Möbius transformations have even more fascinating geometric properties. En route to an example of such, we introduce some terminology. Special cases of Möbius transformations are **translations** $f(z) = z + b$, **dilations** $f(z) = az$, and **inversion** $f(z) = \frac{1}{z}$. The next result says that if we understand those three special Möbius transformations, we understand them all.

Proposition 3.3. Suppose $f(z) = \frac{az+b}{cz+d}$ is a linear fractional transformation. If $c = 0$ then

$$f(z) \;=\; \frac{a}{d}\,z + \frac{b}{d}\,,$$

and if $c \neq 0$ then

$$f(z) \;=\; \frac{bc-ad}{c^2}\,\frac{1}{z+\frac{d}{c}} + \frac{a}{c}\,.$$

In particular, every linear fractional transformation is a composition of translations, dilations, and inversions.

Proof. Simplify. □

Theorem 3.4. Möbius transformations map circles and lines into circles and lines.

Example 3.5. Continuing Example 3.2, consider again $f(z) = \frac{z-1}{iz+i}$. For $\varphi \in \mathbb{R}$,

$$\begin{aligned} f(e^{i\varphi}) \;&=\; \frac{e^{i\varphi}-1}{i\,e^{i\varphi}+i} \;=\; \frac{(e^{i\varphi}-1)(e^{-i\varphi}+1)}{i\,|e^{i\varphi}+1|^2} \\ &=\; \frac{e^{i\varphi}-e^{-i\varphi}}{i\,|e^{i\varphi}+1|^2} \;=\; \frac{2\,\mathrm{Im}(e^{i\varphi})}{|e^{i\varphi}+1|^2} \;=\; \frac{2\sin\varphi}{|e^{i\varphi}+1|^2}\,, \end{aligned}$$

which is a real number. Thus Theorem 3.4 implies that f maps the unit circle to the real line. □

Proof of Theorem 3.4. Translations and dilations certainly map circles and lines into circles and lines, so by Proposition 3.3, we only have to prove the statement of the theorem for the inversion $f(z) = \frac{1}{z}$.

The equation for a circle centered at $x_0 + i\,y_0$ with radius r is $(x-x_0)^2 + (y-y_0)^2 = r^2$, which we can transform to

$$\alpha(x^2+y^2) + \beta x + \gamma y + \delta \;=\; 0 \tag{3.1}$$