$F(z) = \text{Log}(z)$, for all $z \in D$. If C is a contour in D that joins the point z_1 to the point z_2, then Theorem 6.9 implies that

$$\int_C \frac{1}{z}\,dz = \int_{z_1}^{z_2} \frac{1}{z}\,dz = \text{Log}(z_2) - \text{Log}(z_1).$$

Example 6.20. Show that $\int_{C_1^+(0)} \frac{dz}{z} = 2\pi i$.

***Solution*:**

Recall that $C_1^+(0)$ is the unit circle with positive orientation. We let C be that circle with the point -1 omitted, as shown in Figure 6.32(b). The contour C is contained in the simply connected domain D of Example 6.19. We know that $f(z) = \frac{1}{z}$ is analytic in D, and has an antiderivative $F(z) = \text{Log}(z)$, for all $z \in D$. Therefore, if we let z_2 approach -1 on C through the upper half-plane and z_1 approach -1 on C through the lower half-plane,

$$\begin{aligned}
\int_{C_1^+(0)} \frac{1}{z}\,dz &= \lim_{\substack{z_2 \to -1\,(z_2 \in C,\,\text{Im}\,z_2 > 0) \\ z_1 \to -1\,(z_1 \in C,\,\text{Im}\,z_1 < 0)}} \int_{z_1}^{z_2} \frac{1}{z}\,dz \\
&= \lim_{z_2 \to -1\,(z_2 \in C,\,\text{Im}\,z_2 > 0)} \text{Log}(z_2) - \lim_{z_1 \to -1\,(z_1 \in C,\,\text{Im}\,z_1 < 0)} \text{Log}(z_1) \\
&= i\pi - (-i\pi) \\
&= 2\pi i.
\end{aligned}$$

Exercises for Section 6.4 (Selected answers or hints are on page 444.)

For Exercises 1–14, find the value of the definite integral using Theorem 6.9 and explain why you are justified in using it.

1. $\int_C z^2\,dz$, where C is the line segment from $1+i$ to $2+i$.
2. $\int_C \cos z\,dz$, where C is the line segment from $-i$ to $1+i$.
3. $\int_C \exp z\,dz$, where C is the line segment from 2 to $i\frac{\pi}{2}$.
4. $\int_C z\exp z\,dz$, where C is the line segment from $-1-i\frac{\pi}{2}$ to $2+i\pi$.
5. $\int_C \frac{1+z}{z}\,dz$, where C is the line segment from 1 to i.
6. $\int_C \sin\frac{z}{2}\,dz$, where C is the line segment from 0 to $\pi - 2i$.
7. $\int_C (z^2 + z^{-2})dz$, where C is the line segment from i to $1+i$.
8. $\int_C z\exp(z^2)dz$, where C is the line segment from $1-2i$ to $1+2i$.
9. $\int_C z\cos z\,dz$, where C is the line segment from 0 to i.
10. $\int_C \sin^2 z\,dz$, where C is the line segment from 0 to i.
11. $\int_C \text{Log}\,z\,dz$, where C is the line segment from 1 to $1+i$.
12. $\int_C \frac{dz}{z^2-z}$, where C is the line segment from 2 to $2+i$.
13. $\int_C \frac{2z-1}{z^2-z}\,dz$, where C is the line segment from 2 to $2+i$.
14. $\int_C \frac{z-2}{z^2-z}\,dz$, where C is the line segment from 2 to $2+i$.