(with its minus sign) slows the growth down. The solution follows an S-curve that we can compute exactly.

What are the numbers b and c for human population? Ecologists estimate the natural growth rate as $c = .029/\text{year}$. That is not the actual rate, because of b. About 1930, the world population was 3 billion. The cy term predicts a yearly increase of (.029)(3 billion) = 87 million. The actual growth was more like $dy/dt = 60$ million/year. That difference of 27 million/year was by^2:

$$27 \text{ million/year} = b(3 \text{ billion})^2 \text{ leads to } b = 3 \cdot 10^{-12}/\text{year}.$$

Certainly b is a small number (three trillionths) but its effect is not small. It reduces 87 to 60. What is fascinating is to calculate the ***steady state***, when the new term by^2 equals the old term cy. When these terms cancel each other, $dy/dt = cy - by^2$ is zero. The loss from competition balances the gain from new growth: $cy = by^2$ and $y = c/b$. The growth stops at this equilibrium point—the top of the S-curve:

$$y_\infty = \frac{c}{b} = \frac{.029}{3} 10^{12} \approx 10 \text{ billion people.}$$

According to Verhulst's logistic equation, *the world population is converging to* 10 *billion*. That is from the model. From present indications we are growing much faster. We will very probably go beyond 10 billion. The United Nations report in Section 3.3 predicts 11 billion to 14 billion.

Notice a special point halfway to $y_\infty = c/b$. (In the model this point is at 5 billion.) It is the *inflection point* where the S-curve begins to bend down. The second derivative d^2y/dt^2 is zero. The slope dy/dt is a maximum. It is easier to find this point from the differential equation (which gives dy/dt) than from y. Take one more derivative:

$$y'' = (cy - by^2)' = cy' - 2byy' = (c - 2by)y'. \tag{8}$$

The factor $c - 2by$ is zero at the inflection point $y = c/2b$, halfway up the S-curve.

THE S-CURVE

The logistic equation is solved by separating variables y ***and*** t:

$$dy/dt = cy - by^2 \text{ becomes } \int dy/(cy - by^2) = \int dt. \tag{9}$$

The first question is whether we recognize this y-integral. *No*. The second question is whether it is listed in the cover of the book. *No*. The nearest is $\int dx/(a^2 - x^2)$, which can be reached with considerable manipulation (Problem 21). The third question is whether a general method is available. *Yes*. "Partial fractions" is perfectly suited to $1/(cy - by^2)$, and Section 7.4 gives the following integral of equation (9):

$$\ln \frac{y}{c - by} = ct + C \qquad \text{and then} \qquad \ln \frac{y_0}{c - by_0} = C. \tag{10}$$

That constant C makes the solution correct at $t = 0$. The logistic equation is integrated, but the solution can be improved. Take exponentials of both sides to remove the logarithms:

$$\frac{y}{c - by} = e^{ct} \frac{y_0}{c - by_0}. \tag{11}$$

This contains the same growth factor e^{ct} as in linear equations. But the logistic