Verify the compatibility condition $(AB).v = A.(B.v)$ by using the properties of matrix multiplication. ◊

Example 23.1.11. Let G be a group and $\mathcal{E}_n$ be the set of all subsets of G with n elements where n is a positive integer and $n \leq |G|$. Let $S \in \mathcal{E}_n$, meaning S is a subset of G with n elements. Then G acts on $\mathcal{E}_n$ by $g.S := \{gs \mid s \in S\}$. Note that $g.S$ is a subset of G with n elements. Let's verify that this is an action:

(1) Check the identity condition: $e.S = \{es \mid s \in S\} = S$

(2) Check the compatibility condition: Let $g, h \in G$, then $(gh).S = \{(gh)s \mid s \in S\} = \{g(hs) \mid s \in S\} = g.(h.S)$

Parts (1) and (2) verify that $\mathcal{E}_n$ is a G-set. ♦

To show that (G, X) is *not* an action (in other words X is not a G-set), one may show any one of the following:

- $g.x \notin X$ for some $g \in G$ and $x \in X$;
- the identity condition fails $e.x \neq x$ for some $x \in X$; or
- the compatibility condition fails: $(g_1 g_2).x \neq g_1.(g_2.x)$ for some $x \in X$ and some $g_1, g_2 \in G$.

Usually the easiest way to show one of the above items is by a counterexample.

Exercise 23.1.12.

(a) Let $G = 2\mathbb{Z}$ and let $X = \mathbb{Z}$. Show that X is a G-set.

(b) Let $X = 2\mathbb{Z}$. Show that X is *not* a $\mathbb{Z}$-set. (*Hint*)

(c) Let $G = H_6$ (the complex 6-th roots of unity (see Section 4.4.1 in Chapter 4) and let $X = \mathbb{C}$. Show that X is a G-set.

(d) Let $X = H_8$. Is X a $\mathbb{C}$-set? Explain. (*Hint*)

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