NTNU – Trondheim
Norwegian University of
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Department of Engineering Cybernetics

Examination paper for **TTK4115 Linear Systems Theory**

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Examination date: 9. December 2017

Examination time (from–to): 09:00–13:00

Permitted examination support material: D: No printed or handwritten material allowed. Specific simple calculator allowed.

Other information:

1. Note that no parts of this exam assume that you have solved any of the previous parts. The given information from previous parts should be sufficient to move on.
2. You may answer in Norwegian or English. Du kan svare på norsk eller engelsk.

Language: English

Number of pages: 7

Number pages enclosed: 4

Checked by:

Date Signature

Problem 1 (30 %)

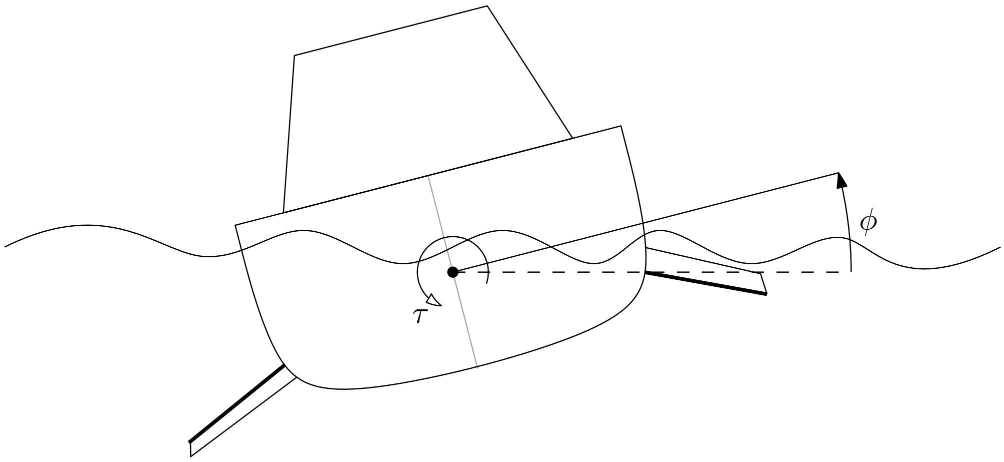
Fin stabilizers are an effective way of reducing roll motion (ϕ) on ships, granted that the ship is traveling sufficiently fast ($V > 0$). The figure shown below depicts the idea in a simplified fashion. By controlling the angle of the fins, a torque $\tau = V^2 u$ can be generated.

The goal of this task is to undertake the initial design steps for an *output-feedback controller* capable of suppressing rolling motion.

The equation of motion for the system is given as a nonlinear mass-spring-damper.

$$j\ddot{\phi}(t) + d|\dot{\phi}(t)|\dot{\phi}(t) + k\phi(t) = V^2 u(t) \tag{1}$$

Here, (j, d, k, V) are positive constants. It is reasonable to assume that the roll rate $\dot{\phi}$ is *small*.



a) (5 %) Demonstrate that the nonlinear dynamics described by (1) can be written on the following *approximate* form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{2}$$

You must here specify the positive constants a and b.

b) (4 %) Is the linear plant (2) controllable? Is it stable, and in what sense? Motivate your answer mathematically.

c) (4 %) Consider the linear plant (2). State-feedback from *estimates* is to be used. For the measurement $y = \mathbf{c}\mathbf{x}$, it is possible to use a gyroscope measuring x_2, or an inclinometer measuring x_1. This leads to the respective measurement matrices

$$\mathbf{c}_{\text{gyro.}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{3a}$$

$$\mathbf{c}_{\text{incl.}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3b}$$

Is it, in principle, possible to estimate x_1 and x_2 with the respective sensors (3a) and (3b)? Motivate your answer mathematically.

d) (5 %) Consider the linear plant (2). Assume (for simplicity) that $a = b = 1$. Find an input signal u that places the closed-loop poles at $\lambda_1 = -1$ and $\lambda_2 = -2$.

e) (8 %) Consider again the linear plant (2). Assume (for simplicity) that $a = 1$. Furthermore, let the gyroscope be used as sensor with measurement matrix (3a). Let $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2]^\mathsf{T}$ denote estimates of the true states described by $\mathbf{x} = [x_1, x_2]^\mathsf{T}$.

Write up the equation for a Luenberger observer ($\dot{\hat{\mathbf{x}}} = \ldots$). Then, choose the gain matrix

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

so that the observer poles are placed at $\lambda'_1 = \lambda'_2 = -2$.

f) (4 %) When using output-feedback, *estimated* states are fed back instead of the states themselves.

Suppose now that a feedback from the *states* ($\mathbf{u} = -\mathbf{K}\mathbf{x}$) has been found that renders a plant ($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$) *stable*. Now suppose, that one replaces the states in the control signal with *estimates* ($\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}}$) from a *stable* observer. Should one, in theory, be concerned with stability of the closed loop system?

Problem 2 (10 %)

Find a *minimal* state-space realization of the transfer-function $\mathbf{g}(s)$ relating a scalar input $u(s)$ to a vectorial output $\mathbf{y}(s) = [y_1(s), y_2(s)]^\mathsf{T}$ so that $\mathbf{y}(s) = \mathbf{g}(s)u(s)$.

$$\mathbf{g}(s) = \begin{bmatrix} \frac{1}{\tau s+1} \\ \frac{\tau s}{\tau s+1} \end{bmatrix}$$

Is the resulting state-space realization observable from the output $y'(t) = y_1(t) + y_2(t)$?

Problem 3 (10 %)

Let a simple dynamic system with two inputs be described by

$$\dot{x} + x = u_1 + u_2$$

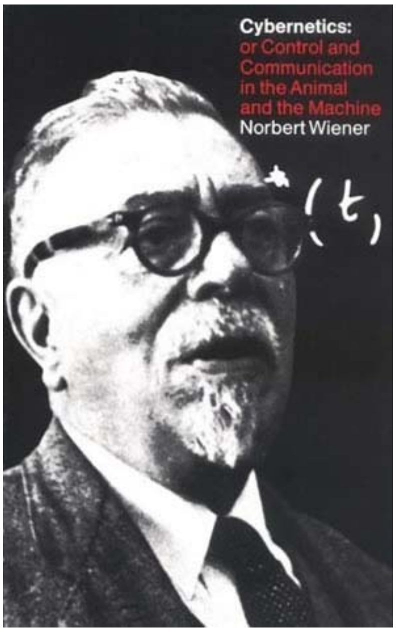
It will be desirable to minimize the following cost function.

$$J = \int_0^\infty x^2 + ru_1^2 + ru_2^2 \; dt$$

where $r > 0$. Identify the feedback gain $\mathbf{k}$ in $\mathbf{u} = -\mathbf{k}x$ that minimizes J.

Does the optimal feedback prefer one of the inputs (u_1, u_2) over the other? What would happen if the cost $ru_1^2 + ru_2^2$ was replaced with $r_1u_1^2 + r_2u_2^2$ in J, where $r_2 \gg r_1 > 0$? An informal answer will suffice.

Problem 4 (20 %)
Norbert Wiener coined the term "Cybernetics" in his famous book *Cybernetics: Or Control and Communication in the Animal and the Machine* published in 1948.



In engineering cybernetics (teknisk kybernetikk), we most often consider machines. However, Wiener's scope was broader. This task considers the cybernetics of biological systems.

An ecosystem often contains prey and predators. Let x_1 quantify the number of prey animals in the ecosystem, for example rabbits. Let x_2 quantify the number of predators in the ecosystem, say foxes. The interaction between predator and prey is governed by the Lotka-Volterra model shown below

$$\dot{x}_1 = \alpha x_1 - \beta x_1 x_2 \tag{4a}$$
$$\dot{x}_2 = \delta x_1 x_2 - \gamma x_2 \tag{4b}$$

Here, $(\alpha, \beta, \gamma, \delta)$ are positive constants.

a) (7 %) Find the *nontrivial*[1] equilibrium of the predator-prey model (4) and denote it by $\bar{x}_1$ and $\bar{x}_2$. Let $\tilde{x}_1 = x_1 - \bar{x}_1$ and $x_2 = \tilde{x}_2 - \bar{x}_2$ denote perturbations away from the equilibrium. Then, show that the following linearized model holds true.

$$\dot{\tilde{x}}_1 = -(\beta\gamma/\delta)\tilde{x}_2 \tag{5a}$$
$$\dot{\tilde{x}}_2 = (\alpha\delta/\beta)\tilde{x}_1 \tag{5b}$$

[1] The *trivial* solution $x_1 = x_2 = 0$ is also an equilibrium.

b) (5 %) Is the system (5) stable, and in what sense? Set all constants to unity so that $\alpha = \beta = \gamma = \delta = 1$. Then, compute and sketch a solution[2] to the equations. A rough sketch is sufficient.

c) (5 %) Consider (5). Is it, in principle, possible to compute the number of predators by counting the prey animals? Find the appropriate principle from linear systems theory and make your case.

d) (3 %) Consider (5). Suppose now that the number of predators can be altered by hunting (to reduce $\tilde{x}_2$) or by introducing animals (to increase $\tilde{x}_2$). This alteration can be included by writing

$$\dot{\tilde{x}}_1 = -\tilde{x}_2 \tag{6a}$$
$$\dot{\tilde{x}}_2 = \tilde{x}_1 + u \tag{6b}$$

Is it, in principle, possible to control the number of animals in the ecosystem through the alterations represented by u?

[2] Any nontrivial solution will suffice.

Problem 5 (30 %)

GPS measurements can be used to determine static positions with high precision. An antenna system for surveying purposes is shown below.

In this task, a static location $\mathbf{x} = [x_1, x_2]^\mathsf{T}$ in the horizontal plane is to be estimated through noisy GPS measurements with a discrete time Kalman filter.

It is assumed that the measurements are approximately normally distributed around the true position $\mathbf{x}$ with a standard deviation $\sigma_v \sim 5[m]$. A discrete-time measurement model is thus

$$\mathbf{y}[k] = \mathbf{x}[k] + \mathbf{v}[k], \quad \mathbf{v}[k] \sim \mathcal{N}\left(\mathbf{0}, \sigma_v^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{7}$$

a) (15 %) Choose the process model $\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{w}[k]$ so that it represents a GPS antenna that is *standing perfectly still.*

The index k runs as $0, 1, 2, \ldots$ Let the Kalman filter be initialized at

$$\hat{\mathbf{x}}^-[0] = \mathbf{0}, \quad \mathbf{P}^-[0] = \sigma_v^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

With $\mathbf{R} = \mathsf{E}[v[k]v[k]^\mathsf{T}] = \sigma_v^2\mathbf{I}$, *show that* for $k \geq 0$, one has

$$\mathbf{L}[k] = \frac{\mathbf{I}}{k+2} \tag{8a}$$

$$\mathbf{P}[k] = \frac{\mathbf{R}}{k+2} \tag{8b}$$

$$\mathbf{P}^-[k+1] = \frac{\mathbf{R}}{k+2} \tag{8c}$$

b) (5 %) How many measurements does it take to reduce the standard deviation of the estimation error by 50% from the a-priori value at $k = 0$?

c) (10 %) You now receive the following measurements:

k	0	1	2	3	4	5
y_1	10	6	1	1	11	-3
y_2	4	0	1	16	8	8

Compute the optimal estimate $\hat{\mathbf{x}}$ for $k = 0 \ldots 5$. Then make a rough sketch that includes the following elements:

1. The measurements $\mathbf{y}[k]$ labeled with k.
2. The estimates $\mathbf{x}[k]$ labeled with k.
3. The initial a-priori estimate $\hat{\mathbf{x}}^-[0] = \mathbf{0}$ surrounded by a circle with radius equal to the standard deviation σ_v.
4. The final a-posteriori estimate $\hat{\mathbf{x}}[5]$ surrounded by a circle with radius equal to the standard deviation of the estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ at $k = 5$.

It is convenient to use the gridlines on the supplied paper for this task.

Formula sheet

Solutions

$$
\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\
\mathbf{x}[k] &= \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m]
\end{aligned}
$$

Controllability/Observability

$$
\begin{aligned}
\mathcal{C} &= [\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ \cdots,\ \mathbf{A}^{n-1}\mathbf{B}] \\
\mathcal{O} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}
\end{aligned}
$$

Realization

$$
\begin{aligned}
\mathbf{G}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\
\mathbf{G}(s) &= \mathbf{G}(\infty) + \mathbf{G}_{sp}(s) \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G}_{sp}(s) &= \frac{1}{d(s)}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}\mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N}_1 \quad \mathbf{N}_2 \quad \cdots \quad \mathbf{N}_{r-1} \quad \mathbf{N}_r\ ]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}
$$

LQR

$$J = \int_0^\infty \mathbf{x}^\mathsf{T}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)\, dt$$

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P} = \mathbf{0}$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{x}(t)$$

Lyapunov equation

$$\mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

Kalman filtering (Discrete time)

Process model

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] + \mathbf{w}[k], \quad \mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{v}[k]$$

The noise and disturbance are unbiased ($\mathsf{E}[\mathbf{v}[k]] = \mathbf{0},\ \mathsf{E}[\mathbf{w}[k]] = \mathbf{0}$) and white

$$\mathsf{E}[\mathbf{v}[k]\mathbf{v}[l]^\mathsf{T}] = \delta[k,l]\mathbf{R}_d, \quad \mathsf{E}[\mathbf{w}[k]\mathbf{w}[l]^\mathsf{T}] = \delta[k,l]\mathbf{Q}_d$$

Algorithm Initialize at $\hat{\mathbf{x}}^-[0] = \mathsf{E}[\mathbf{x}(0)]$ and $\mathbf{P}^-[0] = \mathsf{E}[(\mathbf{x}[0] - \hat{\mathbf{x}}^-[0])(\mathbf{x}[0] - \hat{\mathbf{x}}^-[0])^\mathsf{T}]$. Compute recursively:

1. $\mathbf{L}[k] = \mathbf{P}^-[k]\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{P}^-[k]\mathbf{C}^\mathsf{T} + \mathbf{R}_d)^{-1}$
2. $\hat{\mathbf{x}}[k] = \hat{\mathbf{x}}^-[k] + \mathbf{L}[k](\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}^-[k])$
3. $\mathbf{P}[k] = (\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C})^\mathsf{T} + \mathbf{L}[k]\mathbf{R}_d\mathbf{L}[k]^\mathsf{T}$
4. $\hat{\mathbf{x}}^-[k+1] = \mathbf{A}_d\hat{\mathbf{x}}[k] + \mathbf{B}_d\mathbf{u}[k], \quad \mathbf{P}^-[k+1] = \mathbf{A}_d\mathbf{P}[k]\mathbf{A}_d^\mathsf{T} + \mathbf{Q}_d$

Kalman filtering (Continuous time)

Process model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v}$$

The noise and disturbance are unbiased ($\mathsf{E}[\mathbf{v}(t)] = \mathbf{0},\ \mathsf{E}[\mathbf{w}(t)] = \mathbf{0}$) and white

$$\mathsf{E}[\mathbf{v}(t)\mathbf{v}(\tau)^\mathsf{T}] = \delta(t-\tau)\mathbf{R}, \quad \mathsf{E}[\mathbf{w}(\tau)\mathbf{w}(t)^\mathsf{T}] = \delta(t-\tau)\mathbf{Q}$$

Optimal gain The Kalman gain is given by $\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^\mathsf{T}\mathbf{R}^{-1}$ where

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{G}\mathbf{Q}\mathbf{G}^\mathsf{T} - \mathbf{P}\mathbf{C}^\mathsf{T}\mathbf{R}^{-1}\mathbf{C}\mathbf{P}$$

Set $\dot{\mathbf{P}} = \mathbf{0}$ to find stationary gain.

Stationary processes

Autocorrelation and power spectral density

$$\mathcal{R}_u(\tau) = \mathsf{E}[u(t)u(t+\tau)], \quad \mathcal{S}_u(\omega) = \mathcal{F}\{\mathcal{R}_u(\tau)\}$$

With $y(s) = H(s)w(s)$ where $\mathsf{E}[w(t)] = 0$ and $\mathcal{R}_w(\tau) = \delta(\tau)q$ it holds that

$$\mathcal{S}_y(\omega) = H(\jmath\omega)H(-\jmath\omega)q$$

Laplace transform pairs

$f(t)$	$\Longleftrightarrow$	$F(s)$
$\delta(t)$	$\Longleftrightarrow$	1
1	$\Longleftrightarrow$	$\frac{1}{s}$
e^{-at}	$\Longleftrightarrow$	$\frac{1}{s+a}$
t	$\Longleftrightarrow$	$\frac{1}{s^2}$
t^2	$\Longleftrightarrow$	$\frac{2}{s^3}$
te^{-at}	$\Longleftrightarrow$	$\frac{1}{(s+a)^2}$
$\sin \omega t$	$\Longleftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\Longleftrightarrow$	$\frac{s}{s^2+\omega^2}$