The set of all affine combinations of points in some set $C \subseteq \mathbf{R}^n$ is called the *affine hull* of C, and denoted $\mathbf{aff}\, C$:

$$\mathbf{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \ldots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1\}.$$

The affine hull is the smallest affine set that contains C, in the following sense: if S is any affine set with $C \subseteq S$, then $\mathbf{aff}\, C \subseteq S$.

2.1.3 Affine dimension and relative interior

We define the *affine dimension* of a set C as the dimension of its affine hull. Affine dimension is useful in the context of convex analysis and optimization, but is not always consistent with other definitions of dimension. As an example consider the unit circle in $\mathbf{R}^2$, *i.e.*, $\{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 = 1\}$. Its affine hull is all of $\mathbf{R}^2$, so its affine dimension is two. By most definitions of dimension, however, the unit circle in $\mathbf{R}^2$ has dimension one.

If the affine dimension of a set $C \subseteq \mathbf{R}^n$ is less than n, then the set lies in the affine set $\mathbf{aff}\, C \neq \mathbf{R}^n$. We define the *relative interior* of the set C, denoted $\mathbf{relint}\, C$, as its interior relative to $\mathbf{aff}\, C$:

$$\mathbf{relint}\, C = \{x \in C \mid B(x,r) \cap \mathbf{aff}\, C \subseteq C \text{ for some } r > 0\},$$

where $B(x,r) = \{y \mid \|y - x\| \leq r\}$, the ball of radius r and center x in the norm $\|\cdot\|$. (Here $\|\cdot\|$ is any norm; all norms define the same relative interior.) We can then define the *relative boundary* of a set C as $\mathbf{cl}\, C \setminus \mathbf{relint}\, C$, where $\mathbf{cl}\, C$ is the closure of C.

Example 2.2 Consider a square in the (x_1, x_2)-plane in $\mathbf{R}^3$, defined as

$$C = \{x \in \mathbf{R}^3 \mid -1 \leq x_1 \leq 1,\ -1 \leq x_2 \leq 1,\ x_3 = 0\}.$$

Its affine hull is the (x_1, x_2)-plane, *i.e.*, $\mathbf{aff}\, C = \{x \in \mathbf{R}^3 \mid x_3 = 0\}$. The interior of C is empty, but the relative interior is

$$\mathbf{relint}\, C = \{x \in \mathbf{R}^3 \mid -1 < x_1 < 1,\ -1 < x_2 < 1,\ x_3 = 0\}.$$

Its boundary (in $\mathbf{R}^3$) is itself; its relative boundary is the wire-frame outline,

$$\{x \in \mathbf{R}^3 \mid \max\{|x_1|, |x_2|\} = 1,\ x_3 = 0\}.$$

2.1.4 Convex sets

A set C is *convex* if the line segment between any two points in C lies in C, *i.e.*, if for any $x_1,\ x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta x_1 + (1 - \theta) x_2 \in C.$$