$$
\begin{aligned}
(\sigma\tau)^n &= \sigma^n\tau^n && \text{(by Exercise 14.4.16)}\\
&= \sigma^{j\cdot p+r}\tau^n && \text{(substitution)}\\
&= (\sigma^j)^p\sigma^r\tau^n && \text{(by exponentiation rules)}\\
&= \mathsf{id}^p\sigma^r\tau^n && \text{(by definition of order)}\\
&= \sigma^r\tau^n && \text{(by definition of identity)}
\end{aligned}
$$

Now since $r < k$, and $|\sigma| = k$, it follows that $\sigma^r \neq \mathsf{id}$. Thus there is some x such that $\sigma^r(x) \neq x$. But since σ and τ are disjoint, it must be the case that $\tau(x) = x$. It follows that:

$$\sigma^r\tau^n(x) = \sigma^r(x) \neq x.$$

From this we may conclude that $(\sigma\tau)^n$ is *not* the identity. This completes the proof of (ii). □

What Proposition 14.4.18 establishes for two disjoint cycles is also true for multiple disjoint cycles. We state the proposition without proof, because it is similar to that of Proposition 14.4.18 except with more details.

Proposition 14.4.19. Suppose $\sigma_1, \sigma_2, \ldots, \sigma_n$ are n disjoint cycles, where $k_1, k_2, \ldots, k_n$ are the lengths, respectively, of the n disjoint cycles. Then

$$|\sigma_1\sigma_2\cdots\sigma_n| = \text{ lcm } (k_1, k_2, \ldots, k_n).$$

Now we can find the order of any permutation by first representing it as a product of disjoint cycles.

Exercise 14.4.20. What are all the possible orders for the permutations in each of the following sets (look back at your work for Exercise 14.3.38).

(a) S_6 (b) S_7 (c) S_8

◊

Exercise 14.4.21. Compute the following: