15. Show that $\int_C 1\, dz = z_2 - z_1$, where C is the line segment from z_1 to z_2, by parametrizing C.

16. Let z_1 and z_2 be points in the right half-plane and let C be the line segment joining them. Show that $\int_C \frac{1}{z^2}\, dz = \frac{1}{z_1} - \frac{1}{z_2}$.

17. Let $z^{\frac{1}{2}}$ be the principal branch of the square root function.

    (a) Evaluate $\int_C \frac{1}{2z^{\frac{1}{2}}}\, dz$, where C is the line segment joining 9 to $3 + 4i$.

    (b) Evaluate $\int_C z^{\frac{1}{2}}\, dz$, where C is the right half of the circle $C_2^+(0)$ joining $-2i$ to $2i$.

18. Using partial fraction decomposition, show that if z lies in the right half-plane and C is the line segment joining 0 to z, then

$$\int_C \frac{1}{\xi^2+1}\, d\xi = \operatorname{Arctan}(z) = \frac{i}{2}\operatorname{Log}(z+i) - \frac{i}{2}\operatorname{Log}(z-i) + \frac{\pi}{2}.$$

19. Let f' and g' be analytic for all z, and let C be any contour joining the points z_1 and z_2. Show that

$$\int_C f(z)g'(z)\, dz = f(z_2)g(z_2) - f(z_1)g(z_1) - \int_C f'(z)g(z)\, dz.$$

20. Compare the various methods for evaluating contour integrals. What are the limitations of each method?

21. Explain how the fundamental theorem of calculus studied in complex analysis and the fundamental theorem of calculus studied in calculus are different. How are they similar?

22. Show that $\int_C z^i dz = (i-1)\frac{1+e^{-\pi}}{2}$, where C is the upper half of $C_1^+(0)$.

6.5 Integral Representations

We now present some major results in the theory of functions of a complex variable. The first one is known as Cauchy's integral formula. It shows that the value of an analytic function f can be represented by a certain contour integral. The n^{th} derivative, $f^{(n)}(z)$, has a similar representation. In Chapter 7, we use these results to prove Taylor's theorem and also establish the power series representation for analytic functions. The Cauchy integral formulas are a convenient tool for evaluating certain contour integrals.

Theorem 6.10 (Cauchy's integral formula)**.** *Let f be analytic in the simply connected domain D and let C be a simple closed positively oriented contour that lies in D. If z_0 is a point that lies interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\, dz. \tag{6.44}$$

Proof. Because f is continuous at z_0, if $\varepsilon > 0$ is given there is a $\delta > 0$ such that the positively oriented circle $C_0 = \{z : |z - z_0| = \frac{1}{2}\delta\}$ lies interior to C (as Figure 6.33 shows) and such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta. \tag{6.45}$$