Exercise 4.1.19. Prove that $p^{1/n}$ is irrational, if p is a prime and n is any integer with $|n| > 1$. $\diamondsuit$

Exercise 4.1.20.

(a) Suppose that a, b, c are integers and $(a/b)^2 = c$. Suppose further that a and b have no common factors except 1: that is, any integer $x > 1$ which divides b doesn't divide a. Prove by contradiction that $b = 1$.

(b) Generalize part (a): Suppose that a, b, c are integers and $(a/b)^n = c$, where n is a positive integer. If a and b have no common factors, prove by contradiction that $b = 1$.

(c) Use part (b) to prove the following: Let a and n be integers, both greater than 1. Let x be a real nth root of a. If x is not an integer, then x is irrational.

$\diamondsuit$

The inconvenient truth expressed in Proposition 4.1.10 forced mathematicians to extend the 'real' numbers to include *irrational* as well as *rational* numbers. But complex numbers opened the floodgates by setting a precedent. New generations of mathematicians became so used to working with "unreal" numbers that they became accustomed to making up other number systems whenever it suited their purpose. Within a few centuries after the complex numbers, several new number systems were created. This eventually prompted mathematicians to study the properties of general numbers systems. The outcome of this is what is known today as abstract algebra!

To close this section, here's another exercise to practice using substitution:

Exercise 4.1.21.

(a) Suppose that:

- a is a negative number;
- n is a positive integer;
- the equation $x^n = a$ has a real solution for the unknown x.