# TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

## Solution to homework assignment 5

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### Problem 1: Input-output stability of discrete-time systems

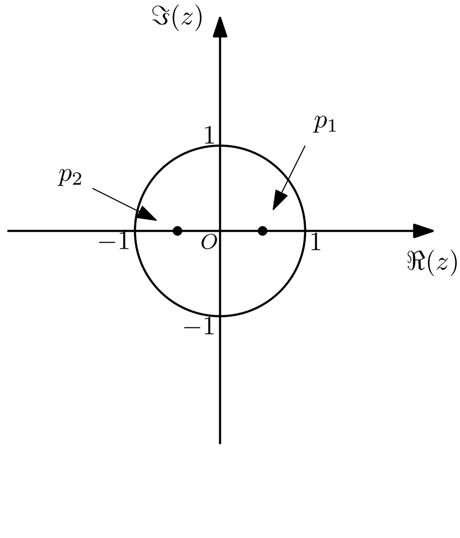
a) The discrete transfer function $\hat{g}(z) = \frac{\hat{y}(z)}{\hat{u}(z)}$ is given by

$$\begin{aligned}
\hat{g}(z) &= \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z + 0.7 & 0.6 \\ -0.4 & z - 0.7 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} - 1 \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{z-0.7}{z^2-0.25} & \frac{-0.6}{z^2-0.25} \\ \frac{0.4}{z^2-0.25} & \frac{z+0.7}{z^2-0.25} \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} - 1 \\
&= \frac{0.1z - 0.25}{z^2 - 0.25} - 1 \\
&= \frac{-z^2 + 0.1z}{z^2 - 0.25}.
\end{aligned}$$

b) The system is BIBO stable if and only if every pole of the discrete transfer function $\hat{g}(z)$ has a magnitude less than 1. Therefore, the boundary of the region of the $z$-plane in which the poles of the discrete transfer function $\hat{g}(z)$ must be for the system to be BIBO stable is a circle of radius one centered around the origin. The poles of the discrete transfer function $\hat{g}(z)$ are equal to the roots of the denominator

$$d(z) = z^2 - 0.25 = (z - 0.5)(z + 0.5).$$

Hence, we obtain the poles $p_1 = 0.5$ and $p_2 = -0.5$. The boundary of the stability region and the poles are visualized as follows:



Because both poles are inside the stability region of the $z$-plane, the system is BIBO stable.

**Problem 2: Stability of continuous-time systems**

a) To find out if the system is marginally stable, asymptotically stable and/or unstable, we compute the eigenvalues of $\mathbf{A}$. The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -2 \\ 0 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 = (1-\lambda)(-1-\lambda).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Because $\lambda_1$ has a positive real part, the system is not marginally stable and not asymptotically stable. Because the system is not marginally stable, it is unstable.

b) The transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$ is given by

$$\begin{aligned} \hat{g}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 2 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{-2}{(s-1)(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \\ &= \frac{1}{s-1} - \frac{2}{(s-1)(s+1)} + 1 \\ &= \frac{s+2}{s+1}. \end{aligned}$$

c) To determine if the system is BIBO stable, we compute the poles of the transfer function. The denominator of the transfer function is given by

$$d(s) = s + 1.$$

The poles of the transfer function are equal to the roots of the denominator. Hence, we obtain that the pole of the system is given by $p = -1$. Because the pole $p$ has a negative real part, we conclude that the system is BIBO stable.

d) The impulse response $g(t)$ of the system can be obtained from the inverse Laplace transform of the transfer function $\hat{g}(s)$:

$$\begin{aligned} g(t) = \mathcal{L}^{-1}[\hat{g}(s)] &= \mathcal{L}^{-1}\left[\frac{s+2}{s+1}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1} + 1\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}[1] = e^{-t} + \delta(t), \end{aligned}$$

where $\delta(t)$ is the Dirac delta function.

e) The system is BIBO stable if and only if the impulse response $g(t)$ is absolutely integrable on the domain $[0, \infty)$. That is, the system is BIBO stable if and only if there exists a constant $M > 0$, such that

$$\int_0^\infty |g(t)|dt \leq M < \infty.$$

By computing this integral, we obtain the following:

$$\int_0^\infty |g(t)|dt = \int_0^\infty e^{-t} + \delta(t)dt = \int_0^\infty e^{-t}dt + \int_0^\infty \delta(t)dt = 1 + 1 = 2.$$

Hence, we obtain that the impulse response $g(t)$ is absolutely integrable on the domain $[0, \infty)$, with constant $M = 2$. Therefore, we conclude that the system is BIBO stable.

**Problem 3: Internal stability**

a) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 \\ -2 & -\lambda \end{vmatrix} = \lambda^2.$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalue $\lambda_1 = 0$ with multiplicity 2. The corresponding eigenvectors can be obtained from the kernel of the matrix $(\mathbf{A} - \lambda_1\mathbf{I})$:

$$\ker(\mathbf{A} - \lambda_1\mathbf{I}) = \ker\left(\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\mathbf{q}_1$ is the corresponding eigenvector. Note that $\mathbf{A}$ has only one eigenvector associated with $\lambda_1$.

b) Because there is only one eigenvalue associated with the eigenvalue $\lambda_1 = 0$ and the (algebraic) multiplicity of the eigenvalue is two, the size of the corresponding Jordan block is $2 \times 2$. Therefore, the matrix $\mathbf{A}$ has an eigenvalue with a zero real part that is <u>not</u> a simple root of the minimal polynomial of $\mathbf{A}$. This implies that the system is neither marinally stable (or Lyapunov stable) nor asymptotically stable. Because the system is not marginally stable, it is unstable.

c) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \lambda^2.$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, similar to a), we obtain the eigenvalue $\lambda_1 = 0$ with multiplicity 2. The corresponding eigenvectors can be obtained from the kernel of the matrix $(\mathbf{A}-\lambda_1\mathbf{I})$:

$$\ker(\mathbf{A} - \lambda_1\mathbf{I}) = \ker\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ are the corresponding eigenvectors. Note that $\mathbf{A}$ has two eigenvectors associated with $\lambda_1$.

d) Because there are two eigenvalues associated with the eigenvalue $\lambda_1 = 0$, which has an (algebraic) multiplicity of two, there are two Jordan blocks of size $1 \times 1$ associated with $\lambda_1 = 0$. Because the size of the Jordan blocks $1 \times 1$, the eigenvalue $\lambda_1 = 0$ (with multiplicity 2) is a simple root of the minimal polynomial of $\mathbf{A}$. Therefore, the eigenvalues of the matrix $\mathbf{A}$ have zero or negative real parts and those with zero real parts are simple root of the minimal polynomial of $\mathbf{A}$, which implies that the system is marginally stable (or Lyapunov stable). Because the eigenvalues of $\mathbf{A}$ do not have negative real parts, the system is not asymptotically stable. Moreover, because the system is marginally stable, it is <u>not</u> unstable.

e) To find the matrix $\mathbf{P}$, we solve the Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{I}.$$

Note that $\mathbf{P}$ is a symmetric matrix, i.e. $\mathbf{P} = \mathbf{P}^T$. Let $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix},$$

where $p_1$, $p_2$ and $p_3$ are constant that are yet to be determined. Substituting the matrices $\mathbf{A}$ and $\mathbf{P}$ in the Lyapunov equation, we obtain

$$\begin{bmatrix} -4 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} -4p_1 + p_2 & -4p_2 + p_3 \\ -2p_1 - 2p_2 & -2p_2 - 2p_3 \end{bmatrix} + \begin{bmatrix} -4p_1 + p_2 & -2p_1 - 2p_2 \\ -4p_2 + p_3 & -2p_2 - 2p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

From this, we obtain the equations

$$\begin{aligned} -8p_1 + 2p_2 &= -1, \\ -2p_1 - 6p_2 + p_3 &= 0, \\ -4p_2 - 4p_3 &= -1, \end{aligned}$$

which can be written in the following form:

$$\begin{bmatrix} -8 & 2 & 0 \\ -2 & -6 & 1 \\ 0 & -4 & -4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

Solving for $p_1$, $p_2$ and $p_3$ yields $p_1 = \frac{1}{8}$, $p_2 = 0$ and $p_3 = \frac{1}{4}$. Hence, we obtain the matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

The eigenvalues of **A** have negative real parts if and only if the matrix **P** is positive definite. The matrix **P** is positive definite if and only if all its leading principle minors are positive. The leading principle minors of **P** are

$$p_1 = \frac{1}{8} \qquad \text{and} \qquad \det(\mathbf{P}) = \begin{vmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{4} \end{vmatrix} = \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{32}.$$

Because all leading principle minors of **P** are positive, we conclude that the matrix **P** is positive definite and that the eigenvalues of **A** all have negative real parts. Because all eigenvalues of **A** have negative real parts, the system with system matrix **A** is asymptotically stable.

## Problem 4: Process classification

a) The probability density function of the variable $\Phi$ is given by

$$f_\Phi(\phi) = \begin{cases} \frac{1}{2\pi}, & \text{if } -\pi \leq \phi < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

The mean $\mu_X(t) = E[X(t)]$ is calculated as follows:

$$\begin{aligned} \mu_X(t) &= E[X(t)] = E[a\sin(\omega t + \Phi)] = aE[\sin(\omega t + \Phi)] \\ &= a\int_{-\infty}^{\infty} \sin(\omega t + \phi) f_\Phi(\phi) d\phi = \frac{a}{2\pi}\int_{-\pi}^{\pi} \sin(\omega t + \phi) d\phi \\ &= \frac{a}{2\pi}\left[-\cos(\omega t + \phi)\right]_{-\pi}^{\pi} = \frac{a}{2\pi}\left[-\cos(\omega t + \pi) + \cos(\omega t + -\pi)\right] \\ &= \frac{a}{2\pi}\left[\cos(\omega t) - \cos(\omega t)\right] = 0. \end{aligned}$$

b) The variance $\sigma_X^2(t) = E[X^2(t)]$ is given by

$$\begin{aligned} \sigma_X^2(t) &= E[X^2(t)] = E[(a\sin(\omega t + \Phi))^2] = a^2E[\sin^2(\omega t + \Phi)] \\ &= a^2E\left[\frac{1 - \cos(2\omega t + 2\Phi)}{2}\right] = \frac{a^2}{2}\left(1 - E\left[\cos(2\omega t + 2\Phi)\right]\right) \\ &= \frac{a^2}{2}\left(1 - \int_{-\infty}^{\infty} \cos(2\omega t + 2\phi) f_\Phi(\phi) d\phi\right) \\ &= \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega t + 2\phi) d\phi\right) = \frac{a^2}{2}\left(1 - \frac{1}{2\pi}\left[\frac{\sin(2\omega t + 2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\ &= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t + 2\pi) - \sin(2\omega t - 2\pi)\right]\right) \\ &= \frac{a^2}{2}\left(1 - \frac{1}{4\pi}\left[\sin(2\omega t) - \sin(2\omega t)\right]\right) = \frac{a^2}{2}, \end{aligned}$$

where we used the probability density function $f_\Phi$ in a).

c) Using the probability density function $f_\Phi$ in a), we obtain the following autocorrelation function $R_X(t_1,t_2) = E[X(t_1)X(t_2)]$:

$$
\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] = E[(a\sin(\omega t_1+\Phi))(a\sin(\omega t_2+\Phi))] \\
&= a^2 E[\sin(\omega t_1+\Phi)\sin(\omega t_2+\Phi)] \\
&= a^2 E\left[\frac{1}{2}\cos(\omega t_1+\Phi-(\omega t_2+\Phi)) - \frac{1}{2}\cos(\omega t_1+\Phi+(\omega t_2+\Phi))\right] \\
&= \frac{a^2}{2} E\left[\cos(\omega(t_1-t_2)) - \cos(\omega(t_1+t_2)+2\Phi)\right] \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - E[\cos(\omega(t_1+t_2)+2\Phi)]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \int_{-\infty}^{\infty}\cos(\omega(t_1+t_2)+2\phi)f_\Phi(\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega(t_1+t_2)+2\phi)d\phi\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{2\pi}\left[\frac{\sin(\omega(t_1+t_2)+2\phi)}{2}\right]_{-\pi}^{\pi}\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1+t_2)+2\pi) - \sin(\omega(t_1+t_2)-2\pi)\right]\right) \\
&= \frac{a^2}{2}\left(\cos(\omega(t_1-t_2)) - \frac{1}{4\pi}\left[\sin(\omega(t_1+t_2)) - \sin(\omega(t_1+t_2))\right]\right) \\
&= \frac{a^2}{2}\cos(\omega(t_1-t_2)).
\end{aligned}
$$

Substituting $t_1 = t$ and $t_2 = t+\tau$, we get

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{a^2}{2}\cos(\omega(t-(t+\tau))) = \frac{a^2}{2}\cos(-\omega\tau) = \frac{a^2}{2}\cos(\omega\tau).$$

d) The process is deterministic. With $\Phi = \Phi_1$ the process becomes $X(t,\Phi_1) = a\sin(\omega t + \Phi_1)$. Knowledge about the process for $t \leq t_0$ makes identification of $\Phi_1$, $\omega$ and $a$ possible, and the process is uniquely defined $\forall t > t_0$.

e) Because the mean $\mu_X(t)$ is not dependent on the time origin (i.e. $\mu_X(t)$ is independent of $t$, see a)) and the autocorrelation function $R_X(t_1,t_2)$ in c) is only dependent on the time difference between sample points (i.e. $R_X(t_1,t_2)$ is dependent only on the time difference $t_2 - t_1$, since we can write $R_X(t_1,t_2) = R_X(\tau)$ for $t_1 = t$ and $t_2 = t+\tau$, see c)), the process is wide-sense stationary. In fact, it can be shown that all density functions associated with the process are independent of time, which implies that the process is stationary, which is a stronger property than wide-sense stationarity.

f) While ergodicity applies to all density functions associated with the process, ergodicity in wide sense only applies to the mean and autocorrelation function of the process. For a process to be ergodic in wide sense, the time mean and the time autocorrelation function must be equivalent to the ensemble mean (i.e. $\mu_X$) and the ensemble autocorrelation function (i.e. $R_X(\tau)$), respectively.

The time mean is given by

$$\begin{aligned}
\mathfrak{m}_X &= \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T a\sin(\omega t+\Phi)dt \\
&= \lim_{T\to\infty} \frac{a}{T}\left[\frac{-\cos(\omega t+\Phi)}{\omega}\right]_0^T = \lim_{T\to\infty} \frac{a}{\omega T}\left[-\cos(\omega T+\Phi)+\cos(\Phi)\right] = 0.
\end{aligned}$$

The time autocorrelation function is given by

$$\begin{aligned}
\mathfrak{R}_X(\tau) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)X(t+\tau)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T (a\sin(\omega t+\Phi))(a\sin(\omega(t+\tau)+\Phi))dt \\
&= \lim_{T\to\infty} \frac{a^2}{T}\int_0^T \sin(\omega t+\Phi)\sin(\omega(t+\tau)+\Phi)dt \\
&= \lim_{T\to\infty} \frac{a^2}{T}\int_0^T \Bigg(\frac{1}{2}\cos(\omega t+\Phi-(\omega(t+\tau)+\Phi)) \\
&\qquad -\frac{1}{2}\cos(\omega t+\Phi+\omega(t+\tau)+\Phi)\Bigg)dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\int_0^T \left(\cos(-\omega\tau)-\cos(2\omega t+\omega\tau+2\Phi)\right)dt \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\left[\cos(\omega\tau)t-\frac{\sin(2\omega t+\omega\tau+2\Phi)}{2\omega}\right]_0^T \\
&= \lim_{T\to\infty} \frac{a^2}{2T}\left[\cos(\omega\tau)T-\frac{\sin(2\omega T+\omega\tau+2\Phi)}{2\omega}+\frac{\sin(\omega\tau+2\Phi)}{2\omega}\right] \\
&= \frac{a^2}{2}\cos(\omega\tau).
\end{aligned}$$

Because the time mean $\mathfrak{m}_X$ and time autocorrelation function $\mathfrak{R}_X(\tau)$ are equal to the ensemble mean $\mu_X$ in a) and the ensemble autocorrelation function $R_X(\tau)$ in c), respectively, we conclude that the process is ergodic in wide sense. In fact, it can be shown that process is ergodic (not only in wide sense).