# RAOP (Risk-Aware Option Pricing) – Theory

## 1. Options

Table summarizing the main options and the attributes that define them:

| Name | Type (put or call) | Underlying price | Strike price | Risk free interest rate | Time to maturity | Volatility | Specific attributes |
|---|---|---|---|---|---|---|---|
| European | X | X | X | X | X | X | - |
| American | X | X | X | X | X | X | - |
| Asian | X | X | X | X | X | X | Averaging period, Averaging method |
| Barrier | X | X | X | X | X | X | Barrier type (up-and-out, up-and-in, down-and-out, or down-and-in), Barrier level, Barrier observation frequency |
| Look back | X | - | - | X | X | X | Look back type (fixed or floating), Look back period |
| Binary | X | X | X | X | X | X | Binary payout |
| Spread | X | - | X | X | X | X | Underlying price of the 1st asset, Underlying price of the 2nd asset |
| Exchange | X | - | X | X | X | X | Underlying price of the 1st asset, Underlying price of the 2nd asset, Exchange rate |
| Chooser | X | X | X | X | X | X | Choice date |
| Quanto | X | X | X | X | X | X | Foreign risk free interest rate, Exchange rate |

## 2. Pricing Methods

*Black-Scholes method*

In its original form, this method is only useful to price **European options**.

This method makes the following assumptions about the assets:

- The **rate of return** on the riskless asset is **constant**;
- The **stock price** follows a **geometric Brownian motion**, and it is assumed that the **drift** and **volatility** of the motion are **constant;**
- The **stock** does **not** pay a **dividend.**

And about the market:

- **No arbitrage** opportunity;
- **Ability to borrow and lend any amount**, even fractional, of cash at the **riskless rate**;
- **Ability to buy and sell any amount**, even fractional, of the **stock** (short selling);
- The above transactions do not incur any fees or costs (**frictionless market**);

Value of a **call** option for a non-dividend-paying underlying stock:

$$\boxed{\boldsymbol{C(S_t,t)=N(d_1)S_t-N(d_2)Ke^{-r(T-t)}}}$$

With $N(x)=\frac{1}{\sqrt{2pi}}\int_{-\infty}^{x}e^{-z^2/2}dz$ the standard normal cumulative distribution (hence, $N'(x)=\frac{1}{\sqrt{2pi}}e^{-x^2/2}$ is the standard normal probability density function), and:

- $d_1=\frac{1}{\sigma\sqrt{T-t}}[\ln(S_t/K)+(r+\sigma^2/2)(T-t)]$
- $d_2=d_1-\sigma\sqrt{(T-t)}$

Price of the corresponding **put** option (based on put-call parity with discount factor $e^{-r(T-t)}$ :

$$\boxed{\boldsymbol{P(S_t,t)=N(-d_2)Ke^{-r(T-t)}-N(-d_1)S_t}}$$

The Greeks given by this method are:

| Greek | Call | Put |
|---|---|---|
| *Delta* $\frac{\partial V}{\partial S}$ | $N(d_1)$ | $N(d_1)-1$ |
| *Gamma* $\frac{\partial^2 V}{\partial S^2}$ | $\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$ | |
| *Vega* $\frac{\partial V}{\partial \sigma}$ | $SN'(d_1)\sqrt{T-t}$ | |
| *Theta* $\frac{\partial V}{\partial t}$ | $-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}}-rKe^{-r(T-t)}N(d_2)$ | $-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}}+rKe^{-r(T-t)}N(-d_2)$ |
| *Rho* $\frac{\partial V}{\partial r}$ | $K(T-t)e^{-r(T-t)}N(d_2)$ | $-K(T-t)e^{-r(T-t)}N(-d_2)$ |

*Binomial method*

This method can be used to price **European and American options**.

The different steps of this method are:

1. **Build a binomial price tree** starting from **valuation date to expiration**. For each node, they are 2 branches: one up and one down.
   If the price at the node $k$ is $S_k$ , the 2 child branches will be: $S_{k+1}^{up}=u\times S_k$ and $S_{k+1}^{down}=d\times S_k$ with $u=e^{\sigma\sqrt{\Delta t}}$ and $d=e^{-\sigma\sqrt{\Delta t}}$ . The tree will have $n$ layers such that $n\Delta t=T$ .

2. Find **option value at each final node** (it corresponds to the pay-off of the option) $g(S_T)$ .

3. Find **option value at earlier nodes**. This values are found by starting at the penultimate time step, and working back to the first node of the tree (the valuation date) computing each value as:
   $$\boxed{\boldsymbol{C_{k-1}=e^{-r\Delta t}\left(pC_k^{up}+(1-p)C_k^{down}\right)}}$$
   Where $p$ is the probability of an up move in the underlying and $1-p$ is the probability of a down move. This probability is given by $p=\frac{e^{(r-q)\Delta t}-d}{u-d}$ with $q$ the dividend yield of the underlying corresponding to the life of the option. We must impose for $p$ to be in $(0,1)$ that $\Delta t<\frac{\sigma^2}{(r-q)^2}$ .

From this method, it is also possible to estimate the Greeks as follows:

| Greek | Numerical Approach (same for Put and Call) |
|---|---|
| *Delta* $\frac{\partial V}{\partial S}$ | Impose a little **perturbation** of $S$ at the corresponding node (maybe 1%) and recompute the binomial tree. Then re-evaluate the value of the option at the node with the previous method and compute the **first order finite difference** with the previous result. |
| *Gamma* $\frac{\partial^2 V}{\partial S^2}$ | Compute the **second order finite difference** (e.g. with central scheme): impose a $+\Delta S$ perturbation and evaluate the option value; impose a $-\Delta S$ perturbation and evaluate the option value; and use these values and the values without perturbation to compute the finite difference. |
| *Vega* $\frac{\partial V}{\partial \sigma}$ | Impose a little **perturbation** of $\sigma$ at the corresponding node (maybe 1%) and recompute the binomial tree. Then re-evaluate the value of the option at the node with the previous method and compute the **first order finite difference** with the previous result. |
| *Theta* $\frac{\partial V}{\partial t}$ | Impose a little **perturbation** of the expiration time $T'=T-\Delta T$ (maybe one day for example) and recompute the binomial tree. Then compute the **first order finite difference** with the price estimate without perturbation. |
| *Rho* $\frac{\partial V}{\partial r}$ | Impose a little **perturbation** of $r$ and re-evaluate the value of the option at the node. Then compute the **first order finite difference** with the previous result. |

*Monte-Carlo method*

This method can be used to price **any option** (with some specifities for American options – see Appendix 2).

The different steps of the method for European options (and more generally, in a very similar way, for Bermudian options) are:

1. Choose a **model** for simulating the **dynamics of the underlying asset's price**. Generally, a stochastic process is chosen to model the evolution of the price over time. For instance we could have: $dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t$ .

2. **Simulate** $N$ **paths** (the higher this number is, the more accurate the estimate tends to be) of the underlying asset's price from $t=0$ to $t=T$ (expiration time) after discretization of the period (time step $\Delta t$ ). For each path, at the expiration time, compute the corresponding **pay-off**.

3. Compute the discounted pay-offs under risk-neutral probability (for each path and average them to obtain an estimate of the option's price. i.e. compute an estimate of:

$$\boxed{\mathrm{E}\left(\boldsymbol{Payoff} \times \boldsymbol{e}^{-rT}\right)}$$

From this method, it is also possible to estimate the Greeks as follows (similar approach than with the binomial method):

| Greek | Numerical Approach (same for Put and Call) |
| --- | --- |
| *Delta* $\frac{\partial V}{\partial S}$ | Impose a little **perturbation** of $S$ at $t=0$ (maybe 1%) and recompute the multiple paths and the average of discounted pay-offs. Then compute the **first order finite difference** with the price estimate without perturbation. |
| *Gamma* $\frac{\partial^2 V}{\partial S^2}$ | Compute the **second order finite difference** (e.g. with central scheme): impose a $+\Delta S$ perturbation and evaluate the option value with the Monte-Carlo method; impose a $-\Delta S$ perturbation and evaluate the option value; and use these values and the values without perturbation to compute the finite difference. |
| *Vega* $\frac{\partial V}{\partial \sigma}$ | Impose a little **perturbation** of $\sigma$ at $t=0$ (maybe 1%) and re-evaluate the value of the option with the Monte-Carlo method and compute the **first order finite difference** with the price estimate without perturbation. |
| *Theta* $\frac{\partial V}{\partial t}$ | Impose a little **perturbation** of the expiration time $T' = T - \Delta T$ (maybe one day for example) and recompute the multiple paths and the average of discounted pay-offs. Then compute the **first order finite difference** with the price estimate without perturbation. |
| *Rho* $\frac{\partial V}{\partial r}$ | Impose a little **perturbation** of $r$ and re-compute the discounted pay-offs with this new $r+\Delta r$ . Then compute the estimate of the option price by averaging and finally compute the **first order finite difference** with the price estimate without perturbation. |

## 3. Risk Evaluation Methods

**Appendix 1:**

Variance Reduction Methods for Convergence Acceleration of Monte-Carlo simulations

**Appendix 2:**

Longstaff-Schwartz Method for American Option Pricing

Since American options can be exercised at any time before maturity, some adaptations need to be done in the Monte-Carlo method to price them. One of the most used method in practice is the Longstaff-Schwartz method (2001).

The following steps must be followed:

1. As in the classical Monte-Carlo method for Bermudean option pricing, we start by generating $N$ random paths for the underlying asset price starting from its initial value $S_0$ assuming that it follows a certain stochastic process (for instance a geometric Brownian motion).
   They will be noted $(S_k^{(i)})_{i=1,\dots,N}$ at $t_k$ the k-th time step.

2. Then we compute the pay-offs for path at every time step.
   They will be noted $h_k^{(i)}=h(S_k^{(i)})$ .

3. Starting from the maturity $t_m=T$ , for each path, the continuation value at each time step
   $$C_{k-1}^{(i)}=C_{k-1}(S_{k-1}^{(i)})=\mathrm{E}(max(C_k^{(i)},h_k^{(i)})|S_{k-1}^{(i)})$$
   is approximated with linear regression by backward induction as follow:

   i. Choose $R$ basis functions (e.g. Laguerre polynomials) $\psi_r$ and approximate the continuation value by $C_{k-1}^{estim}(x)=\sum_{r=1}^{R}\beta_r\psi_r(x)$ with $\beta$ the coefficients that minimize $\mathrm{E}((C_{k-1}-C_{k-1}^{estim})^2)$ .

   ii. Compute the coefficients by solving the following system:
   $$B_{[\psi\psi]}\beta=B_{[V\psi]}$$
   Where:
   $$[B_{[\psi\psi]}]_{rl}=[\mathrm{E}(\psi_r(S_{k-1})\psi_l(S_{k-1}))]_{rl}\approx N^{-1}\sum_{i=1}^{N}\psi_r(S_{k-1}^{(i)})\psi_l(S_{k-1}^{(i)})$$
   $$[B_{V\psi}]_l=\mathrm{E}(max(C_k,h_k)\psi_l(S_{k-1}))\approx N^{-1}\sum_{i=1}^{N}max(C_k^{(i)},h_k^{(i)})\psi_l(S_{k-1}^{(i)})$$

   iii. Finally, the continuation value is approximated by computing:
   $$C_{k-1}^{(i)}\approx C_{k-1}^{estim}(S_{k-1}^{(i)})$$

4. For each path, the continuation value at each time step is compared to the pay-off; if the latter is greater then the option is exercised. The first time step for which this is verified is the optimal stopping time: $\tau=t_{k*}$ , with $k*=min\{k|h_k>C_k\}$ . This time is determined for each path, leading to a list of optimal pay-offs $(h(S_{k*}^{(i)}))_{i=1,\dots,N}$ .

5. Finally, discounted pay-offs are computed as $e^{-r\tau^{(i)}}\times h_{k*}^{(i)}$ $\forall i$ and averaged in order to obtain an approximate of the option's price.