5.4 Integration by Substitution

The integrals encountered so far—whether indefinite or definite—have been the simplest kind, since the antiderivatives had known formulas. For example, $\int \cos x \, dx = \sin x + C$. What if the integral were $\int \cos 2x \, dx$ instead? No formula has been discussed yet for this integral, and the answer is not $\sin 2x + C$, since the derivative of $\sin 2x$ is $2\cos 2x$, not $\cos 2x$. But dividing $\sin 2x$ by 2 first and then taking the derivative *would* yield $\cos 2x$, so that $\int \cos 2x \, dx = \frac{1}{2}\sin 2x + C$.

Evaluating an integral in such a manner is often done when the function is not too complicated, as the one above. Usually it will not be quite that simple, and so a general technique called **substitution** is needed. The idea behind substitution is to replace part of the function being integrated by a new variable—typically u—so that a complicated function of x is now a simpler function of u that you know how to integrate.

Example 5.16

Evaluate $\displaystyle\int \cos 2x \, dx$ by substitution.

Solution: The $2x$ in the cosine function is what makes this integral unknown, so replace it by u: let $u = 2x$. The integral is now

$$\int \cos u \, dx$$

which is a problem because the point of doing substitution is to eliminate all references to the variable x, including in the infinitesimal dx. The entire integral needs to be in terms of u and du, but the dx is still there. So put dx in terms of du:

$$u = 2x \quad \Rightarrow \quad du = 2dx \quad \Rightarrow \quad dx = \frac{1}{2}du$$

The integral now becomes

$$\int \cos u \left(\frac{1}{2}du\right) = \frac{1}{2}\int \cos u \, du = \frac{1}{2}\sin u + C$$

by the formula already known, just with the letter u as the variable instead of x. The original integral was in terms of x, so the final answer—for an indefinite integral—should also be in terms of x. Thus, the final step is to substitute back into the answer what u equals in terms of x, namely $2x$:

$$\int \cos 2x \, dx = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$$

If the procedure in the above example seems similar to making a substitution when using the Chain Rule to take a derivative, that is because it is similar: you are basically doing the same thing only in reverse. Just as for differentiation, it is not always obvious what part of the function is the best candidate for substitution when performing integration. There is one obvious rule: *never* make the substitution $u = x$, because that changes nothing.