Chapter 1

Complex Numbers

Get ready for a treat. You are about to begin studying some of the most beautiful ideas in mathematics. They are ideas with surprises. They evolved over several centuries, yet they greatly simplify extremely difficult computations, making some as easy as sliding a hot knife through butter. They also have applications in a variety of areas, ranging from fluid flow, to electric circuits, to the mysterious quantum world. Generally, they belong to the area of mathematics known as complex analysis, which is the subject of this book. This chapter focuses on the development of entities we now call *complex numbers*.

1.1 The Origin of Complex Numbers

Complex analysis can roughly be thought of as the subject that applies the theory of calculus to imaginary numbers. But what exactly are imaginary numbers? Usually, students learn about them in high school with introductory remarks from their teachers along the following lines: "We can't take the square root of a negative number. But let's *pretend* we can and begin by using the symbol $i = \sqrt{-1}$." Rules are then learned for doing arithmetic with these numbers. At some level the rules make sense: if $i = \sqrt{-1}$, it stands to reason that $i^2 = -1$. However, it is not uncommon for students to wonder whether they are really doing magic rather than mathematics.

If you ever felt that way, congratulate yourself!—you are in the company of some of the great mathematicians from the sixteenth through the nineteenth centuries. They also were perplexed by the notion of roots of negative numbers. Our purpose in this section is to highlight some of the episodes in the very colorful history of how thinking about imaginary numbers developed. We intend to show you that, contrary to popular belief, there is really nothing *imaginary* about "imaginary umbers." They are just as real as "real numbers."

Our story begins in 1545. In that year, the Italian mathematician Girolamo Cardano published *Ars Magna* (*The Great Art*), a 40-chapter masterpiece in which he gave for the first time a method for solving the general cubic equation

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0. \tag{1.1}$$

Cardano did not have at his disposal the power of today's algebraic notation, and he tended to think of cubes or squares as geometric objects rather than algebraic quantities. Essentially, however, his solution began by making the substitution $z = x - \frac{a_2}{3}$. This move transformed Equation (1.1) into a cubic equation without a squared term, which is called a **depressed cubic**.