1.1.1 Exponential Growth

We begin our study of ordinary differential equations by modeling some real world phenomena. For a particular situation that we might wish to investigate, our first task is to write an equation (or equations) that best describes the phenomenon. Suppose that we wish to study how a population $P(t)$ grows with time t. We might make the assumption that a constant fraction of the population is having offspring at any particular time. If we also assume that the population has a constant death rate, the change in the population ΔP during a small time interval Δt will be

$$\Delta P \approx k_{\text{birth}} P(t) \Delta t - k_{\text{death}} P(t) \Delta t,$$

where k_{birth} is the fraction of the population having offspring during the interval and k_{death} is the fraction of the population that dies during the interval. Equivalently, we can write

$$\frac{\Delta P}{\Delta t} \approx kP(t),$$

where $k = k_{\text{birth}} - k_{\text{death}}$. Since the derivative of P is

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t},$$

the rate of change of the population is proportional to the size of the population, or

$$\frac{dP}{dt} = kP. \tag{1.1.1}$$

The equation

$$\frac{dP}{dt} = kP$$

is one of the simplest differential equations that we will consider. The equation tells us that the population grows in proportion to its current size. It is not too difficult to see that $P(t) = Ce^{kt}$ is a solution to this equation, where C is an arbitrary constant. Indeed, if we differentiate $P(t)$, we obtain

$$\frac{d}{dt} P(t) = kCe^{kt} = kP(t).$$

In addition, if we know the value of $P(t)$, say when $t = 0$, we can also determine the value of C. For example, if the population at the time $t = 0$ is $P(0) = P_0$, then

$$P_0 = P(0) = Ce^{k \cdot 0} = C$$

or $P(t) = P_0 e^{kt}$. The differential equation

$$\begin{aligned} P'(t) &= kP(t), \\ P(0) &= P_0 \end{aligned}$$

is an example of an **initial value problem** or **IVP**, and we say that $P(0) = P_0$ is an **initial condition**. Since the solution to equation (1.1.1) is $P(t) = Ce^{kt}$, we say that the population grows **exponentially**.

As an example, suppose that $P(t)$ is a population of a colony of bacteria at time t, whose initial population is 1000 at $t = 0$, where time is measured in hours. Then

$$1000 = P(0) = Ce^0 = C,$$

and our solution becomes $P(t) = 1000e^{kt}$. If the population grows at three percent per hour, then

$$1030 = P(1) = 1000e^k,$$