$$f(x|E) = \begin{cases} 2, & \text{if } 0 \le x < 1/2, \\ 0, & \text{if } 1/2 \le x < 1. \end{cases}$$

Thus the conditional density function is nonzero only on $[0, 1/2]$, and is uniform there. □

Example 4.19 In the dart game (cf. Example 2.8), suppose we know that the dart lands in the upper half of the target. What is the probability that its distance from the center is less than 1/2?

Here $E = \{\, (x, y) : y \ge 0 \,\}$, and $F = \{\, (x, y) : x^2 + y^2 < (1/2)^2 \,\}$. Hence,

$$\begin{aligned} P(F|E) &= \frac{P(F \cap E)}{P(E)} = \frac{(1/\pi)[(1/2)(\pi/4)]}{(1/\pi)(\pi/2)} \\ &= 1/4\ . \end{aligned}$$

Here again, the size of $F \cap E$ is 1/4 the size of E. The conditional density function is

$$f((x, y)|E) = \begin{cases} f(x, y)/P(E) = 2/\pi, & \text{if } (x, y) \in E, \\ 0, & \text{if } (x, y) \notin E. \end{cases}$$

□

Example 4.20 We return to the exponential density (cf. Example 2.17). We suppose that we are observing a lump of plutonium-239. Our experiment consists of waiting for an emission, then starting a clock, and recording the length of time X that passes until the next emission. Experience has shown that X has an exponential density with some parameter λ, which depends upon the size of the lump. Suppose that when we perform this experiment, we notice that the clock reads r seconds, and is still running. What is the probability that there is no emission in a further s seconds?

Let $G(t)$ be the probability that the next particle is emitted after time t. Then

$$\begin{aligned} G(t) &= \int_t^\infty \lambda e^{-\lambda x}\, dx \\ &= -e^{-\lambda x}\Big|_t^\infty = e^{-\lambda t}\ . \end{aligned}$$

Let E be the event "the next particle is emitted after time r" and F the event "the next particle is emitted after time $r + s$." Then

$$\begin{aligned} P(F|E) &= \frac{P(F \cap E)}{P(E)} \\ &= \frac{G(r+s)}{G(r)} \\ &= \frac{e^{-\lambda(r+s)}}{e^{-\lambda r}} \\ &= e^{-\lambda s}\ . \end{aligned}$$