Stellar Structure and Evolution
Forms of Pressure throughout Stellar Evolution

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Stars are massive gaseous objects described by the mass continuity equation,

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

where M is mass, r is radius and ρ is the density. Stellar structures consist mostly of hydrogen(X), helium(Y) and other heavy metals(Z). Gravitational forces are pushing to compress the star inward to its centre. Depending on mass and other properties such as ratio of particle types, an outward pressure counters gravitational pressure until hydrostatic equilibrium is met described by,

$$\frac{dP}{dr} = \frac{-GM(r)\rho(r)}{r^2} \tag{2}$$

where P is pressure and G is the gravitational constant.

Main sequence stars rely heavily on gas pressure given by,

$$P_g = \frac{\rho k_B T}{\mu m_H} \tag{3}$$

where T is temperature, k_B is Boltzmann's constant and μ is mean mass per particle. Gas pressure combined with radiation pressure apposes gravitational pressure for main sequence stars. Gas pressure arises when particles within the star increase in velocity and collision rates, when compressed by gravitational forces and energy is transferred from nuclear fusion outward from the core. This results in average particle kinetic energy increasing, creating an outward pressure.

Momentum from photons colliding with atoms in main sequence stars results in radiation pressure outward given by,

$$P_r = \frac{4\sigma T^4}{3c} = \frac{aT^4}{3} \tag{4}$$

where σ is the gas constant and c is the speed of light. Atoms are inefficient at scattering photons so at this point radiation pressure is minute. However when stellar evolution takes place, radiation pressure becomes more important. Ions and fermions are more efficient at scattering photon momentum. Low temperature, fermion rich stars such as old white dwarfs rely on radiation pressure.

Once fusion fuel sources run out, main sequence stars evolves off main sequence. Radiation pressure from nuclear fusion is still present, but gas pressure is no longer present and gravitational forces dominate, resulting in core collapse. The core of the star collapses until another pressure can counter the effects of gravity.

Brown dwarfs have such little mass that fusion is not possible. Therefore the main pressure apposing gravitational forces is electron degeneracy when core becomes dense enough. Similarly, white dwarfs are not massive enough to produce heavy fusion at the core so when the fuel source is depleting, the core collapses until electron degeneracy pressure halts this process.

Electron degeneracy pressure is an outward pressure resulting from electrons being forced close enough together producing a repulsion force, that can be strong enough to halt core collapse. This occurs when a core collapses to an extremely dense state. The Pauli exclusion principle forbids electrons occupying the same quantum state. This pressure is temperature independent as electrons are in fully occupied quantum states. Pressure is instead dependant on mass of the particles within the star (X, Y, Z).

Low mass white dwarfs experience non-relativistic degeneracy pressure given by,

$$P_{nrd} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} \frac{h^2}{m_e}\left(\frac{\rho}{\mu_e m_H}\right)^{\frac{5}{3}} \tag{5}$$

where μ_e is the mean mass of a particle per electron in a gas and h is Planck's constant. High mass white dwarfs experience relativistic degeneracy pressure given by,

$$P_{rd} = \frac{1}{8}\left(\frac{3}{\pi}\right)^{\frac{1}{3}} hc\left(\frac{\rho}{\mu_e m_H}\right)^{\frac{4}{3}} \tag{6}$$

Clearly degeneracy pressure is independent of temperature and only dependant on density and electron number density. The equation of state for a non-relativistic gas is P $\propto \rho^{5/3}$ and for relativistic gas is P $\propto \rho^{4/3}$. Relativistic particles are present due to extreme densities increasing velocities of particles within the star.

The internal structure of such stars can be represented by polytropes with polytropic index n=1.5 for non-relativistic case and n=3 for relativistic. Using the mass radius relation for polytropes,

$$M \propto R^{\frac{(3-n)}{(1-n)}} \tag{7}$$

we find that non-relativistic degenerate stars $\gamma = 5/3$ and relativistic stars $\gamma = 4/3$, where gamma is the ratio of heat capacities.

Combining the equation for hydrostatic equilibrium, mass continuity and an equation for polytrope gas pressure,

$$P = K\rho^{(1+\frac{1}{n})} \tag{8}$$

where n is the polytropic index and K is a separate equation relating central density and central pressure,

$$K = \frac{P_c}{\rho_c^{\frac{(n+1)}{n}}} \tag{9}$$

the Lane-Emden equation can be derived. The Lane-Emden equation relates density, pressure, temperature, mass and radius of a star. Solutions to the equation are polytropes and are functions of density versus radius. The solutions can be scaled by varying the central density and central pressure. This gives solutions for stars over a range of total mass and radius making it extremely versatile and effective at giving insight to stellar structure.

Mass is independent of central density and radius for highly relativistic stars. Degenerate stars such as white dwarfs are near the end of their life cycle so hydrogen values can be approximated as zero. As the last of the hydrogen fuel burns, density increases as radius decreases and the electrons become more and more relativistic while the mass of the star approaches the Chandrasekhar limit,

$$M_C h = 1.45M\odot \tag{10}$$

where $M\odot$ is solar masses. This results in more massive white dwarfs being smaller in size and states no white dwarf star can be more massive than this limit. A more massive star capable of producing a carbon core would be more massive than the Chandrasekhar limit, resulting in a different evolutionary ending.

A star capable of oxygen-carbon fusion, will have a mass to large for electron degeneracy to appose gravitational pressure. Electron capture ensues due to extreme gravitational pressure resulting in a star with no free electrons. The star now collapses until neutron degeneracy provides enough pressure to appose gravitational collapse forming a neutron star. When enough iron is created at the core, gravitational pressure will overcome neutron degeneracy, resulting in infinite core collapse. Matter is now condensed down into an infinitely dense point, resulting in the formation of a black hole.