$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right) \approx n\left(\frac{1}{n}\right) = 1. \tag{3}$$

As $1/n$ gets smaller, this approximation gets better. The limit is 1. Conclusion: $(1 + 1/n)^n$ approaches the number whose logarithm is 1. Sections 6.2 and 6.4 define the same number (which is e).

2. *Slow method for* $(1 + 1/n)^n$: ***Multiply out all the terms. Then let*** $n \to \infty$.

This is a brutal use of the binomial theorem. It involves nothing smart like logarithms, but the result is a fantastic new formula for e.

$$\text{Practice for } n=3: \quad \left(1+\frac{1}{3}\right)^3 = 1 + 3\left(\frac{1}{3}\right) + \frac{3\cdot 2}{1\cdot 2}\left(\frac{1}{3}\right)^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3.$$

Binomial theorem for any positive integer n:

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n. \tag{4}$$

Each term in equation (4) approaches a limit as $n \to \infty$. Typical terms are

$$\frac{n(n-1)}{1\cdot 2}\left(\frac{1}{n}\right)^2 \to \frac{1}{1\cdot 2} \qquad \text{and} \qquad \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3 \to \frac{1}{1\cdot 2\cdot 3}.$$

Next comes $1/1\cdot 2\cdot 3\cdot 4$. The sum of all those limits in (4) is our new formula for e:

$$\lim\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots = e. \tag{5}$$

In summation notation this is $\Sigma_{k=0}^{\infty} 1/k! = e$. The factorials give fast convergence:

$$1 + 1 + .5 + .16667 + .04167 + .00833 + .00139 + .00020 + .00002 = 2.71828.$$

Those nine terms give an accuracy that was not reached by $n = 365$ compoundings. A limit is still involved (to add up the whole series). ***You never see e without a limit!*** It can be defined by derivatives or integrals or powers $(1 + 1/n)^n$ or by an infinite series. Something goes to zero or infinity, and care is required.

All terms in equation (4) are below (or equal to) the corresponding terms in (5). ***The power*** $(1 + 1/n)^n$ ***approaches e from below.*** There is a steady increase with n. Faster compounding yields more interest. Continuous compounding at 100% yields e, as each term in (4) moves up to its limit in (5).

Remark Change $(1 + 1/n)^n$ to $(1 + x/n)^n$. Now the binomial theorem produces e^x:

$$\left(1+\frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{1\cdot 2}\left(\frac{x}{n}\right)^2 + \cdots \text{ approaches } 1 + x + \frac{x^2}{1\cdot 2} + \cdots. \tag{6}$$

Please recognize e^x on the right side! It is the infinite power series in equation (1). The next term is $x^3/6$ (x can be positive or negative). This is a final formula for e^x:

6L The limit of $(1 + x/n)^n$ is e^x. At $x = 1$ we find e.

The logarithm of that power is $n\ln(1 + x/n) \approx n(x/n) = x$. The power approaches e^x.

To summarize: The quick method proves $(1 + 1/n)^n \to e$ by logarithms. The slow method (multiplying out every term) led to the infinite series. Together they show the agreement of all our definitions of e.