Contents

1 Combinatorics

1.1 Binomial Coefficients

Binomial coefficients $\binom{n}{k}$ are the number of ways to select a set of k elements from n different elements without taking into account the order of arrangement of these elements (i.e., the number of unordered sets).
Binomial coefficients are also the coefficients in the expansion of $(a+b)^n$ (so-called binomial theorem):

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Recurrence formula** (which is associated with the famous “Pascal’s Triangle”):

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1.1.1 Odd numbers in the i-th line

O número de elementos ímpares na n-śeima linha do triangulo de pascal é 2^c, onde c é o número de bits na representação binária de n.

1.1.2 Properties

Binomial coefficients have many different properties. Here are the simplest of them:

- Symmetry rule:

$$\binom{n}{k} = \binom{n}{n-k}$$

- Factoring in:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

- Sum over k:

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

- Sum over n:

$$\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1}$$

- Sum over n and k:

$$\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m+1}{m}$$

- Sum of the squares:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

- Weighted sum:

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}$$

- Connection with the [Fibonacci numbers](../algebra/fibonacci-numbers.md):

$$\binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n-k}{k} + \cdots + \binom{0}{n} = F_{n+1}$$

1.2 Partition

In number theory and combinatorics, a partition of a non-negative integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. (If order matters, the sum becomes a composition.) For example, 4 can be partitioned in five distinct ways:

$$\begin{gathered}4\\3+1\\2+2\\2+1+1\\1+1+1+1\end{gathered}$$

The only partition of zero is the empty sum, having no parts.
An individual summand in a partition is called a *part*. The number of partitions of n is given by the partition function $p(n)$. So $p(4) = 5$. The notation λ n means that λ is a partition of n.

1	1	2	3
5	7	11	15
22	30	42	56
77	101	135	176
231	297	385	490
627	792	1002	1255
1575	1958	2436	3010
3718	4565	5604	6842
8349	10143	12310	14883
17977	21637	26015	31185
37338	44583	53174	63261
75175	89134	105558	124754
147273	173525		

Table 1: Values of a(n) for n from 0 to 49

For instance, whenever the decimal representation of n ends in the digit 4 or 9, the number of partitions of n will be divisible by 5.
$P(n,k)$ denotes the number of ways of writing n as a sum of exactly k terms or, equivalently, the number of partitions into parts of which the largest is exactly k. $P(n,k)$ can be computed from the recurrence relation

$$P(n,k) = P(n-1,k-1) + P(n-k,k) \tag{1}$$

With $P(n,k) = 0$ for $k > n$, $P(n,n) = 1$, and $P(n,0) = 0$. The triangle of P(k,n) is given by :

1.3 4 fundamental problems of distribution

1.3.1 N equal balls in K equal boxes

Considering that no box can be empty, it's given by the partition function when the number of terms is limited by k.

$$P(n,k) = P(n-1,k-1) + P(n-k,k) \tag{2}$$

1.3.2 N equal balls in K distinct boxes

Equivalent to count the number of solutions for the equation:

$$\begin{gathered}x_1 + \ldots + x_K = N\\ \text{where } x_i > 0,\ i \in [1,K],\ \text{N} > 0\end{gathered} \tag{3}$$

It's given by the formula:

$$\tbinom{N+1}{K-1} \tag{4}$$

If some boxes may be empty ($xi \geq 0$), then it's given by:

$$\binom{N+K-1}{N}$$

1.3.3 N distinct balls in K equal boxes (Stirling's Number of Second Kind)

Also known as Stirling's Number of Second Kind let's define $S(N,K)$ as how many distinct ways to distribute N distinguishable balls in K indistinguishable boxes.
Some special cases:

$$\begin{aligned} S(0,0) &= 1 \\ S(n,0) &= 0 \\ S(n,k) &= 0 \text{ if } n < k \\ S(n,1) &= 1 \\ S(n,k) &= 1 \text{ if } n = k \\ S(n,2) &= 2^{n-1} - 1 \\ S(n,3) &= \tfrac{1}{2}(3^{n-1} + 1) - 2^{n-1} \\ S(n,n-1) &= \binom{n}{2} \\ S(n,n-2) &= \binom{n}{3} + 3\binom{n}{4} \end{aligned} \tag{5}$$

Can be found by the following recurrence relationship.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \tag{6}$$

There is also this formula :

$$S(n,k) = \sum_{r=0}^{k} (-1)^r \frac{(k-r)^n}{r!(k-r)!} \tag{7}$$

1.3.4 N distinct balls in K distinct boxes

$$K^N \tag{8}$$

2 Boolean Algebra (Logic)

2.1 Symbols

2.1.1 Implication ($\rightarrow$)

$$a \rightarrow b \Leftrightarrow \neg a \vee b \tag{9}$$

3 Geometry

3.1 Trigonometry

$$\sin(v+w) = \sin v \cos w + \cos v \sin w \tag{10}$$

$$\cos(v+w) = \cos v \cos w - \sin v \sin w \tag{11}$$

$$\tan(v+w) = \frac{\tan v + \tan w}{1 - \tan v \tan w} \tag{12}$$

$$\sin v + \sin w = 2 \sin \frac{v+w}{2} \cos \frac{v-w}{2} \tag{13}$$

$$\cos v + \cos w = 2 \cos \frac{v+w}{2} \cos \frac{v-w}{2} \tag{14}$$

$$\begin{gathered} (V+W)\tan(v-w)/2 = (V-W)\tan(v+w)/2 \\ \text{where } V, W \text{ are lengths of sides opposite angles } v, w. \end{gathered} \tag{15}$$

$$\begin{gathered} a\cos x + b \sin x = r\cos(x-\phi) \\ a \sin x + b\cos x = r\sin(x+\phi) \\ \text{where } r = \sqrt{a^2+b^2}, \phi = \operatorname{atan2}(b,a). \end{gathered} \tag{16}$$

3.2 Line

3.2.1 General equation

$$ax + by + c = 0 \tag{17}$$

Note that a same line can have multiple representations (just multiply the equation by any real number), so to make each line have a single equation divide everyhting by a, or b if a is zero.

3.2.2 General equation from two points

Let P and Q be the points that de˘ne the line.

$$\begin{aligned} a &= P_y - Q_y \\ b &= Q_x - P_x \\ c &= P \times Q = P_x Q_y - P_y Q_x \end{aligned}$$

3.2.3 Line inclination from two points

Let P and Q be two points that belongs to the line, such that $P_x < Q_x$ the inclination m or angular coefficient is given by:

$$m = \frac{Q_y - P_y}{Q_x - P_x} \tag{18}$$

3.2.4 Check if a point belongs to the line

Let r be a line such that $ax + by + c = 0$ and P a point. $P \in r$ if and only if :

$$aP_x + bP_y + c = 0$$

3.2.5 Distance from a point to a line

The distance from a point P and a line r is defined as the shortest distance possible between every point that belongs to r and P. Such distance will be the distance from P and the intersection between r and the orthogonal projection from P to r, and can be found by:

$$\frac{|aP_x + bP_y + c|}{\sqrt{a^2 + b^2}}$$

The coordinates of the point Q are given by:

$$Q_x = \frac{b(bP_x - aP_y) - ac}{a^2 + b^2}$$
$$Q_y = \frac{a(-bP_x + aP_y) - bc}{a^2 + b^2}$$

3.3 Convex and Concave Polygons

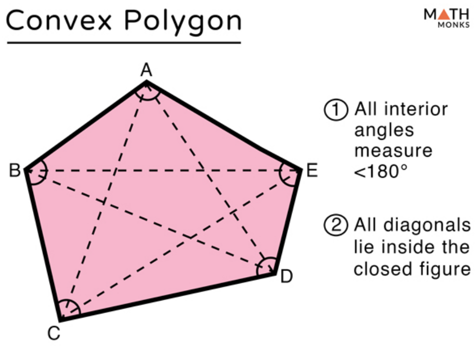


Figure 1: Convex Polygon

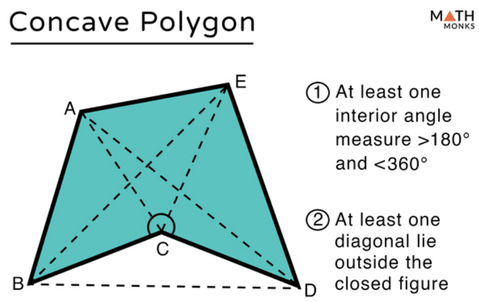
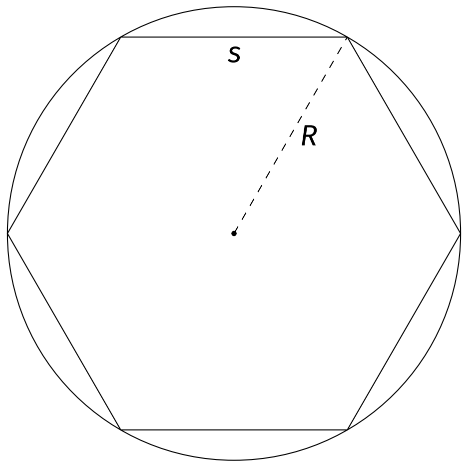


Figure 2: Concave Polygon

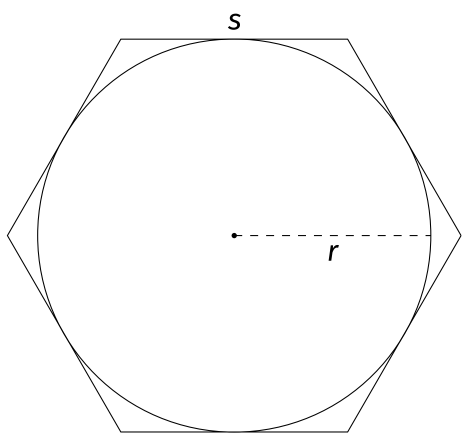
Figure 3: Two Types of Polygons

3.3.1 Regular polygon circumradius



$$R = \frac{s}{2} \csc \frac{\pi}{n}$$

3.3.2 Regular polygon inscribed circle radius



$$r = R\cos\frac{\pi}{n}$$

3.3.3 Area of regular polygons

- Let n be the number of sides of the regular polygon, the area can be found using one of the values below:
  1. the lenght of one of the sides (s)
  2. apothem, the radius of the inscribed circle (r)
  3. the radius of the circumscribed circle (R)

$$A = \frac{1}{2}nrs = \frac{1}{4}ns^2\cot\frac{\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{1}{2}nR^2\sin\frac{2\pi}{n}$$

3.3.4 Sum of internal angle of a regular polygon

A regular polygon with n sides have $(n-2)180$ degrees as sum of it‚s internal angle.

3.4 Triangle

Let the lenght of the sides of the triangle be a, b, c.

3.4.1 Semiperimeter

Let p be the semiperimeter de˘nded as:

$$p = \frac{a+b+c}{2} \tag{19}$$

3.4.2 Area

Let A be the area de˘ned as:

$$\sqrt{p(p-q)(p-b)(p-c)} \tag{20}$$

3.4.3 Circumradius

$$R = \frac{abc}{4A} \tag{21}$$

3.4.4 Inradius

$$r = \frac{A}{p} \tag{22}$$

3.4.5 Lenght of bisector

$$s_a = \sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]} \tag{23}$$

3.4.6 Law of sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{1}{2R} \tag{24}$$

3.4.7 Law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{25}$$

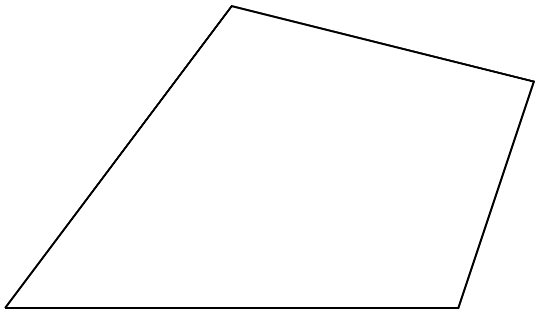
3.4.8 Law of tangents

$$\frac{a+b}{a-b} = \frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} \tag{26}$$

3.5 Quadrilaterals

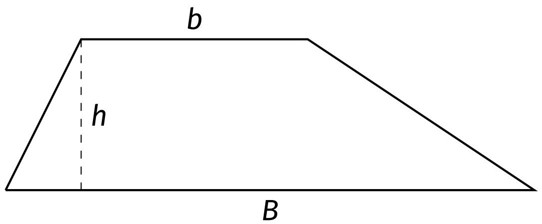
Let it„s sides lenght be a, b, c, d

3.5.1 Intersection of ... ?



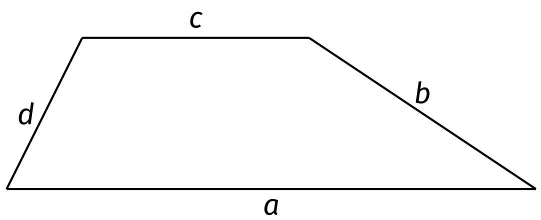
3.6 Trapezium

3.6.1 Area, knowing base and height



$$A = \frac{(B+b)h}{2}$$

3.6.2 Area knowing only sides



$$e = \frac{d^2 - b^2 + a^2 - 2ac + c^2}{2a - 2c}$$

$$h = \sqrt{d^2 - e^2}$$

$$A = \frac{h(a+c)}{2}$$

3.7 Sphere

3.7.1 Equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

3.7.2 Equation in spherical coordinate system

r is the radius, θ is an angle that goes from 0 to 2π, and φ is an angle that goes from 0 to π.

$$x = x_0 + r\cos\theta\sin\varphi$$
$$y = y_0 + r\sin\theta\sin\varphi$$
$$z = z_0 + r\cos\varphi$$

3.7.3 Area

$A = 4\pi r^2$, where r is the radius, comes from :

$$A = \int_0^{\pi}\int_0^{2\pi} r^2 \sin(\varphi)\, d\theta d\varphi$$

3.7.4 Volume

$$V = \frac{4}{3}\pi r^3$$

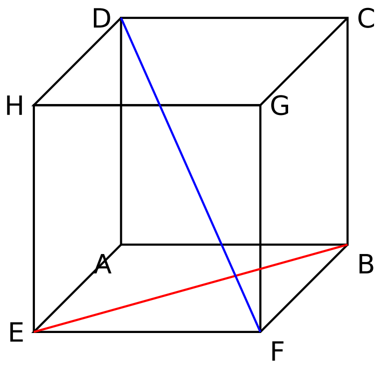
comes from :

$$V = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin(\varphi)\, d\theta d\varphi dR$$

3.8 Cube

3.8.1 Facial & Body diagonal

Facial diagonal join two vertices at the same face.
Body diagonal join two vertices from opposite faces.



Facial diagonal = $L\sqrt{2}$
Body diagonal = $L\sqrt{3}$

3.8.2 Area

$$A = 6L^2$$

3.8.3 Volume

$$V = L^3$$

3.8.4 Circumscribed sphere

Pass through the 8 vertices, radius equal to $L(\frac{\sqrt{3}}{2})$.

3.8.5 Inscribed Sphere

Tangent to the 6 faces, radius equal to $\frac{L}{2}$.

3.8.6 Tangent Sphere

Tangent to the edges, radius equal to $\frac{L}{\sqrt{2}}$.

3.9 Parallelepiped

Let it be de˘ned by three linear independent vectors

$$\vec{u}, \vec{v}, \vec{w}$$

3.9.1 Area delimited by two vectors

$$A = |\vec{u} \times \vec{v}|$$

3.9.2 Volume

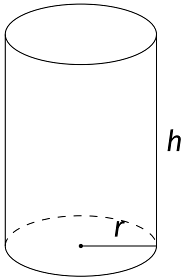
$$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|,$$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \det \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

or if you know the sides a,b, c, and the angles α, β, γ:

$$V = abc\sqrt{1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma}$$

3.10 Cilinder



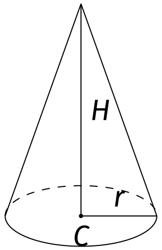
3.10.1 Area

$$A = 2\pi rh + 2\pi r^2 = 2\pi r(h + r),$$

3.10.2 Volume

$$A = \pi r^2 h$$

3.11 Cone



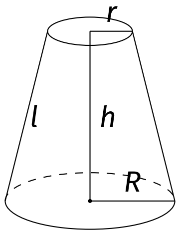
3.11.1 Area

$$A = \pi r^2 + \pi r\sqrt{r^2 + H^2}$$

3.11.2 Volume

$$V = \frac{1}{3}\pi r^2 H$$

3.12 Truncated Cone



3.12.1 Area

$$l = \sqrt{(}h^2 + l^2)$$

$$A = \pi(R + r)l + \pi(R^2 + r^2)$$

3.12.2 Volume

$$V = \frac{1}{3}\pi h(R^2 + Rr + r^2),$$

3.13 Dot product

The dot product of vectors u and v in n dimensions is given by:

$$\langle \mathrm{u}, \mathrm{v}\rangle = u_1 \cdot v_1 + u_2 \cdot v_2 + \ldots + u_n \cdot v_n$$

3.14 Magnitude

The magnitude of a vector v in n dimensions is given by:

$$|\mathrm{v}| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

The dot product of two Euclidean vectors u and v is de˘ned by

$$\langle \mathrm{u}, \mathrm{v}\rangle = |\mathrm{u}| \cdot |\mathrm{v}| \cdot \cos(\theta)$$

4 Algebra

4.1 Absolute Value

4.1.1 Deˇnition

The absolute value of x, denotated by $|x|$ is deˇned by:

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases} \tag{27}$$

4.1.2 Properties and Theorems

$$|x| < a \iff -a < x < a, \text{ where } a > 0 \tag{28}$$

$$|x| \leq a \iff -a \leq x \leq a, \text{ where } a > 0 \tag{29}$$

$$|x| > a \iff (x > a) \vee (x < -a), \text{ where } a > 0 \tag{30}$$

$$|x| \geq a \iff (x \geq a) \vee (x \leq -a), \text{ where } a > 0 \tag{31}$$

$$|ab| = |a| \cdot |b|, \text{ where } a, b \in 3 \tag{32}$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \text{ where } a, b \in 3, \text{ and } b \neq 0 \tag{33}$$

$$|a + b| \leq |a| + |b|, \text{ where } a, b \in 3 \tag{34}$$

$$|a - b| \leq |a| + |b|, \text{ where } a, b \in 3 \tag{35}$$

$$|a| - |b| \leq |a - b|, \text{ where } a, b \in 3 \tag{36}$$

4.2 Sums

$$\begin{aligned} c^a + c^{a+1} + \cdots + c^b &= \frac{c^{b+1} - c^a}{c-1}, c \neq 1z \\ 1 + 2 + 3 + \cdots + n &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + 3^2 + \cdots + n^2 &= \frac{n(2n+1)(n+1)}{6} \\ 1^3 + 2^3 + 3^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4} \\ 1^4 + 2^4 + 3^4 + \cdots + n^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \end{aligned} \tag{37}$$

5 Graphs

5.1 Maximum ‼ow

5.2 Maximum of disjoint nodes paths

Given a directed graph count the maximum number of distinct paths, such that there isn‚t two paths that share a node.
Run a ‼ow on the graph but uses two nodes to represent a single node in the origial graph, let I_u be the •input node of u•, and O_u be the •output node of u•. for each edge (u, v) in the original graph put an edge $(O_u), I_v$ in the ‼ow graph. Also add an edge between I_u, O_u for every u that way even that multiple edges •chooses I_u•, as between I_u and O_u there is only 1 of capacity, O_u will have at most one edge leaving it.

5.2.1 Bipartite Graph

Bipartite Graph is a special graph with the following characteristics: the set of vertices V can be partitioned into two disjoint sets V1 and V2 and all undirected edges (u, v) ∈ E have the property that u ∈ V1 and v ∈ V2 . This makes a Bipartite Graph free from odd-length cycle. Not that by this deˇnition is possible to have isolated vertices.

5.3 Topological Sorting

Deˇnition: You are given a directed graph with n vertices and m edges. You have to ˇnd an order of the vertices, so that every edge leads from the vertex with a smaller index to a vertex with a larger one.

Topological order can be non-unique !
A Topological order may not exist at all. It only exists, if the directed graph contains no cycles. Otherwise because there is a contradiction: if there is a cycle containing the vertices a and b, then a needs to have a smaller index than b (since you can reach b from a) and also a bigger one (as you can reach a from b). Every acyclic directed graph contains at least one topological order.

5.4 Strongly Connected Components

You are given a directed graph G with vertices V and edges E. It is possible that there are loops and multiple edges. Let„s denote n as number of vertices and m as number of edges in G.
Strongly connected component is a maximal subset of vertices C such that any two vertices of this subset are reachable from each other, i.e. for any $u, v \in C$:

$$u \mapsto v, v \mapsto u$$

where $\mapsto$ means reachability, i.e. existence of the path from ˘rst vertex to the second.
The most important property of the condensation graph is that it is a DAG. Indeed, suppose that there is an edge between C and C', let„s prove that there is no edge from C' to C. Suppose that $C' \mapsto C$. Then there are two vertices $u' \in C$ and $v' \in C'$ such that $v' \mapsto u'$. But since u and u' are in the same strongly connected component then there is a path between them; the same for v and v'. As a result, if we join these paths we have that $v \mapsto u$ and at the same time $u \mapsto v$. Therefore u and v should be at the same strongly connected component, so this is contradiction. This completes the proof.

5.5 Minimum spanning tree

5.5.1 Propeties

- A minimum spanning tree of a graph is unique, if the weight of all the edges are distinct. Otherwise, there may be multiple minimum spanning trees. (Speci˘c algorithms typically output one of the possible minimum spanning trees).
- Minimum spanning tree is also the tree with minimum product of weights of edges. (It can be easily proved by replacing the weights of all edges with their logarithms)

5.6 Eulerian path

A Eulerian path is a path in a graph that passes through all of its edges exactly once. A Eulerian cycle is a Eulerian path that is a cycle.
An Eulerian cycle exists if and only if the degrees of all vertices are even
And an Eulerian path exists if and only if the number of vertices with odd degrees is two (or zero, in the case of the existence of a Eulerian cycle).

5.7 Network

A networkis a directed graph G with vertice V and edges E combined with a function c, which assigns each edge $e \in E$ a non-negative integer value, the *capacity* of e.

5.8 Flow network

Is a network with two vertices labeled as source and sink.

5.8.1 Propeties

- The ‼ow of an edge cannot exceed the capacity

$$f(e) <= c(e)$$

- And the sum of the incoming ‼ow of a vertex u has to be equal to the sum of the outgoing ‼ow of u except in the source and sink vertices.
- The source vertex s only has an outgoing ‼ow, and the sink vertex t has only incoming ‼ow.

5.9 Prufer Code

The Prüfer code is a way of encoding a labeled tree with n vertices using a sequence of $n-2$ integers in the interval $[0; n-1]$. This encoding also acts as a bijection between all spanning trees of a complete graph and the numerical sequences.
The Prüfer code is constructed as follows. We will repeat the following procedure $n-2$ times: we select the leaf of the tree with the smallest number, remove it from the tree, and write down the number of the vertex that was connected to it. After $n-2$ iterations there will only remain
2 vertices, and the algorithm ends.
Thus the Prüfer code for a given tree is a sequence of $n-2$ numbers, where each number is the number of the connected vertex, i.e. this number is in the interval
$[0, n-1]$.
The algorithm for computing the Prüfer code can be implemented easily with $O(n \log n)$ time complexity, simply by using a data structure to extract the minimum (for instance set or priority_queue in C++), which contains a list of all the current leafs.
After constructing the Prüfer code two vertices will remain. One of them is the highest vertex $n-1$, but nothing else can be said about the other one.
Each vertex appears in the Prüfer code exactly a ˘xed number of times - its degree minus one. This can be easily checked, since the degree will get smaller every time we record its label in the code, and we remove it once the degree is 1. For the two remaining vertices this fact is also true.

5.10 Prüfer„s Sequence

The Prüfer sequence is a bijection between labeled trees with n vertices and sequences with $n-2$ numbers from 1 to n.
To get the sequence from the tree:

- While there are more than 2 vertices, remove the leaf with smallest label and append it„s neighbour to the end of the sequence.

To get the tree from the sequence:

- The degree of each vertex is 1 more than the number of occurrences of that vertex in the sequence. Compute the degree d, then do the following: for every value x in the sequence (in order), ˘nd the vertex with smallest label y such that $d(y) = 1$ and add an edge between x and y, and also decrease their degrees by 1. At the end of this procedure, there will be two vertices left with degree 1; add an edge between them.

6 Trees

6.1 Centroid

A centroid of a tree is de˘ned as a node such that when the tree is rooted at it, no other nodes have a subtree of size greater than $\frac{N}{2}$.

- if $N \geq 3$ the centroid is never a leaf
- Every tree have a centroid.
- There is at most two centroids (mostly only one)
- All centroids are adjascent
- Let $s(v)$ be the sum of the distance from v to every other node, essentially $\sum_{u \neq v} dist(u, v)$, then if, c is the centroid, it implies that $s(c)$ is the smallest one among every other node.

6.2 Centroid decomposition

The centroid decomposition of a tree is another tree de˘ned recursively as:

- Its root is the centroid of the original tree.
- Its children are the centroid of each tree resulting from the removal of the centroid from the original tree.

Properties:

- The tree formed contains all the N nodes.
- The height of the centroid decomposition is $O() \log N)$.

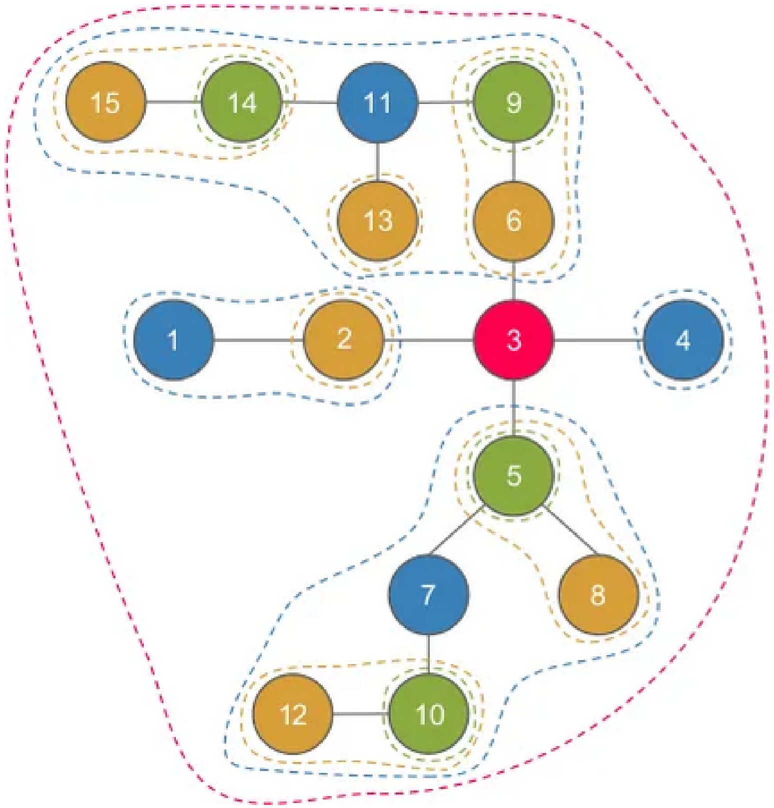
- Let A and B be two arbitrary nodes, let C be the $LCA(A, B)$ in the centroid decomposition tree.

  C lies on the path from A to B in the original tree.

  Every path in the tree can be decomposed in A to C, and C to B.

  As the height of the tree is $O(\log N)$ the LCA can be found in $O(\log N)$ (just like with binary lifting)

Example:



7 Number Theory

7.1 Fermat,s Theorems and Lemmas

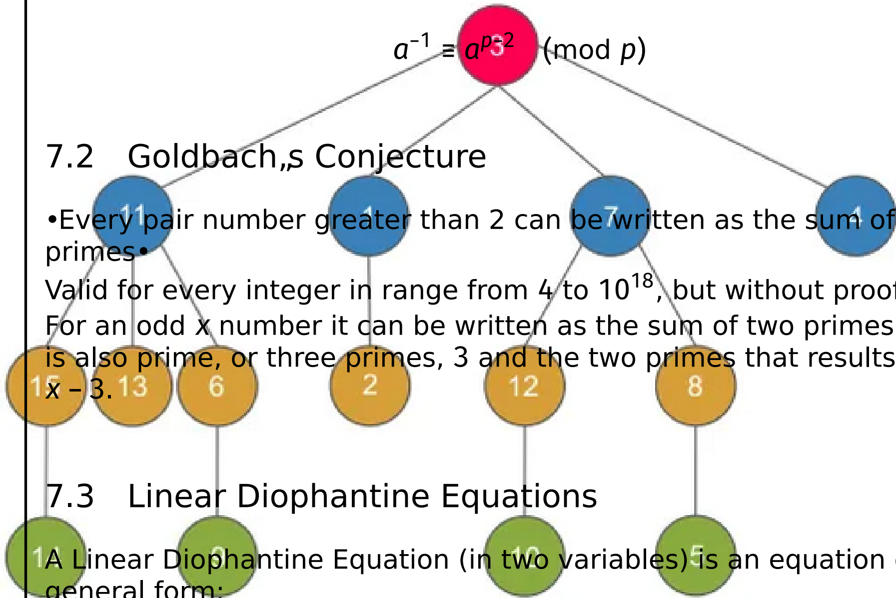
Let p be a prime number and $a, b \in \mathbb{Z}$:

$$a^p \equiv a \pmod{p} \tag{38}$$

$$a^{p-1} \equiv 1 \pmod{p} \tag{39}$$

$$(a+b)^p \equiv a^p + b^p \pmod{p} \tag{40}$$

$$a^{-1} \equiv a^{p-2} \pmod{p} \tag{41}$$



7.2 Goldbach,s Conjecture

•Every pair number greater than 2 can be written as the sum of two primes•
Valid for every integer in range from 4 to 10^{18}, but without proof
For an odd x number it can be written as the sum of two primes if $x-2$ is also prime, or three primes, 3 and the two primes that results in $x-3$.

7.3 Linear Diophantine Equations

A Linear Diophantine Equation (in two variables) is an equation of the general form:

$$ax + by = c$$

Where a, b, c are given integers, and x, y are unknown integers.
If $a = b = 0$, we have in˘nite solutions if $c = 0$, and 0 otherwise.

7.3.1 Solution(s)

Let $g = gcd(a, b)$ such that $ax_g + by_g = g$, then we only have a solution if and only if $g \mid c$, and if it have a solution it have in˘nite.
The solutions will be of the form :

$$\begin{gathered} x_0 = x_g \cdot \tfrac{c}{g}, y_0 = y_g \cdot \tfrac{c}{g}. \\ a \cdot x_0 + b \cdot y_0 = c \end{gathered} \tag{42}$$

With the initial solution, we can can ˘nd every solution, with :

$$x = x_0 + k \cdot \tfrac{b}{g}, y = y_0 - k \cdot \tfrac{a}{g} \tag{43}$$

To ˘nd the solution that minimize $x + y$ we use the fact that:

$$x' = x + k \cdot \frac{b}{g},$$

$$y' = y - k \cdot \frac{a}{g}.$$

Note that
$x + y$ change as follows:

$$x' + y' = x + y + k \cdot \left(\frac{b}{g} - \frac{a}{g}\right) = x + y + k \cdot \frac{b - a}{g}$$

If $a < b$, we need to select smallest possible value of k. If $a > b$, we need to select the largest possible value of k. If $a = b$, all solution will have the same sum $x + y$.

7.4 Wilson‚s theorem

Wilson‚s theorem states that a natural number $n > 1$ is a prime number if and only if the product of all the positive integers less than n is one less than a multiple of n.
That is :

$$(n-1)! \equiv -1 \pmod{n} \tag{44}$$

In other words, any integer $n > 1$ is a prime number if, and only if, $(n-1)! + 1$is divisible by n

7.5 Fundamental theorem of arithmetic

Every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors.

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k} \tag{45}$$

7.5.1 LCM and GCD

$$\begin{aligned} a &= p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k} \\ b &= p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k} \\ (a, b) &= p_1^{\min \alpha_1, \beta_1} p_2^{\min \alpha_2, \beta_2} ... p_k^{\min \alpha_k, \beta_k} \\ [a, b] &= p_1^{\max \alpha_1, \beta_1} p_2^{\max \alpha_2, \beta_2} ... p_k^{\max \alpha_k, \beta_k} \end{aligned} \tag{46}$$

7.6 Taking modulo at the exponent

If $gcd(a, m) = 1$ then:

$$a^m \equiv a^{n \mod \varphi(m)} \pmod{m} \tag{47}$$

8 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$$

9 Linear Algebra

9.1 Matrix Multiplication

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$. The product $C = AB \in \mathbb{R}^{m \times p}$ is deˇned as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

for $i = 1, \dots, m$ and $j = 1, \dots, p$.
Matrix multiplication is associative and distributive, but not commutative: $AB \neq BA$ in general.

10 Probability Theory

Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x.
It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$
Variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation.
If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.
Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

10.1 Discrete distributions

10.1.1 Binomial distribution

The number of successes in n independent yes/no experiments, each which yields success with probability p is $\mathrm{Bin}(n, p)$, $n = 1, 2, \dots$, $0 \leq p \leq 1$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np,\ \sigma^2 = np(1-p)$$

$\mathrm{Bin}(n, p)$ is approximately $\mathrm{Po}(np)$ for small p.

10.1.2 First success distribution

The number of trials needed to get the ˇrst success in independent yes/no experiments, each which yields success with probability p is $\mathrm{Fs}(p)$, $0 \leq p \leq 1$.

$$p(k) = p(1-p)^{k-1},\ k = 1, 2, \dots$$

$$\mu = \frac{1}{p},\ \sigma^2 = \frac{1-p}{p^2}$$

10.1.3 Poisson distribution

The number of events occurring in a ˇxed period of time t if these events occur with a known average rate κ and independently of the time since the last event is $\mathrm{Po}(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!},\ k = 0, 1, 2, \dots$$

$$\mu = \lambda,\ \sigma^2 = \lambda$$

10.2 Continuous distributions

10.2.1 Uniform distribution

If the probability density function is constant between a and b and 0 elsewhere it is $\mathrm{U}(a, b)$, $a < b$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2},\ \sigma^2 = \frac{(b-a)^2}{12}$$

10.2.2 Exponential distribution

The time between events in a Poisson process is Exp(λ), $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

10.2.3 Normal distribution

Most real random values with mean μ and variance σ^2 are well described by $N(\mu, \sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim N(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

10.3 Markov chains

A Markov chain is a discrete random process with the property that the next state depends only on the current state. Let $X_1, X_2, \ldots$ be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathrm{P} = (p_{ij})$, with $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$, and $\mathrm{p}^{(n)} = \mathrm{P}^n \mathrm{p}^{(0)}$ is the probability distribution for X_n (i.e., $p_i^{(n)} = \Pr(X_n = i)$), where $\mathrm{p}^{(0)}$ is the initial distribution.

10.3.1 Stationary distribution

 is a stationary distribution if = P. If the Markov chain is irreducible (it is possible to get to any state from any state), then $\pi_i = \frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state i. π_j / π_i is the expected number of visits in state j between two visits in state i.

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, π_i is proportional to node i„s degree.

10.3.2 Ergodicity

A Markov chain is ergodic if the asymptotic distribution is independent of the initial distribution. A ˘nite Markov chain is ergodic i! it is irreducible and aperiodic (i.e., the gcd of cycle lengths is 1). $\lim_{k\to\infty} \mathrm{P}^k = \mathbf{1}\pi$.

10.3.3 Absorption

A Markov chain is an A-chain if the states can be partitioned into two sets A and G, such that all states in A are absorbing ($p_{ii} = 1$), and all states in G leads to an absorbing state in A. The probability for absorption in state $i \in \mathrm{A}$, when the initial state is j, is $a_{ij} = p_{ij} + \sum_{k \in \mathrm{G}} a_{ik} p_{kj}$. The expected time until absorption, when the initial state is i, is $t_i = 1 + \sum_{k \in \mathrm{G}} p_{ki} t_k$.

11 Polynomial

11.1 Bhaskara

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{48}$$

11.2 Pascal„s Triangle

11.3 N-th ˘rst terms of P-th column in Pascal Triangle

$$\binom{p}{p} + \binom{p+1}{p} + \ldots + \binom{p+n}{p} = \binom{p+n+1}{p+1} \tag{49}$$

11.4 Number of odd numbers in the N-th line of pascal triangle

There is a mathematical relation which gives the number of odd numbers in the N-th row of pascal„s triangle. The theorem states that the number of odd numbers in N-th row is equal to 2 raised to the number of ones in the binary representation of N.

12 Trees

12.1 Heavy-Light Decomposition

Heavy-Light Decomposition (HLD) is a technique to decompose a tree into a set of disjoint paths. This technique is particularly useful to deal with problems which require us to do some path-queries in a tree which seemingly complicated but easy enough to be solved for a line-graph. The idea is to decompose the tree into several paths (line-graph) of disjoint vertices. Then, each path-query in the original tree might be able to be answered by queries in one or more of those paths.
An edge (a, b) is heavy if and only if size(b) size(a)/2; otherwise, it is light

13 Bitwise

13.1 Binary to gray code

$G_n = B_n$
$G_{n-1} = B_n \oplus B_{n-1}$
$G_i = B_i \oplus B_{i-1}$
...
$G_1 = B_2 \oplus B_1$

13.2 Gray code to binary

$B_n = G_n$
$B_{n-1} = B_n \oplus G_{n-1} = G_n \oplus G_{n-1}$
...
$B_1 = B_2 \oplus G_1 = G_n \oplus G_1$

14 Game Theory

14.1 Impartial Games

To be considered a impartial game following rules must be true:

1. The available moves win/loose depends only on the state of the game, in other words, the only di! erence between the two players is that one of them moves ˘rst
2. Additionally, we assume that the game has perfect information, i.e. no information is hidden from the players (they know the rules and the possible moves).
3. It is assumed that the game is ˘nite, i.e. after a certain number of moves, one of the players will end up in a losing position ’ from which they can„t move to another position. On the other side, the player who set up this position for the opponent wins. Understandably, there are no draws in this game.

Such games can be completely described by a directed acyclic graph: the vertices are game states and the edges are transitions (moves). A vertex without outgoing edges is a losing vertex (a player who must make a move from this vertex loses).
Since there are no draws, we can classify all game states as either winning or losing. Winning states are those from which there is a move that causes inevitable defeat of the other player, even with their best response. Losing states are those from which all moves lead to winning states for the other player. Summarizing, a state is winning if there is at least one transition to a losing state and is losing if there isn„t at least one transition to a losing state.
Our task is to classify the states of a given game.

14.2 Sprague-Grundy Theorem

The Sprague-Grundy Theorem states that every impartial game is equivalent to a pile of a certain size in Nim. In other words, every impartial game can be solved as Nim by ˘nding their corresponding game.
Basically, for a game situation A and its SG function value $g(A)$:

1. $g(A) = 0$ if and only if A is a must-lose situation. Otherwise, $g(A) \in \mathbb{Z}*$
2. If A can be divided into n sub-situations $x_1, x_2, ..., x_n$, then $g(A) = g(x_1) \oplus (x_2) \oplus ... \oplus g(x_n)$
3. If A can be converted to situation B_1 or B_2 or ... or B_n by only one operation, then $g(A) = mex(g(B_1), g(B_2), ..., g(B_n))$ where function $mex(S)$ is de˘ned as the smallest non-negative integer that does not appear in S. For example, $mex(0,1,2,4) = 3, mex = 0, mex(0,1,2,4) = 3, mex() = 0$

14.3 Nim variation Subtract game

Work just like nim but instead remove any number of objects you can remove at most K, this game can be seen as a nim game but before computing the num-sum you need to take the size of each pile module $K + 1$, the optimal way to play it is by taking K at each turn.

15 Group Theory

15.1 Permutations

A permutation is an arrangement of elements. A permutation of N elements can be represented by an arrangement of the numbers $1, 2, \dots, N$ in some order. Eg. $5, 1, 4, 2, 3$.

15.1.1 Odd permutations

A permutation is called odd if it can be expressed as a product of odd number of transpositions.
A sorted permutation is an even permutation

15.1.2 Even permutations

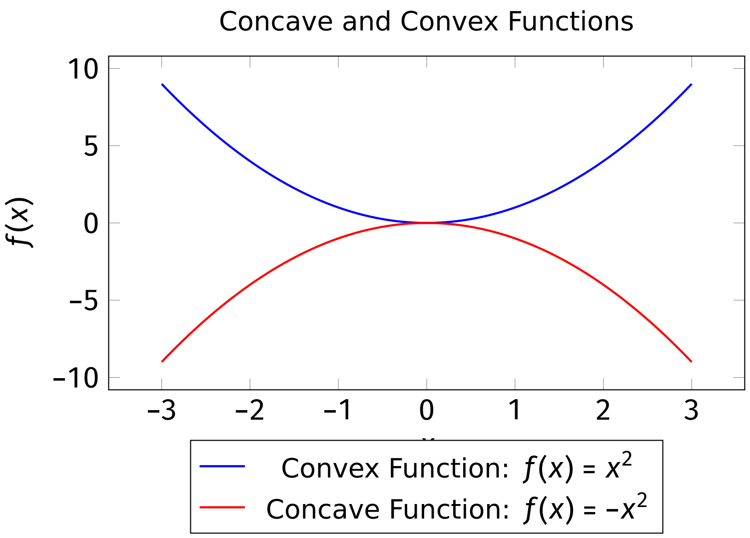
A permutation is called even if it can be expressed as a product of even number of transpositions.

16 Others

16.1 Unimodal Functions

By unimodal function, we mean one of two behaviors of the function:

- The function strictly increases ˘rst, reaches a maximum (at a single point or over an interval), and then strictly decreases.
- The function strictly decreases ˘rst, reaches a minimum, and then strictly increases.



16.2 Critérios de divisibilidade

16.2.1 7

Para veri˘car a divisibilidade de um número por 7, siga a seguinte regra:

1. Pegue o número em questão.
2. Remova o último dígito (unidade) do número.
3. Dobre o valor removido no passo anterior.

4. Subtraia o valor dobrado do número restante.
5. Se o resultado da subtração for divisível por 7, o número original é divisível por 7.

Exemplo:
Suponha que desejamos veri˘car a divisibilidade do número 413 por 7.

1. Remova o último dígito (3) e dobre-o, obtendo 6.
2. Subtraia 6 do número restante (41 - 6 = 35).

16.2.2 11

$$n \text{ é divisível por 11} \iff \sum_{i=1}^{k} a_{2i-1} - \sum_{i=1}^{j} a_{2i} \text{ é divisível por 11}$$

onde a_i é o i-ésimo dígito do número n, k é a quantidade de dígitos ímpares, j é a quantidade de dígitos pares.
Exemplo:
Suponha que desejamos veri˘car a divisibilidade do número $n = 7923$ por 11.

$$k = 2, \quad j = 2$$

$$\text{Soma dos dígitos ímpares: } 7 + 3 = 10$$

$$\text{Soma dos dígitos pares: } 9 + 2 = 11$$

$$\text{Subtração: } 10 - 11 = -1$$

Como −1 não é divisível por 11, o número 7923 não é divisível por 11.

16.2.3 13

$$13 | x \equiv 13 | 4 \cdot (x \% 10) + \lfloor x/10 \rfloor \tag{50}$$

Em outras palavras 13 divide x se o quádruplo do último algarismo somado com o número sem este algarismo for divisível por 13.

16.2.4 17

$$17 | x \equiv 17 | \lfloor x/10 \rfloor - 5 \cdot (x \% 10) \tag{51}$$

Em outras palavras 17 divide x se o a diferença entre o quíntuplo do último algarismo e o número sem este algarísmo for divisível por 17.

16.2.5 19

$$19 | x \equiv 19 | \lfloor x/10 \rfloor + 2 \cdot (x \mod 10) \tag{52}$$

Em outras palavras 19 divide x se o dobro do último algarismo de x somado a o número restante de x é divisível por 19.

16.2.6 23

$$23 | x \equiv 23 | x/10 + 7 \cdot (x \mod 10) \tag{53}$$