(b) Solve the equation $xy' + y = x^4 y^3$.

28. The first-order nonlinear differential equation

$$y' = p(t) + q(t)y + r(t)y^2 \tag{1.5.12}$$

is known as the **Ricatti equation** and has some useful applications in control theory. If one solution $y_1(t)$ of the Ricatti equation is known, then a more general solution containing an arbitrary constant can be found by substituting $y = y_1(t) + 1/v(t)$ into equation (1.5.12) to find a first-order linear equation in v and t, which we can solve to find a general solution to the Ricatti equation.

(a) Show that this first-order linear equation is $v' + [q(t) + 2r(t)y_1(t)]v = -r(t)$.

(b) Find the solution to the Ricatti equation

$$y' = -\frac{1}{t^2} - \frac{1}{t}y + y^2$$

given the particular solution $y_1 = 1/t$.

(c) Find the solution to the Ricatti equation

$$y' = \cos t - y \tan t + y^2 \sec t$$

given the particular solution $y_1 = \sin t$.

(d) Find the solution to the Ricatti equation

$$y' = 2 - 3y + y^2$$

given the particular solution $y_1 \equiv 2$.

Hint.

(a) If $y = y_1 + 1/v$, then $y' = y_1' - v'/v^2$. Substituting into our original equation, we obtain

$$y' = y_1' - \frac{v'}{v^2} = p + qy_1 + ry_1^2 - \frac{v'}{v^2}.$$

On the other hand,

$$\begin{aligned} y' &= p + q\left(y_1 + \frac{1}{v}\right) + r\left(y_1 + \frac{1}{v}\right)^2 \\ &= p + qy_1 + \frac{q}{v} + ry_1^2 + \frac{2ry_1}{v} + \frac{r}{v^2} \\ &= y_1' + \frac{q}{v} + \frac{2ry_1}{v} + \frac{r}{v^2}. \end{aligned}$$

Therefore,

$$-\frac{v'}{v^2} = \frac{q}{v} + \frac{2ry_1}{v} + \frac{r}{v^2},$$

which is just the first-order linear equation

$$v' + [q(t) + 2r(t)y_1(t)]v = -r(t).$$

[2] Polking p. 63