Solution Suggestion
Exam - TTK4115 Linear System Theory
9. December, 2017

MDP

December 11, 2017

Problem 1

The equation of motion for the system is given as a nonlinear mass-spring-damper.

$$j\ddot{\phi}(t) + d|\dot{\phi}(t)|\dot{\phi}(t) + k\phi(t) = V^2u(t) \tag{1}$$

Here, (j, d, k, V) are positive constants. It assumed that the roll rate $\dot{\phi}$ is <u>small</u>.

a) The nonlinear relation (1) linearizes as

$$j\ddot{\phi}(t) + k\phi(t) \simeq V^2u(t) \tag{2}$$

Define

$$x_1 = \phi, \quad x_2 = \dot{\phi} \tag{3}$$

Next, verify that

$$\dot{x}_1 = \dot{\phi} = x_2 \tag{4}$$

$$\dot{x}_2 = \ddot{\phi} = -(k/j)\phi + (V^2/j)u = -(k/j)x_1 + (V^2/j)u \tag{5}$$

On a matrix-vector format, this is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/j & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ V^2/j \end{bmatrix} u \tag{6}$$

In simplified notation with $a = k/j$ and $b = V^2/j$ this is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{7}$$

b) The controllability of (6) is given by

$$\mathcal{C} = [\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}] = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \tag{8}$$

This matrix has full rank, so the system is controllable. It is readily seen that the eigenvalues of the plant are imaginary.

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda & -1 \\ a & \lambda \end{vmatrix} = \lambda^2 + a = 0 \quad \Rightarrow \lambda = \pm i\sqrt{a} \tag{9}$$

This result qualifies the plant as marginally stable, or stable in the sense of Lyapunov.

c) Two measurement matrices are given by

$$\boldsymbol{c}_{\text{gyro.}} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{10a}$$

$$\boldsymbol{c}_{\text{incl.}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{10b}$$

If the plant is observable with a given measurement matrix, it is in principle possible to estimate the states giving rise to the measurement. In our case, the respective observability matrices read as

$$O_{\text{gyro.}} = \begin{bmatrix} \boldsymbol{c}_{\text{gyro.}} \\ \boldsymbol{c}_{\text{gyro.}}\boldsymbol{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} \tag{11a}$$

$$O_{\text{incl.}} = \begin{bmatrix} \boldsymbol{c}_{\text{incl.}} \\ \boldsymbol{c}_{\text{incl.}}\boldsymbol{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11b}$$

Since the observability matrices both have full rank, one may use an inclinometer or a gyroscope. Both will permit stable estimation of $\boldsymbol{x}$.

d) We are to place poles in the simplified system shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{12}$$

Letting $u = -[k_1, k_2]\boldsymbol{x}$ one finds the closed loop dynamics.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{13}$$

The eigenvalues of $\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}$ are readily found as

$$\begin{vmatrix} \lambda & -1 \\ 1 + k_1 & \lambda + k_2 \end{vmatrix} = \lambda^2 + k_2\lambda + (1 + k_1) = 0 \tag{14}$$

The desired characteristic polynomial is given by

$$(\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2 \tag{15}$$

Letting $k_1 = 1$ and $k_2 = 3$ yields the correct feedback gain $\boldsymbol{k} = [1, 3]$.

e) The Luenberger observer follows from the general equation

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}) \tag{16}$$

In our case, one finds that

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{b}u + \boldsymbol{l}(\boldsymbol{y} - \boldsymbol{c}_{\text{gyro.}}\hat{\boldsymbol{x}}) \tag{17}$$

The observer poles specicy the dynamics of the estimation error $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$). Id est

$$\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}_{\text{gyro.}})\boldsymbol{e} \tag{18}$$

Let $\boldsymbol{l} = [l_1, l_2]^{\mathsf{T}}$. Now verify the following characteristic polynomial

$$|\lambda'\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}_{\text{gyro.}})| = \begin{vmatrix} \lambda' & -1 + l_1 \\ 1 & \lambda' + l_2 \end{vmatrix} = \lambda'^2 + l_2\lambda' + (1 - l_1) = 0 \tag{19}$$

The desired polynomial reads as $(\lambda' + 2)^2 = \lambda'^2 + 4\lambda' + 4 = 0$. Comparison fixes the entries in $\boldsymbol{l}$ as $l_1 = -3$ and $l_2 = 4$.

f) Due to the separation principle, one can in theory design the state-feedback in complete independence from the observer. Granted, of course, that both $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ and $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ are stable matrices.

Problem 2

We are given $\boldsymbol{y}(s) = \boldsymbol{g}(s)u(s)$ and

$$\boldsymbol{g}(s) = \begin{bmatrix} \frac{1}{\tau s+1} \\ \frac{\tau s}{\tau s+1} \end{bmatrix}$$

A minimal realization is to be found. Inspecting the common denominator $\tau s + 1$, one state is seen to suffice. As a start, consider

$$\tau \dot{x} + x = u \quad \Rightarrow \frac{x}{u}(s) = \frac{1}{\tau s + 1} \tag{20}$$

This yields an appropriate common denominator. The transfer function $\boldsymbol{g}(s)$ is not strictly proper. The feedthrough term can be identified thus

$$\boldsymbol{d} = \lim_{s\to\infty} \boldsymbol{g}(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{21}$$

This leaves a strictly proper transfer function described by

$$\boldsymbol{g}_{\text{sp}}(s) = \boldsymbol{g}(s) - \boldsymbol{d} = \frac{1}{\tau s + 1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{22}$$

The output equation (of appropriate dimension) can now be chosen as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{23}$$

Summarizing, the (non-unique) minimal realization is described by the system matrices

$$a = -\frac{1}{\tau}, \quad b = \frac{1}{\tau}, \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{24}$$

Where we to use the output $y' = y_1 + y_2$, this would amount to the modified measurement matrix

$$c' = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{c} = 0 \tag{25}$$

This leaves a decidely unobservable system!

Problem 3

A simple dynamic system with two inputs is described by

$$\dot{x} + x = u_1 + u_2$$

It will be desirable to minimize the following cost function.

$$J = \int_0^\infty x^2 + ru_1^2 + ru_2^2 \, dt$$

where $r > 0$. We are to identify the feedback gain $\boldsymbol{k}$ in $\boldsymbol{u} = -\boldsymbol{k}x$ that minimizes J. The LQR is the clearly an appropriate tool.

First note that the system matrices follow from $a = -1$ and $\boldsymbol{b} = [1, 1]$. The weights read as $q = 1$ and $\boldsymbol{R} = r\boldsymbol{I}$. One may proceed to solve the Riccati equation for p, viz.

$$ap + pa + q - p\boldsymbol{bRb}^{\mathsf{T}}p = -2rp^2 - 2p + 1 = 0 \tag{26}$$

With the ABC-formula, it holds that

$$p = \frac{2 \pm \sqrt{2^2 + 8r}}{-2^2 r} = \frac{-1 \pm \sqrt{1 + 2r}}{2r} \tag{27}$$

The positive solution $p_+ = (\sqrt{1+2r}-1)/(2r)$ is now chosen. The desired gain-matrix reads as

$$\boldsymbol{k} = \boldsymbol{R}^{-1}\boldsymbol{b}^\mathsf{T} p = [1, 1]\left(\frac{\sqrt{1+2r}-1}{2r^2}\right) \tag{28}$$

If the second input had been assigned a higher cost through $r_1 u_1^2 + r_2 u_2^2$ with $r_2 \gg r_1 > 0$, the control system would prefer the use of u_1 relative to u_2. In practice, one would observe lower values for u_2 when compared to u_1.

Problem 4

The Lotka-Volterra model shown below is to be analyzed.

$$\dot{x}_1 = \alpha x_1 - \beta x_1 x_2 \tag{29a}$$
$$\dot{x}_2 = \delta x_1 x_2 - \gamma x_2 \tag{29b}$$

Here, $(\alpha, \beta, \gamma, \delta)$ are positive constants.

a) The nontrivial equilibrium of (29) is found be setting $\dot{x}_1 = \dot{x}_2 = 0$. This leads to the equations

$$\alpha \bar{x}_1 - \beta \bar{x}_1 \bar{x}_2 = 0 \quad \bar{x}_2 = \frac{\alpha}{\beta} \tag{30a}$$
$$\delta \bar{x}_1 \bar{x}_2 - \gamma \bar{x}_2 = 0 \quad \bar{x}_1 = \frac{\gamma}{\delta} \tag{30b}$$

The Lotka-Volterra model is now perturbed around the equilibrium. Let

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \delta x_1 x_2 - \gamma x_2 \end{bmatrix} \tag{31}$$

A first order expansion about $\boldsymbol{x} = \bar{\boldsymbol{x}}$ with $\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(t) + \bar{\boldsymbol{x}}$ is given by

$$\dot{\boldsymbol{x}} = \dot{\tilde{\boldsymbol{x}}} = \boldsymbol{f}(\boldsymbol{x}) \simeq \boldsymbol{f}(\bar{\boldsymbol{x}}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\bar{\boldsymbol{x}})\tilde{\boldsymbol{x}} \tag{32}$$

Here, $\boldsymbol{f}(\bar{\boldsymbol{x}})$ vanishes since $\bar{\boldsymbol{x}}$ is an equilibrium. One may proceed to find that

$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\bar{\boldsymbol{x}}) = \begin{bmatrix} \alpha - \beta \bar{x}_2 & -\beta \bar{x}_1 \\ \delta \bar{x}_2 & \delta \bar{x}_1 - \gamma \end{bmatrix} = \begin{bmatrix} \alpha - \beta \frac{\alpha}{\beta} & -\beta \frac{\gamma}{\delta} \\ \delta \frac{\alpha}{\beta} & \delta \frac{\gamma}{\delta} - \gamma \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix} \tag{33}$$

This leads to the following perturbed model, which upon expansion, solves the problem.

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} \tag{34}$$

b) The system matrix $\boldsymbol{A}$ has imaginary eigenvalues, as seen from its structure. The linearized predator-prey model is therefore marginally stable. With unity constants, one must solve

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{35}$$

Clearly, a solution is given by $\tilde{x}_1 = \cos(t)$ and $\tilde{x}_2 = \sin(t)$.

c) It is indeed possible to observe the animal populations described by (34). With $\boldsymbol{c}_{\text{prey}} = [1, 0]$ and $\boldsymbol{A}$ as identified above, the observability matrix becomes

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{\beta\gamma}{\delta} \end{bmatrix} \tag{36}$$

Full rank implies that the number of predators can be be estimated.

c) This task suggests the input matrix $\boldsymbol{b} = [0, 1]^{\mathsf{T}}$. Controllability is the appropriate quality to examine. With $\boldsymbol{A}$ as identified above, The controllability matrix reads as

$$\mathcal{C} = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ 1 & 0 \end{bmatrix} \tag{37}$$

With full rank, it appears possible to actively control the animal populations in the ecosystem.

Problem 5

In this task, a static location $\boldsymbol{x} = [x_1, x_2]^{\mathsf{T}}$ in the horizontal plane is to be estimated through noisy GPS measurements with a discrete time Kalman filter. It is assumed that the measurements are approximately normally distributed around the true position $\boldsymbol{x}$ with a standard deviation $\sigma_v \sim 5[m]$. A discrete-time measurement model is thus

$$\boldsymbol{y}[k] = \boldsymbol{x}[k] + \boldsymbol{v}[k], \quad \boldsymbol{v}[k] \sim \mathcal{N}\left(\boldsymbol{0}, \sigma_v^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{38}$$

a) A process model representing the fact that the GPS antenna is perfectly stationary reads as

$$\boldsymbol{x}[k+1] = \boldsymbol{x}[k] \tag{39}$$

In terms of the general model $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}u[k] + \boldsymbol{w}[k]$ with $\boldsymbol{w}[k] \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q})$, one finds that

$$\boldsymbol{A} = \boldsymbol{I}, \quad \boldsymbol{B} = \boldsymbol{0}, \quad \boldsymbol{Q} = \boldsymbol{0} \tag{40}$$

The measurement equation $\boldsymbol{y}[k] = \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{v}[k]$ is represented by

$$\boldsymbol{C} = \boldsymbol{I}, \quad \boldsymbol{R} = \sigma_v^2 \boldsymbol{I} \tag{41}$$

The Kalman filter is initialized at

$$\hat{\boldsymbol{x}}^{-}[0] = \boldsymbol{0}, \quad \boldsymbol{P}^{-}[0] = \boldsymbol{R} = \sigma_v^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using the formula sheet, this implies the following, simplified, algorithm:

Algorithm Initialize at $\hat{\boldsymbol{x}}^{-}[0] = \boldsymbol{0}$ and $\boldsymbol{P}^{-}[0] = \boldsymbol{R}$. Compute recursively:

1. $\boldsymbol{L}[k] = \boldsymbol{P}^{-}[k](\boldsymbol{P}^{-}[k] + \boldsymbol{R})^{-1}$
2. $\hat{\boldsymbol{x}}[k] = \hat{\boldsymbol{x}}^{-}[k] + \boldsymbol{L}[k](\boldsymbol{y}[k] - \hat{\boldsymbol{x}}^{-}[k])$
3. $\boldsymbol{P}[k] = (\mathbb{I} - \boldsymbol{L}[k])\boldsymbol{P}^{-}[k](\mathbb{I} - \boldsymbol{L}[k])^{\mathsf{T}} + \boldsymbol{L}[k]\boldsymbol{R}\boldsymbol{L}[k]^{\mathsf{T}}$
4. $\hat{\boldsymbol{x}}^{-}[k+1] = \hat{\boldsymbol{x}}[k], \quad \boldsymbol{P}^{-}[k+1] = \boldsymbol{P}[k]$

It must be shown that

$$\boldsymbol{L}[k] = \frac{1}{k+2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{k+2}\boldsymbol{I} \tag{42a}$$

$$\boldsymbol{P}[k] = \frac{\sigma_v^2}{k+2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{k+2}\boldsymbol{R} \tag{42b}$$

$$\boldsymbol{P}^{-}[k+1] = \frac{\sigma_v^2}{k+2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{k+2}\boldsymbol{R} \tag{42c}$$

In the very first time step $k = 0$, the Kalman gain reads as

$$\boldsymbol{L}[0] = \boldsymbol{R}(\boldsymbol{R} + \boldsymbol{R})^{-1} = \frac{1}{2+0}\boldsymbol{I} \tag{43}$$

The update covariance matrix becomes

$$\boldsymbol{P}[0] = (\mathbb{I} - \boldsymbol{L}[0])\boldsymbol{P}^{-}[0](\mathbb{I} - \boldsymbol{L}[0])^{\mathsf{T}} + \boldsymbol{L}[0]\boldsymbol{R}\boldsymbol{L}[0]^{\mathsf{T}} = \frac{1}{2+0}\boldsymbol{R} \tag{44}$$

Finally, the predicted covariance for the next step $k = 1$ becomes

$$\boldsymbol{P}^{-}[1] = \frac{1}{2+0}\boldsymbol{R} \tag{45}$$

These work out as expected.

Having demonstrated the case for $k = 0$, let us now try to insert the general items to be demonstrated into the algorithm and prove equality for $k > 0$. First, the Kalman gain:

$$\boldsymbol{L}[k] = \frac{1}{k+1}\boldsymbol{R}\left(\frac{1}{k+1}\boldsymbol{R} + \boldsymbol{R}\right)^{-1} = \frac{1}{k+1}\boldsymbol{R}\left(\frac{k+2}{k+1}\boldsymbol{R}\right)^{-1} = \frac{1}{k+2}\boldsymbol{I} \tag{46}$$

This is correct. Next, the updated covariance matrix:

$$\begin{aligned}\boldsymbol{P}[k] = \left(\mathbb{I} - \frac{1}{k+2}\boldsymbol{I}\right)\frac{1}{k+1}\boldsymbol{R}\left(\mathbb{I} - \frac{1}{k+2}\boldsymbol{I}\right)^{\mathsf{T}} + \frac{1}{(k+2)^2}\boldsymbol{R}& \\ = \frac{k+1}{(k+2)^2}\boldsymbol{R} + \frac{1}{(k+2)^2}\boldsymbol{R} = \frac{1}{k+2}\boldsymbol{R}&\end{aligned} \tag{47}$$

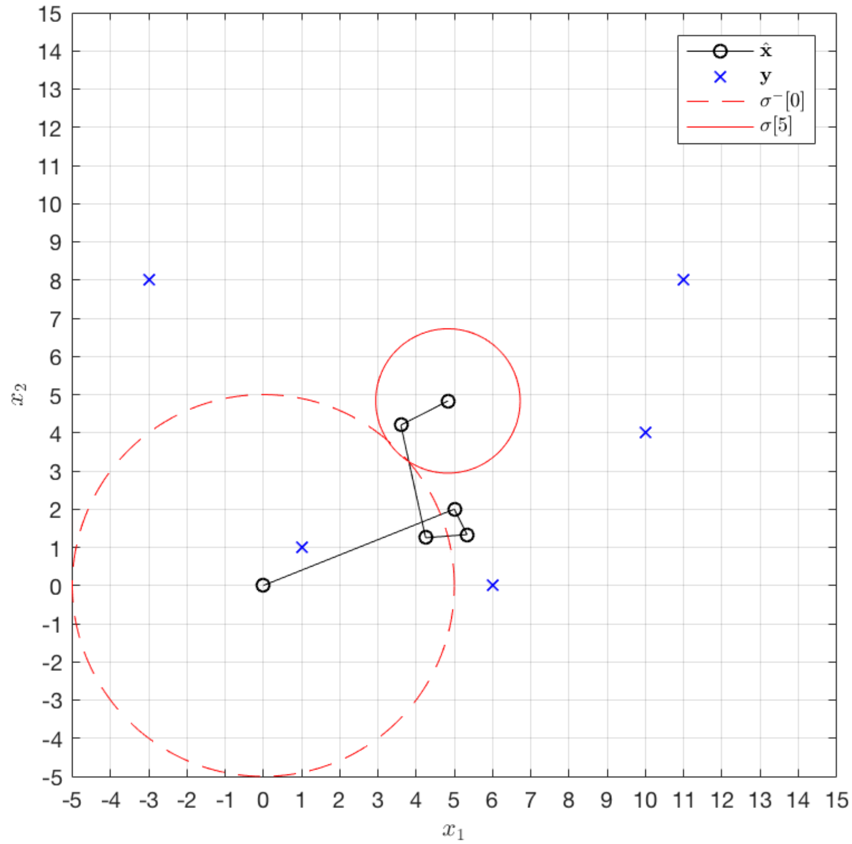
The last step is self-evident from the simplifed algorithm.

b) The standard deviation of the measurement error is found by taking the square-root of the covariance.

$$\sqrt{\boldsymbol{P}[k]} = \sqrt{\frac{\sigma_v^2}{k+2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \frac{\sigma_v}{\sqrt{k+2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{48}$$

The a-priori error covariance at $k = 0$ was $\boldsymbol{P}^{-}[0] = \sigma_v^2\boldsymbol{I}$ with standard deviation $\sqrt{\boldsymbol{P}^{-}[0]} = \sigma_v\boldsymbol{I}$. This deviation is down 50% when $k = 2$, since then $\sqrt{\boldsymbol{P}[k]} = (\sigma_v/2)\boldsymbol{I}$. At $k = 2$, <u>three</u> measurements have been absorbed.

c)


$\hat{\mathbf{x}}$
$\mathbf{y}$
$\sigma^{-}[0]$
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x_1
x_2