11. Recall that the ideal gas law states that $PV = RT$, where R is a constant, P is the pressure, V is the volume, and T is the temperature. It can be shown that the work W done by an ideal gas in expanding the volume from V_a to V_b is

$$W = \int_{V_a}^{V_b} P\,dV\ .$$

Calculate W.

12. Verify that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for the function $f(x)$ in formula (8.20) in Example 8.27 for all $\lambda > 0$.

13. Find $P(X < 300)$ in Example 8.27.

14. The *distribution function* $F(x)$ for a random variable X is defined as $F(x) = P(X \le x)$ for all x. Show that $F'(x) = f(x)$, where $f(x)$ is the probability density function for X.

B

15. Formula (8.18) can be extended to regions over an infinite interval, provided the area is finite. Use that fact to find the center of gravity of the region between $y = e^{-x}$ and the x-axis for $0 \le x < \infty$.

16. The *expected value* (or *mean*) $E[X]$ of a random variable X with probability density function $f(x)$ is

$$E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx\ .$$

Show that $E[X] = \frac{1}{\lambda}$ if X has the exponential distribution with parameter $\lambda > 0$.
Note: The expected value can be thought of as the weighted average of all possible values of X, with weights determined by probability. It is analogous to the idea of a center of gravity.

17. A random variable X is said to have a *normal distribution* if its probability density function $f(x)$ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for all } x$$

where $\sigma > 0$ and μ are constants. This is the famous "bell curve" in statistics.

(a) Verify that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. *(Hint: Use Example 6.25 and a substitution.)*

(b) Show that $E[X] = \mu$.

(c) Use numerical integration to show that $P(-1 < X < 1) \approx 0.6827$ when $\mu = 0$ and $\sigma = 1$.

18. A random variable X has the *beta distribution* if its probability density function $f(x)$ is

$$f(x) = \begin{cases} \frac{1}{B(a,b)}\,x^{a-1}(1-x)^{b-1} & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

for positive constants a and b, where $B(a,b)$ is the Beta function. Show that $E[X] = \frac{a}{a+b}$.

19. Show that any value between $F(x)$ and $F(x+dx)$ for the force over $[x, x+dx]$ gives the same formula $dW = F(x)\,dx$ for the work performed over that interval. *(Hint: Consider $F(x+\alpha\,dx)$ for $0 \le \alpha \le 1$.)*

20. A drop of water of mass M is released from rest at a height sufficient for the drop to evaporate completely, losing mass m each second (i.e. at a constant rate). Ignoring air resistance, show that the work performed by gravity on the drop up to complete evaporation is $\frac{g^2M^2}{6m^2}$.