D-optimal design

The most widely used scalarization is called *D-optimal design*, in which we minimize the determinant of the error covariance matrix E. This corresponds to designing the experiment to minimize the volume of the resulting confidence ellipsoid (for a fixed confidence level). Ignoring the constant factor $1/m$ in E, and taking the logarithm of the objective, we can pose this problem as

$$\begin{array}{ll} \text{minimize} & \log\det\left(\sum_{i=1}^p \lambda_i v_i v_i^T\right)^{-1} \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1, \end{array} \qquad (7.26)$$

which is a convex optimization problem.

E-optimal design

In *E-optimal design*, we minimize the norm of the error covariance matrix, *i.e.*, the maximum eigenvalue of E. Since the diameter (twice the longest semi-axis) of the confidence ellipsoid $\mathcal{E}$ is proportional to $\|E\|_2^{1/2}$, minimizing $\|E\|_2$ can be interpreted geometrically as minimizing the diameter of the confidence ellipsoid. E-optimal design can also be interpreted as minimizing the maximum variance of $q^T e$, over all q with $\|q\|_2 = 1$.

The E-optimal experiment design problem is

$$\begin{array}{ll} \text{minimize} & \left\|\left(\sum_{i=1}^p \lambda_i v_i v_i^T\right)^{-1}\right\|_2 \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1. \end{array}$$

The objective is a convex function of λ, so this is a convex problem.

The E-optimal experiment design problem can be cast as an SDP

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & \sum_{i=1}^p \lambda_i v_i v_i^T \succeq tI \\ & \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1, \end{array} \qquad (7.27)$$

with variables $\lambda \in \mathbf{R}^p$ and $t \in \mathbf{R}$.

A-optimal design

In *A-optimal experiment design*, we minimize $\mathbf{tr}\, E$, the trace of the covariance matrix. This objective is simply the mean of the norm of the error squared:

$$\mathbf{E}\,\|e\|_2^2 = \mathbf{E}\,\mathbf{tr}(ee^T) = \mathbf{tr}\, E.$$

The A-optimal experiment design problem is

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}\left(\sum_{i=1}^p \lambda_i v_i v_i^T\right)^{-1} \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1. \end{array} \qquad (7.28)$$

This, too, is a convex problem. Like the E-optimal experiment design problem, it can be cast as an SDP:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & \begin{bmatrix} \sum_{i=1}^p \lambda_i v_i v_i^T & e_k \\ e_k^T & u_k \end{bmatrix} \succeq 0, \quad k = 1, \ldots, n \\ & \lambda \succeq 0, \quad \mathbf{1}^T\lambda = 1, \end{array}$$