Vectors and index Notations:

Often times we vectors like the following.

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

But here we are going to take a detour from our original convention and use index notation instead.

In index notations our new conventions would look like the following

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$
$$\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + A_3\hat{e}_3$$

Which is a fancy way of writing

$$\vec{A} = \sum_{i=1}^{3} A_i\hat{e}_i$$
$$\vec{B} = \sum_{j=1}^{3} B_j\hat{e}_j$$

Dot products:

By recalling our usual dot products we get that

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i$$

where $\hat{e}_1\cdot\hat{e}_1 = \hat{e}_2\cdot\hat{e}_2 = \hat{e}_3\cdot\hat{e}_3 = 1$

and $\hat{e}_1\cdot\hat{e}_2 = \hat{e}_2\cdot\hat{e}_3 = \hat{e}_3\cdot\hat{e}_1 = 0$

Using index Notations

$$\vec{A}\cdot\vec{B} = \left(\sum_{i=1}^{3} A_i \hat{e}_i\right)\cdot\left(\sum_{j=1}^{3} B_j \hat{e}_j\right)$$

$$\therefore \vec{A}\cdot\vec{B} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j (\hat{e}_i \cdot \hat{e}_j)$$

But from the properties of dot products we can observe the fact that

$$\hat{e}_i \cdot \hat{e}_j = 1 \text{ for } i \neq j$$
$$\hat{e}_i \cdot \hat{e}_j = 0 \text{ for } i \neq j$$

But writing $(\hat{e}_i \cdot \hat{e}_j)$ all the time can be a bit painful so we are going to introduce a more compact way of writing that which is called a Kronecker Delta.

Here $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1; & i=j \\ 0; & i \neq j \end{cases}$

Using that we can rewrite

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j \delta_{ij}$$

There is an even more compact way of writing things down in terms of index notations, and that is known as an Einstein Notation. Here we get rid of the summation notations because the summations are always over fixed indices going from 1 to 3 for three dimensional Spatial co-ordinates

For General Relativity it goes from zero to four and it might be even higher for other branches of theoretical physics such as String Theory/ M Theory But that is beyond the point of this note so we are going to stick with our usual indices running from 1 to 3.

Which means,

$$\vec{A}\cdot\vec{B} = \underbrace{\sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j \delta_{ij}}_{\text{General index Notation}} = \underbrace{A_i B_j \delta_{ij}}_{\text{Einstein Notation}} \quad \text{———} \quad (i)$$

$$\text{But } \vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A_i B_i = A_i B_i \quad \text{———} \quad (ii)$$

$\therefore$ Equating these two equations, results in

$$\cancel{A_i} B_i = \cancel{A_i} B_j \delta_{ij}$$

$$\therefore B_i = B_j \delta_{ij}$$

Properties of Kronecker Delta:

1. $\delta_{ij} = 1 ; i = j$ [Similar basis dot product gives one]
2. $\delta_{ij} = 0 ; i \neq j$ [Perpendicular bases make zero]
3. $\delta_{ij} = \delta_{ji}$ [Dot product is commutative]

④ $\delta_{ij} A_i = A_j$ [Property derived using equation ⓘ and eq ⓘⓘ]

⑤ $\delta_{ij} M_j = M_i$

⑥ $\delta_{ij}\delta_{jk} = \delta_{ik}$ [Extension of Rule 5, 6 to the δ]

This here wraps up dot products, Einstein Summation and Kronecker Delta.

Vector Cross product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= (A_2B_3 - A_3B_2)\hat{e}_1 - (A_1B_3 - A_3B_1)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3$$

in index Notation

$$\vec{A}\times\vec{B} = \left(\sum_{j=1}^{3} A_j\hat{e}_j\right)\times\left(\sum_{k=1}^{3} B_k\hat{e}_k\right)$$

$$\vec{A}\times\vec{B} = \sum_{j=1}^{3}\sum_{k=1}^{3} A_j B_k(\hat{e}_j\times\hat{e}_k) \text{ —— ⓘ}$$

in the cross product, the basis vectors crossing together results in another index popping up.

Just looking at the $\hat{e}_1$ Component, we had

$(A_2B_3 - A_3B_2)\hat{e}_1$

Here $j,k = 2,3$ and $j \neq k$ and $j,k \neq 1$

Using $\hat{e}_2$ Basis and its component

$(A_3B_1 - A_1B_3)\hat{e}_2$

Here $j,k = 1,3$;
$j,k \neq 2$;
$j \neq k$

For the $\hat{e}_3$ component,

$(A_1B_2 - A_2B_1)\hat{e}_3$

Here, $j,k = 1,2$;
$j,k \neq 3$
$j \neq k$

Does anyone see a pattern?

in every case if the basis component was $\hat{e}_i$ then

$j,k \neq i$
$j,k =$ Anything other than i
$j \neq k$.

Introducing another index, we can make a new notation for the cross product like we introduced Kronecker Delta for the dot product (δ_{ij}).

Using another index (i) we get the Levi-Civita Symbol (ε_{ijk}) which is summed over i from 1 to 3 For basis $\hat{e}_i$.

Meaning

$$\vec{A}\times\vec{B} = \sum_{j=1}^{3}\sum_{k=1}^{3} A_j B_k (\hat{e}_j \times \hat{e}_k)$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3} A_j B_k \left(\sum_{i=1}^{3} \varepsilon_{ijk}\right)$$

$$\vec{A}\times\vec{B} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$$

$$\boxed{\therefore\ \vec{A}\times\vec{B} = \varepsilon_{ijk} A_j B_k} \rightarrow \text{Einstein Notation}$$

What about ε_{ijk}?

ε_{ijk} Accounts for Both positive and Negative Components here.

Take

$(A_1B_2 - A_2B_1)\hat{e}_3$ for example.

Here $i = 3$; $j = 1, 2$
$k = 2, 1$

when, $j = 1, k = 2$. A_jB_k was as is.
$j = 2, k = 1$, A_jB_k had a negative sign.

Similarly for $(A_2B_3 - A_3B_2)\hat{e}_1$

$i = 1$; when $j = 2, k = 3$
" $j = 3, k = 2$

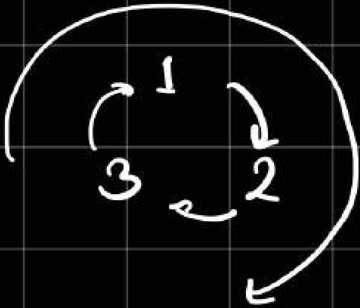
See the pattern So far?

Any situation involving going from j to k in the following manner

$1 \to 2 \to 3 \to 1 \cdots$ yields positive and

$2 \to 1 \to 3 \to 2 \cdots$ yields Negative.

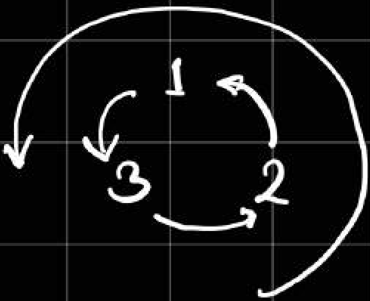
if we made clock using 1, 2, 3 then



Clockwise Rotation

(Cylic permutation)

yields positive



Anti-Clockwise Rotation

(Anti-cyclic permutatio)

yields negative.

Using All thess information above we can deduce the following properties for ε_{ijk}

i) $$\varepsilon_{ijk} = \begin{cases} 1 \,; & i,j,k \text{ Cyclic permutation} \\ -1 \,; & i,j,k \text{ Anti-cyclic permutation} \\ 0 \,; & i=j \text{ or } j=k \text{ or } k=i \end{cases}$$

Some other properties of Levi-Civita:

$$\varepsilon_{ijk}\,\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{jp} & \delta_{kp} \\ \delta_{iq} & \delta_{jq} & \delta_{kq} \\ \delta_{ir} & \delta_{jr} & \delta_{kr} \end{vmatrix}$$

For $i=p$; $\varepsilon_{ijk}=\varepsilon_{iqr}$

$$\varepsilon_{ijk}\varepsilon_{iqr} = \begin{vmatrix} \delta_{ii} & \delta_{ji} & \delta_{ki} \\ \delta_{iq} & \delta_{jq} & \delta_{kq} \\ \delta_{ir} & \delta_{jr} & \delta_{kr} \end{vmatrix}$$

$$= \delta_{ii}\begin{vmatrix} \delta_{jq} & \delta_{kq} \\ \delta_{jr} & \delta_{kr} \end{vmatrix}$$

$$\therefore \varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{kq}\delta_{jr}$$

Now you might be wondering why I didn't Account for the other cofactors of the determinant.

The Answer: That would require other indices to be repeated for the δ entries to be 1 but Repeated indices (i.e ε_{112}, $\varepsilon_{322} = 0$) Meaning the other factors will be zero anyways

For $i=p, j=q$, $\varepsilon_{ijk}=\varepsilon_{ijr}$ and $i \neq j$

$$\therefore \sum_{j=1}^{3} \varepsilon_{ijk}\varepsilon_{ijr} = \sum_{j=1}^{3}\left(\delta_{jj}\delta_{kr} - \delta_{jr}\delta_{jk}\right)$$

$$= \sum_{j=1}^{3} \delta_{jj}\delta_{kr} - \delta_{kr}$$

$$= 3\delta_{kr} - \delta_{kr}$$

$$\therefore \varepsilon_{ijk}\varepsilon_{ijr} = 2\delta_{kr}$$ [Eintein Notation]

And $\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{kj}$

$$= \delta_{jj}\delta_{kk} - \delta_{jj}$$

$$= \delta_{jj}(\delta_{kk} - 1)$$

$$= 3(3-1)$$ [Eintein Notation omits the $\sum$ but $\sum_{k=1}^{3} \delta_{kk} = 3$]

$$\therefore \varepsilon_{ijk}\varepsilon_{ijk} = 6$$

Note: Just like $\delta_{ij}\delta_{jk} = \delta_{ik}$

$$\delta_{ij}\varepsilon_{jkl} = \varepsilon_{ikl}$$

$$\delta_{ij}\varepsilon_{ijl} = \varepsilon_{jjl} = 0$$

Recap of Levi civita:

1. $\varepsilon_{ijk} = 1$ Cyclic permutation $i \to j \to k$
2. $\varepsilon_{ijk} = -1$ Anti-Cyclic permutation $k \leftarrow j \leftarrow i$
3. $\varepsilon_{ijk} = 0$ Repeating Permutation
4. $\varepsilon_{ijk}\varepsilon_{pqr} = \text{Det}\begin{pmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{pmatrix}$
5. $\varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$

⑥ $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$

⑦ $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

⑧ $\varepsilon_{ijk}\delta_{ij} = 0$

⑨ $\varepsilon_{ijk}\delta_{ip} = \varepsilon_{pjk}$

⑩ $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$

⑪ $\varepsilon_{ijk} = -\varepsilon_{jik}$ / $\varepsilon_{ijk} = -\varepsilon_{ikj}$ / $\varepsilon_{ijk} = -\varepsilon_{kji}$

Proving Some important Vector identities:

① $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \delta_{ij}A_i(\vec{B}\times\vec{C})_j$$

$$= \delta_{ij}A_i(\varepsilon_{jkl}B_kC_l)$$

$$= \varepsilon_{ikl}A_iB_kC_l$$

$$= \varepsilon_{ikp}A_iB_kC_l\delta_{lp}$$

$$= \delta_{lp}C_p\varepsilon_{ikp}A_iB_k$$

$$= \vec{C}\cdot(\vec{A}\times\vec{B})$$

Similarly we can prove that

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$ (Proven)

② $$\vec{A}\times(\vec{B}\times\vec{C})$$

$$= \varepsilon_{ijk}A_j(\vec{B}\times\vec{C})_k$$

$$= \varepsilon_{ijk}A_j(\varepsilon_{klm}B_lC_m)$$

$$= \varepsilon_{ijk}\varepsilon_{klm}A_jB_lC_m$$

$$= \varepsilon_{kij}\varepsilon_{klm}A_jB_lC_m$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) A_j B_l C_m$$

$$= \delta_{il}\delta_{jm} A_j B_l C_m - \delta_{im}\delta_{jl} A_j B_l C_m$$

$$= B_i A_m C_m - C_i A_l B_l$$

$$= B_i \delta_{mm} A_m C_m - C_i \delta_{ll} A_l B_l$$

$$\therefore \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\left(\vec{A}\times\vec{C}\right) - \vec{C}\left(\vec{A}\times\vec{B}\right) \text{ (Proven)}$$

3 $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$

$$= \delta_{ii}\left(\vec{A}\times\vec{B}\right)_i\left(\vec{C}\times\vec{D}\right)_i$$

$$= \left(\varepsilon_{ijk} A_j B_k\right)\left(\varepsilon_{imn} C_m D_n\right)$$

$$= \varepsilon_{ijk}\varepsilon_{imn} A_j B_k C_m D_n$$

$$= \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) A_j B_k C_m D_n$$

$$= \delta_{jm}\delta_{kn} A_j B_k C_m D_n - \delta_{jn}\delta_{km} A_j B_k C_m D_n$$

$$= A_m C_m B_k D_k - A_n D_n B_k C_k$$

$$= \delta_{mm} A_m C_m \delta_{kk} B_k D_k - \delta_{nn} A_n D_n \delta_{kk} B_k C_k$$

$$\therefore (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \text{ (Proven)}$$

4 $$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \vec{C}\left[(\vec{A}\times\vec{B})\cdot\vec{D}\right] - \vec{D}\left[(\vec{A}\times\vec{B})\cdot\vec{C}\right]$$

$$= \vec{C}(ABD) - \vec{D}(ABC) \text{ (Proven)}$$

Recap of everything / TLDR:

Properties of Kronecker Delta:

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② $\delta_{ij} = 0 ; i \neq j$ [Perpendicular bases make zero]

③ $\delta_{ij} = \delta_{ji}$ [Dot product is commutative]

④ $\delta_{ij} A_i = A_j$

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[Property derived using equation ⓘ and eq ⓘⓘ]

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④ $\varepsilon_{ijk} \varepsilon_{pqr} = \mathrm{Det}\begin{pmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{pmatrix}$

⑤ $\varepsilon_{ijk} \varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$

⑥ $\varepsilon_{ijk} \varepsilon_{ijm} = 2\delta_{km}$

⑦ $\varepsilon_{ijk} \varepsilon_{ijk} = 6$

⑧ $\varepsilon_{ijk} \delta_{ij} = 0$

⑨ $\varepsilon_{ijk} \delta_{ip} = \varepsilon_{pjk}$

(10) $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$

(11) $\varepsilon_{ijk} = -\varepsilon_{jik}$ / $\varepsilon_{ijk} = -\varepsilon_{ikj}$ / $\varepsilon_{ijk} = -\varepsilon_{kji}$

Vector identities:

(1) $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

(2) $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\times\vec{C}) - \vec{C}(\vec{A}\times\vec{B})$

(3) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

(4) $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \vec{C}(ABD) - \vec{D}(ABC)$