TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 3

Hand-out time:	Monday, September 18, 2017, at 8:00
Hand-in deadline:	Friday, September 29, 2017, at 16:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard, or in the boxes in D238. Please write your name on your answer sheet, should you choose to hand in physically. Any questions regarding the assignment should be directed through Blackboard.

Problem 1: State feedback

Consider the following system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

with state $\mathbf{x}(t)$, input $u(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

a) Calculate the controllability matrix of the system and determine if the system is controllable.

Consider a state-feedback controller of the following form:

$$u(t) = -\mathbf{K}\mathbf{x}(t),$$

where $\mathbf{K} = [k_1, k_2, k_3]$ is a feedback matrix. The closed-loop system (i.e. the system with controller) can be written as

$$\dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t),$$

with $\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}$.

b) Calculate the characteristic polynomial of the closed-loop system matrix $\bar{\mathbf{A}}$ as a function of k_1, k_2 and k_3.

We aim to find the values of k_1, k_2 and k_3 such that the poles of the closed-loop system, which are equal to the eigenvalues of $\bar{\mathbf{A}}$, are given by

$$\bar{\lambda}_1 = -1, \qquad \bar{\lambda}_2 = -2 \qquad \text{and} \qquad \bar{\lambda}_3 = -3.$$

Note that if the eigenvalues of $\bar{\mathbf{A}}$ are given by $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$, the characteristic polynomial of $\bar{\mathbf{A}}$ is given by

$$\det(\bar{\mathbf{A}} - \lambda \mathbf{I}) = (\bar{\lambda}_1 - \lambda)(\bar{\lambda}_2 - \lambda)(\bar{\lambda}_3 - \lambda). \tag{1}$$

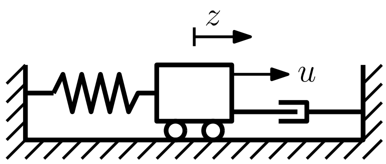
c) Determine the feedback matrix $\mathbf{K} = [k_1, k_2, k_3]$ such that the eigenvalues of $\bar{\mathbf{A}}$ are given by $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ by comparing your result in b) with the characteristic polynomial (1). Show that $\mathbf{K} = [3, 12, 15]$.

Problem 2: Linear quadratic regulator and tracking control

Consider the following mass-spring-damper system:

$$\ddot{y}(t) + \dot{y}(t) + 2y(t) = 2u(t),$$

where $y(t)$ is the displacement of the mass and where the input $u(t)$ is an external force applied to the mass.



a) Derive a state-space equation of the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$.

We aim to design a controller that regulates the displacement of the mass to a desired setpoint value r, i.e. we want to design a controller such that the displacement $y(t)$ converges to the constant value r. The controller can be written as

$$u(t) = u_{eq} + \tilde{u}(t),$$

where u_{eq} correspond to the feedforward part of the controller, and $\tilde{u}(t)$ corresponds to the feedback part of the controller. We design the feedforward part of the controller such that the following equations are satisfied:

$$\begin{aligned}\mathbf{0} &= \mathbf{A}\mathbf{x}_{eq} + \mathbf{B}u_{eq}\\ r &= \mathbf{C}\mathbf{x}_{eq}.\end{aligned} \tag{2}$$

Hence, we choose u_{eq} such that the state $\mathbf{x}(t) = \mathbf{x}_{eq}$ is an equilibrium point of the system that satisfies the condition $y(t) = r$.

b) Determine the state $\mathbf{x}_{eq}$ and the input u_{eq} such that the equations in (2) are satisfied. Show that the state $\mathbf{x}_{eq}$ and the input u_{eq} can be written in the following form:

$$\mathbf{x}_{eq} = \mathbf{F}r \qquad \text{and} \qquad u_{eq} = Gr,$$

with

$$\mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad G = 1.$$

We define the state error $\tilde{\mathbf{x}}(t) = [\tilde{x}_1(t), \tilde{x}_2(t)]^T$ and the output error $\tilde{y}(t)$ as follows:

$$\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_{eq} \qquad \text{and} \qquad \tilde{y}(t) = y(t) - r.$$

c) Show that the state error $\tilde{\mathbf{x}}(t)$ and the output error $\tilde{y}(t)$ satisfy the equations

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) &= \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) \\ \tilde{y}(t) &= \mathbf{C}\tilde{\mathbf{x}}(t). \end{aligned}$$

Note that the state error $\tilde{\mathbf{x}}(t)$ is only influenced by the feedback part $\tilde{u}(t)$ of the controller. If the state error $\tilde{\mathbf{x}}(t)$ is at the origin (i.e. if $\tilde{\mathbf{x}}(t) = \mathbf{0}$), then the output error $\tilde{y}(t)$ is equal to zero, which implies that $y(t) = r$. To control the state error to the origin, we design a linear quadratic regulator for the feedback part of the controller. The linear quadratic regulator minimizes the cost function

$$J = \int_0^\infty \left[\tilde{\mathbf{x}}^T(t)\mathbf{Q}\tilde{\mathbf{x}}(t) + R\tilde{u}^2(t)\right] dt.$$

The control input that minimizes this cost function is given by

$$\tilde{u}(t) = -\mathbf{K}\tilde{\mathbf{x}}(t), \qquad \text{with} \qquad \mathbf{K} = R^{-1}\mathbf{B}^T\mathbf{P},$$

where the real and positive-definite matrix $\mathbf{P}$ is the solution of the algebraic Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

The weighting matrices are given by

$$\mathbf{Q} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad R = 4.$$

d) Calculate the matrix $\mathbf{P}$ and show that $\mathbf{K} = [\frac{1}{2}, \frac{1}{2}]$.

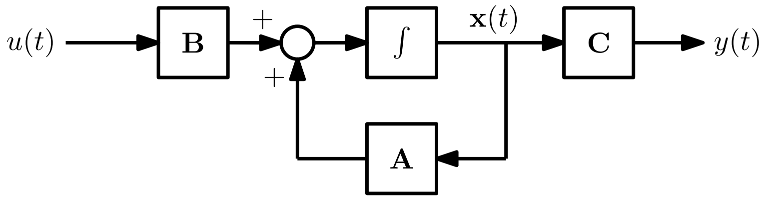
e) Rewrite the controller as

$$u(t) = -\mathbf{K}\mathbf{x}(t) + pr(t)$$

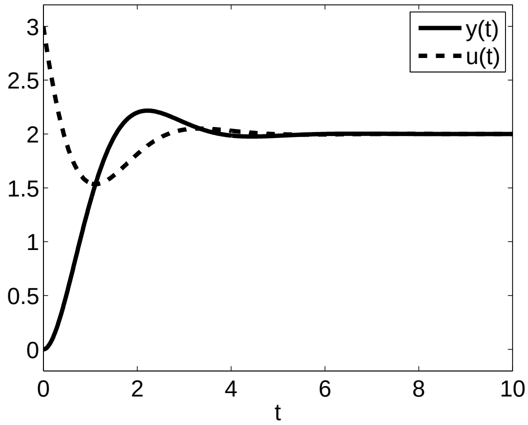
and determine the value of the constant p.

A block diagram of the system (without controller) is given by:

f) Extent the block diagram by adding the controller.



The response of the closed-loop system (i.e. the system with controller) for $r = 2$ and $\mathbf{x}(0) = [0, 0]^T$ is given by:



g) How can the weighting matrices $\mathbf{Q}$ and R be altered such that the output $y(t)$ converges faster to the setpoint value $r = 2$? How will these new values of $\mathbf{Q}$ and R change the input $u(t)$?

Problem 3: State estimator

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t),$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 3 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \text{and} \qquad D = -2.$$

a) Calculate the observability matrix and determine if the system is observable.

Consider the following state estimator for the system:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t) - Du(t)),$$

where $\mathbf{L}$ is a gain matrix that will be determined later.

b) Define the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ and show that

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{LC})\mathbf{e}(t).$$

c) Determine the estimator-gain matrix $\mathbf{L}$ such that the poles of the state estimator (i.e. the eigenvalues of the matrix $\mathbf{A} - \mathbf{LC}$) are equal to -8 and -10, respectively. Show that $\mathbf{L} = [8, 13]^T$.

d) Draw a block diagram of the system with state estimator, similar to f) of Problem 2.