CSc 545: Homework Assignment 4

Assigned: Monday, October 16 2017
Due: 9:30 AM, Monday, October 30 2017

Clear, neat and concise solutions are required in order to receive full credit so revise your work carefully before submission, and consider how your work is presented. If you cannot solve a particular problem, state this clearly in your write-up, and write down only what you know to be correct. For complicated proofs, first outline the argument and the delve into the details.

1. (10 points) These questions are about systems of difference constraints:

   (a) Consider a system of difference constraints with m inequalities and n variables. Describe how to modify the Bellman-Ford algorithm to solve the system in $O(nm)$ time.

   (b) We now would like to allow equalities, not just inequalities. Describe how to do this and how the Bellman-Ford algorithm needs to be modified to handle this type of constraint system.

2. (10 points) Given a graph $G = (V, E)$, $|V| = n$, assign an arbitrary linear order $v_1, v_2, \ldots, v_n$ to the vertices of G. Then partition the edges E into E_f and E_b, where $E_f = \{(v_i, v_j) \in E : i < j\}$ and $E_b = \{(v_i, v_j) \in E : i > j\}$. Assuming that G contains no self-loops, $E = E_f \cup E_b$ and $E_f \cap E_b = \emptyset$. Define $G_f = (V, E_f)$ and $G_b = (V, E_b)$.

   (a) Prove that G_f is acyclic with topological sort $v_1, v_2, \ldots, v_n$ and G_b is acyclic with topological sort $v_n, v_{n-1}, \ldots, v_1$.

   (b) Suppose that we implement each pass of the Bellman-Ford algorithm in the following way. We visit each vertex in the order $v_1, v_2, \ldots, v_n$, relaxing edges of E_f that leave the vertex. We then visit each vertex in the order $v_n, v_{n-1}, \ldots v_1$, relaxing edges of E_b that leave the vertex. Prove that with this scheme, if G contains no negative-weight cycles that are reachable from the source vertex s, then after only $\lceil V/2 \rceil$ passes over the edges we will have computed the shortest paths.

3. (10 points) Show that matrix multiplication defined by EXTEND-SHORTEST-PATHS for the APSP problem is associative (e.g., if L is an $n \times n$ matrix, $L \times L \times L = (L \times L) \times L = L \times (L \times L)$).

4. (10 points) As described in class, the Floyd-Warshall algorithm requires $O(n^3)$ space, since we compute $d_{ij}^{(k)}$ for $i, j, k = 1, 2, \ldots, n$. Show that the following modification which drops all superscripts, is correct, and thus only $O(n^2)$ space is required.

   Algorithm *CLEVER-FLOYD-WARSHALL(W)*
   1. D=W
   2. **for** k = 1 to n
   3.   **for** i = 1 to n
   4.   **for** j = 1 to n
   5. $d_{ij} = min(d_{ij}, d_{ik} + d_{kj})$
   6. return D

5. (10 points) Recall Johnson's algorithm for the APSP problem.

   (a) Suppose that the weight function is defined as $\hat{w} = w(u, v) - \min_{(u,v) \in E}\{w(u, v)\}$. Does this modification yield a correct APSP algorithm? Prove your claim.

   (b) Suppose we modify Johnson's algorithms as follows: do not create a new source vertex but use $G' = G$ and let $s = v$ where v is any vertex in $V(G)$. Does this modification yield a correct APSP algorithm? Prove your claim.

6. (10 points) The questions below are about flow networks.

   (a) Show that splitting an edge in a flow network as described in class yields an equivalent network. More formally, suppose that flow network G contains edge (u, v) and we create a new flow network G' by creating a new vertex x and replacing (u, v) by new edges (u, x) and (x, v) with $c(u, x) = c(x, v) = c(u, v)$. Show that a maximum flow in G' has the same value as a maximum flow in G.

   (b) Suppose that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \leadsto v \leadsto t$ for all vertices $v \in V$. Let u be a vertex for which there is no path $s \leadsto u \leadsto t$. Show that there must exist a maximum flow f in G such that $f(u, v) = f(v, u) = 0$ for all vertices $v \in V$.

7. (10 points) Show that the maximum flow in a network $G = (V, E)$ can be found by a sequence of at most $|E|$ augmenting paths. (Note that this question doesn't ask you to find these paths, but to argue that such paths exist)

8. (20 points) Let $G = (V, E)$ be a directed graph with source s and sink t and with edge capacities given by $c_e \geq 0$ for all $e \in E$. Let f be a maximum flow defined by f_e for all edges $e \in E$.

   (a) Suppose we increase the capacity of a single edge e from c_e to $c_e + 1$. Show how to find a maximum flow in the new graph in time $O(|V| + |E|)$.

   (b) Suppose we decrease the capacity of a single edge e from c_e to $c_e - 1$. Show how to find a maximum flow in the new graph in time $O(|V| + |E|)$.

9. (10 points) Define edge-connectivity of an undirected graph to be the minimum number k of edges that need to be removed in order to disconnect the graph (trees have $k = 1$, cycles have $k = 2$, a complete graph on n vertices has $k = n - 1$). Using a max-flow approach, design and analyze an algorithm to determine the edge connectivity of a given graph $G = (V, E)$.

Extra Credit: It is not difficult to prove that a triangle T with area 1 unit can be divided into three triangles T_1, T_2, T_3 by adding a single point inside it such that the areas of the three triangles realize any three numbers a_1, a_2, a_3 so long as $a_1 + a_2 + a_3 = 1$ and $0 \leq a_i \leq 1$ for all i. Now consider a rectangle R with area 1 unit. Can you divide it into 4 convex quadrilaterals R_1, R_2, R_3, R_4 by adding a single point inside it such that the areas of the four quadrilaterals realize any four numbers a_1, a_2, a_3, a_4 so long as $a_1 + a_2 + a_3 + a_4 = 1$ and $0 \leq a_i \leq 1$ for all i? If so give a proof; otherwise give a counterexample.