*Proof.* Proposition 1.3.1 gives $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Add these two inequalities to obtain

$$-\big(|x| + |y|\big) \leq x + y \leq |x| + |y|.$$

Apply Proposition 1.3.1 again to find $|x + y| \leq |x| + |y|$. □

There are other often applied versions of the triangle inequality.

**Corollary 1.3.3.** *Let $x, y \in \mathbb{R}$.*

(*i*) (*reverse triangle inequality*) $\big|(|x| - |y|)\big| \leq |x - y|$.

(*ii*) $|x - y| \leq |x| + |y|$.

*Proof.* Let us plug in $x = a - b$ and $y = b$ into the standard triangle inequality to obtain

$$|a| = |a - b + b| \leq |a - b| + |b|,$$

or $|a| - |b| \leq |a - b|$. Switching the roles of $a$ and $b$ we find $|b| - |a| \leq |b - a| = |a - b|$. Applying Proposition 1.3.1, we obtain the reverse triangle inequality.

The second item in the corollary is obtained from the standard triangle inequality by just replacing $y$ with $-y$, and noting $|-y| = |y|$. □

**Corollary 1.3.4.** *Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then*

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

*Proof.* We proceed by induction. The conclusion holds trivially for $n = 1$, and for $n = 2$ it is the standard triangle inequality. Suppose the corollary holds for $n$. Take $n + 1$ numbers $x_1, x_2, \ldots, x_{n+1}$ and first use the standard triangle inequality, then the induction hypothesis

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &\leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \\ &\leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|. \end{aligned}$$ □

Let us see an example of the use of the triangle inequality.

**Example 1.3.5:** Find a number $M$ such that $|x^2 - 9x + 1| \leq M$ for all $-1 \leq x \leq 5$.

Using the triangle inequality, write

$$|x^2 - 9x + 1| \leq |x^2| + |9x| + |1| = |x|^2 + 9|x| + 1.$$

The expression $|x|^2 + 9|x| + 1$ is largest when $|x|$ is largest (why?). In the interval provided, $|x|$ is largest when $x = 5$ and so $|x| = 5$. One possibility for $M$ is

$$M = 5^2 + 9(5) + 1 = 71.$$

There are, of course, other $M$ that work. The bound of 71 is much higher than it need be, but we didn't ask for the best possible $M$, just one that works.

The last example leads us to the concept of bounded functions.