At $x=0$ those derivatives are $\frac{1}{2}$, $-\frac{1}{4}$, $\frac{3}{8}$. Dividing by 1!, 2!, 3! gives

$$a_1=\frac{1}{2}\quad a_2=-\frac{1}{8}\quad a_3=\frac{1}{16}\qquad a_n=\frac{1}{n!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right).$$

These are the ***binomial coefficients*** when the power is $p=\frac{1}{2}$.

Notice the difference from the binomials in Chapter 2. For those, the power p was a positive integer. The series $(1+x)^2=1+2x+x^2$ stopped at x^2. The coefficients for $p=2$ were $1, 2, 1, 0, 0, 0, \ldots$. For fractional p or negative p those later coefficients are *not* zero, and we find them from the derivatives of $(1+x)^p$:

$$(1+x)^p\quad p(1+x)^{p-1}\quad p(p-1)(1+x)^{p-2}\qquad f^{(n)}=p(p-1)\cdots(p-n+1)(1+x)^{p-n}.$$

Dividing by $0!, 1!, 2!, \ldots, n!$ at $x=0$, the binomial coefficients are

$$1\quad p\quad \frac{p(p-1)}{2}\quad \cdots\quad \frac{f^{(n)}(0)}{n!}=\frac{p(p-1)\cdots(p-n+1)}{n!}. \tag{5}$$

For $p=n$ that last binomial coefficient is $n!/n!=1$. It gives the final x^n at the end of $(1+x)^n$. For other values of p, the binomial series never stops. ***It converges for*** $|x|<1$:

$$(1+x)^p=1+px+\frac{p(p-1)}{2}x^2+\cdots=\sum_{n=0}^{\infty}\frac{p(p-1)\cdots(p-n+1)}{n!}x^n. \tag{6}$$

When $p=1, 2, 3, \ldots$ ***the binomial coefficient*** $p!/n!(n-p)!$ ***counts the number of ways to select a group of n friends out of a group of p friends.*** If you have 20 friends, you can choose 2 of them in $(20)(19)/2=190$ ways.

Suppose p is not a positive integer. What goes wrong with $(1+x)^p$, to stop the convergence at $|x|=1$? The failure is at $x=-1$. If p is negative, $(1+x)^p$ blows up. If p is positive, as in $\sqrt{1+x}$, the higher derivatives blow up. Only for a positive integer $p=n$ does the convergence radius move out to $r=\infty$. In that case the series for $(1+x)^n$ stops at x^n, and f never fails.

A power series is a function in a new form. It is not a simple form, but sometimes it is the only form. To compute f we have to sum the series. To square f we have to multiply series. But the operations of calculus—derivative and integral—are easier. That explains why power series help to solve differential equations, which are a rich source of new functions. (Numerically the series are not always so good.) I should have said that the derivative and integral are easy *for each separate term* a_nx^n—and fortunately the convergence radius of the whole series is not changed.

If $f(x)=\Sigma a_nx^n$ ***has convergence radius*** r***, so do its derivative and its integral:***

$df/dx=\Sigma na_nx^{n-1}$ and $\int f(x)dx=\Sigma a_nx^{n+1}/(n+1)$ also converge for $|x|<r$.

EXAMPLE 5 The series for $1/(1-x)$ and its derivative $1/(1-x)^2$ and its integral $-\ln(1-x)$ all have $r=1$ (because they all have trouble at $x=1$). The series are Σx^n and Σnx^{n-1} and $\Sigma x^{n+1}/(n+1)$.

EXAMPLE 6 We can integrate e^{x^2} (previously impossible) by integrating every term in its series:

$$\int e^{x^2}dx=\int\left(1+x^2+\frac{1}{2!}x^4+\cdots\right)dx=x+\frac{x^3}{3}+\frac{1}{2!}\left(\frac{x^5}{5}\right)+\frac{1}{3!}\left(\frac{x^7}{7}\right)+\cdots.$$

This always converges ($r=\infty$). The derivative of e^{x^2} was never a problem.