TTK4115 Linear System Theory
Department of Engineering Cybernetics
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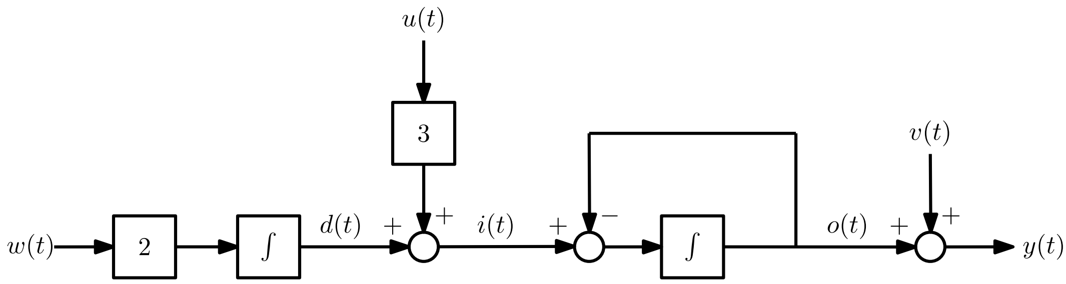
Solution to homework assignment 7

Problem 1: Stationary Kalman filter

a) A block diagram is given by



b) From the transfer function $g(s) = \frac{o(s)}{i(s)} = \frac{1}{s+1}$, it follows that

$$(s+1)o(s) = i(s).$$

By taking the inverse Laplace transform, the following dynamics are obtained:

$$\dot{o}(t) + o(t) = i(t).$$

This can be written as

$$\dot{o}(t) = -o(t) + i(t).$$

Substituting $i(t) = 3u(t) + d(t)$, we get

$$\dot{o}(t) = -o(t) + d(t) + 3u(t).$$

From $d(t) = 2\int_0^t w(\tau)d\tau$, it follows that

$$\dot{d}(t) = 2w(t).$$

Combining these two differential equations and the output equation $y(t) = o(t) + v(t)$, we obtain the following system

$$\begin{bmatrix} \dot{o}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} o(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} o(t) \\ d(t) \end{bmatrix} + v(t).$$

This can be written as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t) + Hv(t),\end{aligned}$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o(t) \\ d(t) \end{bmatrix}$ and matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad H = 1.$$

c) Because the matrix

$$\mathbf{P}(t) = \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}$$

is time invariant, we have

$$\dot{\mathbf{P}}(t) = \mathbf{0}.$$

Therefore, to show that $\mathbf{P}(t) = \mathbf{P}$ is a solution of the Riccati differential equation, we must show that

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T(HRH^T)^{-1}\mathbf{C}\mathbf{P} + \mathbf{G}Q\mathbf{G}^T = \mathbf{0}.$$

By substituting the values of the various matrices, we obtain

$$\begin{aligned}
&\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T(HRH^T)^{-1}\mathbf{C}\mathbf{P} + \mathbf{G}Q\mathbf{G}^T \\
&= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} + \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}(1\cdot 4\cdot 1)^{-1}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} 1 \begin{bmatrix} 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 4(2-\sqrt{3}) & 4(\sqrt{3}-1) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4(2-\sqrt{3}) & 0 \\ 4(\sqrt{3}-1) & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 4(\sqrt{3}-1) \\ 4 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 4(\sqrt{3}-1) & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 8(2-\sqrt{3}) - 4(\sqrt{3}-1)^2 & 4(\sqrt{3}-1) - 4(\sqrt{3}-1) \\ 4(\sqrt{3}-1) - 4(\sqrt{3}-1) & -4+4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Hence, the matrix $\mathbf{P}(t)$ is a solution of the Riccati differential equation.

d) The corresponding Kalman gain is given by

$$\begin{aligned}\mathbf{K}(t) &= \mathbf{P}(t)\mathbf{C}^T(HRH^T)^{-1} \\ &= \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 \cdot 4 \cdot 1)^{-1} \\ &= \begin{bmatrix} 4(\sqrt{3}-1) \\ 4 \end{bmatrix} \frac{1}{4} \\ &= \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}.\end{aligned}$$

e) From the definition of the estimation error $\mathbf{e}(t)$ and the differential equation of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$, it follows that

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t) - (\mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\ &= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\ &= \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}(\mathbf{C}\mathbf{x}(t) + Hv(t) - \mathbf{C}\hat{\mathbf{x}}(t))) \\ &= (\mathbf{A} - \mathbf{K}\mathbf{C})(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) + \mathbf{G}w(t) - \mathbf{K}Hv(t) \\ &= (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).\end{aligned}$$

f) The poles of the state estimator are equal to the eigenvalues of the matrix $\mathbf{A}-\mathbf{K}\mathbf{C}$, which is given by

$$\mathbf{A} - \mathbf{K}\mathbf{C} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{3}-1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A} - \mathbf{K}\mathbf{C}$ can be calculated from the characteristic polynomial of $\mathbf{A} - \mathbf{K}\mathbf{C}$, which is given by

$$\begin{aligned}\det(\mathbf{A} - \mathbf{K}\mathbf{C} - \lambda\mathbf{I}) &= \begin{vmatrix} -\sqrt{3}-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + \sqrt{3}\lambda + 1 \\ &= \left(\lambda + \frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)\left(\lambda + \frac{1}{2}\sqrt{3} - \frac{1}{2}i\right).\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_{1,2} = -\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$. Therefore, the estimator poles are given by $-\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$.

g) From the answer in e), we have that the dynamics of the estimation error dynamics are perturbed by the disturbances $w(t)$ and $v(t)$:

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).$$

We note that the last term in the right-hand side of this equation implies that the contribution of the disturbance $v(t)$ is proportional to the Kalman gain $\mathbf{K}$

(while the contribution of the disturbance $w(t)$ is not). If the covariance of the disturbance $v(t)$ increases by a factor ten, we can expect a larger contribution of $v(t)$ in the error $\mathbf{e}(t)$. To limit the effect of this increase, we should lower the values of the elements in $\mathbf{K}$.