Norwegian University of Science and Technology
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Exercise 5
Open-Loop Optimal Control and MPC

Problem 1 (60 %) Open-Loop Optimal Control

We have the model

$$
\begin{aligned}
x_{t+1} &= \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0.1 & -0.79 & 1.78 \end{bmatrix}}_{A} x_t + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0.1 \end{bmatrix}}_{B} u_t \\
y_t &= \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} x_t
\end{aligned} \tag{1}
$$

where y_t is a measurement, and wish to use this model for control of a process. The process has been at the at the origin $x_t = 0$, $u_t = 0$ for a while, but at $t = -1$ a disturbance moved the process so that $x_0 = [0,\ 0,\ 1]^\top$. We wish to solve a finite horizon ($N < \infty$) optimal control problem with the cost (or objective) function

$$
f(y_1, \ldots, y_N, u_0, \ldots, u_{N-1}) = \sum_{t=0}^{N-1} \left\{ y_{t+1}^2 + r u_t^2 \right\}, \quad r > 0 \tag{2}
$$

Use $r = 1$ unless otherwise noted. We use $N = 30$ for the entire exercise.

a Is (1) a stable system?

b What are the dimensions of x_t and u_t? Rewrite the cost function (2) as

$$
f(z) = \frac{1}{2} \sum_{t=0}^{N-1} \left\{ x_{t+1}^\top Q x_{t+1} + u_t^\top R u_t \right\} \tag{3}
$$

where $z = [x_1^\top, \ldots, x_N^\top, u_0^\top, \ldots, u_{N-1}^\top]^\top$. What are Q and R?

c Is the minimization problem with objective function (3) and constraints (1) convex, strictly convex, or non-convex? Explain. Does convexity depend on A, B, C, Q, R, or N?

d We will now cast the optimal control problem as the equality-constrained QP

$$
\begin{aligned}
\min_{z} \quad & f(z) = \frac{1}{2} z^\top G z \\
\text{s.t.} \quad & A_{\text{eq}} z = b_{\text{eq}}
\end{aligned} \tag{4}
$$

(see equation (16.3) in the textbook) with z defined as above.

Show that the matrix A_{eq} and the vector b_{eq} can be written

$$A_{\text{eq}} = \begin{bmatrix} I & 0 & \cdots & \cdots & 0 & -B & 0 & \cdots & \cdots & 0 \\ -A & I & \ddots & & \vdots & 0 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -A & I & 0 & \cdots & \cdots & 0 & -B \end{bmatrix}, \quad b_{\text{eq}} = \begin{bmatrix} Ax_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

and give the structure of G. Set up the KKT system (equation (16.4) in the textbook) and solve it with MATLAB. Plot y_t and u_t.

We call the sequence u_0, u_1, …, u_{N-1} an optimal control sequence. However, this form of control is open loop. Why do we call this open-loop control? What are the advantages of including feedback and how can this be accomplished?

Hint: The matrices G and A can be constructed in MATLAB using the functions `eye`, `kron`, `diag`, `ones`, and `blkdiag`. One can of course use for loops instead.

e Solve the optimization problem you posed in d) using `quadprog` in MATLAB. Plot y_t and u_t and compare your results with those obtained in d). How many iterations does `quadprog` use to find the solution? Try different values of r, one less than 1 and one greater than 1. Plot y_t and u_t for these cases and comment on the differences.

f We now add the input constraint

$$-1 \leq u_t \leq 1 \quad t \in [0, N-1] \tag{6}$$

Formulate this as a constraint on z and solve with `quadprog`. Plot y_t and u_t and compare your results with those obtained above. How many iterations does `quadprog` use to find the solution? Explain the difference in the number of iterations from d).

Problem 2 (40 %) Model Predictive Control (MPC)

We still use the model (1), the objective function (3), and the input constraints (6). The initial condition on the state vector is also the same.

a Provide a short explanation of the MPC principle. Include a figure in your explanation.

b Assume that full state information is available (as opposed to just the measurement y_t) and control the system using MPC with a control horizon length of $N = 30$. Simulate the MPC-controlled system for 30 time steps, and make a plot that compares the resulting output y_t and control input u_t with the ones obtained in Problem 1.6).

Note that most of the code from Problem 1 can be used here. You need a for loop where every iteration is one discrete time instant. One iteration in the for loop involves solving a QP problem, determine the control input, and "simulate" one time step ahead.

c Now, assume that (1) is an imperfect model of the plant, and that the real plant is described by

$$
\begin{aligned}
x_{k+1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0.1 & -0.855 & 1.85 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_k \\
y_k &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x_k
\end{aligned}
\tag{7}
$$

However, this is not known to the control designer. Repeat Problem 2.2) under these conditions; that is, use system (1) in the control design and system (7) in the simulation. Make a plot that compares the resulting output y_t and control input u_t with the ones obtained in Problem 2.2) and discuss the difference between the results.