TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

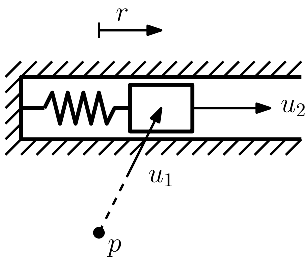
Homework assignment 2

Hand-out time:	Monday, September 4, 2017, at 8:00
Hand-in deadline:	Friday, September 15, 2017, at 16:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard, or in the boxes in D238. Please write your name on your answer sheet, should you choose to hand in physically. Any questions regarding the assignment should be directed through Blackboard.

Problem 1: Discretization

Consider the slider mechanism as shown in the following figure.



In Assignment 1 it was shown that the system could be linearized about $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{u} = \mathbf{u}_0$, with $\mathbf{x}_0 = [4, 2]^T$ and $\mathbf{u}_0 = [5, 0]^T$, by the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

The linearized system is sampled with sampling time $T = \frac{\pi}{2}$. Assuming that the input $\bar{\mathbf{u}}$ remains constant between subsequent samples, we use *exact discretization* to discretize the linearized system. Show that the discretized linearized system can be written in the following form:

$$\bar{\mathbf{x}}[k+1] = \mathbf{A}_d \bar{\mathbf{x}}[k] + \mathbf{B}_d \bar{\mathbf{u}}[k],$$
$$\bar{y}[k] = \mathbf{C}_d \bar{\mathbf{x}}[k] + \mathbf{D}_d \bar{\mathbf{u}}[k],$$

with $\bar{\mathbf{x}}[k] = \bar{\mathbf{x}}(t)$ for all $t = kT$, where k is an integer and the matrices $\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{C}_d$ and $\mathbf{D}_d$ are given by

$$\mathbf{A}_d = \begin{bmatrix} e^{-\frac{\pi}{2}} & e^{-\frac{\pi}{2}} \\ -2e^{-\frac{\pi}{2}} & -e^{-\frac{\pi}{2}} \end{bmatrix}, \quad \mathbf{B}_d = \begin{bmatrix} 0 & \frac{1}{2} - \frac{1}{2}e^{-\frac{\pi}{2}} \\ 0 & e^{-\frac{\pi}{2}} \end{bmatrix}, \quad \mathbf{C}_d = \begin{bmatrix} 4 & 0 \end{bmatrix}, \quad \mathbf{D}_d = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Problem 2: Realizations

a) Give conditions under which a transfer matrix is realizable.

Consider the following transfer matrix:

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{s^2+4s+2}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} \end{bmatrix}.$$

b) Use the conditions in a) to show that the transfer matrix $\hat{\mathbf{G}}(s)$ is realizable.

c) Show that the transfer matrix $\hat{\mathbf{G}}(s)$ can be written as $\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}_{sp}(s) + \mathbf{D}$, where $\hat{\mathbf{G}}_{sp}(s)$ is a strictly proper transfer matrix and $\mathbf{D}$ is a constant matrix.

d) Find the least common denominator

$$d(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

for the transfer functions of the transfer matrix $\hat{\mathbf{G}}(s)$, where n is the degree of the denominator and $\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$ are constants. Moreover, show that $\hat{\mathbf{G}}_{sp}(s)$ can be written in the following form:

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)} \left[\mathbf{N}_1 s^{n-1} + \mathbf{N}_2 s^{n-2} + \cdots + \mathbf{N}_{n-1}s + \mathbf{N}_n\right],$$

with matrices

$$\mathbf{N}_1 = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} -2 & -6 \\ 0 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_3 = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}.$$

e) Find a realization of $\hat{\mathbf{G}}(s)$ using the set of equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\alpha_1\mathbf{I} & -\alpha_2\mathbf{I} & \cdots & -\alpha_{n-1}\mathbf{I} & -\alpha_n\mathbf{I} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \cdots & \mathbf{N}_{n-1} & \mathbf{N}_n \end{bmatrix} \mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

where $\mathbf{I}$ is the identity matrix.

Problem 3: Similarity transforms and equivalent state-space equations

Consider the following system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t), \end{aligned} \tag{1}$$

with state $\mathbf{x}(t)$, input $u(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} -2 & 4 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \text{and} \qquad D = 2.$$

Consider the coordinate transformation

$$\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}, \tag{2}$$

with

$$\mathbf{T} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}.$$

By applying the coordinate transformation (2), the system (1) can be written in the following form:

$$\begin{aligned} \dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}u(t), \\ y(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{D}u(t). \end{aligned} \tag{3}$$

a) Show that the matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$ and $\bar{D}$ are given by

$$\bar{\mathbf{A}} = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}, \quad \bar{\mathbf{C}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \text{and} \quad \bar{D} = 2.$$

b) Are the systems (1) and (3) algebraically equivalent? Are the systems (1) and (3) zero-state equivalent? Motivate your answer.

Consider the system

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\ y(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}u(t), \end{aligned} \tag{4}$$

with

$$\tilde{A} = -1, \qquad \tilde{B} = 2, \qquad \tilde{C} = 3 \qquad \text{and} \qquad \tilde{D} = 2.$$

c) Are the systems (1) and (4) algebraically equivalent? Are the systems (1) and (4) zero-state equivalent? Motivate your answer.

Problem 4: Controllability tests

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

with state $\mathbf{x}(t)$, input $\mathbf{u}(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

a) Calculate the controllability matrix of the system and determine if the system is controllable.

b) Calculate the eigenvalues of $\mathbf{A}$.

The system is controllable if and only if

$$\operatorname{rank}\begin{bmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}\end{bmatrix} = n = 2$$

for every eigenvalue λ of $\mathbf{A}$. This is known as the Popov-Belevitch-Hautus test for controllability.

c) Use the Popov-Belevitch-Hautus test for controllability to determine if the system is controllable.

We want to use the Lyapunov test for controllability to determine if the system is controllable. The corresponding Lyapunov equation is given by

$$\mathbf{A}\mathbf{W} + \mathbf{W}\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T,$$

where $\mathbf{W}$ is a symmetric matrix. If the eigenvalues of $\mathbf{A}$ have strictly negative real parts, then the system is controllable if and only if the matrix $\mathbf{W}$ is positive definite.

d) Calculate the matrix $\mathbf{W}$ from the Lyapunov equation and determine if the system is controllable.