9.4 Power Series

A **power series** is an infinite series whose terms involve constants a_n and powers of $x-c$, where x is a variable and c is a constant: $\sum a_n (x-c)^n$. In many cases c will be 0. For example, the geometric progression

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

converges when $|r| < 1$, i.e. for $-1 < r < 1$, as shown in Section 9.1. Replacing the constant r by a variable x yields the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{9.8}$$

that converges to $\frac{1}{1-x}$ when $-1 < x < 1$. Note that the series diverges for $|x| \ge 1$, by the n-th Term Test.

In general a power series of the form $\sum f_n(x)$, where $f_n(x) = a_n(x-c)^n$ is a sequence of functions, has an **interval of convergence** defined as the set of all x such that the series converges. The interval can be any combination of open or closed, as well as the extreme cases of a single point or all real numbers. On its interval of convergence the power series is thus a function of x. The **radius of convergence** R of a power series is defined as half the length of the interval of convergence. In the case where the interval of convergence is all of $\mathbb{R}$ you would say $R = \infty$.

For example, for the above power series $\sum_{n=0}^{\infty} f_n(x)$, where $f_n(x) = x^n$ for $n \ge 0$, the interval of convergence is $-1 < x < 1$, so the radius of convergence is $R = 1$. Notice that

$$f(x) = \sum_{n=0}^{\infty} x^n \quad \text{for } -1 < x < 1$$

is thus a well-defined function on the interval $(-1, 1)$, where it happens to equal $\frac{1}{1-x}$. This power series can be thought of as a polynomial of infinite degree.

To find the interval of convergence of a power series $\sum f_n(x)$, you typically would use the Ratio Test on the absolute values of the terms (since the Ratio Test requires positive terms):

$$r(x) = \lim_{n\to\infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| \tag{9.9}$$

Note that the limit $r(x)$ in this case is a function of x. When taking the limit, though, treat x as fixed. By the Ratio Test the power series will then converge for all x such that $r(x) < 1$, and diverge when $r(x) > 1$. When $r(x) = 1$ the test is inconclusive, so you would have to check those cases individually to see if those values of x should be added to the interval of convergence (along with the points where $r(x) < 1$).