For example, we might consider an RLC circuit with $R = 1$, $L = 1$, and $C = 1$. At $t = 0$ when both $I(0) = 0$ and $I'(0) = Q(0) = 0$, the impressed voltage on the circuit is given by $E(t) = \sin(t)$. Our equation becomes

$$I'' + I' + I = E'(t) = \cos t.$$

This is an example of a **second-order linear differential equation**.

4.1.2 Second-Order Linear Equations

Suppose that we have a homogeneous second-order linear differential equation with constant coefficients,

$$ax'' + bx' + cx = 0. \tag{4.1.2}$$

The goal of this section is to be able to solve all such equations. However, we did a great deal of work finding unique solutions to systems of first-order linear systems equations in Chapter 3. Our efforts are now rewarded. Since each second-order homogeneous system with constant coefficients can be rewritten as a first-order linear system, we are guaranteed the existence and uniqueness of solutions. Indeed, we can rewrite (4.1.2) as a system of first-order linear equations,

$$\begin{aligned} x' &= y \\ y' &= -\frac{c}{a}x - \frac{b}{a}y, \end{aligned}$$

and then find the general solution by computing the eigenvalues and eigenvectors of the matrix of the corresponding system.

Example 4.1.2 Solutions of a linear system $\mathbf{x}' = A\mathbf{x}$ often include terms of the form e^{rt}. It makes sense that solutions to equation (4.1.2) take the same form. Consider the equation

$$x'' + 3x' - 10x = 0. \tag{4.1.3}$$

If we assume that a solution is of the form e^{rt}, we can substitute this expression into the left-hand side of (4.1.3) to obtain

$$\begin{aligned} \frac{d^2}{dt^2}e^{rt} + \frac{d}{dt}3e^{rt} - 10e^{rt} &= r^2e^{rt} + 3re^{rt} - 10e^{rt} \\ &= (r^2 + 3r - 10)e^{rt} \\ &= (r+5)(r-2)e^{rt}. \end{aligned}$$

Since e^{rt} is never zero, we find that $(r + 5)(r - 2) = 0$ or $r = -5$ or 2. Thus, we have two solutions

$$x_1(t) = e^{-5t} \text{ and } x_2(t) = e^{2t}.$$

By the Principle of Superposition,

$$x(t) = c_1x_1(t) + c_2x_2(t) = c_1e^{-5t} + c_2e^{2t} \tag{4.1.4}$$

is a solution to $x'' + 3x' - 10x = 0$.

Indeed, this is the general solution of our second-order equation since we have a one-to-one correspondence between the solutions of

$$x'' + 3x' - 10x = 0$$