**Theorem 3.6** Given $n$ Bernoulli trials with probability $p$ of success on each experiment, the probability of exactly $j$ successes is

$$b(n,p,j) = \binom{n}{j} p^j q^{n-j}$$

where $q = 1 - p$.

**Proof.** We construct a tree measure as described above. We want to find the sum of the probabilities for all paths which have exactly $j$ successes and $n - j$ failures. Each such path is assigned a probability $p^j q^{n-j}$. How many such paths are there? To specify a path, we have to pick, from the $n$ possible trials, a subset of $j$ to be successes, with the remaining $n - j$ outcomes being failures. We can do this in $\binom{n}{j}$ ways. Thus the sum of the probabilities is

$$b(n,p,j) = \binom{n}{j} p^j q^{n-j} \ .$$

$\square$

**Example 3.8** A fair coin is tossed six times. What is the probability that exactly three heads turn up? The answer is

$$b(6,.5,3) = \binom{6}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 = 20 \cdot \frac{1}{64} = .3125 \ .$$

$\square$

**Example 3.9** A die is rolled four times. What is the probability that we obtain exactly one 6? We treat this as Bernoulli trials with *success* = "rolling a 6" and *failure* = "rolling some number other than a 6." Then $p = 1/6$, and the probability of exactly one success in four trials is

$$b(4,1/6,1) = \binom{4}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 = .386 \ .$$

$\square$

To compute binomial probabilities using the computer, multiply the function choose$(n,k)$ by $p^k q^{n-k}$. The program **BinomialProbabilities** prints out the binomial probabilities $b(n,p,k)$ for $k$ between $kmin$ and $kmax$, and the sum of these probabilities. We have run this program for $n = 100$, $p = 1/2$, $kmin = 45$, and $kmax = 55$; the output is shown in Table 3.8. Note that the individual probabilities are quite small. The probability of exactly 50 heads in 100 tosses of a coin is about .08. Our intuition tells us that this is the most likely outcome, which is correct; but, all the same, it is not a very likely outcome.