



Problem 1 (50 %) Quadratic Programming

a A quadratic program

$$\min_x q(x) = \frac{1}{2}x^\top Gx + x^\top c \quad (1a)$$

$$\text{s.t. } a_i^\top x = b_i, \quad i \in \mathcal{E} \quad (1b)$$

$$a_i^\top x \geq b_i, \quad i \in \mathcal{I} \quad (1c)$$

is convex if the Hessian matrix G is positive semidefinite. If G is positive definite, the QP is strictly convex.

Convexity is important because in the case of convex QPs, the problem is often similar in difficulty to a linear program, and the problem has only one solution. A nonconvex QP can be more challenging due to the possibility of several local minima and stationary points.

b If the condition in Theorem 16.2 is changed from $Z^\top GZ > 0$ to $Z^\top GZ \geq 0$, all we can say about x^* is that it is a *(local) solution* of the equality constrained QP (16.3).

The proof is identical up until the equation

$$q(x) = \frac{1}{2}u^\top Z^\top GZ u + q(x^*) \quad (2)$$

Now, since we have $Z^\top GZ \geq 0$, we can only conclude $q(x) \geq q(x^*)$, which shows that x^* is a local minimizer (not strict!).

c We will apply Algorithm 16.3 to the quadratic program with

$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad c^\top = [-2 \quad -5] \quad (3a)$$

$$a_1^\top = [1 \quad -2] \quad a_2^\top = [-1 \quad -2] \quad (3b)$$

$$a_3^\top = [-1 \quad 2] \quad a_4^\top = [1 \quad 0] \quad (3c)$$

$$a_5^\top = [0 \quad 1] \quad b^\top = [-2 \quad -6 \quad -2 \quad 0 \quad 0] \quad (3d)$$

and $\{1, 2, 3, 4, 5\} \in \mathcal{I}$. Note that the objective function also has a constant element 7.25, but this has no effect on the solution. We start the algorithm at $x = [2, 0]^\top$, where constraints 3 and 5 are active. However, we set the initial working set \mathcal{W}_0 to only contain constraint 3.

Let x_k , λ_k , p_k denote the variables x , λ , p at iteration k ; let A_k the matrix $[A_i]_{i \in \mathcal{W}_k}$, that is, containing the vectors a_i which are in the working set at iteration k .

Iteration $k = 0$

Since only constraint 3 is active, A_0 will contain a_3 . To find the direction p , we solve the equality-constrained QP

$$\min_p \quad q(x) = \frac{1}{2}p^\top Gp + g_0^\top p \quad (4a)$$

$$\text{s.t.} \quad a_3^\top p = 0 \quad (4b)$$

where $g_0 = Gx_0 + c$. (See equation (16.39) in the textbook.) From Chapter 16.1, we know that the solution to this problem can be found by solving the equation set

$$\begin{bmatrix} G & -A_0^\top \\ A_0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} -g_0 \\ 0 \end{bmatrix} \quad (5a)$$

which gives

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ \lambda_0 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} \quad (5b)$$

This equation set has the solution $p_0 = [0.2, 0.1]^\top$ and $\lambda_0 = -2.4$. With this direction we get

$$x_1 = x_0 + p_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 2.2 \\ 0.1 \end{bmatrix} \quad (6)$$

This point is feasible with respect to all constraints, so we do not need to find an α_0 at this iteration. Note that there also are no blocking constraints. Hence, the working set at the next iteration will be the same as in this iteration. We set $k = 1$ and proceed to the next iteration.

Iteration $k = 1$

We now solve the same QP as above, except that we now have $g_1 = Gx_1 + c = [2.4, -4.8]^\top$. This gives the equation set

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} -2.4 \\ 4.8 \\ 0 \end{bmatrix} \quad (7)$$

with the solution $p_1 = [0, 0]^\top$ and $\lambda_1 = -2.4$. Since $p_1 = 0$ and $\lambda_1 < 0$, we remove the constraint with the most negative multiplier from the working set. With only one constraint in the working set, the working set will now be empty. Furthermore, $x_2 = x_1 + 0 = x_1$. We set $k = 2$ and proceed to the next iteration.

Iteration $k = 2$

Now that we have an empty working set, so A_2 is an empty matrix. The direction p is then found from the unconstrained QP

$$\min_p \quad q(x) = \frac{1}{2}p^\top Gp + g_2^\top p \quad (8)$$

where $g_2 = Gx_2 + c = [2.4, -4.8]^\top$. Solving the equation set

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} p_2 = \begin{bmatrix} -2.4 \\ 4.8 \end{bmatrix} \quad (9)$$

gives the solution $p_2 = [-1.2, 2.4]^\top$. (There are no multipliers due to the absence of constraints in the working set.) Note that the solution to the unconstrained problem is $p_2 = G^{-1}(-g_2)$, that $\nabla^2 q(x_2) = G$, and that $\nabla q(x_2) = Gx_2 + c$. Hence, we can write $p_2 = -(\nabla^2 q(x_2))^{-1} \nabla q(x_2)$, which is the Newton direction (see equation (2.15) in the textbook). Setting $x_3 = x_2 + p_2 = [1, 2.5]^\top$ gives an infeasible point (constraint 1 is violated), so a step-length parameter α_2 must be found. $a_i^\top p_2 < 0$ for $i = 1, 2$, and 4, so

$$\begin{aligned} \alpha_2 &= \min \left(1, \frac{b_1 - a_1^\top x_2}{a_1^\top p_2}, \frac{b_2 - a_2^\top x_2}{a_2^\top p_2}, \frac{b_4 - a_4^\top x_2}{a_4^\top p_2} \right) \\ &= \min \left(1, \frac{2}{3}, 1, \frac{11}{6} \right) = \frac{2}{3} \end{aligned} \quad (10)$$

As the minimum value corresponds to corresponds to constraint 1 (a blocking constraint), this constraint is added to the working set. That is, $\mathcal{W}_3 = \{1\}$. x at the next iteration is then found from

$$x_3 = x_2 + \alpha_2 p_2 = \begin{bmatrix} 1.4 \\ 1.7 \end{bmatrix} \quad (11)$$

We set $k = 3$ and proceed to the next iteration.

Iteration $k = 3$

We now have $g_3 = Gx_3 + c = [0.8, -1.6]^\top$ and $A_3 = a_1$. Then, the KKT system that solves the QP for the direction p_3 becomes

$$\begin{bmatrix} G & -A_3^\top \\ A_3 & 0 \end{bmatrix} \begin{bmatrix} p_3 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -g_3 \\ 0 \end{bmatrix} \quad (12a)$$

which gives

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} p_3 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1.6 \\ 0 \end{bmatrix} \quad (12b)$$

The solution to this equation set is $p_3 = [0, 0]^\top$ and $\lambda_0 = 0.8$. Hence, we can conclude that we have found the solution. That is, $x^* = [1.4, 1.7]^\top$.

The problem and iteration sequence is illustrated in Figure 1.

d We write the problem in Example 16.4 as

$$\min_x \quad q(x) = \frac{1}{2} x^\top Gx + x^\top c \quad (13a)$$

$$\text{s.t.} \quad Ax - b \geq 0 \quad (13b)$$

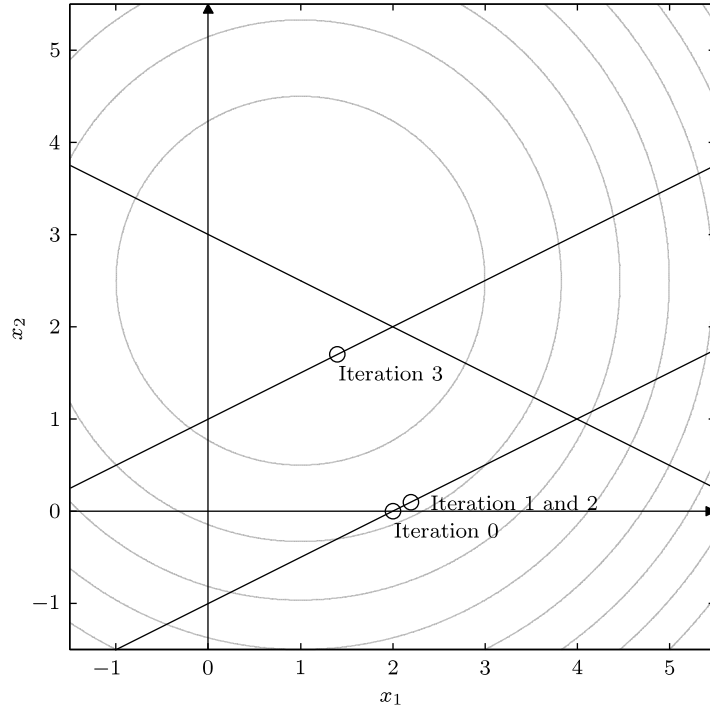


Figure 1: Contour plot with constraints and iterations for Problem 1.

where

$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \quad (14a)$$

$$A = \begin{bmatrix} 1 & -2 \\ -1 & -2 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ -6 \\ -2 \\ 0 \\ 0 \end{bmatrix} \quad (14b)$$

The Lagrangian for this problem is

$$\mathcal{L}(x, \lambda) = \frac{1}{2}x^\top Gx + x^\top c - \lambda^\top (Ax - b) \quad (15)$$

Hence, the dual objective is

$$f(\lambda) = \inf_x \mathcal{L}(x, \lambda) = \inf_x \frac{1}{2}x^\top Gx + x^\top c - \lambda^\top (Ax - b) \quad (16)$$

Since $G > 0$ and the Lagrangian is a strictly convex quadratic function, the infimum with respect to x is where $\nabla_x \mathcal{L}(x, \lambda) = 0$. That is:

$$\begin{aligned} \nabla_x \mathcal{L}(x, \lambda) &= \frac{1}{2}G^\top x + \frac{1}{2}Gx + c - (\lambda^\top A)^\top, \quad G = G^\top \\ &= Gx + c - A^\top \lambda = 0 \end{aligned} \quad (17)$$

We can now write the the dual problem in three different forms. One form is obtained by writing x as $x = G^{-1}(A^\top \lambda - c)$ and substitute in the dual objective to get

$$\begin{aligned} f(\lambda) = & \frac{1}{2}(A^\top \lambda - c)^\top G^{-1} G G^{-1} (A^\top \lambda - c) \\ & + (A^\top \lambda - c)^\top G^{-1} c - \lambda^\top (A G^{-1} (A^\top \lambda - c) - b) \end{aligned} \quad (18)$$

After some rearranging, we can write

$$\begin{aligned} f(\lambda) = & \frac{1}{2}(A^\top \lambda - c)^\top G^{-1} (A^\top \lambda - c) \\ & + A^\top \lambda G^{-1} c - c^\top G^{-1} c \\ & - \lambda^\top A G^{-1} A^\top \lambda + \lambda^\top A G^{-1} c + \lambda^\top b \\ \Rightarrow f(\lambda) = & \frac{1}{2}(A^\top \lambda - c)^\top G^{-1} (A^\top \lambda - c) \\ & - (A^\top \lambda - c)^\top G^{-1} (A^\top \lambda - c) + \lambda^\top b \\ \Rightarrow f(\lambda) = & -\frac{1}{2}(A^\top \lambda - c)^\top G^{-1} (A^\top \lambda - c) + \lambda^\top b \end{aligned} \quad (19)$$

The dual problem can then be formulated as

$$\max_{\lambda} \quad f(\lambda) = -\frac{1}{2}(A^\top \lambda - c)^\top G^{-1} (A^\top \lambda - c) + \lambda^\top b \quad (20a)$$

$$\text{s.t.} \quad \lambda \geq 0 \quad (20b)$$

Alternatively, the dual problem can be formulated in both x and λ as

$$\max_{x, \lambda} \quad \frac{1}{2} x^\top G x + x^\top c - \lambda^\top (A x - b) \quad (21a)$$

$$\text{s.t.} \quad G x + c - A^\top \lambda = 0 \quad (21b)$$

$$\lambda \geq 0 \quad (21c)$$

A third option is to reformulate the constraint as $(c - A^\top \lambda)^\top x = -x^\top G x$, and substitute into the objective function to get

$$\max_{x, \lambda} \quad -\frac{1}{2} x^\top G x + \lambda^\top b \quad (22a)$$

$$\text{s.t.} \quad G x + c - A^\top \lambda = 0 \quad (22b)$$

$$\lambda \geq 0 \quad (22c)$$

- e The dual optimization problem can be used to give an over-estimate of $q(\bar{x}) - q(x^*)$ (q is the primal objective), when x^* is not known. Any feasible \bar{x} and any $\bar{\lambda} \geq 0$ will give $f(\bar{\lambda}) \leq q(x^*)$ (f is the dual objective). Therefore,

$$q(\bar{x}) - q(x^*) \leq q(\bar{x}) - f(\bar{\lambda}) \quad (23)$$

Problem 2 (50 %) Production Planning and Quadratic Programming

Two reactors, R_I and R_{II} , produce two products A and B . To make 1000 kg of A , 2 hours of R_I and 1 hour of R_{II} are required. To make 1000 kg of B , 1 hour of R_I and 3 hours of R_{II} are required. The order of R_I and R_{II} does not matter. R_I and R_{II} are available for 8 and 15 hours, respectively. We want to maximize the total profit from the two products.

The profit now depends on the production rate:

- the profit from A is $3 - 0.4x_1$ per tonne produced,
- the profit from B is $2 - 0.2x_2$ per tonne produced,

where x_1 is the production of product A and x_2 is the production of product B (both in tonnes).

a The total profit from selling the two products is

$$p(x) = (3 - 0.4x_1)x_1 + (2 - 0.2x_2)x_2 \quad (24)$$

Since we want to maximize $p(x)$, we minimize the negative of $p(x)$. Hence, the objective function for the minimization problem is

$$f(x) = -p(x) = -(3 - 0.4x_1)x_1 - (2 - 0.2x_2)x_2 = \frac{1}{2}x^\top \underbrace{\begin{bmatrix} 0.8 & 0 \\ 0 & 0.4 \end{bmatrix}}_G x + \underbrace{[-3 \quad -2]}_{c^\top} x \quad (25)$$

Note that $G > 0$. The availability constraints can be formulated as $2x_1 + x_2 \leq 8$ and $x_1 + 3x_2 \leq 15$; nonnegative production rates are formulated $x_1 \geq 0$ and $x_2 \geq 0$. We then have the optimization problem

$$\min_x \quad q(x) = \frac{1}{2}x^\top Gx + x^\top c \quad (26a)$$

$$\text{s.t.} \quad -2x_1 - x_2 \geq -8 \quad (26b)$$

$$-x_1 - 3x_2 \geq -15 \quad (26c)$$

$$x_1 \geq 0 \quad (26d)$$

$$x_2 \geq 0 \quad (26e)$$

b A contour plot with constraints and iterations indicated is illustrated in Figure 2. Code for producing a contour plot in MATLAB is attached at the end of this solution.

c The modifications that have to be made to the file `qp_prodplan.m` so that it solves the problem formulated in a) is attached at the end of this solution. We see in Figure 2 that as opposed to the linear case in exercise 3, the solution is not at a point of intersection between constraints. The solution and all iterations are indicated in Figure 2.

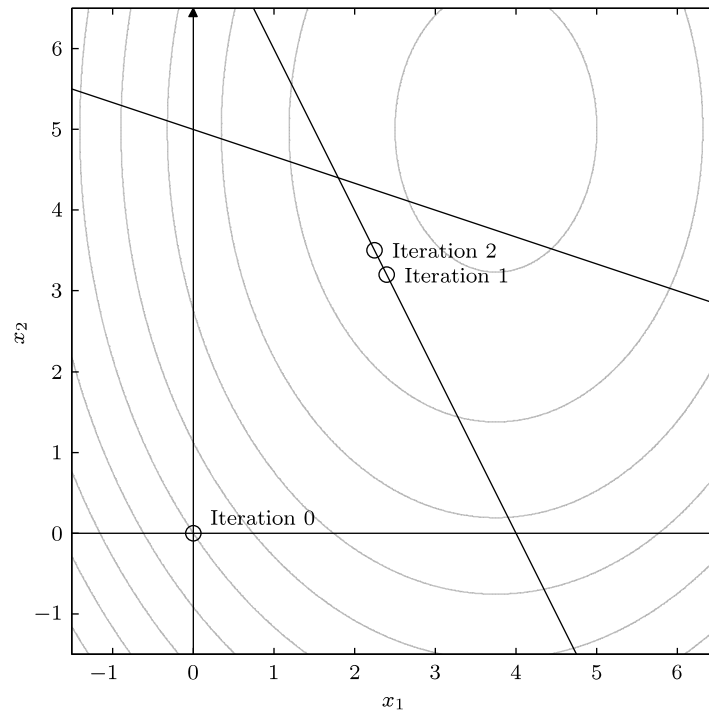


Figure 2: Contour plot with constraints and iterations for Problem 2.

- d** See Problem 1 c) and the textbook for an explanation of the active set algorithm.
- e** As mentioned, the solution is not found at a corner point of the feasible area. For linear programs, the solution is always at a point of intersection between constraints. For quadratic programs, the solution may be at any point in the feasible area, including the interior (where no constraints are active).

```

1 % Code for making a contour plot for Problem 2.
2 x1_l = -1.5; x1_h = 6.5;
3 x2_l = -1.5; x2_h = 6.5;
4 res = 0.01;
5 [x1, x2] = meshgrid(x1_l:res:x2_h, x1_l:res:x2_h);
6 f = -(3-0.4*x1).*x1 - (2-0.2*x2).*x2;
7 levels = (-12:2:8)';
8 [C, h] = contour(x1, x2, f, levels, 'Color', .7*[1 1 1]);
9 set(h, 'ShowText', 'on', 'LabelSpacing', 300); % For text labels

1 % Changes made to the file qp_prodplan.m
2 % Quadratic objective (MODIFY THESE)
3 G = [0.8  0   ;
4       0   0.4]; % Remember the factor 1/2 in the objective
5 c = [-3 ; -2];
6 % Linear constraints (MODIFY THESE)
7 A = [2 1 ;
8       1 3];
9 b = [8 ; 15];

```