Notes on differential Equations

Topics to Cover are:

1. Bernoulli Equations
2. Special Integrating factors and transformations
3. RL Circuit Solution with complex numbers

1 Bernoulli Equation:

Bernoulli equation is a type of nonlinear 1st order differential equation for which a solution is obtainable.

it is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \quad \text{———} \quad (i)$$

it is non-linear because of y^n term

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

if we let $y^{1-n} = v$

$$\Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore\ y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$$

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Let $(1-n)P(x) = P_1(x)$

$(1-n)Q(x) = Q_1(x)$

$$\therefore\ \frac{dv}{dx} + P_1(x)v = Q_1(x)$$

$$v(x) = \frac{1}{\mu(x)}\int \mu(x)Q_1(x)\,dx$$

where $\mu(x) = e^{\int P_1(x)dx}$

Worked Example:

$$\frac{dy}{dx} + \underset{\uparrow}{\overset{P(x)=1}{y}} = \underset{Q(x)=x}{x}y^3 \qquad y(x) = ?$$

$$\therefore\ y^{-3}\frac{dy}{dx} + y^{-2} = x$$

Let $y^{-2} = v$

$$-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

$$-\frac{1}{2}\frac{dv}{dx} + v = x$$

$$\therefore \frac{dv}{dx} - 2v = -2x$$

$$\mu(x) = e^{\int(-2)dx} = e^{-2x}$$

$$v(x) = \frac{1}{e^{-2x}}\int e^{-2x}\cdot(-2x)\,dx$$

$$= -2e^{2x}\int xe^{-2x}dx$$

	D	I
		e^{-2x}
+	x	$-\frac{1}{2}e^{-2x}$
−	1	$\frac{1}{4}e^{-2x}$
+	0	

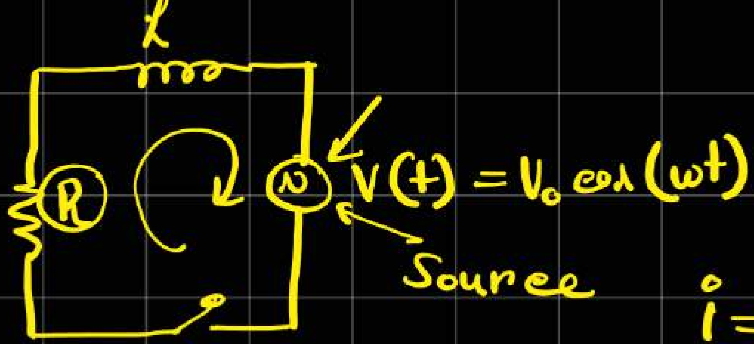
$$= -2e^{2x}\left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C\right]$$

$$v(x) = x + \frac{1}{2} - 2Ce^{2x}$$

$$y^{-3} = v$$

$$\frac{1}{y^3} = x + \frac{1}{2} - 2Ce^{2x} \quad \text{(Ans).}$$

RL Circuit with AC Source



i = current (time dependent)

Kirchoff's voltage law says that

Voltage of source = Voltage of the electrical component

$$\left.\begin{aligned} &\text{inductor voltage } V_L = \mathcal{L}\frac{di}{dt} \\ &\text{Resistor } \quad '' \quad V_R = iR \end{aligned}\right\} +$$

Source $\quad '' \quad V_S = V_0 \cos(\omega t) \}$

$$V_L + V_R = V_S$$

$$\therefore \mathcal{L}\frac{di}{dt} + iR = V_0 \cos(\omega t)$$

Complex Number: $e^{ix} = \cos x + i \sin x$

$$Re(e^{ix}) = \cos x$$

$$Im(e^{ix}) = \sin x$$

i = current

$j = \sqrt{-1}$

$$\mathcal{L}\frac{di}{dt} + Ri = V_0 e^{j\omega t}$$

$$\therefore \frac{di}{dt} + \left(\frac{R}{\mathcal{L}}\right) i = \frac{V_0}{\mathcal{L}} e^{j\omega t}$$

$$\mu(t) = e^{\int \frac{R}{\mathcal{L}} dt} = e^{Rt/\mathcal{L}}$$

$$i(t) = \frac{1}{\mu(t)} \int \mu(t)\, e^{j\omega t}\, dt$$

$$= e^{-Rt/L} \int e^{Rt/L} \frac{V_0}{L} e^{j\omega t}\, dt$$

$$= \frac{V_0}{L} e^{-Rt/L} \int e^{(R/L + j\omega)t}\, dt$$

$$= \frac{V_0}{L} e^{-Rt/L} \left[\frac{e^{(R/L + j\omega)t}}{\frac{R}{L} + j\omega} + C \right]$$

$$= \frac{V_0}{L} e^{-Rt/L} \left[\frac{e^{Rt/L}\, e^{j\omega t}}{\frac{R + \omega L j}{L}} + C \right]$$

$$= \frac{V_0}{L} \left[\frac{L\, e^{j\omega t}}{R + \omega L j} \, \frac{R - \omega L j}{R - \omega L j} + C e^{-Rt/L} \right]$$

$$= \frac{V_0}{L} \left[\frac{L\left(\cos(\omega t) + j\sin(\omega t)\right)\left(R - \omega L j\right)}{R^2 + \omega^2 L^2} + C e^{-Rt/L} \right]$$

$$i(t) = \frac{V_0}{L} \left[\frac{L\left\{R\cos(\omega t) + \omega L \sin(\omega t)\right\}}{R^2 + \omega^2 L^2} + C e^{-Rt/L} \right]$$

$i(0) = 0$

$t = 0;\ i = 0$

$$i(0) = \frac{V_0}{L} \left[\frac{RL}{R^2 + \omega^2 L^2} + C \right] = 0$$

$$\therefore C = \frac{-RL}{R^2 + \omega^2 L^2}$$

$$i(t) = \frac{V_0 R \cos(\omega t) + \omega L V_0 \sin(\omega t) - V_0 R e^{-Rt/L}}{R^2 + \omega^2 L^2}$$