TTK4115 Linear System Theory

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Solution to homework assignment 4

Problem 1: Output-feedback controllers

a) Substituting the output equation $y(t) = \mathbf{C}\mathbf{x}(t)$ in the equation for the controller $u(t) = -k_p y(t)$, we obtain

$$u(t) = -k_p \mathbf{C}\mathbf{x}(t).$$

Substituting this in the equation for the system dynamics yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}k_p\mathbf{C}\mathbf{x}(t) = (\mathbf{A} - \mathbf{B}k_p\mathbf{C})\mathbf{x}(t).$$

Hence, we have

$$\dot{\mathbf{x}}(t) = \mathbf{A}_p\mathbf{x}(t),$$

with

$$\mathbf{A}_p = \mathbf{A} - \mathbf{B}k_p\mathbf{C} = \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} k_p \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 - 2k_p & -2 \end{bmatrix}.$$

b) The eigenvalues of $\mathbf{A}_p$ can be calculated from the roots of the characteristic polynomial of $\mathbf{A}_p$:

$$\begin{aligned} \det(\mathbf{A}_p - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 2 \\ -1-2k_p & -2-\lambda \end{vmatrix} &= \lambda^2 - 2\lambda + 4k_p - 6 \\ &= (1 + \sqrt{7 - 4k_p} - \lambda)(1 - \sqrt{7 - 4k_p} - \lambda). \end{aligned}$$

Hence, we obtain the eigenvalues

$$\lambda_{1,2} = 1 \pm \sqrt{7 - 4k_p}.$$

From this, we conclude that there exists no value of k_p such that both eigenvalues of $\mathbf{A}_p$ have a negative real part. Therefore, there exists no value of k_p such that the closed-loop system is asymptotically stable.

c) First, we compute the time-derivative of the output:

$$\dot{y}(t) = \mathbf{C}\dot{\mathbf{x}}(t) = \mathbf{C}\mathbf{A}\mathbf{x}(t) + \mathbf{C}\mathbf{B}u(t).$$

We substitute this equation and the output equation $y(t) = \mathbf{C}\mathbf{x}(t)$ in the expression for the PD-controller:

$$u(t) = -k_p\mathbf{C}\mathbf{x}(t) - k_d\mathbf{C}\mathbf{A}\mathbf{x}(t) - k_d\mathbf{C}\mathbf{B}u(t).$$

From this, it follows that

$$(1 + k_d \mathbf{CB})u(t) = -(k_p \mathbf{C} + k_d \mathbf{CA})\mathbf{x}(t),$$

which implies that

$$u(t) = -\frac{k_p \mathbf{C} + k_d \mathbf{CA}}{1 + k_d \mathbf{CB}} \mathbf{x}(t).$$

By substituting this in the equation for the system dynamics, we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}(k_p \mathbf{C} + k_d \mathbf{CA})\mathbf{x}(t) = \left(\mathbf{A} - \mathbf{B}\frac{k_p \mathbf{C} + k_d \mathbf{CA}}{1 + k_d \mathbf{CB}} \right) \mathbf{x}(t).$$

Hence, we have

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{pd}\mathbf{x}(t),$$

with

$$\begin{aligned}
\mathbf{A}_{pd} &= \mathbf{A} - \mathbf{B}\frac{k_p \mathbf{C} + k_d \mathbf{CA}}{1 + k_d \mathbf{CB}} \\
&= \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix} - \frac{1}{1 + k_d \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \left(k_p \begin{bmatrix} 1 & 0 \end{bmatrix} + k_d \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 2k_p + 8k_d & 4k_d \end{bmatrix} \\
&= \begin{bmatrix} 4 & 2 \\ -1 - 2k_p - 8k_d & -2 - 4k_d \end{bmatrix}.
\end{aligned}$$

d) The characteristic polynomial of $\mathbf{A}_{pd}$ is given by

$$\det(\mathbf{A}_{pd} - \lambda \mathbf{I}) = \begin{vmatrix} 4 - \lambda & 2 \\ -1 - 2k_p - 8k_d & -2 - 4k_d - \lambda \end{vmatrix} = \lambda^2 + (4k_d - 2)\lambda + 4k_p - 6.$$

If the eigenvalues of $\mathbf{A}_{pd}$ are given by $\lambda_{1,2} = -1 \pm i$, the characteristic polynomial of $\mathbf{A}_{pd}$ is given by

$$\det(\mathbf{A}_{pd} - \lambda \mathbf{I}) = (-1 + i - \lambda)(-1 - i - \lambda) = \lambda^2 + 2\lambda + 2.$$

By comparing both expression for the characteristic polynomial of $\mathbf{A}_{pd}$, we obtain the equations

$$\begin{aligned} 4k_d - 2 &= 2, \\ 4k_p - 6 &= 2. \end{aligned}$$

Solving for k_p and k_d yields $k_p = 2$ and $k_d = 1$.

Problem 2: Separation principle

a) By combining the equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$, we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t).$$

From $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$, it follows that

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}(\mathbf{e}(t) + \mathbf{x}(t)) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{e}(t).$$

Taking the time derivative of $\mathbf{e}(t)$ yields

$$\dot{\mathbf{e}}(t) = \dot{\hat{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t).$$

By substituting the equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ and $\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t))$, we get

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)) - \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)).\end{aligned}$$

By combining this with $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$, we obtain

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}(\mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) - \mathbf{C}\hat{\mathbf{x}}(t) - \mathbf{D}\mathbf{u}(t)) \\ &= \mathbf{A}\mathbf{e}(t) + \mathbf{L}\mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).\end{aligned}$$

From $\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{e}(t)$ and $\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t)$, we get

$$\begin{bmatrix}\dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t) \\ \mathbf{e}(t)\end{bmatrix}.$$

b) The characteristic polynomial of the matrix $\mathbf{H}$ is given by

$$\begin{aligned}\det(\mathbf{H} - \lambda\mathbf{I}) &= \det\left(\begin{bmatrix}\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}\end{bmatrix}\right) \\ &= \det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I})\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}).\end{aligned}$$

For any eigenvalue λ of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$, we have $\det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I}) = 0$ or $\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = 0$. This implies that $\det(\mathbf{H} - \lambda\mathbf{I}) = \det(\mathbf{A} - \mathbf{B}\mathbf{K} - \lambda\mathbf{I})\det(\mathbf{A} - \mathbf{L}\mathbf{C} - \lambda\mathbf{I}) = 0$. Because $\det(\mathbf{H} - \lambda\mathbf{I})$ is zero, λ must be an eigenvalue of $\mathbf{H}$. Hence, any eigenvalue of the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}$ or the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$ is an eigenvalue of the matrix $\mathbf{H}$. Moreover, because $\det(\mathbf{H} - \lambda\mathbf{I})$ is only zero if λ is an eigenvalue of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$, we have that all eigenvalues of $\mathbf{H}$ are eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ or $\mathbf{A} - \mathbf{L}\mathbf{C}$. Hence, the eigenvalues of $\mathbf{H}$ are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$.

c) If the system is controllable, then the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ can be assigned arbitrarily by choosing $\mathbf{K}$. Moreover, if the system is observable, then the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ can be assigned arbitrarily by choosing $\mathbf{L}$. Because the poles of the closed-loop system (i.e. the eigenvalues of $\mathbf{H}$) are the union of the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ and $\mathbf{A} - \mathbf{L}\mathbf{C}$, we conclude that the poles of the closed-loop system can be assigned arbitrarily.

Problem 3: Controllable decompositions

a) The transfer matrix of the system is given by

$$
\begin{aligned}
\hat{\mathbf{G}}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\
&= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+4} & \frac{-4}{(s+2)(s+4)} & \frac{-10}{(s+2)(s-3)} \\ 0 & \frac{1}{s+2} & \frac{5}{(s+2)(s-3)} \\ 0 & 0 & \frac{1}{s-3} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{s+4} & \frac{-4}{(s+2)(s+4)} + \frac{2}{s+2} & 0 \\ 0 & 0 & \frac{-1}{s-3} \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{s+4} + \frac{8}{(s+2)(s+4)} - \frac{4}{s+2} \\ \frac{1}{s-3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4s+8}{(s+2)(s+4)} + \frac{8}{(s+2)(s+4)} - \frac{4s+16}{(s+2)(s+4)} \\ \frac{1}{s-3} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{1}{s-3} \end{bmatrix}.
\end{aligned}
$$

b) The controllability matrix is given by

$$
\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 4 & 2 & 26 \\ -2 & -1 & -13 \\ -1 & -3 & -9 \end{bmatrix}.
$$

Because the controllability matrix has two linearly independent columns (the third column is equal to six times the first column plus the second column), the column rank of the controllability matrix is two. Because the controllability matrix does not have full row rank, i.e. $\text{rank}(\mathcal{C}) = 2 < n$, we conclude that the system is <u>not</u> controllable.

c) The transformation matrix is given by

$$
\mathbf{P} = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{bmatrix}.
$$

From the similarity transformation $\mathbf{x}(t) = \mathbf{P}\hat{\mathbf{x}}(t)$, it follows that the controllable canonical decomposition of the system is given by

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}}(t) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t) \\
\mathbf{y}(t) &= \hat{\mathbf{C}}\hat{\mathbf{x}}(t),
\end{aligned}
$$

with

$$
\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \frac{1}{5}\begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & -2 \\ 5 & 10 & 0 \end{bmatrix} \begin{bmatrix} -4 & -4 & -10 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix},
$$

$$\hat{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \frac{1}{5}\begin{bmatrix} 0 & -3 & 1 \\ 0 & 1 & -2 \\ 5 & 10 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{\mathbf{C}} = \mathbf{C}\mathbf{P} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \end{bmatrix}.$$

d) From straightforward computation, we obtain

$$\begin{aligned}
\hat{\mathbf{C}}_c(s\mathbf{I} - \hat{\mathbf{A}}_c)^{-1}\hat{\mathbf{B}}_c &= \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} s & -6 \\ -1 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \frac{s-1}{s^2-s-6} & \frac{6}{s^2-s-6} \\ \frac{1}{s^2-s-6} & \frac{s}{s^2-s-6} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{s+2}{s^2-s-6} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{s+2}{(s+2)(s-3)} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{1}{s-3} \end{bmatrix}.
\end{aligned}$$

From a), we have

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} 0 \\ \frac{1}{s-3} \end{bmatrix}.$$

Hence, it follows that

$$\hat{\mathbf{G}}(s) = \hat{\mathbf{C}}_c(s\mathbf{I} - \hat{\mathbf{A}}_c)^{-1}\hat{\mathbf{B}}_c.$$

Problem 4: Minimal realizations

a) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -5-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = \lambda^2 + \lambda - 30 = (\lambda+6)(\lambda-5).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = -6$ and $\lambda_2 = 5$.

b) For $\lambda = \lambda_1 = -6$, we have

$$\operatorname{rank}\begin{bmatrix} \mathbf{A} - \lambda_1\mathbf{I} & \mathbf{B} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 2 & 1 \\ 5 & 10 & -2 \end{bmatrix} = 2.$$

Similarly, for $\lambda = \lambda_2 = 5$, we have

$$\operatorname{rank}\begin{bmatrix} \mathbf{A} - \lambda_2\mathbf{I} & \mathbf{B} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -10 & 2 & 1 \\ 5 & -1 & -2 \end{bmatrix} = 2.$$

Because the matrix $\begin{bmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}\end{bmatrix}$ has full row rank for every eigenvalue λ of $\mathbf{A}$ (i.e. rank $\begin{bmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}\end{bmatrix} = n = 2$ for every eigenvalue λ of $\mathbf{A}$), we conclude that the system is controllable.

c) For $\lambda = \lambda_1 = -6$, we have

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda_1\mathbf{I}\\ \mathbf{C}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}1 & 2\\ 5 & 10\\ -5 & 1\\ 10 & -2\end{bmatrix} = 2.$$

For $\lambda = \lambda_2 = 5$, we have

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda_2\mathbf{I}\\ \mathbf{C}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}-10 & 2\\ 5 & -1\\ -5 & 1\\ 10 & -2\end{bmatrix} = 1.$$

Because the matrix $\begin{bmatrix}\mathbf{A}-\lambda\mathbf{I}\\ \mathbf{C}\end{bmatrix}$ does not have full row rank for every eigenvalue λ of $\mathbf{A}$ (i.e. rank $\begin{bmatrix}\mathbf{A}-\lambda\mathbf{I}\\ \mathbf{C}\end{bmatrix}$ is not $n = 2$ for every eigenvalue λ of $\mathbf{A}$), we conclude that the system is <u>not</u> observable.

d) The system is a minimal realization if and only if it is controllable and observable. From b) and c), we have that the system is controllable, but not observable. Therefore, the system is <u>not</u> a minimal realization.

e) The transfer matrix of the system is given by

$$\begin{aligned}
\hat{\mathbf{G}}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\\
&= \begin{bmatrix}-5 & 1\\ 10 & -2\end{bmatrix}\begin{bmatrix}s+5 & -2\\ -5 & s-4\end{bmatrix}^{-1}\begin{bmatrix}1\\ -2\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}\\
&= \begin{bmatrix}-5 & 1\\ 10 & -2\end{bmatrix}\begin{bmatrix}\frac{s-4}{s^2+s-30} & \frac{2}{s^2+s-30}\\ \frac{5}{s^2+s-30} & \frac{s+5}{s^2+s-30}\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}\\
&= \begin{bmatrix}\frac{-5s+25}{s^2+s-30} & \frac{s-5}{s^2+s-30}\\ \frac{10s-50}{s^2+s-30} & \frac{-2s+10}{s^2+s-30}\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}\\
&= \begin{bmatrix}\frac{-5(s-5)}{(s-5)(s+6)} & \frac{s-5}{(s-5)(s+6)}\\ \frac{10(s-5)}{(s-5)(s+6)} & \frac{-2(s-5)}{(s-5)(s+6)}\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}\\
&= \begin{bmatrix}\frac{-5}{s+6} & \frac{1}{s+6}\\ \frac{10}{s+6} & \frac{-2}{s+6}\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}\\
&= \begin{bmatrix}\frac{-7}{s+6}\\ \frac{14}{s+6}\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}\\
&= \begin{bmatrix}\frac{s-1}{s+6}\\ \frac{-2s+2}{s+6}\end{bmatrix}.
\end{aligned}$$

f) The least common denominator of the transfer function $\hat{\mathbf{G}}(s)$ is $d(s) = s + 6$. Therefore, the degree of $\hat{\mathbf{G}}(s)$ is one (i.e. $\deg(\hat{\mathbf{G}}(s)) = 1$). The system is a minimal realization if and only if $\dim(\mathbf{A}) = \deg(\hat{\mathbf{G}}(s))$. Because $\mathbf{A}$ is a matrix of size 2×2, we have $\dim(\mathbf{A}) = 2$. Hence, because $\dim(\mathbf{A}) \neq \deg(\hat{\mathbf{G}}(s))$, we conclude that the system is <u>not</u> a minimal realization.