(d) $x(t) = Ce^{2t} - 5/2$; $x' = 2x + 5$

(e) $x(t) = \dfrac{7}{3}e^{3t^2/2} - \dfrac{1}{3}$; $x' = 3tx + t$, $x(0) = 2$

1.1.2 Logistic Growth

Not all populations grow exponentially; otherwise, a bacteria culture in a petri dish would grow unbounded and soon be much larger than the size of the laboratory. To see what happens if there are limiting factors to population growth, let us consider the population of fish in a children's trout pond. The number of trout will be limited by the available resources such as food supply as well as by spawning habitat. A small population of fish might grow exponentially if the pond is large and food is abundant, but the growth rate will decline as the population increases and the availability of resources declines. We can use the **logistic equation** to model population growth in a resource limited environment.[2]

To see how the logistic model works, let us try to adjust our model of exponential growth to account for the limited resources of the pond. We will make the following assumptions.

- If the population of trout is small and the pond is large with abundant resources, the rate of growth will be approximately exponential,

$$\frac{dP}{dt} \approx kP.$$

- If N is the maximum population of trout that the pond can support, then any population larger than N will decrease. In other words,

$$\frac{dP}{dt} < 0$$

  for $P > N$. We say that N is the **carrying capacity** for the population.

Our assumptions suggest that we might try an equation of the form

$$\frac{dP}{dt} = kf(P)P,$$

where $f(P)$ is a function of P that is close to 1 if the population is small but negative if the population is greater than N. The simplest function satisfying these conditions is

$$f(P) = \left(1 - \frac{P}{N}\right).$$

Thus, the **logistic population model** is given by the differential equation

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P.$$

Example 1.1.2 Suppose we have a pond that will support 1000 fish, and the initial population is 100 fish. In order to determine the number of fish in the

[2]The logistic model was first used by the Belgian mathematician and physician Pierre François Verhulst in 1836 to predict the populations of Belgium and France.