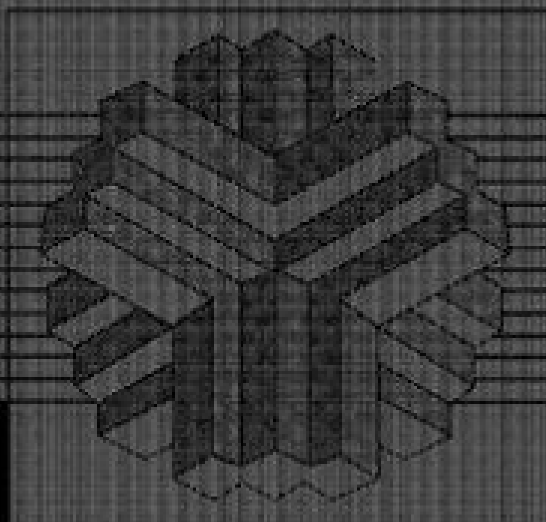
ENVIRONMENTAL AND INTELLIGENT MANUFACTURING SYSTEMS SERIES *Mohammad Jamshidi, Series Editor*



FUZZY LOGIC and CONTROL

Software and Hardware Applications

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FUZZY RULE-BASED EXPERT SYSTEMS I

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This chapter originally appeared in the journal ***Intelligent and Fuzzy Systems: Applications in Engineering and Technology***, March 1993. Used here by permission.

The general problems of system identification and control system design are formulated and the concept of control or decision surface is described. It is shown that a collection of fuzzy conditional restrictive rules could be used to model complex or ill-defined processes (fuzzy systems), and some simple canonical formats for the production rule-sets are given. It is shown that a system of fuzzy relational equations could be obtained from a set of IF-THEN production rules. A fuzzy system transfer relation based on a system of fuzzy relational equations is defined. It is also described how a variety of fuzzy implication relations or Zadeh's extension principle might be used to derive the fuzzy relational equations. Various methods for obtaining the solution of these equations based on a number of composition of relations techniques are studied. Finally, two most commonly used solution techniques in fuzzy control applications, i.e., Max.-Min. and Max.-Product methods, are described.

4.1 INTRODUCTION

A mathematical model that describes a wide variety of physical systems is an nth-order ordinary differential equation of the type:

$$\frac{d^n y(t)}{dt^n} = w\left[t, y(t), \dot{y}(t), \cdots, \frac{d^{n-1}y(t)}{dt^{n-1}}, u(t)\right] \quad (1)$$

where t is the time parameter, u(.) is the input function, and y(.) is the output or response function. If we define the auxiliary functions:

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

$$\vdots$$

$$x_n(t) = \frac{d^{n-1}y(t)}{dt^{n-1}} \quad (2)$$

then the single *n*th-order equation (1) can be equivalently expressed as a system of *n* first-order equations:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\vdots$$

$$\dot{x}_{n-1}(t) = x_n(t) \quad (3)$$

Finally, if we define *n*-vector-valued functions **x**(.) and **f**(.) by

$$\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_n(t)]' \quad (4)$$

$$\mathbf{f}(t, \mathbf{x}, u) = [x_2, x_3, \cdots, x_n, w(t, x_1, \cdots, x_n, u)]' \quad (5)$$

where **x**(t) is the system state vector at time t, then the *n* first-order equations (3) can be combined into a first-order vector differential equation, i.e.,

$$\dot{\mathbf{x}}(t) = \mathbf{f}\,[t, \mathbf{x}(t), u(t)] \quad (6)$$

and the output y(t) is given from (2) as:

$$y(t) = [1, 0, \cdots, 0]\,\mathbf{x}(t) \quad (7)$$

Similarly, a system with p inputs, m outputs, and n states, will be described, in general, as:

$$\dot{\mathbf{x}}(t) = \mathbf{f}\ [t, \mathbf{x}(t), \mathbf{u}(t)] \tag{8}$$

$$\mathbf{y}(t) = \mathbf{g}\ [t, \mathbf{x}(t), \mathbf{u}(t)] \tag{9}$$

where $\mathbf{u}(t)$ and $\mathbf{y}(t)$ vectors defined as:

$$\mathbf{u}(t) = [u_1(t), u_2(t), \cdots, u_p(t)]' \tag{10}$$

$$\mathbf{y}(t) = [y_1(t), y_2(t), \cdots, y_m(t)]' \tag{11}$$

are input vector and output vector, respectively. Physical systems descriptions based on equations (8) and (9) are known as state-space representations. In the case of time-invariant systems, equations (8) and (9) become:

$$\dot{\mathbf{x}}(t) = \mathbf{f}\ [\mathbf{x}(t), \mathbf{u}(t)] \tag{12}$$

$$\mathbf{y}(t) = \mathbf{g}\ [\mathbf{x}(t), \mathbf{u}(t)] \tag{13}$$

and for a linear time-invariant system

$$\dot{\mathbf{x}}(t) = A \,.\, \mathbf{x}(t) + B \,.\, \mathbf{u}(t) \tag{14}$$

$$\mathbf{y}(t) = C \,.\, \mathbf{x}(t) + D \,.\, \mathbf{u}(t) \tag{15}$$

where constants A, B, C, and D are known as system matrices.

A first-order single-input and single-output system is described using a discrete-time equation as:

$$x_{k+1} = f\,(x_k, u_k) \tag{16}$$

where x_{k+1}, x_k are the values of state at time moments kth and $(k+1)th$ and u_k is the input at the kth moment. An nth order single-input and single-output system can be put in the form of:

$$y_{k+n} = f\,(y_k, y_{k+1}, \cdots, y_{k+n-1}, u_k) \tag{17}$$

and for an nth-order, multiple-input and single-output discrete system:

Chapter 4. Fuzzy Rule-Based Expert Systems I. Fuzzy Logic and Control: Software and Hardware Applications, Volume 2, ISBN: 9780132442183
Prepared for safaribooksonline@lorenz.kiefner.de, Lorenz Kiefner

$$y(k+n) = f\,[(y(k), y(k+1), \cdots, y(k+n-1),\; u_1(k), u_2(k), \cdots, u_p(k)] \tag{18}$$

4.1.1 System Identification Problem

The general problem of identifying a physical system based on the measurements of the input, output, and state variables is defined as obtaining functions **f** and **g**, in the case of a nonlinear system, and system matrices A, B, C, and D, in the case of a linear system. There exist algorithms that adaptively converge to these system parameters based on numerical data taken from input and output variables. Fuzzy systems and artificial neural network paradigms are two evolving disciplines for nonlinear system identification problems.

4.1.2 Control System Design Problem

The general problem of feedback control system design is defined as obtaining a generally nonlinear vector-valued function **h**(.) defined as:

$$\mathbf{u}(t) = \mathbf{h}\,[t, \mathbf{x}(t), \mathbf{r}(t)] \tag{19}$$

where **u**(t) is the input to the plant or process, **r**(t) is the reference input, and **x**(t) is the state vector. The feedback control law **h** is supposed to stabilize the feedback control system and result in a satisfactory performance.

In the case of a time-invariant system with a regulatory type of controller, the control command could be stated as either a state-feedback or an output feedback as shown in the following:

$$\mathbf{u}(t) = \mathbf{h}\,[\mathbf{x}(t)] \tag{20}$$

$$\mathbf{u}(t) = \mathbf{h}\,[y(t), \dot{y}, \int y\,dt] \tag{21}$$

In the case of a simple single-input and single-output system the function **h** takes one of the following forms:

$$\mathbf{u}(t) = K_p \,.\, e(t)$$

for a proportional or **P** controller;

$$\mathbf{u}(t) = K_p \,.\, e(t) + K_I \,.\, \int e(t)dt$$

for a proportional plus integral or PI controller;

$$\mathbf{u}(t) = K_p \,.\, e(t) + K_D \,.\, \dot{e}(t)$$

for a proportional plus derivative or PD controller;

$$\mathbf{u}(t) = K_P \, . \, e(t) + K_I \, . \int e(t)dt + K_D \, . \, \dot{e}(t)$$

for a proportional plus derivative plus integral or PID controller, where e(t), $\dot{e}(t)$, and $\int$ e(t)dt are the output error, error derivative, and error integral, respectively; and

$$\mathbf{u}(t) = -\,[k_1 \, . \, x_1(t) + k_2 \, . \, x_2(t) + \cdots + k_n \, . \, x_n(t)]$$

for a full state-feedback controller.

The problem of control system design is defined as obtaining the generally nonlinear function **h**(.) in the case of nonlinear systems, coefficients K_P, K_I, and K_D in the case of an output-feedback, and coefficients k_1, k_2, . . . , and k_n, in the case of state-feedback policies for linear system models.

4.1.3 Control Surface

The function **h** as defined in equations (19), (20), and (21) is, in general, defining a nonlinear hypersurface in an *n*-dimensional space. For the case of linear systems with output feedback or state feedback it is a hyperplane in an *n* dimensional space. This surface is known as the control or decision surface.

The control surface describes the dynamics of the controller and is generally a time-varying non-linear surface. Due to unmodeled dynamics present in the design of any controller, techniques should exist for adaptively tuning and modifying the control surface shape. Fuzzy logic rule-based expert systems use a collection of fuzzy conditional statements derived from an expert knowledge-base to approximate and construct the control surface. This paradigm of control system design is based on interpolative and approximate reasoning. Fuzzy rule-based controllers or system identifiers are generally model-free paradigms. Fuzzy logic rule-based expert systems are universal nonlinear function approximators and any nonlinear function of *n* independent variables and one dependent variable can be approximated to any desired precision.

On the other hand, artificial neural networks are based on analogical learning and try to learn the nonlinear function through adaptive and converging techniques and based on numerical data available from input-output measurements on the system variables and some performance criteria.

4.1.4 Control System Design Stages

The seven basic steps in designing a complex or ill-defined physical system are:

i) Large scale systems are decentralized and decomposed into a collection of decoupled sub-systems.

ii) The temporal variations of plant dynamics are assumed to be "slowly varying."

iii) The nonlinear plant dynamics are locally linearized about a set of operating points.

iv) A set of state variables, control variables, or output features are made available.

v) A simple P, PD, PID (output feedback), or state-feedback controller is designed for each de coupled system. The controllers are of regulatory type and are fast enough to perform satisfactorily under tracking control situations. Optimal controllers can also be tried using LQR or LQG techniques.

vi) The first five steps mentioned above introduce uncertainties. There are also uncertainties due to external environment. The controller design should be made as close as possible to the optimal one based on the control engineer's all best available knowledge, in the form of I/O numerical observations data, analytic, linguistic, intuitive, and etc., information regarding the plant dynamics and external world.

vii) A supervisory control system, either automatic or a human expert operator, forms an additional feedback control loop to tune and adjust the controller's parameters in order to compensate the effects of uncertainties and variations due to unmodeled dynamics.

4.1.5 Assumptions in a Fuzzy Control System Design

Six basic assumptions are commonly made whenever a fuzzy logic-based control policy is selected. These assumptions are outlined below:

i) The plant is observable and controllable: State, input, and output variables are available for observation and measurement or computation.

ii) There exists a body of knowledge in the form of expert production linguistic rules, engineering common sense, intuition, or an analytic model that can be fuzzified and the rules be extracted.

iii) A solution exists.

vi) The control engineer is looking for a good enough solution and not necessarily the optimum one.

v) We desire to design a controller to the best of our available knowledge and within an acceptable precision range.

vi) The problems of stability and optimality are open problems.

The following sections discuss the problem of obtaining the control surface **h**(.), i.e., approximations based on a collection of fuzzy IF-THEN rules which describe the dynamics of the controller. Fuzzy rule-based expert models can also be used to obtain acceptable approximations for the functions **f**(.) and **g**(.) in the case of a system identification problem.

A fuzzy production rule system consists of four structures:

i) A set of rules which represents the policies and heuristic strategies of the expert decision-maker.

ii) A set of input data assessed immediately prior to the actual decision.

iii) A method for evaluating any proposed action in terms of its conformity to the expressed rules, given the available data.

iv) A method for generating promising actions and for determining when to stop searching for better ones.

The input data, rules, and output action or consequence are generally fuzzy sets expressed by means of appropriate membership functions defined on a proper space. The method of evaluation of rules is known as *approximate reasoning* or *interpolative reasoning*, and is commonly represented by composition of fuzzy relations applied to a fuzzy relational equation.

4.2 FUZZY RULE-BASED EXPERT SYSTEMS (CANONICAL FORMS)

Consider an n-input and m-output system shown in Figure 4-1. Let **X** be a Cartesian product of n universes $\mathbf{x}_i$, for i = 1, 2, . . . , n, i.e.,$\mathbf{X} = \mathbf{x}_1 \times \mathbf{x}_2 \times \cdots \times \mathbf{x}_n$; and **Y** be a Cartesian product of m universes $\mathbf{y}_j$ for j = 1, 2, . . . , m, i.e., $\mathbf{Y} = \mathbf{y}_1 \times \mathbf{y}_2 \times \cdots \times \mathbf{y}_m$. $\mathbf{x} = (x_1, x_2, \cdots, x_n)'$ is the input vector to the system defined on real space R^n and$\mathbf{y} = (y_1, y_2, \cdots, y_m)'$ is the output vector of the system defined on real space R^m. The system S could represent any general static nonlinear ng from **X** to **Y**, an industrial control system, a dynamic system identification mapping model, a pattern recognition system, or a decision-making process.

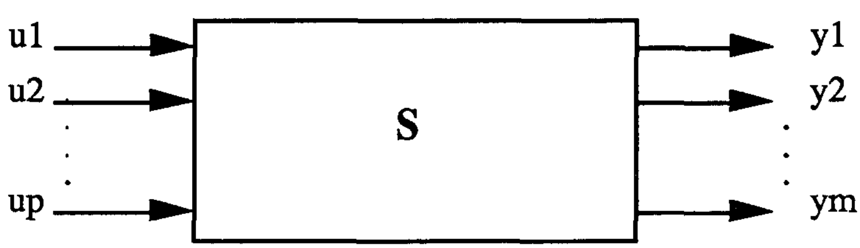

Figure 4-1. A block diagram for a p-input and m-output system.

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There are three general spaces present in all these systems and processes, i.e.,

i) the space of possible conditions at the inputs to the system and, in general, could be represented by a collection of fuzzy subsets A^K, for k= 1, 2, . . . , which are fuzzy partitions of space **X**, expressed by means of membership functions

$$A^k \varepsilon F(X); \mu_{A^k}(x): X \rightarrow [0, 1], \text{for } k = 1, 2, \cdots \quad (22)$$

ii) the space of possible output consequences, control commands from a controller, or the decisions recommended to or made by a decision-maker, based on some specific conditions at the inputs, and, in general, could be represented by a collection of fuzzy subsets B^K, for k = 1, 2, . . . , which are fuzzy partitions of space **Y**, expressed by means of membership functions

$$B^k \varepsilon F(Y); \mu_{B^k}(y): Y \rightarrow [0, 1], \text{for } k = 1, 2, \cdots \quad (23)$$

and,

iii) the space of possible mapping relations from the input space **X** onto the output space **Y**. The mapping relations are, in general, represented by fuzzy relations R^K, for k = 1, 2, . . . , and expressed by means of membership functions

$$R^k \varepsilon F(X \text{ x } Y); \mu_{R^k}(x, y): X \text{ x } Y \rightarrow [0, 1], \text{for } k = 1, 2, \cdots \quad (24)$$

where F(.), in equations (22), (23), and (24), denotes a family of fuzzy sets on a proper space. A human perception of the system S is based on experience and expertise, intuition, a knowledge of the physics of the system, or a set of subjective preferences and goals. This type of knowledge is usually put in the form of a set of unconditional as well as conditional propositions in natural language, by the human expert. Our understanding of complex humanistic and non-humanistic systems is at a qualitative and declarative level, based on vague linguistic terms. This is called a fuzzy level of understanding of a physical system.

The unconditional as well as conditional statements, in general, place some restrictions on the consequent of the process based on certain immediate as well as past conditions. These restrictions are usually vague natural language terms and words that could be modeled using fuzzy mathematics.

Consider the problem of the coloring of a landscape and some expert restrictive statements:

If the season is spring, then the color is rather light green.
If the season is summer, then the color is deep green.
If the season is fall, then the color is bright and deep yellow.
.
.
.
, etc.

The vague term "rather light green" in the first statement places a fuzzy restriction on the color, based on a fuzzy "spring" condition in the antecedent. A similar case is a pattern recognition problem such as "If the color is red, the tomato is ripe." A room temperature control problem might contain the following expert rules:

If the room temperature is very hot,
then
If the heat is on
then turn the heat lower.

In this case restrictions are placed on the actions taken by a controller, i.e., "turn the heat lower."

In summary, the fuzzy level of understanding and describing a complex system basically is put in the form of a set of restrictions on the output based on certain conditions at the input. Restrictions are generally modeled by fuzzy sets and relations.

4.2.1 Canonical Fuzzy Rule-Based Expert System

The canonical form of a set of expert production rules is defined as a set of unconditional restrictions followed by a set of conditional restrictions. These restriction statements, unconditional as well as conditional, are usually connected by linguistic connectives such as "and," "or," or "else." The canonical form of an expert production rule-set is given in Figure 4-2.

R^1	Restriction R^1
R^2	Restriction R^2
.	
.	
.	
R^k	Restriction R^k
R^{k+1}	IF condition C^1, THEN restriction R^{k+1}
R^{k+2}	IF condition C^2, THEN restriction R^{k+2}
.	
.	
.	
R^r	IF condition C^{r-k}, THEN restriction R^r

Figure 4-2. The canonical form of a set of expert rules.

The restrictions $R^1, R^2, \ldots, R^r$ apply to the output actions taken on the output, or the decisions made for a desired performance.

In general, there exist three general statement forms in any linguistic algorithm or expert production rule-set.

i) Assignment statements, e.g.:

$x \cong s$
x = small
season = spring
room temperature = hot
tomato's color = red
x is large
x is not large and not very small

ii) Conditional statements, e.g.:
IF x is small THEN y is large ELSE y is not large
IF x is positive THEN decrease y slightly
IF the tomato is red THEN the tomato is ripe
IF x is very small THEN stop

iii) Unconditional action statements, e.g.:
multiply by x
turn the heat lower
delete first few terms
go to 7
stop

The unconditional propositions, equivalently, may be thought of as conditional restrictions with their IF clause condition being the universe of discourse of the input conditions, which is always true and satisfied. An unconditional restriction such as "output is low" could, equivalently, be written as:

IF any conditions THEN output is low

or

IF anything THEN low.

Hence, the system under consideration could be described using a collection of conditional restrictive statements. These statements may also be modeled as fuzzy conditional statements, such as:

IF condition C^1 THEN restriction R^1.

The unconditional restrictions might be in the form:

R^1: The output is B^1
AND
R^2: The output is B^2
AND
.
.
.

where $B^1, B^2, \ldots$ are fuzzy subsets defined in equation (23).

Figure 4-3 is the expert system comprised of a set of conditional rules. Hence, the canonical rule set may be put in the following form:

Rule 1:	IF condition C^1, THEN restriction R^1
Rule 2:	IF condition C^2, THEN restriction R^2
.	
.	
.	
Rule R^r	IF condition C^r, THEN restriction R^r

Figure 4-3. The canonical fuzzy rule-based expert system.

For the case of n-input and m-output system S, described earlier, the canonical fuzzy rule-based expert system (FRBES) could be put in the form shown in Figure 4-4.

Rule 1:	IF **x** is A^1, THEN **y** is B^1
Rule 2:	IF **x** is A^2, THEN **y** is B^2
.	
.	
.	
Rule r:	IF **x** is A^r, THEN **y** is B^r

Figure 4-4. FRBES describing the system S.

and in a more compact form:

(IF A^1 THEN B^1 "α" IF A^2 THEN B^2 "α" . . . "α " IF A^r THEN B^r)

where "α" could be any of "and," "or," or "else" linguistic connectives. In Figure 4-5, **x** and **y** are the input and output vectors, respectively. The input to the system could have crisp and sharp as well as fuzzy values. It is noted that crisp values $x = \bar{x}$, or $y = \bar{y}$, known as fuzzy singletons, are expressed by a membership function as follows:

$$\mu_A(x) = \delta_{x,\bar{x}} = \delta(x - \bar{x}) = \begin{cases} 1, \text{ for } x = \bar{x} \\ 0, \text{ otherwise} \end{cases} \tag{25}$$

and, similarly,

$$\mu_B(y) = \delta_{y,\bar{y}} = \delta(y - \bar{y}) = \begin{cases} 1, \text{ for } y = \bar{y} \\ 0, \text{ otherwise} \end{cases} \tag{26}$$

or a crisp interval defined as $a \le x \le b$, or $c \le y \le d$ may be defined as fuzzy sets expressed by the membership functions

$$\mu_A(x) = \begin{cases} 1, \text{ for } a \le x \le b \\ 0, \text{ otherwise} \end{cases} \tag{27}$$

and

$$\mu_B(y) = \begin{cases} 1, \text{ for } c \le y \le d \\ 0, \text{ otherwise} \end{cases} \tag{28}$$

Therefore, crisp and fuzzy conditions and consequences could be treated in a common setting.

It is very true that IF A THEN B

are transformed to a new meaning using equations (29) and (30). In the case of conditional statements, the fuzzy set describing the fuzzy relation A → B will be transformed to a new fuzzy relation.

As an example: if x has a membership value equal to 0.85 in the fuzzy set A, then its membership values of:

x is A is true
x is A is false
x is A is fairly true
x is A is very false

are 0.85, 0.15, 0.92 and 0.22, respectively, which clearly is based on Figure 4-5.

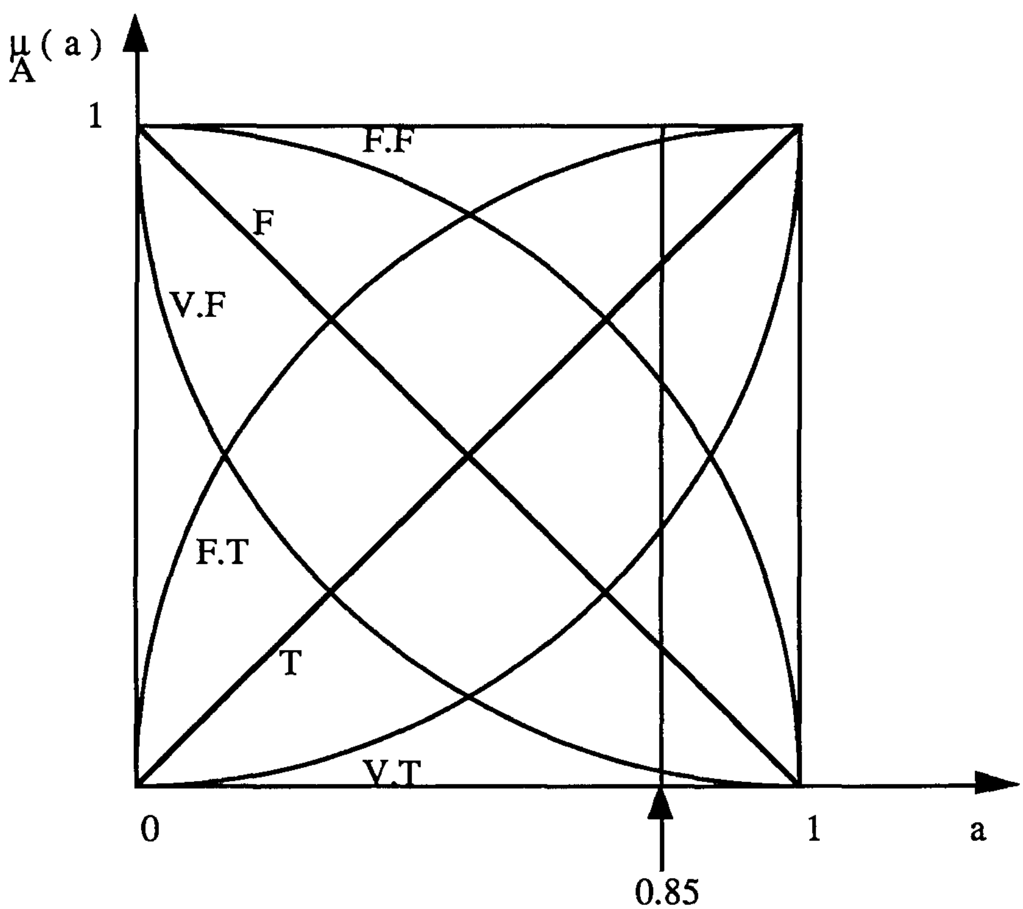

Figure 4-5. Truth value qualification graphs.

4.2.2 Decomposition of Compound Rules into Simple Canonical Forms

In a linguistically-expressed algorithmic or descriptive form given by a human expert, more compound rule structures could be present. As an example, consider an algorithm for the control of temperature in a room:

```
IF it is raining hard
        THEN close the window.
IF the room temperature is very hot,
        THEN
                IF the heat is on
                        THEN turn the heat lower
                ELSE
                        IF (the window is closed) AND (the air
                                conditioner is off)

        AND (it is not raining hard)
                        THEN open the window
                ELSE
                        IF (the window is closed)
                        AND (the air conditioner is on)
                                THEN open the window.
.
.
.
IF "the temperature is cold" is fairly true
        THEN
                IF the air conditioner is on
                        THEN turn the air conditioner off.
etc.
```

4.2.3 Basic Linguistic Terms

In general, a value of a linguistic variable is a composite term which is a concatenation of atomic terms. These atomic terms may be divided into four categories:

i) primary terms which are labels of specified fuzzy subsets of the universe of discourse (e.g., hot, cold, hard, lower, etc., in the preceding example).

ii) The negation NOT and connectives "AND" and "OR."

iii) Hedges, such as "very," "much," "slightly," "more or less," etc.

iv) Markers, such as parentheses.

The primary as well as composite terms may also be followed by linguistic variable labeled likelihood, such as "likely," "very likely," "highly likely," "unlikely," etc. In addition, primary and composite terms might also be modified semantically by truth qualification statements such as "true," "fairly true," "very true," "false," fairly false," and "very false." Likelihood labels are based on probability. The primary terms, as well as the rules, may also be restricted by the linguistic variable labeled certainty, such as "indefinite," "unknown," and "definite." The conditional rules might as well be simple or compound. Compound conditional rules may be in the form of nested IF-THEN rules or rules with linguistic terms such as "unless" or "else."

Figure 4-4 depicts the basic canonical form of FRBES that we deal with. Based on basic properties and operations defined for fuzzy sets, it can be shown that any compound rule structure may be decomposed and reduced to a number of simple canonical rules as given in Figure 4-4. These rules are based on natural language representations and models which are themselves based on fuzzy sets and fuzzy logic. The following illustrates a number of the most common techniques for decomposition of compound linguistic rules.

4.2.4 Primary Terms Preceded by Linguistic Hedges

In general, the linguistic hedges such as "very," "more or less," "slightly," "sort of," "more than," "less than," "essentially," etc., whenever operating on a primary fuzzy term, are equivalent to a specified nonlinear transformation of the membership function of the primary term into a new membership function of the composite term.

Consider a universe of discourse on the set of integer numbers in the interval [1, 5]. The fuzzy subset "small" may be expressed by means of membership function:

$$\text{"small"} = \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5}$$

and the fuzzy subset "large" may be defined as

$$\text{"large"} = \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5}$$

"Very small" could then be defined as a concentration of "small," or

$$\text{"very small"} = (\text{small})^2$$

$$= \int \frac{[m_{small}(x)]^2}{x}$$

$$= \frac{1}{1} + \frac{0.64}{2} + \frac{0.36}{3} + \frac{0.16}{4} + \frac{0.04}{5}$$

and in the same manner:

"very, very small" = $(\text{small})^4 = (\text{very small})^2$

$$= \int \frac{[\mu_{\text{very small}}(x)]^2}{x}$$

$$= \frac{1}{1} + \frac{0.4}{2} + \frac{0.1}{3}$$

where small terms have been neglected.

Based on the definition of fuzzy complement operation, "not very small" may be defined as:

$$= \int \frac{1 - \mu_{\text{very small}}(x)}{x}$$

$$\cong \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5}$$

As a more complicated example, consider the composite term

x = not very small and not very large, where "very large" is defined as a concatenation of "very" and "large," as:

very large = $(\text{large})^2$

$$= \frac{0.04}{1} + \frac{0.16}{2} + \frac{0.36}{3} + \frac{0.64}{4} + \frac{1}{5}$$

and "not very large" as:

not very large = NOT (very large)

$$\cong \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4}$$

Based on fuzzy intersection operation for "AND," we define the fuzzy subset "not very small and not very large" as:

not very small and not very large
= (not very small) $\cap$ (not very large)

$$= \left(\frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{0.96}{5}\right) \cap \left(\frac{1}{1} + \frac{0.9}{2} + \frac{0.6}{3} + \frac{0.4}{4}\right)$$

$$= \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.4}{4}$$

Here are some other linguistic hedge operations:

$$\text{"plus A"} \triangleq A^{1.25}$$

$$\text{"minus A"} \triangleq A^{0.75}$$

$$\text{"more than A"} \triangleq \begin{cases} 0, \text{ for } x \text{ £ } x_0 \\ 1 - \mu_A(x), \text{ for } x > x_0 \end{cases}$$

where X_0 is the element attaining the maximum grade of membership in A, and likewise:

$$\text{"less than A"} \triangleq \begin{cases} 0, \text{ for } x \geq x_0 \\ 1 - \mu_A(x), \text{ for } x < x_0 \end{cases}$$

and,

"plus plus A" = minus very A

"highly A" = minus very very A

and equivalently:

"highly A" = plus plus very A

4.2.5 Likelihood Linguistic Labels

An example of a different nature is provided by the values of a linguistic variable labeled likelihood. In this case, we assume that the universe of discourse is given by

$$U = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$$

in which the elements of U represent probabilities.

Suppose that we wish to compute the meaning of the value

x = highly unlikely

in which "highly" is defined as

highly = minus very very

and

unlikely = not likely.

With the meaning of the primary term "likely" given by

$$\text{likely} = \frac{1}{1} + \frac{1}{0.9} + \frac{1}{0.8} + \frac{0.8}{0.7} + \frac{0.6}{0.6} + \frac{0.5}{0.5} + \frac{0.3}{0.4} + \frac{0.2}{0.3}$$

we obtain

$$\text{unlikely} = \frac{1}{0} + \frac{1}{0.1} + \frac{1}{0.2} + \frac{0.8}{0.3} + \frac{0.7}{0.4} + \frac{0.5}{0.5} + \frac{0.4}{0.6} + \frac{0.2}{0.7}$$

and hence

$$\text{very very unlikely} = (\text{unlikely})^4$$

$$\cong \frac{1}{0} + \frac{1}{0.1} + \frac{1}{0.2} + \frac{0.4}{0.3} + \frac{0.2}{0.4}.$$

Finally,

$$\text{highly unlikely} = \text{minus very very unlikely}$$

$$\cong \left(\frac{1}{0} + \frac{1}{0.1} + \frac{1}{0.2} + \frac{0.4}{0.3} + \frac{0.2}{0.4}\right)^{0.75}$$

$$\cong \frac{1}{0} + \frac{1}{0.1} + \frac{1}{0.2} + \frac{0.5}{0.3} + \frac{0.3}{0.4}.$$

The primary terms "yes," "maybe," and "no" may also be assigned meanings based on membership functions given by "very very likely," "likely," and "very very unlikely."

It is also noted that the atomic term "anything" is equivalent to the universe of discourse and given by

$$\mu_{\text{anything}}(x) = 1; \text{ for all } x \in X$$

It should be noted that in computing the meaning of the composite terms in the preceding examples we have made implicit use of the usual precedence rules governing the evaluation of Boolean expressions. With the addition of hedges, these precedence rules may be expressed as follows:

Precedence	Operation
First	hedges, not
Second	and
Third	or

As usual, parentheses may be used to change the precedence order and ambiguities may be resolved by the use of association to the right. Thus "plus very minus very tall" should be interpreted as

plus (very(minus(very(tall))))

In summary, every atomic term as well as every composite term has a syntax represented by its linguistic label and a semantics or meaning which is given by a membership function. The concept of membership function gives an elastic and flexible meaning to a linguistic term. Based on this elasticity and flexibility, it is possible to incorporate subjectivity and bias into the meaning of a linguistic term. This is one of the most important benefits of fuzzy mathematics introduction into the modeling of linguistic propositions. Another important benefit is the straightforward and simple computational tools for processing and manipulating the meaning of composite linguistic terms and compound linguistic propositions. These capabilities and tools of fuzzy mathematics are used to encode and automate human expert knowledge, which is often expressed by natural language propositions, for implementation on computers.

The subjectivity that exists in fuzzy modeling is a blessing rather than a curse. The subjectivity present in the definition of terms is balanced by the subjectivity in the conditional rules used by an expert. As far as the set of variables and their meanings are compatible and consistent with the set of conditional rules used, the overall outcome turns out to be objective and meaningful. Fuzzy mathematical tools and the calculus of fuzzy IF-THEN rules opened the way for automation and implementation of a huge body of human expert knowledge. It provided a means of sharing, communicating, or transferring of human expert subjective knowledge of systems and processes.

4.2.6 Truth Qualification Rate

Let τ be a fuzzy truth value, for example, "very true," "true," "fairly true," "fairly false," "false," etc. Such a truth value may be regarded as a fuzzy subject of the unit interval that is characterized by a membership function $x_t : [0, 1] \to [0 \times 1]$. A truth qualification proposition can be expressed as "x is A is τ." The translation rule for such propositions can be given by:

$$\text{x is A is } \tau \to \mu = \mu_{X^+} \tag{29}$$

$$\mu_{X^+}(x) = x_\tau[\mu_X(x)] \tag{30}$$

Figure 4-5 illustrates the meaning of true (T), fairly true (FT), very true (VT), false (F), fairly false (FF), and very false (VF). The fuzzy assignment statements such as:

x is A is very true

y is B is very false

and fuzzy conditional statements such as

4.2.7 Multiple Antecedents or Consequents Connected by "AND" or "OR"

The following discussion covers a number of general compound rules structures with more than one antecedent or consequent and illustrates methods for transforming these compound rules into simple canonical forms.

i) IF x is A^1 and A^2 . . . and A^L THEN y is B^s
Assuming a new fuzzy subset A^s as

$$A^s = A^1 \cap A^2 \cap \cdots \cap A^L$$

expressed by means of membership function

$$\mu_{A^s}(x) = \text{Min}\,[\mu_{A^1}(x), \mu_{A^2}(x), \cdots, \mu_{A^L}(x)]$$

based on the definition of fuzzy intersection operation, the compound rule may be rewritten as

IF A^s THEN B^s

ii) IF x is A^1 OR x is A^2 . . . OR x is A^L THEN y is B^s
could be rewritten as

IF x is A^s THEN y is B^s

where the fuzzy set A^s is defined as

$$A^s = A^1 \cup A^2 \cup \cdots \cup A^L$$

$$\mu_{A^s}(x) = \text{Max}\,[\mu_{A^1}(x), \mu_{A^2}(x), \cdots, \mu_{A^L}(x)]$$

which clearly is based on the definition of fuzzy union operation.

4.2.8 Conditional Statements With "ELSE," "UNLESS"

i) IF A^1 THEN (B^1 ELSE B^2) may be decomposed into two simple canonical form rules connected by "OR":

IF A^1 THEN B^1
OR
IF NOT A^1 THEN B^2

ii) IF A^1 (THEN B^1) UNLESS A^2 could be decomposed as

IF $\mathbf{A}^1$ THEN $\mathbf{B}^1$

OR

IF $\mathbf{A}^2$ THEN NOT $\mathbf{B}^1$

iii) IF $\mathbf{A}^1$ THEN ($\mathbf{B}^1$ ELSE IF $\mathbf{A}^2$ THEN ($\mathbf{B}^2$)) may be put into the following form:

IF $\mathbf{A}^1$ THEN $\mathbf{B}^1$

OR

IF NOT $\mathbf{A}^1$ AND $\mathbf{A}^2$ THEN $\mathbf{B}^2$

4.2.9 Nested IF-THEN Rules

i) IF $\mathbf{A}^1$ THEN (IF $\mathbf{A}^2$ THEN ($\mathbf{B}^1$)) may be put into the form:

IF $\mathbf{A}^1$ AND $\mathbf{A}^2$ THEN $\mathbf{B}^1$

ii) IF $\mathbf{A}^1$ THEN (IF $\mathbf{A}^2$ THEN (IF . . . ($\mathbf{B}^1$) . . .) is the general case which is rewritten as:

IF $\mathbf{A}^1$ AND $\mathbf{A}^2$ AND . . . THEN $\mathbf{B}^1$

iii) IF $\mathbf{A}^1$

THEN

IF $\mathbf{A}^2$

THEN

IF $\mathbf{A}^3$

THEN $\mathbf{B}^1$ ELSE $\mathbf{B}^2$

ELSE $\mathbf{B}^3$

may be transformed to:

IF $\mathbf{A}^1$ AND $\mathbf{A}^2$ AND $\mathbf{A}^3$ THEN $\mathbf{B}^1$

OR

IF $\mathbf{A}^1$ AND $\mathbf{A}^2$ AND NOT $\mathbf{A}^3$ THEN $\mathbf{B}^2$

OR

IF NOT $\mathbf{A}^1$ THEN $\mathbf{B}^3$

Using parentheses this compound rule might also be written as:

IF $\mathbf{A}^1$ (THEN (IF $\mathbf{A}^2$ (THEN IF $\mathbf{A}^3$ THEN ($\mathbf{B}^1$ ELSE $\mathbf{B}^2$)) ELSE $\mathbf{B}^3$)

As another example of nested IF-THEN rule structures, consider the following compound rule:

IF $\mathbf{A}^1$
THEN $\mathbf{B}^1$ AND $\mathbf{B}^2$
IF $\mathbf{A}^2$
THEN $\mathbf{B}^3$
IF $\mathbf{A}^3$
THEN $\mathbf{B}^4$

which may be put into the following form:

IF $\mathbf{A}^1$ THEN $\mathbf{B}^1$

IF $\mathbf{A}^1$ THEN $\mathbf{B}^2$

IF $\mathbf{A}^1$ AND $\mathbf{A}^2$ THEN $\mathbf{B}^3$

IF $\mathbf{A}^1$ AND $\mathbf{A}^2$ AND $\mathbf{A}^3$ THEN $\mathbf{B}^4$

4.2.10 Canonical FRBES Forms for Multiple-input Multiple-output Physical Systems

In the following we discuss two more common canonical fuzzy rule-based expert systems.

i) For the n-output and m-output system described earlier, if it can be assumed that input fuzzy sets $\mathbf{A}^1, \mathbf{A}^2, \ldots,$ are comprised of n non-interactive fuzzy sets defined on universes x_i for $i = 1, 2, \ldots, n$ and also that the output fuzzy sets $\mathbf{B}^1, \mathbf{B}^2, \ldots,$ are composed of m non-interactive fuzzy sets defined as universes y_j for $j = 1, 2, \ldots, m$, then the canonical rule-based expert system given in Figure 4-4 could be put into the canonical form shown in Figure 4-4.

R^1: IF x_1 is $\mathbf{A}_1{}^1$ AND x_2 is $\mathbf{A}_2{}^1$ ... AND x_n is $\mathbf{A}_n{}^1$
THEN y_1 is $\mathbf{B}_1{}^1$ AND y_2 is $\mathbf{B}_2{}^1$... AND y_m is $\mathbf{B}_m{}^1$

R^2: IF x_1 is $\mathbf{A}_1{}^2$ AND x_2 is $\mathbf{A}_2{}^2$ ... AND x_n is $\mathbf{A}_n{}^2$
THEN y_1 is $\mathbf{B}_1{}^2$ AND y_2 is $\mathbf{B}_2{}^2$ ... AND y_m is $\mathbf{B}_m{}^2$.

.
.
.

R^r: IF x_1 is $\mathbf{A}_1{}^r$ AND x_2 is $\mathbf{A}_2{}^r$ ... AND x_n is $\mathbf{A}_n{}^r$
THEN y_1 is $\mathbf{B}_1{}^r$ AND y_2 is $\mathbf{B}_2{}^r$ ... AND y_m is $\mathbf{B}_m{}^r$.

Figure 4-6. Canonical FRBES for multi-input multi-output system.

In Figure 4-6, the fuzzy sets A_i^k (i = 1, 2, . . . n and k = 1, 2, . . . , r) and fuzzy sets B_j^k (j = 1, 2, . . . , m and k = 1, 2, . . . , r) are expressed as

$$A_i^K \in F(x_i);\ \mu_{A_i^K}(x_i) : x_i \rightarrow [0, 1] \tag{31}$$

and

$$B_j^k \in F(y_j);\ \mu_{B_j^k}(y_j) : y_j \rightarrow [0, 1] \tag{32}$$

ii) Systems with more than one output, e.g., *m* outputs, may be described by a collection of FRBES, each rule set dealing with only one output. Hence, the canonical FRBES describing an *n*-input and single-output system, with non-interactive input fuzzy sets, could be put in the form given in Figure 4-7.

R^1: IF x_1 is A_1^1 and x_2 is A_2^1 . . . AND x_n is A_n^1
THEN y is B^1

R^2: IF x_1 is A_1^2 and x_2 is A_2^2 . . . AND x_n is A_n^2
THEN y is B^2.

.
.
.

R^r: IF x_1 is A_1^r and x_2 is A_2^r . . . AND x_n is A_n^r
THEN y is B^r.

Figure 4-7. FRBES for an *n*-input and single-output system.

The fuzzy rule-based expert system given in Figure 4-7 is the most common canonical form in system identification and control problems.

4.3 SYSTEMS OF FUZZY RELATIONAL EQUATIONS

Fuzzy conditional proposition IF **A** THEN **B** is known as the *generalized modus ponen.* There are numerous techniques for obtaining a fuzzy relation R which will represent the generalized modus ponen in the form of a fuzzy relational equation given by

$$B = A \bullet R$$

where "•" represents a general method for composition of fuzzy relations. Some common techniques for obtaining the fuzzy relation R from the expert rules as well as other forms of knowledge regarding the system will be discussed in the next section.

Using fuzzy relational equations corresponding to each single rule, the FRBES given in Figure 4-5 may be described in the following form:

$$
\begin{array}{ll}
R^1: & y^1 = x \bullet R^1 \\
R^2: & y^2 = x \bullet R^2 \\
\cdot & \\
\cdot & \\
\cdot & \\
R^r: & y^r = x \bullet R^r
\end{array}
$$

Figure 4-8. System of fuzzy relational equations.

where y^k, for $k = 1, 2, \ldots, r$, is the output of the system contributed by the kth rule, and defined as:

$$y^k \in F(Y);\ \mu_{y^k}(y) : Y \rightarrow [0, 1],\ k = 1, 2, \cdots, r \tag{33}$$

and x is the input fuzzy set to the system. Both x and y^k ($k = 1, 2, \ldots, r$) are written as unary fuzzy relations, of dimensions 1 x n and 1 x m, respectively. The unary relations, in this case, are actually similarity relations between the elements of the fuzzy set and a most typical or prototype element with membership value equal to unity. As an example, for the case where x is defined as:

$$x = \frac{0}{-3} + \frac{0.5}{-2} + \frac{0.8}{-1} + \frac{0.1}{0} + \frac{0.8}{1} + \frac{0.5}{2} + \frac{0}{3}$$

on a universe of discourse

$$U = \{-3, -2, -1, 0, +1, +2, +3\}$$

it may be put in the form of a unary fuzzy relation as:

$$x = [0 \quad 0.5 \quad 0.8 \quad 0.1 \quad 0.8 \quad 0.5 \quad 0\,].$$

Similarly, the case of a crisp input $X = \overline{X} = -1$, or a fuzzy singleton, will be

$$x = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0].$$

The system of fuzzy relational equations given in Figure 4-8 describes a general fuzzy system. The system could be, equivalently, described by a crisp set of r fuzzy relations as:

$$R = \{R^1, R^2, \cdots, R^r\}$$

4.3.1 Aggregation of Rules

The problem of obtaining the overall output fuzzy set y from individual outputs contributed by individual rules or fuzzy relational equation is known as aggregation of rules problem.

Two simple extreme cases exist:

i) *Conjunctive system of rules*: In the case of a system of rules which have to be jointly satisfied, the rules are connected by "and" connectives. In this case the aggregated output is found by the fuzzy intersection of all individual rule outputs as:

$$y = (y^1) \text{ AND } (y^2) \text{ AND } \cdots \text{ AND } (y^r)$$

or,

$$y = (y^1) \cap (y^2) \cap \cdots \cap (y^r) \tag{34}$$

which is defined by the membership function

$$m_y(y) = \text{Min}\,[\mu_{y^1}(y), \mu_{y^2}(y), \cdots, \mu_{y^r}(y)], \text{ for } y \in Y \tag{35}$$

The fuzzy system, in the case of system of conjunctive rules, could be described by a single aggregated fuzzy relational equation:

$$y = (x \bullet R^1) \text{ AND } (x \bullet R^2) \text{ AND} \ldots \text{AND } (x \bullet R^r),$$

and equivalently:

$$y = x \bullet (R^1 \text{ AND } R^2 \ldots \text{AND } R^r)$$

and finally

$$y = x \bullet R \tag{36}$$

where R is defined as

$$R = R^1 \cap R^2 \cap \cdots \cap R^r \tag{37}$$

The aggregated fuzzy relation R is called the *fuzzy system transfer relation.* For the case of a system with *n* non-interactive fuzzy inputs and single output described in Figure 4-7, the fuzzy relational equation (36) will be written in the form of

$$y = x_1 \bullet x_2 \bullet \ldots \bullet x_n \bullet R \tag{38}$$

ii) *Disjunctive system of rules*: For the case of disjunctive system of rules where the satisfaction of at least one rule is required, the rules are connected by the "OR" connectives. In this case the aggregated output is found by the fuzzy union of all individual rule contributions, as

$$\mathbf{y} = (\mathbf{y}^1) \text{ OR } (\mathbf{y}^2) \text{ OR} \cdots \text{ OR } (\mathbf{y}^r)$$

or:

$$\mathbf{y} = (\mathbf{y}^1) \cup (\mathbf{y}^2) \cup \cdots \cup (\mathbf{y}^r) \tag{39}$$

which is defined by the membership function

$$\mu_y(y) = \text{Max}\, [\mu_{y^1}(y), \mu_{y^2}(y), \cdots, \mu_{y^r}(y)], \text{ for } y \in Y \tag{40}$$

The fuzzy system, in the case of system of disjunctive rules, could be described by a single aggregated fuzzy relational equation as:

$$y = (x \bullet R^1) \text{ OR } (x \bullet R^2) \text{ OR} \ldots \text{OR } (x \bullet R^r)$$

and equivalently

$$y = x \bullet (R^1 \text{ OR } R^2 \text{ OR} \ldots \text{OR } R^r)$$

and finally

$$y = x \bullet R \tag{41}$$

where R is defined as

$$R = R^1 \cup R^2 \cup \cdots \cup R^r \tag{42}$$

The aggregated fuzzy relation, i.e., R, is called the fuzzy system transfer relation.

For the case of a system with *n* non-interactive fuzzy inputs and single output, described as in Figure 4-7, the fuzzy relational equation (41) will be written in the following form:

$$y = x_1 \bullet x_2 \bullet \ldots \bullet x_n \bullet R \tag{43}$$

the same as equation (37).

There is an interesting interpretation for the way that the fuzzy system transfer relation R is described. The fuzzy system transfer relation R is given by a crisp set of fuzzy relations representing the rules:

$$R = \{R^1, R^2, \cdots, R^r\}$$

and in its aggregated form is given by equations (37) and (42). Each individual relation R^k, for k = 1, 2, . . . , r, represents a fuzzy data point in the Cartesian product space X • Y. The fuzzy system transfer relation, i.e., R, is being approximated by r fuzzy input-output data point R^k, for k = 1, 2, . . . , and r. It is analogous to the case where a crisp fraction y = f(x) is approximately described by r numerical input-output values. Each $A^k \rightarrow B^k$ implication gives a fuzzy data point for approximating the overall fuzzy system transfer relation R. The more the number of these fuzzy data points with overlapping supports, a better approximation of the system input-output mapping is obtained.

The fuzzy system transfer relation R, described in this section, is a parallel computational process. The output of the fuzzy system, described by y = x. **R**, is the aggregated outcome of r parallel fuzzy relations R1, R2, . . . , Rr. Figure 4-9 illustrates this parallel operation where " a" represents a general aggregation method such as union (Max.) or intersection (Min.).

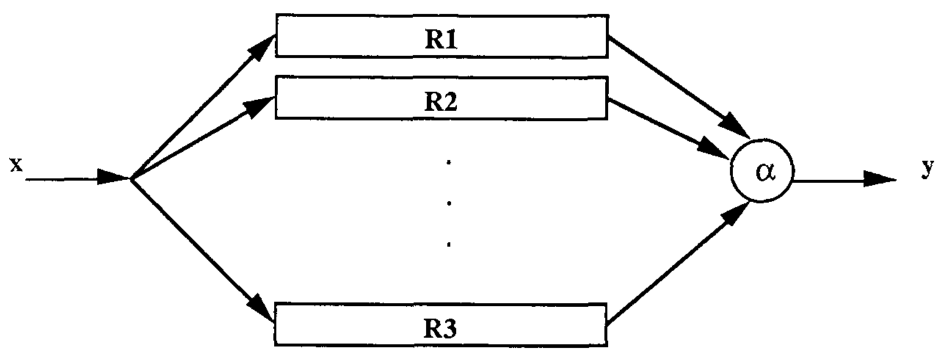

Figure 4-9. Parallel computation in FRBES models.

Fuzzy systems described by fuzzy transfer relations could be combined in series or in parallel. The corresponding equivalent fuzzy system, in the case of series connection of the fuzzy systems, could be found using composition of fuzzy relations. Figure 4-10 shows the series connection of two fuzzy systems and their corresponding equivalent system.

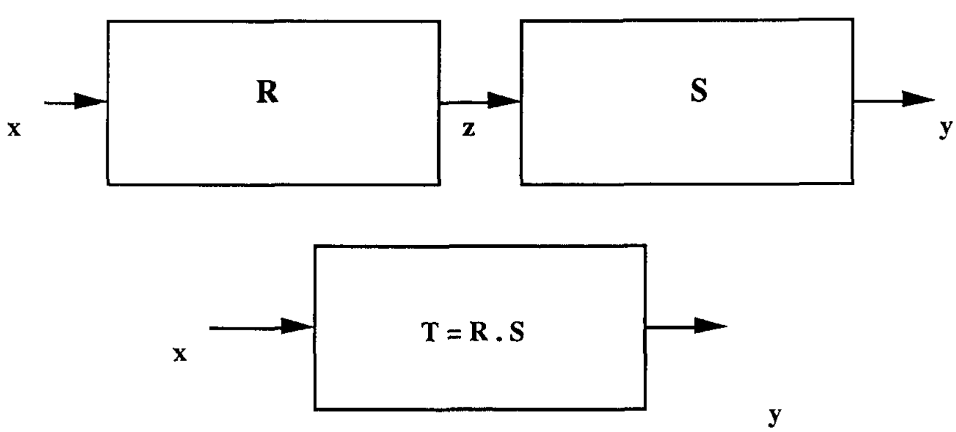

Figure 4-10. Series connection of two fuzzy systems.

Chapter 4. Fuzzy Rule-Based Expert Systems I. Fuzzy Logic and Control: Software and Hardware Applications, Volume 2, ISBN: 9780132442183
Prepared for safaribooksonline@lorenz.kiefner.de, Lorenz Kiefner

Fuzzy systems in parallel connection could also be aggregated into a single equivalent fuzzy system by using an appropriate aggregation method. Figure 4-11 shows two general cases,

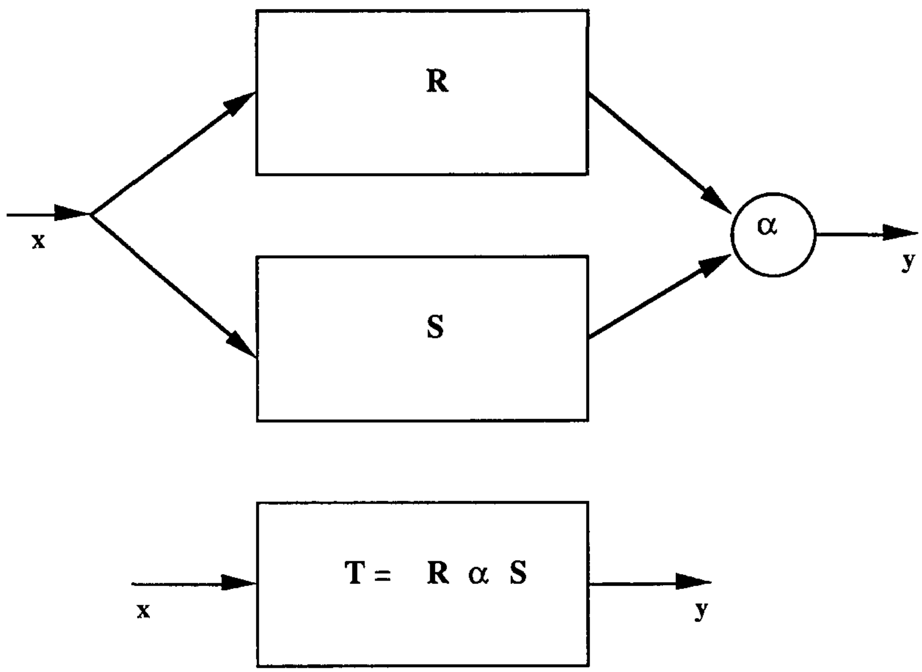

Figure 4-11. Parallel connection of fuzzy systems.

4.3.2 Obtaining the Fuzzy Relation R^k

The set of fuzzy restrictions, describing a real world dynamic system or process, is, actually, the perceptual or cognitive level of understanding and describing it. These fuzzy restrictions could be found based on one of the following methods:

i) Human expert knowledge expressed in the form of linguistic IF . . . THEN. . . rules. These rules can be extracted from interviews conducted or protocols obtained when gathering the human expert or operator knowledge regarding identification or control of an existing real world process or system.

ii) Common sense and intuitive knowledge of the design engineer about the real world process or system under investigation.

iii) Using the general physical principles and laws governing the dynamics of the process or the system under study.

iv) Using pattern classification, clustering, and statistical analysis of some available set of input-output numerical results obtained from measurements carried on an existing system or process.

v) Starting with some available closed form analytical equations which describe the process or the system and using the Zadeh's extension principles to come up with a set of fuzzy restrictions on the input-output mapping of the process or the system.

The following describes two of the most common techniques for obtaining the fuzzy relation R representing the generalized modus ponen IF **A** THEN **B**. Based on either one of these techniques a fuzzy relational equation may be found for each canonical rule of the fuzzy rule-based expert system given in Figures 4-4, 4-6, and 4-7.

i) *Fuzzy implications*: There are many different techniques for obtaining the fuzzy relation **R** based on the fuzzy sets of the IF-part and THEN-part of the fuzzy conditional proposition IF **A** THEN **B**. These are known as fuzzy implication relations. In the following we mention nine different techniques for obtaining the membership function values of fuzzy relation R defined on the Cartesian product space **X** x **Y**.

$$\mu_R(x, y) = \text{Max}\left\{\text{Min}\left[\mu_A(x), \mu_B(y)\right], 1 - \mu_A(x)\right\} \tag{44}$$

$$\mu_R(x, y) = \text{Max}\left\{\left[\mu_B(y), \left[1 - \mu_A(x)\right]\right]\right\} \tag{45}$$

$$\mu_R(x, y) = \text{Min}\left[\mu_A(x), \mu_B(y)\right] \tag{46}$$

$$\mu_R(x, y) = \text{Min}\left\{1, \left[1 - \mu_A(x) + \mu_B(y)\right]\right\} \tag{47}$$

$$\mu_R(x, y) = \text{Min}\left\{1, \left[\mu_A(x) + \mu_B(y)\right]\right\} \tag{48}$$

$$\mu_R(x, y) = \text{Min}\left\{1, \left[\frac{\mu_B(y)}{\mu_A(x)}\right]\right\} \tag{49}$$

$$\mu_R(x, y) = \text{Max}\left\{\mu_A(x) \bullet \mu_B(y), \left[1 - \mu_A(x)\right]\right\} \tag{50}$$

$$\mu_R(x, y) = \mu_A(x) \bullet \mu_B(y) \tag{51}$$

$$\mu_R(x, y) = \left[\mu_B(y)\right]^{\mu_A(x)} \tag{52}$$

$$\mu_R(x, y) = \begin{cases} \mu_B(y) \text{ for } \mu_B(y) < \mu_A(x) \\ 1 \text{ otherwise} \end{cases} \tag{53}$$

Equations (44) to (53) are valid for all values of $x \in X$ and $y \in Y$. For the case where the input universe of discourse is represented by p discrete elements and the output universe of discourse is represented by q discrete elements, then the fuzzy relation R will be a $p \times q$ matrix.

Equation (44) is known as the classical implication equation or also as *Zadeh's implication*.. For $\mu_B(y) < \mu_A(x)$ for all $(x, y) \in (x \bullet y)$, the equation given by (44) reduces to the one given by (45). Equation (46) is called *correlation-minimum* and known as *Mamdani's implication.* This implication may also be found by the fuzzy cross-product of sets **A** and **B**, i.e., R = **A** x **B**. For $\mu_A(x) \geq 0.5$ and $\mu_B(y) \geq 0.5$ Zadeh's implication reduces to Mamdani's implication. The implication defied by equation (47) is known as *Luckawics implication*.. The fuzzy implication relation defined by equation (48) is known as the *bounded sum implication*. Equations (50) and (51) describe two forms of *correlation-product* and are based on the notions of conditioning and reinforcement and also Hebbian type of learning in neuropsychology. This implication method is similar to the one used in artificial neural network computations. Equation (53) is known as *Gödel's implication.* or " α " implication. Equation (50) is suggested by the author and is equally valid for the crisp non-fuzzy cases. The choice of implication equation basically depends on the meaning behind the membership functions defined for the fuzzy sets **A** and **B** and also the mechanism by which the (If **A** THEN **B**) fact was learned in the first place.

ii) *Extension principle*: Let f be a mapping from $\mathbf{x}_1 \times \mathbf{x}_2 \times \ldots \times \mathbf{x}_n$ to the universe **y** such that $y = f(x_1, x_2, \ldots, x_n)$. The fuzzy extension principle allows us to induce from *n* fuzzy sets $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, a fuzzy set **B** on **y** through f such that

$$\mu_B(y) = \mathrm{Sup}_{(x_1, x_2, \cdots, x_n) = f^{-1}(y)} \left\{ \mathrm{Min}\left[\mu_{A_1}(x_1), \mu_{A_2}(x_1), \cdots, \mu_{A_r}(x_n)\right]\right\}$$

$$\mu_B(y) = 0 \text{ if } f^{-1}(y) = \varnothing$$

where $f^{-1}(y)$ is the inverse image of y, and $\mu_B(y)$ is the largest among the membership values $\mu_{A_1 \times A_2 \times \cdots \times A_n}(x_1, x_2, \cdots, x_n)$ of the realization of y using n-tuples $(x_1, x_2, \ldots, x_n)$.

For the cases where the system could be described by a closed form function $y = f(x_1, x_2, \ldots, x_n)$, each application of the extension principle to the system will result in a canonical fuzzy rule in the form of

IF x_1 is A_1 and x_2 is A_2 . . . and x_n is A_n

THEN y is **B**

Therefore, r fuzzy rules may be found by r times application of the extension principle to the system under consideration. Then the implication relations given by equations (44) to (53) can be used to derive the fuzzy relations representing the rules.

There also exist other techniques for obtaining the fuzzy relations which are based on learning and adaptive algorithms. These techniques start from a set of numerical data derived from observation of the input-output mapping going on in the system. The numerical input-output data is used to come up with linguistic rules or fuzzy relations which correctly fit the available data.

4.3.3 Composition of Fuzzy Relations

Max-Min and Max-Prod methods of composition of fuzzy relation are the two most commonly used techniques for the solution of fuzzy relational equations. In addition, Max-Min, Max-Max, Min-Min, (p, q) composition, Sum-Prod, and Max-Ave techniques are mentioned in literature. Each method of composition of fuzzy relations reflects a special inference machine and has its own significance and applications. Max-Min method is the one used by Zadeh in his approximate reasoning based on linguistic IF-THEN rules. It is claimed that this method of composition of fuzzy relations correctly reflects the approximate and interpolative reasoning used by humans when using natural language propositions for deductive reasoning.

Approximate reasoning involves the following general situation in deductive reasoning:

IF **A** THEN **B**

IF **A**^{1} THEN ?

The fuzzy relation describing the antecedent **A** → **B** is used to compute the consequent resulting from the application of **A**^{1} as in the following:

$$\mathbf{B} = \mathbf{A} \bullet \mathbf{R}$$

$$\mathbf{B}^1 = \mathbf{A}^1 \bullet \mathbf{R}$$

In the following we describe eight different methods for composition of fuzzy relations.

i) Max-Min composition

$$y^k = x \bullet R^k$$

$$\mu_{y^k}(y) = \underset{x \in X}{\text{Max}} \left\{ \text{Min}\left[\mu_x(x), \mu_{R^k}(x, y)\right]\right\} \tag{54}$$

ii) Max-Prod composition

$$y^k = x * R^k$$

$$\mu_{y^k}(y) = \underset{x \in X}{\text{Max}} \left[\mu_x(x) \cdot \mu_{R^k}(x, y)\right] \tag{55}$$

iii) Min-Max composition

$$y^k = x \dagger R^k$$

$$\mu_{y^k}(y) = \underset{x \in X}{\mathrm{Min}} \{\mathrm{Max}[\mu_x(x), \mu_{R^k}(x, y)]\} \tag{56}$$

iv) Max-Max composition

$$y^k = x \circ R^k$$

$$\mu_{y^k}(y) = \underset{x \in X}{\mathrm{Max}} \{\mathrm{Max}[\mu_x(x), \mu_{R^k}(x, y)]\} \tag{57}$$

v) Min-Min composition

$$y^k = x \Delta R^k$$

$$\mu_{y^k}(y) = \underset{x \in X}{\mathrm{Min}} \{\mathrm{Min}[\mu_x(x), \mu_{R^k}(x, y)]\} \tag{58}$$

vi) (p, q) composition

$$y^k = x \bullet^{pq} R^k$$

$$\mu_{y^k}(y) = \underset{x \in X}{\mathrm{Max}}(p) \{\mathrm{Min}(q)[\mu_x(x), \mu_{R^k}(x, y)]\} \tag{59}$$

where

$$\underset{x \in X}{\mathrm{Max}}(p)\ a(x) = \inf._{(x_1, x_2, \cdots, x_n) \in x} \left\{1, [[a(x_1)]^p + [a(x_2)]^p + \cdots + [a(x_n)]^p]^{\frac{1}{p}}\right\}$$

and

$$\mathrm{Min}_q [a(x), b(x)] = 1 - \min\left\{[1, [1 - a(x)]^q + [1 - b(x)]^q]^{\frac{1}{q}}\right\}$$

vii) Sum-Prod composition

$$y^k = x \times R^k$$

$$\mu_{y^k}(y) = f\left\{\sum_{x \in X}[\mu_x(x) \cdot \mu_{R^k}(x, y)]\right\} \tag{60}$$

where f(•) is a logistic function that limits the value of the function within the interval [0, 1]. This is a composition method commonly used in the artificial neural networks for mapping between parallel layers in a multi-layer network.

viii) Max-Ave composition

$$y^k = x \cdot_{av.} R^k$$

$$\mu_{y^k}(y) = \tfrac{1}{2} \max_{x \in X} [\mu_x(x) + \mu_{R^k}(x, y)] \tag{61}$$

4.4 SOLUTION OF A SYSTEM OF FUZZY RELATIONAL EQUATIONS

Once a fuzzy rule-based expert system is put in the canonical form given in Figure 4-4, it is always possible to describe the system under consideration by a system of relational equations, as shown in Figure 4-8. In the following, the general solution of a system of fuzzy relational equations is discussed. The general solution is derived based on two most common techniques for composition of fuzzy relations, i.e., the Max-Min and Max. Prod techniques given in equations (54) and (55).

i) Max-Min Method

For a system of disjunctive fuzzy relations, i.e., connected by "or" or "else," the aggregated output y is found based on equations (40) and (55) as in the following:

$$\mu_y(y) = \max_k \left\{ \max_{x \in X} \left\{ \min[\mu_x(x), \mu_{R^k}(x, y)] \right\} \right\} \tag{62}$$

where $\mu_y(y)$ is the fuzzy membership function describing the overall output response to the fuzzy input x. For the case of a system with n non-interactive fuzzy inputs, the aggregated output will be in the following form:

$$\mu_y(y) = \max_k \left\{ \max_{x \in X} \left\{ \min[\mu_{x_1}(x_1), \mu_{x_2}(x_2), \cdots, \mu_{x_n}(x_n), \mu_{R^n}(x_1, x_2, \cdots, x_n, y)] \right\} \right\} \tag{63}$$

where $x = [x_1, x_2, \ldots, x_n]'$ is the vector of n non-interactive inputs to the fuzzy system.

For a conjunctive set of fuzzy relations, equations, i.e., connected by "and," equations (35) and (55) are used to find the aggregated output.

ii) Max. Prod Method

For a system of conjunctive fuzzy relational equations, the aggregated output y is found based on equations (40) and (56) as illustrated in the following:

$$\mu_y(y) = \max_k \left\{ \max_{x \in X} [\mu_x(x) \bullet \mu_{R^k}(x, y)] \right\} \tag{64}$$

where $\mu_y(y)$ is the fuzzy membership function describing the overall output response to the fuzzy input x.

For the case of a system with n non-interactive fuzzy inputs, the aggregated output will be in the following form:

$$\mu_y(y) = \underset{k}{\text{Max}} \left\{ \underset{x \in X}{\text{Max}} \left[\mu_{x_1}(x_1) \bullet \mu_{x_2}(x_2) \bullet \cdots \bullet \mu_{x_n}(x_n) \bullet \mu_{R^k}(x_1, x_2, \cdots, x_n, y)\right]\right\} \tag{65}$$

and, also, sometimes given in the form:

$$\mu_y(y) = \underset{k}{\text{Max}} \left\{ \underset{x \in X}{\text{Max}} [\text{Min}\left[\mu_{x_1}(x_1),\ \mu_{x_2}(x_2),\ \cdots,\ \mu_{x_n}(x_n)\right] \bullet \mu_{R^k}(x_1, x_2, \cdots, x_n, y)]\right\} \tag{66}$$

For a disjunctive set of fuzzy relations, equations (35) and (56) are used to find the aggregated output.

4.5 CONCLUSION

As this chapter shows, a set of fuzzy restrictions stated as fuzzy conditional propositions can be put into simple canonical rule-sets. Canonical fuzzy rule-based expert systems may also be represented by a system of fuzzy relational equations that are either conjunctively or disjunctively connected. The chapter presented some techniques for the solution of a system of fuzzy relational equations.

REFERENCES

Dubois, D. and Prade (1980a), *Fuzzy Sets and Systems: Theory and Applications.* Academic Press, New York.

Gupta, M.M., R.K. Ragade, and R.R. Yager, eds. (1979), *Advances in Fuzzy Set Theory and Applications*. North-Holland, New York.

Kaufmann, A. and M.M. Gupta (1985), *Introduction to Fuzzy Arithmetic: Theory and Applications*. Van Nostrand Reinhold, New York.

Klir, G.J. and T.A. Folger (1988), *Fuzzy Sets, Uncertainty, and Information*. Prentice Hall, New Jersey.

Mamdani, E.H. and R.R. Gaines, eds. (1981), *Fuzzy Reasoning and Its Applications.* Academic Press, London.

Pedrycs, W. (1883c), "Some applicational aspects of fuzzy relational equations in systems analysis." *Inern J. of General Systems*, 9 125-132.

Sugeno, M. ed. (1985a), *Industrial Application of Fuzzy Control*. North-Holland, New York.

Zadeh, L.A. (1973), "Outline of a new approach to the analysis of complex systems and decision processes." IEEE *Trans. on Systems, Man, and Cybernetics*, SMC-1, pp. 28-44.