3.3 *Inverse of an increasing convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse, *i.e.*, the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?

3.4 [RV73, page 15] Show that a continuous function $f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if for every line segment, its average value on the segment is less than or equal to the average of its values at the endpoints of the segment: For every $x,\ y \in \mathbf{R}^n$,

$$\int_0^1 f(x + \lambda(y - x))\, d\lambda \le \frac{f(x) + f(y)}{2}.$$

3.5 [RV73, page 22] *Running average of a convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, with $\mathbf{R}_+ \subseteq \mathbf{dom}\, f$. Show that its *running average* F, defined as

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt, \qquad \mathbf{dom}\, F = \mathbf{R}_{++},$$

is convex. *Hint.* For each s, $f(sx)$ is convex in x, so $\int_0^1 f(sx)\, ds$ is convex.

3.6 *Functions and epigraphs.* When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

3.7 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex with $\mathbf{dom}\, f = \mathbf{R}^n$, and bounded above on $\mathbf{R}^n$. Show that f is constant.

3.8 *Second-order condition for convexity.* Prove that a twice differentiable function f is convex if and only if its domain is convex and $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbf{dom}\, f$. *Hint.* First consider the case $f : \mathbf{R} \to \mathbf{R}$. You can use the first-order condition for convexity (which was proved on page 70).

3.9 *Second-order conditions for convexity on an affine set.* Let $F \in \mathbf{R}^{n \times m}$, $\hat{x} \in \mathbf{R}^n$. The *restriction* of $f : \mathbf{R}^n \to \mathbf{R}$ to the affine set $\{Fz + \hat{x} \mid z \in \mathbf{R}^m\}$ is defined as the function $\tilde{f} : \mathbf{R}^m \to \mathbf{R}$ with

$$\tilde{f}(z) = f(Fz + \hat{x}), \qquad \mathbf{dom}\, \tilde{f} = \{z \mid Fz + \hat{x} \in \mathbf{dom}\, f\}.$$

Suppose f is twice differentiable with a convex domain.

(a) Show that $\tilde{f}$ is convex if and only if for all $z \in \mathbf{dom}\, \tilde{f}$

$$F^T \nabla^2 f(Fz + \hat{x}) F \succeq 0.$$

(b) Suppose $A \in \mathbf{R}^{p \times n}$ is a matrix whose nullspace is equal to the range of F, *i.e.*, $AF = 0$ and $\mathbf{rank}\, A = n - \mathbf{rank}\, F$. Show that $\tilde{f}$ is convex if for all $z \in \mathbf{dom}\, \tilde{f}$ there exists a $\lambda \in \mathbf{R}$ such that

$$\nabla^2 f(Fz + \hat{x}) + \lambda A^T A \succeq 0.$$

Hint. Use the following result: If $B \in \mathbf{S}^n$ and $A \in \mathbf{R}^{p \times n}$, then $x^T B x \ge 0$ for all $x \in \mathcal{N}(A)$ if there exists a λ such that $B + \lambda A^T A \succeq 0$.

3.10 *An extension of Jensen's inequality.* One interpretation of Jensen's inequality is that randomization or dithering hurts, *i.e.*, raises the average value of a convex function: For f convex and v a zero mean random variable, we have $\mathbf{E}\, f(x_0 + v) \ge f(x_0)$. This leads to the following conjecture. If f is convex, then the larger the variance of v, the larger $\mathbf{E}\, f(x_0 + v)$.

(a) Give a counterexample that shows that this conjecture is false. Find zero mean random variables v and w, with $\mathbf{var}(v) > \mathbf{var}(w)$, a convex function f, and a point x_0, such that $\mathbf{E}\, f(x_0 + v) < \mathbf{E}\, f(x_0 + w)$.