NTNU – Trondheim
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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Thursday, May 24 2018
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

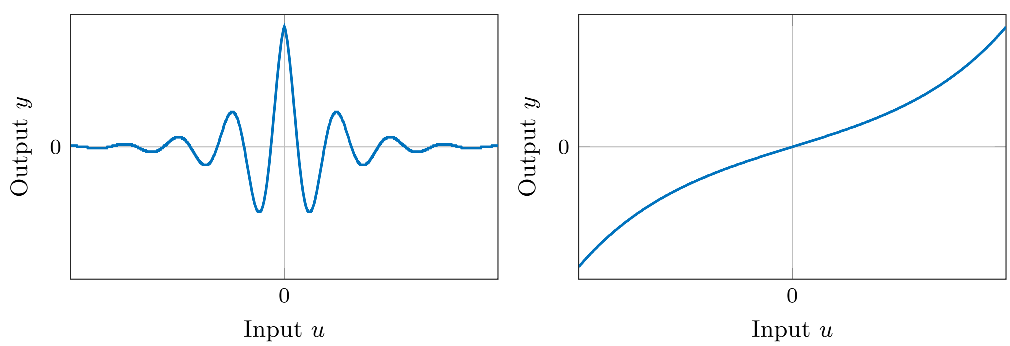
Answers in English, Norwegian, or a mixture of the two are accepted.

Grades available: As specified by regulations.

Problem 1 (16 %)

A memoryless system defines a simple system, that cannot store energy and where the output signal at each time depends only on the input at that time. Consequently, it cannot remember previous values of the input in order to determine the current value of the output. An example of such system is a electrical circuit with only an ideal resistor, where the relationship between current and voltage is given by Ohm's law.

(4 %) (a) In Figure 1 the relationship between input u and output y of two different memoryless systems are shown. Please explain briefly, if the systems are passive or not.



(a) System 1. (b) System 2.

Figure 1: Relationship between input u and output y of system 1 and system 2.

(4 %) (b) The model *ModSim* was implemented previously in Modelica. Now you would like to create a new model *ModSimExpert* that adds the following equation to the model *ModSim*

$$\frac{dp}{dt} = \frac{c^2}{V}(\omega_1 - \omega_2).$$

The variables ω_1 and ω_2 and the parameter V were already defined in the model *ModSim*.
Write the Modelica code of the new model *ModSimExpert*. You can choose yourself values for possibly needed parameter and initial values.
Hint: You do not have to implement the model ModSim to solve the task.

(8 %) (c) The following Modelica code has unfortunately several errors. Find the errors in the implementation of the model and explain briefly what is wrong

```
model PredatorPrey
  type birthrate = Real (unit="years-1");
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```
  type deathrate = Real (unit="years-1");
  parameter deathrate b2 = 0.5 "deathrate [years-1]";
  parameter deathrate c = 0.3 "deathrate [years-1]";
  parameter birthrate b1 = 2.0;
  parameter Real p "PredatorInfluence [year-1.predator-1]";
  parameter Real r = 0.004 "PreyInfluence [year-1.predator-1]"

  Real S(initial=2000) "PreyPopulation";
  Real L(initial=80) "PredatorPopulation";

  d_dt(S) = (b1-b2-p*L); // Change of PreyPopulation
  d_dt(L) = (r*S-c)*L;
  0 = S - b1*L;
end PredatorPray;
```

Problem 2 (26 %)

We have an electrical circuit with two resistors, an inductor and a capacitor (Fig. 2). The resistors follow Ohm's law ($V = IR$). The voltage source provides a voltage V_i.
Hint: $exp(x) = \lim_{n\to\infty}(1 + x/n)^n$

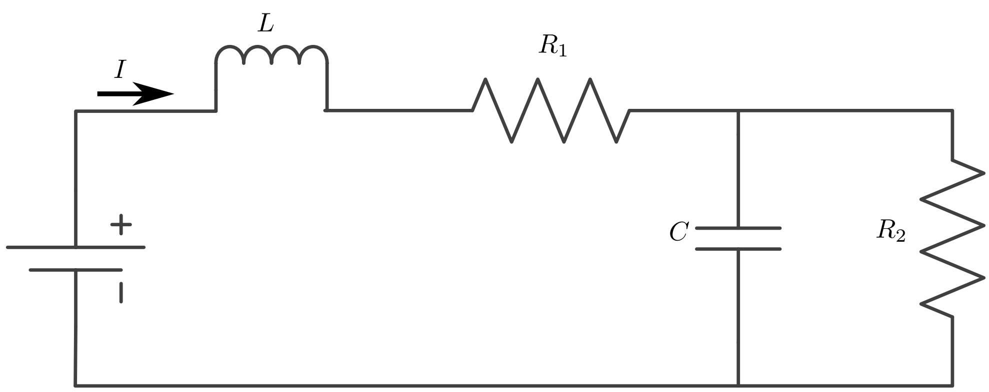

Figure 2: Electrical Circuit.

(4 %) (a) Derive an ODE model of the circuit for the current I and voltage in the capacitor V_c.

If you were not able to find an ODE model in a) continue with the following model [which is not necessarily the correct answer to a)]:

$$\frac{dI}{dt} = \frac{1}{R_1^2}V_i + \frac{1}{CR_2}V_c + \frac{R_1}{C}I,$$
$$\frac{dV_c}{dt} = -\frac{R_1^2}{L}I + \frac{1}{L}V_c - \frac{C}{LR_2}V_i.$$

(2 %) (b) Transfer your model into a state-space form.

Assume the resistor has the resistance $R = R_1 = R_2 = 1000\,\Omega$, the inductor the inductance $L = 10^{-2}$ H, and the capacitor the capacitance $C = 10^{-4}$ F.

(4 %) (c) Explain why it may be problematic to use an explicit Runge-Kutta method to simulate the system.

(4 %) (d) Use the modified Euler method to simulate the system (Tab. 1). What is the largest step-size you can use for a stable simulation?

$$\begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}.$$

Table 1: Butcher array of modified Euler method

(4 %) (e) In order to choose a larger step-size the system should be simulated with the Gauss method of order four (Tab. 2). What is the maximum step-size you can choose to guarantee a stable simulation of the system if you use this method? What kind of other problem can you encounter in the simulations?

$$\begin{array}{c|cc} \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\ \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}.$$

Table 2: Butcher array of Gauss method of order four.

The exact solution of the differential equation

$$\frac{dI}{dt} = \frac{U}{L} - \frac{R}{L}I,$$

is given by

$$I(t) = I_0 e^{-\frac{R}{L}t} + \frac{U}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

(5 %) (f) Show that the explicit Euler method applied to the differential equation of I with the initial value $x_0 = I_0$ and the step-size h results in

$$I_n = \left(1 - h\frac{R}{L}\right)^n I_0 + \left(1 - \left(1 - h\frac{R}{L}\right)^n\right)\frac{U}{R},$$

where I_n is the approximation at time $t_n = nh$.

(3 %) (g) Can you find a guideline to choose h such that the approximation I_n at least qualitatively represent the behaviour of the exact solution $I(t_n)$?

Problem 3 (14 %)

The thickness of a homogenous right triangular plate of mass m varies linearly with the perpendicular distance from the vertex at $z = h_0$ toward the base (Fig. 3). The thickness at the base of the plate is l_0, the length of the side in x-direction is b_0 and in z-direction h_0. The mass m of the triangular plate is

$$m = \frac{1}{3}\rho l_0 b_0 h_0,$$

where ρ is the density of the plate. The center of mass in x-direction is at $x_c = {}^3/_8 b_0$ and in y-direction at $y_c = {}^3/_8 l_0$.

(2 %) (a) Find a dependency between the thickness l of the triangular plate and the coordinate axes.

(2 %) (b) Find an equation that describes how the length $b(z)$ varies in z-direction.

(6 %) (c) Find the position of the center of mass z_c along the z-axis of the plate (calculation required).

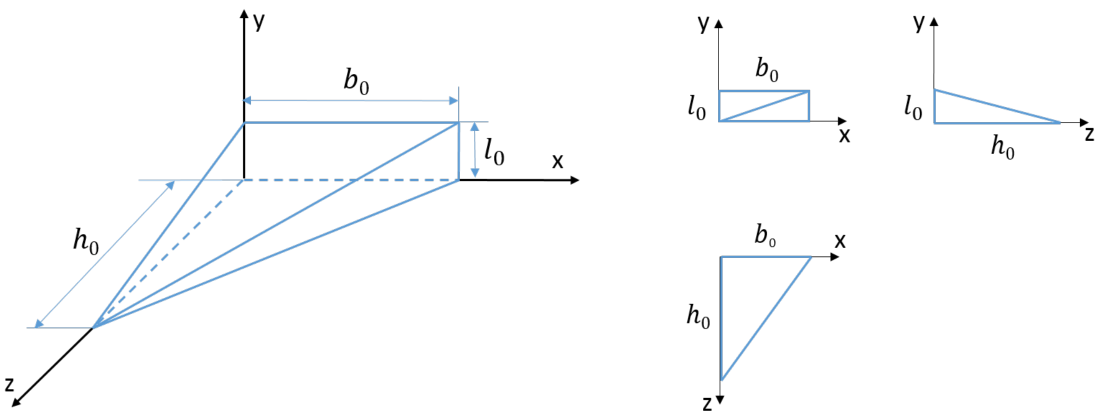

Figure 3: 3D drawing of triangular plate and projections from each side.

(4 %) (d) The moment of inertia I_x with respect to the x-axis in the given coordinate system with the origin in the vertex of the triangular plate is

$$I_{x,0} = \int z^2 dm = \frac{1}{10} m h_0^2.$$

What is the moment of inertia $I_{x,c}$ about the x-axis with respect to the center of mass? Assume that $l_0 << b_0$ and $l_0 << h_0$.

Problem 4 (24 %)

A particle with mass m rolls frictionless down the inner surface of a circular cone (Fig. 4). The cones angle is denoted by α.

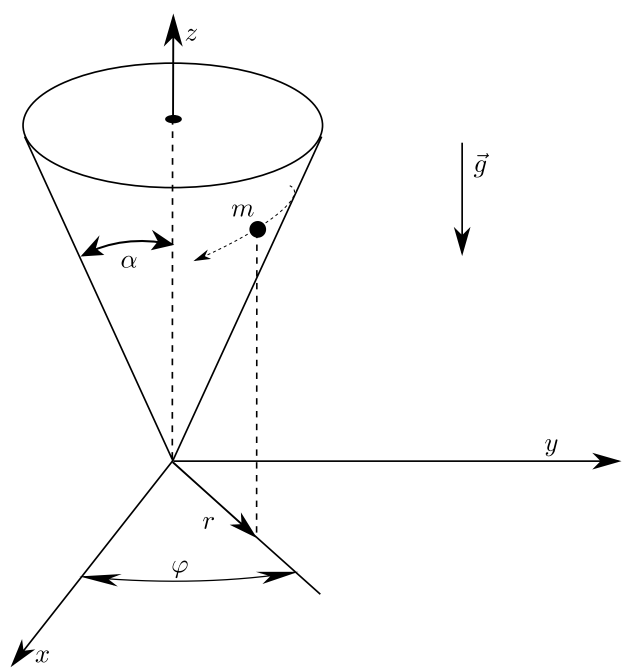

Figure 4: Circular cone with mass particle.

(4 %) (a) The system is best described using the cylindrical coordinates. Write down expressions for Cartesian coordinates (x, y, z) as function of the cylindrical coordinates. How many degrees of freedom does the system have?

(10 %) (b) Derive the equations of motion of the mass particle m in the coordinates r and φ with the Lagrange approach.

(3 %) (c) Find the sum of forces acting on the mass particle in the cylindrical coordinate system.

(7 %) (d) Derive the equations of motion of the mass particle m in the coordinates r and φ with the Newton-Euler approach.
The following you may find useful

$$\vec{\omega}_{ib} \times \vec{e}_z = \vec{0}, \quad \vec{\omega}_{ib} \times \vec{e}_\varphi = -\dot{\varphi}\vec{e}_r,$$

where $\vec{\omega}_{ib}, \vec{e}_r, \vec{e}_\varphi$ and $\vec{e}_z$ are the angular velocity and the unit vectors of the coordinate axes, respectively.
Hint: It is not necessary, but it is recommended to operate in the cylindrical coordinate system.

Problem 5 (20 %)

The cascade if three tanks is used to mix liquids (Fig. 6). The cross-sectional areas A_1, A_2, and A_3 of the tanks, the resistance of the valves R_1, R_2, the density ρ are parameters and the inflow ω_0 and its derivatives $\dot{\omega}_0, \ddot{\omega}_0, \ldots$ are known inputs. The mass flow between the tanks and the outflow of tank 3 is given by ω_1, ω_2 and ω_3, respectively.

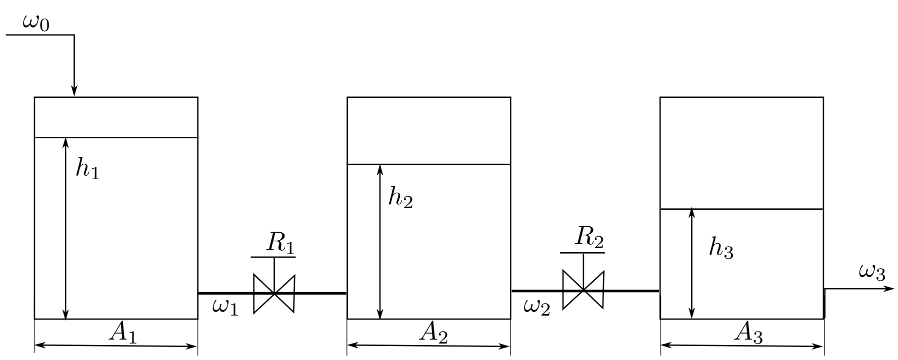

Figure 6: Mixing cascade of three tanks.

(3 %) (a) Derive ordinary differential equations for the change of the fluid height h_i for the tanks $i = 1, 2, 3$.

In addition, the following equations are applied

$$\omega_1 R_1 = f_1(h_1, h_2), \tag{34a}$$
$$\omega_2 R_2 = f_2(h_2, h_3), \tag{34b}$$
$$\omega_3 = f_3(h_3), \tag{34c}$$

where f_1, f_2 and f_3 are smooth functions.

(5 %) (b) Give the vectors of differential and algebraic variables of the system. Check if the DAE-system has a differential index of one.

In the following assume that the tank cross-sections are identical ($A_i = A$ for $i = 1, 2, 3$) and the resistance of the valves can be neglected ($R_1 = R_2 = 0$). Consequently, (34) can be replaced with (39).

$$h_1 = h_2, \tag{39a}$$

$$h_2 = h_3, \tag{39b}$$

$$\omega_3 = A\rho\sqrt{2gh_3}, \tag{39c}$$

(7 %) (c) Determine the differential index of the new DAE-system by transferring the system into an ODE-system.

(5 %) (d) Use your result of the previous sub-task and write down a suitable index-1 system. Check your result by examine if your new system is structurally regular.