RAOP (**R**isk-**A**ware **O**ption **P**ricing) **– Theory**

1. Options

Table summarizing the main options and the attributes that define them:

Name	**Type (put or call)**	**Underlying price**	**Strike price**	**Risk free interest rate**	**Time to maturity**	**Volatility**	**Specific attributes**
European	X	X	X	X	X	X	-
American	X	X	X	X	X	X	-
Asian	X	X	X	X	X	X	Averaging period, Averaging method
Barrier	X	X	X	X	X	X	Barrier type (up-and-out, up-and-in, down-and-out, or down-and-in), Barrier level, Barrier observation frequency
Look back	X	-	-	X	X	X	Look back type (fixed or floating), Look back period
Binary	X	X	X	X	X	X	Binary payout
Spread	X	-	X	X	X	X	Underlying price of the 1st asset, Underlying price of the 2nd asset
Exchange	X	-	X	X	X	X	Underlying price of the 1st asset, Underlying price of the 2nd asset, Exchange rate
Chooser	X	X	X	X	X	X	Choice date
Quanto	X	X	X	X	X	X	Foreign risk free interest rate, Exchange rate

2. Pricing Methods

Black-Scholes method

In its original form, this method is only useful to price **European options**.

This method makes the following assumptions about the assets:

- The **rate of return** on the riskless asset is **constant**;
- The **stock price** follows a **geometric Brownian motion**, and it is assumed that the **drift** and **volatility** of the motion are **constant;**
- The **stock** does **not** pay a **dividend.**

And about the market:

- **No arbitrage** opportunity;
- **Ability to borrow and lend any amount**, even fractional, of cash at the **riskless rate**;
- **Ability to buy and sell any amount**, even fractional, of the **stock** (short selling);
- The above transactions do not incur any fees or costs (**frictionless market**);

Value of a **call** option for a non-dividend-paying underlying stock:

$$\boxed{\boldsymbol{C(S_t,t)=N(d_1)S_t-N(d_2)Ke^{-r(T-t)}}}$$

With $N(x)=\frac{1}{\sqrt{2pi}}\int_{-\infty}^{x}e^{-z^2/2}dz$ the standard normal cumulative distribution (hence, $N'(x)=\frac{1}{\sqrt{2pi}}e^{-x^2/2}$ is the standard normal probability density function), and:

- $d_1=\frac{1}{\sigma\sqrt{T-t}}[\ln(S_t/K)+(r+\sigma^2/2)(T-t)]$
- $d_2=d_1-\sigma\sqrt{(T-t)}$

Price of the corresponding **put** option (based on put-call parity with discount factor $e^{-r(T-t)}$:

$$\boxed{\boldsymbol{P(S_t,t)=N(-d_2)Ke^{-r(T-t)}-N(-d_1)S_t}}$$

The Greeks given by this method are:

Greek	Call	Put
Delta $\frac{\partial V}{\partial S}$	$N(d_1)$	$N(d_1)-1$
Gamma $\frac{\partial^2 V}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	
Vega $\frac{\partial V}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	
Theta $\frac{\partial V}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}}-rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}}+rKe^{-r(T-t)}N(-d_2)$
Rho $\frac{\partial V}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

Binomial method

This method can be used to price **European and American options**.

The different steps of this method are:

1. **Build a binomial price tree** starting from **valuation date to expiration**. For each node, they are 2 branches: one up and one down.
   If the price at the node k is S_k , the 2 child branches will be: $S_{k+1}^{up}=u\times S_k$ and $S_{k+1}^{down}=d\times S_k$ with $u=e^{\sigma\sqrt{\Delta t}}$ and $d=e^{-\sigma\sqrt{\Delta t}}$. The tree will have n layers such that $n\Delta t=T$.

2. Find **option value at each final node** (it corresponds to the pay-off of the option) $g(S_T)$.

3. Find **option value at earlier nodes**. This values are found by starting at the penultimate time step, and working back to the first node of the tree (the valuation date) computing each value as:
   $$\boxed{\boldsymbol{C_{k-1}=e^{-r\Delta t}\left(pC_k^{up}+(1-p)C_k^{down}\right)}}$$
   Where p is the probability of an up move in the underlying and $1-p$ is the probability of a down move. This probability is given by $p=\frac{e^{(r-q)\Delta t}-d}{u-d}$ with q the dividend yield of the underlying corresponding to the life of the option. We must impose for p to be in $(0,1)$ that $\Delta t<\frac{\sigma^2}{(r-q)^2}$.

From this method, it is also possible to estimate the Greeks as follows:

Greek	Numerical Approach (same for Put and Call)
Delta $\frac{\partial V}{\partial S}$	Impose a little **perturbation** of S at the corresponding node (maybe 1%) and recompute the binomial tree. Then re-evaluate the value of the option at the node with the previous method and compute the **first order finite difference** with the previous result.
Gamma $\frac{\partial^2 V}{\partial S^2}$	Compute the **second order finite difference** (e.g. with central scheme): impose a $+\Delta S$ perturbation and evaluate the option value; impose a $-\Delta S$ perturbation and evaluate the option value; and use these values and the values without perturbation to compute the finite difference.
Vega $\frac{\partial V}{\partial \sigma}$	Impose a little **perturbation** of σ at the corresponding node (maybe 1%) and recompute the binomial tree. Then re-evaluate the value of the option at the node with the previous method and compute the **first order finite difference** with the previous result.
Theta $\frac{\partial V}{\partial t}$	Impose a little **perturbation** of the expiration time $T'=T-\Delta T$ (maybe one day for example) and recompute the binomial tree. Then compute the **first order finite difference** with the price estimate without perturbation.
Rho $\frac{\partial V}{\partial r}$	Impose a little **perturbation** of r and re-evaluate the value of the option at the node. Then compute the **first order finite difference** with the previous result.

Monte-Carlo method

This method can be used to price **any option** (with some specifities for American options – see Appendix 2).

The different steps of the method for European options (and more generally, in a very similar way, for Bermudian options) are:

1. Choose a **model** for simulating the **dynamics of the underlying asset's price**. Generally, a stochastic process is chosen to model the evolution of the price over time. For instance we could have: $dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t$.

2. **Simulate** N **paths** (the higher this number is, the more accurate the estimate tends to be) of the underlying asset's price from $t=0$ to $t=T$ (expiration time) after discretization of the period (time step Δt). For each path, at the expiration time, compute the corresponding **pay-off**.

3. Compute the discounted pay-offs under risk-neutral probability (for each path and average them to obtain an estimate of the option's price. i.e. compute an estimate of:
$$\boxed{\mathrm{E}\left(\boldsymbol{Payoff} \times \boldsymbol{e}^{-rT}\right)}$$

From this method, it is also possible to estimate the Greeks as follows (similar approach than with the binomial method):

Greek	Numerical Approach (same for Put and Call)
Delta $\frac{\partial V}{\partial S}$	Impose a little **perturbation** of S at $t=0$ (maybe 1%) and recompute the multiple paths and the average of discounted pay-offs. Then compute the **first order finite difference** with the price estimate without perturbation.
Gamma $\frac{\partial^2 V}{\partial S^2}$	Compute the **second order finite difference** (e.g. with central scheme): impose a $+\Delta S$ perturbation and evaluate the option value with the Monte-Carlo method; impose a $-\Delta S$ perturbation and evaluate the option value; and use these values and the values without perturbation to compute the finite difference.
Vega $\frac{\partial V}{\partial \sigma}$	Impose a little **perturbation** of σ at $t=0$ (maybe 1%) and re-evaluate the value of the option with the Monte-Carlo method and compute the **first order finite difference** with the price estimate without perturbation.
Theta $\frac{\partial V}{\partial t}$	Impose a little **perturbation** of the expiration time $T'=T-\Delta T$ (maybe one day for example) and recompute the multiple paths and the average of discounted pay-offs. Then compute the **first order finite difference** with the price estimate without perturbation.
Rho $\frac{\partial V}{\partial r}$	Impose a little **perturbation** of r and re-compute the discounted pay-offs with this new $r+\Delta r$. Then compute the estimate of the option price by averaging and finally compute the **first order finite difference** with the price estimate without perturbation.

3. Risk Evaluation Methods

Appendix 1:

Variance Reduction Methods for Convergence Acceleration of Monte-Carlo simulations

Appendix 2:

Longstaff-Schwartz Method for American Option Pricing

Since American options can be exercised at any time before maturity, some adaptations need to be done in the Monte-Carlo method to price them. One of the most used method in practice is the Longstaff-Schwartz method (2001).
The following steps must be followed:

1. As in the classical Monte-Carlo method for Bermudean option pricing, we start by generating N random paths for the underlying asset price starting from its initial value S_0 assuming that it follows a certain stochastic process (for instance a geometric Brownian motion).
   They will be noted $(S_k^{(i)})_{i=1,\dots,N}$ at t_k the k-th time step.

2. Then we compute the pay-offs for path at every time step.
   They will be noted $h_k^{(i)} = h(S_k^{(i)})$.

3. Starting from the maturity $t_m = T$, for each path, the continuation value at each time step
   $$C_{k-1}^{(i)} = C_{k-1}(S_{k-1}^{(i)}) = \mathrm{E}(max(C_k^{(i)}, h_k^{(i)}) | S_{k-1}^{(i)})$$
   is approximated with linear regression by backward induction as follow:

   i. Choose R basis functions (e.g. Laguerre polynomials) ψ_r and approximate the continuation value by $C_{k-1}^{estim}(x) = \sum_{r=1}^{R} \beta_r \psi_r(x)$ with β the coefficients that minimize $\mathrm{E}((C_{k-1} - C_{k-1}^{estim})^2)$.

   ii. Compute the coefficients by solving the following system:
   $$B_{[\psi\psi]} \beta = B_{[V\psi]}$$
   Where:
   $$[B_{[\psi\psi]}]_{rl} = [\mathrm{E}(\psi_r(S_{k-1}) \psi_l(S_{k-1}))]_{rl} \approx N^{-1} \sum_{i=1}^{N} \psi_r(S_{k-1}^{(i)}) \psi_l(S_{k-1}^{(i)})$$
   $$[B_{V\psi}]_l = \mathrm{E}(max(C_k, h_k) \psi_l(S_{k-1})) \approx N^{-1} \sum_{i=1}^{N} max(C_k^{(i)}, h_k^{(i)}) \psi_l(S_{k-1}^{(i)})$$

   iii. Finally, the continuation value is approximated by computing:
   $$C_{k-1}^{(i)} \approx C_{k-1}^{estim}(S_{k-1}^{(i)})$$

4. For each path, the continuation value at each time step is compared to the pay-off; if the latter is greater then the option is exercised. The first time step for which this is verified is the optimal stopping time: $\tau = t_{k*}$, with $k* = min\{k | h_k > C_k\}$. This time is determined for each path, leading to a list of optimal pay-offs $(h(S_{k*}^{(i)}))_{i=1,\dots,N}$.

5. Finally, discounted pay-offs are computed as $e^{-r\tau^{(i)}} \times h_{k*}^{(i)}$ $\forall i$ and averaged in order to obtain an approximate of the option's price.