Norwegian University of Science and Technology
Department of Engineering Cybernetics

Exercise 3
LP, QP, and KKT Conditions

Problem 1 (35 %) LP and KKT conditions (Exam August 2000)

Consider the following LP in standard form:

$$\min_{x} c^\top x \qquad \text{s.t.} \qquad Ax = b, \quad x \geq 0 \tag{1}$$

with $c,\ x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. State the KKT conditions for this problem (copy them from your last homework or the textbook).

a Show that the Newton direction (see p. 22) cannot be defined for problem (1).

b Show that (1) is a convex problem by using the definition of a convex function and the definition of a convex optimization problem.

c The dual problem for (1) is defined as

$$\max_{\lambda} b^\top \lambda \qquad \text{s.t.} \qquad A^\top \lambda \leq c \tag{2}$$

Show that the KKT-conditions for the dual problem (2) equals the KKT-conditions for problem (1).

d What is the relation between the optimal objective $c^\top x^*$ of problem (1) and the optimal objective $b^\top \lambda^*$ of problem (2)? (You do not have to derive the relation if you did so in the previous assignment.)

e Define the term *basic feasible point* for problem (1).

f We always assume that A in (1) has full (row) rank (see page 362 in the textbook). What does this mean for satisfying the LICQ (Definition 12.4 in the textbook)?

Problem 2 (40 %) LP

Two reactors, R_I and R_{II}, produce two products A and B. To make 1000 kg of A, 2 hours of R_I and 1 hour of R_{II} are required. To make 1000 kg of B, 1 hour of R_I and 3 hours of R_{II} are required. The order of R_I and R_{II} does not matter. R_I and R_{II} are available for 8 and 15 hours, respectively. The selling price of A is $\frac{3}{2}$ of the selling price of B (i.e., 50 % higher). We want to maximize the total selling price of the two products.

a Formulate this problem as an LP in standard form.

b Make a contour plot (use the MATLAB functions `contour` and `meshgrid`) and sketch the constraints (i.e., use a pen for the constraints if you prefer).

c Calculate the production of A and B that maximizes the total selling price. Use the MATLAB function `simplex` published on Blackboard (an example of use is also published). Start the algorithm at $x_1 = x_2 = 0$. Is the solution at a point of intersection between the constraints? Are all constraints active? (DO NOT attach a printout of the algorithm output.)

d Mark all iterations on the plot made in **b)**, as well as the iteration number.

e Look at the iterations on the plot and the algorithm output. Does everything agree with the theory in Chapter 13.3?

Problem 3 (25 %) QP and KKT Conditions (Exam May 2000)

A quadratic program (QP) can be formulated as

$$\min_x \quad q(x) = \frac{1}{2}x^\top Gx + x^\top c \tag{3a}$$

$$\text{s.t.} \quad a_i^\top x = b_i, \qquad i \in \mathcal{E} \tag{3b}$$

$$a_i^\top x \geq b_i, \qquad i \in \mathcal{I} \tag{3c}$$

where G is a symmetric $n \times n$ matrix, $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices, and c, x and $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}$, are vectors in $\mathbb{R}^n$.

a Define the active set $\mathcal{A}(x^*)$ for problem (3).

b Derive the KKT conditions for problem (3), using the active set in the formulation.