Topics we will cover are:

1. Types of differential equations
2. Seperation of variables
3. Exact & Inexact differential equations
4. Integrating Factor
5. Some Real World Applications.

Textbook: Differential equations by Shepley Ross.

1 Types of Differential equations

Linear/Non-linear, Homogenenceous/Inhomogeneous, Orders and Degrees.

Linear Differential Equations:

Any differential equation of the following form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(x)\,y = f(x)$$

is called a Linear-Differential equation.

Non-linear Differential Equations:

Any differential equation of the following form

$$y\,\frac{d^n y}{dx^n} + (\tan y)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0(y)\,y^n = f(x)$$

is called a Non-linear-Differential equation.

Homogeneous Differential Equations:

Any differential equations of the following form

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = f(x)$$ is called

a homogeneous differential equations

Homogeneous Differential Equations:

Any differential equations of the following form

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0$$ is called

a homogeneous differential equations

Order:

The order of a differential equation is the highest derivative available.

Example: $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + 5y = \sin x$

Degree:

A degree refers to the highest exponent power of the leading order derivative term.

Example: $\left(\frac{d^4y}{dx^4}\right)^{2} + 2y\left(\frac{dy}{dx}\right)^3 + y = 0$

Note: The Red Markers signify the details.

2 Seperation of variables:

$$\frac{dy}{dx} = \frac{5y^2}{1+x^2}$$

$$\rightarrow \int \frac{dy}{5y^2} = \int \frac{dx}{1+x^2}$$

$$\therefore -\frac{5}{y} = \tan^{-1}(x) + C$$

The General Form of which looks like the following

$$\frac{dy}{dx} = \frac{f(y)}{g(x)}$$

$$\therefore \int \frac{dy}{f(y)} = \int g(x)\, dx$$

Now by performing the integration we can get our algebraic expression.

3 Exact & Inexact differential equations

Before jumping into formal definitions, let's take a detour and consider the expression (x^2y^3)

$$\Rightarrow \frac{d}{dx}\left(x^2y^3\right) = \frac{dc}{dx}$$

$$\Rightarrow 2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$$

$$\therefore 2xy^3\,dx + 3x^2y^2\,dy = 0 \quad \text{——— (i)}$$

$$F(x,y) = x^2y^3 = C$$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(x^2y^3\right) = 2xy^3$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x^2y^3\right) = 3x^2y^2$$

So we can rewrite it as

$$\therefore 2xy^3\,dx + 3x^2y^2\,dy = 0 \quad \text{——— (i)}$$

$$\Rightarrow \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

Now we have to check if it is Exact or Inexact.

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)$$

We apply the same for our original function.

and if $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$

Then it is an exact Differential Equation (DE)

$F(x,y) = x^2y^3$

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) &= \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2 \\ \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) &= \frac{\partial}{\partial y}(2xy^3) = 6xy^2 \end{aligned}\right\}$$ Conclusion : Exact

Now we rewrite $\frac{\partial F}{\partial x} = M(x,y)$

and $\frac{\partial F}{\partial y} = N(x,y)$

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

$$\Rightarrow M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ [Condition For Exact]

Now

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} M$$

$$\Rightarrow \int d\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\int M\,dx$$

$$\Rightarrow \frac{\partial F}{\partial y} = \left(\frac{\partial}{\partial y}\int M\,dx\right)$$

$$\Rightarrow \int dF = \int\left(\frac{\partial}{\partial y}\int M\,dx\right)dy$$

With a bit of Mathematical Sacrilege we get.

$$\boxed{F(x,y) = \int M\,dx + \phi(y)}$$

$$\boxed{\text{or } F(x,y) = \int N\,dy + \alpha(x)}$$

Worked Examples

① $(3x+2y)\,dx + (2x+y)\,dy = 0$

$M(x,y) = 3x+2y$
$N(x,y) = 2x+y$

The condition for exactness is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x+2y) = 2 \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x+y) = 2 \end{aligned}\right\} \text{Exact}$$

$\therefore$ This is an exact ODE

Now $F(x,y) = \int M\,dx + \phi(y)$

$$= \int (3x+2y)\,dx + \phi(y)$$

$$F(x,y) = \frac{3}{2}x^2 + 2xy + \phi(y)$$

$$\frac{\partial F}{\partial y} = N(x,y)$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\frac{3}{2}x^2 + 2xy + \phi(y)\right] = N(x,y)$$

$$\Rightarrow 0 + 2x + \frac{\partial \phi}{\partial y} = N(x,y)$$

$$\Rightarrow \cancel{2x} + \frac{\partial \phi}{\partial y} = \cancel{2x} + y$$

$$\Rightarrow \int d\phi = \int y\, dy$$

$$\therefore \phi = \frac{1}{2}y^2 + C$$

$$F(x,y) = \frac{3}{2}x^2 + 2xy + \frac{1}{2}y^2 + C \text{ (Am)}$$

$$\frac{dF}{dx} = 3x + 2y + 2x\frac{dy}{dx} + y\frac{dy}{dx}$$

$$dF = (3x+2y)\,dx + (2x+y)\,dy$$

Now doing it Again with

$$F(x,y) = \int N\,dy + \alpha(x)$$

$$\Rightarrow F(x,y) = \int (2x+y)\,dy + \alpha(x)$$

$$F(x,y) = 2xy + \frac{1}{2}y^2 + \alpha(x)$$

$$\frac{\partial F}{\partial x} = M(x,y)$$

$$\Rightarrow (3x+2y) = \frac{\partial}{\partial x}\left[2xy + \frac{1}{2}y^2 + \alpha(x)\right]$$

$$3x + \cancel{2y} = \cancel{2y} + 0 + \frac{\partial \alpha}{\partial x}$$

$$\Rightarrow \int d\alpha = \int 3x\,dx$$

$$\therefore \alpha(x) = \frac{3}{2}x^2 + C$$

$$F(x,y) = 2xy + \frac{1}{2}y^2 + \frac{3}{2}x^2 + C \quad \text{(Ans)}$$

4 Integrating Factor:

Not All first order ODE's are going to be exact on their own. However, some can be algebraically manipulated into following the condition of exactness.
The multiplicative factor that imposes an exact condition here is called the integrating factor. Let us consider

Problem :

$$(3y+4xy^2)\,dx+(2x+3x^2y)\,dy=0$$

$$\frac{\partial F}{\partial x}=M(x,y)=(3y+4xy^2)$$

$$\frac{\partial F}{\partial y}=N(x,y)=(2x+3x^2y)$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(3y+4xy^2)=3+8xy\\ \frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2x+3x^2y)=2+6xy\end{aligned}\right\}\text{inexact}$$

Integrating factor.

Let us consider integrating factor $\mu(x,y)=x^m y^n$

$$=x^2y$$

$$(3y+4xy^2)\,dx+(2x+3x^2y)\,dy=0$$

$$(3x^2y^2+4x^3y^3)\,dx+(2x^3y+3x^4y^2)\,dy=0$$

$$\frac{\partial F}{\partial x}=M(x,y)=3x^2y^2+4x^3y^3$$

$$\frac{\partial F}{\partial y}=N(x,y)=2x^3y+3x^4y^2$$

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(3x^2y^2+4x^3y^3)$$

$$=6x^2y+12x^3y^2$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2x^3y+3x^4y^2)$$

$= 6x^2y + 12x^3y^2$

$$F(x,y) = \int M\,dx + \alpha(x)$$

$$= \int (3x^2y^2 + 4x^3y^3)\,dx + \phi(y)$$

$$F(x,y) = x^3y^2 + x^4y^3 + \phi(y)$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[x^3y^2 + x^4y^3 + \phi(y)\right]$$

$$\Rightarrow \cancel{2x^3y} + \cancel{3x^4y^2} = \cancel{2x^3y} + \cancel{3x^4y^2} + \frac{\partial\phi}{\partial y}$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = 0$$

$$\Rightarrow \int \partial\phi = \int 0\,dy$$

$$\therefore \phi = C$$

$$F(x,y) = x^3y^2 + x^4y^3 + C$$

But how do we know $\mu(x,y) = x^2y$? We don't!
I just got lucky but you have to assume $\mu(x,y) = x^m y^n$

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0$$

$$\Rightarrow \left(3x^m y^{n+1} + 4x^{m+1}y^{n+2}\right)dx + \left(2x^{m+1}y^n + 3x^{m+2}y^{n+1}\right)dy = 0$$

$$M(x,y) = \left(3x^m y^{n+1} + 4x^{m+1}y^{n+2}\right)$$

$$N(x,y) = \left(2x^{m+1}y^n + 3x^{m+2}y^{n+1}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(3x^m y^{n+1} + 4x^{m+1}y^{n+2}\right)$$

$$\boxed{\frac{\partial M}{\partial y} = \underline{3(n+1)}\,\underline{x^m y^n} + \underline{4(n+2)}\,x^{m+1}y^{n+1}}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(2x^{m+1}y^n + 3x^{m+2}y^{n+1}\right)$$

$$\boxed{\frac{\partial N}{\partial x} = \underline{2(m+1)}\,\underline{x^m y^n} + 3(m+2)\,x^{m+1}y^{n+1}}$$

$$3(n+1) = 2(m+1) \quad \text{—— (i)}$$

$$4(n+2) = 3(m+2) \quad \text{—— (ii)}$$

$$4n+8 = 3m+6$$

$$\rightarrow 3m-4n-2=0$$

$$\rightarrow 3n+3 = 2m+2$$

$$\left.\begin{array}{l} \therefore\ 2m-3n-1=0 \\ 3m-4n-2=0 \end{array}\right\}$$

$$m, n = 2, 1$$

$\therefore$ integrating factor was indeed $\mu(x,y) = x^2 y$

We will later use integrating factors for first order linear ODE's but this it for now.

⑤ Some Real World Examples:

Problem: (First Order drag). Let an object fall down to the ground. There is air resistence with linear drag proportional to the velocity of the falling object.

Ans:

The Boundary conditions for the given system are $v(0) = 0$ [initial velocity] and $x(0) = 0$ [Assuming the point it is dropped from is the reference point].

All Acting forces are:

Gravity (mg) [downwards]

Air drag $(-bv)$ [Upwards]

Using Newton's second law,

$$F = ma \quad \text{——— (i)}$$

$$F = mg - bv$$

$$\Rightarrow ma = mg - bv$$

$$\Rightarrow m\frac{dv}{dt} = mg - bv$$

$$\Rightarrow \frac{dv}{dt} = \left(g - \frac{bv}{m}\right)$$

$$\Rightarrow \int \frac{dv}{\left(g - \frac{bv}{m}\right)} = \int dt$$

$$\Rightarrow -\frac{m}{b}\ln\left(g - \frac{bv}{m}\right) = t + C_1$$

$$\Rightarrow \cancel{\ln}\left(g - \frac{bv}{m}\right) = e^{\left(\frac{-bt}{m} - C_2\right)}$$

$$\Rightarrow g - \frac{bv}{m} = Ce^{-\frac{bt}{m}}$$

$$\Rightarrow \frac{bv}{m} = g - Ce^{-\frac{bt}{m}}$$

$$\therefore v(t) = \frac{mg}{b} - \frac{mC}{b} e^{-bt/m}$$

When the ball is dropped from rest.

$$v(t=0) = 0$$

$$v(0) = 0$$

$$\Rightarrow \frac{mg}{b} - \frac{mC}{b} = 0$$

$$\Rightarrow \frac{\cancel{m}g}{\cancel{b}} = \frac{\cancel{m}C}{\cancel{b}}$$

$$\therefore C = g$$

$$v(t) = \frac{mg}{b} - \frac{mg}{b} e^{-bt/m}$$

$$v(t) = \frac{mg}{b}\left(1 - e^{-bt/m}\right) \quad \text{(Ans)}$$

$$\int dx = \int \frac{mg}{b}\left(1 - e^{-bt/m}\right) dt$$

$$\therefore x(t) = \frac{mg}{b}\left(t + \frac{m}{b} e^{-bt/m}\right) + C$$

at $x(0) = 0$

$$\frac{mg}{b}\left(0 + \frac{m}{b}\right) + C = 0$$

$$\Rightarrow C = -\frac{gm^2}{b^2}$$

$$\therefore x(t) = \frac{mg}{b}\left(t + \frac{m}{b} e^{-bt/m}\right) - \frac{m^2 g}{b^2}$$

$$= \frac{mg}{b}\left(t + \frac{m}{b} e^{-bt/m} - \frac{m}{b}\right)$$

This is the equation of motion for an object falling towards ground accounting for air resistance.

Problem: (Radioactive decay). A sample Radioactive element A decays into B with decay constant λ_A and B decays into C with decay constant λ_B. C is the Stable Element. Construct the Algebraic form of this whole phenomenon usin the above parameters.

Sol^n:

The Whole decay diagram looks like the following

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C$$

We know that from A to B.

$$\frac{dN_A}{dt} = -\lambda_A N_A$$

$$\int \frac{dN_A}{N_A} = \int -\lambda_A \, dt$$

$$\therefore \ln N_A = -\lambda_A t + C$$

When $t = 0$; $N_A = N_{oA}$

$$\Rightarrow \ln N_{oA} = -\lambda_A \cdot 0 + C$$

$$\therefore C = \ln N_{oA}$$

Meaning $N_A(t) = N_{oA} e^{-\lambda_A t}$

Now whatever amount of A gets decayed turns into B.

$\therefore N_B = N_{0A} - N_A = N_{0A}(1-e^{-\lambda_A t})$

Now in case of B. the amount of B material present depends on how much of it is decayed into C and how much it is produced from A.

$$\therefore \frac{dN_B}{dt} = -\lambda_B N_B + \lambda_A N_A$$

$$= -\lambda_B N_B + \lambda_A N_{0A} e^{-\lambda_A t}$$

$$\therefore \frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_{0A} e^{-\lambda_A t}$$

Here the integrating factor is λ_B, so

$$e^{\int \lambda_B dt} = e^{\lambda_B t}$$

$$\therefore e^{\lambda_B t} \frac{dN_B}{dt} + e^{\lambda_B t} \lambda_B N_B = \lambda_A N_{0A} e^{-(\lambda_A - \lambda_B)t}$$

$$\Rightarrow \frac{d}{dt}\left[N_B e^{\lambda_B t}\right] = \lambda_A N_{0A} e^{-(\lambda_A - \lambda_B)t}$$

$$\Rightarrow \int d\left(N_B e^{\lambda_B t}\right) = \lambda_A N_{0A} \int e^{(\lambda_B - \lambda_A)t}\, dt$$

$$\Rightarrow N_B e^{\lambda_B t} = \frac{\lambda_A N_{0A}}{\lambda_B - \lambda_A} e^{(\lambda_B - \lambda_A)t} + C$$

$$\therefore N_B(t) = \frac{\lambda_A N_{0A}}{\lambda_B - \lambda_A} e^{-\lambda_A t} + C e^{-\lambda_B t}$$

Now at $t = 0$; $N_B = 0$ giving us

$$0 = \frac{\lambda_A N_{0A}}{\lambda_B - \lambda_A} + C \cdot 0$$

$$\therefore C = -\frac{\lambda_A N_{0A}}{\lambda_B - \lambda_A}$$

Plugging it back in gives us the following

$$N_B(t) = \frac{\lambda_A N_{0A}}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right)$$