10. Suppose that v is a harmonic conjugate of u and that u is a harmonic conjugate of v. Show that u and v must be constant functions.

11. Let $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ be analytic on a domain D that does not contain the origin. Use the polar form of the Cauchy-Riemann equations $u_\theta = -rv_r$, and $v_\theta = ru_r$. Differentiate them first with respect to θ and then with respect to r. Use the results to establish the polar form of Laplace's equation:

$$r^2 u_{rr}(r,\theta) + ru_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0.$$

12. Use the polar form of Laplace's equation given in Exercise 11 to show that the following functions are harmonic.

    (a) $u(r,\theta) = (r + \frac{1}{r})\cos\theta$ and $v(r,\theta) = (r - \frac{1}{r})\sin\theta$.

    (b) $u(r,\theta) = r^n \cos n\theta$ and $v(r,\theta) = r^n \sin n\theta$.

13. The function $F(z) = \frac{1}{z}$ is used to determine a field known as a dipole.

    (a) Express $F(z)$ in the form $F(z) = \phi(x,y) + i\psi(x,y)$.

    (b) Sketch the equipotentials $\phi = 1, \frac{1}{2}, \frac{1}{4}$ and streamlines $\psi = 1, \frac{1}{2}, \frac{1}{4}$.

14. Assume that $F(z) = \phi(x,y) + i\psi(x,y)$ is analytic on the domain D and that $F'(z) \neq 0$ on D. Consider the families of level curves $\{\phi(x,y) = \text{ constant}\}$ and $\{\psi(x,y) = \text{ constant}\}$, which are the equipotentials and streamlines for the fluid flow $\mathbf{V}(x,y) = \overline{F'(z)}$. Prove that the two families of curves are orthogonal.
    Hint: Suppose that (x_0, y_0) is a point common to the two curves $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$. Use the gradients of ϕ and ψ to show that the normals to the curves are perpendicular.

15. We introduce the logarithmic function in Chapter 5. For now, let $F(z) = \operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$. Here we have $\phi(x,y) = \ln|z|$ and $\psi(x,y) = \operatorname{Arg} z$. Sketch the equipotentials $\phi = 0$, $\ln 2$, $\ln 3$, $\ln 4$ and streamlines $\psi = \frac{k\pi}{8}$ for $k = 0, 1, \ldots, 7$.

16. Theorem 3.9 claims that it is possible to prove that $C'(x)$ is a function of x alone. Prove this assertion.

17. Discuss and compare the statements "$u(x,y)$ is harmonic" and "$u(x,y)$ is the imaginary part of an analytic function."