Suppose that if T is a Cayley table of a group G. Then $g \in G$ appears on the diagonal of T if and only if there is an element $g' \in G$ such that $g' \cdot g' = g$. It turns out that this property is preserved under isomorphism:

Proposition 20.3.20. Given a Cayley table T for a finite group G, and let $g \in G$ appears on the diagonal of T. Let $\phi : G \to H$ be an isomorphism, and let T' be a Cayley table of H. Then $\phi(g)$ appears on the diagonal of T'.

PROOF. As stated above, g appears on the diagonal of T if and only if there exists $g' \in G$ such that $g' \cdot g' = g$. Since ϕ is an isomorphism, this implies $\phi(g') \cdot \phi(g') = \phi(g)$, which in turn implies that $\phi(g)$ appears on the diagonal of T'. □

Proposition 20.3.21. Given a Cayley table T for a finite group G, and suppose the element $g \in G$ appears m times on the diagonal of T. Let $\phi : G \to H$ be an isomorphism, and let T' be a Cayley table of H. Then $\phi(g)$ appears m times on the diagonal of T'.

Exercise 20.3.22. Prove Proposition 20.3.21. ◊

Proposition 20.3.23. Given a Cayley table T for a finite group G, and suppose n distinct elements of G appear on the diagonal of T. Let $\phi : G \to H$ be an isomorphism, and let T' be a Cayley table of H. Then n distinct elements of H appear on the diagonal of T'.

Exercise 20.3.24. Prove Proposition 20.3.23. ◊

Exercise 20.3.25. By using the preceding propositions and comparing diagonal elements of Cayley tables, prove that $\mathbb{Z}_4 \not\cong U(12)$. ◊

Exercise 20.3.26. Prove or disprove: $U(8) \cong \mathbb{Z}_4$. ◊

Exercise 20.3.27. Let σ be the permutation (12), and let τ be the permutation (34). Let G be the set $\{\mathsf{id}, \sigma, \tau, \sigma\tau\}$ together with the operation of composition.