Example 6.26 Let X be an exponentially distributed random variable with parameter λ. Then the density function of X is

$$f_X(x) = \lambda e^{-\lambda x} \ .$$

From the definition of expectation and integration by parts, we have

$$\begin{aligned} E(X) &= \int_0^\infty x f_X(x)\, dx \\ &= \lambda \int_0^\infty x e^{-\lambda x}\, dx \\ &= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx \\ &= 0 + \frac{e^{-\lambda x}}{-\lambda}\Big|_0^\infty = \frac{1}{\lambda} \ . \end{aligned}$$

Similarly, using Theorems 6.11 and 6.15, we have

$$\begin{aligned} V(X) &= \int_0^\infty x^2 f_X(x)\, dx - \frac{1}{\lambda^2} \\ &= \lambda \int_0^\infty x^2 e^{-\lambda x}\, dx - \frac{1}{\lambda^2} \\ &= -x^2 e^{-\lambda x}\Big|_0^\infty + 2\int_0^\infty x e^{-\lambda x}\, dx - \frac{1}{\lambda^2} \\ &= -x^2 e^{-\lambda x}\Big|_0^\infty - \frac{2xe^{-\lambda x}}{\lambda}\Big|_0^\infty - \frac{2}{\lambda^2} e^{-\lambda x}\Big|_0^\infty - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \ . \end{aligned}$$

In this case, both $E(X)$ and $V(X)$ are finite if $\lambda > 0$. $\square$

Example 6.27 Let Z be a standard normal random variable with density function

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \ .$$

Since this density function is symmetric with respect to the y-axis, then it is easy to show that

$$\int_{-\infty}^\infty x f_Z(x)\, dx$$

has value 0. The reader should recall however, that the expectation is defined to be the above integral only if the integral

$$\int_{-\infty}^\infty |x| f_Z(x)\, dx$$

is finite. This integral equals

$$2\int_0^\infty x f_Z(x)\, dx \ ,$$