Senier Solutions:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

Taylor enpanding y about x_0 gives us

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

Plugging them Back into the equation we get

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^{n-1} + \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} \cdot p(x)$$

$$+ \; q(x)\sum_{n=0}^{\infty} a_n (x-x_0)^n = r(x)$$

By treating

i) $p(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$

ii) $q(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$

iii) $r(x) = \sum_{n=0}^{\infty} d_n (x-x_0)^n$

So the differential equation becomes

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-x_0)^n + \left[\sum_{n=0}^{\infty} b_n (x-x_0)^n\right]\sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}$$

$$+ \left[\sum_{n=0}^{\infty} c_n (x-x_0)^n\right]\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} d_n (x-x_0)^n$$

The unknown is a_n

Legendre ODE Problem

The equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+k(k+1)y=0,$$

$$\Rightarrow \frac{d^2y}{dx^2}-\underbrace{\frac{2x}{(1-x^2)}}_{x=0,\,p(x)=0}\frac{dy}{dx}+\underbrace{\frac{k(k+1)}{(1-x^2)}}_{x=0,\,q(x)=k(k+1)}y=0$$

Using the series about $x_0=0$

$$y=\sum_{n=0}^{\infty}a_n x^n,$$

$$y'=\sum_{n=0}^{\infty}n a_n x^{n-1}$$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_n e^{n-2}$$

So the equation Becomes

$$\sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)\, a_n x^n$$

$$- \sum_{n=0}^{\infty} 2n a_n x^n$$

$$+ \sum_{n=0}^{\infty} k(k+1) a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} n(n-1)\, a_n x^n$$

$$- \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} k(k+1) a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2n a_n + k(k+1) a_n \right] x^n$$

$$= 0$$

Meaning

$$a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2n a_n + k(k+1) a_n$$

$$= 0$$

$$\therefore\ a_{n+2} = \frac{a_n[n(n-1)+2n-k(k+1)]}{(n+2)(n+1)}$$

$$\therefore\ a_{n+2} = \frac{a_n(n-k)(n+k+1)}{(n+2)(n+1)}$$

Even index

$$a_1,\ a_3 = \frac{a_1(1-k)(2+k)}{3\cdot 2}$$

$$a_5 = \frac{a_3(3-k)(4+k)}{5\cdot 4}$$

$$= \frac{a_1(k-3)(k-1)(k+2)(k+4)}{5!}$$

Odd index

$$a_0,\ a_2 = \frac{a_0\cdot(-k)(k+1)}{2\cdot 1}$$

$$a_4 = \frac{a_2\cdot(2-k)(3+k)}{4\cdot 3}$$

$$= \frac{(k-2)k(k+1)(k+3)\,a_0}{4!}$$

for both solutions.

$$y_{odd} = a_1 x = a_1 P(x)$$

For $k=2$

$$a_2 = \frac{a_0 \cdot (-2)(3)}{2} = -3a_0 \,,\quad a_4 = 0$$

$$y_{even} = a_0 - 3a_0 x^2 = a_0(1-3x^2) = \frac{1}{2}(3x^2-1)$$

for $a_0 = -\frac{1}{2}$

For $k=3$

$$y_{odd} = -\frac{5}{3}a_1 x^3 + a_1 x = \frac{1}{2}(5x^3 - 3x)$$

$= P_3(x)$ for

$a_1 = -\frac{3}{2}$

Ground Rule.

$\therefore$ Even $k \rightarrow$ Even Series Terminates

Odd $k \rightarrow$ Odd Series Terminates

Here P_1, P_2, P_3 are Legendre Polynomials.

Rodrigues' Formula:

$$a_{n+2} = \frac{a_n(n-k)(n+k+1)}{(n+2)(n+1)}$$

Two possible solution for positive integer k:

i) infinite Series Solution

ii) Polynomial Solution: $y = a_0 + a_1 x + \cdots + a_{k-1} x^{k-1} + a_k x^k$

Related to Legendre Polynomials.

To find these values:

Set a_k to Calculate a_{k-2}

$$a_k = \frac{(2k)!}{2^k (k!)^2}$$

Simplifies the Legendre polynomial

$$P_k(x=1) = 1$$

$$a_n = \frac{a_{n+2}(n+2)(n+1)}{(n-k)(n+k+1)}$$

$$a_{k-2} = \frac{a_k k(k-1)}{(-2)(2k-1)}$$

Plug in a_k

$$a_{k-2} = \frac{(-1)(2k-2)!}{2^k (k-1)!(k-2)!}$$

induced Surface Charge:

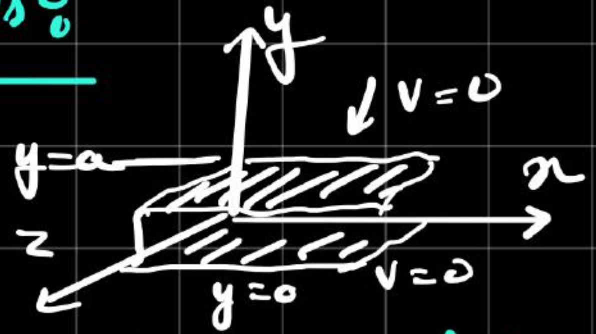
$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$ where n is the normal direction.

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \frac{\partial V}{\partial z}$$

if normal direction is in the z direction

Seperation of Variables:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$



with the following Boundary Conditions:

$$\left.\begin{array}{l} \text{i) } V=0, \text{ when } y=0, \\ \text{ii) } V=0, \text{ when } y=a, \\ \text{iii) } V=V_0(y) \text{ when } x=0, \\ \text{iv) } V\to 0 \text{ as } x\to\infty \end{array}\right\}$$

Seperating the varibles gives us

$$V(x,y) = X(x)Y(y)$$

So $\quad Y\dfrac{\partial^2 X}{\partial x^2} + X\dfrac{\partial^2 Y}{\partial y^2} = 0$

$$\therefore \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2$$

Which means

$$\frac{\partial^2 X}{\partial x^2} = k^2 X$$

$$\frac{\partial^2 Y}{\partial y^2} = -k^2 Y$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

$\therefore X(x) = Ae^{kx} + Be^{-kx}$

So $V(x,y) = X(x)\,Y(y)$

$$= \underline{\left(Ae^{kx} + Be^{-kx}\right)}\left(C\sin(ky) + D\cos(ky)\right)$$

Now $V = 0$ as $x \to \infty$

Meaning $Ae^{kx} + Be^{-kx} = 0$

Now $0 + \infty \neq 0$ unless $A = 0$.

Meaning

$$V(x) = e^{-kx}\left(C\cos(ky) + D\sin(ky)\right)$$

$V = 0$ when $y = 0$.

$\therefore C\cos 0 + D\sin 0 = 0$

$\Rightarrow C = 0$

$\therefore V(x,y) = De^{-kx}\sin ky.$

And ii) yields the fact that

$$k = \frac{\pi n}{a}\,;\ n \in \mathbb{Z}$$

Since Laplace's Equation is linear

it can make any linear combination

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \cdot + \cdot \cdots = 0$$

$$\nabla(\nabla V) = -\vec{E}$$

$$\Rightarrow \nabla^2 V = -\cancel{\nabla \vec{E}}^{\,0}$$

$$\therefore \nabla^2 V = 0$$

By enploiting the above relation we can say that

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$\text{and } V(0,y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$$

Now $\int_0^a V_0 \sin(n'\pi y/a)\,dy$

$$= \int_0^a \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) \sin(n'\pi y/a)\,dy$$

$$= \sum_{n=1}^{\infty} C_n \left(\frac{a}{2}\right) \delta_{nn'}$$

$$\Rightarrow \therefore C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\,dy$$

Symmetry :

Laplace's equation yields

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

Azimuthal Symmetry means V is independant of ϕ

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0.$$

The solution of which looks like

$$V(r,\theta)=R(r)\,\Theta(\theta)$$

We get the following by differentiating

V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$

and $$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -l(l+1).$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

$$\Theta(\theta) = P_l(\cos\theta)$$ Legendre Polynomial.

where $$P_l(x) = \frac{1}{2^l\, l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

$$\therefore V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$