Notably, there exist conditionally convergent series where the absolute values of the terms go to zero arbitrarily slowly. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

converges for arbitrarily small $p > 0$, but it does not converge absolutely when $p \leq 1$.

2.6.3 Rearrangements

Absolutely convergent series behave as we imagine they should. For example, absolutely convergent series can be summed in any order whatsoever. Nothing of the sort holds for conditionally convergent series (see Example 2.6.4 and Exercise 2.6.3).

Consider a series

$$\sum_{n=1}^{\infty} x_n .$$

Given a bijective function $\sigma \colon \mathbb{N} \to \mathbb{N}$, the corresponding rearrangement is the series:

$$\sum_{k=1}^{\infty} x_{\sigma(k)}.$$

We simply sum the series in a different order.

Proposition 2.6.3. *Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series converging to a number x. Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is absolutely convergent and converges to x.*

In other words, a rearrangement of an absolutely convergent series converges (absolutely) to the same number.

Proof. Let $\epsilon > 0$ be given. As $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, take M such that

$$\left| \left(\sum_{n=1}^{M} x_n \right) - x \right| < \frac{\epsilon}{2} \qquad \text{and} \qquad \sum_{n=M+1}^{\infty} \left| x_n \right| < \frac{\epsilon}{2}.$$

As σ is a bijection, there exists a number K such that for each $n \leq M$, there exists $k \leq K$ such that $\sigma(k) = n$. In other words $\{1, 2, \ldots, M\} \subset \sigma\bigl(\{1, 2, \ldots, K\}\bigr)$.