\!\left(\omega/2 ight)$
$\frac{\sin(Wt)}{\pi t}$	$\begin{cases} 1, & \omega \le W \\ 0, & \omega > W \end{cases}$	$\frac{\sin(Wn)}{\pi n}$	$\begin{cases} 1, & \omega \le W \\ 0, & \omega > W \end{cases}$

Table 3: Properties of Fourier Series

Property	CT Time Domain	CT Frequency Domain (a_k)	DT Time Domain	DT Frequency Domain (a_k)
Linearity	$Ax_1(t) + Bx_2(t)$	$Aa_k + Bb_k$	$Ax_1[n] + Bx_2[n]$	$A a_k + B b_k$
Time Shifting	$x(t-t_0)$	$a_k e^{-jk\omega_0 t_0}$	$x[n-n_0]$	$a_k e^{-j\frac{2\pi}{N}kn_0}$
Frequency Shifting	$x(t)e^{jM\omega_0t}$	a_{k-M}	$x[n] e^{j\frac{2\pi}{N}Mn}$	$a_{(k-M) \bmod N}$
Time Reversal	x(-t)	a_{-k}	x[-n]	a_{-k} (indices mod N)
Conjugation	$x^*(t)$	a_{-k}^*	$x^*[n]$	a_{-k}^* (indices mod N)
Periodic Convolution	(x*y)(t)	Ta_kb_k	$(x \circledast y)[n]$	Na_kb_k
Multiplication	x(t)y(t)	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$	x[n] y[n]	$\sum_{m=0}^{N-1} a_m b_{(k-m) \bmod N}$
Differentiation	$\frac{d}{dt}x(t)$	$jk\omega_0 a_k$	x[n] - x[n-1]	$a_k \left(1 - e^{-j\frac{2\pi}{N}k}\right)$
Integration	$\int x(t)dt$	$\frac{a_k}{jk\omega_0} \ (a_0 = 0)$	_	
Parseval's Theorem	$\frac{1}{T} \int_T x(t) ^2 dt$	$\sum_{k=-\infty}^{\infty} a_k ^2$	$\frac{1}{N} \sum_{n=0}^{N-1} x[n] ^2$	$\sum_{k=0}^{N-1} a_k ^2$
Running Sum	$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$	$\frac{a_k}{jk\omega_0} (k \neq 0, \ a_0 = 0)$	$y[n] = \sum_{m = -\infty}^{n} x[m]$	$\frac{a_k}{1 - e^{-j\frac{2\pi}{N}k}} (k \neq 0, \ a_0 = 0)$
Symmetry (Real)	x(t) real	$a_k = a_{-k}^*$	x[n] real	$a_k = a_{-k}^*$
Symmetry (Real+Even)	x(t) real and even	a_k real and even	x[n] real and even	a_k real and even
Symmetry (Real+Odd)	x(t) real and odd	a_k purely imaginary and odd	x[n] real and odd	a_k purely imaginary and odd