Series Solution Methods

We know for a fact that any elementary function defined within an interval can be expressed as a taylor Series of the form:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots c_n x^n + \cdots$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{———} \quad (i)$$

and if $f(x)$ is a solution to any given differential equations then we can expand out its differential form to solve the differential equation.

Let's suppose a simple harmonic oscillator given by

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{——————} \quad (ii)$$

if the Solution is given by $x(t) = \sum_{n=0}^{\infty} c_n t^n$

then

$$\rightarrow x'(t) = \sum_{n=0}^{\infty} n c_n t^{n-1} = 0 + \sum_{n=1}^{\infty} n c_n t^{n-1}$$ and

$$x''(t) = \sum_{n=1}^{n} n(n-1) c_n t^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n t^{n-2}$$

We can re-inden it by setting

$p = n-2$ which means

$$\left.\begin{array}{l} n = p+2 \\ n-2 = p \end{array}\right\} \text{ and when } \begin{array}{l} n=2;\ p=0 \\ n=\infty;\ p=\infty \end{array}$$

Therefor $f''(t)$ can be rewritten interms of p given by the following

$$x''(t) = \sum_{p=0}^{\infty} (p+2)(p+1) c_{p+2} t^p$$

By p is just a dummy inden in that it doesn't matter whether we use p or n as long as we are consistent with our indexing, So we are gonna switch back from p to n just by setting $p=n$ in this case. which yields the result

$$x''(t)=\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}t^{n} \quad \text{——— (iii)}$$

Then our original equation becomes

$$x''(t)+\omega^2 x(t)=0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}t^{n}+\omega^2\sum_{n=0}^{\infty}C_n t^{n}=0$$

$$\therefore \sum_{n=0}^{\infty}\left[(n+2)(n+1)C_{n+2}+\omega^2 C_n\right]t^{n}=0$$

Now the Sum itself has no reason to be zero for $t\neq 0$, which leaves us with the only choice to set the coefficient of t^n to be equal to zero which would yield the result.

$$(n+2)(n+1)c_{n+2} + \omega^2 c_n = 0$$

$$\therefore c_{n+2} = \frac{-\omega^2}{(n+2)(n+1)} c_n$$

This gives us a recursion relationship between any specific index and then two more. Which means for odd indices we can trace them back to c_1 and for even c_2.

Lets take the odd case:

for $n=1$; $c_{n+2} = c_3$ and $c_n = c_1$ and

$$c_3 = \frac{-\omega^2}{3 \cdot 2} c_1 = -\frac{\omega^2}{3!} c_1$$

for $n=3$;

$$c_5 = -\frac{\omega^2}{5 \cdot 4} \cdot c_3 = -\frac{\omega^2}{5 \cdot 4}\left(-\frac{\omega^2}{3!} c_1\right) = \frac{\omega^4}{5!} c_1$$

you can go through the algebra steps to figure out $c_7, c_9, c_{11} \cdots$ which I'm writing down here.

$$C_7 = -\frac{\omega^6}{7!} c_1$$

$$C_9 = \frac{\omega^8}{9!} c_1$$

$$C_{11} = -\frac{\omega^{10}}{11!} c_1$$

$$\vdots$$

For the even case:

for $n=0$; $C_{n+2} = C_2$ and $C_n = C_0$ and

$$C_2 = \frac{-\omega^2}{2\cdot 1} c_0 = -\frac{\omega^2}{2!} c_0$$

for $n=3$;

$$C_4 = -\frac{\omega^2}{4\cdot 3} \cdot c_2 = -\frac{\omega^2}{4\,3}\left(-\frac{\omega^2}{2!} c_0\right) = \frac{\omega^4}{4!} c_0$$

$$c_6 = \frac{-\omega^6}{6!} c_0$$

$$c_8 = \frac{\omega^8}{8!} c_0$$

$$c_{10} = \frac{\omega^{10}}{10!} c_0$$

$$\vdots$$

I think you can find the pattern. Since our original

Solution was

$$x(t) = \sum_{t=0}^{\infty} c_n t^n$$

$$= c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t^6 + c_7 t^7 + \cdots$$

$$= c_0 + c_1 t - \frac{\omega^2 t^2}{2!} c_0 - \frac{\omega^2 t^3}{3!} c_1 + \frac{\omega^4 t^4}{4!} c_0 + \frac{\omega^4 t^5}{5!} c_1 - \frac{\omega^6 t^6}{6!} c_0 - \frac{\omega^6 t^7}{7!} c_1$$

$$+ \cdots - \quad -$$

$$= c_0 \left(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \frac{\omega^6 t^6}{6!} + \cdots\right)$$

$$+ \frac{c_1}{\omega} \left(\omega t - \frac{\omega^3 t^3}{3!} + \frac{\omega^5 t^5}{5!} - \frac{\omega^7 t^7}{7!} + \cdots\right)$$

$$= c_0 \cos(\omega t) + \frac{c_1}{\omega} \sin(\omega t)$$

$$= A\cos(\omega t) + B\sin(\omega t)$$ [Setting $c_0 = A$ and $\frac{c_1}{\omega} = B$]

(Ans)

Note: Now that I've done this, I wouldn't demonstrate the process any further and jump straight into the solution process.

Problem 1:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Let $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=1}^{\infty} c_n x^n$

Therefore $x\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^n$

and $\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$

$\therefore \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Now $x\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1}$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\therefore (2c_2 + c_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) c_{n+2} + n c_n + c_n\right] x^n = 0$$

Therefore $2c_2 + c_0 = 0$

and $(n+2)(n+1)\, c_{n+2} + n\, c_n + c_n = 0$

$$\Rightarrow (n+2)\,(n+1)\, c_{n+2} + (n+1)\, c_n = 0$$

$$\Rightarrow (n+2)\, c_{n+2} + c_n = 0$$

$$\therefore\ c_{n+2} = \frac{-c_n}{(n+2)}$$

For $n = 1, 3, 5, 7, 9 \ldots$

$$c_3 = -\frac{c_1}{3\cdot 1}$$

$$c_5 = -\frac{c_3}{5} = \frac{c_1}{5\cdot 3\cdot 1}$$

$$c_7 = -\frac{c_1}{7\cdot 5\cdot 3\cdot 1}$$

$$c_9 = \frac{c_1}{9\cdot 7\cdot 5\cdot 3\cdot 1}$$

$\vdots$

$$c_{2k+1} = \frac{(-1)^k 2k\,(2k-2)(2k-4)\cdots}{(2k+1)(2k)(2k-1)\cdots 1}\, c_1$$

$$c_{2k+1} = \frac{(-1)^k\, 2^k k!}{(2k+1)!}\, c_1$$

For $n = 0, 2, 4, 6, 8 \ldots$

$$c_2 = \frac{-c_0}{2\cdot 1}$$

$$c_4 = -\frac{c_2}{4} = \frac{c_0}{4\cdot 2}$$

$$c_6 = \frac{-c_0}{6\cdot 4\cdot 2}$$

$\vdots$

$$c_{2k} = \frac{(-1)^{k+1}\, c_0}{2^k\, k!}$$

$$\therefore\ y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$= C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + \cdots$$

$$= C_0 + C_1 x - \frac{x^2}{2} C_0 - \frac{x^3}{3} C_1 + \frac{x^4}{4\cdot 2} C_0 + \frac{x^5}{5\cdot 3} C_1 - \frac{x^6}{6\cdot 4\cdot 2} C_0$$

$$- \frac{x^7}{7\cdot 5\cdot 3} C_1 + \cdots$$

$$= C_0\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots\right) + C_1\left(x - \frac{x^3}{3} + \frac{x^5}{15} - \frac{x^7}{105} + \cdots\right)$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1} x^{2n}}{2^n n!}\right] C_0 + \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}\, 2^n\, n!}{(2n+1)!}\right] C_1$$

(Ans)

Case in point:

$$\prod_{k=0}^{n}(2k+1) = 1\cdot 3\cdot 5\cdot 7\cdot 9\cdot \cdots (2n-3)\cdot(2n-1)\cdot(2n+1)$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9 \cdots (2n+1)}{2\cdot 4\cdot 6\cdot 8\cdots 2n}$$

$$= \frac{(2n+1)!}{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot(2\cdot 4)\cdots(2\cdot n)}$$

$$\therefore \prod_{k=0}^{n}(2k+1) = \frac{(2n+1)!}{2^n\, n!}$$ (Proven)

and $$\prod_{k=0}^{n}(2k) = 2\cdot 4\cdot 6\cdot 8\cdots 2n$$

$$= (2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdots(2\cdot n)$$

$$\therefore \prod_{k=0}^{n} = 2^n n! \quad \text{(Proven)}$$

I will be using this frequently without any algebraic Steps Showing how things went from one to another.

Problem ③

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (2x^2+1)\,y = 0$$

Let $y(x) = \sum_{n=0}^{\infty} c_n x^n =$

$$(2x^2+1)\,y = (2x^2+1)\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} 2c_n x^{n+2} + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} 2c_{n-2}x^n + \sum_{n=0}^{\infty} c_n x^n$$

$$(2x^2+1)\,y = c_0 + c_1 x + \sum_{n=2}^{\infty}\left[2c_{n-2} + c_n\right]x^n$$

$$xy' = \sum_{n=0}^{\infty} n c_n x^n = c_1 x + \sum_{n=2}^{\infty} n c_n x^n$$

$$y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)\,c_{n+2}\,x^n.$$

$$= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+1)(n+2)c_{n+2}x^n$$

$$\therefore \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (2x^2+1)y = 0$$

$$(c_0+2c_2)+(2c_1+6c_3)x+\sum_{n=2}^{\infty}\left[(n+1)(n+2)c_{n+2}+nc_n+2c_{n-2}+c_n\right]$$

$$\therefore (c_0+2c_2)+(2c_1+6c_3)x+\sum_{n=2}^{\infty}\left[(n+1)(n+2)c_{n+2}+(n+1)c_n+2c_{n-2}\right]x^n=0$$

Which means the co-efficients add up to zero

$$c_0+2c_2=0$$

$$2c_1+6c_3=0$$

$$(n+1)(n+2)c_{n+2}+(n+1)c_n+2c_{n-2}=0$$

$$\therefore c_{n+2}=-\frac{c_n}{(n+2)}-\frac{2c_{n-2}}{(n+2)(n+1)}$$

For even indices

$$c_2=-\frac{c_0}{2}$$

Now if $n=2$ then

$$c_4=-\frac{c_2}{4}-\frac{2c_0}{12}=\frac{c_0}{8}-\frac{c_0}{6}=-\frac{c_0}{24}$$

For odd indices

$$C_3 = -\frac{C_1}{3}$$

when $n=3$

$$C_5 = -\frac{C_3}{5} - \frac{2C_1}{4\cdot 5} = \frac{C_1}{15} - \frac{C_1}{10} = -\frac{1}{30}$$

$$\therefore\ y(x) = C_0 + C_1 x - \frac{C_0}{2}x^2 - \frac{C_1}{3}x^3 - \frac{C_0}{24}x^4 - \frac{C_1}{30}x^5$$

$$= C_0\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots\right) + C_1\left(x - \frac{x^3}{3} - \frac{x^5}{30} + \cdots\right)$$

(Ans)

Problem 4:

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-4)y = 0$$

Let $y(x) = \sum_{n=0}^{\infty} C_n x^n$

$$\therefore (x^2-4)y = \sum_{n=0}^{\infty} C_n x^{n+2} - 4\sum_{n=0}^{\infty} C_n x^n$$

$$= \sum_{n=2}^{\infty} C_{n-2}x^n - 4C_0 - 4C_1 x + \sum_{n=2}^{\infty} 4C_n x^n$$

$$= -4C_0 - 4C_1 x + \sum_{n=2}^{\infty}(C_{n-2} - 4C_n)x^n$$

$$xy'(x) = \sum_{n=1}^{\infty} nC_n x^n = C_1 + \sum_{n=2}^{\infty} nC_n x^n$$

$$y''(x) = 2C_2 + 6C_3 x + \sum_{n=2}^{\infty} (n+1)(n+2) C_{n+2} x^n$$

$$\therefore \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-4)y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} \left[(n+1)(n+2)C_{n+2} + nC_n - 4C_n + C_{n-2}\right]x^n$$

$$+(-4C_0+2C_2) + (-4C_1+6C_3)x = 0$$

Therefore the coefficients add upto zero

$$-4C_0+2C_2=0 \Longrightarrow C_2 = 2C_0$$
$$-4C_1+6C_3=0 \Longrightarrow C_3 = \frac{2}{3}C_0$$

$$(n+1)(n+2)C_{n+2} + (n-4)C_n + C_{n-2} = 0$$

$$\therefore C_{n+2} = -\frac{(n-4)C_n + C_{n-2}}{(n+1)(n+2)}$$

To be continued.