TTK4115 Linear System Theory

Department of Engineering Cybernetics

NTNU

Homework assignment 5

Hand-out time:	Monday, October 16, 2017, at 8:00
Hand-in deadline:	Friday, October 27, 2017, at 16:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard, or in the boxes in D238. Please write your name on your answer sheet, should you choose to hand in physically. Any questions regarding the assignment should be directed through Blackboard.

Problem 1: Input-output stability of discrete-time systems

Consider the following discrete-time system:

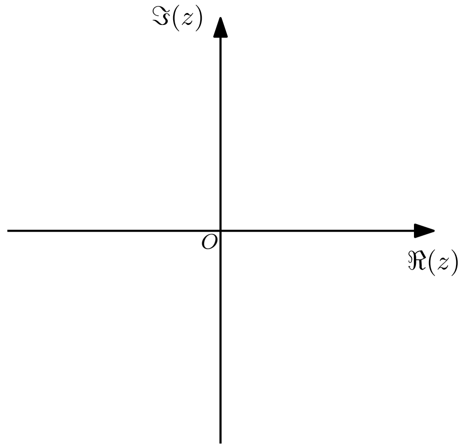
$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}u[k] \\ y[k] &= \mathbf{C}\mathbf{x}[k] + Du[k], \end{aligned}$$

with state $\mathbf{x}[k]$, input $u[k]$, output $y[k]$ and matrices

$$\mathbf{A} = \begin{bmatrix} -0.7 & -0.6 \\ 0.4 & 0.7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad D = -1.$$

a) Show that the discrete transfer function $\hat{g}(z) = \frac{\hat{y}(z)}{\hat{u}(z)}$ is given by $\hat{g}(z) = \frac{-z^2+0.1z}{z^2-0.25}$.

A graphical representation of the z-plane is given by:



where $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of z and O is the origin (i.e. the point $z = 0$).

b) Draw the boundary of the region of the z-plane in which the poles of the discrete transfer function $\hat{g}(z)$ must be for the system to be BIBO stable. Moreover, draw the poles of the discrete transfer function $\hat{g}(z)$ in the z-plane. Determine if the system is BIBO stable.

Problem 2: Stability of continuous-time systems

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t),\end{aligned}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad D = 1.$$

a) Determine from the system matrix $\mathbf{A}$ if the system is marginally stable (or Lyapunov stable), asymptotically stable and/or unstable.

b) Show that the transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ of the system is given by $\hat{g}(s) = \frac{s+2}{s+1}$.

c) Determine from the transfer function if the system is BIBO stable.

d) Show that the impulse response of the system is given by $g(t) = e^{-t} + \delta(t)$, where $\delta(t)$ is the Dirac delta function.

e) Determine from the impulse response if the system is BIBO stable.

Problem 3: Internal stability

Consider the autonomous system described by the state-space equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t), \quad \text{with} \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}.$$

a) Calculate the eigenvalues and eigenvectors of $\mathbf{A}_1$.

b) Determine if the system is marginally stable (or Lyapunov stable), asymptotically stable and/or unstable.

Now, consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_2\mathbf{x}(t), \quad \text{with} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

c) Calculate the eigenvalues and eigenvectors of $\mathbf{A}_2$.

d) Determine if the system is marginally stable (or Lyapunov stable), asymptotically stable and/or unstable.

Next, consider the system matrix:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_3\mathbf{x}(t), \quad \text{with} \quad \mathbf{A}_3 = \begin{bmatrix} -4 & -2 \\ 1 & -2 \end{bmatrix}.$$

e) Compute the symmetric matrix $\mathbf{P}$ from the Lyapunov equation

$$\mathbf{A}_3^T\mathbf{P} + \mathbf{P}\mathbf{A}_3 = -\mathbf{Q},$$

where $\mathbf{Q} = \mathbf{I}$ is the identity matrix. Conclude from the matrix $\mathbf{P}$ if the system is asymptotically stable.

Problem 4: Process classification

Consider the following process:

$$X(t) = a\sin(\omega t + \Phi),$$

where a and ω are constants and the variable Φ is uniformly distributed in the interval $[-\pi, \pi]$ (i.e. $\Phi \sim \mathcal{U}(-\pi, \pi)$).

a) Show that the mean $\mu_X(t) = E[X(t)]$ is given by $\mu_X(t) = 0$.

b) Show that the variance $\sigma_X^2(t) = E[X^2(t)]$ is given by $\sigma_X^2(t) = \frac{a^2}{2}$.

Note that for any real numbers b and c, we have

$$\sin(b)\sin(c) = \frac{1}{2}\cos(b-c) - \frac{1}{2}\cos(b+c).$$

c) Show that the autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$ is given by $R_X(t_1, t_2) = \frac{a^2}{2}\cos(\omega(t_1 - t_2))$. Write the autocorrelation function as $R_X(\tau) = E[X(t)X(t+\tau)]$ if possible.

d) Is the process deterministic? Motivate your answer.

e) Is the process wide-sense stationary? Motivate your answer.

f) Is the process ergodic (in wide sense)? Motivate your answer.