Collatz ghost cycles and multi-level sequences

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Abstract

Let $\mathrm{Col}: \mathbb{N} \to \mathbb{N}$ the well known *Collatz map* by $\mathrm{Col}(N) := 3N+1$ if N is odd, and $\mathrm{Col}(N) := \frac{1}{2}N$ if N is even. We present concatenations of Syracuse maps by parameters $k_a \in \mathbb{N}$, by $a \in \mathbb{N}$ being the number of even steps between two odd numbers, to make our investigations mostly independent of any original parameter, and let us present two main results about regular Collatz sequences. The first one limits the set of k_a values on $\{0,1,2\}$ or $k_{a_i} = k_{a_{i+1}}$ for all k_{a_i}, $k_{a_{i+1}}$, for the existence of possible Collatz cycles by introducing so-called *Collatz ghost cycles*. The second one shows *multi-level sequence* properties for k_a parameters in such a way that, among other things, for the concatenation of two subsequently Syracuse steps in infinity Collatz sequences with a_1 and $a_2 = 1$ for two odd numbers $N_{a_1}(k_{a_1}), N_{a_2}(k_{a_2}) \in \mathbb{N}$, $k_{a_1}, k_{a_2} \in \mathbb{N}_0$, with $\mathrm{Syr}(N_{a_1}(k_{a_1})) = N_{a_2}(k_{a_2})$, the parameter k_{a_1} performs one Syracuse map itself by $\mathrm{Syr}(k_{a_1}) = k_{a_2}$ parameterized on lower ranges.

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1 Introduction

In 1937, Lother Collatz introduced the *Collatz conjecture* [6, 9]. Be $\mathbb{N} := \{1,2,\dots\}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then the *Collatz map* $\mathrm{Col} : \mathbb{N} \to \mathbb{N}$ is given by $\mathrm{Col}(N) := 3N+1$ if N is odd, and $\mathrm{Col}(N) := \frac{1}{2}N$ if N is even (OEIS A006370) [8]. We call $\mathrm{Col}^{\mathbb{N}_0}(N) := \{N, \mathrm{Col}(N), \mathrm{Col}^2(N), \dots\}$ the *Collatz orbit*, and $\mathrm{Col}_{\min}(N) := \min \mathrm{Col}^{\mathbb{N}_0}(N) = \inf_{n\in\mathbb{N}_0} \mathrm{Col}^n(N)$, for any $N \in \mathbb{N}$. Collatz' conjecture is then stated by

Conjecture 1 (Collatz conjecture)**.** *For all $N \in \mathbb{N}$, it is $\mathrm{Col}_{\min}(N) = 1$.*

The sequence given by $1 \to 4 \to 2 \to 1$ is to be called the *trivial cycle*, the only known cycle so far. The conjecture can be splitted into two subquestions:

1. Does exist also an other cycle apart from the trivial one?
2. Does exist at least one number which gives an infinity sequence, not ending in any cycle at all?

Over the years, several approaches for answering this questions have been given. So, N. Yoneda [6] verified by computer the conjecture up to $N = 2^{40} \approx 1.1 \cdot 10^{12}$, as well as K. Ishihata up to $N = 3 \cdot 10^{12}$. In 1981, Garner [3] showed that any non trivial cycle must have at least $35,400$ terms, which got later improved by Eliahou [2] to $17,087,915$ elements, in 1993. Simons and de Weger investigated in m-cycles, by m, also called *local minima*, being the numbers in it appearing in m sequences [10]. They showed that there do not exist non trivial m-cycles for $1 \le m \le 75$. Finally, Hercher raised this range to $m \ge 92$ [4], in 2023. Finally, Tao [11] improved a former result of Korec [5], who proved that for any $\theta > \frac{\log 3}{\log 4} \approx 0.7924$, one has $\mathrm{Col}_{\min}(N) \le N^{\theta}$ for almost all $N \in \mathbb{N}+1$ (in the sense of natural density), by showing that for any function $f : \mathbb{N}+1 \to \mathbb{R}$ with $\lim_{N\to\infty} f(N) = +\infty$, one has $\mathrm{Col}_{\min}(N) \le f(N)$ for almost all $N \in \mathbb{N}+1$ (in the sense of logarithmic density). For more information and a complete survey of the history of research in the Collatz conjecture, see also [6].

Looking at the mentioned work above, we notice that all of them investigate, in one way or another, in the evolution of the numbers of the Collatz sequence, their amount of odd steps in ratio to the amount of even steps between two odd steps, based on their values, or in their related ratio or density properties. In our work, we want to investigate in the two given questions by looking at this mentioned parameters as less as possible, and instead by focusing our attention on new introduced k_a parameters, which makes the proofs of our main statements mostly independent of any of the original parameters.

In section 2 (*Collatz maps and concatenations*), we will build up the necessary fundamentals by using Syracuse maps and establishing their concatenations, by introducing our new parameters $k_a \in \mathbb{N}_0$ for each odd map step with the maximum number of follow-up even steps $a \in \mathbb{N}$. This construction gives us in section 3 (*Collatz ghost cycles*) the chance to constitute so-called *Collatz ghost cycles*, a concept for Collatz sequences which share some properties with Collatz cycles without being cycles themselves. Finally, in section 4 (*Multi-level sequences*), we will use our approach to go by infinity Collatz sequences and prove that some of our introduced k_a parameters also show some Collatz map behaviors, which helps us to make statements about their overall evolution within sequences and the original Collatz sequence itself.

Here, we can sum up our final main achievements of this paper by:

- The set of $k_{a_i}, k_{a_{i+1}} \in \mathbb{N}_0$ values of $N_{a_i}(k_{a_i})$, $N_{a_{i+1}}(k_{a_{i+1}})$ forming Collatz cycles is limited to $\{0,1,2\}$ or being $k_{a_i} = k_{a_{i+1}}$, for all $k_{a_i}, k_{a_{i+1}}$ of the cycle.
- Be $a_1, a_2 \in \mathbb{N}$, with $a_2 = 1$, the number of even steps of the concatenation of two subsequently Syracuse steps for two odd numbers $N_{a_1}(k_{a_1}), N_{a_2}(k_{a_2}) \in \mathbb{N}$, $k_{a_1}, k_{a_2} \in \mathbb{N}_0$, with $\mathrm{Syr}(N_{a_1}(k_{a_1})) = N_{a_2}(k_{a_2})$, then the parameter k_{a_1} performs one Syracuse map itself by $\mathrm{Syr}(k_{a_1}) = k_{a_2}$, with $k_{a_1}(\tilde{k}_{a_2})$, $\tilde{k}_{a_2} \in \mathbb{N}_0$, such that $\tilde{k}_{a_2} < k_{a_1}$.

2 Collatz maps and concatenations

Over the time, several formularization of concatenations of the Collatz map have been made [6, 3, 10]. In our work, we are going by so-called Syracuse maps like e.g. already used in works of Crandall [1] and Tao [11].

2.1 Syracuse maps

Let $2\mathbb{N}$ be the set of all even, and $2\mathbb{N}-1$ the set of all odd natural numbers. Let us go with the *Syracuse map* $\mathrm{Syr}\colon 2\mathbb{N}-1 \to 2\mathbb{N}-1$ (OEIS A075677) [7], the largest odd natural number dividing $3N+1$, like for example $\mathrm{Syr}(3)=5$, and $\mathrm{Syr}(7)=11$. We define the affine map $\mathrm{Aff}_a : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{N}$, by

$$\mathrm{Aff}_a(x) := \frac{3x+1}{2^a},$$

so that

$$\mathrm{Syr}(N) = \mathrm{Col}^{\nu_2(3N+1)+1}(N) = \mathrm{Aff}_{\nu_2(3N+1)}(N), \tag{1}$$

if $\nu_p(M)$, for each integer M, and each prime p, gives the *p-valuation*, defined as the largest natural number a such that p^a divides M (with the convention $\nu_p(0) = +\infty$) [11]. Let $\mathbb{N}_a$ be the set of all $N_a \in 2\mathbb{N}-1$ for which $\mathrm{Syr}(N_a) = \mathrm{Aff}_{\nu_2(3N_a+1)}(N_a) = \mathrm{Aff}_a(N_a)$. Furthermore, be $\overline{\mathbb{N}}_a$ the set of all $\overline{N}_a \in 2\mathbb{N}-1$, for which $\mathrm{Syr}\left(\overline{N}_a\right)$, with $\nu_2\left(3\overline{N}_a+1\right) > \nu_2(3N_a+1)$, and $\mathrm{Syr_i}\colon 2\mathbb{N}-1 \to 2\mathbb{N}$, $i \in \mathbb{N}_0$ with $i < \nu_2\left(3\overline{N}_a+1\right)$, by $\mathrm{Syr}_i\left(\overline{N}_a\right) := \mathrm{Aff}_i\left(\overline{N}_a\right)$. Then, we can put this together to interpret this as a binary tree.

Definition 1 (The Syracuse tree)**.** *Be $SyrT = (V,E)$ a directed and rooted binary tree, with root $v_0 := 2\mathbb{N}-1$, $v_0 \in V$ at depth $d=0$. For all $d=a$, be the left child, a leaf vertex, $v_{2d-1} := \mathrm{Syr}(N_a)$, and the right child, a inner vertex, $v_{2d} := \left(\mathrm{Syr}_d\left(\overline{N}_a\right), N^{a+1}\right)$, with $N^{a+1} := \overline{N}_{a+1} \cup N_{a+1}$, $v_{2d-1}, v_{2d} \in V$, for all $N_a \in \mathbb{N}_a$, $\overline{N}_a \in \overline{\mathbb{N}}_a$, for all $a \in \mathbb{N}$.*

Lemma 2.1 (Syracuse tree values)**.** *Let $N^{a+1} \in \mathbb{N}$ and $a \in \mathbb{N}$. Then with $d=a$, for all $a \in 2\mathbb{N}-1$ it is*

$$\overline{N}_a(k_a) = 2^{a+1}k_a + \frac{2^2}{3}\left(2^{a-1}-1\right)+1, \qquad \mathrm{Syr}_d\left(\overline{N}_a(k_a)\right) = 3\cdot 2k_a + 2 \tag{2}$$

$$N_a(k_a) = 2^{a+1}k_a + 2^a + \frac{2^2}{3}\left(2^{a-1}-1\right)+1, \quad \mathrm{Syr}(N_a(k_a)) = 3\cdot 2k_a + 5, \tag{3}$$

and for all $a \in 2\mathbb{N}$ it is

$$N_a(k_a) = 2^{a+1}k_a + \frac{2^2}{3}\left(2^{a-2}-1\right)+1, \qquad \mathrm{Syr}(N_a(k_a)) = 3\cdot 2k_a + 1 \tag{4}$$

$$\overline{N}_a(k_a) = 2^{a+1}k_a + 2^a + \frac{2^2}{3}\left(2^{a-2}-1\right)+1, \quad \mathrm{Syr}_d\left(\overline{N}_a(k_a)\right) = 3\cdot 2k_a + 2^2, \tag{5}$$

with $k_a \in \mathbb{N}_0$.

Proof. Let $N(k_0) = 2k_0+1$ be $N \in 2\mathbb{N}-1$, with $k \in \mathbb{N}_0$, and we get $\mathrm{Col}(N(k_0)) = 3(2k_0+1)+1 = 3\cdot 2k_0 + 2^2$. Now, be $a=1$, we receive $\mathrm{Col}^2(N(k_0)) = \frac{1}{2}\left(3\cdot 2k_0 + 2^2\right) = 3k_0+2$, and we split $N(k_0)$ into two cases, with $k_1 \in \mathbb{N}_0$.

(a) Be $k_0 := 2k_1$:
Then we have $N(k_1) = 2(2k_1)+1 = 2^2k_1+1$, and $\mathrm{Col}^2(N(k_1)) = 3(2k_1)+2 = 3\cdot 2k_1+2$.

(b) Be $k_0 := 2k_1+1$:
Then we have $N(k_1) = 2(2k_1+1)+1 = 2^2k_1+2+1$, and $\mathrm{Col}^2(N(k_1)) = 3(2k_1+1)+2 = 3\cdot 2k_1+5$.

After this, we go by $a=2$, get $\mathrm{Col}^3(N(k_1)) = \frac{1}{2}(3\cdot 2k_1+2) = 3k_1+1$, which we can also split by introducing $k_2 \in \mathbb{N}_0$, anyway.

(a) Be $k_1 := 2k_2$:
Then we have $N(k_2) = 2^2(2k_2)+1 = 2^3k_2+1$, and $\mathrm{Col}^3(N(k_2)) = 3(2k_2)+1 = 3\cdot 2k_2+1$.

(b) Be $k_1 := 2k_2 + 1$:
Then we have $N(k_2) = 2^2(2k_2+1)+1 = 2^3k_2+2^2+1$, and $\mathrm{Col}^3(N(k_2)) = 3(2k_2+1)+1 = 3\cdot 2k_2 + 2^2$.

With the even result of $a = 2$, we are back at the case with which we started. Hence, we can now generalize it for $a \in 2\mathbb{N}-1$ to

(a) $\overline{N}_a(k_a) = 2^{a+1}k_a + \sum_{j=1}^{\frac{a-1}{2}} 2^{2j} + 1 = 2^{a+1}k_a + \frac{2^2}{3}\left(2^{a-1}-1\right)+1$
$\mathrm{Syr}_d\left(\overline{N}_a(k_a)\right) = 3\cdot 2k_a + 2$

(b) $N_a(k_a) = 2^{a+1}k_a + 2^a + \sum_{j=1}^{\frac{a-1}{2}} 2^{2j} + 1 = 2^{a+1}k_a + 2^a + \frac{2^2}{3}\left(2^{a-1}-1\right)+1$
$\mathrm{Syr}(N_a(k_a)) = 3\cdot 2k_a + 5$

and for $a \in 2\mathbb{N}$ we receive

(a) $N_a(k_a) = 2^{a+1}k_a + \sum_{j=1}^{\frac{a-2}{2}} 2^{2j} + 1 = 2^{a+1}k_a + \frac{2^2}{3}\left(2^{a-2}-1\right)+1$
$\mathrm{Syr}(N_a(k_a)) = 3\cdot 2k_a + 1$

(b) $\overline{N}_a(k_a) = 2^{a+1}k_a + 2^a + \sum_{j=1}^{\frac{a-2}{2}} 2^{2j} + 1 = 2^{a+1}k_a + 2^a + \frac{2^2}{3}\left(2^{a-2}-1\right)+1$
$\mathrm{Syr}_d\left(\overline{N}_a(k_a)\right) = 3\cdot 2k_a + 2^2$,

with $k_a \in \mathbb{N}_0$. $\square$

Lemma 2.2 (Syracuse tree density). *Let $N^{a+1} \in \mathbb{N}$, then for all $a \in \mathbb{N}$, N_a have a density of $\rho(N_a) = 2^{-a}$ within the set $2\mathbb{N}-1$.*

Proof. From the given proof 2.1 of Syracuse tree values, it is obvious that for each $a \to a+1$, we have done a split of the remaining set of numbers $\overline{N}_{a-1}$ by 2, which leads to the density for the numbers N_a within $2\mathbb{N}-1$ by $\rho(N_a) = 2^{-a}$. $\square$

2.2 Affine map concatenations

Going by $\mathrm{Syr}(N_a)$ for a single odd-to-odd sequence step, we follow up with the concatenation $\mathrm{Syr}^n(N)$, $n \in \mathbb{N}_0$, which can be also described by the composition $\mathrm{Aff}_{\vec{a}} = \mathrm{Aff}_{a_1,\ldots,a_n} : \mathbb{R} \to \mathbb{R}$, with

$$\mathrm{Aff}_{a_1,\ldots,a_n}(x) := \mathrm{Aff}_{a_n}\left(\mathrm{Aff}_{a_{n-1}}\left(\ldots\left(\mathrm{Aff}_{a_1}(x)\right)\ldots\right)\right), \tag{6}$$

and the finite tuple $\vec{a} = (a_1,\ldots,a_n) \in \mathbb{N}^n$, to

$$\mathrm{Syr}^n(N) = \mathrm{Aff}_{\vec{a}^{(n)}(N)}(N), \tag{7}$$

with the *tuple size* $|\vec{a}|$ of a tuple $\vec{a}$ as

$$|\vec{a}| := a_1 + \cdots + a_n, \tag{8}$$

and

$$\vec{a}^{(n)}(N) := \left(\nu_2(3N+1), \nu_2(3\,\mathrm{Syr}(N)+1), \ldots, \nu_2\left(3\,\mathrm{Syr}^{n-1}(N)+1\right)\right). \tag{9}$$

We call $\vec{a}^{(n)}(N)$ the *n-Syracuse valuation*, with $\vec{a}^{(n)}(N) \in \mathbb{N}^n$ [11]. Finally, Tao already proved that $\vec{a}^{(n)}(N)$ is unique.

Lemma 2.3 (Description of n-Syracuse valuation [11] (Lemma 2.1)). *Let $N \in 2\mathbb{N}-1$ and $n \in \mathbb{N}_0$. Then $\vec{a}^{(n)}(N)$ is the unique tuple $\vec{a}$ in $\mathbb{N}^n$ for which $\mathrm{Aff}_{\vec{a}}(N) \in 2\mathbb{N}-1$.*

With this we define the *Syracuse orbit* by $\mathrm{Syr}^{\mathbb{N}_0}(N) := \{N, \mathrm{Syr}(N), \mathrm{Syr}^2(N), \dots\}$, for any $N \in \mathbb{N}$, and receive the alternative formulation of the Collatz conjecture with the identity

$$\mathrm{Col}_{\min}(N) = \mathrm{Syr}_{\min}\left(N/2^{\nu_2(N)}\right), \tag{10}$$

and the rephrasing of the Collatz conjecture by

Conjecture 2 (Collatz conjecture in Syracuse formulation)**.** *For all $N \in 2\mathbb{N}-1$, it is $\mathrm{Syr}_{\min}(N) = 1$.*

We merge the equations (3) and (4) together by writing in short, for all $a \in \mathbb{N}$,

$$\begin{aligned} N_a(k_a) &= 2^{a+1}k_a + M_a(a) + 1 \\ \mathrm{Syr}\left(N_a^{-1}(k_a)\right) &= 3 \cdot 2k_a + S_a(a) + 1, \end{aligned} \tag{11}$$

with

$$S_a(a) := \begin{cases} 2^2 & \text{if } a \in 2\mathbb{N}-1 \\ 0 & \text{if } a \in 2\mathbb{N}, \end{cases} \tag{12}$$

$$M_a(a) := \begin{cases} 2^a + \frac{2^2}{3}\left(2^{a-1}-1\right) & \text{if } a \in 2\mathbb{N}-1 \\ \frac{2^2}{3}\left(2^{a-2}-1\right) & \text{if } a \in 2\mathbb{N}, \end{cases} \tag{13}$$

and know from lemma 2.1, that for all a it is $M_a(a) \in \mathbb{N}_0$, and notice that $\frac{1}{2}S_a(a), \frac{1}{2}M_a(a) \in \mathbb{N}_0$, too. We are interested in the residue classes of this values in $\mathbb{Z}/3\mathbb{Z}$.

Proposition 2.1 (Residue classes of reduced merging terms)**.** *Be $\frac{1}{2}S_a(a), \frac{1}{2}M_a(a) \in \mathbb{N}_0$, then for all $a \in \mathbb{N}$, it is*

$$\frac{1}{2}S_a(a) \bmod 3 = \begin{cases} 2 & \text{if } a \in 2\mathbb{N}-1 \\ 0 & \text{if } a \in 2\mathbb{N}, \end{cases} \tag{14}$$

and, if $a \in 2\mathbb{N}-1$, we have

$$\frac{1}{2}M_{2\mathbb{N}-1}(a) \bmod 3 = \begin{cases} 1 & \text{if } a \in 6\mathbb{N}_0+1 \\ 0 & \text{if } a \in 6\mathbb{N}_0+3 \\ 2 & \text{if } a \in 6\mathbb{N}_0+5, \end{cases} \tag{15}$$

and, if $a \in 2\mathbb{N}$,

$$\frac{1}{2}M_{2\mathbb{N}}(a) \bmod 3 = \begin{cases} 0 & \text{if } a \in 6\mathbb{N}_0+2 \\ 2 & \text{if } a \in 6\mathbb{N}_0+4 \\ 1 & \text{if } a \in 6\mathbb{N}_0+6, \end{cases} \tag{16}$$

for all $j \in \mathbb{N}_0$.

Proof. For $\frac{1}{2}S_a(a)$ the residue classes can be trivial seen, so we directly move on to $\frac{1}{2}M_a(a)$. Looking at the powers of two, we can steady that $\mathscr{R}(x) := 2^x \bmod 3$ goes alternating by $\mathscr{R}(2j) = 1$, and $\mathscr{R}(2j+1) = 2$, $j \in \mathbb{N}_0$. Hence, we can see that we have to split the set of all $x \in \mathbb{N}_0$ for $\mathscr{M}\mathscr{R}(x) := \frac{1}{3}(2^x - 2) \bmod 3$ into three discrete uniform distributed subsets for which we receive $\mathscr{M}\mathscr{R}(6j+1) = 0$, $\mathscr{M}\mathscr{R}(6j+3) = 2$, and $\mathscr{M}\mathscr{R}(6j+5) = 1$. So, we conclude for $a \in 2\mathbb{N}-1$, for which we have $\frac{1}{2}M_{2\mathbb{N}-1}(a) \bmod 3 = \left(2^{a-1} + \frac{2}{3}\left(2^{a-1}-1\right)\right) \bmod 3 = \left(2^{a-1} + \frac{1}{3}\left(2^a - 2\right)\right) \bmod 3 = \left(2^{a-1} \bmod 3 + \frac{1}{3}\left(2^a - 2\right) \bmod 3\right) \bmod 3$ the mentioned values. Analog, we receive the values for $a \in 2\mathbb{N}$, and $\frac{1}{2}M_{2\mathbb{N}}(a) \bmod 3 = \frac{2}{3}\left(2^{a-2}-1\right) \bmod 3 = \frac{1}{3}\left(2^{a-1}-2\right) \bmod 3$. ∎

Further, we notice one discrete uniform distribution for residue classes.

Proposition 2.2 (Distribution of residue classes)**.** *Be* $D_{a_j,a_i}(a_j,a_i) := \frac{1}{2}M_{a_j}(a_j) - \frac{1}{2}S_{a_i}(a_i)$*, and write in short* $D_{a_j,a_i}(a_j,a_i)_3 := D_{a_j,a_i}(a_j,a_i) \bmod 3$*, then, for all* $a_j,a_i \in \mathbb{N}$*,* $i,j \in \mathbb{N}$*, with* $i \neq j$*, we have*

$$\mathbb{P}\left(D_{a_j,a_i}(a_j,a_i)_3 = 0\right) = \mathbb{P}\left(D_{a_j,a_i}(a_j,a_i)_3 = 1\right) = \mathbb{P}\left(D_{a_j,a_i}(a_j,a_i)_3 = 2\right) = \frac{1}{3}. \tag{17}$$

Proof. We determine all possible solutions for $D_{a_j,a_i}(a_j,a_i) \bmod 3$ and receive $\left(\frac{1}{2}M_{a_j}(a_j) - \frac{1}{2}S_{a_i}(a_i)\right) \bmod 3 = \left(\frac{1}{2}M_{a_j}(a_j) \bmod 3 - \frac{1}{2}S_{a_i}(a_i) \bmod 3\right) \bmod 3 = \{(0-0) \bmod 3, (0-2) \bmod 3, (1-0) \bmod 3, (1-2) \bmod 3, (2-0) \bmod 3, (2-2) \bmod 3\} = \{0,1,1,2,2,0\}$, which includes each possible remainder exactly twice. Additionally, since each possible remainder has the same propability according to proposition 2.1, the statement gets proved. ☐

For the Collatz conjecture, we are, of course, interested in the concatenation $\mathrm{Aff}_{\vec{a}^{(n)}(N)}(N)$. Assuming such concatenation, we search for the connection between parameters from the first to the last element of an arbitrary Collatz sequence.

Lemma 2.4 (Concatenations)**.** *Let* $N \in 2\mathbb{N}-1$*, and* $n \in \mathbb{N}$*. Then* $\mathrm{Aff}_{\vec{a}^{(n)}(N)}(N) \in 2\mathbb{N}-1$ *for all* $\vec{a} \in \mathbb{N}^n$*, if* $N_{a_n}(k_{a_n})$ *with* $k_{a_n} = 3^{n-1}k_{a_n}^{(n-1)} + \sum_{j=2}^{n} 3^{j-2} r_{a_n}^{(j-1)}\left(a_{n-(j-1)},\ldots,a_n\right)$*, for all* $k_{a_n}^{(n-1)} \in \mathbb{N}_0$*, and with* $r_{a_n}^{(j-1)}\left(a_{n-(j-1)},\ldots,a_n\right) \in \mathbb{Z}/3\mathbb{Z}$*, for all* $j \in \{2,\ldots,n\}$*.*

Proof. We consider $\mathrm{Syr}(N_{a_i}(k_{a_i}))$ and $N_{a_{i+1}}\left(k_{a_{i+1}}\right)$, and are searching for the general solution of the intersection for all (a_i,a_{i+1}) pairs.

$$\begin{aligned}
\mathrm{Syr}(N_{a_i}(k_{a_i})) &= N_{a_{i+1}}\left(k_{a_{i+1}}\right) \\
3\cdot 2k_{a_i} + S_{a_i}(a_i) + 1 &= 2^{a_{i+1}+1}k_{a_{i+1}} + M_{a_{i+1}}(a_{i+1}) + 1 \\
k_{a_i} &= \frac{1}{3\cdot 2}\left(2^{a_{i+1}+1}k_{a_{i+1}} + M_{a_{i+1}}(a_{i+1}) - S_{a_i}(a_i)\right) \\
&= \frac{1}{3}\left(2^{a_{i+1}}k_{a_{i+1}} + \frac{1}{2}M_{a_{i+1}}(a_{i+1}) - \frac{1}{2}S_{a_i}(a_i)\right)
\end{aligned} \tag{18}$$

It is clear, we can go further with $k_{a_{i+1}} := 3k'_{a_{i+1}} + r'_{a_{i+1}}$, $k'_{a_{i+1}} \in \mathbb{N}_0$, and $r'_{a_{i+1}} \in \mathbb{Z}/3\mathbb{Z}$, so that we receive

$$\begin{aligned}
k_{a_i} &= \frac{1}{3}\left(2^{a_{i+1}}\left(3k'_{a_{i+1}} + r'_{a_{i+1}}\right) + \frac{1}{2}M_{a_{i+1}}(a_{i+1}) - \frac{1}{2}S_{a_i}(a_i)\right) \\
&= 2^{a_{i+1}}k'_{a_{i+1}} + \frac{1}{3}\left(2^{a_{i+1}}r'_{a_{i+1}} + \frac{1}{2}M_{a_{i+1}}(a_{i+1}) - \frac{1}{2}S_{a_i}(a_i)\right).
\end{aligned} \tag{19}$$

We take the next step and add an attional concatenation with a_{i-1}, combined with our results so far,

$$\begin{aligned}
\mathrm{Syr}\left(N_{a_{i-1}}^{-1}\left(k_{a_{i-1}}\right)\right) &= N_{a_i}(k_{a_i}) \\
3\cdot 2k_{a_{i-1}} + S_{a_{i-1}}(a_{i-1}) + 1 &= 2^{a_i+1}k_{a_i} + M_{a_i}(a_i) + 1 \\
k_{a_{i-1}} &= \frac{1}{3}\left(2^{a_i}k_{a_i} + \frac{1}{2}M_{a_i}(a_i) - \frac{1}{2}S_{a_{i-1}}(a_{i-1})\right),
\end{aligned} \tag{20}$$

and make use of equation (19), so that we receive

$$
\begin{aligned}
k_{a_{i-1}} &= \frac{1}{3}\left(2^{a_i}\left(2^{a_{i+1}}k'_{a_{i+1}}+\frac{1}{3}\left(2^{a_{i+1}}r'_{a_{i+1}}+\frac{1}{2}M_{a_{i+1}}\left(a_{i+1}\right)-\frac{1}{2}S_{a_i}\left(a_i\right)\right)\right)+\frac{1}{2}M_{a_i}\left(a_i\right)-\frac{1}{2}S_{a_{i-1}}\left(a_{i-1}\right)\right)\\
&= \frac{1}{3}\left(2^{a_i}\left(2^{a_{i+1}}\left(3k''_{a_{i+1}}+r''_{a_{i+1}}\right)+\frac{1}{3}\left(2^{a_{i+1}}r'_{a_{i+1}}+\frac{1}{2}M_{a_{i+1}}\left(a_{i+1}\right)-\frac{1}{2}S_{a_i}\left(a_i\right)\right)\right)\right.\\
&\quad\left.+\frac{1}{2}M_{a_i}\left(a_i\right)-\frac{1}{2}S_{a_{i-1}}\left(a_{i-1}\right)\right)\\
&= 2^{a_i}2^{a_{i+1}}k''_{a_{i+1}}+\frac{1}{3}\left(2^{a_i}2^{a_{i+1}}r''_{a_{i+1}}+2^{a_i}\frac{1}{3}\left(2^{a_{i+1}}r'_{a_{i+1}}+\frac{1}{2}M_{a_{i+1}}\left(a_{i+1}\right)-\frac{1}{2}S_{a_i}\left(a_i\right)\right)\right.\\
&\quad\left.+\frac{1}{2}M_{a_i}\left(a_i\right)-\frac{1}{2}S_{a_{i-1}}\left(a_{i-1}\right)\right).
\end{aligned}
\tag{21}
$$

Here, we also used the repeating Ansatz $k'_{a_{i+1}} := 3k''_{a_{i+1}} + r''_{a_{i+1}}$, with $k''_{a_{i+1}} \in \mathbb{N}_0$, and $r''_{a_{i+1}} \in \mathbb{Z}/3\mathbb{Z}$. It is easy to see that this approach will go on in the same way for further steps, so that we can quickly conclude the formulation for valid results by $k_{a_{i+1}} = 3^{n-1}k^{(n-1)}_{a_{i+1}} + \sum_{j=2}^{n} 3^{j-2} r^{(j-1)}_{a_{i+1}}$, for $\vec{a} := (a_1, \ldots, a_{i-1}, a_i, a_{i+1})$, with $i := n-1$. □

3 Collatz ghost cycles

We want to present so-called *Collatz ghost cycles*, Collatz cycles hidden within Collatz sequences.

3.1 k-Collatz cycles

By a *k-Collatz cycle*, we mean an affine map concatenation such that $\mathrm{Syr}^k(N) = N$, $k \in \mathbb{N}$. We investigate in their properties along our k_a connections so far. For this, we are starting with the $k=1$ case, a single Syracuse step.

Lemma 3.1 (1-Collatz cycles)**.** *Let $N \in 2\mathbb{N}-1$, then it exists exactly one 1-Collatz cycle. This cycle is given by $k_{a=2} = 0$ and $N := N_2(0) = 1$.*

Proof. This kind of cycles meets the case of $N_a(k_a) = \mathrm{Syr}(N_a(k_a))$ in equation (3) and (4), respectively. We check this for all $a \in 2\mathbb{N}-1$ by

$$
\begin{aligned}
0 &\stackrel{!}{=} N_a(k_a) - \mathrm{Syr}(N_a(k_a))\\
&\stackrel{!}{=} 2^{a+1}k_a + 2^a + \frac{2^2}{3}\left(2^{a-1}-1\right) - 3\cdot 2k_a - 2^2\\
&\stackrel{!}{=} \left(2^a-3\right)k_a + 2^{a-1} + \frac{2}{3}\left(2^{a-1}-1\right) - 2.
\end{aligned}
$$

Be $a=1$, then we receive for equation (3.1) the result $0 = -k_a - 1$, and hence $k_a = -1$. For all $a > 1$, we notice that all terms, apart from -2, always get positive for $k_a \geq 0$ and sum up to values greater than 2. Hence, it can be never solved for any $k_a \in \mathbb{N}_0$. Now, we check this also for $a \in 2\mathbb{N}$ by

$$
\begin{aligned}
0 &\stackrel{!}{=} 2^{a+1}k_a + \frac{2^2}{3}\left(2^{a-2}-1\right) - 3\cdot 2k_a\\
&\stackrel{!}{=} \left(2^a-3\right)k_a + \frac{2}{3}\left(2^{a-2}-1\right).
\end{aligned}
$$

Be $a=2$, then we receive for equation (3.1) the result $0 = k_a$. With this we receive the 1-Collatz cycle with starting number $N_2(k_2=0) = \mathrm{Syr}\left(N_2^{-1}(k_2=0)\right) = 1$. For all $a > 2$, also here, we notice that all terms always get positive for $k_a > 0$ and sums up to positive values. Hence, it can be never solved for any $k_a \in \mathbb{N}$. □

We move over to general k-Collatz cycles. An arbitrary number of concatenations of the inverse Collatz map can be derived from equation (21) from the proof of lemma 2.4. For this we write in short $a_{[j,k]} := \sum_{i=j}^{k} a_i$, for any $1 \leq j \leq k \leq n$.

$$\begin{aligned} k_{a_{n-j+1}} = 2^{a_{[n-j+2,n]}} k_{a_n}^{(j-1)} + \frac{1}{3} \Big(& 2^{a_{[n-j+2,n]}} r_{a_n}^{(j-1)} \left(a_{n-(j-1)}, \ldots, a_n\right) + 2^{a_{n-j+2}} k_{a_{n-j+2}} \\ & + \frac{1}{2} M_{a_{n-j+2}} \left(a_{n-j+2}\right) - \frac{1}{2} S_{a_{n-j+1}} \left(a_{n-j+1}\right) \Big), \end{aligned} \tag{22}$$

with $k_{a_{n-j+2}=a_n} := 0$, $k_{a_n}^{(j-2)} := 3k_{a_n}^{(j-1)} + r_{a_n}^{(j-1)} \left(a_{n-(j-1)}, \ldots, a_n\right)$, for all $k_{a_n}^{(j-1)} \in \mathbb{N}_0$, $r_{a_n}^{(j-1)} \left(a_{n-(j-1)}, \ldots, a_n\right) \in \mathbb{Z}/3\mathbb{Z}$, and $j \in \{2, \ldots, n\}$, and hence $k_{a_n} = 3^{n-1} k_{a_n}^{(n-1)} + \sum_{j=2}^{n} 3^{j-2} r_{a_n}^{(j-1)} \left(a_{n-(j-1)}, \ldots, a_n\right)$.

Alternatively, we can also go the other direction, the regular Collatz map, concatenations by

$$\begin{aligned} k_{a_j} = 3^{j-1} k_{a_1}^{(j-1)} + \frac{1}{2^{a_j}} \Big(& 3^{j-1} r_{a_1}^{(j-1)} \left(a_1, \ldots, a_j\right) + 3k_{a_{j-1}} \\ & - \frac{1}{2} M_{a_j} \left(a_j\right) + \frac{1}{2} S_{a_{j-1}} \left(a_{j-1}\right) \Big), \end{aligned} \tag{23}$$

with $k_{a_{j-1}=a_1} := 0$, $k_{a_1}^{(j-2)} := 2^{a_j} k_{a_1}^{(j-1)} + r_{a_1}^{(j-1)} \left(a_1, \ldots, a_j\right)$, for all $k_{a_1}^{(j-1)} \in \mathbb{N}_0$, $r_{a_1}^{(j-1)} \left(a_1, \ldots, a_j\right) \in \mathbb{Z}/2^{a_j}\mathbb{Z}$, and $j \in \{2, \ldots, n\}$, and hence $k_{a_1} = 2^{a_{[2,n]}} k_{a_1}^{(n-1)} + \sum_{j=2}^{n} 2^{a_{[2,j-1]}} r_{a_1}^{(j-1)} \left(a_1, \ldots, a_j\right)$, with $2^{a_{[2,1]}} := 1$.

In fact, equation (22) shows us, that there exists a direct dependency of each concatenation step from the remainders $r_{a_n}^{(j-1)}$ of k_{a_n} in base 3, and equation (23) a direct dependency of each step from the remainders $r_{a_1}^{(j-1)}$ of k_{a_1} in terms of remainders of 2^{a_j}, respectively. For the rest of this work, we will refer to the second case representation simple as base $2^{\vec{a}}$. With this, we can also state the condition for a k-Collatz cycle.

Definition 2 (k-Collatz cycles)**.** *If $k_{a_1} = k_{a_n}$ in equation (22) with $a_1 = a_n$, and $k_{a_n} = k_{a_1}$ in equation (23) with $a_n = a_1$, respectively, we call it a Collatz cycle of size $k = n-1$, for all $n \in N$ with $n \geq 2$.*

Because of lemma (2.3), we know that this definition is sufficient to identify one Collatz cycle uniquely. So, keep attention, that just the equality of k_{a_1} and k_{a_n} is not a sufficient enough statement for being a real Collatz cycle. Finally, we can reformulate the conjecture for cycles to:

Conjecture 3 (Collatz cycle conjecture in k-formulation)**.** *For all $k_{a_1}, k_{a_n} \in \mathbb{N}_0$, it is $k_{a_1} = k_{a_n}$ only for $k_{a_1} = k_{a_n} = 0$, with $a_1 = a_2 = \cdots = a_{n-1} = a_n = 2$.*

The repeating appearing of $a = 2$ can be seen as the circulating within the 1-Collatz circle for n times.

3.2 The ghost in the machine

In the following, we want to introduce *k-Collatz ghost cycles*, which are called 'ghosts' because of their hidden existence beneath the origin Collatz sequence.

Given be a vector $\vec{v} := (v_1, \ldots, v_n)$ of size $n \in \mathbb{N}$. Then we will denote the substring of $\vec{v}$ by $\vec{v}_{(i,j)} := (v_i, \ldots, v_j)$, for $i, j \in \{1, \ldots, n\}$, $i \leq j$.

Definition 3 (Collatz ghost cycles)**.** *Be $N_a(k_a)$ with $k_a(\vec{r}_a) \in \mathbb{N}_0$, and $\vec{r}_{a_n}(\vec{a}) \in (\mathbb{Z}/3\mathbb{Z})^n$, respectively $\vec{r}_{a_1}(\vec{a})$ with $r_a^{(j-1)} \in \mathbb{Z}/2^{a_j}\mathbb{Z}$ for all $j \in \{2, \ldots, n\}$, and $\vec{a} \in \mathbb{N}^n$. If there exists some $l \in \{1, \ldots, n\}$, and $l + m - 1 \leq n$, $m \in \mathbb{N}$, with $\vec{r}_{\bar{a}}(\vec{a})_{(l,l+m-1)}$ and $\vec{a}_{(l,l+m-1)}$, such that $\mathrm{Syr}^m\left(N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)\right) = N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)$, with $k_{\bar{a}}^g\left(\vec{r}_{\bar{a}}(\vec{a})_{(l,l+m-1)}\right)$, and $\bar{a} = a_{l+m-1}$, respectively $\bar{a} = a_l$, then we call the sequence given by $k_{\bar{a}}^g\left(\vec{r}_{\bar{a}}(\vec{a})_{(l,l+m-1)}\right)$ and containing $N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)$, a Collatz ghost cycle.*

Collatz ghost cycles seems to be a native part of some natural numbers, thinking about their representation possibilities in different bases. So, it makes sense to check out in which context they play their part in Collatz sequence evolutions. Knowing the trivial cycle for example, fixed by $\vec{r} = (0)$ and $\vec{a} = (2)$, one could say that it is a ghost cycle of each $N_a(k_a)$, which $\vec{r}_a$ has at least one $r_i = 0$ with $a_i = 2$. Looking at this more generally in equations (22) and (23) leads us to the following lemma.

Lemma 3.2 (Ghost cycling concatenations components). *If $k_{n-j+2} \bmod 3^k = 0$, then $k_{a_{n-j+1-(k-1)}} = 2^{a[n-j+2-(k-1),n]}k_{a_n}^{(j-1-(k-1))} + 2^{a_{n-j+2}} \cdot 3^{-k}k_{n-j+2} + k_{\bar{a}}^g$, $k \in \mathbb{N}$, for some $k_{\bar{a}}^g$ of a Collatz cycle of size $m \leq k$, $m \in \mathbb{N}$. Analog, if $k_{j-1} \bmod 2^{a[j,j+(k-1)]} = 0$, then $k_{a_{j-1-(k-1)}} = 3^{j-1+(k-1)}k_{a_1}^{(j-1+(k-1))} + 3 \cdot 2^{-a[j,j+(k-1)]}k_{a_{j-1}} + k_{\bar{a}}^g$, $k \in \mathbb{N}$, for some $k_{\bar{a}}^g$ of a Collatz cycle of size $m \leq k$, $m \in \mathbb{N}$.*

Proof. Looking at equation (22), it is obviously that if $k_{n-j+2} \bmod 3^k$, we can factor it out, together with the co-efficient, k-times. Then the remaining part can be solved for some collatz cycle of size $m \leq k$. Since the starting number of a Collatz cycle is not well-defined, it reaches one $k_{\bar{a}}^g$ which belongs to the cycle. For equation (23), we see that if $k_{j-1} \bmod 2^{a[j,j+(k-1)]}$, we can also factor it out with its co-efficient, again k-times. Analog to the first equation, for the remaining part, we receive one $k_{\bar{a}}^g$ of a cycle of size $m \leq k$, too. ☐

Some interesting note is the trivial cycle, because as $k_a^g = 0$ it can be always get factored out without limitations. Hence, if $a = 2$, we call it an *universal k-component*.

Since, it is clear that $k_{n-j+2} \bmod 3^k = 0$ and $k_{j-1} \bmod 2^{a[j,j+(k-1)]} = 0$ are true for some cases, we will also go by two further definitions, which are less restrictive like definition 3 and which we will call overall *semi Collatz ghost cycles*. The first one only forces $\vec{r}$ to be mandatory identically to the ghost cycle,

Definition 4 ($\vec{r}$-Collatz ghost cycles). *Be $N_a(k_a)$ with $k_a(\vec{r}_a) \in \mathbb{N}_0$, and $\vec{r}_{a_n}(\vec{a}) \in (\mathbb{Z}/3\mathbb{Z})^n$, respectively $\vec{r}_{a_1}(\vec{a})$ with $r_a^{(j-1)} \in \mathbb{Z}/2^{a_j}\mathbb{Z}$ for all $j \in \{2,\ldots,n\}$, and $\vec{a} \in \mathbb{N}^n$. If there exists some $l \in \{1,\ldots,n\}$, and $l+m-1 \leq n$, $m \in \mathbb{N}$, with $\vec{r}_{\bar{a}}(\vec{a}')_{(l,l+m-1)}$ and a vector $\vec{a}' \in \mathbb{N}^m$, such that $\mathrm{Syr}^m\left(N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)\right) = N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)$, with $k_{\bar{a}}^g\left(\vec{r}_{\bar{a}}(\vec{a}')_{(l,l+m-1)}\right)$, and $\bar{a} = a'_m$, respectively $\bar{a} = a'_1$, then we call the sequence given by $k_{\bar{a}}^g\left(\vec{r}_{\bar{a}}(\vec{a}')_{(l,l+m-1)}\right)$ and containing $N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)$, a $\vec{r}$-Collatz ghost cycle.*

and the second one, which only forces $\vec{a}$ to be mandatory identically.

Definition 5 ($\vec{a}$-Collatz ghost cycles). *Be $N_a(k_a)$ with $k_a(\vec{r}_a) \in \mathbb{N}_0$, and $\vec{r}_{a_n}(\vec{a}) \in (\mathbb{Z}/3\mathbb{Z})^n$, respectively $\vec{r}_{a_1}(\vec{a})$ with $r_a^{(j-1)} \in \mathbb{Z}/2^{a_j}\mathbb{Z}$ for all $j \in \{2,\ldots,n\}$, and $\vec{a} \in \mathbb{N}^n$. If there exists some $l \in \{1,\ldots,n\}$, and $l+m-1 \leq n$, $m \in \mathbb{N}$, with a vector $\vec{r}'_{\bar{a}} \in \mathbb{N}^m$ and $\vec{a}_{(l,l+m-1)}$, such that $\mathrm{Syr}^m\left(N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)\right) = N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)$, with $k_{\bar{a}}^g(\vec{r}'_{\bar{a}})$, and $\bar{a} = a_{l+m-1}$, respectively $\bar{a} = a_l$, then we call the sequence given by $k_{\bar{a}}^g(\vec{r}'_{\bar{a}})$ and containing $N_{\bar{a}}^g\left(k_{\bar{a}}^g\right)$, an $\vec{a}$-Collatz ghost cycle.*

With this two definitions, we can now also handle cases with remainders other than 0.

Lemma 3.3 (Semi Ghost cycling concatenations components). *If $k_{n-j+2} = 3^k k'_{n-j+2} + r'_{n-j+2}$ with $r'_{n-j+2} \in \mathbb{Z}/3^k\mathbb{Z}$, $r'_{n-j+2} \neq 0$, then $k_{a_{n-j+1-(k-1)}} = 2^{a[n-j+2-(k-1),n]}k_{a_n}^{(j-1-(k-1))} + 2^{a_{n-j+2}} \cdot 3^{-k}k'_{n-j+2} + \delta k_{\bar{a}}^g$, $k \in \mathbb{N}$, for some $\delta k_{\bar{a}}^g$ of an $\vec{r}$- or $\vec{a}$-Collatz ghost cycle of size $m \leq k$, $m \in \mathbb{N}$. Analog, if $k_{j-1} = 2^{a[j,j+(k-1)]}k'_{j-1} + r'_{j-1}$, with $r'_{j-1} \in \mathbb{Z}/2^{a[j,j+(k-1)]}\mathbb{Z}$, $r'_{j-1} \neq 0$, then $k_{a_{j-(k-1)}} = 3^{j-1+(k-1)}k_{a_1}^{(j-1+(k-1))} + 3 \cdot 2^{-a[j,j+(k-1)]}k'_{a_{j-1}} + \delta k_{\bar{a}}^g$, $k \in \mathbb{N}$, for some $\delta k_{\bar{a}}^g$ of an $\vec{r}$- or $\vec{a}$-Collatz ghost cycle of size $m \leq k$, $m \in \mathbb{N}$.*

Proof. In equation (22), if $k_{n-j+2} = 3^k k'_{n-j+2} + r'_{n-j+2}$ with $r'_{n-j+2} \in \mathbb{Z}/3^k\mathbb{Z}$, with $r'_{n-j+2} \neq 0$, then we can factor out the $3^k k'_{n-j+2}$ part again, but with a remaining $2^{a_{n-j+2}}r'_{n-j+2}$ within the bracket part. So, we can not solve it anymore with the strong restriction for ghost cycles of definition 4, but it is solvable if we either stay constant with our given $\vec{r}$ and vary $\vec{a}$, or with vary $\vec{r}$ and stay constant with $\vec{a}$, which corresponds with our definition of semi Collatz ghost cycles. The argumentation for the equation (23) case is analog. ☐

Remark 3.1 (Collatz ghost sequences). We only introduced the special case of ghost cycles. But of course, the 'ghost' concept can be also applied on general Collatz sequences, independently of if being a cycle or not. Since, in this work, we mainly utilize cycles, the author is not interested in investigating in ghost sequences further, but happily invites the motivated reader to do so in his place.

3.3 Cycle concatenations

Till now, we introduced ghost cycles and only presented some basics. Next, we want to take advantage of this concept for analyzing the possibilities of the existence of other Collatz cycles apart from the trivial one. Assuming of an existing Collatz cycle for $N_{a_m}(k_{a_m}(\vec{r}_{a_m}))$ with size m, we have two possibilities to look at its Collatz sequence evolution. We start with $N_1 := N_{a_m}(k_{a_m}(\vec{r}_{a_m}))$, then we get

$$C_{l=1}(N_1): N_1 \to N_2 \to \cdots \to N_m \to N_1, \tag{24}$$

but also

$$C_{l\geq 2}(N_1): M_1 \to M_2 \to \cdots \to M_m \to M_{m+1} \to M_{m+2} \to \cdots \to M_{2m} \to M_{2m+1} \to M_{2m+2} \to \cdots \tag{25}$$

$$N_1 \to N_2 \to \cdots \to N_m \to \quad N_1 \to \quad N_2 \to \cdots \to N_m \to N_1 \quad \to N_2 \quad \to \cdots \tag{26}$$

with $M_1, M_2, \ldots, M_{2m+1}, M_{2m+2}, N_1, N_2, \ldots, N_{m-1}, N_m \in 2\mathbb{N}-1$, and $l \in \mathbb{N}$.

In equation (24), the first interpretation $C_{l=1}(N_1)$ of a Collatz cycles is just a single loop through it, with exactly one element appearing two times, and everything else each appearing exactly on time. But alternatively, we can also go by the second interpretation $C_{l\geq 2}(N_1)$ in (26), in which the Collatz cycle looping more than ones. Hence, we can always identify each Collatz cycle with an infinity number of sequences by C_l. Since this sequences only represent an infinity loop through a Collatz cycle, we can state their properties.

C_l Collatz sequence properties.

1. The starting value N_1 isn't well-defined, since no element of a cycle is exclusive.
2. If $C_{l=1}$ have size $m \in \mathbb{N}$, then we define C_l always by full loops, leading to a size of lm, $l \in \mathbb{N}$.
3. If $\vec{M} := (M_1, \ldots, M_m)$ of $C_{l=1}$, then $\vec{M}_l = (M_1, \ldots, M_m, M_{m+1}, \ldots, M_{2m}, M_{2m+1}, \ldots, M_{lm})$, with $M_i = M_{(l-1)m+i}$, for all $i \in \{1, \ldots, m\}$, $l \in \mathbb{N}$.
4. If $\vec{a} := (a_1, \ldots, a_m)$ of $C_{l=1}$, then $\vec{a}_l = (a_1, \ldots, a_m, a_{m+1}, \ldots, a_{2m}, a_{2m+1}, \ldots, a_{lm})$, with $a_i = a_{(l-1)m+i}$ for all $i \in \{1, \ldots, m\}$, $l \in \mathbb{N}$.
5. If $\vec{k} := (k_1, \ldots, k_m)$ of $C_{l=1}$, then $\vec{k}_l = (k_1, \ldots, k_m, k_{m+1}, \ldots, k_{2m}, k_{2m+1}, \ldots, k_{lm})$, with $k_i = k_{(l-1)m+i}$, for all $i \in \{1, \ldots, m\}$, $l \in \mathbb{N}$.
6. If $\vec{r} := (r_1, \ldots, r_m)$ of $C_{l=1}$, then $\vec{r}_l = \left(r_1, \ldots, r_m, r'_{m+1}, \ldots, r'_{2m}, r'_{2m+1}, \ldots, r'_{lm}\right)$, with $r'_{lm+i} = r'_{(l+1)m+i}$, for all $i \in \{1, \ldots, m\}$, $l \in \mathbb{N}$.

Knowing that $C_{l=1}$ as well as C_l going through the same subsequence of numbers, we can leverage this situation to deduce a constraint for Collatz cycles by looking at the reverse Collatz map steps of the first loop in equation (22), and comparing this to the reverse Collatz map steps of further loops in C_l, with $l > 1$.

For C_l and the first reverse Collatz map step at all with a_{lm}, we have

$$\begin{aligned} k_{a_{lm-1}} &= \frac{1}{3}\left(2^{a_{lm}} k_{a_{lm}} + \frac{1}{2} M_{a_{lm}}(a_{lm}) - \frac{1}{2} S_{a_{lm-1}}(a_{lm-1})\right) \\ &= 2^{a_{lm}} k^{(1)}_{a_{lm}} + \frac{1}{3}\left(2^{a_{lm}} r^{(1)}_{a_{lm}} + \frac{1}{2} M_{a_{lm}}(a_{lm}) - \frac{1}{2} S_{a_{lm-1}}(a_{lm-1})\right), \end{aligned} \tag{27}$$

which has, compared to equation (22), a term $2^{a_{n-j+2}} k_{a_{n-j+2}} = 0$. For all C_l, $l \geq 2$, let be $l \geq L+1$, $L \in \mathbb{N}$, with $L \geq 1$, then we can rewrite equation (22) to an arbitrary number of cycle loops, for the $a_{(l-L)m+i}$ step, with $i \in \{0, \ldots, m-1\}$, by

$$k_{a_{(l-L)m+i-1}} = 2^{a_{[(l-L)m+i,lm]}} k_{a_{lm}}^{(Lm-i+1)} + \frac{1}{3}\left(2^{a_{[(l-L)m+i,lm]}} r_{a_{lm}}^{(Lm-i+1)} + 2^{a_{(l-L)m+i}} k_{a_{(l-L)m+i}} \right.$$
$$\left. + \frac{1}{2} M_{a_{(l-L)m+i}}\left(a_{(l-L)m+i}\right) - \frac{1}{2} S_{a_{(l-L)m+i-1}}\left(a_{(l-L)m+i-1}\right)\right), \tag{28}$$

knowing that $a_{(l-L)m+i} = a_{(l-L')m+i}$, as well as $k_{a_{(l-L)m+i-1}} = k_{a_{(l-L')m+i-1}}$, for all $L, L' \in \mathbb{N}$.

Although, the set of natural numbers consists of infinite much elements, it is also known that each of this numbers has finite much digits. This can be used in the folowing way. From our considerations so far, we know for the Collatz map (23), the representation in base $2^{|\vec{a}|}$ by $k_{a_1} = 2^{a_{[2,n]}} k_{a_1}^{(n-1)} + \sum_{j=2}^{n} 2^{a_{[2,j-1]}} r_{a_1}^{(j-1)}(a_1, \ldots, a_j)$, and for the reverse Collatz map (22), the representation in base 3 by $k_{a_n} = 3^{n-1} k_{a_n}^{(n-1)} + \sum_{j=2}^{n} 3^{j-2} r_{a_n}^{(j-1)}\left(a_{n-(j-1)}, \ldots, a_n\right)$. In both mappings, the forward and the reverse Collatz map, we go through the r remainders from the last to the first one in the related base. Here, we map each $r_{a_1}^{(j-1)}$ to $r_{a_n}^{(j-1)}$, and vice versa. After going through d remainders $r_{a_1}^{(j-1)}$ of base $2^{|\vec{a}|}$, we receive $\tilde{k}_{a_j} := k_{a_j} - 3^{j-1} k_{a_1}^{(j-1)}$ with up to d digits in base 3, and after going through d remainders $r_{a_n}^{(j-1)}$ of base 3, we receive $\tilde{k}_{a_{n-j+1}} := k_{a_{n-j+1}} - 2^{a_{[n-j+2,n]}} k_{a_n}^{(j-1)}$ with up to d digits in base $2^{|\vec{a}|}$. Since we know that each k_a is a natural number with a finite amount of digits, we can say, there exists some $d < n$, $d \in \mathbb{N}$, for $n \to \infty$, so that $k_{a_1} = \sum_{j=2}^{d} 2^{a_{[2,j-1]}} r_{a_1}^{(j-1)}(a_1, \ldots, a_j)$, and $k_{a_n} = \sum_{j=2}^{d} 3^{j-2} r_{a_n}^{(j-1)}\left(a_{n-(j-1)}, \ldots, a_n\right)$.

For each Collatz cycle $C_{l=1}$ it is possible to loop endlessly by C_l with $l \to \infty$. Let $k_a^{\max}$ be the largest k_{a_i} of all $N_{a_i}^{-1}(k_{a_i})$, for all N_{a_i}, which are element of the Collatz cycle $C_{l=1}$, and knowing that $k_a^{\max}$ having an finite amount of digits in each base, then we know there exists some L', so that $k_{a_{lm}}^{(Lm-i+1)} = 0$, and $r_{a_{lm}}^{(Lm-i+1)} = 0$, for all $L \geq L'$, and hence (28) remains valid by

$$k_{a_{(l-L')m+i-1}} = \frac{1}{3}\left(2^{a_{(l-L')m+i}} k_{a_{(l-L')m+i}} + \frac{1}{2} M_{a_{(l-L')m+i}}\left(a_{(l-L')m+i}\right) - \frac{1}{2} S_{a_{(l-L')m+i-1}}\left(a_{(l-L')m+i-1}\right)\right), \tag{29}$$

If applying $k_{a_{(l-L')m+i}} := 3\tilde{k}_{a_{(l-L')m+i}} + \tilde{r}_{a_{(l-L')m+i}}$, we finally receive

$$k_{a_{(l-L')m+i-1}} = 2^{a_{(l-L')m+i}} \tilde{k}_{a_{(l-L')m+i}} + \frac{1}{3}\left(2^{a_{(l-L')m+i}} \tilde{r}_{a_{(l-L')m+i}} \right.$$
$$\left. + \frac{1}{2} M_{a_{(l-L')m+i}}\left(a_{(l-L')m+i}\right) - \frac{1}{2} S_{a_{(l-L')m+i-1}}\left(a_{(l-L')m+i-1}\right)\right). \tag{30}$$

Some interesting results can be pointed out when comparing this equation for any Collatz loop cycle step by $L \geq L'$, with the first step at all given by equation (27). We already mentioned, that for the first step, we simply set $2^{a_{(l-L)m+i}} k_{a_{(l-L)m+i}} = 0$ in equation (22). We did this some heuristically, without questioning, if it not could also make sense to set the terms $2^{a_{[(l-L)m+i,lm]}} k_{a_{lm}}^{(Lm-i+1)} = 0$ and $2^{a_{[(l-L)m+i,lm]}} r_{a_{lm}}^{(Lm-i+1)} = 0$ instead. We want to catch up on this, now. For this we will call hypothetical predecessors, in relation to the first step at all, *virtual*.

Lemma 3.4 (Virtual predecessors of first step)**.** *Be $2^{a_{lm}} k_{a_{lm}}^{(1)} = 0$ and $2^{a_{lm}} r_{a_{lm}}^{(1)} = 0$ and $2^{a_{lm}} k_{a_{lm}} \neq 0$, then the first step $k_{a_{lm-1}}\left(k_{a_{lm}}^{(1)}, r_{a_{lm}}^{(1)}, k_{a_{lm}}\right)$ can have the virtual predecessors $k_{a_{lm}} \in \{0,1,2\}$.*

Proof. Assuming for the first step $s = 0$ with $k_{a_{lm-1}}\left(k_{a_{lm}}^{(1)}, r_{a_{lm}}^{(1)}, k_{a_{lm}}\right)$, at all, be $2^{a_{lm}} k_{a_{lm}}^{(1)} = 0$, $2^{a_{lm}} r_{a_{lm}}^{(1)} = 0$, and $2^{a_{lm}} k_{a_{lm}} \neq 0$. Then there exists, in relation to step $s = 0$, a virtual predecessor step $s = -1$ with $k_{a_{lm}}\left(k_{a_{lm+1}}^{(1)}, r_{a_{lm+1}}^{(1)}, k_{a_{lm+1}}\right)$ having one digit in base 3, and a virtual predecessor step $s = -2$ with $k_{a_{lm+1}}\left(k_{a_{lm+2}}^{(1)}, r_{a_{lm+2}}^{(1)}, k_{a_{lm+2}}\right)$ having two digits in base 3, and so on. Now, if the former step $s = -1$ value becomes the new step $s' = 0$, then the former step $s = -2$ becomes the new step $s' = -1$, and so on. If we always define first steps according to our first assumption, then it follows that for $s' = -1$, the value of this step has to have one digit, which is a contradiction to our first run in which $s' = -1$ was $s = -2$ before and had two digits in base 3, and so on. From this follows that virutal predecessors can only consists of one digit numbers in base 3. $\square$

So, we can finally state our final theorem regarding the existence of Collatz cycles.

Theorem 3.1 (k_{a_i} limitations for Collatz cycles)**.** *If $C_{l=1}$ being a Collatz cycle for $N_{a_i}(k_{a_i}) \in 2\mathbb{N}-1$, then all k_{a_i} of $N_{a_i}(k_{a_i})$ of this cycle are either in $\{0,1,2\}$ or $k_{a_i} = k_{a_{i+1}}$ for all $k_{a_i}, k_{a_{i+1}}$ of $C_{l=1}$.*

Proof. From the previous lemma 3.4, we know that in this interpretation all k_a's have to be in $\{0,1,2\}$. Now, we will compare our first step equation (27) with equation (29) for $L \geq L'$'s of infinity looping cycles.

At first, one can argue that equation (29) equals the same case like our first step equation in lemma 3.4. Keep attention, that in this lemma, we assumed that we always apply the first step equation with $2^{a_{lm}} k_{a_{lm}}^{(1)} = 0$, $2^{a_{lm}} r_{a_{lm}}^{(1)} = 0$, when shifiting the steps. This is in contrast to our case now, in which we don't[!] mandatory also have to go by the case of leading $2^{a_{[(l-L)m+i,lm]}} k_{a_{lm}}^{(Lm-i+1)} = 0$ and $2^{a_{[(l-L)m+i,lm]}} r_{a_{lm}}^{(Lm-i+1)} = 0$ all the time, so that also predecessors with more digits remain possible, here.

So, we go by the other interpretation, the original heuristic one in (27) by $k_{a_{lm-1}}\left(k_{a_{lm}}^{(1)}, r_{a_{lm}}^{(1)}, k_{a_{lm}} = 0\right)$. By a comparison with equation (29), we notice that $k_{a_{lm}}$ of the first step equation, and $k_{a_{(l-L')m+i}}$ of (29) has to be the same value if $i = 0$, since $a_{lm} = a_{(l-L')m}$ and hence also $M_{a_{lm}}(a_{lm}) = M_{a_{(l-L')m}}\left(a_{(l-L')m}\right)$ as well as $S_{a_{lm-1}}(a_{lm-1}) = S_{a_{(l-L')m}}\left(a_{(l-L')m}\right)$. But in fact $k_{a_{lm}}$ in (27) being a references to the current step, during at the same time $k_{a_{(l-L')m}}$ of (29) being the result of the previous step of equation (29), meaning that the same value gets referenced for steps differing by $(l-(L'-1))m-1$ steps from each other by an index position jump, and hence not being in period. Since the starting number of a Collatz cycle isn't well-defined, this leads to the consequence that all $k_{a_{(l-L')m}}$ of a cycle, matches always also the previous value, and hence $k_{a_i} = k_{a_{i+1}}$ for all k_{a_i} of elements $N_{a_i}(k_{a_i})$ of a cycle $C_{l=1}$.

In the end, we could ask for the difference of the just discussed situation of an infinity cycle, in comparison to the one of a simple, non cycle, Collatz sequence? Here, in contrast to of having an infinity cycle, the step position of the appearance of a specific k_{a_i} value along with the a_i's is not bounded to a specific position forced by period in relation to an arbitrary starting point of the sequence, in a general description, hence we have here a index position shifting for powers $a_i \to a_j$, with $i \neq j$, without limitations. Hence, we can finally conclude the given statement.

$\square$

4 Multi-level sequences

Collatz concatenations can, theoretically, run forewer, without ending in any cycle, at all. We want to check their properties out, and show that they can be described through several levels of sequences.

4.1 Fundamentals of infinity sequences

The generalization of the Collatz map concatenations in equation (23) leads to three different situations during running to infinity much steps, starting by one specific number k_{a_1}, with m digits in base $2^{|\vec{a}|}$. During our first steps $j-1 < m$, we have $k_{a_1}^{(j-1)} \neq 0$ for sure. When reaching the step $j-1 = m$, we reached the point at which $k_{a_1}^{(j-1)} = 0$, but $r_{a_1}^{(j-1)} \neq 0$. Finally, for all steps $j-1 > m$, we always have $k_{a_1}^{(j-1)} = 0$ and $r_{a_1}^{(j-1)} = 0$. We can sum this up to the conclusion, that the first $j-1 \leq m$ steps leading to results, mixed by the value of the direct predecessor, and the remainders of the starting number k_{a_1}, and for all $j-1 > m$ steps leading to pure results, only generated under the influence of its direct predecessor.

Definition 6 (Mixed and pure steps)**.** *Be k_{a_1} with m digits in base $2^{|\vec{a}|}$, the we call k_{a_j} for all steps $j-1 \leq m$, $j \in \mathbb{N}_{\geq 2}$, a mixed step, and for all $j-1 > m$, we call it a pure step.*

Now, going by pure steps, we receive

$$k_{a_j} = \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} - \frac{1}{2}M_{a_j}(a_j) + \frac{1}{2}S_{a_{j-1}}(a_{j-1})\right), \tag{31}$$

which equals the one step case, starting by $k_{a_{j-1}}$. For investigating infinity Collatz sequences, we quickly want to state two points. At first, we are interested in the question of remainders $R(a_j, a_{j-1}) := -\frac{1}{2}M_{a_j}(a_j) + \frac{1}{2}S_{a_{j-1}}(a_{j-1}) \bmod 3$, which can be directly retrieved from proposition (2.1).

$$R(a_j, a_{j-1}) = \begin{cases} 0 & \text{if } a_j \in 6\mathbb{N}_0+4 \cup 6\mathbb{N}_0+5 \wedge a_{j-1} \in 2\mathbb{N}-1 \\ & \text{if } a_j \in 6\mathbb{N}_0+2 \cup 6\mathbb{N}_0+3 \wedge a_{j-1} \in 2\mathbb{N} \\ 1 & \text{if } a_j \in 6\mathbb{N}_0+1 \cup 6\mathbb{N}_0+6 \wedge a_{j-1} \in 2\mathbb{N}-1 \\ & \text{if } a_j \in 6\mathbb{N}_0+4 \cup 6\mathbb{N}_0+5 \wedge a_{j-1} \in 2\mathbb{N} \\ 2 & \text{if } a_j \in 6\mathbb{N}_0+2 \cup 6\mathbb{N}_0+3 \wedge a_{j-1} \in 2\mathbb{N}-1 \\ & \text{if } a_j \in 6\mathbb{N}_0+1 \cup 6\mathbb{N}_0+6 \wedge a_{j-1} \in 2\mathbb{N} \end{cases} \tag{32}$$

Furthermore, we are asking for the situation of getting $k_{a_j} = 0$ from which we know that it always leads into the trivial cycle. It can be easily identified by

$$k_{a_{j-1}} = \frac{1}{3}\left(\frac{1}{2}M_{a_j}(a_j) - \frac{1}{2}S_{a_{j-1}}(a_{j-1})\right), \tag{33}$$

which is solved for all (a_j, a_{j-1})-tuple of residue class 0 according to equation (32).

4.2 k-level evolutions

Infinity Collatz sequences never end in any Collatz cycle at all. We already know that $k_a = 0$ always leads into the trivial cycle, so in the following, we want to look at sequences from which we assume that they never get one $k_a = 0$. Of course, this doesn't automatically mean, that they are infinity, since they still could also end in an other, unknown, Collatz cycle, apart from the trivial one.

Definition 7 (Probably infinity Collatz sequence)**.** *Let $S(k_1)$ be a Collatz sequence starting with $N(k_1)$ with $k_1 \in \mathbb{N}_0$. If for all $k_{a_i} \neq 0$, for all $i \in \mathbb{N}$, we call $S(k_1)$ a probably infinity Collatz sequence.*

By equation (31) we stated the pure step case, valid for all steps starting by k_{a_1} in case of going infinity, and by definition 7 the constraint of not ending in the trivial cycle. This fixes the assumptions for our considerations.

Taking the specific formulation of equation (31) for $a_j \in 2\mathbb{N}-1$,

$$\begin{aligned} k_{a_j} &= \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} - \frac{1}{2}\left(2^{a_j} + \frac{2^2}{3}\left(2^{a_j-1} - 1\right)\right) + \frac{1}{2}S_{a_{j-1}}(a_{j-1})\right) \\ &= \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} + \frac{4}{6} + \frac{1}{2}S_{a_{j-1}}(a_{j-1})\right) - \frac{5}{6} \end{aligned} \tag{34}$$

and

$$k_{a_j} = \begin{cases} \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} + \frac{16}{6}\right) - \frac{5}{6} & \text{if } a_{j-1} \in 2\mathbb{N}-1 \\ \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} + \frac{4}{6}\right) - \frac{5}{6} & \text{if } a_{j-1} \in 2\mathbb{N}, \end{cases} \tag{35}$$

as well as for $a_j \in 2\mathbb{N}$,

$$
\begin{aligned}
k_{a_j} &= \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} - \frac{1}{2}\left(\frac{2^2}{3}\left(2^{a_j-2}-1\right)\right) + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right)\right) \\
&= \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} + \frac{4}{6} + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right)\right) - \frac{1}{6}
\end{aligned}
\tag{36}
$$

and

$$
k_{a_j} = \begin{cases} \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} + \frac{16}{6}\right) - \frac{1}{6} & \text{if } a_{j-1} \in 2\mathbb{N}-1 \\ \frac{1}{2^{a_j}}\left(3k_{a_{j-1}} + \frac{4}{6}\right) - \frac{1}{6} & \text{if } a_{j-1} \in 2\mathbb{N}, \end{cases}
\tag{37}
$$

we discuss some evolutions.

Lemma 4.1 (Values of pure steps)**.** *The values of equation (31):*

1. *If $a_j \in 2\mathbb{N}-1$ and $a_{j-1} \in 2\mathbb{N}-1$, then for all $a_j > 1$, it is $k_{a_j} < k_{a_{j-1}}$.*
2. *If $a_j \in 2\mathbb{N}-1$ and $a_{j-1} \in 2\mathbb{N}$, then for $a_j = 1$ and $k_{a_{j-1}} > 1$, as well as for all $a_j > 1$, it is $k_{a_j} < k_{a_{j-1}}$.*
3. *If $a_j \in 2\mathbb{N}$ and $a_{j-1} \in 2\mathbb{N}-1$, then for $a_j = 2$ and $k_{a_{j-1}} > 2$, as well as for all $a_j > 2$, it is $k_{a_j} < k_{a_{j-1}}$.*
4. *If $a_j \in 2\mathbb{N}$ and $a_{j-1} \in 2\mathbb{N}$, then for all a_j, it is $k_{a_j} < k_{a_{j-1}}$.*

Proof. If $a_j \in 2\mathbb{N}-1$ and $a_{j-1} \in 2\mathbb{N}-1$, we take the smallest power $a_j = 1$ to receive $k_{a_j=1} = \frac{1}{2}\left(3k_{a_{j-1}} + \frac{16}{6}\right) - \frac{5}{6} = \frac{3}{2}k_{a_{j-1}} + \frac{1}{2}$, where $k_{a_j} > k_{a_{j-1}}$ for all $k_{a_{j-1}}$. For the next larger power we get $k_{a_j=3} = \frac{1}{8}\left(3k_{a_{j-1}} + \frac{16}{6}\right) - \frac{5}{6} = \frac{3}{8}k_{a_{j-1}} - \frac{1}{2}$, and hence also for all larger powers than one, we have $k_{a_j} < k_{a_{j-1}}$.

If $a_j \in 2\mathbb{N}-1$ and $a_{j-1} \in 2\mathbb{N}$, then $k_{a_j=1} = \frac{1}{2}\left(3k_{a_{j-1}} + \frac{4}{6}\right) - \frac{5}{6} = \frac{3}{2}k_{a_{j-1}} - \frac{1}{2}$, where $k_{a_j} > k_{a_{j-1}}$ if $k_{a_{j-1}} = 0$, $k_{a_j} = k_{a_{j-1}}$ if $k_{a_{j-1}} = 1$, and $k_{a_j} > k_{a_{j-1}}$ else. The next larger power $k_{a_j=3} = \frac{1}{8}\left(3k_{a_{j-1}} + \frac{4}{6}\right) - \frac{5}{6} = \frac{3}{8}k_{a_{j-1}} - \frac{3}{4}$, shows that for all powers larger than one, we have here $k_{a_j} < k_{a_{j-1}}$, too.

If $a_j \in 2\mathbb{N}$ and $a_{j-1} \in 2\mathbb{N}-1$, then $k_{a_j=2} = \frac{1}{4}\left(3k_{a_{j-1}} + \frac{16}{6}\right) - \frac{1}{6} = \frac{3}{4}k_{a_{j-1}} + \frac{1}{2}$, where $k_{a_j} > k_{a_{j-1}}$ if $k_{a_{j-1}} = 0$ or $k_{a_{j-1}} = 1$, $k_{a_j} = k_{a_{j-1}}$ if $k_{a_{j-1}} = 2$, and $k_{a_j} < k_{a_{j-1}}$ else. The next larger power $k_{a_j=4} = \frac{1}{16}\left(3k_{a_{j-1}} + \frac{16}{6}\right) - \frac{1}{6} = \frac{3}{16}k_{a_{j-1}}$ again gives $k_{a_j} < k_{a_{j-1}}$.

Finally, if $a_j \in 2\mathbb{N}$ and $a_{j-1} \in 2\mathbb{N}$, then $k_{a_j=2} = \frac{1}{4}\left(3k_{a_{j-1}} + \frac{4}{6}\right) - \frac{1}{6} = \frac{3}{4}k_{a_{j-1}}$, and $k_{a_j=4} = \frac{1}{16}\left(3k_{a_{j-1}} + \frac{4}{6}\right) - \frac{1}{6} = \frac{3}{16}k_{a_{j-1}} - \frac{1}{8}$, showing that for all powers it is $k_{a_j} < k_{a_{j-1}}$. ☐

Lemma 4.1 tells us that for the most cases of possible a_j, a_{j-1}, and $k_{a_{j-1}}$ combinations we can already prove the not existing infinity of Collatz sequences just by induction over k_a's, because of $k_{a_j} < k_{a_{j-1}}$. So, we are getting left with some exceptions. The cases $a_j = 1 \wedge a_{j-1} \in 2\mathbb{N}$ for $k_{a_{j-1}} \in \{0,1\}$, and $a_j = 2 \wedge a_{j-1} \in 2\mathbb{N}-1$ for $k_{a_{j-1}} \in \{0,1,2\}$, can be checked manually with equations $\mathrm{Syr}\left(N_a\left(k_a\right)\right) = 3 \cdot 2k_a + 5$ (3) and $\mathrm{Syr}\left(N_a\left(k_a\right)\right) = 3 \cdot 2k_a + 1$ (4) to be already shown of leading into the trivial cycle. So, we get left with $a_j = 1 \wedge a_{j-1} \in 2\mathbb{N}-1$.

Taking a look at the residue classes in (32) again, we want to define retroactive the discussion of equation (31) so far, as the case of having $R\left(a_j, a_{j-1}\right) = 0$, and identifying the situation of having $a_j = 1$, with the residue classes 1 and 2, respectively. Now, with rewriting of equation (31) to

$$
k_{a_j} = \frac{1}{2^{a_j}}\left(3\left(\underbrace{k_{a_{j-1}} + \frac{1}{3}\left(-\frac{1}{2}M_{a_j}\left(a_j\right) + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right) - R\left(a_j, a_{j-1}\right)\right)}_{(*)}\right) + R\left(a_j, a_{j-1}\right)\right),
\tag{38}
$$

and knowing that for each $k_{a_{j-1}}$ exists tuple $\left(a_j, a_{a_{j-1}}\right)$ according to (32), such that $k_{a_j} \in \mathbb{N}_0$, we can identify some Collatz maps for k_a's.

Definition 8 (Multi-level Collatz maps). *If the composition* $\mathrm{Syr}^n(N) = \mathrm{Aff}_{\vec{a}^{(n)}(N)}(N)$*,* $n \in \mathbb{N}_0$*,* $\vec{a} \in \mathbb{N}^n$*, can be rewritten to* $\mathrm{Syr}^i(X_i \circ k_i) = \mathrm{Aff}_{\vec{a}^{(i)}(X_i \circ k_i)}(X_i \circ k_i)$ *for some* $i \in \{1, \dots, n\}$*,* $k_i := k_{a_{j-1}}$*,* $X_i \in \mathbb{Z}$*, we call it a multi-level Collatz map.*

Multi-level Collatz maps can appear under the following conditions:

Theorem 4.1 (Appearance of multi-level Collatz maps). *If* $R\left(a_j, a_{j-1}\right) = 1$ *or* $R\left(a_j, a_{j-1}\right) = 2$ *for a given* $\left(a_j, a_{j-1}\right)$*-tuple, then* $\mathrm{Syr}^i(X_i \circ k_i) = \mathrm{Aff}_{\vec{a}^{(i)}(X_i \circ k_i)}(X_i \circ k_i)$ *forms a multi-level Collatz map.*

At first, let be $R\left(a_j, a_{j-1}\right) = 1$.

Lemma 4.2 (Multi-level Collatz maps for $R = 1$). *If* $R\left(a_j, a_{j-1}\right) = 1$*, a multi-level Collatz map is given by*

$$k_{a_j} = \frac{1}{2^{a_j}}\left(3N_{a_j}\left(\tilde{k}_{a_j}\right) + 1\right), \tag{39}$$

with

$$\tilde{k}_{a_j} = \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} + \frac{1}{3}\left(-\frac{1}{2}M_{a_j}(a_j) + \frac{1}{2}S_{a_{j-1}}(a_{j-1}) - 1\right) - M_{a_j}(a_j) - 1\right), \tag{40}$$

$\tilde{k}_{a_j} \in \mathbb{N}_0$.

For the sake of completeness, equation (40) solved for $k_{a_{j-1}}$:

$$k_{a_{j-1}} = 2^{a_j+1}\tilde{k}_{a_j} + M_{a_j}(a_j) - \frac{1}{3}\left(-\frac{1}{2}M_{a_j}(a_j) + \frac{1}{2}S_{a_{j-1}}(a_{j-1}) - 1\right) + 1. \tag{41}$$

Proof. Looking at equation (38) for $R\left(a_j, a_{j-1}\right) = 1$, we see that $(*)$ is always odd. Furthermore, we know that $(*) \in \mathbb{N}_0$, else k_{a_j} would become negative. With this we can identify $(*)$ with N_a of equations (3) and (4), respectively,

$$N_{a_j}\left(\tilde{k}_{a_j}\right) = k_{a_{j-1}} + \frac{1}{3}\left(-\frac{1}{2}M_{a_j}(a_j) + \frac{1}{2}S_{a_{j-1}}(a_{j-1}) - 1\right), \tag{42}$$

from which we can directly follow $\tilde{k}_{a_j}$. □

The specific values for $a_j \in 2\mathbb{N} - 1$ of (40) gives us

$$\begin{aligned} \tilde{k}_{a_j} &= \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} - \frac{7}{6}\left(2^{a_j} + \frac{2^2}{3}\left(2^{a_j-1} - 1\right)\right) + \frac{1}{6}S_{a_{j-1}}(a_{j-1}) - \frac{4}{3}\right) \\ &= \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} + \frac{28}{18} + \frac{1}{6}S_{a_{j-1}}(a_{j-1}) - \frac{4}{3}\right) - \frac{35}{36}, \end{aligned} \tag{43}$$

with

$$\tilde{k}_{a_j} = \begin{cases} \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} + \frac{16}{18}\right) - \frac{35}{36} & \text{if } a_{j-1} \in 2\mathbb{N} - 1 \\ \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} + \frac{4}{18}\right) - \frac{35}{36} & \text{if } a_{j-1} \in 2\mathbb{N}, \end{cases} \tag{44}$$

and for $a_j \in 2\mathbb{N}$, we receive

$$\begin{aligned}\tilde{k}_{a_j} &= \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} - \frac{7}{6}\left(\frac{2^2}{3}\left(2^{a_j-2}-1\right)\right) + \frac{1}{6}S_{a_{j-1}}\left(a_{j-1}\right) - \frac{4}{3}\right) \\ &= \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} + \frac{28}{18} + \frac{1}{6}S_{a_{j-1}}\left(a_{j-1}\right) - \frac{4}{3}\right) - \frac{7}{36},\end{aligned} \tag{45}$$

with

$$\tilde{k}_{a_j} = \begin{cases} \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} + \frac{16}{18}\right) - \frac{7}{36} & \text{if } a_{j-1} \in 2\mathbb{N}-1 \\ \frac{1}{2^{a_j+1}}\left(k_{a_{j-1}} + \frac{4}{18}\right) - \frac{7}{36} & \text{if } a_{j-1} \in 2\mathbb{N}, \end{cases} \tag{46}$$

which gives us some evolutions.

Lemma 4.3 ($R\left(a_j, a_{j-1}\right) = 1$ evolutions)**.** *If $R\left(a_j, a_{j-1}\right) = 1$, for all $a_j \in \mathbb{N} \wedge a_{j-1} \in \mathbb{N}$, it is $\tilde{k}_{a_j} < k_{a_{j-1}}$.*

Proof. If $a_j \in 2\mathbb{N}-1$, we take the smallest power $a_j = 1$, then $\tilde{k}_{a_j} = \frac{1}{4}\left(k_{a_{j-1}} + \frac{16}{18}\right) - \frac{35}{36} = \frac{1}{4}k_{a_{j-1}} - \frac{27}{36}$, as well as $\tilde{k}_{a_j} = \frac{1}{4}\left(k_{a_{j-1}} + \frac{4}{18}\right) - \frac{35}{36} = \frac{1}{4}k_{a_{j-1}} - \frac{33}{36}$, so that obviously always $\tilde{k}_{a_j} < k_{a_{j-1}}$. If $a_j \in 2\mathbb{N}$, we take the smallest power $a_j = 2$, then $\tilde{k}_{a_j} = \frac{1}{8}\left(k_{a_{j-1}} + \frac{16}{18}\right) - \frac{7}{36} = \frac{1}{8}k_{a_{j-1}} - \frac{3}{36}$, and $\tilde{k}_{a_j} = \frac{1}{8}\left(k_{a_{j-1}} + \frac{4}{18}\right) - \frac{7}{36} = \frac{1}{8}k_{a_{j-1}} - \frac{6}{36}$, which also always leads to $\tilde{k}_{a_j} < k_{a_{j-1}}$. □

Secondly, let be $R\left(a_j, a_{j-1}\right) = 2$.

Lemma 4.4 (Multi-level Collatz maps for $R = 2$)**.** *If $R\left(a_j, a_{j-1}\right) = 2$, a multi-level Collatz map is given by*

$$k_{a_j} = \frac{1}{2^{a_j+1}}\left(3N_{a_j+1}\left(\tilde{k}_{a_j+1}\right) + 1\right), \tag{47}$$

with

$$\tilde{k}_{a_j+1} = \frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{2}{3}\left(-\frac{1}{2}M_{a_j}\left(a_j\right) + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right) - 2\right) + 1 - M_{a_j+1}\left(a_j+1\right) - 1\right), \tag{48}$$

$\tilde{k}_{a_j+1} \in \mathbb{N}_0$.

For the sake of completeness, equation (48) solved for $k_{a_{j-1}}$:

$$k_{a_{j-1}} = \frac{1}{2}\left(2^{a_j+2}\tilde{k}_{a_j} + M_{a_j+1}\left(a_j+1\right) - 1 - \frac{2}{3}\left(-\frac{1}{2}M_{a_j}\left(a_j\right) + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right) - 2\right) + 1\right). \tag{49}$$

Proof. Looking at equation (38) for $R\left(a_j, a_{j-1}\right) = 2$, we have the general form $k_a = \frac{3(*)+2}{2^a}$. For receiving the usual Collatz map, we do $k_a = \frac{3(2(*))+4}{2^{a+1}} = \frac{3(2(*)+1)+1}{2^{a+1}}$. We know that $(*)$ has to be even, and $2(*)+1$ is odd, to receive one integer k_a. Furthermore, we know that $2(*)+1 \in \mathbb{N}$, else k_a would become negative. With this we can identify $2(*)+1$ with N_{a+1} of equations (3) and (4), respectively,

$$N_{a_j+1}\left(\tilde{k}_{a_j+1}\right) = 2\left(k_{a_{j-1}} + \frac{1}{3}\left(-\frac{1}{2}M_{a_j}\left(a_j\right) + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right) - 2\right)\right) + 1, \tag{50}$$

from which we can directly follow $\tilde{k}_{a_j+1}$. □

The specific values for $a_j \in 2\mathbb{N}-1$ of (48) gives us

$$\begin{aligned}\tilde{k}_{a_j+1} = &\frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{2}{3}\left(-\frac{1}{2}\left(2^{a_j} + \frac{2^2}{3}\left(2^{a_j-1}-1\right)\right) + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right) - 2\right) + 1\right.\\ &\left.-\left(\frac{2^2}{3}\left(2^{\left(a_j+1\right)-2}-1\right)\right) - 1\right)\\ = &\frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{8}{18} + \frac{6}{18}S_{a_{j-1}}\left(a_{j-1}\right)\right) - \frac{11}{36},\end{aligned} \tag{51}$$

with

$$\tilde{k}_{a_j-1} = \begin{cases} \frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{32}{18}\right) - \frac{11}{36} & \text{if } a_{j-1} \in 2\mathbb{N}-1 \\ \frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{8}{18}\right) - \frac{11}{36} & \text{if } a_{j-1} \in 2\mathbb{N}, \end{cases} \tag{52}$$

and for $a_j \in 2\mathbb{N}$, we receive

$$\begin{aligned}\tilde{k}_{a_j+1} = &\frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{2}{3}\left(-\frac{1}{2}\left(\frac{2^2}{3}\left(2^{a_j-2}-1\right)\right) + \frac{1}{2}S_{a_{j-1}}\left(a_{j-1}\right) - 2\right) + 1\right.\\ &\left.-\left(2^{a_{1+j-1}+1} + \frac{2^2}{3}\left(2^{\left(a_{1+j-1}+1\right)-1}-1\right)\right) - 1\right)\\ = &\frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{8}{18} + \frac{6}{18}S_{a_{j-1}}\left(a_{j-1}\right)\right) - \frac{31}{36},\end{aligned} \tag{53}$$

with

$$\tilde{k}_{a_j-1} = \begin{cases} \frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{32}{18}\right) - \frac{31}{36} & \text{if } a_{j-1} \in 2\mathbb{N}-1 \\ \frac{1}{2^{a_j+2}}\left(2k_{a_{j-1}} + \frac{8}{18}\right) - \frac{31}{36} & \text{if } a_{j-1} \in 2\mathbb{N}, \end{cases} \tag{54}$$

which gives us some evolutions.

Lemma 4.5 ($R\left(a_j, a_{j-1}\right) = 2$ evolutions)**.** *If $R\left(a_j, a_{j-1}\right) = 2$, then for all $a_j \in \mathbb{N} \wedge a_{j-1} \in \mathbb{N}$, it is $\tilde{k}_{a_j} < k_{a_{j-1}}$.*

Proof. If $a_j \in 2\mathbb{N}-1$, we take the smallest power $a_j = 1$, then $\tilde{k}_{a_j-1} = \frac{1}{8}\left(2k_{a_{j-1}} + \frac{32}{18}\right) - \frac{11}{36} = \frac{1}{4}k_{a_{j-1}} - \frac{3}{36}$, as well as $\tilde{k}_{a_j-1} = \frac{1}{8}\left(2k_{a_{j-1}} + \frac{8}{18}\right) - \frac{11}{36} = \frac{1}{4}k_{a_{j-1}} - \frac{9}{36}$, so that obviously always $\tilde{k}_{a_j} < k_{a_{j-1}}$. If $a_j \in 2\mathbb{N}$, we take the smallest power $a_j = 2$, then $\tilde{k}_{a_j-1} = \frac{1}{16}\left(2k_{a_{j-1}} + \frac{32}{18}\right) - \frac{31}{36} = \frac{1}{8}k_{a_{j-1}} - \frac{27}{36}$, and $\tilde{k}_{a_j-1} = \frac{1}{16}\left(2k_{a_{j-1}} + \frac{8}{18}\right) - \frac{31}{36} = \frac{1}{8}k_{a_{j-1}} - \frac{30}{36}$, which also always leads to $\tilde{k}_{a_j} < k_{a_{j-1}}$. □

Putting things together, we can finally prove theorem 4.1.

Proof (Appearance of multi-level Collatz maps). Proved by lemma 4.2 and lemma 4.4. □

4.3 The $\left(N_a, k_{a'}\right)_{a=1}$-Duality

Multi-level Collatz maps leads to some interesting properties, which we want to recap in the following. At first, we look at the general results, so far.

Theorem 4.2 ($\left(N_a, k_a\right)$-Switching)**.** *If $R\left(a_j, a_{j-1}\right) = 1$ or $R\left(a_j, a_{j-1}\right) = 2$ for a given $\left(a_j, a_{j-1}\right)$-tuple, then $N_{a_j}\left(k_{a_j}\right)$ switch their interpretation as Collatz map domain with $N_{a_j}\left(\tilde{k}_j\left(k_{a_{j-1}}\right)\right)$ mapping to k_{a_j} values instead of $N_{a_j}\left(a_j\right)$ values.*

Proof. This directly follows from theorem 4.1. $\square$

So in fact, we no longer directly map $\mathrm{Syr}(N_{a_1}(k_{a_1})) = N_{a_2}(k_{a_2})$, with $\nu_2(3N_{a_1}(k_{a_1})+1) = a_1$, but instead $\mathrm{Syr}(X_{a_1} \circ k_{a_1}) = k_{a_2}$, with $\nu_2(3(X_{a_1} \circ k_{a_1})+1) = a_2$, now. We will call this switching from $\nu_2(3N_{a_1}(k_{a_1})+1) = a_1$ to $\nu_2(3(X_{a_1} \circ k_{a_1})+1) = a_2$, the $(N_a, k_{a'})$-*anomaly*.

In lemma 4.1, we already saw that we only got left with the case $a_j \in 2\mathbb{N}-1$ and $a_{j-1} \in 2\mathbb{N}-1$, for which, if $a_j = 1$, it is $k_{a_j} \geq k_{a_{j-1}}$, so that it is not directly possible to argue with a prove by induction by dropping down to smaller values of k_a. But if we look at equation (32), we can identify this case with $R(a_j, a_{j-1}) = 1$, which gives us

$$k_{a_j=1} = \frac{1}{2}\left(3N_{a_j=1}\left(\tilde{k}_{a_j=1}\right)+1\right) \tag{55}$$

with

$$\begin{aligned} \tilde{k}_{a_j=1} &= \frac{1}{2^2}\left(k_{a_{j-1}} + \frac{1}{3}\left(-\frac{1}{2}2 + \frac{1}{2}2^2 - 1\right) - 2 - 1\right) \\ &= \frac{1}{2^2}\left(k_{a_{j-1}} - 3\right), \end{aligned} \tag{56}$$

and

$$k_{a_{j-1}} = 2^2 \tilde{k}_{a_j=1} + 3, \tag{57}$$

respectively. But with some interesting observation, which is

$$N_{a_j=1}\left(\tilde{k}_{a_j=1}\right) = k_{a_{j-1}}, \tag{58}$$

in particular for our $(N_a, k_{a'})$-anomaly, in which it becomes from $\nu_2(3N_{a_1}(k_{a_1})+1) = a_1$ to $\nu_2(3(k_{a_1})+1) = a_2$ for $\mathrm{Syr}(k_{a_1}) = k_{a_2}$, since X_{a_1} turned out to get neutral for $a_2 = 1$.

So, we can state our final theorem:

Theorem 4.3 ($(N_a, k_{a'})_{a=1}$-Duality)**.** *If $a_j = 1$, then for all $a_{j-1} \in 2\mathbb{N}-1$, the Syracuse map $\mathrm{Syr}\left(N_{a_{j-1}}\left(k_{a_{j-1}}\right)\right) = N_{a_j}\left(k_{a_j}\right)$ can be represented by the Syracuse map $\mathrm{Syr}\left(k_{a_{j-1}}\right) = k_{a_j}$, where $k_{a_{j-1}}\left(\tilde{k}_{a_j}\right)$, with $\tilde{k}_{a_j} < k_{a_{j-1}}$, $\tilde{k}_{a_j} \in \mathbb{N}_0$, and call this switching of the interpretation of $k_{a_{j-1}}$ as new $N_{a_j=1}\left(\tilde{k}_{a_j=1}\right)$, as well as k_{a_j} as new $N_{a_{j-1}}\left(\tilde{k}_{a_{j-1}}\right)$, $\tilde{k}_{a_{j-1}} \in \mathbb{N}_0$, the $(N_a, k_{a'})_{a=1}$-duality.*

Proof. Equation (58) gives a switching of the interpretation between $N_{a_j=1}\left(\tilde{k}_{a_j=1}\right)$ and $k_{a_{j-1}}$ as number on which $\mathrm{Col}(N_a)$ got applied, and the parameter $k_{a'}$, meaning the parameter $k_{a_{j-1}}$ becomes the new $N_{a_j=1}$. Furthermore, lemma 4.3 already stated that for all $\tilde{k}_{a_j} < k_{a_{j-1}}$. Following from this, for k_{a_j}, we can take the analog interpretation as new $N_{a_{j-1}}\left(\tilde{k}_{a_{j-1}}\right)$. $\square$

5 Conclusion

Final words can be a difficult issue. Often they never really fit the intention of the author and the greater good of the topic. Our paper showed some new points of view of the well known conjecture, gave some insights in the concept of ghost cycles and the interpretation of the parameters k_a's as Collatz numbers themselves. In which way this can open even additional designs for research on number sequences and the question if sequences should also considered in different ways of multi-level or also named meta sequences, sequences developing 'in-parallel' out of the original sequence, states an exciting question which we want to leave here for the curious reader. So, I close with the famous typewriting contribution of my cat to this work: iwksik hdkfjie!

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