for all $n \ge N$,

$$\sum_{k\ge1} M_k - \sum_{k=1}^{n} M_k = \sum_{k>n} M_k < \varepsilon .$$

Thus for all $z \in G$ and $n \ge N$,

$$\left| \sum_{k\ge1} f_k(z) - \sum_{k=1}^{n} f_k(z) \right| = \left| \sum_{k>n} f_k(z) \right| \le \sum_{k>n} |f_k(z)| \le \sum_{k>n} M_k < \varepsilon ,$$

which proves uniform convergence. Replace f_k with $|f_k|$ in this argument to see that $\sum_{k\ge1} |f_k|$ also converges uniformly. □

Example 7.29. We revisit Example 7.8 and consider the geometric series $\sum_{k\ge1} z^k$ as a series of functions in z. We know from Example 7.8 that this function series converges pointwise for $|z| < 1$:

$$\sum_{k\ge1} z^k = \frac{z}{1-z} .$$

To study uniform convergence, we apply Proposition 7.28 with $f_k(z) = z^k$. We need a series of upper bounds that converges, so fix a real number $0 < r < 1$ and let $M_k = r^k$. Then

$$|f_k(z)| = |z|^k \le r^k \qquad \text{for } |z| \le r ,$$

and $\sum_{k\ge1} r^k$ converges by Example 7.8. Thus, Proposition 7.28 says that $\sum_{k\ge1} z^k$ converges uniformly for $|z| \le r$.

We note the subtle distinction of domains for pointwise/uniform convergence: $\sum_{k\ge1} z^k$ converges (absolutely) for $|z| < 1$, but to force *uniform* convergence, we need to shrink the domain to $|z| \le r$ for some (arbitrary but fixed) $r < 1$. □

7.4 Regions of Convergence

For the remainder of this chapter (indeed, this book) we concentrate on some very special series of functions.

Definition. A **power series centered at** z_0 is a series of the form

$$\sum_{k\ge0} c_k \, (z - z_0)^k$$

where $c_0, c_1, c_2, \ldots \in \mathbb{C}$.