PLANES Documentation

Olivier DAZEL
LAUM UMR CNRS 6613
olivier.dazel@univ-lemans.fr

May 10, 2017

Contents

Chapter 1

General overview

1.1 Presentation

PLANES (Porous LAum NumErical Simulator) project is a collection of Matlab/Fortran scripts to simulate the vibroacoustics response of coupled systems including acoustic, elastic, porous materials, PML...

Chapter 2

Tables

- interface(n_1,n_2,e_1,e_2,n_{middle})
- boundary(n_1,n_2,typ(e),typ(boundary),0,n_{middle})

Chapter 3

Figures

The number of the figure is xxxyyy. xxx is associated to the the type of display and yyy corresponds to the type of field

Type of display xxx

- 1yyy: Modulus of the projection on x axis
- 2yyy: Modulus of the projection on y axis
- 3yyy: Angle of the projection on x axis
- 4yyy: Angle of the projection on y axis
- 5000: Shape of the displacement
- 10yyy: Map of the modulus for FEM
- 11yyy: Map of the angle for FEM
- 20yyy: Map of the modulus for DGM
- 21yyy: Map of the angle for DGM
- 30yyy: Map of the modulus in case of a FEM/DGM model
- 31yyy: Map of the angle in case of a FEM/DGM model

Type of fields xxx

- xxx1: v_x of the fluid or v_x^{eq} for an equivalent fluid or u_x^t for a Biot PEM.
- xxx2: v_y of the fluid or v_y^{eq} for an equivalent fluid or v_y^t for a Biot PEM
- xxx3: $|v|$ of the fluid or $|v^{eq}|$ for an equivalent fluid or u_y^t for a Biot PEM

- xxx4: v_x^s for a Biot PEM.
- xxx5: v_y^s for a Biot PEM
- xx10: p

Chapter 4

Weak forms

4.1 Finite-Elements

4.1.1 Fluid

Unknown: the pressure field p. ρ is the density

$$\forall\, q, \quad \int_\Omega \frac{\boldsymbol{\nabla} p\ \boldsymbol{\nabla} q}{\rho\,\omega^2} - \frac{p\,q}{\rho\,c^2}\,\mathrm{d}\Omega = \oint_{\partial\Omega} \frac{1}{\rho\,\omega^2}\frac{\partial p}{\partial n}\,\mathrm{d}\Gamma \tag{4.1}$$

4.1.2 Elastic solid

Unknown: the solid displacement $\mathbf{u}$.

$$\forall\, \mathbf{v}, \quad \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \omega^2 \rho_s \mathbf{u}.\mathbf{v}\,\mathrm{d}\Omega = \oint_{\partial\Omega} [\boldsymbol{\sigma}(\mathbf{u}).\mathbf{n}].\mathbf{v}\,\mathrm{d}\Gamma \tag{4.2}$$

4.2 Discontinuous Galerkin

4.2.1 Abstract weak forms

$$\left(j\omega[\mathbf{M}] + [\mathbf{B}_x]\frac{\partial}{\partial x} + [\mathbf{B}_y]\frac{\partial}{\partial y} \right)\mathbf{S} = \mathbf{0} \text{ on } \Omega. \tag{4.3}$$

$\mathbf{S}$ is the State Vector of the medium. In a conservative form:

$$\left(j\omega + [\mathbf{A}_x]\frac{\partial}{\partial x} + [\mathbf{A}_y]\frac{\partial}{\partial y} \right)\mathbf{S} = \mathbf{0} \text{ on } \Omega. \tag{4.4}$$

$$[\mathbf{A}_x] = [\mathbf{M}]^{-1}[\mathbf{B}_x], \quad [\mathbf{A}_y] = [\mathbf{M}]^{-1}[\mathbf{B}_y]. \tag{4.5}$$

Ω is partitioned in n elements Ω_e with $e = 1, ..., n$.

$$\sum_{e=1}^{n} \int_{\Omega_e} \mathbf{T}_e^T \left(j\omega[\mathbf{M}]\mathbf{S}_e + [\mathbf{B}_x]\frac{\partial \mathbf{S}_e}{\partial x} + [\mathbf{B}_y]\frac{\partial \mathbf{S}_e}{\partial y} \right) \,\mathrm{d}\Omega = 0. \tag{4.6}$$

$\mathbf{T}_e$ is the test field on each element.

$$-\sum_{e=1}^{n} \int_{\Omega_e} \left(j\omega[\mathbf{M}]^T\mathbf{T}_e + [\mathbf{B}_x]^T\frac{\partial \mathbf{T}_e}{\partial x} + [\mathbf{B}_y]^T\frac{\partial \mathbf{T}_e}{\partial y} \right)^T \mathbf{S}_e \,\mathrm{d}\Omega + \sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{G}_e]\mathbf{S}_e \,\mathrm{d}\Gamma = 0.$$

where we have introduced the matrix $[\mathbf{G}_e] = [\mathbf{B}_x]n_x + [\mathbf{B}_y]n_y$ which represents the normal fluxes across the boundary of the element Ω_e. The unit normal $\mathbf{n} = (n_x, n_y)$ on the element boundary $\partial\Omega_e$ points out of element e. A key aspect of the wave-based DGM is to use test functions $\mathbf{T}$ whose restrictions $\mathbf{T}_e$ on each elements are solutions of the adjoint problem defined on each element:

$$\left(j\omega[\mathbf{M}]^T\mathbf{T}_e + [\mathbf{B}_x]^T\frac{\partial \mathbf{T}_e}{\partial x} + [\mathbf{B}_y]^T\frac{\partial \mathbf{T}_e}{\partial y} \right) = \mathbf{0} = 0. \tag{4.7}$$

With this choice of test functions the integral over each element Ω_e vanishes and one is left with integrals on the interfaces between elements and on the boundary of the domain.

$$\sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{G}_e]\mathbf{S}_e \,\mathrm{d}\Gamma = 0. \tag{4.8}$$

$$[\mathbf{F}_e] = [\mathbf{A}_x]n_x + [\mathbf{A}_y]n_y \implies [\mathbf{G}_e] = [\mathbf{M}_e][\mathbf{F}_e] \tag{4.9}$$

$$\sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e]\mathbf{S}_e \,\mathrm{d}\Gamma = 0. \tag{4.10}$$

$$\sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{V}_e[\mathbf{F}_e]\mathbf{S}_e \,\mathrm{d}\Gamma = 0, \quad \mathbf{V}_e = [\mathbf{M}_e]\mathbf{T}_e. \tag{4.11}$$

4.2.2 Characteristics

$$[\mathbf{F}_e][\mathbf{P}_e] = [\mathbf{P}_e][\mathbf{\Lambda}_e], \quad [\mathbf{F}_e] = [\mathbf{P}_e][\mathbf{\Lambda}_e][\mathbf{Q}_e], \quad [\mathbf{Q}_e] = [\mathbf{P}_e]^{-1} \tag{4.12}$$

$$[\mathbf{\Lambda}_e] = \mathrm{diag}([\mathbf{\Lambda}_e^{in}], [\mathbf{\Lambda}_e^{out}], [\mathbf{0}]). \tag{4.13}$$

$$[\mathbf{F}_e] = [\mathbf{P}_e^{in}][\mathbf{\Lambda}_e^{in}][\mathbf{Q}_e^{in}] + [\mathbf{P}_e^{out}][\mathbf{\Lambda}_e^{out}][\mathbf{Q}_e^{out}] \tag{4.14}$$

4.2.3 Boundary term

$$\mathbf{S}_e = [\mathbf{P}_e^{in}]\mathbf{S}_e^{in} + [\mathbf{P}_e^{out}]\mathbf{S}_e^{out} \tag{4.15}$$

$$[\mathbf{C}_e]\mathbf{S}_e = \mathbf{E}_e. \tag{4.16}$$

$$[\mathbf{C}_e][\mathbf{P}^{out}]\mathbf{S}_e^{out} = \mathbf{E}_e - [\mathbf{C}_e][\mathbf{P}^{in}]\mathbf{S}_e^{in} \tag{4.17}$$

$$\mathbf{S}_e^{out} = [\widetilde{\mathbf{R}}_e]\mathbf{S}_e^{in} + \widetilde{\mathbf{E}}_e, \tag{4.18}$$

$$[\widetilde{\mathbf{R}}_e] = -\left([\mathbf{C}_e][\mathbf{P}^{in}]\right)^{-1}[\mathbf{C}_e][\mathbf{P}^{in}], \quad \widetilde{\mathbf{E}}_e = \left([\mathbf{C}_e][\mathbf{P}^{in}]\right)^{-1}\mathbf{E}_e. \tag{4.19}$$

$$\mathbf{S}_e = \Big([\mathbf{P}^{in}] + [\mathbf{P}^{out}][\widetilde{\mathbf{R}}_e]\Big)\mathbf{S}_e^{in} + [\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e \tag{4.20}$$

$$\mathbf{S}_e = \underbrace{\Big([\mathbf{P}^{in}] + [\mathbf{P}^{out}][\widetilde{\mathbf{R}}_e]\Big)[\mathbf{Q}_e^{in}]}_{[\widetilde{\mathbf{P}}_e]}\mathbf{S}_e + [\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e \tag{4.21}$$

$$\begin{aligned}\int_{\partial\Omega_e}\mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e]\mathbf{S}_e\,\mathrm{d}\Gamma &= \int_{\partial\Omega_e}\mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e]\Big[[\widetilde{\mathbf{P}}_e]\mathbf{S}_e + [\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e\Big]\,\mathrm{d}\Gamma && (4.22)\\ &= \int_{\partial\Omega_e}\mathbf{T}_e^T[\widetilde{\mathbf{F}}_e]\mathbf{S}_e\,\mathrm{d}\Gamma + \int_{\partial\Omega_e}\mathbf{T}_e^T\widetilde{\mathbf{S}}_e\,\mathrm{d}\Gamma && (4.23)\end{aligned}$$

$$[\widetilde{\mathbf{F}}_e] = [\mathbf{M}_e][\mathbf{F}_e][\widetilde{\mathbf{P}}_e], \quad \widetilde{\mathbf{S}}_e = [\mathbf{M}_e][\mathbf{F}_e][\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e \tag{4.24}$$

4.2.4 Interfaces terms for DGM/DGM

$$I_{12} = \int_{\Gamma_{12}} \mathbf{T}_1[\mathbf{M}_1][\mathbf{F}_1]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\mathbf{M}_2][\mathbf{F}_2]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.25}$$

$$\mathbf{S}_1 = [\mathbf{P}_1^{in}]\mathbf{S}_1^{in} + [\mathbf{P}_1^{out}]\mathbf{S}_1^{out} \tag{4.26}$$
$$\mathbf{S}_2 = [\mathbf{P}_2^{in}]\mathbf{S}_2^{in} + [\mathbf{P}_2^{out}]\mathbf{S}_2^{out}. \tag{4.27}$$

$$[\mathbf{C}_1]\mathbf{S}_1 = [\mathbf{C}_2]\mathbf{S}_2. \tag{4.28}$$

$$[\mathbf{C}_1][\mathbf{P}_1^{out}]\mathbf{S}_1^{out} - [\mathbf{C}_2][\mathbf{P}_2^{out}]\mathbf{S}_2^{out} = [\mathbf{C}_2][\mathbf{P}_2^{in}]\mathbf{S}_2^{in} - [\mathbf{C}_1][\mathbf{P}_1^{in}]\mathbf{S}_1^{in} \tag{4.29}$$

$$\left\{ \begin{array}{c} \mathbf{S}_1^{out} \\ \mathbf{S}_2^{out} \end{array} \right\} = [\widetilde{\mathbf{R}}] \left\{ \begin{array}{c} \mathbf{S}_1^{in} \\ \mathbf{S}_2^{in} \end{array} \right\}. \tag{4.30}$$

$$[\widetilde{\mathbf{R}}] = \left[[\mathbf{C}_1][\mathbf{P}_1^{out}] \,|\, -[\mathbf{C}_2][\mathbf{P}_2^{out}]\right]^{-1} \left[-[\mathbf{C}_1][\mathbf{P}_1^{in}] \,|[\mathbf{C}_2][\mathbf{P}_2^{in}]\right] \tag{4.31}$$
$$= \left[\begin{array}{cc} [\widetilde{\mathbf{R}}_{11}] & [\widetilde{\mathbf{R}}_{12}] \\ [\widetilde{\mathbf{R}}_{21}] & [\widetilde{\mathbf{R}}_{22}] \end{array} \right]. \tag{4.32}$$

$$\mathbf{S}_1 = \underbrace{\left([\mathbf{P}_1^{in}] + [\mathbf{P}_1^{out}][\widetilde{\mathbf{R}}_{11}]\right)[\mathbf{Q}_1^{in}]}_{[\widetilde{\mathbf{P}}_{11}]} \mathbf{S}_1 + \underbrace{[\mathbf{P}_1^{out}][\widetilde{\mathbf{R}}_{12}][\mathbf{Q}_2^{in}]}_{[\widetilde{\mathbf{P}}_{12}]} \mathbf{S}_2 \tag{4.33}$$
$$\mathbf{S}_2 = \underbrace{[\mathbf{P}_2^{out}][\widetilde{\mathbf{R}}_{21}][\mathbf{Q}_1^{in}]}_{[\widetilde{\mathbf{P}}_{21}]} \mathbf{S}_1 + \underbrace{\left([\mathbf{P}_2^{in}] + [\mathbf{P}_2^{out}][\widetilde{\mathbf{R}}_{22}]\right)[\mathbf{Q}_2^{in}]}_{[\widetilde{\mathbf{P}}_{22}]} \mathbf{S}_2 \tag{4.34}$$

$$I_{12} = \int_{\Gamma_{12}} \mathbf{T}_1[\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_1[\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.35}$$
$$+ \int_{\Gamma_{12}} \mathbf{T}_2[\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\widetilde{\mathbf{F}}_{22}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.36}$$

$$[\widetilde{\mathbf{F}}_{11}] = [\mathbf{M}_1][\mathbf{F}_1][\widetilde{\mathbf{P}}_{11}], \quad [\widetilde{\mathbf{F}}_{12}] = [\mathbf{M}_1][\mathbf{F}_1][\widetilde{\mathbf{P}}_{12}] \tag{4.37}$$
$$[\widetilde{\mathbf{F}}_{21}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{21}], \quad [\widetilde{\mathbf{F}}_{22}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{22}] \tag{4.38}$$

4.2.5 Interfaces terms for FEM/FEM

$$I_{12} = -\int_{\Gamma_{12}} q_1 \frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega}\,\mathrm{d}\Gamma - \int_{\Gamma_{12}} q_2 \frac{\mathbf{v}_2.\mathbf{n}_2}{j\omega}\,\mathrm{d}\Gamma \tag{4.39}$$

$-$ comes from the transposition of the right hand side integral.

Like in the previous section

$$\hat{\mathbf{S}}_1 = [\widetilde{\mathbf{P}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{12}]\hat{\mathbf{S}}_2 \tag{4.40}$$

$$\hat{\mathbf{S}}_2 = [\widetilde{\mathbf{P}}_{21}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{22}]\mathbf{S}_2 \tag{4.41}$$

$$-\frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega} = -\frac{1}{j\omega}\left[\hat{\mathbf{S}}_1(1)\mathbf{n}_1(1) + \hat{\mathbf{S}}_1(2)\mathbf{n}_1(2)\right] \tag{4.42}$$

$$= [\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \tag{4.43}$$

with

$$[\widetilde{\mathbf{F}}_{11}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{11}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{11}(2,:)]\mathbf{n}_1(2)\right] \tag{4.44}$$

$$[\widetilde{\mathbf{F}}_{12}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{12}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{12}(2,:)]\mathbf{n}_1(2)\right] \tag{4.45}$$

Concerning the second term

$$-\frac{\mathbf{v}_2.\mathbf{n}_2}{j\omega} = \frac{1}{j\omega}\left[\hat{\mathbf{S}}_2(1)\mathbf{n}_2(1) + \hat{\mathbf{S}}_2(2)\mathbf{n}_2(2)\right] \tag{4.46}$$

$$= [\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 + [\widetilde{\mathbf{F}}_{22}]\mathbf{S}_2 \tag{4.47}$$

with

$$[\widetilde{\mathbf{F}}_{21}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{21}(1,:)]\mathbf{n}_2(1) + [\widetilde{\mathbf{P}}_{21}(2,:)]\mathbf{n}_2(2)\right] \tag{4.48}$$

$$[\widetilde{\mathbf{F}}_{22}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{22}(1,:)]\mathbf{n}_2(1) + [\widetilde{\mathbf{P}}_{22}(2,:)]\mathbf{n}_2(2)\right] \tag{4.49}$$

$$p_1(x,y) \approx [\mathbf{N}_1(x,y)]\mathbf{p}_1, \quad \mathbf{p}_2(x,y) \approx [\mathbf{N}_2(x,y)]\mathbf{p}_2 \tag{4.50}$$

$$\mathbf{S}_i \approx \begin{bmatrix} [\mathbf{V}_i^x(x,y)] \\ [\mathbf{V}_i^y(x,y)] \\ [\mathbf{N}_i(x,y)] \end{bmatrix} \mathbf{p}_i \tag{4.51}$$

$$[\mathbf{V}_i^x(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_i(x,y)}{\partial x}\right], \quad [\mathbf{V}_i^y(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_i(x,y)}{\partial y}\right] \tag{4.52}$$

The boundary terms

$$\int_{\Gamma_{12}} q_i[\widetilde{\mathbf{F}}_{ij}]\mathbf{S}_j \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_i^T \left[\mathbf{N}_i(x,y)\right]^T [\widetilde{\mathbf{I}}_{ij}]\mathbf{p}_j \,\mathrm{d}\Gamma \tag{4.53}$$

with

$$[\widetilde{\mathbf{I}}_{ij}] = \widetilde{\mathbf{F}}_{ij}(1)\left[\mathbf{V}_j^x(x,y)\right] + \widetilde{\mathbf{F}}_{ij}(2)\left[\mathbf{V}_j^y(x,y)\right] + \widetilde{\mathbf{F}}_{ij}(3)\left[\mathbf{N}_j(x,y)\right] \tag{4.54}$$

4.2.6 Interfaces terms for FEM/DGM

$$I_{12} = -\int_{\Gamma_{12}} q_1 \frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega}\,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\mathbf{M}_2][\mathbf{F}_2]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.55}$$

$-$ for the first comes from the transposition of the right hand side integral.

Like in the previous section

$$\hat{\mathbf{S}}_1 = [\widetilde{\mathbf{P}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{12}]\hat{\mathbf{S}}_2 \tag{4.56}$$

$$\hat{\mathbf{S}}_2 = [\widetilde{\mathbf{P}}_{21}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{22}]\mathbf{S}_2 \tag{4.57}$$

$$-\frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega} = \frac{1}{j\omega}\left[\hat{\mathbf{S}}_1(1)\mathbf{n}_1(1) + \hat{\mathbf{S}}_1(2)\mathbf{n}_1(2)\right] \tag{4.58}$$

$$= [\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \tag{4.59}$$

with

$$[\widetilde{\mathbf{F}}_{11}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{11}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{11}(2,:)]\mathbf{n}_1(2)\right] \tag{4.60}$$

$$[\widetilde{\mathbf{F}}_{12}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{12}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{12}(2,:)]\mathbf{n}_1(2)\right] \tag{4.61}$$

$$[\widetilde{\mathbf{F}}_{21}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{21}], \quad [\widetilde{\mathbf{F}}_{22}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{22}] \tag{4.62}$$

$$I_{12} = \int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.63}$$

$$+ \int_{\Gamma_{12}} \mathbf{T}_2^T[\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2^T[\widetilde{\mathbf{F}}_{22}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.64}$$

$$p_1(x,y) \approx [\mathbf{N}_1(x,y)]\mathbf{p}_1, \quad \mathbf{S}_2(x,y) \approx [\mathbf{N}_2(x,y)]\mathbf{X}_2 \tag{4.65}$$

$$\mathbf{S}_1 \approx \begin{bmatrix} [\mathbf{V}_1^x(x,y)] \\ [\mathbf{V}_1^y(x,y)] \\ [\mathbf{N}_1(x,y)] \end{bmatrix} \mathbf{p}_1 \tag{4.66}$$

$$[\mathbf{V}_1^x(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_1(x,y)}{\partial x}\right], \quad [\mathbf{V}_1^y(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_1(x,y)}{\partial y}\right] \tag{4.67}$$

$$\int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_1^T \left[\mathbf{N}_1(x,y)\right]^T [\widetilde{\mathbf{I}}_{11}]\mathbf{p}_1 \,\mathrm{d}\Gamma \tag{4.68}$$

with

$$[\widetilde{\mathbf{I}}_{11}] = \widetilde{\mathbf{F}}_{11}(1)\,[\mathbf{V}_1^x(x,y)] + \widetilde{\mathbf{F}}_{11}(2)\,[\mathbf{V}_1^y(x,y)] + \widetilde{\mathbf{F}}_{11}(3)\,[\mathbf{N}_1(x,y)] \tag{4.69}$$

$$\int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_1^T[\widetilde{\mathbf{I}}_{12}]\mathbf{X}_2 \,\mathrm{d}\Gamma \tag{4.70}$$

with

$$[\widetilde{\mathbf{I}}_{12}] = \left[\mathbf{N}_1(x,y)\right]^T \widetilde{\mathbf{F}}_{12}\,[\mathbf{N}_2(x,y)] \tag{4.71}$$

$$\int_{\Gamma_{12}} \mathbf{T}_2^T[\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{X}_2^T \left[\mathbf{N}_2(x,y)\right]^T [\widetilde{\mathbf{I}}_{21}]\mathbf{p}_1 \,\mathrm{d}\Gamma \tag{4.72}$$

with

$$[\widetilde{\mathbf{I}}_{21}] = \widetilde{\mathbf{F}}_{11}(:,1)\,[\mathbf{V}_1^x(x,y)] + \widetilde{\mathbf{F}}_{11}(:,2)\,[\mathbf{V}_1^y(x,y)] + \widetilde{\mathbf{F}}_{11}(:,3)\,[\mathbf{N}_1(x,y)] \tag{4.73}$$

$$\int_{\Gamma_{12}} \mathbf{T}_2^T [\widetilde{\mathbf{F}}_{22}] \mathbf{S}_2 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{X}_2^T [\widetilde{\mathbf{I}}_{22}] \mathbf{X}_2 \,\mathrm{d}\Gamma \tag{4.74}$$

with

$$[\widetilde{\mathbf{I}}_{22}] = [\mathbf{N}_2(x,y)]^T \, [\widetilde{\mathbf{F}}_{22}] \, [\mathbf{N}_2(x,y)] \tag{4.75}$$

Chapter 5

Plane waves

5.1 Fluid

$$j\omega\rho v_x = -\frac{\partial p}{\partial x} \tag{5.1}$$

$$j\omega\rho v_y = -\frac{\partial p}{\partial y} \tag{5.2}$$

$$j\omega p = -\rho c^2 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right). \tag{5.3}$$

$$\mathbf{S} = \left\{ \begin{array}{c} v_x \\ v_y \\ p \end{array} \right\}. \tag{5.4}$$

$[\mathbf{M}]$, $[\mathbf{A}_x]$ and $[\mathbf{A}_y]$ are square matrices of size m defined as

$$[\mathbf{M}] = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \dfrac{1}{\rho c^2} \end{bmatrix}, \tag{5.5}$$

and

$$[\mathbf{B}_x] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad [\mathbf{B}_y] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.6}$$

Chapter 6

Finite-Element Matrices

6.1 Global system

6.1.1 Unknown vector

1. $\mathbf{X}_{TR6}$
2. $\mathbf{X}_{H12}$
3. $\mathbf{X}_{DGM}$
4. $\mathbf{X}_{Bloch}$

$$
\left[\begin{array}{cccc|cc}
[\mathbf{K}_e]-\omega^2[\mathbf{M}_e] & & & & \mathbf{I}_R^e & \mathbf{I}_T^e \\
 & [\mathbf{K}]-\omega^2\widetilde{\rho}[\mathbf{M}] & -\widetilde{\gamma}[\mathbf{C}] & & \mathbf{I}_R^s & \mathbf{I}_T^s \\
 & -\widetilde{\gamma}[\mathbf{C}]^t & \dfrac{[\mathbf{H}]}{\omega^2\widetilde{\rho}_{eq}}-\dfrac{[\mathbf{Q}]}{\widetilde{K}_{eq}} & & \mathbf{I}_R^p & \mathbf{I}_T^p \\
 & & & \dfrac{[\mathbf{H}]}{\omega^2\rho_0}-\dfrac{[\mathbf{Q}]}{K_0} & \mathbf{I}_R^a & \mathbf{I}_T^a \\
\hline
\mathbf{J}_R^e & \mathbf{J}_R^s & \mathbf{J}_R^p & \mathbf{J}_R^a & & \\
\mathbf{J}_T^e & \mathbf{J}_T^s & \mathbf{J}_T^p & \mathbf{J}_T^a & &
\end{array}\right]
\left\{\begin{array}{c}
\mathbf{u}^e \\ \mathbf{u}^s \\ \mathbf{p} \\ \mathbf{p}^a \\ \hline \mathbf{R} \\ \mathbf{T}
\end{array}\right\}
=
\left\{\begin{array}{c}
\mathbf{f}^e \\ \mathbf{f}^s \\ \mathbf{u} \\ \mathbf{u}^a \\ \hline \mathbf{R}' \\ \mathbf{T}'
\end{array}\right\}
\tag{6.1}
$$

The last part of the matrix concerns Bloch coefficients that are added in the DtN framework. Even if this doc states they appear as two separated blocks, the reality is... somewhat different. The blocks are not separated between reflection and transmission coefficients but laid in order sequence:

$$R_0,\ T_0,\ R_{-N},\ T_{-N},\ \dots,\ R_{-1},\ T_{-1},\ R_1,\ T_1,\ \dots,\ R_N,\ T_N \tag{6.2}$$

6.2 Number of coefficient in matrices

6.3 DtN operators

6.3.1 Fields properties in incident and transmission media

$$p^{inc}(x,y,t) = 1\, e^{j(\omega t - k_x x - k_y y)} + \sum_{l\in\mathbb{Z}} e^{j(\omega t - k_x x + k_y(l) y)} R_l \tag{6.3}$$

$$k_x = k_0\ sin(\theta), \quad k_y = k_0 \cos(\theta), \quad \theta \in \left[0; \frac{\pi}{2}\right[, \quad k_x(l) = k_x^i + \frac{2\pi l}{D} \tag{6.4}$$

$$u^{inc}(x,y,t) = \frac{\boldsymbol{\nabla} p^{\,inc}}{\rho_0\omega^2} = \left| \begin{array}{l} -\dfrac{jk_x}{\rho\omega^2} \\ -\dfrac{jk_z}{\rho\omega^2} \end{array} \right. e^{j(\omega t - k_x x - k_y y)} + \sum_{l\in\mathbb{Z}} \left| \begin{array}{l} -\dfrac{jk_x}{\rho\omega^2} \\ \dfrac{jk_y(l)}{\rho\omega^2} \end{array} \right. e^{j(\omega t - k_x x + k_y(l) y)} R_l \tag{6.5}$$

$$p^{tr}(x,y,t) = \sum_{l\in\mathbb{Z}} e^{j(\omega t - k_x x - k_y(l) y)} T_l \tag{6.6}$$

$$u^{tr}(x,y,t) = \frac{\boldsymbol{\nabla} p^{\,tr}}{\rho_0\omega^2} = \sum_{l\in\mathbb{Z}} \left| \begin{array}{l} -\dfrac{jk_x}{\rho\omega^2} \\ -\dfrac{jk_y(l)}{\rho\omega^2} \end{array} \right. e^{j(\omega t - k_x x - k_y(l) y)} T_l \tag{6.7}$$

6.3.2 Formalisation of the fields

Let $\mathbf{X}_l$ be an information vector associated to Bloch wave l.

$$\mathbf{Y} = \left\{ \begin{array}{c} 1 \\ \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_n \end{array} \right\} \tag{6.8}$$

On each interface with the FEM model a physical field φ can be written

$$\varphi(x,y) = [\mathbf{\Omega}_F^{\varphi}]e^{-jk_x} + \sum_l [\mathbf{\Omega}_l^{\varphi}]\mathbf{X}_l e^{-jk_x(l)}. \tag{6.9}$$

6.4 Case of a interface with fluid/Biot 1998

$$I_{FEM} = \int_{|\Gamma_R} \frac{1}{\rho\omega^2}\frac{\partial p^{inc}}{\partial n} q \; \mathrm{d}\Gamma = \int_{|\Gamma_R} - u_y(x)q(x) \; \mathrm{d}\Gamma \tag{6.10a}$$

$$= \int_{|\Gamma_R} [\mathbf{\Omega}_F^{u_y}]e^{-jk_x} q \; \mathrm{d}\Gamma \boxed{-} \sum_l \int_{|\Gamma_R} [\mathbf{\Omega}_l^{u_y}]\mathbf{X}_l e^{-jk_x(l)} q \; \mathrm{d}\Gamma \tag{6.10b}$$

- is highlighted as this term will end in the matrix system.

The information vector is scalar $\mathbf{X}_l = R_l$ and correspond to the reflexion coefficient.

$$[\mathbf{\Omega}_F^{u_y}] = \frac{jk_z}{\rho_0\omega^2}, \quad [\mathbf{\Omega}_l^{u_y}] = \frac{jk_z(l)}{\rho_0\omega^2} \tag{6.10c}$$

The additional equation is the projection of the pressure

$$\int_{\Gamma_R} p(x)e^{jk_x(n)x} \, \mathrm{d}\Gamma = \underbrace{\int_{\Gamma_R} e^{j\frac{2\pi n x}{D}} \, \mathrm{d}\Gamma}_{D\delta_{n0}} + \underbrace{\int_{\Gamma_R} \sum_{l\in\mathbb{Z}} e^{-\frac{j2\pi(l-n)x}{D}} \, \mathrm{d}\Gamma \, R_l}_{D\delta_{nl}} \tag{6.11}$$

6.5 Case of a interface with an elastic medium

6.5.1 Reflexion side

$$I_{FEM} = \int_{|\Gamma_R} \sigma_{xy}v_x - \sigma_{yy}v_y \; \mathrm{d}\Gamma = \int_{|\Gamma_R} p v_y \; \mathrm{d}\Gamma \tag{6.12a}$$

$$= \int_{|\Gamma_R} [\mathbf{\Omega}_F^{p}]e^{-jk_x} q \; \mathrm{d}\Gamma - \sum_l \int_{|\Gamma_R} [\mathbf{\Omega}_l^{p}]\mathbf{X}_l e^{-jk_x(l)} v_y \; \mathrm{d}\Gamma \tag{6.12b}$$

The information vector is scalar $\mathbf{X}_l = R_l$ and correspond to the reflexion coefficient.

$$[\mathbf{\Omega}_F^{p}] = 1, \quad [\mathbf{\Omega}_l^{p}] = -1 \tag{6.12c}$$

The additional equation is the projection of the pressure

$$\int_{\Gamma_R} u_y(x) e^{jk_x(n)x}\,\mathrm{d}\Gamma = \frac{-jk_y}{\rho_0\omega^2} D\delta_{n0} + \frac{jk_y(l)}{\rho_0\omega^2} D\delta_{nl} R_l \tag{6.13}$$

6.5.2 Transmission side

$$I_{FEM} = \int_{|\Gamma_R} \sigma_{xy} v_x + \sigma_{yy} v_y \;\mathrm{d}\Gamma = \int_{|\Gamma_R} p v_y \;\mathrm{d}\Gamma \tag{6.14a}$$

$$= \int_{|\Gamma_R} [\mathbf{\Omega}_F^p] e^{-jk_x} q \;\mathrm{d}\Gamma - \sum_l \int_{|\Gamma_R} [\mathbf{\Omega}_l^p] \mathbf{X}_l e^{-jk_x(l)} v_y \;\mathrm{d}\Gamma \tag{6.14b}$$

The information vector is scalar $\mathbf{X}_l = R_l$ and correspond to the reflexion coefficient.

$$[\mathbf{\Omega}_F^p] = -1, \quad [\mathbf{\Omega}_l^p] = 1 \tag{6.14c}$$

The additional equation is the projection of the pressure

$$\int_{\Gamma_T} u_y(x) e^{jk_x(n)x}\,\mathrm{d}\Gamma = -\frac{jk_y(l)}{\rho_0\omega^2} D\delta_{nl} R_l \tag{6.15}$$

6.6 Case of integration of a plate medium

6.6.1 Reflexion side

$$I_{FEM} = \int_{|\Gamma_R} -\sigma_{xz} v_x - \sigma_{zz} v_z \;\mathrm{d}\Gamma \tag{6.16}$$

$$\left\{ \begin{array}{c} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{array} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ -\dfrac{jk_z}{\rho\omega^2} & \dfrac{jk_z}{\rho\omega^2} & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} 1 \\ R \\ u_x \end{array} \right\} \tag{6.17}$$

$$\left\{ \begin{array}{c} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{array} \right\}_{\mathbf{F}} = [\mathbf{T}] \left\{ \begin{array}{c} 0 \\ -\dfrac{jk_z}{\rho\omega^2} \\ -1 \\ 0 \end{array} \right\}, \quad \left\{ \begin{array}{c} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{array} \right\}_{\mathbf{R}} = [\mathbf{T}] \begin{bmatrix} 0 & 0 \\ \dfrac{jk_z}{\rho\omega^2} & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} R \\ u_x \end{array} \right\} \tag{6.18}$$

$$p_{|\Gamma_R}(x,z) = e^{-j(k_x^i x + k_z^i z)} + \sum_{l\in\mathbb{Z}} e^{-j(k_x(l)x - k_z(l)z)} R_l \tag{6.19}$$

$$\frac{\partial p}{\partial z}_{|\Gamma_R}(x,z) = -jk_z^i e^{-j(k_x^i x + k_z^i z)} + \sum_{l\in\mathbb{Z}} jk_y(l) e^{-j(k_x(l)x - k_z(l)z)} R_l \tag{6.20}$$

$$p_{|\Gamma_R}(x,z) = \sum_{l\in\mathbb{Z}} e^{-j(k_x(l)x + k_z(l)z)} T_l \tag{6.21}$$

$$\frac{\partial p}{\partial z}_{|\Gamma_T}(x,z) = \sum_{l\in\mathbb{Z}} - jk_z(l) e^{-j(k_x(l)x + k_z(l)z)} T_l \tag{6.22}$$

Termes de surface

$$\frac{1}{\rho_a \omega^2} \int_{\Gamma_R} \frac{\partial p}{\partial n} q \,\mathrm{d}\Gamma = ... \tag{6.23}$$

Equation supplmentaires

$$\int_{\Gamma_R} p(x) e^{jk_x(n)x} \,\mathrm{d}\Gamma = \underbrace{\int_{\Gamma_R} e^{j\dfrac{2\pi n x}{D}} \,\mathrm{d}\Gamma}_{D\delta_{n0}} + \underbrace{\int_{\Gamma_R} \sum_{l\in\mathbb{Z}} e^{-\dfrac{j2\pi(l-n)x}{D}} \,\mathrm{d}\Gamma}_{D\delta_{nl}} R_l \tag{6.24}$$

6.7 Elementary matrix

6.7.1 I

$$\hat{I} = \int_a^b (AX^2 + BX + C) e^{-jk_x X} \,\mathrm{d}X \tag{6.25}$$

$x = X - a$

$$\hat{I} = e^{-jk_x a} \int_0^h \Phi(x) e^{-jk_x x} \,\mathrm{d}x \tag{6.26}$$

$\xi = -1 + \dfrac{2x}{h}, \quad x = \dfrac{h}{2}(\xi + 1)$

$$\hat{I} = e^{-jk_x a}\frac{h}{2}\int_{-1}^{1}\Phi(\xi)e^{-jk_x\frac{h(\xi+1)}{2}}\,\mathrm{d}\xi = e^{-jk_x\left(a+\frac{h}{2}\right)}\frac{h}{2}\int_{-1}^{1}\Phi(\xi)e^{-jk_x\frac{h\xi}{2}}\,\mathrm{d}\xi \tag{6.27}$$

Chapter 7

Benchmarks

7.1 Kundt Tube

Sous projets

- 0: TR6
- 1: H12
- 2: TR6/H12
- 3: DGM on TR
- 4: DGM on H
- 5: DGM on TR / DGM on H
- 6: H12 / DGM on H
- 7: TR6 / DGM on H

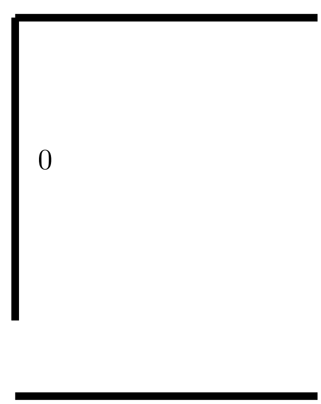

Figure 7.1: caption

Chapter 8

Hermite Finite-Elements

$$\Psi_1(x) = 1 - 3\left(\frac{x}{l_x}\right)^2 + 2\left(\frac{x}{l_x}\right)^3 \tag{8.1}$$

$$\Psi_2(x) = \left(\frac{x}{l_x}\right) - 2\left(\frac{x}{l_x}\right)^2 + \left(\frac{x}{l_x}\right)^3 \tag{8.2}$$

$$\Psi_3(x) = -\left(\frac{x}{l_x}\right)^2 + \left(\frac{x}{l_x}\right)^3 \tag{8.3}$$

$$\Psi_4(x) = 3\left(\frac{x}{l_x}\right)^2 - 2\left(\frac{x}{l_x}\right)^3 \tag{8.4}$$

$$\Psi_{ij}(x, y) = \Psi_i(x)\Psi_j(y) \tag{8.5}$$

Valeurs

	Ψ_1	Ψ_2	Ψ_3	Ψ_4
$f(0)$	1	0	0	0
$f(l_x)$	0	0	0	1
$f'(0)$	0	$1/l$	0	0
$f'(1_x)$	0	0	$1/l$	0

Ordre

	p	$\dfrac{\partial p}{\partial x}$	$\dfrac{\partial p}{\partial y}$	$\emptyset$
$(0,0)$	Ψ_{11}	Ψ_{21}	Ψ_{12}	Ψ_{22}
	$1//1$	$2//2$	$3//3$	$4//\emptyset$
$(1,0)$	Ψ_{41}	Ψ_{31}	Ψ_{42}	Ψ_{23}
	$5//4$	$6//5$	$7//6$	$8//\emptyset$
$(1,1)$	Ψ_{44}	Ψ_{34}	Ψ_{43}	Ψ_{33}
	$9//7$	$10//8$	$11//9$	$12//\emptyset$
$(0,1)$	Ψ_{14}	Ψ_{24}	Ψ_{13}	Ψ_{32}
	$13//10$	$14//11$	$15//12$	$16//\emptyset$

Chapter 9

Miscellaneous

9.1 Integration of an exponential on one element

$$\int_{\Omega} e^{j\mathbf{k}.(\mathbf{x}-\mathbf{x}_c)}\,\mathrm{d}\Omega = \oint \nabla. \left(- \left| \begin{array}{c} \dfrac{jk_x}{k_x^2+k_y^2} \\ \dfrac{jk_y}{k_x^2+k_y^2} \end{array} \right. \quad e^{j\mathbf{k}.(\mathbf{x}-\mathbf{x}_c)} \right) \mathrm{d}\Omega \tag{9.1}$$

$$= - \oint_{\partial\Omega} \left(\frac{jk_x n_x}{k_x^2+k_y^2} + \frac{jk_y n_y}{k_x^2+k_y^2} \right) e^{j\mathbf{k}.(\mathbf{x}-\mathbf{x}_c)}\,\mathrm{d}\Omega \tag{9.2}$$

Chapter 10

Plane Waves 3D

10.1 Fluid medium

10.1.1 Compressional wave

$$\delta_P^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.1}$$

$$u^{\varepsilon}(x,y,z,t) = \begin{vmatrix} k_x \\ k_y \\ \varepsilon k_z \end{vmatrix} e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.2}$$

$$\begin{aligned} p &= (-K_0)(-j)(k_x^2 + k_y^2 + \varepsilon^2 k_z^2) && (10.3) \\ &= jK_0\delta_P^2 && (10.4) \end{aligned}$$

10.2 Elastic Medium

10.2.1 Compressional wave

$$\delta_P^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.5}$$

$$u^{+}(x,y,z,t) = \begin{vmatrix} k_x \\ k_y \\ \varepsilon k_z \end{vmatrix} e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.6}$$

$$\sigma_{zz} = \lambda\left[u_{x,x} + u_{y,y} + u_{z,z}\right] + 2\mu u_{z,z} \tag{10.7}$$
$$= -j\lambda(k_x^2 + k_y^2 + \varepsilon^2 k_z^2) + 2\mu(-j\varepsilon^2 k_z^2) \tag{10.8}$$
$$= -j\left[\lambda(k_x^2 + k_y^2) + (\lambda + 2\mu)k_z^2\right] \tag{10.9}$$

$$\sigma_{yz} = \mu\left[u_{y,z} + u_{z,y}\right] = -j\left[\varepsilon k_y k_z + \varepsilon k_z k_y\right] = -2j\mu\varepsilon k_y k_z \tag{10.10}$$

$$\sigma_{xz} = \mu\left[u_{x,z} + u_{z,x}\right] = -j\mu\left[\varepsilon k_x k_z + \varepsilon k_z k_x\right] = -2j\mu\varepsilon k_x k_z \tag{10.11}$$

10.2.2 Shear wave 1

$$\delta_S^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.12}$$

$$u^{\varepsilon}(x,y,z,t) = \left| \begin{array}{c} k_z \\ 0 \\ -\varepsilon k_x \end{array} \right. e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.13}$$

$$\sigma_{zz} = \lambda\left[u_{x,x} + u_{y,y} + u_{z,z}\right] + 2\mu u_{z,z} \tag{10.14}$$
$$= 2\mu j k_x k_z \tag{10.15}$$

$$\sigma_{yz} = \mu\left[u_{y,z} + u_{z,y}\right] = -j\mu\varepsilon k_x k_y \tag{10.16}$$

$$\sigma_{xz} = \mu\left[u_{x,z} + u_{z,x}\right] = -j\mu\left[\varepsilon k_x k_z - \varepsilon k_z k_x\right] = -j\mu\varepsilon(k_z^2 - k_x^2) \tag{10.17}$$

10.2.3 Shear wave 2

$$\delta_S^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.18}$$

$$u^\varepsilon(x,y,z,t) = \left| \begin{array}{c} 0 \\ k_z \\ -\varepsilon k_y \end{array} \right. e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.19}$$

$$\sigma_{zz} = \lambda \left[u_{x,x} + u_{y,y} + u_{z,z} \right] + 2\mu u_{z,z} \tag{10.20}$$

$$= 2\mu j k_y k_z \tag{10.21}$$

$$\sigma_{yz} = \mu \left[u_{y,z} + u_{z,y} \right] = -j\mu\varepsilon(k_z^2 - k_y^2) \tag{10.22}$$

$$\sigma_{xz} = \mu \left[u_{x,z} + u_{z,x} \right] = -j\varepsilon k_x k_y \tag{10.23}$$

Chapter 11

Rotations

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = [\mathbf{P}] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = [\mathbf{P}] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{11.1}$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial x_1} + \frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial x_2} + \frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial x_3} \tag{11.2}$$

$$= \mathbf{P}(1,:) \begin{Bmatrix} \dfrac{\partial}{\partial X_1} \\ \dfrac{\partial}{\partial X_2} \\ \dfrac{\partial}{\partial X_3} \end{Bmatrix} F(X_1, X_2, X_3) \tag{11.3}$$

$$\begin{Bmatrix} \dfrac{\partial}{\partial X_1} \\ \dfrac{\partial}{\partial X_2} \\ \dfrac{\partial}{\partial X_3} \end{Bmatrix} F(X_1, X_2, X_3) = [\mathbf{Q}] \begin{Bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{Bmatrix} f(x_1, x_2, x_3) \tag{11.4}$$

$$\frac{\partial U_i}{\partial X_j} = \mathbf{Q}(i,:) \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \mathbf{Q}(j,:) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{11.5}$$

$$= \mathbf{Q}(i,:) \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \left(\mathbf{Q}_{j1} u_1 + \mathbf{Q}_{j2} u_2 + \mathbf{Q}_{j3} u_3\right) \tag{11.6}$$

$$= \mathbf{Q}(i,:) \begin{Bmatrix} \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_1} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_1} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_1} \\ \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_2} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_2} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_2} \\ \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_3} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_3} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_3} \end{Bmatrix} \tag{11.7}$$

$$\frac{\partial U_i}{\partial X_j} = \sum_{k=1}^{3} \sum_{k'=1}^{3} Q_{ik} Q_{jk'} u_{k',k} \tag{11.8}$$

$$E_{ii} = Q_{i1}^2 \varepsilon_{11} + Q_{i2}^2 \varepsilon_{22} + Q_{i3}^2 \varepsilon_{33} + Q_{i1} Q_{i2} 2\varepsilon_{12} + +Q_{i1} Q_{i3} 2\varepsilon_{13} + Q_{i2} Q_{i3} 2\varepsilon_{23} \tag{11.9}$$

$$2E_{ij} = \left(Q_{i1} Q_{j1} + Q_{j1} Q_{i1}\right) \varepsilon_{11} + ... \tag{11.10}$$

$$+ \left(Q_{i1} Q_{j2} + Q_{j1} Q_{i2}\right) 2\varepsilon_{12} + ... \tag{11.11}$$