9. Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b), and that $f'(x) > g'(x)$ for all $a < x < b$. Show that $f(b) - g(b) > f(a) - g(a)$.

10. Prove the Extended Mean Value Theorem, by applying Rolle's Theorem to the function

$$F(x) \;=\; f(x) \;-\; f(a) \;-\; \frac{f(b)-f(a)}{g(b)-g(a)}\,(g(x)-g(a))\;.$$

11. Show that $e^x \ge 1 + x$ for all x. *(Hint: Consider $f(x) = e^x - x$.)*

12. Show that $\ln(1+x) < x$ for all $x > 0$.

13. Show that $\tan^{-1} x \,<\, x$ for all $x > 0$.

14. Show that for $0 < \alpha \le \beta < \frac{\pi}{2}$,

$$\frac{\beta-\alpha}{\cos^2\alpha} \;\le\; \tan\beta \;-\; \tan\alpha \;\le\; \frac{\beta-\alpha}{\cos^2\beta}\;.$$

15. Show that for $0 < a \le b$,

$$\frac{b-a}{b} \;\le\; \ln\frac{b}{a} \;\le\; \frac{b-a}{a}\;.$$

16. Show that for $n > 1$ and $a > b$,

$$nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)\;.$$

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17. Show that $\sqrt{a^2+b} \;<\; a + \dfrac{b}{2a}$ for all positive numbers a and b.

18. Show that $f(x) \;=\; \cos^2 x \;+\; \cos^2\left(\frac{\pi}{3}+x\right) \;-\; \cos x \cos\left(\frac{\pi}{3}+x\right)$ is a constant function. What is its value?

19. Suppose that $f(x)$ is a differentiable function and that $f(0) = 0$ and $f(1) = 1$. Show that $f'(x_0) = 2x_0$ for some x_0 in the interval $(0,1)$.

20. Prove the inequality

$$\left|\frac{x_1+x_2}{1+x_1x_2}\right| \;<\; 1 \quad \text{for } -1 \;<\; x_1,x_2 \;<\; 1$$

as follows:

(a) First prove the special case where $x_1 = x_2$.

(b) For the case $x_1 < x_2$ define

$$f(x) = \frac{x+a}{1+ax}$$

for $-1 \le x \le 1$, where $-1 < a < 1$. Show that f is increasing on $[-1,1]$, then use $a = x_2$ and $x = x_1$.

Note that proving the case $x_2 < x_1$ is unnecessary (why?).
This inequality is a generalization of the same inequality for $0 \le x_1, x_2 < 1$ in the *relativistic velocity addition law* from the theory of special relativity: if object 1 has velocity v_1 relative to a frame of reference F, and if object 2 has a velocity v_2 relative to object 1, so that $x_1 = v_1/c$ and $x_2 = v_2/c$ represent the fractions of the speed of light c at which the objects are moving, then the fraction of the speed of light at which object 2 is moving with respect to F is $x = (x_1+x_2)/(1+x_1x_2)$. So it should be true that $0 \le x < 1$, since nothing can move faster than the speed of light.