other state s_k, the expected number of steps required is m_{kj} plus 1 for the step already taken. Thus,

$$m_{ij} = p_{ij} + \sum_{k \neq j} p_{ik}(m_{kj} + 1) ,$$

or, since $\sum_k p_{ik} = 1$,

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj} . \tag{11.2}$$

Similarly, starting in s_i, it must take at least one step to return. Considering all possible first steps gives us

$$\begin{aligned} r_i &= \sum_k p_{ik}(m_{ki} + 1) & (11.3) \\ &= 1 + \sum_k p_{ik} m_{ki} . & (11.4) \end{aligned}$$

Mean First Passage Matrix and Mean Recurrence Matrix

Let us now define two matrices $\mathbf{M}$ and $\mathbf{D}$. The ijth entry m_{ij} of $\mathbf{M}$ is the mean first passage time to go from s_i to s_j if $i \neq j$; the diagonal entries are 0. The matrix $\mathbf{M}$ is called the *mean first passage matrix.* The matrix $\mathbf{D}$ is the matrix with all entries 0 except the diagonal entries $d_{ii} = r_i$. The matrix $\mathbf{D}$ is called the *mean recurrence matrix.* Let $\mathbf{C}$ be an $r \times r$ matrix with all entries 1. Using Equation 11.2 for the case $i \neq j$ and Equation 11.4 for the case $i = j$, we obtain the matrix equation

$$\mathbf{M} = \mathbf{PM} + \mathbf{C} - \mathbf{D} , \tag{11.5}$$

or

$$(\mathbf{I} - \mathbf{P})\mathbf{M} = \mathbf{C} - \mathbf{D} . \tag{11.6}$$

Equation 11.6 with $m_{ii} = 0$ implies Equations 11.2 and 11.4. We are now in a position to prove our first basic theorem.

Theorem 11.15 For an ergodic Markov chain, the mean recurrence time for state s_i is $r_i = 1/w_i$, where w_i is the ith component of the fixed probability vector for the transition matrix.

Proof. Multiplying both sides of Equation 11.6 by $\mathbf{w}$ and using the fact that

$$\mathbf{w}(\mathbf{I} - \mathbf{P}) = \mathbf{0}$$

gives

$$\mathbf{wC} - \mathbf{wD} = \mathbf{0} .$$

Here $\mathbf{wC}$ is a row vector with all entries 1 and $\mathbf{wD}$ is a row vector with ith entry $w_i r_i$. Thus

$$(1, 1, \ldots, 1) = (w_1 r_1, w_2 r_2, \ldots, w_n r_n)$$

and

$$r_i = 1/w_i ,$$

as was to be proved. $\square$