TTK4115 Linear System Theory

Department of Engineering Cybernetics
NTNU

Homework assignment 6

Hand-out time: Monday, October 30, 2017, at 8:00
Hand-in deadline: Friday, November 10, 2017, at 16:00

The problems should be solved by hand, but feel free to use MATLAB to verify your results. Hand in the assignment through Blackboard, or in the boxes in D238. Please write your name on your answer sheet, should you choose to hand in physically. Any questions regarding the assignment should be directed through Blackboard.

Problem 1: Linear systems with white input noise

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}w(t),\\ y(t) &= \mathbf{C}\mathbf{x}(t),\end{aligned}$$

with state $\mathbf{x}(t)$, output $y(t)$ and matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The disturbance $w(t)$ is a white noise process with autocorrelation function

$$R_w(\tau) = 4\delta(\tau),$$

where $\delta(\tau)$ is the Dirac delta function. Assume zero initial conditions for the state $\mathbf{x}(t)$ (i.e. $\mathbf{x}(0) = \mathbf{0}$).

a) Calculate the mean μ_w of the disturbance $w(t)$.

b) Calculate the variance σ_w^2 of the disturbance $w(t)$.

c) Show that the power spectral density function $S_w(j\omega)$ of the disturbance $w(t)$ is given by $S_w(j\omega) = 4$.

d) Show that the transfer function $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{w}(s)}$ is given by $\hat{g}(s) = \frac{s+8}{s^2+6s+8}$.

Note that the transfer function can be written as $\hat{g}(s) = \frac{\alpha_1}{s-\lambda_1} + \frac{\alpha_2}{s-\lambda_2}$, where λ_1 and λ_2 are the poles of the system and α_1 and α_2 are constants.

e) Calculate the impulse response of the system $g(t) = \mathcal{L}^{-1}\{\hat{g}(s)\}$, where $\mathcal{L}^{-1}$ is the inverse Laplace transform. Show that $g(t) = -2e^{-4t} + 3e^{-2t}$.

Note that (for zero initial conditions) the output of the system is given by $y(t) = \int_0^t g(\tau)w(t-\tau)d\tau$.

f) Calculate the stationary mean $\bar{\mu}_y$ of the output $y(t)$ (i.e. $\bar{\mu}_y = \lim_{t\to\infty}\mu_y(t)$). Show that $\bar{\mu}_y = 0$

g) Calculate the stationary variance $\bar{\sigma}_y^2$ of the output $y(t)$ (i.e. $\bar{\sigma}_y^2 = \lim_{t\to\infty}\sigma_y^2(t)$). Show that $\bar{\sigma}_y^2 = 3$.

h) Show that the power spectral density function $S_y(j\omega)$ of the output $y(t)$ is given by $S_y(j\omega) = \frac{20}{\omega^2+4} - \frac{16}{\omega^2+16}$.

Problem 2: Kalman-filter derivation

Consider the following discrete-time system:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{w}_k, \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k + \mathbf{H}\mathbf{v}_k, \end{aligned} \tag{1}$$

with state $\mathbf{x}_k$, input $\mathbf{u}_k$ and output $\mathbf{y}_k$, where $\mathbf{w}_k$ and $\mathbf{v}_k$ are white noise disturbances that satisfy

$$E[\mathbf{w}_k] = \mathbf{0},$$

$$E[\mathbf{v}_k] = \mathbf{0},$$

$$E[\mathbf{w}_k\mathbf{w}_i^T] = \begin{cases} \mathbf{Q}, & \text{if } k = i, \\ \mathbf{0}, & \text{if } k \neq i, \end{cases}$$

$$E[\mathbf{v}_k\mathbf{v}_i^T] = \begin{cases} \mathbf{R}, & \text{if } k = i, \\ \mathbf{0}, & \text{if } k \neq i, \end{cases}$$

$$E[\mathbf{w}_k\mathbf{v}_i^T] = \mathbf{0}.$$

Although the input $\mathbf{u}_k$ is known and the output $\mathbf{y}_k$ is available via measurement, the state $\mathbf{x}_k$ is unknown due to the unknown disturbances. The measurement $\mathbf{y}_k$ contains information about the state $\mathbf{x}_k$. The state estimate that is generated without information of the measurement $\mathbf{y}_k$ is called the *a priori* state estimate and is denoted by $\hat{\mathbf{x}}_k^-$. Hence, we have $\hat{\mathbf{x}}_k^- = E[\mathbf{x}_k]$ if the measurement $\mathbf{y}_k$ is not taken into account. Without the actual measurement $\mathbf{y}_k$, the corresponding output estimate is given by $\hat{\mathbf{y}}_k^- = E[\mathbf{y}_k]$.

a) Use the output equation in (1) to determine $\hat{\mathbf{y}}_k^- = E[\mathbf{y}_k]$, where $E[\mathbf{x}_k] = \hat{\mathbf{x}}_k^-$. Show that

$$\hat{\mathbf{y}}_k^- = \mathbf{C}\hat{\mathbf{x}}_k^- + \mathbf{D}\mathbf{u}_k. \tag{2}$$

We use the measurement $\mathbf{y}_k$ to update the state estimate $\hat{\mathbf{x}}_k^-$. The *a posteriori* (or updated) state estimate is denoted $\hat{\mathbf{x}}_k$. We use the following linear blending to extract the information about the state $\mathbf{x}_k$ that is hidden in the measurement $\mathbf{y}_k$:

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k^-), \tag{3}$$

where $\mathbf{K}_k$ is a blending factor that is yet to be determined. Let the *a priori* and *a posteriori* error covariance matrices be denoted by

$$\mathbf{P}_k^- = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T]$$

and

$$\mathbf{P}_k = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T],$$

respectively.

b) Use the equations (1)-(3) to show that

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{C})\mathbf{P}_k^-(\mathbf{I} - \mathbf{K}_k\mathbf{C})^T + \mathbf{K}_k\mathbf{H}\mathbf{R}\mathbf{H}^T\mathbf{K}_k^T, \tag{4}$$

where it should be noted that the error $\mathbf{x}_k - \hat{\mathbf{x}}_k^-$ is uncorrelated with the disturbance $\mathbf{v}_k$ (i.e. $E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{v}_k^T] = \mathbf{0}$).

For later convenience, we write the expression for $\mathbf{P}_k$ in a different form.

c) Show that $\mathbf{P}_k$ in (4) can be rewritten as

$$\mathbf{P}_k = \mathbf{P}_k^- - \mathbf{K}_k\mathbf{C}\mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{C}^T\mathbf{K}_k^T + \mathbf{K}_k(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T. \tag{5}$$

To obtain an accurate state estimate $\hat{\mathbf{x}}_k$, we want to find the blending factor $\mathbf{K}_k$ that minimizes the expected value of $\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2$ (i.e. the squared magnitude of the estimation error $\mathbf{x}_k - \hat{\mathbf{x}}_k$). Here, we note that

$$E[\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2] = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T(\mathbf{x}_k - \hat{\mathbf{x}}_k)] = \mathrm{tr}(E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]) = \mathrm{tr}(\mathbf{P}_k),$$

where tr is the matrix trace. To obtain the blending factor $\mathbf{K}_k$ that minimizes $\mathrm{tr}(\mathbf{P}_k)$, we determine the derivative $\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k}$ and set it to zero. Note that for any matrices $\mathbf{M}$ and $\mathbf{N}$, such that $\mathbf{MN}$ is a square matrix, we have[1]

$$\frac{d\,\mathrm{tr}(\mathbf{MN})}{d\mathbf{M}} = \frac{d\,\mathrm{tr}(\mathbf{N}^T\mathbf{M}^T)}{d\mathbf{M}} = \mathbf{N}.$$

Moreover, for any matrices $\mathbf{M}$ and $\mathbf{S}$, we have

$$\frac{d\,\mathrm{tr}(\mathbf{M}\mathbf{S}\mathbf{M}^T)}{d\mathbf{M}} = (\mathbf{S} + \mathbf{S}^T)\mathbf{M}^T.$$

d) Use the expression for $\mathbf{P}_k$ in (5) to show that

$$\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = -2\mathbf{C}\mathbf{P}_k^- + 2(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)\mathbf{K}_k^T.$$

e) Assuming that $\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T$ is an invertible matrix, show that from $\frac{d\,\mathrm{tr}(\mathbf{P}_k)}{d\mathbf{K}_k} = \mathbf{0}$, it follows that

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{C}^T(\mathbf{C}\mathbf{P}_k^-\mathbf{C}^T + \mathbf{H}\mathbf{R}\mathbf{H}^T)^{-1}.$$

[1] Here, we use a different matrix layout than in Brown & Hwang.

The *a priori* state estimate for the next time step $\hat{\mathbf{x}}_{k+1}^-$ is the expected value of $\mathbf{x}_{k+1}$ without information of the measurement $\mathbf{y}_{k+1}$.

f) Use the state equation in (1) to determine $\hat{\mathbf{x}}_{k+1}^- = E[\mathbf{x}_{k+1}]$, where $E[\mathbf{x}_k] = \hat{\mathbf{x}}_k$. Show that

$$\hat{\mathbf{x}}_{k+1}^- = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k. \tag{6}$$

We note that

$$\mathbf{P}_{k+1}^- = E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^-)^T].$$

g) Use the equations (1) and (6) to show that

$$\mathbf{P}_{k+1}^- = \mathbf{A}\mathbf{P}_k\mathbf{A}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T,$$

where it should be noted that the error $\mathbf{x}_k - \hat{\mathbf{x}}_k$ is uncorrelated with the disturbance $\mathbf{w}_k$ (i.e. $E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{w}_k^T] = \mathbf{0}$).

Let the matrices of the system (1) be given by

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 4 \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

with covariance matrices

$$Q = 2 \qquad \text{and} \qquad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consider the initial conditions

$$\hat{\mathbf{x}}_0^- = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P}_0^- = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

the inputs

$$u_0 = 1 \qquad \text{and} \qquad u_1 = -1,$$

and the outputs

$$y_0 = 3 \qquad \text{and} \qquad y_1 = -4.$$

h) Show that the corresponding *a priori* and *a posteriori* state estimates are given by

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{x}}_1^- = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \hat{\mathbf{x}}_1 = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{4} \end{bmatrix}.$$