It appeared as the limit of $(1 + 1/n)^n$. This seems an unnatural way to write down such an important number.

I want to show how $(1 + 1/n)^n$ and $(1 + x/n)^n$ arise naturally. They give ***discrete growth in finite steps***—with applications to compound interest. Loans and life insurance and money market funds use the discrete form of $y' = cy + s$. (We include extra information about bank rates, hoping this may be useful some day.) The applications in science and engineering are equally important. Scientific computing, like accounting, has *difference equations* in parallel with differential equations.

Knowing that this section will be full of formulas, I would like to jump ahead and tell you the best one. It is an infinite series for e^x. What makes the series beautiful is that ***its derivative is itself.***

Start with $y = 1 + x$. This has $y = 1$ and $y' = 1$ at $x = 0$. But y'' is zero, not one. Such a simple function doesn't stand a chance! No polynomial can be its own derivative, because the highest power x^n drops down to nx^{n-1}. The only way is *to have no highest power*. We are forced to consider infinitely many terms—***a power series***—to achieve "derivative equals function."

To produce the derivative $1 + x$, we need $1 + x + \frac{1}{2}x^2$. Then $\frac{1}{2}x^2$ is the derivative of $\frac{1}{6}x^3$, which is the derivative of $\frac{1}{24}x^4$. The best way is to write the whole series at once:

$$\textit{\textbf{Infinite series}} \quad e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots. \tag{1}$$

This must be the greatest power series ever discovered. Its derivative is itself:

$$de^x/dx = 0 + 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = e^x. \tag{2}$$

The derivative of each term is the term before it. The integral of each term is the one after it (so $\int e^x dx = e^x + C$). The approximation $e^x \approx 1 + x$ appears in the first two terms. Other properties like $(e^x)(e^x) = e^{2x}$ are not so obvious. (Multiplying series is hard but interesting.) *It is not even clear why the sum is* 2.718... *when* $x = 1$. Somehow $1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots$ equals e. That is where $(1 + 1/n)^n$ will come in.

Notice that x^n is divided by the product $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. This is "*n factorial.*" Thus x^4 *is divided by* $1 \cdot 2 \cdot 3 \cdot 4 = 4! = 24$, and x^5 is divided by $5! = 120$. The derivative of $x^5/120$ is $x^4/24$, because 5 from the derivative cancels 5 from the factorial. In general $x^n/n!$ has derivative $x^{n-1}/(n-1)!$ Surprisingly $0!$ is 1.

Chapter 10 emphasizes that $x^n/n!$ becomes extremely small as n increases. The infinite series adds up to a finite number—which is e^x. We turn now to discrete growth, which produces the same series in the limit.

This headline was on page one of the New York Times for May 27, 1990.

213 Years After Loan, Uncle Sam is Dunned

San Antonio, May 26—More than 200 years ago, a wealthy Pennsylvania merchant named Jacob DeHaven lent \$450,000 to the Continental Congress to rescue the troops at Valley Forge. That loan was apparently never repaid.

So Mr. DeHaven's descendants are taking the United States Government to court to collect what they believe they are owed. The total: \$141 billion if the interest is compounded daily at 6 percent, the going rate at the time. If compounded yearly, the bill is only \$98 billion.

The thousands of family members scattered around the country say they are not being greedy. "It's not the money—it's the principle of the thing," said Carolyn Cokerham, a DeHaven on her father's side who lives in San Antonio.