There are many benefits to using differentials—i.e. infinitesimals—in calculus.[10] For example, recall Example 3.31 in Section 3.5 on related rates, where the volume V of a right circular cylinder with radius r and height h changes with time t as

$$\frac{dV}{dt} = \left(2\pi r \cdot \frac{dr}{dt}\right) h + \pi r^2 \cdot \frac{dh}{dt} .$$

The above equation forces you to consider only the derivative with respect to the time variable t. What if you wanted to see the rates of change with respect to another variable, such as r, h, or some other quantity? In that case using the differential version of the above equation, namely

$$dV = 2\pi r h \, dr + \pi r^2 \, dh$$

provides more flexibility—you are free to divide both sides by any differential, not just by dt. Many related rates problems would likely benefit from this approach.

Present-day calculus textbooks confuse the notion of a differential (infinitesimal) dx with the idea of a small but real value Δx. The two are *not* the same. An infinitesimal is *not* a real number and *cannot* be assigned a real value, no matter how small; Δx *can* be assigned real values. Using dx and Δx interchangeably is a source of much confusion for students (likewise for dy and Δy). This confusion rears its head in exercises involving the linear approximation of a curve by its tangent line near a point x_0, namely $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ when $x - x_0$ is "small" (e.g. $\sqrt{63} \approx 7.9375$, by using $f(x) = \sqrt{x}$, $x = 63$, $x_0 = 64$, and $x - x_o = \Delta x = -1$). Such exercises have nothing to do with differentials, not to mention having dubious value nowadays. They are remnants of a bygone era, before the advent of modern computing obviated the need for such (generally) poor approximations.

Exercises

A

1. Find the differential df of $f(x) = x^2 - 2x + 5$.

2. Find the differential df of $f(x) = \sin^2(x^2)$.

3. Show that $d\left(\tan^{-1}(y/x)\right) = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$

4. Given $y^2 - xy + 2x^2 = 3$, find dy.

5. The *elasticity* of a function $y = f(x)$ is $E(y) = \dfrac{x}{y} \cdot \dfrac{dy}{dx}$. Show that $E(y) = \dfrac{d(\ln y)}{d(\ln x)}$.

6. Prove the differential version of the Quotient Rule:

$$d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$$

[10] For an excellent overview on this subject, see DRAY, T. AND C.A. MANOGUE, Putting Differentials Back into Calculus, *College Math. J.* **41** (2010), 90-100. Some of the material in this section is indebted to that paper, which is available at `http://www.math.oregonstate.edu/bridge/papers/differentials.pdf`