Differentiation of Vectors - Polar

Position Vectors

$$\vec{r} = r\hat{e}_r = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

Velocity Vector:

$$\vec{v} = \frac{d}{dt}(r\hat{e}_r) = \frac{d}{dt}\left[r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\right]$$

$$\therefore\ \vec{v} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \dot{r}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + r\dot{\theta}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

which means $\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

And $\dot{\hat{e}}_r = \dot{\theta}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$$= \dot{\theta}\hat{e}_\theta$$

and $\dot{\hat{e}}_\theta = \frac{d}{dt}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$$= -\dot{\theta}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= -\dot{\theta}\hat{e}_r$$

Summary: $\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$

$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$

Acceleration Vector:

$$a = \frac{dv}{dt} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\hat{e}}_r)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\hat{e}}_r + r\ddot{\hat{e}}_r$$

$$= \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta + r\frac{d}{dt}(\dot{\theta}\hat{e}_\theta)$$

$$= \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$

$$\therefore \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

Summary:

$$\boxed{\begin{aligned} \vec{r} &= r\hat{e}_r \\ \vec{v} &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta \end{aligned}}$$

Note: You are now going to delve deep into all possible vector co-ordinate system the derivations of which requires days worth of handwork and patience but is worth every moment if you can find the time for that.

I may make mistakes so please cross check everything using google or your own mathematical skills.

Vector Calculus:

We know the three operations named Gradient, Divergence and Curl.

Curl Acts on a vector and spits out vector.
Divergence Acts on a vector and spits out scalar.
Gradient Acts on a scalar and spits out vector.

We have Three operations and if we can perform two operations on a vector/scalar field then the combination of those would yield 9 total Compound operations.

Basic operators in index notations

Gradient: For Any Scalar A

$$Grad(A) = \vec{\nabla} A = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)A$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)A$$ [index Notation]

$$= \left(\hat{e}_1\frac{\partial}{\partial x_1} + \hat{e}_2\frac{\partial}{\partial x_2} + \hat{e}_3\frac{\partial}{\partial z}\right)A$$

$$= \sum_{i=1}^{3}\hat{e}_i\frac{\partial A}{\partial x_i}$$

$$= \frac{\partial A}{\partial x_i}$$

$$\therefore \vec{\nabla} A = \nabla_i A$$

Einstein Notation

Divergence: For Any Vector $\vec{A}$

$$\text{Div}(\vec{A}) = \vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{A}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} \nabla_i A_j \delta_{ij} \rightarrow \text{index notation}$$

$$\therefore \vec{\nabla}\cdot\vec{A} = \nabla_i A_j \delta_{ij} \longrightarrow \text{Einstein Notation}$$

Curl: For Any Vector $\vec{A}$

$$\text{Curl}(\vec{A}) = \vec{\nabla}\times\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{A}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\nabla_j A_k \rightarrow \text{index notation}$$

$$\therefore \vec{\nabla}\times\vec{A} = \epsilon_{ijk}\nabla_j A_k \longrightarrow \text{Einstein Notation}$$

All possible vector/scalar operations:

1. Curl of Curl
2. Curl of Gradient
3. ~~Curl of Divergence~~ [invalid operation]
4. ~~Gradient of Curl~~ "
5. ~~Gradient of Gradient~~ "
6. Gradient of Divergence

(7) Divergence of Curl

(8) Divergence of Gradient

~~(9) Divergence of Divergence~~ "

So out of all possible operations only 5 are viable and 4 aren't because it is not possible to take a curl / Divergence of scalars and Gradient of vectors.

Leaving us with the five.

(1) Curl of Curl:

For the vector field $\vec{A}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

$$= \varepsilon_{ijk} \nabla_j (\vec{\nabla} \times \vec{A})_k$$

$$= \varepsilon_{ijk} \nabla_j (\varepsilon_{klm} \nabla_l A_m)$$

$$= \varepsilon_{kij} \varepsilon_{klm} \nabla_j \nabla_l A_m \quad [\varepsilon_{ijk} = \varepsilon_{kij}]$$

$$= \left(\delta_{i1}\delta_{jm} - \delta_{im}\delta_{j1}\right)\nabla_j \nabla_1 A_m$$

$$= \left(\delta_{i1}\delta_{jm}\nabla_j\nabla_1 A_m - \delta_{im}\delta_{j1}\nabla_j\nabla_1 A_m\right)$$

$$= \nabla_i \nabla_m A_m - \nabla_1 \nabla_1 A_i$$

$$= \sum_{i=1}^{3}\nabla_i \hat{e}_i \sum_{m=1}^{3}\nabla_m A_m \delta_{mm} - \sum_{1=1}^{3}\nabla_1^2 \sum_{i=1}^{3} A_i \hat{e}_i$$

$$= \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A}$$

$$\therefore\ \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A}$$

(2) Curl of Gradient:

For the scalar field A

$$\vec{\nabla}\times\left(\vec{\nabla}A\right)$$

$$= \vec{\nabla}\times\sum_{k=1}^{3}\hat{e}_k \frac{\partial A}{\partial x_k}$$

$$= \epsilon_{ijk}\nabla_j \frac{\partial A}{\partial x_k}$$

$$= \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial A}{\partial x_k}$$

$$= \frac{1}{2}\left[\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial A}{\partial x_k} + \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial A}{\partial x_k}\right]$$

$$= \frac{1}{2}\left[\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial A}{\partial x_k} - \epsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial A}{\partial x_j}\right]$$

$$= \frac{1}{2}\times 0$$

$$= 0$$

$\therefore \vec{\nabla} \times \vec{\nabla} A = 0$

(3) Gradient of Divergence

This is trivial and doesn't need further computation.

(4) Divergence of Curl

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})$$

$$= \nabla_i (\vec{\nabla} \times \vec{A})_j \delta_{ij}$$

$$= \nabla_i \delta_{ij} (\varepsilon_{jkl} \nabla_k A_l)$$

$$= \varepsilon_{ikl} \nabla_i \nabla_k A_l$$

$$= \frac{1}{2} \left[\varepsilon_{ikl} \nabla_i \nabla_k A_l + \varepsilon_{ikl} \nabla_i \nabla_k A_l \right]$$

$$= \frac{1}{2} \left[\varepsilon_{ikl} \nabla_i \nabla_k A_l - \varepsilon_{kil} \nabla_i \nabla_k A_l \right]$$

$$= \frac{1}{2} \times 0$$

$$= 0$$

(5) Divergence of Gradient:

$$\vec{\nabla} \cdot \vec{\nabla} A$$

$$= \frac{\partial}{\partial x_i} \frac{\partial A}{\partial x_j} \delta_{ij}$$

$$= \frac{\partial^2 A}{\partial x_i^2}$$

$$= \nabla^2 A$$

$\therefore \quad \vec{\nabla}\cdot\vec{\nabla} A = \nabla^2 A \rightarrow$ Laplacian of A

Summary:

① $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

② $\vec{\nabla}\times\vec{\nabla} A = 0$

③ $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

④ $\vec{\nabla}\cdot\vec{\nabla} A = \nabla^2 A$

Cartesian Co-ordinates:

The vector is in terms of x, y, z.

Here $\vec{x} = x\hat{x}$, $y = y\hat{y}$, $z = z\hat{z}$,

$$\vec{r} = \vec{x} + \vec{y} + \vec{z}$$

$$\Rightarrow \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\therefore \quad \frac{\partial\vec{r}}{\partial x} = \frac{\partial}{\partial x}(x\hat{x} + y\hat{y} + z\hat{z}) = \hat{x}$$

$$\frac{\partial\vec{r}}{\partial y} = \frac{\partial}{\partial y}(x\hat{x} + y\hat{y} + z\hat{z}) = \hat{y}$$

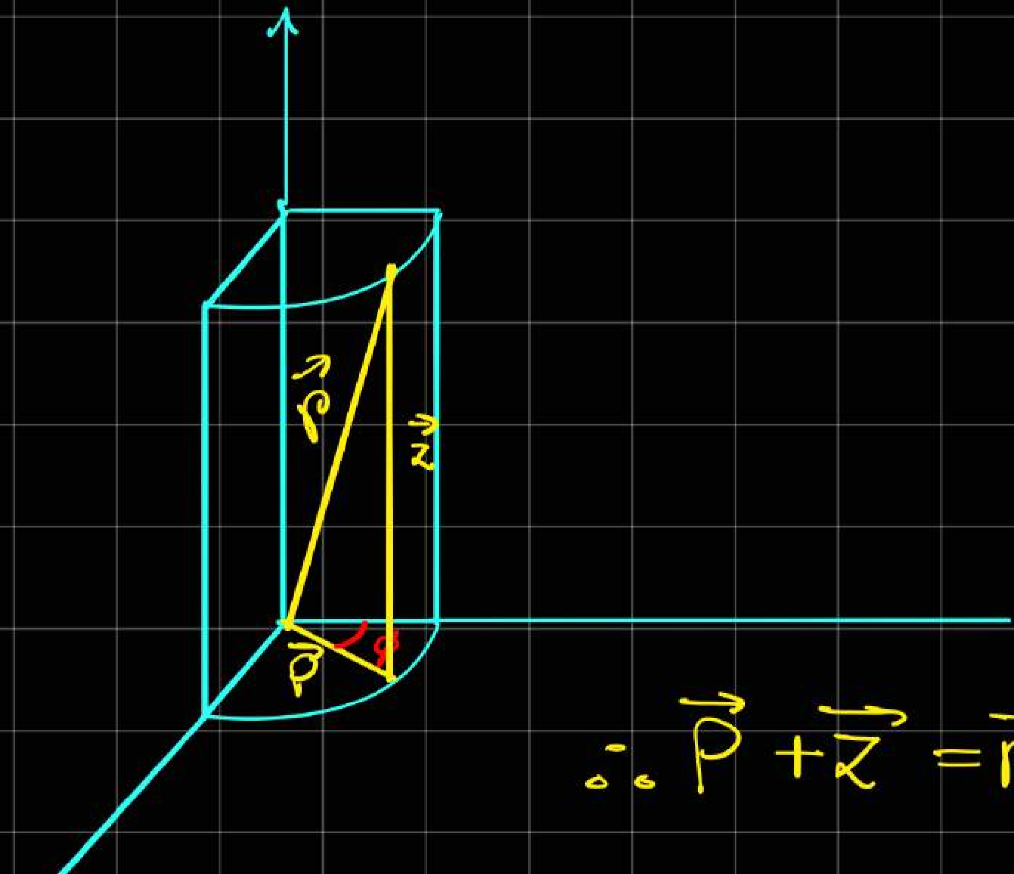
$$\frac{\partial\vec{r}}{\partial z} = \frac{\partial}{\partial z}(x\hat{x} + y\hat{y} + z\hat{z}) = \hat{z}$$

Meaning $\vec{r} = r\hat{r} = x\frac{\partial\vec{r}}{\partial x} + y\frac{\partial\vec{r}}{\partial y} + z\frac{\partial\vec{r}}{\partial z}$

Note : Any unit vector $\hat{e}_i$ can be expressed as

$$\hat{e}_i = \frac{\left(\frac{\partial \vec{r}}{\partial e_i}\right)}{\left|\frac{\partial \vec{r}}{\partial e_i}\right|}$$

Cylindrical Co-ordinates



$\therefore \vec{\rho} + \vec{z} = \vec{r}$

The Cylindrical Co-ordinates are defined in term of (ρ, ϕ, z)

Here $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$\therefore \vec{r} = \vec{x} + \vec{y} + \vec{z} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}$

Now $\dfrac{\partial \vec{r}}{\partial \rho} = \dfrac{\partial}{\partial \rho}\left(\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}\right) = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\dfrac{\partial \vec{r}}{\partial \phi} = \dfrac{\partial}{\partial \phi}\left(\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}\right) = -\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}$

$\dfrac{\partial \vec{r}}{\partial z} = \dfrac{\partial}{\partial z}\left(\rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}\right) = \hat{z}$

And $|\vec{r}| = \sqrt{\rho^2\cos^2\phi + \rho^2\sin^2\phi + z^2}$

$= \sqrt{\rho^2 + z^2}$

$\therefore r^2 = \rho^2 + z^2 = x^2 + y^2 + z^2$

Which means $\rho^2 = x^2 + y^2$

And $\vec{\rho} = \rho\hat{\rho} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y}$

$\therefore \hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

And we know that $\dfrac{\partial \vec{r}}{\partial \phi} = \rho(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$

$\therefore \dfrac{\partial \vec{r}}{\partial \phi} = \rho\dfrac{\partial \hat{\rho}}{\partial \phi} = \rho\hat{\phi}$

Therefore $\left.\begin{aligned}\hat{\rho} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y}\\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\end{aligned}\right\} \Rightarrow \begin{aligned}\hat{x} &= \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}\\ \hat{y} &= \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}\end{aligned}$

But $\vec{r}(\rho, \phi, z)$ Meaning $d\vec{r} = \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \phi} d\phi + \frac{\partial \vec{r}}{\partial z} dz$

And $\rho(x,y)$ Meaning $\vec{\rho} = \rho\hat{\rho} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y}$

$$\therefore \hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

And $\frac{\partial \hat{\rho}}{\partial \phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} = -\hat{\phi}$

$$\left.\begin{aligned} \hat{\rho} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{aligned}\right\} \begin{aligned} \hat{x} &= \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi} \\ \hat{y} &= \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi} \end{aligned}$$

Now Since $\vec{r}(\rho, \phi, z) \Leftrightarrow d\vec{r} = \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \phi} d\phi + \frac{\partial \vec{r}}{\partial z} dz$

And $\frac{\partial \vec{r}}{\partial \rho} = \hat{\rho}$; $\frac{\partial \vec{r}}{\partial \phi} = \rho\hat{\phi}$; $\frac{\partial \vec{r}}{\partial z} = \hat{z}$

$\therefore d\vec{r} = \hat{\rho}\,d\rho + \rho\hat{\phi}\,d\phi + \hat{z}\,dz$

Similarly the infinitesimally small line integral can be written as the following

$$\vec{dl} = d\rho\,\hat{\rho} + \rho\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

Gradient in cylindrical:

For any scalar field

$f(x,y,z)$ $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

And $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

The cartesian line vector $\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$$\therefore \vec{\nabla} f \cdot \vec{dl} = \left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\therefore \vec{\nabla} f \cdot \vec{dl} = df$$

Which holds true for all co-ordinates. Meaning in Cylindrical co-ordinates it would be the same but in cylindrical form.

$\vec{\nabla} f \cdot \vec{dl} = df$ Assuming $\vec{\nabla} f = (\nabla f)_\rho \hat{\rho} + (\nabla f)_\phi \hat{\phi} + (\nabla f)_z \hat{z}$

$$\Rightarrow \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial z}dz = \vec{\nabla} f\cdot\left(d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}\right)$$

$$\Rightarrow \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial z}dz = \left((\nabla f)_\rho\hat{\rho} + (\nabla f)_\phi\hat{\phi} + (\nabla f)_z\hat{z}\right)\cdot\left(d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}\right)$$

$$\therefore \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \phi}d\phi + \frac{\partial f}{\partial z}dz = (\nabla f)_\rho d\rho + \rho(\nabla f)_\phi d\phi + (\nabla f)_z dz$$

By comparing them we get

$$(\nabla f)_\rho = \frac{\partial f}{\partial \rho}$$
$$(\nabla f)_\phi = \frac{1}{\rho}\frac{\partial f}{\partial \phi}$$
$$(\nabla f)_z = \frac{\partial f}{\partial z}$$

$\rightarrow$ Plugging them yields

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

$$\therefore \vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{z}\frac{\partial f}{\partial z}$$

Divergence in Cylindrical:

For any vector field $\vec{v} = v_\rho \hat{\rho} + v_\phi \hat{\phi} + v_z \hat{z}$

$$\vec{\nabla} \cdot \vec{v} = \left(\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}\right) \cdot \left(v_\rho \hat{\rho} + v_\phi \hat{\phi} + v_z \hat{z}\right)$$

Recalling that $\hat{\phi} = \frac{\partial \hat{\rho}}{\partial \phi}$ and $\hat{\rho} = -\frac{\partial \hat{\phi}}{\partial \phi}$

$\hat{\rho}, \hat{\phi}, \hat{z}$ are orthogonal basis vectors meaning

$\hat{\rho} \cdot \hat{\rho} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1$ and

$\hat{\rho} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{z} = \hat{z} \cdot \hat{\rho} = 0$

WARNING: unlike in Carterian Co-ordinates, the orthogonal basis vectors aren't necessalily constant meaning they change direction so we have to take the dot product by bruteforcing our way through all terms :(

Summary: The basis basis vectors are orthogonal but not necessarily independent

Now ① $\left(\hat{\rho} \frac{\partial}{\partial \rho}\right) \cdot \left(v_\rho \hat{\rho}\right)$

$$= \hat{\rho} \cdot \left(\frac{\partial}{\partial \rho} v_\rho \hat{\rho}\right)$$

$$= \hat{\rho} \cdot \left(\hat{\rho} \frac{\partial v_\rho}{\partial \rho} + v_\rho \frac{\partial \hat{\rho}}{\partial \rho}\right)$$

$$= \hat{\rho} \cdot \left(\hat{\rho} \frac{\partial v_\rho}{\partial \rho} + v_\rho \cdot 0\right)$$

$$= \frac{\partial v_\rho}{\partial \rho}$$

② $\left(\hat{\rho} \frac{\partial}{\partial \rho}\right) \cdot \left(v_\phi \hat{\phi}\right)$

$$= \hat{\rho} \cdot \frac{\partial}{\partial \rho}\left(v_\phi \hat{\phi}\right)$$

$$= \hat{\rho} \cdot \left(\hat{\phi} \frac{\partial v_\phi}{\partial \rho} + v_\phi \frac{\partial \hat{\phi}}{\partial \rho}\right)$$

$$= 0 \cdot \frac{\partial v_\phi}{\partial \rho} + v_\phi \cdot 0$$

$$= 0$$

③ $\hat{\rho}\frac{\partial}{\partial\rho}\cdot(v_z\hat{z})$

$$= \hat{\rho}\cdot\left(\hat{z}\frac{\partial v_z}{\partial\rho} + v_z\frac{\partial\hat{z}}{\partial\rho}\right)$$

$$= 0\cdot\frac{\partial v_z}{\partial\rho} + v_z\cdot 0$$

$$= 0$$

④ $\frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi}\cdot(v_\rho\hat{\rho})$

$$= \hat{\phi}\frac{1}{\rho}\cdot\left(\hat{\rho}\frac{\partial v_\rho}{\partial\phi} + v_\rho\frac{\partial\hat{\rho}}{\partial\phi}\right)$$

$$= 0\cdot\frac{1}{\rho}\frac{\partial v_\rho}{\partial\phi} + \hat{\phi}\cdot\hat{\phi}\frac{v_\rho}{\rho}$$

$$= \frac{v_\rho}{\rho}$$

⑤ $\frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi}\cdot(v_\phi\hat{\phi})$

$$= \frac{\hat{\phi}}{\rho}\cdot\left(v_\phi\frac{\partial\hat{\phi}}{\partial\phi} + \hat{\phi}\frac{\partial v_\phi}{\partial\phi}\right)$$

$$= \frac{\hat{\phi}}{\rho}\cdot\left(-v_\phi\hat{\rho} + \hat{\phi}\frac{\partial v_\phi}{\partial\phi}\right)$$

$$= \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi}$$

⑥ $\hat{z}\frac{\partial}{\partial z}\cdot(v_\rho\hat{\rho})$

$$= \hat{z}\cdot\left(v_\rho\frac{\partial\hat{\rho}}{\partial z} + \hat{\rho}\frac{\partial v_\rho}{\partial z}\right)$$

$$= 0$$

⑦ $\hat{z}\frac{\partial}{\partial z}\cdot(v_\phi\hat{\phi})$

$$= \hat{z}\cdot\left(v_\phi\frac{\partial\hat{\phi}}{\partial z} + \hat{\phi}\frac{\partial v_\phi}{\partial z}\right)$$

$$= 0$$

⑧ $\hat{z}\frac{\partial}{\partial z}\cdot(v_z\hat{z})$

$$= \hat{z}\cdot\left(v_z\frac{\partial\hat{z}}{\partial z} + \hat{z}\frac{\partial v_z}{\partial z}\right)$$

$$= \frac{\partial v_z}{\partial z}$$

⑨

I forgot $\frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi}\cdot(v_z\hat{z})$

$$= \frac{\hat{\phi}}{\rho}\left(v_z\frac{\partial\hat{z}}{\partial\phi} + \hat{z}\frac{\partial v_z}{\partial\phi}\right)$$

$$= 0$$

which means $\vec{\nabla}\cdot\vec{v} = \frac{\partial v_\rho}{\partial\rho} + \frac{v_\rho}{\rho} + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

But $\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) = \frac{1}{\rho}\left(\rho \frac{\partial V_\rho}{\partial \rho} + V_\rho \frac{\partial \rho}{\partial \rho}\right) = \frac{\partial V_\rho}{\partial \rho} + \frac{V_\rho}{\rho}$

$$\therefore \quad \vec{\nabla}\cdot\vec{V} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$$

Curl in Cylindrical:

$$\vec{V}(\rho,\phi,z) = V_\rho \hat{\rho} + V_\phi \hat{\phi} + V_z \hat{z}$$

$$\vec{\nabla}\times\vec{V} = \left(\hat{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right)\times\left(V_\rho \hat{\rho} + V_\phi \hat{\phi} + V_z \hat{z}\right)$$

$$= \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ V_\rho & V_\phi & V_z \end{vmatrix}$$

$$= \left(\frac{1}{\rho}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)\hat{\rho} - \left(\frac{\partial V_z}{\partial \rho} - \frac{\partial V_\rho}{\partial z}\right)\hat{\phi} + \left(\frac{\partial V_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial V_\rho}{\partial \phi}\right)\hat{z}$$

Laplacian in Cylindrical Co-ordinate:

$$\nabla^2 f = \vec{\nabla}\cdot\vec{\nabla} f$$

$$= \left(\hat{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}\right)$$

① $\hat{\rho}\frac{\partial}{\partial\rho}\cdot\left(\frac{\partial f}{\partial\rho}\hat{\rho}\right)$

$= \hat{\rho}\cdot\left(\frac{\partial^2 f}{\partial\rho^2}\hat{\rho}+\frac{\partial f}{\partial\rho}\frac{\partial\hat{\rho}}{\partial\rho}\right)$

$= \frac{\partial^2 f}{\partial\rho^2}$

③ $\hat{\rho}\frac{\partial}{\partial\rho}\cdot\left(\frac{\partial f}{\partial z}\hat{z}\right)$

$=0$

④ $\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}\right)\cdot\left(\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi}\right)$

$= \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi}\right)$

$= \frac{\hat{\phi}}{\rho^2}\cdot\left(\hat{\phi}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial f}{\partial\phi}\frac{\partial\hat{\phi}}{\partial\phi}\right)$

$= \frac{1}{\rho^2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$

⑤ $\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}\right)\cdot\left(\frac{\partial f}{\partial\rho}\hat{\rho}\right)$

$= \frac{\hat{\phi}}{\rho}\cdot\left(\frac{\partial^2 f}{\partial\phi\partial\rho}\hat{\rho}+\frac{\partial f}{\partial\rho}\frac{\partial\hat{\rho}}{\partial\phi}\right)$

$= \frac{1}{\rho}\frac{\partial f}{\partial\rho}$

⑥ $\left(\frac{1}{\rho}\frac{\partial}{\partial\phi}\hat{\phi}\right)\left(\hat{z}\frac{\partial f}{\partial z}\right)$

$=0$

② $\hat{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi}\right)$

$= \hat{\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\partial f}{\partial\phi}\hat{\phi}\right)+\frac{\partial f}{\partial\phi}\cancel{\frac{\partial}{\partial\rho}\left(\frac{\hat{\phi}}{\rho}\right)}^{0}\right.$

$\left.+\hat{\phi}\cancel{\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial\rho}\right)}^{0}\right)$

$= \hat{\rho}\cdot\left(\frac{1}{\rho}\left(\frac{\partial^2 f}{\partial\rho\partial\phi}\hat{\phi}+\cancel{\frac{\partial f}{\partial\phi}\frac{\partial\hat{\phi}}{\partial\rho}}^{0}\right)\right)$

$=0$

⑦ $\left(\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial\rho}\hat{\rho}\right)=0$

⑧ $\left(\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi}\right)=0$

⑨ $\left(\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial f}{\partial z}\hat{z}\right)$

$= \hat{z}\cdot\left(\hat{z}\frac{\partial^2 f}{\partial z^2}+\frac{\partial f}{\partial z}\frac{\partial\hat{z}}{\partial z}\right)$

$= \frac{\partial^2 f}{\partial z^2}$

$$\nabla\cdot\nabla f = \left(\frac{\partial^2 f}{\partial\rho^2}\right)+\frac{1}{\rho}\left(\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\left(\frac{\partial^2 f}{\partial\phi^2}\right)+\left(\frac{\partial^2 f}{\partial z^2}\right)$$

$$\therefore \nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}$$

Summary:

$$\vec{V}(\rho, \phi, z) = V_\rho \hat{\rho} + V_\phi \hat{\phi} + V_z \hat{z}$$

orthogonality conditions

$$\hat{\rho} \cdot \hat{\rho} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1$$

$$\hat{\rho} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{z} = \hat{z} \cdot \hat{\rho} = 0$$

Cartesian ⇔ Cylindrical

$x = \rho \cos\phi$	$\rho = \sqrt{x^2 + y^2}$
$y = \rho \sin\phi$	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z$	$z = z$

Cylindrical unit vectors

$$\hat{\rho} = \cos\phi\, \hat{x} + \sin\phi\, \hat{y}$$

$$\hat{\phi} = -\sin\phi\, \hat{x} + \cos\phi\, \hat{y}$$

Cartesian unit vectors

$$\hat{x} = \cos\phi\, \hat{\rho} - \sin\phi\, \hat{\phi}$$

$$\hat{y} = \sin\phi\, \hat{\rho} + \cos\phi\, \hat{\phi}$$

Vector / Scalar operations:

① $\vec{\nabla} = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$ [Del operator]

② $\vec{\nabla} f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}$ [Gradient of f]

③ $\vec{\nabla} \cdot \vec{V} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho} \frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$ [Divergence of $\vec{V}(\rho, \phi, z)$]

④ $$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho} \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ V_\rho & V_\phi & V_z \end{vmatrix} = \left(\frac{1}{\rho} \frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right) \hat{\rho} - \left(\frac{\partial V_z}{\partial} - \frac{\partial V_\rho}{\partial z}\right) \hat{\phi} + \left(\frac{\partial V_\phi}{\partial \rho} - \frac{1}{\rho} \frac{\partial V_\rho}{\partial \phi}\right) \hat{z}$$

[Curl of $\vec{V}(\rho, \phi, z)$]

⑤ $\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$ [Laplacian of f]

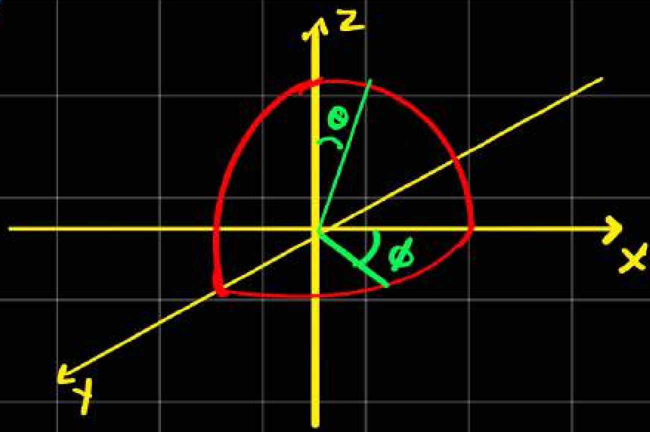
⑥ $\vec{dl} = d\rho\, \hat{\rho} + \rho\, d\phi\, \hat{\phi} + dz\, \hat{z}$ [Line element]

⑦ $dV = \rho\, d\rho\, d\phi\, dz$ [Volume Element/Product of $\hat{\rho}, \hat{\phi}, \hat{z}$ co-efficients]

We are done with Cylindrical Co-ordinates :)

SPHERICAL CO-ORDINATES:

Transformation from cartesian:



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Now $\vec{S} = x\hat{x} + y\hat{y} + z\hat{z}$

① $\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ ② $\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ ③ $\frac{\partial z}{\partial r} = \cos\theta$

④ $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$ ⑤ $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$ ⑥ $\frac{\partial z}{\partial \theta} = -r\sin\theta$

⑦ $\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$ ⑧ $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$ ⑨ $\frac{\partial z}{\partial \phi} = 0$

→ $S^2 = x^2 + y^2 + z^2$

① $\frac{\partial \vec{S}}{\partial x} = \hat{x}$ ② $\frac{\partial \vec{S}}{\partial y} = \hat{y}$

③ $\frac{\partial \vec{S}}{\partial z} = \hat{z}$ ④ $\frac{\partial \vec{S}}{\partial r} = \hat{r}$

⑤ $\frac{\partial \vec{S}}{\partial \theta} = r\hat{\theta}$ ⑥ $\frac{\partial \vec{S}}{\partial \phi} = r\sin\theta\hat{\phi}$

Now

① $\frac{\partial \vec{S}}{\partial r} = \hat{r}\frac{\partial r}{\partial r} = \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

② $\frac{\partial \vec{S}}{\partial \theta} = r\frac{\partial \hat{r}}{\partial \theta} = r\hat{\theta} = r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} - r\sin\theta\,\hat{z}$

③ $\frac{\partial \vec{S}}{\partial \phi} = r\sin\theta\frac{\partial \hat{r}}{\partial \phi} = r\sin\theta\,\hat{\phi} = -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}$

---

① $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

② $\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

③ $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

④ $\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$

⑤ $\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$

⑥ $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

Putting them in a matrix we get

$$\underbrace{\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}}_{J} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \hat{r} \\ r\hat{\theta} \\ r\sin\theta\,\hat{\phi} \end{pmatrix}$$

The Determinant of J is given by

$|J| = r^2\sin\theta$

which is Surprisingly the Jacobian from Cartesian to Spherical

$(x,y,z) \to (r,\theta,\phi)$

Now $\vec{s}(r,\theta,\phi) \to d\vec{s} = \frac{\partial\vec{s}}{\partial r}dr + \frac{\partial\vec{s}}{\partial\theta}d\theta + \frac{\partial\vec{s}}{\partial\phi}d\phi$

$\therefore\ d\vec{l} = d\vec{s} = \hat{r}\,dr + r\hat{\theta}\,d\theta + r\sin\theta\,\hat{\phi}\,d\phi$

But $\vec{\nabla}f \cdot d\vec{l} = df$

And $\vec{\nabla}f = (\nabla f)_r\hat{r} + (\nabla f)_\theta\hat{\theta} + (\nabla f)_\phi\hat{\phi}$

which means $\vec{\nabla}f \cdot d\vec{l} = df$

$\therefore (\nabla f)_r\,dr + r(\nabla f)_\theta\,d\theta + r\sin\theta(\nabla f)_\phi\,d\phi = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta + \frac{\partial f}{\partial\phi}d\phi$

By Comparing the Co-efficients we get;

$$(\nabla f)_r = \frac{\partial f}{\partial r}$$

$$(\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$(\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Which Means $\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial f}{\partial \phi}$

Therefore the Spherical Del operator is given by

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Important relations Check them yourselves

For Shortcuts exploit their orthogonality And Google your results

$$\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1$$

$$\hat{r}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{r} = 0$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \quad \frac{\partial \hat{\theta}}{\partial \theta} = \sin\theta\,\hat{\phi}$$

$$\frac{\partial \hat{r}}{\partial r} = \frac{\partial \hat{\theta}}{\partial r} = \frac{\partial \hat{\phi}}{\partial r} = 0$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}\,; \quad \frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\,\hat{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0\,; \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta$$

Divergence in Spherical:

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}\right)$$

Now

① $$\left(\hat{r}\frac{\partial}{\partial r}\right)\cdot\left(V_r\hat{r}\right)$$
$$= \hat{r}\cdot\left(V_r\frac{\partial\hat{r}}{\partial r} + \hat{r}\frac{\partial V_r}{\partial r}\right)$$
$$= \hat{r}\cdot\left(0 + \hat{r}\frac{\partial V_r}{\partial r}\right)$$
$$= \frac{\partial V_r}{\partial r}$$

② $$\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\cdot\left(V_r\hat{r}\right)$$
$$= \frac{\hat{\theta}}{r}\cdot\left(\hat{r}\frac{\partial V_r}{\partial\theta} + V_r\frac{\partial\hat{r}}{\partial\theta}\right)$$
$$= \frac{\hat{\theta}}{r}\cdot\left(\hat{r}\frac{\partial V_r}{\partial\theta} + V_r\hat{\theta}\right)$$
$$= \frac{V_r}{r}$$

③ $$\frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\cdot\left(V_r\vec{r}\right)$$
$$= \frac{\hat{\phi}}{r\sin\theta}\cdot\left(V_r\frac{\partial\hat{r}}{\partial\phi} + \hat{r}\frac{\partial V_r}{\partial\phi}\right)$$
$$= \frac{\hat{\phi}}{r\sin\theta}\cdot\left(V_r\sin\theta\hat{\phi} + \hat{r}\frac{\partial V_r}{\partial\phi}\right)$$
$$= \frac{V_r}{r}$$

④ $$\left(\hat{r}\frac{\partial}{\partial r}\right)\cdot\left(V_\theta\hat{\theta}\right)$$
$$= \hat{r}\cdot\left(\hat{\theta}\frac{\partial V_\theta}{\partial r} + V_\theta\frac{\partial\hat{\theta}}{\partial r}\right)$$
$$= 0$$

⑤ $$\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\cdot\left(V_\theta\hat{\theta}\right)$$
$$= \frac{\hat{\theta}}{r}\cdot\left(V_\theta\frac{\partial\hat{\theta}}{\partial\theta} + \hat{\theta}\frac{\partial V_\theta}{\partial\theta}\right)$$
$$= \frac{\hat{\theta}}{r}\cdot\left(-\hat{r}V_\theta + \hat{\theta}\frac{\partial V_\theta}{\partial\theta}\right)$$
$$= \frac{1}{r}\frac{\partial V_\theta}{\partial\theta}$$

⑥ $$\frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\cdot\left(V_\theta\hat{\theta}\right)$$
$$= \frac{\hat{\phi}}{r\sin\theta}\cdot\left(\hat{\theta}\frac{\partial V_\theta}{\partial\phi} + V_\theta\frac{\partial\hat{\theta}}{\partial\phi}\right)$$
$$= \frac{1}{r\sin\theta}\left(0 + V_\theta\cos\theta\right)$$
$$= \frac{V_\theta}{r\tan\theta}$$

⑦ $$\left(\hat{r}\frac{\partial}{\partial r}\right)\cdot\left(V_\phi\hat{\phi}\right)$$
$$= \hat{r}\cdot\left(\hat{\phi}\frac{\partial V_\phi}{\partial r} + V_\phi\frac{\partial\hat{\phi}}{\partial r}\right)$$
$$= 0$$

⑧ $$\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\cdot\left(V_\phi\hat{\phi}\right)$$
$$= \frac{\hat{\theta}}{r}\cdot\left(\hat{\phi}\frac{\partial V_\phi}{\partial\theta} + V_\phi\frac{\partial\hat{\phi}}{\partial\theta}\right)$$
$$= 0$$

⑨ $$\frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}\cdot\left(V_\phi\hat{\phi}\right)$$
$$= \frac{\hat{\phi}}{r\sin\theta}\cdot\left(V_\phi\frac{\partial\hat{\phi}}{\partial\phi} + \hat{\phi}\frac{\partial V_\phi}{\partial\phi}\right)$$
$$= \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

which means:

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_r}{\partial r} + \frac{2V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta \sin\theta}{r\cos\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(V_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

Curl in Spherical:

$$\vec{\nabla}\times\vec{V} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \times \left(V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}\right)$$

$$= \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\ V_r & V_\theta & V_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial \theta} r\sin\theta V_\phi - \frac{\partial}{\partial \phi} rV_\theta\right)\hat{r} - r\left(\frac{\partial}{\partial r} r\sin\theta V_\phi - \frac{\partial V_r}{\partial \phi}\right)\hat{\theta}\right.$$

$$\left. + r\sin\theta\left(\frac{\partial}{\partial r} rV_\theta - \frac{\partial V_r}{\partial \theta}\right)\hat{\phi}\right]$$

Laplacian in Sphericals:

$$\nabla^2 f = \vec{\nabla}\cdot\vec{\nabla} f$$

$$= \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(\hat{r}\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)$$

This is oddly similar to the divergence if you replace the following

$$V_r \Longleftrightarrow \frac{\partial f}{\partial r}$$

$$V_\theta \Longleftrightarrow \frac{\partial f}{\partial \theta}$$

$$V_\phi \Longleftrightarrow \frac{\partial f}{\partial \phi}$$

And then plug the additional r term and the $r\sin\theta$ term in the $\hat{\theta}$, $\hat{\phi}$ denominators respectively.

$$\therefore \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$$

Summary:

① $x = r\sin\theta\cos\phi$

② $y = r\sin\theta\sin\phi$

③ $z = r\cos\theta$

④ $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

⑤ $\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

⑥ $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

⑦ $\vec{dl} = \underbrace{dr}_{1}\,\hat{r} + \underbrace{r\,d\theta}\,\hat{\theta} + \underbrace{r\sin\theta\,d\phi}\,\hat{\phi}$ [line Element]

⑧ $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$ [Volume Element]

⑨ Surface Element can be taken by any two from the line Element

$$\begin{cases} dA = r\,dr\,d\theta \\ dA = r^2\sin\theta\,d\theta\,d\phi \\ dA = r\sin\theta\,dr\,d\phi \end{cases}$$

(10) $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$ [Del operator]

(11) $\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial f}{\partial \phi}$ [Gradient]

(12) $\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, V_\theta\right) + \frac{1}{r\sin^2\theta}\frac{\partial V_\phi}{\partial \phi}$

(13) $\vec{\nabla}\times\vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & r\sin\theta\, V_\phi \end{vmatrix}$ [Curl]

$$= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta} r\sin\theta V_\phi - \frac{\partial}{\partial\phi} rV_\theta\right)\hat{r} - r\left(\frac{\partial}{\partial r} r\sin\theta V_\phi - \frac{\partial V_r}{\partial\phi}\right)\hat{\theta} + r\sin\theta\left(\frac{\partial}{\partial r} rV_\theta - \frac{\partial V_r}{\partial\theta}\right)\hat{\phi}\right]$$

(14) $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 f}{\partial\phi^2}\right)$

[Laplacian]

We are done with spherical Co-ordinates.

Curvilinear Co-ordinates:

I won't Explain Much here nor will I derive any but I will set the ground rules and show their similarities to others

In cylindrical co-ordinates; (ρ, ϕ) were functions of x, y. In Spherical Co-ordinates; (r, θ, ϕ) were all functions of (x, y, z)

in curvilinear it is (u_1, u_2, u_3) as function of (x, y, z).

Co-ordinates	Variables	Scale factors	Basis Vectors
Curvilinear	(u_1, u_2, u_3)	(h_1, h_2, h_3)	$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$
Cartesian	(x, y, z)	$(1, 1, 1)$	$(\hat{x}, \hat{y}, \hat{z})$
Cylindrical	(ρ, ϕ, z)	$(1, \rho, 1)$	$(\hat{\rho}, \hat{\phi}, \hat{z})$
Spherical	(r, θ, ϕ)	$(1, r, r\sin\theta)$	$(\hat{r}, \hat{\theta}, \hat{\phi})$

Line Element: $d\vec{l} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$

Surface Element: $dA = h_1 h_2 du_1 du_2$ or $h_2 h_3 du_2 du_3$ or $h_3 h_1 du_3 du_1$

Volume Element: $dV = h_1 h_2 h_3 du_1 du_2 du_3$

Del operator: $\vec{\nabla} = \hat{e}_1 \frac{1}{h_1} \frac{\partial}{\partial u_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial}{\partial u_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial}{\partial u_3}$

Gradient: $\vec{\nabla} f = \hat{e}_1 \frac{1}{h_1} \frac{\partial f}{\partial u_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial f}{\partial u_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial f}{\partial u_3}$

Divergence: $$\vec{\nabla}\cdot\vec{V} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(V_1h_2h_3) + \frac{\partial}{\partial u_2}(V_2h_3h_1) + \frac{\partial}{\partial u_3}(V_3h_1h_2)\right]$$

Curl: $$\vec{\nabla}\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix}$$

Laplacian: $$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_2h_1}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

See their Similarities???

if you have a hard time memorizing the vector/scalar operations for cylindrical/spherical. Just memorize it for curvilinear and replace the corresponding parameters for the co-ordinate systems.

	Basis Vectors			Scale factors			Variables		
Curvilinear	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$	h_1	h_2	h_3	u_1	u_2	u_3
Cartesian	$\hat{x}$	$\hat{y}$	$\hat{z}$	1	1	1	x	y	z
Cylindrical	$\hat{\rho}$	$\hat{\phi}$	$\hat{z}$	1	ρ	1	ρ	ϕ	z
Spherical	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$	1	r	$r\sin\theta$	r	θ	ϕ

Caveat Here

And here

Warning: I might have wrote them on the same row/column in order to signify their equivalence between co-ordinates while writing equations it doesn't mean the variable/basis vectors are equivalent. to illustrate it better have a look at the following example.

$$\vec{dl}_{Curvi} = h_1\, du_1\, \hat{e}_1 + h_2\, du_2\, \hat{e}_2 + h_3\, du_3\, \hat{e}_3$$

$$\vec{dl}_{Spheri} = 1\, dr\, \hat{r} + r\, d\theta\, \hat{\theta} + r\sin\theta\, d\phi\, \hat{\phi}$$

Red Markers indicate equivalence between co-ordinate system

White Marker means they are same but in different sytems

If you can make out these differences then you are prepared to perform vector / Scalar operations in every co-ordinate system and Here ends our vector transformations.

I will tackle two more topics and Call it a day. Don't worry they are going to be physically explaine. They are called divergence theorem and Stokes theorem.

Divergence Theorems

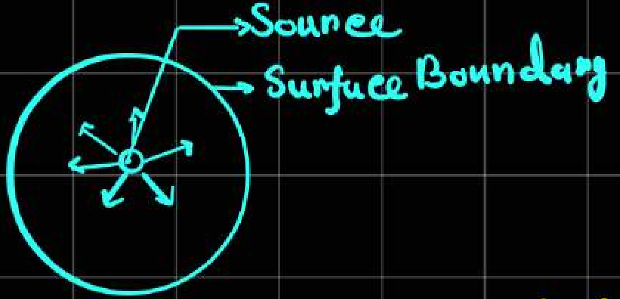
Suppose a region in 3D-Space with a source/sink contrained by some imaginary boundary of 2 Dimension aka a surface. If there is a source/sink then whatever imaginary fluid (This fluid can be physical fluid such as liquid or gas, any other fluid like vector field such as Electric field / Magnetic field or even probability density in Quantum Mechanics. I used the phrase "imaginary fluid" as a generalization of all these possible fluids even if some of may not be physical fluids) comes out of the source or goes in to the sink must pass through the 2D surface enclosing the source/sink.

In short the flux through the surface enclosing the source/sink is the same as the flux coming out of the source/going into the sink. With All that wording in mind. Lets say our fluid creates a vector field $\vec{V}$ then

Flux through Surface = flux within the 3D Region in the Surface

$$\int_{\text{Surface}} \vec{V} \cdot \vec{dS} = \int_{\text{Volume}} (\vec{\nabla} \cdot \vec{V})\, d\tau$$

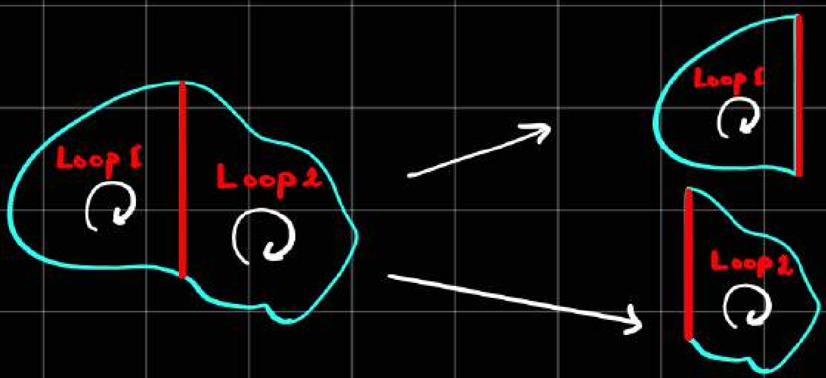
τ = Volume $\qquad$ S = Surface Area



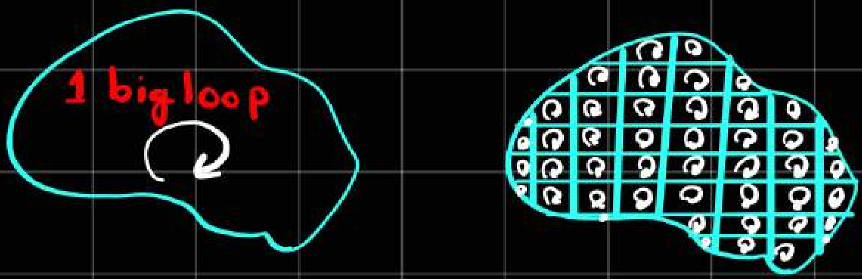
whatever flux comes out of the Source has to come out of the boundary therefore the flux inside the space contained by the surface would be the same as the flux through the surface

Stokes Theorem:

For the given boundary divided in the middle like this. If something passed through both loops in the same orientation (Clockwise or Counterclockwise)



Then the resulting displacement by adding the displacement-s of each loop would yield the same result as it would without any divisions like this.



Similarly if we divide it into infinitesimally small segments then the resulting rotational vector field would be the same as though it is one big loop covering some 2D Surface in other words

Microcirculation of the Vector field = One big loop by the Vector field

$$\int_{\text{Entire Surface Bounded By the Big loop}} (\vec{\nabla} \times \vec{V}) \cdot \vec{dS} = \int_{\text{The loop}} \vec{V} \cdot \vec{dl}$$

Here l = line Element along the loop

Warning: this was all very heuristic explanation without proper mathematical justification so don't take my words for granted and definitely look into further readings for it. For the two vector theorems of course.

Last But Not Least:

There will be mistakes throughout the note so please cross reference everything once again.