**21.** This exercise is related to Einstein's famous law $E = mc^2$. The *relativistic momentum* $p$ of a particle of mass $m$ moving at a speed $v$ along a straight line (say, the $x$-axis) is

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} ,$$

where $c$ is the speed of light. The *relativistic force* on the particle along that line is

$$F = \frac{dp}{dt} ,$$

which is the same formula as Newton's Second Law of motion in classical mechanics. Assume that the particle starts at rest at position $x_1$ and ends at position $x_2$ along the $x$-axis. The work done by the force $F$ on the particle is:

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx$$

**(a)** Show that

$$\frac{dp}{dv} = \frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} .$$

**(b)** Use the Chain Rule formula

$$\frac{dp}{dt} = \frac{dp}{dv}\,\frac{dv}{dx}\,\frac{dx}{dt}$$

to show that

$$F\,dx = v\,\frac{dp}{dv}\,dv .$$

**(c)** Use parts (a) and (b) to show that

$$W = \int_0^v \frac{dp}{dv}\,v\,dv = \int_0^v \frac{mv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}\,dv .$$

**(d)** Use part (c) to show that

$$W = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 .$$

**(e)** Define the *relativistic kinetic energy* $K$ of the particle to be $K = W$, and define the *total energy* $E$ to be

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

So by part (d), $K = E - mc^2$. Show that

$$E^2 = p^2c^2 + (mc^2)^2 .$$

*(Hint: Expand the right side of that equation.)*

**(f)** What is $E$ when the particle is at rest?

**22.** A *median* of a triangle is a line segment from a vertex to the midpoint of the opposite side, and the three medians intersect at a common point. Show that this point is a triangle's center of gravity.