That Taylor series (which stops at $n = 5$) should agree with $(1+x)^5$. It does. We could rewrite $1 + x$ as $2 + (x - 1)$, and take its fifth power directly. Then 32, 16, 8, 4, 2, 1 will multiply the usual coefficients 1, 5, 10, 10, 5, 1 to give our Taylor coefficients 32, 80, 80, 40, 10, 1. The series stops as it will stop for any polynomial—because the high derivatives are zero.

EXAMPLE 6 Find the Taylor series for $f(x) = e^x$ around the basepoint $x = 1$.

Solution At $x = 1$ the function and all its derivatives equal e. Therefore the Taylor series has that constant factor (note the powers of $x - 1$, not x):

$$e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots. \tag{6}$$

DEFINING THE FUNCTION BY ITS SERIES

Usually, we define $\sin x$ and $\cos x$ from the sides of a triangle. But we could start instead with the series. Define $\sin x$ by equation (2). The logic goes backward, but it is still correct:

First, prove that the series converges.
Second, prove properties like $(\sin x)' = \cos x$.
Third, connect the definitions by series to the sides of a triangle.

We don't plan to do all this. The usual definition was good enough. But note first: There is no problem with convergence. The series for $\sin x$ and $\cos x$ and e^x all have terms $\pm x^n/n!$. The factorials make the series converge for all x. The general rule for e^x times e^y can be based on the series. Equation (6) is typical: e is multiplied by powers of $(x - 1)$. Those powers add to e^{x-1}. So the series proves that $e^x = ee^{x-1}$. That is just one example of the multiplication $(e^x)(e^y) = e^{x+y}$:

$$\left(1 + x + \frac{x^2}{2} + \ldots\right)\left(1 + y + \frac{y^2}{2} + \ldots\right) = 1 + x + y + \frac{x^2}{2} + xy + \frac{y^2}{2} + \ldots. \tag{7}$$

Term by term, multiplication gives the series for e^{x+y}. Term by term, differentiating the series for e^x gives e^x. Term by term, the derivative of $\sin x$ is $\cos x$:

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots. \tag{8}$$

We don't need the famous limit $(\sin x)/x \to 1$, by which geometry gave us the derivative. The identities of trigonometry become identities of infinite series. We could even define π as the first positive x at which $x - \frac{1}{6}x^3 + \cdots$ equals zero. But it is certainly not obvious that this sine series returns to zero—much less that the point of return is near 3.14.

The function that ***will*** *be defined by infinite series is* $e^{i\theta}$. This is the exponential of the ***imaginary number*** $i\theta$ (a multiple of $i = \sqrt{-1}$). The result $e^{i\theta}$ is a ***complex number***, and our goal is to identify it. (We will be confirming Section 9.4.) The technique is to treat $i\theta$ like all other numbers, real or complex, and simply put it into the series:

DEFINITION $e^{i\theta}$ is the sum of $1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \cdots$. (9)

Now use $i^2 = -1$. The even powers are $i^4 = +1$, $i^6 = -1$, $i^8 = +1, \ldots$. We are just multiplying -1 by -1 to get 1. The odd powers are $i^3 = -i$, $i^5 = +i, \ldots$. There-