Density Functions of Continuous Random Variables

Definition 2.1 Let X be a continuous real-valued random variable. A *density function* for X is a real-valued function f which satisfies

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

for all $a,\ b \in \mathbf{R}$. □

We note that it is *not* the case that all continuous real-valued random variables possess density functions. However, in this book, we will only consider continuous random variables for which density functions exist.

In terms of the density $f(x)$, if E is a subset of $\mathbf{R}$, then

$$P(X \in E) = \int_E f(x)\,dx\ .$$

The notation here assumes that E is a subset of $\mathbf{R}$ for which $\int_E f(x)\,dx$ makes sense.

Example 2.10 (Example 2.7 continued) In the spinner experiment, we choose for our set of outcomes the interval $0 \le x < 1$, and for our density function

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

If E is the event that the head of the spinner falls in the upper half of the circle, then $E = \{\, x : 0 \le x \le 1/2 \,\}$, and so

$$P(E) = \int_0^{1/2} 1\,dx = \frac{1}{2}\ .$$

More generally, if E is the event that the head falls in the interval $[a, b]$, then

$$P(E) = \int_a^b 1\,dx = b - a\ .$$

□

Example 2.11 (Example 2.8 continued) In the first dart game experiment, we choose for our sample space a disc of unit radius in the plane and for our density function the function

$$f(x, y) = \begin{cases} 1/\pi, & \text{if } x^2 + y^2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The probability that the dart lands inside the subset E is then given by

$$\begin{aligned} P(E) &= \int\int_E \frac{1}{\pi}\,dx\,dy \\ &= \frac{1}{\pi} \cdot (\text{area of } E)\ . \end{aligned}$$

□