involving powers of z. In the punctured disk $D_1^*(0)$, $g(z) = \sum\limits_{n=0}^{\infty}[(-1)^n + \frac{1}{2^{n+1}}]z^{n-1}$. Computing the first few coefficients, we obtain

$$g(z) = \frac{3}{2}\frac{1}{z} - \frac{3}{4} + \frac{9}{8}z - \frac{15}{16}z^2 + \cdots .$$

Therefore, $\text{Res}[g, 0] = a_{-1} = \frac{3}{2}$.

Recall that, for a function f analytic in $D_R^*(z_0)$ and for any r with $0 < r < R$, the Laurent series coefficients of f are given by

$$a_n = \frac{1}{2\pi i} \int\limits_{C_r^+(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad \text{for} \quad n = 0,\, \pm 1,\, \pm 2,\, \ldots, \tag{8.1}$$

where $C_r^+(z_0)$ denotes the circle $\{z : |z - z_0| = r\}$ with positive orientation. This result gives us an important fact concerning $\text{Res}[f, z_0]$. If we set $n = -1$ in Equation (8.1) and replace $C_r^+(z_0)$ with any positively oriented simple closed contour C containing z_0, provided z_0 is the still only singularity of f that lies inside C, then we obtain

$$\int\limits_C f(z)\, dz = 2\pi i a_{-1} = 2\pi i \text{Res}[f, z_0]. \tag{8.2}$$

If we are able to find the Laurent series expansion for f, then Equation (8.2) gives us an important tool for evaluating contour integrals.

Example 8.3. Evaluate $\int\limits_{C_1^+(0)} \exp(\frac{2}{z})\, dz$.

Solution:

Example 8.1 showed that the residue of $f(z) = \exp\left(\frac{2}{z}\right)$ at $z_0 = 0$ is $\text{Res}[f, 0] = 2$. Using Equation (8.2), we get

$$\int\limits_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz = 2\pi i \text{Res}[f, 0] = 4\pi i.$$

Theorem 8.1 (Cauchy's residue theorem)**.** *Let D be a simply connected domain and let C be a simple closed positively oriented contour that lies in D. If f is analytic inside C and on C, except at the points $z_1,\, z_2, \ldots, z_n$ that lie inside C, then*

$$\int\limits_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f, z_k].$$

The situation is illustrated in Figure 8.1.

Proof. Since there are a finite number of singular points inside C, there exists an $r > 0$ such that the positively oriented circles $C_k = C_r^+(z_k)$, for $k = 1, 2, \ldots, n$, are mutually disjoint and all lie inside C. From the extended Cauchy-Goursat theorem (Theorem 6.7 on page 189), it follows that

$$\int\limits_C f(z)\, dz = \sum_{k=1}^{n} \int\limits_{C_k} f(z)\, dz.$$