(b) *Minimizing a linear function over an ellipsoid.*

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & (x - x_c)^T A(x - x_c) \le 1, \end{array}$$

where $A \in \mathbf{S}^n_{++}$ and $c \neq 0$.

(c) *Minimizing a quadratic form over an ellipsoid centered at the origin.*

$$\begin{array}{ll} \text{minimize} & x^T Bx \\ \text{subject to} & x^T Ax \le 1, \end{array}$$

where $A \in \mathbf{S}^n_{++}$ and $B \in \mathbf{S}^n_{+}$. Also consider the nonconvex extension with $B \notin \mathbf{S}^n_{+}$. (See §B.1.)

4.22 Consider the QCQP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & x^T x \le 1, \end{array}$$

with $P \in \mathbf{S}^n_{++}$. Show that $x^\star = -(P + \lambda I)^{-1} q$ where $\lambda = \max\{0, \bar{\lambda}\}$ and $\bar{\lambda}$ is the largest solution of the nonlinear equation

$$q^T (P + \lambda I)^{-2} q = 1.$$

4.23 *ℓ_4-norm approximation via QCQP.* Formulate the ℓ_4-norm approximation problem

$$\text{minimize} \quad \|Ax - b\|_4 = (\textstyle\sum_{i=1}^m (a_i^T x - b_i)^4)^{1/4}$$

as a QCQP. The matrix $A \in \mathbf{R}^{m\times n}$ (with rows a_i^T) and the vector $b \in \mathbf{R}^m$ are given.

4.24 *Complex ℓ_1-, ℓ_2- and ℓ_∞-norm approximation.* Consider the problem

$$\text{minimize} \quad \|Ax - b\|_p,$$

where $A \in \mathbf{C}^{m\times n}$, $b \in \mathbf{C}^m$, and the variable is $x \in \mathbf{C}^n$. The complex ℓ_p-norm is defined by

$$\|y\|_p = \left(\sum_{i=1}^m |y_i|^p \right)^{1/p}$$

for $p \ge 1$, and $\|y\|_\infty = \max_{i=1,\dots,m} |y_i|$. For $p = 1$, 2, and ∞, express the complex ℓ_p-norm approximation problem as a QCQP or SOCP with real variables and data.

4.25 *Linear separation of two sets of ellipsoids.* Suppose we are given $K + L$ ellipsoids

$$\mathcal{E}_i = \{P_i u + q_i \mid \|u\|_2 \le 1\}, \quad i = 1, \dots, K + L,$$

where $P_i \in \mathbf{S}^n$. We are interested in finding a hyperplane that strictly separates $\mathcal{E}_1, \dots, \mathcal{E}_K$ from $\mathcal{E}_{K+1}, \dots, \mathcal{E}_{K+L}$, *i.e.*, we want to compute $a \in \mathbf{R}^n$, $b \in \mathbf{R}$ such that

$$a^T x + b > 0 \text{ for } x \in \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_K, \qquad a^T x + b < 0 \text{ for } x \in \mathcal{E}_{K+1} \cup \cdots \cup \mathcal{E}_{K+L},$$

or prove that no such hyperplane exists. Express this problem as an SOCP feasibility problem.

4.26 *Hyperbolic constraints as SOC constraints.* Verify that $x \in \mathbf{R}^n$, $y, z \in \mathbf{R}$ satisfy

$$x^T x \le yz, \qquad y \ge 0, \qquad z \ge 0$$

if and only if

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \le y + z, \qquad y \ge 0, \qquad z \ge 0.$$

Use this observation to cast the following problems as SOCPs.