When $f: A \to B$ is a bijection, then the inverse image of a single element, $f^{-1}(\{y\})$, is always a unique element of A. We then consider f^{-1} as a function $f^{-1}: B \to A$ and we write simply $f^{-1}(y)$. In this case, we call f^{-1} the *inverse function* of f. For instance, for the bijection $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^3$, we have $f^{-1}(x) = \sqrt[3]{x}$.

Definition 0.3.18. Consider $f: A \to B$ and $g: B \to C$. The *composition* of the functions f and g is the function $g \circ f: A \to C$ defined as

$$(g \circ f)(x) := g\bigl(f(x)\bigr).$$

For example, if $f: \mathbb{R} \to \mathbb{R}$ is $f(x) := x^3$ and $g: \mathbb{R} \to \mathbb{R}$ is $g(y) = \sin(y)$, then $(g \circ f)(x) = \sin(x^3)$. It is left to the reader as an easy exericise to show that composition of one-to-one maps is one-to-one and composition of onto maps is onto. Therefore, composition of bijections is a bijection.

0.3.4 Relations and equivalence classes

We often compare two objects in some way. We say $1 < 2$ for natural numbers, or $1/2 = 2/4$ for rational numbers, or $\{a, c\} \subset \{a, b, c\}$ for sets. The '$<$', '$=$', and '$\subset$' are examples of relations.

Definition 0.3.19. Given a set A, a *binary relation* on A is a subset $\mathcal{R} \subset A \times A$, which are those pairs where the relation is said to hold. Instead of $(a, b) \in \mathcal{R}$, we write $a \mathrel{\mathcal{R}} b$.

Example 0.3.20: Take $A := \{1, 2, 3\}$.

Consider the relation '$<$'. The corresponding set of pairs is $\bigl\{(1,2), (1,3), (2,3)\bigr\}$. So $1 < 2$ holds as $(1, 2)$ is in the corresponding set of pairs, but $3 < 1$ does not hold as $(3, 1)$ is not in the set.

Similarly, the relation '$=$' is defined by the set of pairs $\bigl\{(1,1), (2,2), (3,3)\bigr\}$.

Any subset of $A \times A$ is a relation. Let us define the relation $\dagger$ via $\bigl\{(1,2), (2,1), (2,3), (3,1)\bigr\}$, then $1 \dagger 2$ and $3 \dagger 1$ are true, but $1 \dagger 3$ is not.

Definition 0.3.21. Let $\mathcal{R}$ be a relation on a set A. Then $\mathcal{R}$ is said to be

(i) *Reflexive* if $a \mathrel{\mathcal{R}} a$ for all $a \in A$.

(ii) *Symmetric* if $a \mathrel{\mathcal{R}} b$ implies $b \mathrel{\mathcal{R}} a$.

(iii) *Transitive* if $a \mathrel{\mathcal{R}} b$ and $b \mathrel{\mathcal{R}} c$ implies $a \mathrel{\mathcal{R}} c$.

If $\mathcal{R}$ is reflexive, symmetric, and transitive, then it is said to be an *equivalence relation*.

Example 0.3.22: Let $A := \{1, 2, 3\}$. The relation '$<$' is transitive, but neither reflexive nor symmetric. The relation '$\leq$' defined by $\bigl\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\bigr\}$ is reflexive and transitive, but not symmetric. Finally, a relation '$\star$' defined by $\bigl\{(1,1), (1,2), (2,1), (2,2), (3,3)\bigr\}$ is an equivalence relation.

Equivalence relations are useful in that they divide a set into sets of "equivalent" elements.