$S := \{0,1,2\}$ to $T := \{0,2\}$ can be defined by assigning $f(0) := 2$, $f(1) := 2$, and $f(2) := 0$. That is, a function $f : A \to B$ is a black box, into which we stick an element of A and the function spits out an element of B. Sometimes f is called a *mapping* or a *map*, and we say *f maps A to B*.

Often, functions are defined by some sort of formula; however, you should really think of a function as just a very big table of values. The subtle issue here is that a single function can have several formulas, all giving the same function. Also, for many functions, there is no formula that expresses its values.

To define a function rigorously, first let us define the Cartesian product.

Definition 0.3.10. Let A and B be sets. The *Cartesian product* is the set of tuples defined as

$$A \times B := \{(x,y) : x \in A, y \in B\}.$$

For instance, $\{a,b\} \times \{c,d\} = \{(a,c),(a,d),(b,c),(b,d)\}$. A more complicated example is the set $[0,1] \times [0,1]$: a subset of the plane bounded by a square with vertices $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. When A and B are the same set we sometimes use a superscript 2 to denote such a product. For example, $[0,1]^2 = [0,1] \times [0,1]$ or $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (the Cartesian plane).

Definition 0.3.11. A *function* $f : A \to B$ is a subset f of $A \times B$ such that for each $x \in A$, there exists a unique $y \in B$ for which $(x,y) \in f$. We write $f(x) = y$. Sometimes the set f is called the *graph* of the function rather than the function itself.

The set A is called the *domain* of f (and sometimes confusingly denoted $D(f)$). The set

$$R(f) := \{y \in B : \text{there exists an } x \in A \text{ such that } f(x) = y\}$$

is called the *range* of f. The set B is called the *codomain* of f.

It is possible that the range $R(f)$ is a proper subset of the codomain B, while the domain of f is always equal to A. We generally assume that the domain of f is nonempty.

Example 0.3.12: From calculus, you are most familiar with functions taking real numbers to real numbers. However, you saw some other types of functions as well. The derivative is a function mapping the set of differentiable functions to the set of all functions. Another example is the Laplace transform, which also takes functions to functions. Yet another example is the function that takes a continuous function g defined on the interval $[0,1]$ and returns the number $\int_0^1 g(x)\,dx$.

Definition 0.3.13. Consider a function $f : A \to B$. Define the *image* (or *direct image*) of a subset $C \subset A$ as

$$f(C) := \{f(x) \in B : x \in C\}.$$

Define the *inverse image* of a subset $D \subset B$ as

$$f^{-1}(D) := \{x \in A : f(x) \in D\}.$$

In particular, $R(f) = f(A)$, the range is the direct image of the domain A.