In his 1572 treatise *L'Algebra,* Rafael Bombelli showed that roots of negative numbers have great utility. Consider the depressed cubic $x^3 - 15x - 4 = 0$. Using Formula (1.3), we compute

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}.$$

Simplifying this expression would have been very difficult if Bombelli had not come up with what he called a "wild thought." He suspected that if the original depressed cubic had real solutions, then the two parts of x in the preceding equation could be written as $u + v\sqrt{-1}$ and $u - v\sqrt{-1}$ for some real numbers u and v. That is, Bombelli believed $u + v\sqrt{-1} = \sqrt[3]{2 + 11\sqrt{-1}}$ and $u - v\sqrt{-1} = \sqrt[3]{2 - 11\sqrt{-1}}$, which would mean

$$(u + v\sqrt{-1})^3 = 2 + 11\sqrt{-1}, \quad \text{and} \quad (u - v\sqrt{-1})^3 = 2 - 11\sqrt{-1}.$$

Then, using the well-known algebraic identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, and assuming that roots of negative numbers obey the rules of algebra, he obtained

$$\begin{aligned}
(u + v\sqrt{-1})^3 &= u^3 + 3(u^2)v\sqrt{-1} + 3(u)(v\sqrt{-1})^2 + (v\sqrt{-1})^3 \\
&= u^3 + 3(u)(v\sqrt{-1})^2 + 3(u^2)v\sqrt{-1} + (v\sqrt{-1})^3 \\
&= (u^3 - 3uv^2) + (3u^2v - v^3)\sqrt{-1} \\
&= u(u^2 - 3v^2) + v(3u^2 - v^2)\sqrt{-1} && (1.4) \\
&= 2 + 11\sqrt{-1}. && (1.5)
\end{aligned}$$

By equating like parts of Equations (1.4) and (1.5) Bombelli reasoned that $u(u^2 - 3v^2) = 2$ and $v(3u^2 - v^2) = 11$. Perhaps thinking even more wildly, Bombelli then supposed that u and v were integers. The only integer factors of 2 are 2 and 1, so the equation $u(u^2 - 3v^2) = 2$ led Bombelli to conclude that $u = 2$ and $u^2 - 3v^2 = 1$. From this conclusion it follows that $v^2 = 1$, or $v = \pm 1$. Amazingly, $u = 2$ and $v = 1$ solve the second equation $v(3u^2 - v^2) = 11$, so Bombelli declared the values for u and v to be $u = 2$ and $v = 1$, respectively.

Since $(2 + \sqrt{-1})^3 = 2 + 11\sqrt{-1}$, we clearly have $2 + \sqrt{-1} = \sqrt[3]{2 + 11\sqrt{-1}}$. Similarly, Bombelli showed that $2 - \sqrt{-1} = \sqrt[3]{2 - 11\sqrt{-1}}$, so that

$$\sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4, \tag{1.6}$$

which was a proverbial bombshell. Prior to Bombelli, mathematicians could easily scoff at imaginary numbers when they arose as solutions to quadratic equations. With cubic equations, they no longer had this luxury. That $x = 4$ was a correct solution to the equation $x^3 - 15x - 4 = 0$ was indisputable, as it could be checked easily. However, to arrive at this very real solution, mathematicians had to take a detour through the uncharted territory of "imaginary numbers." Thus, whatever else might have been said about these numbers (which, today, we call *complex numbers*), their utility could no longer be ignored.

1.1.1 Geometric Progress of John Wallis

As significant as Bombelli's work was his results left many issues unresolved. For example, his technique applied only to a few specialized cases. Could it be extended? Even if it could be extended a larger question remained: What possible physical representation could complex numbers have? That question remained unanswered for more than two centuries. Paul J. Nahin's book *An Imaginary Tale: the Story of* $\sqrt{-1}$ describes the progress in answering it as