Topics to Cover are:

1. 2^{nd} Order linear differential equations with constants coefficients.
2. Superposition Theorem
3. Method of solving Homogeneous differential Equations
4. Method of solving Inhomogeneous differential Equations

Intro to 2^{nd} order differential equations

Second order differential equations with constant coefficients take the form of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Quick note on Superposition Theorem:

The number of solutions of any n^{th} order differential equation is n. Meaning a third order differential equation would have three solutions and n^{th} order would have n solutions in total.

Superposition Theorem

The superposition theorem states that if x_1 and x_2 are two linearly independent solutions of any given differential equation then the linear combination of x_1 and x_2 are also solutions to the DE.

$\therefore$ $(Ax_1 + Bx_2)$ is the general solution.

The damped harmonic oscillator.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Guess $= ce^{pt}$

$x(t) = ce^{pt}$

$x'(t) = Cpe^{pt}$

$x''(t) = Cp^2e^{pt}$

$mCp^2e^{pt} + bCpe^{pt} + kCe^{pt} = 0$

$\therefore Ce^{pt}(mp^2 + bp + k) = 0$

$$mp^2+bp+k=0$$

$$p=\frac{-b\pm\sqrt{b^2-4mk}}{2m}$$

$$p_1=\frac{-b+\sqrt{b^2-4mk}}{2m}$$

$$p_2=\frac{-b-\sqrt{b^2-4mk}}{2m}$$

$$mp^2+bp+k=0$$

$\therefore (p-p_1)(p-p_2)=0$ Solution

initial Guess $x(t)=c\,e^{pt}$ ——— (i)

$$\frac{dx}{dt}=c\,p\,e^{pt} \quad \text{——— (ii)}$$

(ii) $\div$ (i)

$$\frac{\frac{dx}{dt}}{x}=\frac{c\,p\,e^{pt}}{c\,e^{pt}}$$

$$\therefore \frac{1}{x}\frac{dx}{dt}=p$$

$$\therefore \left(\frac{1}{x}\frac{dx}{dt}-p_1\right)\left(\frac{1}{x}\frac{dx}{dt}-p_2\right)=0$$

$$\frac{1}{x}\frac{dx}{dt}=p_1$$

$$\Rightarrow \int\frac{dx}{x}=\int p_1\,dt$$

$$\Rightarrow e^{\ln x} = e^{(P_1 t + C_1')}$$

$$\Rightarrow x = e^{P_1 t} e^{C_1'}$$

$$\therefore \quad x = C_1 e^{P_1 t}$$

$$x = C_2 e^{P_2 t}$$

According to Superposition theorem

$$x(t) = C_1 e^{P_1 t} + C_2 e^{P_2 t}$$

But P_1, P_2 can be real, complex or equal

$$P_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m} \qquad P_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

Real Sol^n: $\quad b^2 - 4mk > 0$

$$x(t) = C_1 e^{P_1 t} + C_2 e^{P_2 t}$$

Complex Sol^n: $\quad b^2 - 4mk < 0$

$$P_1 = -\frac{b}{2m} + \left(\frac{\sqrt{b^2 - 4mk}}{2m}\right) = A + Bi$$

$$P_2 = -\frac{b}{2m} - \left(\frac{\sqrt{b^2 - 4mk}}{2m}\right) = A - Bi$$

$$x(t) = C_1 e^{P_1 t} + C_2 e^{P_2 t}$$

$$= C_1 e^{(A+Bi)t} + C_2 e^{(A-Bi)t}$$

$$= C_1 e^{At} e^{Bit} + C_2 e^{At} e^{-Bit}$$

$$= e^{At}\left\{C_1 e^{Bit} + C_2 e^{-Bit}\right\}$$

$$= e^{At}\left\{C_1(\cos Bt + i\sin Bt) + C_2(\cos Bt - i\sin Bt)\right\}$$

$$= e^{At}\left\{\underline{(C_1+C_2)}\cos Bt + i(C_1-C_2)\sin Bt\right\}$$

$$x(t) = e^{At}(D\cos Bt + E\sin Bt)$$

Equal Solution: $P_1 = P_2$

$$x(t) = C_1 e^{pt} + C_2 e^{pt}$$

$$= (C_1+C_2)e^{pt}$$

$$= Ce^{pt} +$$

Note: this doesn't give you the full picture.

$$x(t) = C_1 e^{pt} + C_2 t e^{pt} \quad [\text{Guess}]$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$x'(t) = C_1 pe^{pt} + C_2 e^{pt} + C_2 pte^{pt}$$

$$x''(t) = C_1 p^2 e^{pt} + C_2 pe^{pt} + C_2 pe^{pt} + C_2 p^2 te^{pt}$$

$$m\left\{C_1 p^2 e^{pt} + C_2 pe^{pt} + C_2 pe^{pt} + C_2 p^2 te^{pt}\right\}$$

$$+ b\left\{C_1 pe^{pt} + C_2 e^{pt} + C_2 pte^{pt}\right\} + k\left\{C_1 e^{pt} + C_2 te^{pt}\right\}$$

$$= e^{pt}\left\{mc_1p^2 + 2mc_2p + mc_2p^2t\right\} + e^{pt}\left\{bc_1p + bc_2 + bc_2pt\right\}$$

$$+ e^{pt}\left\{kc_1 + kc_2t\right\}$$

$$= te^{pt}\left\{mc_2p^2 + bc_2p + kc_2\right\} + e^{pt}\left\{mc_1p^2 + 2mc_2p + bc_1p + bc_2 + kc_1\right\}$$

$$= c_2te^{pt}\left\{mp^2 + bp + k\right\} + c_1e^{pt}\left\{mp^2 + bp + k\right\} + c_2e^{pt}\left\{2mp + b\right\}$$

$$= 0 + 0 + c_2e^{pt}\left\{\frac{d}{dp}(mp^2 + bp + k)\right\}$$

$$= 0$$

Worked example:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$$ this DE

$y = e^{px}$; $\frac{dy}{dx} = pe^{px}$; $\frac{d^2y}{dx^2} = p^2e^{px}$

$$p^2e^{px} + 3pe^{px} - 4e^{px} = 0$$

$$\Rightarrow e^{px}\underbrace{(p^2 + 3p - 4)}_{\text{Characteristic eq of}} = 0$$

Characteristic eq of

$p^2+4p-p-4=0$

$\therefore p=1,-4. \in \mathbb{R}$

$y(x)=c_1e^{x}+c_2e^{-4x}$

__Worked example.__

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$$

CE $\to$ $p^2+4p+4=0$

$\therefore (p+2)^2=0$

$p=-2,-2$

$y(x)=c_1e^{-2x}+c_2xe^{-2x}$

Roots of a characteristic eq

$\underline{4\ \ 4\ \ 4}$ $\underline{2+3i}$ $\underline{2-3i}$ $\underline{5}$

$$y(x)=c_1e^{4x}+c_2xe^{4x}+c_3x^2e^{4x}+\underline{e^{2x}(c_4\cos 3x+c_5\sin 3x)}$$
$$+c_6e^{5x}$$

Roots:

4, 4, 4, 2+3i, 2-3i, 2+3i, 2-3i, 5, 5

$$y(x)=c_1e^{4x}+c_2xe^{4x}+c_3x^2e^{4x}+e^{2x}(c_4\cos 3x+c_5\sin 3x)$$
$$+xe^{2x}(c_6\cos 3x+c_7\sin 3x)+c_8e^{5x}+c_9xe^{5x}$$