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Exercise 8

TTK4130 Modeling and Simulation

Problem 1 (Implementation of friction models)

In this problem we will implement different *static* and *dynamic* friction models. To test the developed models, we will use a simple *testbench* as illustrated in Figure 1. The testbench is based on a model based on Newton's law of a box of mass $m = 1$kg sliding on a table due to application of an external force F_a (assumed to be a ramp function). A friction force F_f opposes motion. We will use the parameters

$F_c = 1$	Coulomb friction
$F_s = 1.2$	Stiction (static friction)
$F_v = 0.05$	Viscous friction
$v_s = 0.1$	Characteristic Stribeck velocity

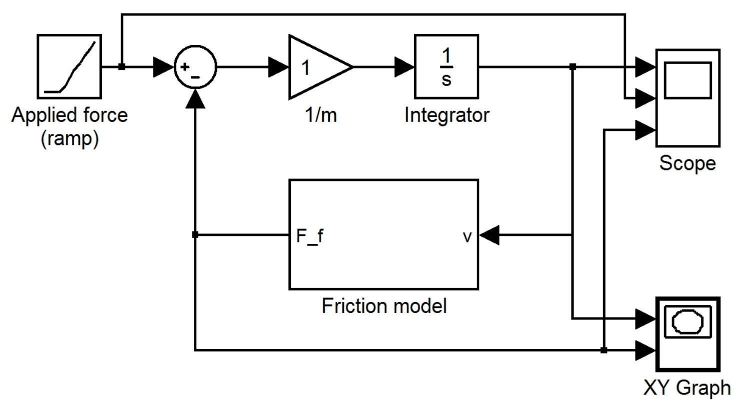


Figure 1: Testbench for friction models

(a) We start with static friction models, more specifically *Coulomb's model*, given by

$$F_f = F_c \operatorname{sign}(v), \quad v \neq 0 \tag{1}$$

where v is velocity. Implement this model in Simulink using a sign-block (do not use the built-in Coulomb friction block). Simulate the model over 10s, with ramp slope of 0.5 and 2.0. Use first a variable step solver. Explain what happens (hint: does the sign block include zero-crossing detection?). Choose the fixed-step Euler solver with sample time 0.01 instead – how does the model simulate now, for both values of the ramp?

Solution: The implementation is shown in Figure 2.

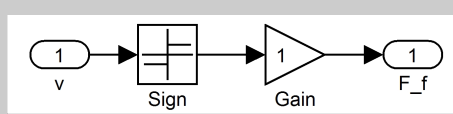


Figure 2: Implementation of Coulomb's friction model in Simulink

The variable-step solver detects an event (a zero-crossing in the sign-function, v goes from $v = 0$ to $v > 0$) already on the first step, and tries to locate the time of the zero-crossing. However, no matter how small step is chosen, the zero-crossing happens during the first step. Due to settings in the solver, it finally chooses a very small step-size, and tries to continue. Since initially F_a is small, $F_s > F_a$ and $\dot{v} = F_a - F_s < 0$ for $v > 0$, and $\dot{v} = F_a + F_s > 0$ for $v < 0$, and v will (unphysically) oscillate between a negative and positive value, generating a lot of zero-crossings (one per step). Due to the small steps taken, the maximum number of zero-crossings is quickly reached.

For a fixed-step solver you might get similar oscillations (depending on the algorithm and the time-step chosen). For the settings in this problem, we see that we have oscillations as long as $F_a < F_s$, and they disappear as $F_a > F_s$. That is, the oscillations last longer for small slopes of the ramp.

The above (Coulomb) model has the disadvantage that it is not defined at $v = 0$. The Karnopp friction model (for Coulomb friction) remedies this by defining

$$F_f = \begin{cases} \text{sat}(F_a, F_c), & v = 0 \\ F_c \,\text{sign}(v), & v \neq 0 \end{cases}$$

where the sat-function is as defined in the book (the Simulink saturation function can be used for this saturation function by "hard-coding" the upper and lower limit at the second argument (F_c)). For this model to work properly, we must either use variable-step methods with event-detection to detect exactly when $v = 0$, or we have to use some kind of dead-zone around zero-velocity to treat the velocity as zero when it is small. Here, we will implement both of these approaches.

(b) Implement Karnopp's friction model in the setup in Figure 3, that is, fill in the two If Action Subsystems. Note that the if-block (by default) generate events when the if-clause changes. The Merge-block can be found under Signal Routing, and the If Action Subsystems under Ports & Subsystems.

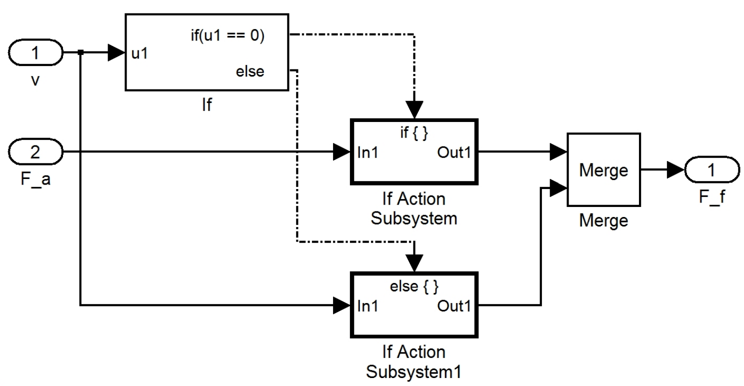


Figure 3: Setup for implementing Karnopp's friction model with event detection

Simulate using a variable step method (for example ode45) and comment. How does event-detection help? Note: It might be necessary to turn off event-detection for the sign-function inside the If Action Subsystem (why?). On the other hand, event-detection for the saturation function inside the other If Action Subsystem block should be turned on (why?).

Solution: The contents of the textsfIf Action Subsystems is seen in Figure 4. Note that F_c must be hardcoded in the saturation function. (Alternatively, the sat-function in the book could be implemented.)

(a) Subsystem 1 (b) Subsystem 2

Figure 4: The If Action Subsystems

We now see that $v = 0$ until F_a becomes larger than F_c, which is what we expect. The event-detection avoids executing the signum-function unless $v \neq 0$.

(c) Implement Karnopp's model without relying on event-detection, by using dead-zone:

$$F_f = \begin{cases} \mathrm{sat}(F_a, F_c), & |v| \leq \delta \\ F_c\,\mathrm{sign}(v), & |v| > \delta \end{cases}$$

Choose $\delta = 0.5$. Simulate using a fixed-step solver, and comment. Test also using an initial velocity of $v(0) = -1$.

Solution: This can be implemented in several ways, for instance using if/else blocks or Relay-blocks. In Figure 5 we have reused the framework from the previous problem.

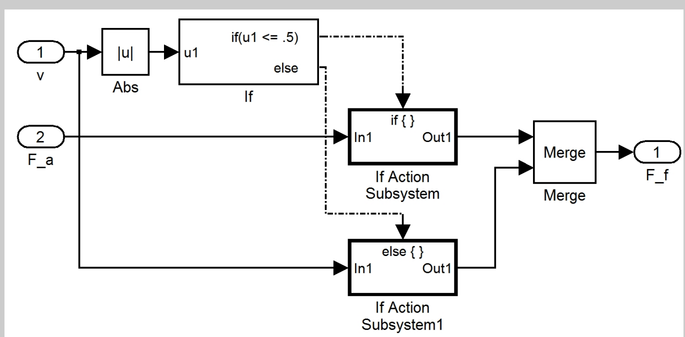


Figure 5: Setup for implementing Karnopp's friction model with deadzone.

If we use initial velocity $v(0) = 0$ we get identical results as above. However, with initial velocity $v(0) = -1$ we see that the velocity remains $v = -\delta$ when inside the deadzone (where we ideally want $v = 0$). If this is not acceptable behavior, we should choose δ smaller, but if we choose it too small, we will get back the oscillations/non-physical behavior from (a).
In other words, if we can use a solver with event-detection, that is preferable.

(d) Extend the model in (b) with sticking, Stribeck-effect (see 5.2.6 in book) and linear viscous friction,

$$F_f = \begin{cases} \mathrm{sat}(F_a, F_s), & v = 0 \\ \left[F_c + (F_s - F_c)e^{-(v/v_s)^2}\right]\mathrm{sign}(v) + F_v v, & v \neq 0 \end{cases}$$

(note slight error in book). Compare with Coulomb friction. Use ramp slope 0.5.

Solution: The second If Action Subsystem is shown in Figure 6. In the first If Action Subsystem, the limit on the saturation function is changed to F_s (and note that zero-crossing detection should be enabled for this saturation function).

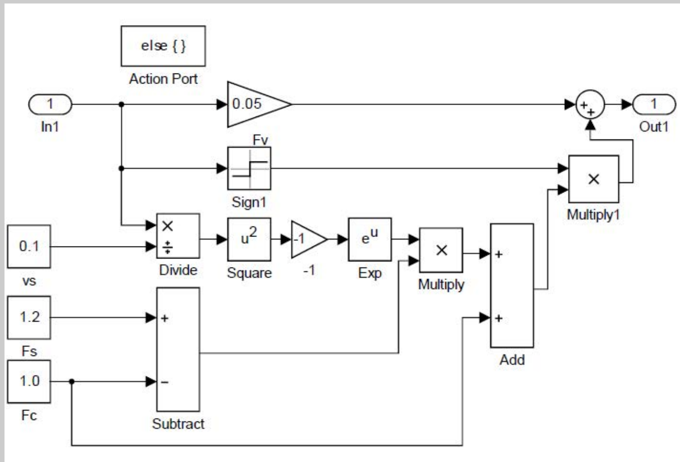
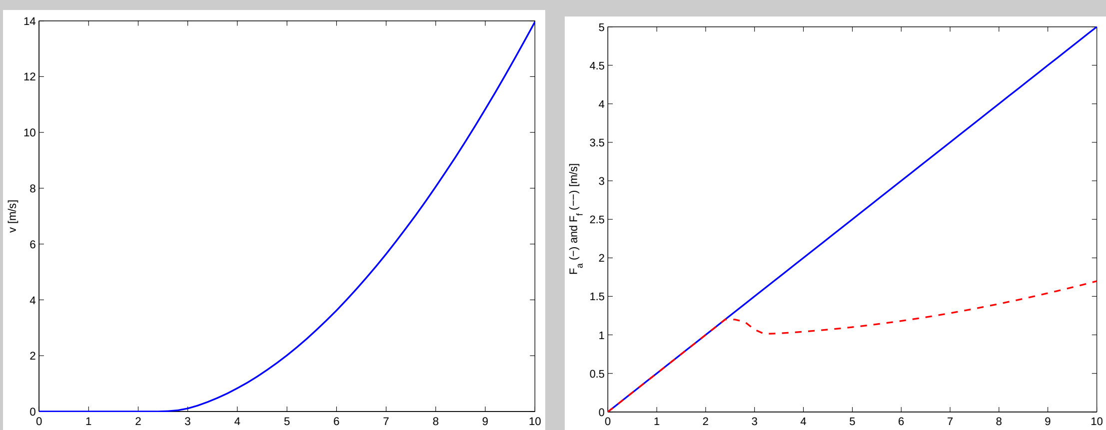


Figure 6: The If Action Subsystem for Stribeck friction.

A plot of velocity and forces is shown in Figure 7. We see that in the sticking region, the friction force is equal to the applied force (due to Karnopp's model), while in the sliding region we have an initial larger sticking force, which then reduces before it increases again due to viscous friction for higher velocities.



(a) Velocity

(b) F_a (blue, whole) and F_f (red, dashed).

Figure 7: Velocity and forces, Karnopp model of Stribeck friction

(e) Finally, we shall implement the LuGre dynamic friction model,

$$\dot{z} = (v - \sigma_0 \frac{|v|}{g(v)} z),$$
$$F_f = \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v,$$

where z may represent a small displacement in the stick-zone and σ_0 "spring-stiffness" of asperities. Set $\sigma_0 = 500$, $\sigma_1 = 0$ and $\sigma_2 = F_v$, and and implement $g(v)$ to get the same steady-state solution as in (d). Simulate and compare with solution in (d). Play around with the parameters σ_0 and σ_1, and comment.

Solution: The implementation of the dynamic LuGre friction model is shown in Figure 8. Now, we do not have to worry about discontinuities and events, and can use both variable-step and fixed-step solvers. Note, however, that the model can become stiff if too large σ_0 and σ_1 are used, so that we might have to use small step lengths in fixed-step solvers (and variable-step solvers may use long time to simulate).

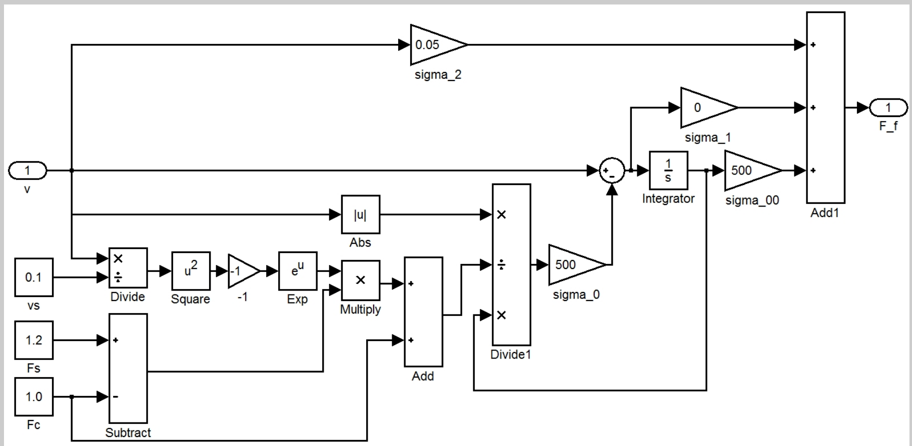
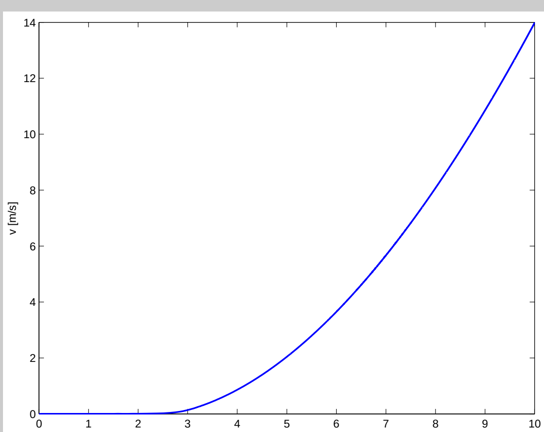
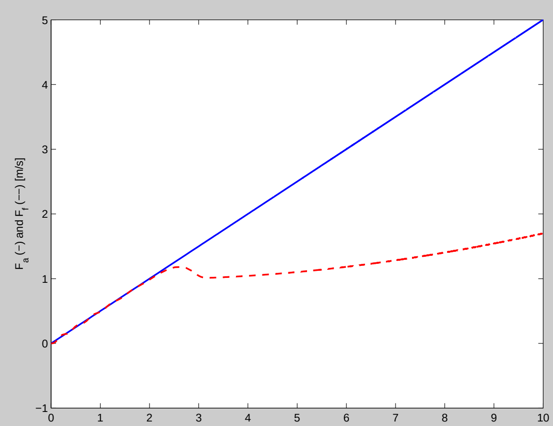


Figure 8: Simulink implementation of the LuGre friction model.

As can be seen in Figure 9, the results are similar to those obtained above (Figure 7). However, if we look closely, we see that we have a small oscillation in the friction force in the sticking region, which will lead to small oscillations in the velocity, which again might lead to (unphysical) drift (integral of velocity) in the sticking region. The oscillations can be reduced by using a smaller σ_0, but this increases the “time constant” and may therefore not be a good solution. Increasing σ_1 is another way to reduce oscillations in the sticking region, but may introduce significant stiffness in the model.



(a) Velocity



(b) F_a (blue, whole) and F_f (red, dashed).

Figure 9: Velocity and forces, LuGre friction model of Stribeck friction

Problem 2 (Transmission line)

A lossless transmission line is described by

$$\frac{\partial u}{\partial t} = -cz_0 \frac{\partial y}{\partial x} \tag{2a}$$

$$\frac{\partial y}{\partial t} = -\frac{c}{z_0} \frac{\partial u}{\partial x} \tag{2b}$$

where c is wave velocity (speed of sound) and z_0 is line (characteristic) impedance. The wave variables are

$$a = u + z_0 y,$$
$$b = u - z_0 y.$$

(a) We first assume that the equations describe a lossless hydraulic transmission line filled with water. What physical variables are u and y in this case?
The transmission line is a pipe with circular cross section with radius $R = 10\,\mathrm{cm}$. Water has speed of sound

$$c = \sqrt{\frac{\beta}{\rho}} = 1500\,\mathrm{m/s} \tag{3}$$

and density $\rho = 980\,\mathrm{kg/m^3}$. What is the bulk modulus of water, and what is (characteristic) line impedance for the transmission line? What are the units?

Solution: The variables u and y are

$$u = p \quad \text{(pressure)}$$
$$y = q \quad \text{(volume flow)}$$

The equation for the speed of sound gives

$$c = \sqrt{\frac{\beta}{\rho}} \Rightarrow \beta = c^2\rho = (1500\,\mathrm{m/s})^2\, 980\,\mathrm{kg/m^3} = 2.205 \cdot 10^9\,\mathrm{Pa} = 22050\,\mathrm{bar},$$

while the impedance z_0 for the transmission line is (see (4.146) in book)

$$z_0 = \frac{\rho c}{A} = \frac{980\,\mathrm{kg/m^3} 1500\,\mathrm{m/s}}{\pi\,(0.1\,\mathrm{m})^2} = 4.68 \cdot 10^7\,\mathrm{kg/m^4/s} = 468\,\mathrm{bar/m^3/s}.$$

(b) We now assume the equations describe an electrical transmission line in the form of a high voltage line in air. The wave velocity is the speed of light $c = 3 \cdot 10^8\,\mathrm{m/s}$ and the characteristic impedance is $z_0 = 300\,\Omega$. What are the line inductance L and capacitance C per unit length? What physical variables are u and y in this case?

Solution: We have that

$$u = v \quad \text{(voltage)},$$
$$y = i \quad \text{(electric current)}.$$

Moreover,

$$z_0 = \sqrt{\frac{L}{C}} \Rightarrow L = z_0^2 C$$
$$c = \frac{1}{\sqrt{LC}} \Rightarrow c^2 = \frac{1}{LC}$$

Combining these,

$$c^2 = \frac{1}{z_0^2 C^2} \Rightarrow C = \sqrt{\frac{1}{z_0^2 c^2}} = \frac{1}{z_0 c} = \frac{1}{300\,\mathrm{V/A}\,(3\cdot 10^8\,\mathrm{m/s})} = 1.1\cdot 10^{-11} = 11\,\mathrm{pF/m}$$

$$L = z_0^2\sqrt{\frac{1}{z_0^2 c^2}} = \sqrt{\frac{z_0^2}{c^2}} = \frac{z_0}{c} = \frac{300\,\mathrm{V/A}}{3\cdot 10^8\,\mathrm{m/s}} = 1.0\cdot 10^{-6}\,\mathrm{H/m}$$

Problem 3 (Passivity of transmission line)

A lossless transmission line is described by (2). We use index 1 to denote input-side and index 2 to denote output-side. That is, u_1 is input and y_2 is the output/measurement. On the output side we have a load impedance $Z_2(s)$ such that

$$U_2(s) = Z_2(s)Y_2(s). \tag{4}$$

The corresponding wave variables fulfil

$$B_2(s) = H_2(s)A_2(s) \tag{5}$$

where

$$H_2(s) = \frac{G_2(s)-1}{G_2(s)+1} \tag{6}$$

and

$$G_2(s) := \frac{Z_2(s)}{z_0}. \tag{7}$$

See e.g. Chapter 1.6.8 in book.

A linear system with input u and output y is passive if the transfer function $Y(s)/U(s) = G(s)$ is positive real.

(a) Show that if $G_2(s)$ is positive real, then $H_2(s)$ is bounded real (hint: see book Chapter 2.4.11).

Solution:

1. $H_2(s)$ is analytic since $G_2(s)$ is analytic, and $G_2(s)+1$ (the denominator of $H_2(s)$) cannot be zero in $\mathrm{Re}\,[s] > 0$ since $G_2(s)$ is positive real ($\mathrm{Re}\,[G(s)] \geq 0$ in $\mathrm{Re}\,[s] > 0$).

2. We have that

$$\begin{aligned}|H_2(s)|^2 &= \frac{G_2(s)-1}{G_2(s)+1}\cdot\frac{G_2^*(s)-1}{G_2^*(s)+1}\\ &= \frac{|G_2(s)|^2 - [G_2(s)+G_2^*(s)] + 1}{|G_2(s)|^2 + [G_2(s)+G_2^*(s)] + 1}\end{aligned}$$

Since $2\,\mathrm{Re}\,[G_2(s)] = G_2(s) + G_2^*(s)$, we get

$$\begin{aligned}|H_2(s)|^2 &= \frac{|G_2(s)|^2 - 2\,\mathrm{Re}\,[G_2(s)] + 1}{|G_2(s)|^2 + 2\,\mathrm{Re}\,[G_2(s)] + 1}\\ &= 1 - \frac{4\,\mathrm{Re}\,[G_2(s)]}{|G_2(s)+1|^2}\end{aligned}$$

which implies that

$$|H_2(s)| \leq 1 \quad \text{for all} \quad \mathrm{Re}\,[s] > 0,$$

since $G_2(s)$ is positive real,

$$\mathrm{Re}\,[G_2(s)] \geq 0 \quad \text{for all} \quad \mathrm{Re}\,[s] > 0.$$

That is, $H_2(s)$ is bounded real.

(b) Define $H_1(s)$ by

$$B_1(s) = H_1(s)A_1(s). \tag{8}$$

Show that if $H_2(s)$ is bounded real, then $H_1(s)$ is also bounded real. Use that

$$H_1(s) = \frac{B_1(s)}{A_1(s)} = e^{-2Ts}H_2(s).$$

Solution: We see straight away H_1 is just as analytic as $H_2(s)$, as e^{-2Ts} is analytic everywhere. Moreover, for $\mathrm{Re}\,[s] > 0$ we have that

$$|H_1(s)| < |e^{-2Ts}| \cdot |H_2(s)| \leq 1,$$

implying that $H_1(s)$ is bounded real.

(c) We have the load impedance $Z_1(s)$ such that

$$Z_1(s) = \frac{Y_1(s)}{U_1(s)}. \tag{9}$$

Further, we define the transfer function

$$G_1(s) = Z_1(s)z_0. \tag{10}$$

Show that

$$G_1(s) = \frac{1 - H_1(s)}{1 + H_1(s)} \tag{11}$$

and that if $H_1(s)$ is bounded real, then $G_1(s)$ is positive real. (Hint 1: Check how Eq. 6 is found. Hint 2: $2\,\mathrm{Re}\,[G_1(s)] = G_1(s) + G_1^*(s)$).

Solution: First,

$$\begin{aligned} H_1(s) &= \frac{B_1(s)}{A_1(s)} \\ &= \frac{U_1(s) - z_0 Y_1(s)}{U_1(s) + z_0 Y_1(s)} \\ &= \frac{1 - z_0 Y_1(s)/U_1(s)}{1 + z_0 Y_1(s)/U_1(s)} \\ &= \frac{1 - G_1(s)}{1 + G_1(s)} \end{aligned}$$

In the same way an impression for Eq. 6 can be found (check if you want). Second,

$$\begin{aligned} H_1(s) &= \frac{1 - G_1(s)}{1 + G_1(s)} \\ G_1(s) &= \frac{1 - H_1(s)}{1 + H_1(s)} \end{aligned}$$

(in the last step the solution is shortened, please show all calculation steps!)
Since $H_1(s)$ is bounded real, $H_1(s)$ is analytic and $|H_1(s)| < 1$ (from (b)) for $\mathrm{Re}\,[s] > 0$, we see that $G_1(s)$ is analytic.
To show that $\mathrm{Re}\,[H(s)] \geq 0$ for all $\mathrm{Re}\,[s] > 0$, we use the hint:

$$\begin{aligned}
2\,\mathrm{Re}\,[G_1(s)] &= G_1(s) + G_1^*(s) \\
&= \frac{1 - H_1(s)}{1 + H_1(s)} + \frac{1 - H_1^*(s)}{1 + H_1^*(s)} \\
&= \frac{\left[1 + H_1^*(s)\right]\left[1 - H_1(s)\right] + \left[1 + H_1(s)\right]\left[1 - H_1^*(s)\right]}{\left[1 + H_1(s)\right]\left[1 + H_1^*(s)\right]} \\
&= 2\frac{1 - H_1^*(s)H_1(s)}{\left[1 + H_1(s)\right]\left[1 + H_1^*(s)\right]} \\
&= 2\frac{1 - |H_1(s)|^2}{|1 + H_1(s)|^2}.
\end{aligned}$$

That is, if $H_1(s)$ is bounded real, then $\mathrm{Re}\,[G(s)] \geq 0$ for all $\mathrm{Re}\,[s] > 0$.

(d) Show that if $Z_2(s)$ is positive real, then the system with input u_1 and output y_1 is passive.

Solution:

$$\begin{aligned}
&Z_2(s) \quad \text{positive real} \\
\Rightarrow &G_2(s) = \frac{Z_2(s)}{z_0} \quad \text{positive real} \\
\Rightarrow &H_2(s) \quad \text{bounded real} \\
\Rightarrow &H_1(s) \quad \text{bounded real} \\
\Rightarrow &G_1(s) \quad \text{positive real} \\
\Rightarrow &u_1 \mapsto y_1 \quad \text{is passive.}
\end{aligned}$$