Solution Suggestion
Exam - TTK4115 Linear System Theory
December 17, 2012

MDP (2012-12-06)

Problem 1

a) We start by finding the eigenvalues of the system. The characteristic polynomial of $\mathbf{A}$ is:

$$\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda+1 & 1 \\ -1 & \lambda+3 \end{vmatrix} = \lambda^2 + 4\lambda + 4$$

The eigenvalues of $\mathbf{A}$ can be found solving the characteristic polynomial:

$$\begin{aligned} \Delta(\lambda) &= \lambda^2 + 4\lambda + 4 = 0 \\ &\Rightarrow \lambda_{1,2} = \{-2, -2\} \end{aligned}$$

We have two similar eigenvalues. Now, the eigenvectors must be found. The first eigenvector(s) are found by solving:

$$\begin{aligned} \mathbf{A}\mathbf{v} &= \lambda\mathbf{v} \\ \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} &= \mathbf{0} \\ \Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

The rank of the matrix is:

$$\rho(\mathbf{A} - \lambda\mathbf{I}) = 1$$

It follows that the nullity is $\text{null}(\mathbf{A} - \lambda\mathbf{I}) = 1$. We can therefore only find <u>one</u> linearly independent eigenvector. This eigenvector, which is easily gleaned by inspection, is:

$$\mathbf{v} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $t \neq 0$ may take any real value. Choose e.g. $t = 1$ to obtain the first eigenvector:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The second eigenvector is of the generalized kind. We find this by solving the equation:

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1 \\ \Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

We may use Gauss elimination to find the reduced row echelon form:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We find that:

$$\begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_{2,2}$$

Here $v_{2,2}$ is a free variable. This is to be expected, as the vector $[1,1]^T$ is an eigenvector! Letting e.g. $v_{2,2} = 0$, the second eigenvector is:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Arraying the two eigenvectors in a matrix produces the eigenvector matrix:

$$\mathbf{M} = [\mathbf{v}_1, \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

The similarity transformation:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\bar{\mathbf{x}} + \mathbf{M}^{-1}\mathbf{b}\mathbf{u} \\ \mathbf{y} &= \mathbf{c}\mathbf{M}\bar{\mathbf{x}} \end{aligned}$$

brings the system to Jordan form. The inverse of $\mathbf{M}$ may be found via Gaussian elimination or other methods:

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

The various items in the system on Jordan form are:

$$\begin{aligned} \mathbf{J} &= \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \\ \bar{\mathbf{b}} &= \mathbf{M}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ \bar{\mathbf{c}} &= \mathbf{c}\mathbf{M} = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \end{aligned}$$

The system on Jordan form looks like:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} -1 & 1 \end{bmatrix} \bar{\mathbf{x}} \end{aligned}$$

b) The system has eigenvalues with negative real parts. It is therefore marginally stable and asymptotically stable. Because the eigenvalues correspond to the poles of the system, the poles of the system have negative real parts. As a consequence, the system is BIBO stable. It is not unstable, as this would require at least one pole with a positive real part or a pole with a zero real part that is not a simple root of the minimal polynomial. **Conclusion**: Properties **A**, **B** and **D** match, whereas **C** does not.

c) The controllability of the system is examined via the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$$

We require full row rank, i.e. $\rho(\mathcal{C}) = 2$. For a SISO system, one can simply ascertain whether the determinant is $\neq 0$. In this case $\det(\mathcal{C}) = 1$. The system is therfore controllable. We note that this may be done in both the original and Jordan form.

c) The observability of the system is examined via the observability matrix:

$$\mathcal{O} = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$$

We require full column rank, i.e. $\rho(\mathcal{O}) = 2$. For a SISO system, one can simply ascertain whether the determinant is $\neq 0$. In this case $\det(\mathcal{O}) = -1$. The system is therfore observable. We note that this may be done in both the original and Jordan form.

Problem 2

The exact discretization of:

$$\begin{aligned} \dot{x} &= -2x + u = ax + bu \\ y &= x = cx \end{aligned}$$

is calculated as:

$$\begin{aligned} x[k+1] &= a_d x[k] + b_d u[k] \\ y[k] &= c_d x[k] \end{aligned}$$

where:

$$a_d = e^{aT}, \quad b_d = \int_0^T e^{a\tau} b \, d\tau, \quad c_d = c$$

Inserting parameters yield:

$$a_d = e^{aT} = e^{-2} \simeq 0.135, \quad b_d = \int_0^1 e^{-2\tau} d\tau = \frac{1}{2} - \frac{1}{2e^2} \simeq 0.432, \quad c_d = 1$$

The discretized equation reads:

$$\begin{aligned} x[k+1] &= 0.135x[k] + 0.432u[k] \\ y[k] &= x[k] \end{aligned}$$

Problem 3

a) To realize the transfer function:

$$h(s) = \frac{7s^3 + 11s^2 + 19s + 27}{s^3 + s^2 + 2s + 3}$$

we must first separate out its strictly proper part, $h(s) = h_{sp}(s) + h(\infty)$. Taking the limit:

$$h(\infty) = \lim_{s \to \infty} h(s) = 7$$

yields the infinite frequency response. We note that $d = h(\infty)$. Subtracting this from $h(s)$:

$$h_{sp}(s) = h(s) - h(\infty) = \frac{7s^3 + 11s^2 + 19s + 27}{s^3 + s^2 + 2s + 3} - 7 = \frac{6 + 5s + 4s^2}{3 + 2s + s^2 + s^3}$$

yields the strictly proper part. The denominator is monic and the formula in the appendix may be applied:

$$\frac{n_1 s^2 + n_2 s + n_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3} = \frac{4s^2 + 5s + 6}{s^3 + s^2 + 2s + 3}$$

Inserting gives:

$$\bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{\mathbf{c}} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

Finally,

$$\bar{\mathbf{b}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The realization then reads:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \bar{\mathbf{x}} + 7u \end{aligned}$$

b) The realization is <u>minimal</u>. The dimension of the system $n = 3$ is equal to the the order of the denominator (the term in the denominator with highest power is s^3). One may alternatively check controllability and observability, where a controllable and observable system implies that the realization is minimal.

c) Pole placement proceeds by specifying a new denominator with the desired poles:

$$d(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) = (s + 1)(s + 2)(s + 3) = s^3 + 6s^2 + 11s + 6$$

The new coefficients read:

$$s^3 + \alpha_1' s^2 + \alpha_2' s + \alpha_3' = s^3 + 6s^2 + 11s + 6$$

Choosing a state feedback $\bar{\mathbf{k}}$ such that:

$$\bar{\mathbf{k}} = \begin{bmatrix} \alpha_1' - \alpha_1 & \alpha_2' - \alpha_2 & \alpha_3' - \alpha_3 \end{bmatrix}$$

The altered system reads:

$$(\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}}) = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \alpha_1' - \alpha_1 & \alpha_2' - \alpha_2 & \alpha_3' - \alpha_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\alpha_1' & -\alpha_2' & -\alpha_3' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The final statefeedback should be:

$$\bar{\mathbf{k}} = \begin{bmatrix} 6 - 1 & 11 - 2 & 6 - 3 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 3 \end{bmatrix}$$

Hence, we obtain:

$$u = -\begin{bmatrix} 5 & 9 & 3 \end{bmatrix} \bar{\mathbf{x}}$$

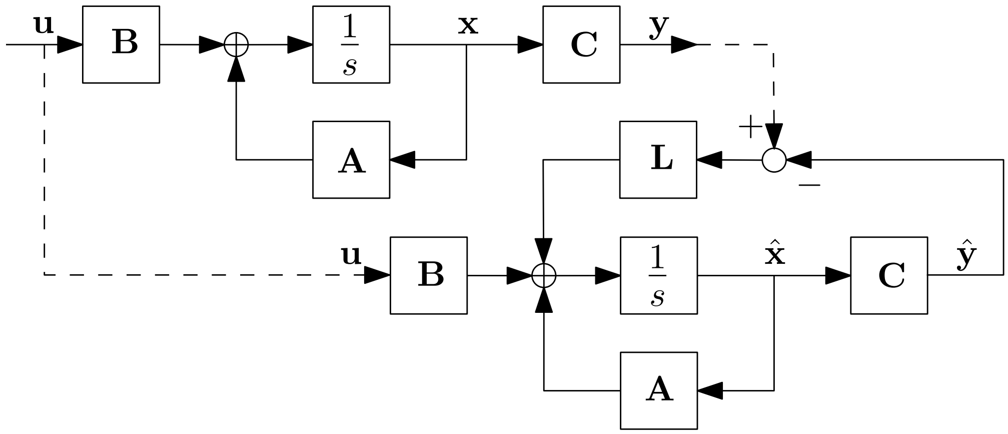

Figure 1: Tasks a) and b).

Problem 4

a,b) The block diagrams should look like Figure 1. Task b) is added on via dashed lines.

c) For two states and one measurement, the correct form of **L** is:

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

The poles of the observer are placed by assigning the eigenvalues of:

$$\mathbf{A} - \mathbf{LC} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 - l_1 & 1 \\ -l_2 & -2 \end{bmatrix}$$

The eigenvalues are computed as:

$$\det(\lambda \mathbf{I} - \mathbf{A} + \mathbf{LC}) = \lambda^2 + \lambda\,(3 + l_1) + (2l_1 + l_2 + 2) = 0$$

The solution is:

$$\lambda = \frac{-(3 + l_1) \pm \sqrt{(3 + l_1)^2 - 4(2l_1 + l_2 + 2)}}{2} = \frac{-3 - l_1 \pm \sqrt{l_1^2 - 2l_1 - 4l_2 + 1}}{2}$$

The elements of **L** are selected such that:

$$\lambda_1 = -7 = \frac{-3 - l_1 \pm \sqrt{l_1^2 - 2l_1 - 4l_2 + 1}}{2}$$
$$\lambda_2 = -6 = \frac{-3 - l_1 \pm \sqrt{l_1^2 - 2l_1 - 4l_2 + 1}}{2}$$

The solution to these are:

$$l_1 = 10, \quad l_2 = 20$$

Thus, selecting the matrix:

$$\mathbf{L} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

will place the poles of the observer at the specified values.

Problem 5

a) The proof of asymptotic stability requires that there is a symmetric positive definite matrix $\mathbf{M} > 0$, for any symmetric positive definite matrix $\mathbf{N} > 0$, that solves:

$$\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

We choose $\mathbf{N} = \mathbf{I}$. Then the above equation reads:

$$\begin{bmatrix} -5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 4 & -3 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I.e.:

$$\begin{bmatrix} -10m_{11} + 8m_{12} & 2\left(m_{11} - 4m_{12} + 2m_{22}\right) \\ 2\left(m_{11} - 4m_{12} + 2m_{22}\right) & 4m_{12} - 6m_{22} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is three equations in three unknowns. The matrix that solves this is:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} = \begin{bmatrix} 2/7 & 13/56 \\ 13/56 & 9/28 \end{bmatrix}$$

If this matrix is positive definite, the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable. The check for positive definiteness may be done by verifying that the leading principal minors are positive:

$$2/7 > 0, \qquad \begin{vmatrix} 2/7 & 13/56 \\ 13/56 & 9/28 \end{vmatrix} = \frac{17}{448} \simeq 0.0379 > 0$$

One may also check that the eigenvalues of $\mathbf{M}$ are positive:

$$\lambda(\mathbf{M}) = \frac{\frac{17}{28} \pm \sqrt{\left(\frac{17}{28}\right)^2 - \frac{17}{112}}}{2} \simeq 0.3036 \pm 0.2328 > 0$$

Problem 6

a) We recognize that the process is a Gauss-Markov process, which means that the spectral density function can be written in the following form:

$$S_X(j\omega) = \frac{2\sigma^2\beta}{\omega^2 + \beta^2}$$

Comparing this with the given spectral density function in the problem description, we obtain:

$$\begin{aligned} 2\sigma^2\beta &= 6 \\ \beta^2 &= 4 \end{aligned}$$

From this, we get the values:

$$\beta = 2, \quad \sigma^2 = \frac{3}{2}$$

The corresponding autocorrelation function is:

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|} = \frac{3}{2} e^{-2|\tau|}$$

b) The transfer function of the filter is defined as:

$$G(s) = \frac{X(s)}{W(s)}$$

Therefore, we have:

$$X(s) = G(s)W(s)$$

The spectral density function of $X(t)$ in Laplace domain is given by

$$S_X(s) = X(s)X(-s) = G(s)G(-s)W(s)W(-s) = G(s)G(-s)S_W(s)$$

Substituting $S_X(s) = \frac{6}{4-s^2}$ and $S_W(s) = 1$ gives:

$$G(s)G(-s) = \frac{\sqrt{6}}{2+s} \cdot \frac{\sqrt{6}}{2-s} = \frac{6}{4-s^2}$$

Therefore, the transfer function of the filter is given by:

$$G(s) = \frac{\sqrt{6}}{2+s}$$

Problem 7

a) The system can be written in the following form:

$$\begin{aligned} x_{k+1} &= \Phi x_k + v_k \\ z_k &= H x_k + w_k \end{aligned}$$

with $\Phi = 0.8$ and $H = 1$. The Kalman filter estimates $\hat{x}_0, \hat{x}_1, \hat{x}_2$ are obtained by repeating the following steps:

Step 1: Compute the Kalman gain

$$K_k = P_k^- H(HP_k^- H + R)^{-1}$$

Step 2: Update the estimate

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

Step 3: Update the error covariance

$$P_k = (I - K_k H)P_k^-(I - K_k H) + K_k R K_k$$

Step 4: Project ahead to $k+1$

$$\begin{aligned} \hat{x}_{k+1}^- &= \Phi \hat{x}_0 \\ P_{k+1}^- &= \Phi P_k \Phi + Q \end{aligned}$$

For $k = 0$, we obtain:

$$\begin{aligned} K_0 &= P_0^- H(HP_0^- H + R)^{-1} = 1 \cdot 1(1 \cdot 1 \cdot 1 + 0.1)^{-1} \simeq 0.9091 \\ \hat{x}_0 &= \hat{x}_0^- + K_0(z_0 - H\hat{x}_0^-) \simeq 1 + 0.9091(2 - 1 \cdot 1) = 1.9091 \\ P_0 &= (I - K_0 H)P_0^-(I - K_0 H) + K_0 R K_0 \\ &\simeq (1 - 0.9091 \cdot 1)1(1 - 0.9091 \cdot 1) + 0.9091 \cdot 0.1 \cdot 0.9091 \simeq 0.0909 \\ \hat{x}_1^- &= \Phi \hat{x}_0 \simeq 0.8 \cdot 1.9091 \simeq 1.5273 \\ P_1^- &= \Phi P_0 \Phi + Q \simeq 0.8 \cdot 0.0909 \cdot 0.8 + 4 \simeq 4.0582 \end{aligned}$$

Subsequently, for $k = 1$, we get:

$$\begin{aligned}
K_1 &= P_1^- H(HP_1^- H + R)^{-1} = 4.0582 \cdot 1(1 \cdot 4.0582 \cdot 1 + 0.1)^{-1} \simeq 0.9760 \\
\hat{x}_1 &= \hat{x}_1^- + K_1(z_1 - H\hat{x}_1^-) \simeq 1.5273 + 0.9760(2.4 - 1 \cdot 1.5273) \simeq 2.3790 \\
P_1 &= (I - K_1H)P_1^-(I - K_1H) + K_1RK_1 \\
&\simeq (1 - 0.9760 \cdot 1)4.0582(1 - 0.9760 \cdot 1) + 0.9760 \cdot 0.1 \cdot 0.9760 \simeq 0.0976 \\
\hat{x}_2^- &= \Phi\hat{x}_1 \simeq 0.8 \cdot 2.3790 \simeq 1.5097 \\
P_2^- &= \Phi P_1 \Phi + Q \simeq 0.8 \cdot 0.0976 \cdot 0.8 + 4 \simeq 4.0625
\end{aligned}$$

Finally, for $k = 2$, we obtain:

$$\begin{aligned}
K_2 &= P_2^- H(HP_2^- H + R)^{-1} = 4.0625 \cdot 1(1 \cdot 4.0625 \cdot 1 + 0.1)^{-1} \simeq 0.9760 \\
\hat{x}_2 &= \hat{x}_2^- + K_2(z_2 - H\hat{x}_2^-) \simeq 1.5097 + 0.9760(1.5 - 1 \cdot 1.5097) \simeq 1.5097
\end{aligned}$$

To summarize, we have obtained:

$$\hat{x}_0 \simeq 1.9091, \quad \hat{x}_1 \simeq 2.3790, \quad \hat{x}_2 \simeq 1.5097$$

b) The system can be written in the following form:

$$\begin{aligned}
x_{k+1} &= f(x_k) + w_k \\
z_k &= h(x_k) + v_k
\end{aligned}$$

with $f(x_k) = 0.1x_k^3 + 0.7x_k$ and $h(x_k) = x_k$. For the extended Kalman filter, the same equations as for the Kalman filter are used. To obtain Φ_k and H_k, the functions f and h are linearized about the points $\hat{x}_k$ and $\hat{x}_k^-$, respectively. The extended Kalman filter estimates $\hat{x}_0, \hat{x}_1, \hat{x}_2$ are obtained by repeating the following steps:

Step 1: Compute the Kalman gain

$$\begin{aligned}
H_k &= \left.\frac{dh}{dx_k}\right|_{x_k = \hat{x}_k^-} \\
K_k &= P_k^- H_k(H_k P_k^- H_k + R)^{-1}
\end{aligned}$$

Step 2: Update the estimate

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - h(\hat{x}_k^-))$$

Step 3: Update the error covariance

$$P_k = (I - K_kH_k)P_k^-(I - K_kH_k) + K_kRK_k$$

Step 4: Project ahead to $k + 1$

$$\begin{aligned}
\Phi_k &= \left.\frac{df}{dx_k}\right|_{x_k = \hat{x}_k} \\
\hat{x}_{k+1}^- &= f(\hat{x}_k) \\
P_{k+1}^- &= \Phi_k P_k \Phi_k + Q
\end{aligned}$$

For $k = 0$, we obtain:

$$\begin{aligned}
H_0 &= \left.\frac{dh}{dx_k}\right|_{x_k=\hat{x}_0^-} = 1 \\
K_0 &= P_0^- H_0 (H_0 P_0^- H_0 + R)^{-1} = 1 \cdot 1(1 \cdot 1 \cdot 1 + 0.1)^{-1} \simeq 0.9091 \\
\hat{x}_0 &= \hat{x}_0^- + K_0(z_0 - h(\hat{x}_0^-)) \simeq 1 + 0.9091(2 - 1) = 1.9091 \\
P_0 &= (I - K_0 H_0) P_0^- (I - K_0 H_0) + K_0 R K_0 \\
&\simeq (1 - 0.9091 \cdot 1)1(1 - 0.9091 \cdot 1) + 0.9091 \cdot 0.1 \cdot 0.9091 \simeq 0.0909 \\
\Phi_0 &= \left.\frac{df}{dx_k}\right|_{x_k=\hat{x}_0} \simeq 0.3 \cdot 1.9091^2 + 0.7 \simeq 1.7934 \\
\hat{x}_1^- &= f(\hat{x}_0) \simeq 0.1 \cdot 1.9091^3 + 0.7 \cdot 1.9091 \simeq 2.0322 \\
P_1^- &= \Phi_0 P_0 \Phi_0 + Q \simeq 1.7934 \cdot 0.0909 \cdot 1.7934 + 4 \simeq 4.2924
\end{aligned}$$

Subsequently, for $k = 1$, we get:

$$\begin{aligned}
H_1 &= \left.\frac{dh}{dx_k}\right|_{x_k=\hat{x}_1^-} = 1 \\
K_1 &= P_1^- H_1 (H_1 P_1^- H_1 + R)^{-1} = 4.2924 \cdot 1(1 \cdot 4.2924 \cdot 1 + 0.1)^{-1} \simeq 0.9772 \\
\hat{x}_1 &= \hat{x}_1^- + K_1(z_1 - h(\hat{x}_1^-)) \simeq 3.4237 + 0.9772(2.4 - 2.0322) = 2.3916 \\
P_1 &= (I - K_1 H_1) P_1^- (I - K_1 H_1) + K_1 R K_1 \\
&\simeq (1 - 0.9772 \cdot 1)4.2924(1 - 0.9772 \cdot 1) + 0.9772 \cdot 0.1 \cdot 0.9772 \simeq 0.0977 \\
\Phi_1 &= \left.\frac{df}{dx_k}\right|_{x_k=\hat{x}_1} \simeq 0.3 \cdot 2.4233^2 + 0.7 \simeq 2.4617 \\
\hat{x}_2^- &= f(\hat{x}_1) \simeq 0.1 \cdot 2.3916^3 + 0.7 \cdot 2.3916 \simeq 3.0421 \\
P_2^- &= \Phi_1 P_1 \Phi_1 + Q \simeq 2.4617 \cdot 0.0977 \cdot 2.4617 + 4 \simeq 4.5922
\end{aligned}$$

Finally, for $k = 2$, we obtain:

$$\begin{aligned}
H_2 &= \left.\frac{dh}{dx_k}\right|_{x_k=\hat{x}_2^-} = 1 \\
K_2 &= P_2^- H_2 (H_2 P_2^- H_2 + R)^{-1} = 4.5922 \cdot 1(1 \cdot 4.5922 \cdot 1 + 0.1)^{-1} \simeq 0.9787 \\
\hat{x}_2 &= \hat{x}_2^- + K_2(z_2 - h(\hat{x}_2^-)) \simeq 5.9655 + 0.9787(1.5 - 3.0421) = 1.5330
\end{aligned}$$

To summarize, we have obtained:

$$\hat{x}_0 \simeq 1.9091, \quad \hat{x}_1 \simeq 2.3916, \quad \hat{x}_2 \simeq 1.5330$$