TTK4115 Linear System Theory

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Solution to homework assignment 1

Problem 1: State-space equation, transfer function and impulse response

a) Using the differential equation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} - \dot{u} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -2\dot{y} + 4u \end{bmatrix} = \begin{bmatrix} \dot{y} - u + u \\ -2(\dot{y} - u) + 2u \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -2x_2 + 2u \end{bmatrix}.$$

Therefore, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u.$$

Note that $y = x_1$. Therefore, we have

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hence, we obtain

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u, \\ y &= \mathbf{C}\mathbf{x} + Du, \end{aligned}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 \end{bmatrix}.$$

b) First, we compute $(s\mathbf{I} - \mathbf{A})^{-1}$ as follows:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+2s} \\ 0 & \frac{1}{s+2} \end{bmatrix}.$$

Using $\hat{g}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$, it follows that

$$\hat{g}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+2s} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{s} + \frac{2}{s^2 + 2s} = \frac{s+4}{s^2 + 2s}.$$

c) Applying the Laplace transform to the differential equation while assuming zero initial conditions yields

$$s^2\hat{y}(s) + 2s\hat{y}(s) = s\hat{u}(s) + 4\hat{u}(s).$$

It follows that

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{s+4}{s^2+2s},$$

which is the same result as obtained in the previous question.

d) To compute the constants α_1 and α_2, note that

$$\hat{g}(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2} = \frac{\alpha_1(s+2)}{s(s+2)} + \frac{s\alpha_2}{s(s+2)} = \frac{(\alpha_1+\alpha_2)s + 2\alpha_1}{s^2+2s} = \frac{s+4}{s^2+2s}.$$

From this, we obtain the equations

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad 2\alpha_1 = 4.$$

Solving for α_1 and α_2 yields $\alpha_1 = 2$ and $\alpha_2 = -1$. Therefore, we have

$$\hat{g}(s) = \frac{2}{s} - \frac{1}{s+2}.$$

The corresponding impulse response is given by

$$g(t) = \mathcal{L}^{-1}[\hat{g}(s)] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{1}{s+2}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = 2 - e^{-2t}.$$

Problem 2: Solutions of state-space equations

a) First, $(s\mathbf{I} - \mathbf{A})^{-1}$ can be computed as

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2+3s}\begin{bmatrix} s+3 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2+3s} \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

Note that

$$\frac{1}{s^2+3s} = \frac{1}{3s} - \frac{1}{3(s+3)}.$$

Therefore, we have

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{3s} - \frac{1}{3(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

Taking the inverse Laplace transform leads to

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I}-\mathbf{A})^{-1}] = \begin{bmatrix} \mathcal{L}^{-1}\left[\frac{1}{s}\right] & \mathcal{L}^{-1}\left[\frac{1}{3s} - \frac{1}{3(s+3)}\right] \\ 0 & \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}.$$

b) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda-3 \end{vmatrix} = \lambda^2 + 3\lambda = \lambda(\lambda+3).$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -3$. The corresponding eigenvectors $\mathbf{q}_i$ can be obtained from the kernel of the matrix $(\lambda_i \mathbf{I} - \mathbf{A})$ for $i = 1, 2$:

$$\ker(\lambda_1 \mathbf{I} - \mathbf{A}) = \ker\left(\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \implies \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\ker(\lambda_2 \mathbf{I} - \mathbf{A}) = \ker\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

c) The matrices $\hat{\mathbf{A}}$ and $\mathbf{Q}$ are given by

$$\hat{\mathbf{A}} = \operatorname{diag}\{\lambda_1, \lambda_2\} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2] = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}.$$

Note that

$$\mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} = \mathbf{A}.$$

d) The matrix $e^{\mathbf{A}t}$ can be computed as follows

$$e^{\mathbf{A}t} = \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}.$$

We note that this the same result as obtained in a).

e) The output $y(t)$ is given by

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau + \mathbf{D}u(t).$$

Substitution yields

$$\begin{aligned} y(t) &= \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & \frac{1}{3} - \frac{1}{3}e^{-3(t-\tau)} \\ 0 & e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} d\tau + 5 \\ &= \begin{bmatrix} 3 & 1 - e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 3 & 0 \end{bmatrix} \int_0^t \begin{bmatrix} -2e^{-3(t-\tau)} \\ 6e^{-3(t-\tau)} \end{bmatrix} d\tau + 5 \\ &= 3x_1(0) + (1 - e^{-3t})x_2(0) + \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3}e^{-3(t-\tau)} - \frac{2}{3} \\ 2 - 2e^{-3(t-\tau)} \end{bmatrix} + 5 \\ &= 3x_1(0) + (1 - e^{-3t})x_2(0) + 2e^{-3t} + 3. \end{aligned}$$

f) From the solution of question e), we have

$$\begin{aligned} y(1) &= 3x_1(0) + (1 - e^{-3})x_2(0) + 2e^{-3} + 3, \\ y(2) &= 3x_1(0) + (1 - e^{-6})x_2(0) + 2e^{-6} + 3. \end{aligned}$$

By substituting $y(1) = y(2) = 4$, we obtain

$$3x_1(0) + (1 - e^{-3})x_2(0) = 1 - 2e^{-3},$$
$$3x_1(0) + (1 - e^{-6})x_2(0) = 1 - 2e^{-6}.$$

Solving for $x_1(0)$ and $x_2(0)$ yields $x_1(0) = -\frac{1}{3}$ and $x_2(0) = 2$, or equivalently $\mathbf{x}(0) = [-\frac{1}{3}, 2]^T$.

Problem 3: Linearization

a) Using the equation of motion, we have

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \begin{bmatrix} \dot{r} \\ \ddot{r} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ -\frac{2u_1}{\sqrt{r^2+9}}\dot{r} - 3r - 4 + \frac{r}{\sqrt{r^2+9}}u_1 + u_2 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} x_2 \\ -\frac{2x_2u_1}{\sqrt{x_1^2+9}} - 3x_1 - 4 + \frac{x_1u_1}{\sqrt{x_1^2+9}} + u_2 \end{bmatrix}}_{\mathbf{h}(\mathbf{x},\mathbf{u})}.$$

From $y = 5\sqrt{r^2+9}$, it follows that

$$y = \underbrace{5\sqrt{x_1^2+9}}_{f(\mathbf{x},\mathbf{u})}.$$

b) The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are computed as follows

$$\mathbf{A} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0) = \left.\begin{bmatrix} 0 & 1 \\ \frac{2x_1x_2u_1}{(x_1^2+9)^{\frac{3}{2}}} - 3 + \frac{u_1}{\sqrt{x_1^2+9}} - \frac{x_1^2u_1}{(x_1^2+9)^{\frac{3}{2}}} & \frac{-2u_1}{\sqrt{x_1^2+9}} \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}}$$
$$= \begin{bmatrix} 0 & 1 \\ \frac{80}{125} - 3 + \frac{5}{5} - \frac{80}{125} & \frac{-10}{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix},$$

$$\mathbf{B} = \frac{\partial \mathbf{h}}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) = \left.\begin{bmatrix} 0 & 0 \\ \frac{-2x_2}{\sqrt{x_1^2+9}} + \frac{x_1}{\sqrt{x_1^2+9}} & 1 \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}} = \begin{bmatrix} 0 & 0 \\ \frac{-4}{5} + \frac{4}{5} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0) = \left.\begin{bmatrix} \frac{5x_1}{\sqrt{x_1^2+9}} & 0 \end{bmatrix}\right|_{\substack{\mathbf{x}=\mathbf{x}_0 \\ \mathbf{u}=\mathbf{u}_0}} = \begin{bmatrix} \frac{20}{5} & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix},$$

$$\mathbf{D} = \frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Problem 4: Jordan forms

a) The eigenvalues of $\mathbf{A}$ can be calculated from the characteristic polynomial of $\mathbf{A}$, which is given by

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & -9 \\ 1 & -6-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2.$$

The eigenvalues of $\mathbf{A}$ are equal to the roots the characteristic polynomial of $\mathbf{A}$. Hence, we obtain the eigenvalue $\lambda = -3$ with multiplicity 2. The corresponding eigenvectors can be obtained from the kernel of the matrix $(\mathbf{A} - \lambda\mathbf{I})$:

$$\ker(\mathbf{A} - \lambda\mathbf{I}) = \ker\left(\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\right) \implies \mathbf{q} = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

where $\mathbf{q}$ is the corresponding eigenvector. Note that $\mathbf{A}$ has only one eigenvector associated with λ.

b) Because the eigenvalue $\lambda = -3$ of $\mathbf{A}$ has multiplicity 2, and $\mathbf{A}$ has only one eigenvector associated with λ, the eigenvalues of $\mathbf{A}$ are not (all) distinct. Therefore, the system cannot be transformed into a diagonal form using a similarity transformation.

c) In order to transform the system into a Jordan form, we have to find the generalized eigenvectors of $\mathbf{A}$. The chain of generalized eigenvectors satisfies the following equalities:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0},$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the generalized eigenvectors. Note that we can choose $\mathbf{v}_1 = \mathbf{q} = [3, 1]^T$, since $\mathbf{q}$ is an eigenvector associated with λ, and therefore $(\mathbf{A}-\lambda\mathbf{I})\mathbf{q} = \mathbf{0}$. The generalized eigenvector $\mathbf{v}_2$ can be obtained from the second equality:

$$\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that the choice for $\mathbf{v}_2$ is not unique. We define

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using the similarity transformation $\mathbf{x} = \mathbf{Q}\hat{\mathbf{x}}$, the system is transformed to Jordan form:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t),$$
$$y(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t),$$

with matrices

$$\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & -9 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix},$$

$$\hat{\mathbf{B}} = \mathbf{Q}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix},$$

$$\hat{\mathbf{C}} = \mathbf{C}\mathbf{Q} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

$$\hat{\mathbf{D}} = \mathbf{D} = 2.$$