19. Show that $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0.$

20. Show that $\displaystyle\int_0^\infty \frac{x^a}{a^x}\,dx = \frac{\Gamma(a+1)}{(\ln a)^{a+1}}$ for $a > 1$.

21. Use the result from Example 6.28 to show that

$$\frac{d^{1/2}}{dx^{1/2}}\left(\frac{d^{1/2}}{dx^{1/2}}(x)\right) = 1 = \frac{d}{dx}(x)\,.$$

22. Calculate $\dfrac{d^{1/2}}{dx^{1/2}}(c)$ for all constants c.

23. Calculate $\dfrac{d^{1/3}}{dx^{1/3}}(x)$.

24. Show that $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^n}}\,dx = \tfrac{1}{n}B\left(\tfrac{1}{n},\tfrac{1}{2}\right)$ for $n \ge 1$.

25. Show that the Gamma function $\Gamma(t)$ can be written as

$$\Gamma(t) = p^t \int_0^\infty u^{t-1}e^{-pu}\,du \quad \text{for all } t>0 \text{ and } p>0.$$

26. Show that the Gamma function $\Gamma(t)$ can be written as

$$\Gamma(t) = \int_0^1 \left(\ln\left(\frac{1}{u}\right)\right)^{t-1} du \quad \text{for all } t>0.$$

27. Using the result from Exercise 27 in Section 6.1 that

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

for all constants a and $b \neq 0$, differentiate under the integral sign to show that for all $\alpha > 0$

$$\int_0^\infty x e^{-x}\sin\alpha x\,dx = \frac{2\alpha}{(1+\alpha^2)^2}\,.$$

C

28. Use the Leibniz rule and formula (6.8) from Section 6.3 to show that for all $a>0$,

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right| + C\,.$$

29. Use Example 6.27 to show that the Beta function satisfies the relation

$$B(x,1-x) = \int_0^1 \frac{t^{-x}+t^{x-1}}{1+t}\,dt \quad \text{for all } 0<x<1.$$

(Hint: First use a substitution to show that $\displaystyle\int_0^\infty \frac{u^{x-1}}{1+u}\,du = \int_0^\infty \frac{t^{-x}}{1+t}\,dt$.*)*

30. Show that for all $a > -1$,

$$\int_0^{\pi/2} \frac{d\theta}{1+a\sin^2\theta} = \frac{\pi}{2\sqrt{1+a}}\,.$$