To illustrate, begin with $z^3 + 9z^2 + 24z + 20 = 0$ and substitute $z = x - \frac{a_2}{3} = x - \frac{9}{3} = x - 3$. The equation then becomes $(x-3)^3 + 9(x-3)^2 + 24(x-3) + 20 = 0$, which simplifies to $x^3 - 3x + 2 = 0$.

You need not worry about the computational details here, but in general the substitution $z = x - \frac{a_2}{3}$ transforms Equation (1.1) into

$$x^3 + bx + c = 0, \tag{1.2}$$

where $b = a_1 - \frac{1}{3}a_2^2$, and $c = -\frac{1}{3}a_1a_2 + \frac{2}{27}a_2^3 + a_0$.

If Cardano could get any value of x that solved a depressed cubic, he could easily get a corresponding solution to Equation (1.1) from the identity $z = x - \frac{a_2}{3}$. Happily, Cardano knew how to solve a depressed cubic. The technique had been communicated to him by Niccolo Fontana who, unfortunately, came to be known as Tartaglia (the *stammerer*) due to a speaking disorder that was caused when he was 12 years old. (Evidently, during the Italian wars, French troops sacked his home in Brescia, Italy in 1512, and struck Tartaglia in the face with a saber.) The procedure was also independently discovered some 30 years earlier by Scipione del Ferro of Bologna. Ferro and Tartaglia showed that one of the solutions to Equation (1.2) is

$$x = \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} + \sqrt[3]{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}. \tag{1.3}$$

Although Cardano would not have reasoned in the following way, today we can take this value for x and use it to factor the depressed cubic into a linear and quadratic term. The remaining roots can then be found with the quadratic formula. For example, to solve the (full) cubic equation $z^3 + 9z^2 + 24z + 20 = 0$, use the substitution $z = x - 3$ to get $x^3 - 3x + 2 = 0$, which is a depressed cubic in the form of Equation (1.2). Next, apply the "Ferro-Tartaglia" formula with $b = -3$ and $c = 2$ to get

$$x = \sqrt[3]{-\frac{2}{2} + \sqrt{\frac{2^2}{4} + \frac{(-3)^3}{27}}} + \sqrt[3]{-\frac{2}{2} - \sqrt{\frac{2^2}{4} + \frac{(-3)^3}{27}}} = \sqrt[3]{-1} + \sqrt[3]{-1} = -2.$$

Since $x = -2$ is a root, $x + 2$ must be a factor of $x^3 - 3x + 2$. Dividing $x + 2$ into $x^3 - 3x + 2$ gives $x^2 - 2x + 1 = (x-1)^2$, so that the remaining (duplicate) roots are $x = 1, x = 1$. The solutions to $z^3 + 9z^2 + 24z + 20 = 0$ are obtained by recalling $z = x - 3$, which yields the three roots $z_1 = -2 - 3 = -5$, and $z_2 = z_3 = 1 - 3 = -2$.

So, by using Tartaglia's work and a clever transformation technique, Cardano was able to crack what had seemed to be the impossible task of solving the general cubic equation. Surprisingly, this development played a significant role in helping to establish the legitimacy of imaginary numbers. Roots of negative numbers, of course, had come up earlier in the simplest of quadratic equations, such as $x^2 + 1 = 0$. The solutions we know today as $x = \pm\sqrt{-1}$, however, were easy for mathematicians to ignore. In Cardano's time, negative numbers were still being treated with some suspicion, as it was difficult to conceive of any physical reality corresponding to them. Taking square roots of such quantities was surely all the more ludicrous. Nevertheless, Cardano made some genuine attempts to deal with $\sqrt{-1}$. Unfortunately, his geometric thinking made it hard to make much headway. At one point he commented that the process of arithmetic that deals with quantities such as $\sqrt{-1}$ "involves mental tortures and is truly sophisticated." At another point he concluded that the process is "as refined as it is useless." Many mathematicians held this view, but finally there was a breakthrough.