TTK4115 Linear System Theory

Department of Engineering Cybernetics
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Homework assignment 4

Hand-out time:	Monday, October 2, 2017, at 8:00
Hand-in deadline:	Friday, October 13, 2017, at 16:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard, or in the boxes in D238. Please write your name on your answer sheet, should you choose to hand in physically. Any questions regarding the assignment should be directed through Blackboard.

Problem 1: Output-feedback controllers

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Moreover, consider the following P-controller:

$$u(t) = -k_p y(t),$$

where k_p is a constant.

a) Show that the closed-loop system (i.e. the system with the P-controller) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_p\mathbf{x}(t),$$

with

$$\mathbf{A}_p = \begin{bmatrix} 4 & 2 \\ -1 - 2k_p & -2 \end{bmatrix}.$$

b) For which values of k_p is the closed-loop system asymptotically stable (i.e. for which values of k_p do the eigenvalues of $\mathbf{A}_p$ have negative real parts)?

Next, we add derivative-output feedback to the controller, which results in the following PD-controller:

$$u(t) = -k_p y(t) - k_d\dot{y}(t),$$

where k_p and k_d are constants.

c) Show that the closed-loop system (i.e. the system with the PD-controller) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{pd}\mathbf{x}(t),$$

with

$$\mathbf{A}_{pd} = \begin{bmatrix} 4 & 2 \\ -1 - 2k_p - 8k_d & -2 - 4k_d \end{bmatrix}.$$

d) Determine the constants k_p and k_d such that the poles of the closed-loop system (i.e. the eigenvalues of the matrix $\mathbf{A}_{pd}$) are equal to $-1 \pm i$, with $i = \sqrt{-1}$. Show that $k_p = 2$ and $k_d = 1$.

Problem 2: Separation principle

Consider the system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{aligned}$$

the feedback controller

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$$

and the state estimator

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)).$$

We define the estimation error $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$.

a) Show that the closed-loop system (i.e. the system with feedback controller and state estimator) can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix}}_{=\mathbf{H}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}.$$

Note that for any square matrix $\mathbf{F}$, its characteristic polynomial $\det(\mathbf{F} - \lambda\mathbf{I})$ is equal to zero if and only if λ is an eigenvalue of $\mathbf{F}$. Moreover, for any block-triangular matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{0} & \mathbf{G}_{22} \end{bmatrix},$$

it holds that $\det(\mathbf{G}) = \det(\mathbf{G}_{11})\det(\mathbf{G}_{22})$.

b) Show that any eigenvalue of $\mathbf{H}$ is an eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$ (i.e. the eigenvalues of $\mathbf{H}$ are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$).

c) Assuming that the system is controllable and observable, argue that the poles of the closed-loop system (i.e. the eigenvalues of $\mathbf{H}$) can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs).

Problem 3: Controllable decompositions

Consider the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $\mathbf{y}(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} -4 & -4 & -10 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

a) Use the inverse matrix

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s+4} & \frac{-4}{(s+2)(s+4)} & \frac{-10}{(s+2)(s-3)} \\ 0 & \frac{1}{s+2} & \frac{5}{(s+2)(s-3)} \\ 0 & 0 & \frac{1}{s-3} \end{bmatrix}$$

to show that the transfer matrix of the system is given by

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} 0 \\ \frac{1}{s-3} \end{bmatrix}.$$

b) Calculate the controllability matrix and determine if the system is controllable.

We want to use an equivalence transformation to obtain a controllable canonical decomposition of the system. Consider the following transformation matrix:

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix},$$

where the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ are the first and second column of the controllability matrix, respectively. The vector $\mathbf{p}_3$ is chosen such that the matrix $\mathbf{P}$ is invertible. For simplicity, we define

$$\mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

c) Use the similarity transform $\mathbf{x}(t) = \mathbf{P}\hat{\mathbf{x}}(t)$ to transform the system to the controllable canonical decomposition

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}u(t)\\ \mathbf{y}(t) &= \hat{\mathbf{C}}\hat{\mathbf{x}}(t),\end{aligned}$$

with matrices

$$\hat{\mathbf{A}} = \left[\begin{array}{c:c} \hat{\mathbf{A}}_c & \hat{\mathbf{A}}_{12} \\ \hdashline \mathbf{0} & \hat{A}_{\bar{c}} \end{array}\right] = \left[\begin{array}{cc:c} 0 & 6 & 0 \\ 1 & 1 & 0 \\ \hdashline 0 & 0 & -4 \end{array}\right], \qquad \hat{\mathbf{B}} = \left[\begin{array}{c} \hat{\mathbf{B}}_c \\ \hdashline 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \\ \hdashline 0 \end{array}\right]$$

and

$$\hat{\mathbf{C}} = \left[\hat{\mathbf{C}}_c \; \vdots \; \hat{\mathbf{C}}_{\bar{c}}\right] = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 1 & 3 & 0 \end{array}\right],$$

where the subscripts c and $\bar{c}$ indicate the controllable and uncontrollable parts of the matrices, respectively.

d) Show that the transfer matrix $\hat{\mathbf{G}}(s)$ in a) is equal to $\hat{\mathbf{C}}_c(s\mathbf{I} - \hat{\mathbf{A}}_c)^{-1}\hat{\mathbf{B}}_c$.

Problem 4: Minimal realizations

Consider the following system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t), \end{aligned}$$

with matrices

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 5 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} -5 & 1 \\ 10 & -2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

a) Determine the eigenvalues of the system.

From the Popov-Belevitch-Hautus test for controllability, it follows that the system is controllable if and only if

$$\text{rank}\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B} \end{bmatrix} = n = 2$$

for every eigenvalue λ of $\mathbf{A}$.

b) Use the Popov-Belevitch-Hautus test for controllability to determine if the system is controllable.

From the Popov-Belevitch-Hautus test for observability, it follows that the system is observable if and only if

$$\text{rank}\begin{bmatrix} \mathbf{A} - \lambda\mathbf{I} \\ \mathbf{C} \end{bmatrix} = n = 2$$

for every eigenvalue λ of $\mathbf{A}$.

c) Use the Popov-Belevitch-Hautus test for observability to determine if the system is observable.

d) Determine from your answers in b) and c) if the system is a minimal realization.

e) Show that the transfer matrix of the system is given by

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s-1}{s+6} \\ \frac{-2s+2}{s+6} \end{bmatrix}.$$

f) Determine from the transfer matrix in e) if the system is a minimal realization.