The reasoning is direct. When $f(x)$ is multiplied by c, so is $f(x+h)$. The difference Δf is also multiplied by c. All averages $\Delta f/h$ contain c, so their limit is cf'. The only incomplete step is the last one (the limit). ***We still have to say what*** *"limit"* ***means.***

Rule 2D is similar. Adding $f+g$ means adding $\Delta f + \Delta g$. Now divide by h. In the limit as $h \to 0$ we reach $f' + g'$—because a limit of sums is a sum of limits. Any example is easy and so is the proof—it is the definition of limit that needs care (Section 2.6).

You can now find the derivative of every polynomial. A "polynomial" is a combination of $1, x, x^2, \ldots, x^n$—for example $9 + 2x - x^5$. That particular polynomial has slope $2 - 5x^4$. Note that the derivative of 9 is zero! A constant just raises or lowers the graph, without changing its slope. It alters the mileage before starting the car.

The disappearance of constants is one of the nice things in differential calculus. The reappearance of those constants is one of the headaches in integral calculus. When you find v from f, the starting mileage doesn't matter. The constant in f has no effect on v. (*Δf is measured by a trip meter; Δt comes from a stopwatch.*) To find distance from velocity, you need to know the mileage at the start.

A LOOK AT DIFFERENTIAL EQUATIONS (FIND y FROM dy/dx)

We know that $y = x^3$ has the derivative $dy/dx = 3x^2$. Starting with the function, we found its slope. Now reverse that process. ***Start with the slope and find the function.*** This is what science does all the time—and it seems only reasonable to say so.

Begin with $dy/dx = 3x^2$. The slope is given, the function y is not given.

Question Can you go backward to reach $y = x^3$?
Answer Almost but not quite. You are only entitled to say that $y = x^3 + C$. The constant C is the starting value of y (when $x = 0$). Then the ***differential equation*** $dy/dx = 3x^2$ is solved.

Every time you find a derivative, you can go backward to solve a differential equation. The function $y = x^2 + x$ has the slope $dy/dx = 2x + 1$. In reverse, the slope $2x + 1$ produces $x^2 + x$—and all the other functions $x^2 + x + C$, shifted up and down. After going from distance f to velocity v, we return to $f + C$. But there is a lot more to differential equations. Here are two crucial points:

1. We reach dy/dx by way of $\Delta y/\Delta x$, but we have no system to go backward. With $dy/dx = (\sin x)/x$ we are lost. What function has this derivative?
2. Many equations have the same solution $y = x^3$. Economics has $dy/dx = 3y/x$. Geometry has $dy/dx = 3y^{2/3}$. These equations involve y as well as dy/dx. Function and slope are mixed together! This is typical of differential equations.

To summarize: Chapters 2–4 compute and use derivatives. Chapter 5 goes in reverse. Integral calculus discovers the function from its slope. Given dy/dx we find $y(x)$. Then Chapter 6 solves the differential equation $dy/dt = y$, function mixed with slope. Calculus moves from *derivatives* to *integrals* to *differential equations*.

This discussion of the purpose of calculus should mention a specific example. Differential equations are applied to an epidemic (like AIDS). In most epidemics the number of cases grows exponentially. The peak is quickly reached by e^t, and the epidemic dies down. Amazingly, exponential growth is not happening with AIDS—the best fit to the data through 1988 is a ***cubic polynomial*** (*Los Alamos Science*, 1989):

The number of cases fits a cubic within 2%: $y = 174.6(t - 1981.2)^3 + 340$.