Sum of Two Independent Normal Random Variables

Example 7.5 It is an interesting and important fact that the convolution of two normal densities with means μ_1 and μ_2 and variances σ_1 and σ_2 is again a normal density, with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. We will show this in the special case that both random variables are standard normal. The general case can be done in the same way, but the calculation is messier. Another way to show the general result is given in Example 10.17.

Suppose X and Y are two independent random variables, each with the standard *normal* density (see Example 5.8). We have

$$f_X(x) = f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} ,$$

and so

$$\begin{aligned} f_Z(z) &= f_X * f_Y(z) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(z-y)^2/2} e^{-y^2/2}\, dy \\ &= \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{+\infty} e^{-(y-z/2)^2}\, dy \\ &= \frac{1}{2\pi} e^{-z^2/4} \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(y-z/2)^2}\, dy \right] . \end{aligned}$$

The expression in the brackets equals 1, since it is the integral of the normal density function with $\mu = 0$ and $\sigma = \sqrt{2}$. So, we have

$$f_Z(z) = \frac{1}{\sqrt{4\pi}} e^{-z^2/4} .$$

□

Sum of Two Independent Cauchy Random Variables

Example 7.6 Choose two numbers at random from the interval $(-\infty, +\infty)$ with the Cauchy density with parameter $a = 1$ (see Example 5.10). Then

$$f_X(x) = f_Y(x) = \frac{1}{\pi(1 + x^2)} ,$$

and $Z = X + Y$ has density

$$f_Z(z) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \frac{1}{1 + (z-y)^2} \frac{1}{1 + y^2}\, dy .$$