Robust linear programming

We consider a linear program in inequality form,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

in which there is some uncertainty or variation in the parameters c, a_i, b_i. To simplify the exposition we assume that c and b_i are fixed, and that a_i are known to lie in given ellipsoids:

$$a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\},$$

where $P_i \in \mathbf{R}^{n \times n}$. (If P_i is singular we obtain 'flat' ellipsoids, of dimension $\mathbf{rank}\, P_i$; $P_i = 0$ means that a_i is known perfectly.)

We will require that the constraints be satisfied for all possible values of the parameters a_i, which leads us to the *robust linear program*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m. \end{array} \tag{4.37}$$

The robust linear constraint, $a_i^T x \leq b_i$ for all $a_i \in \mathcal{E}_i$, can be expressed as

$$\sup\{a_i^T x \mid a_i \in \mathcal{E}_i\} \leq b_i,$$

the lefthand side of which can be expressed as

$$\begin{aligned} \sup\{a_i^T x \mid a_i \in \mathcal{E}_i\} &= \bar{a}_i^T x + \sup\{u^T P_i^T x \mid \|u\|_2 \leq 1\} \\ &= \bar{a}_i^T x + \|P_i^T x\|_2. \end{aligned}$$

Thus, the robust linear constraint can be expressed as

$$\bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i,$$

which is evidently a second-order cone constraint. Hence the robust LP (4.37) can be expressed as the SOCP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m. \end{array}$$

Note that the additional norm terms act as *regularization terms*; they prevent x from being large in directions with considerable uncertainty in the parameters a_i.

Linear programming with random constraints

The robust LP described above can also be considered in a statistical framework. Here we suppose that the parameters a_i are independent Gaussian random vectors, with mean $\bar{a}_i$ and covariance Σ_i. We require that each constraint $a_i^T x \leq b_i$ should hold with a probability (or confidence) exceeding η, where $\eta \geq 0.5$, *i.e.*,

$$\mathbf{prob}(a_i^T x \leq b_i) \geq \eta. \tag{4.38}$$