ONE-DIMENSIONAL RANDOM WALKS

1. Simple Random Walk

Definition 1. A *random walk* on the integers $\mathbb{Z}$ with step distribution F and initial state $x \in \mathbb{Z}$ is a sequence S_n of random variables whose increments are independent, identically distributed random variables ξ_i with common distribution F, that is,

$$S_n = x + \sum_{i=1}^{n} \xi_i. \tag{1}$$

The definition extends in an obvious way to random walks on the d−dimensional integer lattice $\mathbb{Z}^d$: the increments are then random d−vectors. *Simple random walk* on $\mathbb{Z}^d$ is the particular case where the step distribution is the uniform distribution on the $2d$ nearest neighbors of the origin; in one dimension, this is the *Rademacher*-$\frac{1}{2}$ distribution, the distribution that puts mass $1/2$ at each of the two values ± 1. The moves of a simple random walk in 1D are determined by independent fair coin tosses: For each Head, jump one to the right; for each Tail, jump one to the left.

1.1. Gambler's Ruin.

Simple random walk describes (among other things) the fluctuations in a speculator's wealth when he/she is fully invested in a risky asset whose value jumps by either ± 1 in each time period. Although this seems far too simple a model to be of any practical value, when the unit of time is small (e.g., seconds) it isn't so bad, at least over periods on the order of days or weeks, and in fact it is commonly used as the basis of the so-called *tree models* for valuing options.

Gambler's Ruin Problem: Suppose I start with x dollars. What is the probability that my fortune will grow to A dollars before I go broke? More precisely, if

$$T = T_{[0,A]} := \min\{n : S_n = 0 \text{ or } A\} \tag{2}$$

then what is $P^x\{S_T = A\}$?[1] Before we try to answer this, we need to verify that $T < \infty$ with probability 1. To see that this is so, observe that if at any time during the course of the game there is a run of A consecutive Heads, then the game must end, because my fortune will have increased by at least A dollars. But if I toss a fair coin forever, a run of A consecutive Heads will certainly occur. (Why?)

Difference Equations: To solve the gambler's ruin problem, we'll set up and solve a *difference equation* for the quantity of interest

$$u(x) := P^x\{S_T = A\}. \tag{3}$$

First, if I start with A dollars then I have already reached my goal, so $u(A) = 1$; similarly, $u(0) = 0$. Now consider what happens on the very first play, if $0 < x < A$: either I toss a Head, in which case

[1] Here and throughout the course, the superscript x denotes the initial state of the process S_n. When there is no superscript, the initial state is $x = 0$. Thus, $P = P^0$.

my fortune increases by 1, or I toss a tail, in which case it decreases by 1. At this point, it is like starting the game from scratch, but with initial fortune either $x+1$ or $x-1$. Hence, u satisfies the difference equation

$$u(x) = \frac{1}{2}u(x+1) + \frac{1}{2}u(x-1) \quad \forall\, 1 \le x \le A-1 \tag{4}$$

and the boundary conditions

$$\begin{aligned} u(A) &= 1; \\ u(0) &= 0. \end{aligned} \tag{5}$$

How do we solve this? The most direct approach is to translate the difference equation into a relation between the successive differences $u(x+1) - u(x)$ and $u(x) - u(x-1)$:

$$\frac{1}{2}(u(x+1) - u(x)) = \frac{1}{2}(u(x) - u(x-1)). \tag{6}$$

This equation says that the successive differences in the function u are all the same, and it is easy to see (exercise!) that the only functions with this property are *linear* functions $u(x) = Bx + C$. Conversely, any linear function solves (4). To determine the coefficients B, C, use the boundary conditions: these imply $C = 0$ and $B = 1/A$. This proves

Proposition 1. $P^x\{S_T = A\} = x/A$.

Remark 1. We will see later in the course that first-passage problems for Markov chains and continuous-time Markov processes are, in much the same way, related to boundary value problems for other difference and differential operators. This is the basis for what has become known as *probabilistic potential theory*. The connection is also of practical importance, because it leads to the possibility of *simulating* the solutions to boundary value problems by running random walks and Markov chains on computers.

Remark 2. In solving the difference equation (4) , we used it to obtain a relation (6) between successive differences of the unknown function u. This doesn't always work. However, in general, if a difference equation is of order m, then it relates $u(x)$ to the last m values $u(x-1), \dots, u(x-m)$. Thus, it relates the *vector*

$$\begin{aligned} U(x) &:= (u(x), u(x-1), \dots, u(x-m+1)) \quad \text{to the vector} \\ U(x-1) &:= (u(x-1), u(x-2), \dots, u(x-m)). \end{aligned}$$

If the difference equation is *linear*, as is usually the case in Markov chain problems, then this relation can be formulated as a matrix equation $MU(x-1) = U(x)$. This can then be solved by matrix multiplication. Following is a simple example where this point of view is helpful.

Expected Duration of the Game: Now that we know the probabilities of winning and losing, it would be nice to know how long the game will take. This isn't a well-posed problem, because the duration T of the game is random, but we can at least calculate $E^x T$. Once again, we will use difference equations: Set

$$v(x) := E^x T; \tag{7}$$

then $v(0) = v(A) = 0$ and, by reasoning similar to that used above,

$$v(x) = 1 + \frac{1}{2}v(x-1) + \frac{1}{2}v(x+1) \quad \forall\, 1 \le x \le A-1. \tag{8}$$