TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 7

Hand-out time: Monday, November 13, 2017, at 8:00
Hand-in deadline: Monday, November 21, 2017, at 16:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard, or in the boxes in D238. Please write your name on your answer sheet, should you choose to hand in physically. Any questions regarding the assignment should be directed through Blackboard.

Problem 1: Stationary Kalman filter

Consider a plant where the relation between the input $i(t)$ and the output $o(t)$ is given by the transfer function

$$\hat{g}(s) = \frac{\hat{o}(s)}{\hat{i}(s)} = \frac{1}{s+1}.$$

The input $i(t)$ consists of the control input $u(t)$ and an input disturbance $d(t)$ and is given by

$$i(t) = 3u(t) + d(t).$$

The disturbance $d(t)$ is the output of the following Wiener process:

$$d(t) = 2\int_0^t w(\tau)d\tau,$$

where $w(t)$ is Gaussian white noise. The output $o(t)$ of the plant is measured. The corresponding output measurement is given by

$$y(t) = o(t) + v(t),$$

where $v(t)$ is Gaussian white noise.

a) Draw a block diagram of the system.

b) Show that the system can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}w(t),$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + Hv(t),$$

with state $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o(t) \\ d(t) \end{bmatrix}$, and matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad H = 1.$$

Let the covariance of the white noise processes $w(t)$ and $v(t)$ be given by

$$E[w(t)w(\tau)] = Q\delta(t-\tau)$$

and

$$E[v(t)v(\tau)] = R\delta(t-\tau),$$

with $Q = 1$ and $R = 4$, where δ is the Dirac delta function. Moreover, assume that the processes are independent, such that

$$E[w(t)v(\tau)] = 0$$

for all t and τ. For the initial expected value $\hat{\mathbf{x}}(0) = E[\mathbf{x}(0)]$ and the initial covariance matrix $\mathbf{P}(0) = E[(\mathbf{x}(0) - \hat{\mathbf{x}}(\mathbf{0}))(\mathbf{x}(0) - \hat{\mathbf{x}}(\mathbf{0}))^T]$, the state estimate $\hat{\mathbf{x}}(t)$ that minimizes the expected value of $\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\|^2$ is given by the solution of the differential equation

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{K}(t)(y(t) - \mathbf{C}\hat{\mathbf{x}}(t)),$$

with Kalman gain

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{C}^T(HRH^T)^{-1},$$

where $\mathbf{P}(t)$ is the solution of the Riccati differential equation

$$\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T - \mathbf{P}(t)\mathbf{C}^T(HRH^T)^{-1}\mathbf{C}\mathbf{P}(t) + \mathbf{G}Q\mathbf{G}^T.$$

c) Show that the constant matrix

$$\mathbf{P}(t) = \begin{bmatrix} 4(\sqrt{3}-1) & 4 \\ 4 & 4\sqrt{3} \end{bmatrix}$$

is a solution of the Riccati differential equation.

d) Show that the corresponding Kalman gain is given by

$$\mathbf{K}(t) = \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}.$$

Because the matrix $\mathbf{P}(t)$ and the Kalman gain $\mathbf{K}(t)$ are time invariant for these values, we drop the time index and write $\mathbf{P}$ and $\mathbf{K}$ instead. We define the estimation error

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t).$$

e) Show that the following differential equation holds

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) + \mathbf{G}w(t) - \mathbf{K}Hv(t).$$

f) Determine the poles of the state estimator (i.e. the eigenvalues of the matrix $\mathbf{A} - \mathbf{K}\mathbf{C}$). Show that the poles are given by $-\frac{1}{2}\sqrt{3} \pm \frac{1}{2}i$, with $i = \sqrt{-1}$.

g) Suppose that the variance of $v(t)$ becomes ten times larger (the value of R changes from 4 to 40). Reason if the elements of $\mathbf{K}$ should be increased or decreased to minimize the expected value of $\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\|^2$.