Main point: ***The equilibrium price P is a function of s and t.*** Reducing s by better technology lowers the supply line to $p = .3q + 10$. The demand line has not changed. The customer is as eager or stingy as ever. But the price P and quantity Q are different. The new equilibrium is at $Q = 60$ and $P = \$28$, where the new line XX crosses DD.

If the technology is expensive, the supplier will raise t when reducing s. Line YY is $p = .3q + 20$. That gives a higher equilibrium $P = \$32$ at a lower quantity $Q = 40$—the demand was too weak for the technology.

Calculus question Find $\partial P/\partial s$ and $\partial P/\partial t$. The difficulty is that P is not given as a function of s and t. So take implicit derivatives of the supply = demand equations:

$$\begin{aligned} \text{supply} = \text{demand:} \quad & P = -.2Q + 40 = sQ + t && (16) \\ s \text{ derivative:} \quad & P_s = -.2Q_s = sQ_s + Q \quad (\text{note } t_s = 0) \\ t \text{ derivative:} \quad & P_t = -.2Q_t = sQ_t + 1 \quad (\text{note } t_t = 1) \end{aligned}$$

Now substitute $s = .4$, $t = 10$, $P = 30$, $Q = 50$. That is the starting point, around which we are finding a linear approximation. The last two equations give $P_s = 50/3$ and $P_t = 1/3$ (Problem 25). The linear approximation is

$$P = 30 + 50(s - .4)/3 + (t - 10)/3. \tag{17}$$

Comment This example turned out to be subtle (so is economics). I hesitated before including it. The equations are linear and their derivatives are easy, but something in the problem is hard—there is no explicit formula for P. The function $P(s, t)$ is not known. Instead of a point on a surface, we are following the intersection of two lines. *The solution changes as the equation changes.* ***The derivative of the solution comes from the derivative of the equation.***

Summary The foundation of this section is equation (1) for the tangent plane. Everything builds on that—total differential, linear approximation, sensitivity to small change. Later sections go on to the chain rule and "directional derivatives" and "gradients." The central idea of differential calculus is $\Delta f \approx f_x \Delta x + f_y \Delta y$.

NEWTON'S METHOD FOR TWO EQUATIONS

Linear approximation is used ***to solve equations***. To find out where a function is zero, look first to see where its approximation is zero. To find out where a graph crosses the xy plane, look to see where its tangent plane crosses.

Remember Newton's method for $f(x) = 0$. The current guess is x_n. Around that point, $f(x)$ is close to $f(x_n) + (x - x_n)f'(x_n)$. This is zero at the next guess $x_{n+1} = x_n - f(x_n)/f'(x_n)$. That is where the tangent line crosses the x axis.

With two variables the idea is the same— but two unknowns x and y require *two equations*. We solve $g(x, y) = 0$ and $h(x, y) = 0$. Both functions have linear approximations that start from the current point (x_n, y_n)—where derivatives are computed:

$$\begin{aligned} g(x, y) &\approx g(x_n, y_n) + (\partial g/\partial x)(x - x_n) + (\partial g/\partial y)(y - y_n) \\ h(x, y) &\approx h(x_n, y_n) + (\partial h/\partial x)(x - x_n) + (\partial h/\partial y)(y - y_n). \end{aligned} \tag{18}$$

The natural idea is to ***set these approximations to zero***. That gives linear equations for $x - x_n$ and $y - y_n$. Those are the steps Δx and Δy that take us to the next guess