

矩阵求导

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标量、向量、矩阵间的求导共有 9 种可能：

$\partial \text{标量} / \partial \text{标量}$	$\partial \text{标量} / \partial \text{向量}$	$\partial \text{标量} / \partial \text{矩阵}$
$\partial \text{向量} / \partial \text{标量}$	$\partial \text{向量} / \partial \text{向量}$	$\partial \text{向量} / \partial \text{矩阵}$
$\partial \text{矩阵} / \partial \text{标量}$	$\partial \text{矩阵} / \partial \text{向量}$	$\partial \text{矩阵} / \partial \text{矩阵}$

表 1: 9 种求导情形

$\partial \text{标量} / \partial \text{标量}$ 就是我们熟悉的单变量微积分， $\partial \text{向量} / \partial \text{矩阵}$ 、 $\partial \text{矩阵} / \partial \text{向量}$ 、 $\partial \text{矩阵} / \partial \text{矩阵}$ 会涉及高阶张量，处理更为麻烦，因此本文只考虑剩下的 5 种情形。

设 $\mathbf{u} \in \mathbb{R}^l$ ， $\mathbf{U} \in \mathbb{R}^{m \times n}$ ，则向量、矩阵对标量求导的定义为

$$\frac{\partial \mathbf{u}}{\partial x} \triangleq \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial u_2}{\partial x} \\ \vdots \\ \frac{\partial u_l}{\partial x} \end{bmatrix}, \quad \frac{\partial \mathbf{U}}{\partial x} \triangleq \begin{bmatrix} \frac{\partial u_{11}}{\partial x} & \frac{\partial u_{12}}{\partial x} & \cdots & \frac{\partial u_{1n}}{\partial x} \\ \frac{\partial u_{21}}{\partial x} & \frac{\partial u_{22}}{\partial x} & \cdots & \frac{\partial u_{2n}}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{m1}}{\partial x} & \frac{\partial u_{m2}}{\partial x} & \cdots & \frac{\partial u_{mn}}{\partial x} \end{bmatrix}$$

设 $\mathbf{x} \in \mathbb{R}^l$ ， $\mathbf{X} \in \mathbb{R}^{m \times n}$ ，则标量对向量、矩阵求导的定义为

$$\frac{\partial u}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_l} \end{bmatrix}, \quad \frac{\partial u}{\partial \mathbf{X}} \triangleq \begin{bmatrix} \frac{\partial u}{\partial x_{11}} & \frac{\partial u}{\partial x_{21}} & \cdots & \frac{\partial u}{\partial x_{m1}} \\ \frac{\partial u}{\partial x_{12}} & \frac{\partial u}{\partial x_{22}} & \cdots & \frac{\partial u}{\partial x_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_{1n}} & \frac{\partial u}{\partial x_{2n}} & \cdots & \frac{\partial u}{\partial x_{mn}} \end{bmatrix}$$

即向量、矩阵对标量求导的结果与分子尺寸相同，标量对向量、矩阵求导的结果与分母的转置尺寸相同。

向量对向量求导的定义为雅可比矩阵：

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_l} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_l} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_l}{\partial x_1} & \frac{\partial u_l}{\partial x_2} & \cdots & \frac{\partial u_l}{\partial x_l} \end{bmatrix}$$

即行数与分子尺寸相同、列数与分母尺寸相同。

以上即为分子布局, 其好处是链式法则跟单变量微积分中的顺序一样, 坏处是计算标量值函数 $f(x)$ 关于向量变量 x 的梯度时要多做做一个转置: $\nabla f = (\partial f / \partial x)^\top$, 否则梯度下降优化变量和梯度没法直接相减。分母布局的结果均是分子布局的转置, 好处就是算梯度时不用做转置, 坏处就是链式法则的顺序要完全反过来。

1 基本结果

以下结果根据定义和单变量微积分的求导法则都是显然的。

单变量微积分中常量的导数为零

$$\frac{\partial a}{\partial x} = 0$$

类似的这里有

$$\frac{\partial \mathbf{a}}{\partial x} = \mathbf{0}, \quad \frac{\partial a}{\partial \mathbf{x}} = \mathbf{0}^\top, \quad \frac{\partial \mathbf{a}}{\partial x} = \mathbf{0}, \quad \frac{\partial \mathbf{A}}{\partial x} = \mathbf{0}, \quad \frac{\partial a}{\partial \mathbf{X}} = \mathbf{0}^\top$$

单变量微积分中常数标量乘的求导法则为

$$\frac{\partial (au)}{\partial x} = a \frac{\partial u}{\partial x}$$

类似的这里有

$$\frac{\partial (au)}{\partial x} = a \frac{\partial u}{\partial x}, \quad \frac{\partial (au)}{\partial x} = a \frac{\partial u}{\partial x}, \quad \frac{\partial (au)}{\partial x} = a \frac{\partial u}{\partial x}, \quad \frac{\partial (a\mathbf{U})}{\partial x} = a \frac{\partial \mathbf{U}}{\partial x}, \quad \frac{\partial (au)}{\partial \mathbf{X}} = a \frac{\partial u}{\partial \mathbf{X}}$$

单变量微积分中加法的求导法则为

$$\frac{\partial (u+v)}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

类似的这里有

$$\begin{aligned} \frac{\partial (u+v)}{\partial x} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, & \frac{\partial (u+v)}{\partial \mathbf{x}} &= \frac{\partial u}{\partial \mathbf{x}} + \frac{\partial v}{\partial \mathbf{x}}, & \frac{\partial (u+v)}{\partial x} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \\ \frac{\partial (\mathbf{U} + \mathbf{V})}{\partial x} &= \frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{V}}{\partial x}, & \frac{\partial (u+v)}{\partial \mathbf{X}} &= \frac{\partial u}{\partial \mathbf{X}} + \frac{\partial v}{\partial \mathbf{X}} \end{aligned}$$

单变量微积分中乘法的求导法则为

$$\frac{\partial (uv)}{\partial x} = \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x}$$

类似的这里有

$$\begin{aligned} \frac{\partial (uv)}{\partial x} &= \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x}, & \frac{\partial (uv)}{\partial \mathbf{x}} &= \frac{\partial u}{\partial \mathbf{x}} v + u \frac{\partial v}{\partial \mathbf{x}} \\ \frac{\partial (\mathbf{U}\mathbf{V})}{\partial x} &= \frac{\partial \mathbf{U}}{\partial x} \mathbf{V} + \mathbf{U} \frac{\partial \mathbf{V}}{\partial x}, & \frac{\partial (uv)}{\partial \mathbf{X}} &= \frac{\partial u}{\partial \mathbf{X}} v + u \frac{\partial v}{\partial \mathbf{X}} \end{aligned}$$

其中第二行是因为

$$\begin{aligned}\left[\frac{\partial(\mathbf{UV})}{\partial x}\right]_{ij} &= \frac{\partial(\sum_k u_{ik}v_{kj})}{\partial x} = \sum_k \frac{\partial u_{ik}}{\partial x} v_{kj} + \sum_k u_{ik} \frac{\partial v_{kj}}{\partial x} = \left[\frac{\partial \mathbf{U}}{\partial x} \mathbf{V}\right]_{ij} + \left[\mathbf{U} \frac{\partial \mathbf{V}}{\partial x}\right]_{ij} \\ \left[\frac{\partial(uv)}{\partial \mathbf{X}}\right]_{ij} &= \frac{\partial(uv)}{\partial x_{ji}} = \frac{\partial u}{\partial x_{ji}} v + u \frac{\partial v}{\partial x_{ji}} = \left[\frac{\partial u}{\partial \mathbf{X}}\right]_{ij} v + u \left[\frac{\partial v}{\partial \mathbf{X}}\right]_{ij}\end{aligned}$$

第一行可看作第二行的特例。 $\partial(uv)/\partial x$ 有两种可能，一是 uv 为标量，即两者的内积，这里暂且不表，后文再讲；二是 uv 为矩阵，这属于我们不考虑的 ∂ 矩阵/ ∂ 向量情形。

单变量微积分中有 $\partial x/\partial x = 1$ ，类似的这里有

$$\frac{\partial x_i}{\partial x} = \mathbf{e}_i^\top, \quad \frac{\partial x}{\partial x_i} = \mathbf{e}_i, \quad \frac{\partial x}{\partial x} = \mathbf{I}, \quad \frac{\partial x_{ij}}{\partial \mathbf{X}} = \mathbf{E}_{ji}, \quad \frac{\partial \mathbf{X}}{\partial x_{ij}} = \mathbf{E}_{ij}$$

其中 \mathbf{E}_{ij} 是 (i, j) 处为 1 其余为 0 的矩阵。

单变量微积分中的链式法则为

$$\frac{\partial g(u)}{\partial x} = \frac{\partial g(u)}{\partial u} \frac{\partial u}{\partial x}$$

类似的，

- 只涉及向量：设 $x \in \mathbb{R}^n$, $u = u(x) \in \mathbb{R}^m$, $g: \mathbb{R}^m \mapsto \mathbb{R}^l$ ，则

$$\underbrace{\frac{\partial g(u)}{\partial x}}_{l \times n} = \underbrace{\frac{\partial g(u)}{\partial u}}_{l \times m} \underbrace{\frac{\partial u}{\partial x}}_{m \times n}$$

这是因为

$$\begin{aligned}\left[\frac{\partial g(u)}{\partial x}\right]_{ij} &= \frac{\partial [g(u)]_i}{\partial x_j} = \sum_{k \in [m]} \frac{\partial [g(u)]_i}{\partial u_k} \frac{\partial u_k}{\partial x_j} = \frac{\partial [g(u)]_i}{\partial u} \frac{\partial u}{\partial x_j} \\ &= \left[\frac{\partial g(u)}{\partial u}\right]_{i,:} \left[\frac{\partial u}{\partial x}\right]_{:,j} = \left[\frac{\partial g(u)}{\partial u} \frac{\partial u}{\partial x}\right]_{i,j}\end{aligned}$$

注意若 $n = m = l = 1$ ，就退化成了单变量的链式法则。

- 自变量是矩阵：设 $u = u(\mathbf{X})$, $g: \mathbb{R} \mapsto \mathbb{R}$ ，则

$$\frac{\partial g(u)}{\partial \mathbf{X}} = \frac{\partial g(u)}{\partial u} \frac{\partial u}{\partial \mathbf{X}}$$

这是因为

$$\left[\frac{\partial g(u)}{\partial \mathbf{X}}\right]_{ij} = \frac{\partial g(u)}{\partial x_{ji}} = \frac{\partial g(u)}{\partial u} \frac{\partial u}{\partial x_{ji}} = \frac{\partial g(u)}{\partial u} \left[\frac{\partial u}{\partial \mathbf{X}}\right]_{ij}$$

- 中间变量是矩阵：设 $\mathbf{U} = \mathbf{U}(x) \in \mathbb{R}^{m \times n}$, $g: \mathbb{R}^{m \times n} \mapsto \mathbb{R}$ ，则

$$\frac{\partial g(\mathbf{U})}{\partial x} = \sum_p \sum_q \frac{\partial g(\mathbf{U})}{\partial u_{pq}} \frac{\partial u_{pq}}{\partial x} = \sum_q \sum_p \left[\frac{\partial g(\mathbf{U})}{\partial \mathbf{U}}\right]_{qp} \left[\frac{\partial \mathbf{U}}{\partial x}\right]_{pq} = \text{tr} \left(\frac{\partial g(\mathbf{U})}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial x} \right) \quad (1)$$

2 向量对标量求导

矩阵和向量的乘积是向量，若 \mathbf{A} 与 x 无关，易知有

$$\left[\frac{\partial(\mathbf{A}\mathbf{u})}{\partial x} \right]_i = \frac{\partial[\mathbf{A}\mathbf{u}]_i}{\partial x} = \frac{\partial(\sum_k a_{ik}u_k)}{\partial x} = \sum_k a_{ik} \frac{\partial u_k}{\partial x} = \left[\mathbf{A} \frac{\partial \mathbf{u}}{\partial x} \right]_i \Rightarrow \frac{\partial(\mathbf{A}\mathbf{u})}{\partial x} = \mathbf{A} \frac{\partial \mathbf{u}}{\partial x}$$

$$\frac{\partial(\mathbf{u}^\top \mathbf{A})}{\partial x} = \left[\frac{\partial(\mathbf{A}^\top \mathbf{u})}{\partial x} \right]^\top = \left[\mathbf{A}^\top \frac{\partial \mathbf{u}}{\partial x} \right]^\top = \frac{\partial \mathbf{u}^\top}{\partial x} \mathbf{A}$$

向量的外积也是向量，记 $\mathbf{u} = [u_1(x); u_2(x); u_3(x)]$, $\mathbf{v} = [v_1(x); v_2(x); v_3(x)]$, 则

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

于是

$$\frac{\partial(\mathbf{u} \times \mathbf{v})}{\partial x} = \begin{bmatrix} \frac{\partial u_2}{\partial x} v_3 - \frac{\partial u_3}{\partial x} v_2 + u_2 \frac{\partial v_3}{\partial x} - u_3 \frac{\partial v_2}{\partial x} \\ \frac{\partial u_3}{\partial x} v_1 - \frac{\partial u_1}{\partial x} v_3 + u_3 \frac{\partial v_1}{\partial x} - u_1 \frac{\partial v_3}{\partial x} \\ \frac{\partial u_1}{\partial x} v_2 - \frac{\partial u_2}{\partial x} v_1 + u_1 \frac{\partial v_2}{\partial x} - u_2 \frac{\partial v_1}{\partial x} \end{bmatrix} = \left(\frac{\partial \mathbf{u}}{\partial x} \right) \times \mathbf{v} + \mathbf{u} \times \frac{\partial \mathbf{v}}{\partial x}$$

3 标量对向量求导

二次型是标量，设 \mathbf{A} 与 x 无关，易知有

$$\left[\frac{\partial(\mathbf{u}^\top \mathbf{A} \mathbf{v})}{\partial x} \right]_i = \frac{\partial(\mathbf{u}^\top \mathbf{A} \mathbf{v})}{\partial x_i} = \frac{\partial(\sum_j \sum_k u_j a_{jk} v_k)}{\partial x_i} = \sum_j \sum_k u_j a_{jk} \frac{\partial v_k}{\partial x_i} + \sum_j \sum_k \frac{\partial u_j}{\partial x_i} a_{jk} v_k$$

$$= \mathbf{u}^\top \mathbf{A} \frac{\partial \mathbf{v}}{\partial x_i} + \mathbf{v}^\top \mathbf{A}^\top \frac{\partial \mathbf{u}}{\partial x_i} = \left[\mathbf{u}^\top \mathbf{A} \frac{\partial \mathbf{v}}{\partial x} \right]_i + \left[\mathbf{v}^\top \mathbf{A}^\top \frac{\partial \mathbf{u}}{\partial x} \right]_i$$

$$\Rightarrow \frac{\partial(\mathbf{u}^\top \mathbf{A} \mathbf{v})}{\partial x} = \mathbf{u}^\top \mathbf{A} \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v}^\top \mathbf{A}^\top \frac{\partial \mathbf{u}}{\partial x}$$

特别的,

- 取 $\mathbf{A} = \mathbf{I}$, 则

$$\frac{\partial(\mathbf{u}^\top \mathbf{v})}{\partial x} = \mathbf{u}^\top \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v}^\top \frac{\partial \mathbf{u}}{\partial x}$$

进一步若 $\mathbf{u} = \mathbf{a}$ 与 x 无关, 则

$$\frac{\partial(\mathbf{a}^\top \mathbf{v})}{\partial x} = \mathbf{a}^\top \frac{\partial \mathbf{v}}{\partial x}, \quad \frac{\partial(\mathbf{a}^\top \mathbf{x})}{\partial x} = \mathbf{a}^\top \frac{\partial \mathbf{x}}{\partial x} = \mathbf{a}^\top, \quad \frac{\partial(\mathbf{b}^\top \mathbf{A} \mathbf{x})}{\partial x} = \mathbf{b}^\top \mathbf{A}$$

- 取 $\mathbf{u} = \mathbf{v} = \mathbf{x}$, 则

$$\frac{\partial(\mathbf{x}^\top \mathbf{A} \mathbf{x})}{\partial x} = \mathbf{x}^\top \mathbf{A} \frac{\partial \mathbf{x}}{\partial x} + \mathbf{x}^\top \mathbf{A}^\top \frac{\partial \mathbf{x}}{\partial x} = \mathbf{x}^\top (\mathbf{A} + \mathbf{A}^\top)$$

进一步若 $\mathbf{A} = \mathbf{I}$, 则

$$\frac{\partial(\mathbf{x}^\top \mathbf{x})}{\partial x} = \frac{\partial \|\mathbf{x}\|^2}{\partial x} = 2\mathbf{x}^\top$$

- 若 $\mathbf{A} = \mathbf{b}\mathbf{a}^\top$, 则

$$\frac{\partial(\mathbf{x}^\top \mathbf{b}\mathbf{a}^\top \mathbf{x})}{\partial \mathbf{x}} = \frac{\partial(\mathbf{a}^\top \mathbf{x}\mathbf{x}^\top \mathbf{b})}{\partial \mathbf{x}} = \mathbf{x}^\top (\mathbf{a}\mathbf{b}^\top + \mathbf{b}\mathbf{a}^\top)$$

- 更一般的有

$$\begin{aligned} \frac{\partial[(\mathbf{A}\mathbf{x} + \mathbf{b})^\top \mathbf{C}(\mathbf{D}\mathbf{x} + \mathbf{e})]}{\partial \mathbf{x}} &= \frac{\partial(\mathbf{x}^\top \mathbf{A}^\top \mathbf{C}\mathbf{D}\mathbf{x} + \mathbf{b}^\top \mathbf{C}\mathbf{D}\mathbf{x} + \mathbf{x}^\top \mathbf{A}^\top \mathbf{C}\mathbf{e} + \mathbf{b}^\top \mathbf{e})}{\partial \mathbf{x}} \\ &= \mathbf{x}^\top (\mathbf{A}^\top \mathbf{C}\mathbf{D} + \mathbf{D}^\top \mathbf{C}^\top \mathbf{A}) + \mathbf{b}^\top \mathbf{C}\mathbf{D} + \mathbf{e}^\top \mathbf{C}^\top \mathbf{A} \\ &= (\mathbf{D}\mathbf{x} + \mathbf{e})^\top \mathbf{C}^\top \mathbf{A} + (\mathbf{A}\mathbf{x} + \mathbf{b})^\top \mathbf{C}\mathbf{D} \end{aligned}$$

范数也是标量, 若 \mathbf{a} 与 \mathbf{x} 无关, 则

$$\begin{aligned} \left[\frac{\partial \|\mathbf{x} - \mathbf{a}\|}{\partial \mathbf{x}} \right]_i &= \frac{\partial \|\mathbf{x} - \mathbf{a}\|}{\partial x_i} = \frac{\partial \sqrt{\sum_j (x_j - a_j)^2}}{\partial x_i} = \frac{1}{2} \frac{2(x_i - a_i)}{\sqrt{\sum_j (x_j - a_j)^2}} = \frac{x_i - a_i}{\|\mathbf{x} - \mathbf{a}\|} \\ &\Rightarrow \frac{\partial \|\mathbf{x} - \mathbf{a}\|}{\partial \mathbf{x}} = \frac{(\mathbf{x} - \mathbf{a})^\top}{\|\mathbf{x} - \mathbf{a}\|} \end{aligned} \quad (2)$$

4 向量对向量求导

若 \mathbf{A} 与 \mathbf{x} 无关, 易知有

$$\left[\frac{\partial(\mathbf{A}\mathbf{u})}{\partial \mathbf{x}} \right]_{ij} = \frac{\partial[\mathbf{A}\mathbf{u}]_i}{\partial x_j} = \frac{\partial(\sum_k a_{ik} u_k)}{\partial x_j} = \sum_k a_{ik} \frac{\partial u_k}{\partial x_j} = \left[\mathbf{A} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right]_{ij} \Rightarrow \frac{\partial(\mathbf{A}\mathbf{u})}{\partial \mathbf{x}} = \mathbf{A} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$$

特别的, 若 $\mathbf{u} = \mathbf{x}$, 则

$$\frac{\partial(\mathbf{A}\mathbf{x})}{\partial \mathbf{x}} = \mathbf{A} \frac{\partial \mathbf{x}}{\partial \mathbf{x}} = \mathbf{A}$$

若 $v = v(\mathbf{x})$, 则

$$\left[\frac{\partial(v\mathbf{u})}{\partial \mathbf{x}} \right]_{ij} = \frac{\partial(vu_i)}{\partial x_j} = v \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial v}{\partial x_j} = v \left[\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right]_{ij} + \left[\mathbf{u} \frac{\partial v}{\partial \mathbf{x}} \right]_{ij} \Rightarrow \frac{\partial(v\mathbf{u})}{\partial \mathbf{x}} = v \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{u} \frac{\partial v}{\partial \mathbf{x}}$$

注意第一项是标量乘以雅可比矩阵, 第二项是列向量乘以行向量。

若 \mathbf{a} 与 \mathbf{x} 无关, 结合式 (2) 可得

$$\begin{aligned} \left[\frac{\partial}{\partial \mathbf{x}} \frac{\mathbf{x} - \mathbf{a}}{\|\mathbf{x} - \mathbf{a}\|} \right]_{ij} &= \frac{\partial}{\partial x_j} \frac{x_i - a_i}{\|\mathbf{x} - \mathbf{a}\|} = \frac{\delta_{ij} \|\mathbf{x} - \mathbf{a}\|}{\|\mathbf{x} - \mathbf{a}\|^2} - \frac{x_i - a_i}{\|\mathbf{x} - \mathbf{a}\|^2} \frac{\partial \|\mathbf{x} - \mathbf{a}\|}{\partial x_j} \\ &= \frac{\delta_{ij}}{\|\mathbf{x} - \mathbf{a}\|} - \frac{x_i - a_i}{\|\mathbf{x} - \mathbf{a}\|^2} \frac{x_j - a_j}{\|\mathbf{x} - \mathbf{a}\|} \\ &\Rightarrow \frac{\partial}{\partial \mathbf{x}} \frac{\mathbf{x} - \mathbf{a}}{\|\mathbf{x} - \mathbf{a}\|} = \frac{\mathbf{I}}{\|\mathbf{x} - \mathbf{a}\|} - \frac{(\mathbf{x} - \mathbf{a})(\mathbf{x} - \mathbf{a})^\top}{\|\mathbf{x} - \mathbf{a}\|^3} \end{aligned}$$

5 矩阵对标量求导

若 $u = u(x)$, $\mathbf{V} = \mathbf{V}(x)$, 则

$$\left[\frac{\partial(u\mathbf{V})}{\partial x} \right]_{ij} = \frac{\partial(uv_{ij})}{\partial x} = \frac{\partial u}{\partial x} v_{ij} + u \frac{\partial v_{ij}}{\partial x} = \frac{\partial u}{\partial x} [\mathbf{V}]_{ij} + u \left[\frac{\partial \mathbf{V}}{\partial x} \right]_{ij} \implies \frac{\partial(u\mathbf{V})}{\partial x} = \frac{\partial u}{\partial x} \mathbf{V} + u \frac{\partial \mathbf{V}}{\partial x}$$

若乘积求导法则中的 \mathbf{U} 或 \mathbf{V} 可继续分解为 x 相关项的乘积, 例如 $\mathbf{V} \leftarrow \mathbf{V}\mathbf{W}$, 则

$$\frac{\partial(\mathbf{U}\mathbf{V}\mathbf{W})}{\partial x} = \frac{\partial \mathbf{U}}{\partial x} \mathbf{V}\mathbf{W} + \mathbf{U} \frac{\partial(\mathbf{V}\mathbf{W})}{\partial x} = \frac{\partial \mathbf{U}}{\partial x} \mathbf{V}\mathbf{W} + \mathbf{U} \left(\frac{\partial \mathbf{V}}{\partial x} \mathbf{W} + \mathbf{V} \frac{\partial \mathbf{W}}{\partial x} \right) = \frac{\partial \mathbf{U}}{\partial x} \mathbf{V}\mathbf{W} + \mathbf{U} \frac{\partial \mathbf{V}}{\partial x} \mathbf{W} + \mathbf{U}\mathbf{V} \frac{\partial \mathbf{W}}{\partial x} \quad (3)$$

由此可知若 \mathbf{A} 、 \mathbf{B} 与 x 无关, 则

$$\frac{\partial(\mathbf{A}\mathbf{U}\mathbf{B})}{\partial x} = \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} \mathbf{B}$$

当 \mathbf{U} 为方阵、 n 为正整数时有

$$\frac{\partial \mathbf{U}^n}{\partial x} = \mathbf{U}^{n-1} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{U}^{n-2} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U} + \cdots + \mathbf{U} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{n-2} + \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{n-1} = \sum_{i \in [n]} \mathbf{U}^{i-1} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{n-i} \quad (4)$$

令乘积求导法则中的 $\mathbf{V} = \mathbf{U}^{-1}$ 可得

$$\mathbf{0} = \frac{\partial \mathbf{I}}{\partial x} = \frac{\partial(\mathbf{U}\mathbf{U}^{-1})}{\partial x} = \mathbf{U} \frac{\partial \mathbf{U}^{-1}}{\partial x} + \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{-1} \implies \frac{\partial \mathbf{U}^{-1}}{\partial x} = -\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{-1} \quad (5)$$

由此可知

$$\begin{aligned} \frac{\partial[\mathbf{X}^{-1}]_{kl}}{\partial x_{ij}} &= \text{tr} \left(\frac{\partial[\mathbf{X}^{-1}]_{kl}}{\partial \mathbf{X}^{-1}} \frac{\partial \mathbf{X}^{-1}}{\partial x_{ij}} \right) = -\text{tr} \left(\mathbf{E}_{lk} \mathbf{X}^{-1} \frac{\partial \mathbf{X}}{\partial x_{ij}} \mathbf{X}^{-1} \right) = -\text{tr}(\mathbf{X}^{-1} \mathbf{E}_{lk} \mathbf{X}^{-1} \mathbf{E}_{ij}) \\ &= -[\mathbf{X}^{-1} \mathbf{E}_{lk} \mathbf{X}^{-1}]_{ji} = -\sum_p \sum_q [\mathbf{X}^{-1}]_{jp} [\mathbf{E}_{lk}]_{pq} [\mathbf{X}^{-1}]_{qi} = -[\mathbf{X}^{-1}]_{jl} [\mathbf{X}^{-1}]_{ki} \end{aligned}$$

结合式 (3) 还可得海森矩阵

$$\begin{aligned} \frac{\partial^2 \mathbf{U}^{-1}}{\partial x \partial y} &= \frac{\partial}{\partial y} \left(-\mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{-1} \right) = -\frac{\partial \mathbf{U}^{-1}}{\partial y} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{-1} - \mathbf{U}^{-1} \frac{\partial^2 \mathbf{U}}{\partial x \partial y} \mathbf{U}^{-1} - \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial x} \frac{\partial \mathbf{U}^{-1}}{\partial y} \\ &= \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial y} \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{-1} - \mathbf{U}^{-1} \frac{\partial^2 \mathbf{U}}{\partial x \partial y} \mathbf{U}^{-1} + \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial y} \mathbf{U}^{-1} \\ &= \mathbf{U}^{-1} \left(\frac{\partial \mathbf{U}}{\partial y} \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial x} - \frac{\partial^2 \mathbf{U}}{\partial x \partial y} + \frac{\partial \mathbf{U}}{\partial x} \mathbf{U}^{-1} \frac{\partial \mathbf{U}}{\partial y} \right) \mathbf{U}^{-1} \end{aligned}$$

矩阵除了常规的乘积外, 还有克罗内克积和哈达玛积。设 $\mathbf{U} \in \mathbb{R}^{m \times n}$, $\mathbf{V} \in \mathbb{R}^{p \times q}$, 则

$$\frac{\partial(\mathbf{U} \otimes \mathbf{V})}{\partial x} = \begin{bmatrix} \frac{\partial u_{11} \mathbf{V}}{\partial x} & \frac{\partial u_{12} \mathbf{V}}{\partial x} & \cdots & \frac{\partial u_{1n} \mathbf{V}}{\partial x} \\ \frac{\partial u_{21} \mathbf{V}}{\partial x} & \frac{\partial u_{22} \mathbf{V}}{\partial x} & \cdots & \frac{\partial u_{2n} \mathbf{V}}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{m1} \mathbf{V}}{\partial x} & \frac{\partial u_{m2} \mathbf{V}}{\partial x} & \cdots & \frac{\partial u_{mn} \mathbf{V}}{\partial x} \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} \frac{\partial u_{11}}{\partial x} \mathbf{V} + u_{11} \frac{\partial \mathbf{V}}{\partial x} & \frac{\partial u_{12}}{\partial x} \mathbf{V} + u_{12} \frac{\partial \mathbf{V}}{\partial x} & \cdots & \frac{\partial u_{1n}}{\partial x} \mathbf{V} + u_{1n} \frac{\partial \mathbf{V}}{\partial x} \\ \frac{\partial u_{21}}{\partial x} \mathbf{V} + u_{21} \frac{\partial \mathbf{V}}{\partial x} & \frac{\partial u_{22}}{\partial x} \mathbf{V} + u_{22} \frac{\partial \mathbf{V}}{\partial x} & \cdots & \frac{\partial u_{2n}}{\partial x} \mathbf{V} + u_{2n} \frac{\partial \mathbf{V}}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{m1}}{\partial x} \mathbf{V} + u_{m1} \frac{\partial \mathbf{V}}{\partial x} & \frac{\partial u_{m2}}{\partial x} \mathbf{V} + u_{m2} \frac{\partial \mathbf{V}}{\partial x} & \cdots & \frac{\partial u_{mn}}{\partial x} \mathbf{V} + u_{mn} \frac{\partial \mathbf{V}}{\partial x} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial u_{11}}{\partial x} \mathbf{V} & \frac{\partial u_{12}}{\partial x} \mathbf{V} & \cdots & \frac{\partial u_{1n}}{\partial x} \mathbf{V} \\ \frac{\partial u_{21}}{\partial x} \mathbf{V} & \frac{\partial u_{22}}{\partial x} \mathbf{V} & \cdots & \frac{\partial u_{2n}}{\partial x} \mathbf{V} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{m1}}{\partial x} \mathbf{V} & \frac{\partial u_{m2}}{\partial x} \mathbf{V} & \cdots & \frac{\partial u_{mn}}{\partial x} \mathbf{V} \end{bmatrix} + \begin{bmatrix} u_{11} \frac{\partial \mathbf{V}}{\partial x} & u_{12} \frac{\partial \mathbf{V}}{\partial x} & \cdots & u_{1n} \frac{\partial \mathbf{V}}{\partial x} \\ u_{21} \frac{\partial \mathbf{V}}{\partial x} & u_{22} \frac{\partial \mathbf{V}}{\partial x} & \cdots & u_{2n} \frac{\partial \mathbf{V}}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} \frac{\partial \mathbf{V}}{\partial x} & u_{m2} \frac{\partial \mathbf{V}}{\partial x} & \cdots & u_{mn} \frac{\partial \mathbf{V}}{\partial x} \end{bmatrix} \\
&= \frac{\partial \mathbf{U}}{\partial x} \otimes \mathbf{V} + \mathbf{U} \otimes \frac{\partial \mathbf{V}}{\partial x}
\end{aligned}$$

设 $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{m \times n}$, 则

$$\begin{aligned}
\frac{\partial(\mathbf{U} \odot \mathbf{V})}{\partial x} &= \begin{bmatrix} \frac{\partial u_{11} v_{11}}{\partial x} & \frac{\partial u_{12} v_{12}}{\partial x} & \cdots & \frac{\partial u_{1n} v_{1n}}{\partial x} \\ \frac{\partial u_{21} v_{21}}{\partial x} & \frac{\partial u_{22} v_{22}}{\partial x} & \cdots & \frac{\partial u_{2n} v_{2n}}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{m1} v_{m1}}{\partial x} & \frac{\partial u_{m2} v_{m2}}{\partial x} & \cdots & \frac{\partial u_{mn} v_{mn}}{\partial x} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial u_{11}}{\partial x} v_{11} & \frac{\partial u_{12}}{\partial x} v_{12} & \cdots & \frac{\partial u_{1n}}{\partial x} v_{1n} \\ \frac{\partial u_{21}}{\partial x} v_{21} & \frac{\partial u_{22}}{\partial x} v_{22} & \cdots & \frac{\partial u_{2n}}{\partial x} v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_{m1}}{\partial x} v_{m1} & \frac{\partial u_{m2}}{\partial x} v_{m2} & \cdots & \frac{\partial u_{mn}}{\partial x} v_{mn} \end{bmatrix} + \begin{bmatrix} u_{11} \frac{\partial v_{11}}{\partial x} & u_{12} \frac{\partial v_{12}}{\partial x} & \cdots & u_{1n} \frac{\partial v_{1n}}{\partial x} \\ u_{21} \frac{\partial v_{21}}{\partial x} & u_{22} \frac{\partial v_{22}}{\partial x} & \cdots & u_{2n} \frac{\partial v_{2n}}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} \frac{\partial v_{m1}}{\partial x} & u_{m2} \frac{\partial v_{m2}}{\partial x} & \cdots & u_{mn} \frac{\partial v_{mn}}{\partial x} \end{bmatrix} \\
&= \frac{\partial \mathbf{U}}{\partial x} \odot \mathbf{V} + \mathbf{U} \odot \frac{\partial \mathbf{V}}{\partial x}
\end{aligned}$$

设多项式函数 $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, 则 $g'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$, 若 \mathbf{A} 为与 x 无关的方阵, 记

$$\begin{aligned}
g(x\mathbf{A}) &= a_0 \mathbf{I} + a_1 x \mathbf{A} + a_2 x^2 \mathbf{A}^2 + a_3 x^3 \mathbf{A}^3 + \cdots \\
g'(x\mathbf{A}) &= a_1 \mathbf{I} + 2a_2 x \mathbf{A} + 3a_3 x^2 \mathbf{A}^2 + \cdots
\end{aligned}$$

易知有

$$\begin{aligned}
\frac{\partial g(x\mathbf{A})}{\partial x} &= a_1 \mathbf{A} + 2a_2 x \mathbf{A}^2 + 3a_3 x^2 \mathbf{A}^3 + \cdots \\
&= \mathbf{A}(a_1 \mathbf{I} + 2a_2 x \mathbf{A} + 3a_3 x^2 \mathbf{A}^2 + \cdots) = \mathbf{A} g'(x\mathbf{A}) \\
&= (a_1 \mathbf{I} + 2a_2 x \mathbf{A} + 3a_3 x^2 \mathbf{A}^2 + \cdots) \mathbf{A} = g'(x\mathbf{A}) \mathbf{A}
\end{aligned}$$

对于 e^x 、 $\ln x$ 、 $\sin x$ 、 $\cos x$, 上式依然适用, 例如

$$\frac{\partial e^{x\mathbf{A}}}{\partial x} = \mathbf{A} e^{x\mathbf{A}} = e^{x\mathbf{A}} \mathbf{A}$$

6 标量对矩阵求导

矩阵常见的标量函数有迹和行列式，二次型可以归为迹来处理。

6.1 迹对矩阵求导

若 a 与 \mathbf{X} 无关， $\mathbf{U} = \mathbf{U}(\mathbf{X})$ ， $\mathbf{V} = \mathbf{V}(\mathbf{X})$ ，则以下结论是显然的：

$$\frac{\partial \text{tr}(\mathbf{X})}{\partial \mathbf{X}} = \mathbf{I}, \quad \frac{\partial \text{tr}(\mathbf{U} + \mathbf{V})}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{U})}{\partial \mathbf{X}} + \frac{\partial \text{tr}(\mathbf{V})}{\partial \mathbf{X}}, \quad \frac{\partial \text{tr}(a\mathbf{U})}{\partial \mathbf{X}} = a \frac{\partial \text{tr}(\mathbf{U})}{\partial \mathbf{X}}$$

对于乘积有

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{UV})}{\partial \mathbf{X}} \right]_{ij} &= \frac{\partial \text{tr}(\mathbf{UV})}{\partial x_{ji}} = \frac{\partial (\sum_p \sum_q u_{pq} v_{qp})}{\partial x_{ji}} = \sum_p \sum_q \left(\frac{\partial u_{pq}}{\partial x_{ji}} v_{qp} + u_{pq} \frac{\partial v_{qp}}{\partial x_{ji}} \right) \\ &= \text{tr} \left(\frac{\partial \mathbf{U}}{\partial x_{ji}} \mathbf{V} \right) + \text{tr} \left(\mathbf{U} \frac{\partial \mathbf{V}}{\partial x_{ji}} \right) = \text{tr} \left(\frac{\partial (\mathbf{UV})}{\partial x_{ji}} \right) \end{aligned}$$

由此可知迹和求导的顺序可以交换。特别的，

- 取 $\mathbf{U} = \mathbf{BA}$ 与 \mathbf{X} 无关， $\mathbf{V} = \mathbf{X}$ ，则

$$\left[\frac{\partial \text{tr}(\mathbf{BAX})}{\partial \mathbf{X}} \right]_{ij} = \text{tr} \left(\mathbf{BA} \frac{\partial \mathbf{X}}{\partial x_{ji}} \right) = \text{tr}(\mathbf{BA} \mathbf{E}_{ji}) = [\mathbf{BA}]_{ij} \implies \frac{\partial \text{tr}(\mathbf{BAX})}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{AXB})}{\partial \mathbf{X}} = \mathbf{BA}$$

- 取 $\mathbf{U} = \mathbf{BA}$ 与 \mathbf{X} 无关， $\mathbf{V} = \mathbf{X}^\top$ ，则

$$\frac{\partial \text{tr}(\mathbf{BAX}^\top)}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{XA}^\top \mathbf{B}^\top)}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{A}^\top \mathbf{B}^\top \mathbf{X})}{\partial \mathbf{X}} = \mathbf{A}^\top \mathbf{B}^\top$$

- 取 $\mathbf{U} = \mathbf{A}$ 与 \mathbf{X} 无关， $\mathbf{V} = \mathbf{XX}^\top$ ，则

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{AXX}^\top)}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\mathbf{A} \frac{\partial \mathbf{XX}^\top}{\partial x_{ji}} \right) = \text{tr} \left(\mathbf{A} \frac{\partial \mathbf{X}}{\partial x_{ji}} \mathbf{X}^\top \right) + \text{tr} \left(\mathbf{AX} \frac{\partial \mathbf{X}^\top}{\partial x_{ji}} \right) \\ &= \text{tr}(\mathbf{AE}_{ji} \mathbf{X}^\top) + \text{tr}(\mathbf{AXE}_{ji}) \\ &= [\mathbf{X}^\top \mathbf{A}]_{ij} + [\mathbf{AX}]_{ji} \end{aligned}$$

从而

$$\frac{\partial \text{tr}(\mathbf{AXX}^\top)}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{X}^\top \mathbf{AX})}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{XX}^\top \mathbf{A})}{\partial \mathbf{X}} = \mathbf{X}^\top \mathbf{A} + \mathbf{X}^\top \mathbf{A}^\top = \mathbf{X}^\top (\mathbf{A} + \mathbf{A}^\top)$$

- 取 $\mathbf{U} = \mathbf{A}$ 与 \mathbf{X} 无关， $\mathbf{V} = \mathbf{X}^\top \mathbf{X}$ ，则

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{AX}^\top \mathbf{X})}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\mathbf{A} \frac{\partial \mathbf{X}^\top \mathbf{X}}{\partial x_{ji}} \right) = \text{tr} \left(\mathbf{A} \frac{\partial \mathbf{X}^\top}{\partial x_{ji}} \mathbf{X} \right) + \text{tr} \left(\mathbf{AX}^\top \frac{\partial \mathbf{X}}{\partial x_{ji}} \right) \\ &= \text{tr}(\mathbf{AE}_{ij} \mathbf{X}) + \text{tr}(\mathbf{AX}^\top \mathbf{E}_{ji}) \\ &= [\mathbf{XA}]_{ji} + [\mathbf{AX}^\top]_{ij} \end{aligned}$$

从而

$$\frac{\partial \text{tr}(\mathbf{AX}^\top \mathbf{X})}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{XAX}^\top)}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{X}^\top \mathbf{XA})}{\partial \mathbf{X}} = (\mathbf{A} + \mathbf{A}^\top) \mathbf{X}^\top$$

- 取 $\mathbf{U} = \mathbf{BA}$ 与 \mathbf{X} 无关, $\mathbf{V} = \mathbf{X}^{-1}$, 结合式 (5) 可得

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{BA}\mathbf{X}^{-1})}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\mathbf{BA} \frac{\partial \mathbf{X}^{-1}}{\partial x_{ji}} \right) = \text{tr} \left(-\mathbf{BA}\mathbf{X}^{-1} \frac{\partial \mathbf{X}}{\partial x_{ji}} \mathbf{X}^{-1} \right) \\ &= -\text{tr} (\mathbf{X}^{-1} \mathbf{BA}\mathbf{X}^{-1} \mathbf{E}_{ji}) = -[\mathbf{X}^{-1} \mathbf{BA}\mathbf{X}^{-1}]_{ij} \\ &\implies \frac{\partial \text{tr}(\mathbf{BA}\mathbf{X}^{-1})}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{AX}^{-1}\mathbf{B})}{\partial \mathbf{X}} = -\mathbf{X}^{-1} \mathbf{BA}\mathbf{X}^{-1} \end{aligned}$$

- 取 $\mathbf{U} = \mathbf{I}$, $\mathbf{V} = (\mathbf{X} + \mathbf{A})^{-1}$, 结合式 (5) 可得

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{X} + \mathbf{A})^{-1}}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\frac{\partial (\mathbf{X} + \mathbf{A})^{-1}}{\partial x_{ji}} \right) = -\text{tr} \left((\mathbf{X} + \mathbf{A})^{-1} \frac{\partial (\mathbf{X} + \mathbf{A})}{\partial x_{ji}} (\mathbf{X} + \mathbf{A})^{-1} \right) \\ &= -\text{tr} ((\mathbf{X} + \mathbf{A})^{-1} (\mathbf{X} + \mathbf{A})^{-1} \mathbf{E}_{ji}) = -[(\mathbf{X} + \mathbf{A})^{-1} (\mathbf{X} + \mathbf{A})^{-1}]_{ij} \\ &\implies \frac{\partial \text{tr}(\mathbf{X} + \mathbf{A})^{-1}}{\partial \mathbf{X}} = -(\mathbf{X} + \mathbf{A})^{-1} (\mathbf{X} + \mathbf{A})^{-1} \end{aligned}$$

- 取 $\mathbf{U} = \mathbf{AXB}$, $\mathbf{V} = \mathbf{X}^\top \mathbf{C}$, 其中 \mathbf{A} 、 \mathbf{B} 、 \mathbf{C} 与 \mathbf{X} 无关, 则

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{AXB}\mathbf{X}^\top \mathbf{C})}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\frac{\partial (\mathbf{AXB})}{\partial x_{ji}} \mathbf{X}^\top \mathbf{C} \right) + \text{tr} \left(\mathbf{AXB} \frac{\partial (\mathbf{X}^\top \mathbf{C})}{\partial x_{ji}} \right) \\ &= \text{tr} (\mathbf{AE}_{ji} \mathbf{BX}^\top \mathbf{C}) + \text{tr} (\mathbf{AXB} \mathbf{E}_{ij} \mathbf{C}) \\ &= [\mathbf{BX}^\top \mathbf{CA}]_{ij} + [\mathbf{CAXB}]_{ji} \\ &\implies \frac{\partial \text{tr}(\mathbf{AXB}\mathbf{X}^\top \mathbf{C})}{\partial \mathbf{X}} = \mathbf{BX}^\top \mathbf{CA} + \mathbf{B}^\top \mathbf{X}^\top \mathbf{A}^\top \mathbf{C}^\top \end{aligned}$$

- 取 $\mathbf{U} = \mathbf{AX}^\top \mathbf{B}$, $\mathbf{V} = \mathbf{XC}$, 其中 \mathbf{A} 、 \mathbf{B} 、 \mathbf{C} 与 \mathbf{X} 无关, 则

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{AX}^\top \mathbf{BXC})}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\frac{\partial (\mathbf{AX}^\top \mathbf{B})}{\partial x_{ji}} \mathbf{XC} \right) + \text{tr} \left(\mathbf{AX}^\top \mathbf{B} \frac{\partial (\mathbf{XC})}{\partial x_{ji}} \right) \\ &= \text{tr} (\mathbf{AE}_{ij} \mathbf{BXC}) + \text{tr} (\mathbf{AX}^\top \mathbf{BE}_{ji} \mathbf{C}) \\ &= [\mathbf{BXC}\mathbf{A}]_{ji} + [\mathbf{CAX}^\top \mathbf{B}]_{ij} \\ &\implies \frac{\partial \text{tr}(\mathbf{AX}^\top \mathbf{BXC})}{\partial \mathbf{X}} = \mathbf{CAX}^\top \mathbf{B} + \mathbf{A}^\top \mathbf{C}^\top \mathbf{X}^\top \mathbf{B}^\top \end{aligned}$$

- 取 $\mathbf{U} = \mathbf{BA}$ 与 \mathbf{X} 无关, $\mathbf{V} = \mathbf{X}^n$, 其中 n 是正整数, 结合式 (4) 可得

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{BA}\mathbf{X}^n)}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\mathbf{BA} \frac{\partial \mathbf{X}^n}{\partial x_{ji}} \right) = \text{tr} \left(\mathbf{BA} \sum_{k \in [n]} \mathbf{X}^{k-1} \frac{\partial \mathbf{X}}{\partial x_{ji}} \mathbf{X}^{n-k} \right) = \sum_{k \in [n]} \text{tr} \left(\mathbf{BA}\mathbf{X}^{k-1} \frac{\partial \mathbf{X}}{\partial x_{ji}} \mathbf{X}^{n-k} \right) \\ &= \sum_{k \in [n]} \text{tr} (\mathbf{X}^{n-k} \mathbf{BA}\mathbf{X}^{k-1} \mathbf{E}_{ji}) = \sum_{k \in [n]} [\mathbf{X}^{n-k} \mathbf{BA}\mathbf{X}^{k-1}]_{ij} \\ &\implies \frac{\partial \text{tr}(\mathbf{BA}\mathbf{X}^n)}{\partial \mathbf{X}} = \frac{\partial \text{tr}(\mathbf{AX}^n \mathbf{B})}{\partial \mathbf{X}} = \sum_{k \in [n]} \mathbf{X}^{n-k} \mathbf{BA}\mathbf{X}^{k-1} \end{aligned}$$

进一步若 $\mathbf{A} = \mathbf{B} = \mathbf{I}$, 则

$$\frac{\partial \text{tr}(\mathbf{X}^n)}{\partial \mathbf{X}} = \sum_{k \in [n]} \mathbf{X}^{n-k} \mathbf{X}^{k-1} = \sum_{k \in [n]} \mathbf{X}^{n-1} = n\mathbf{X}^{n-1}$$

不难发现形式上和单变量的求导公式 $\partial x^n / \partial x = nx^{n-1}$ 是一样的。类似的记

$$\begin{aligned} e^{\mathbf{X}} &= \mathbf{I} + \mathbf{X} + \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^3}{3!} + \cdots \\ \sin \mathbf{X} &= \mathbf{X} - \frac{\mathbf{X}^3}{3!} + \frac{\mathbf{X}^5}{5!} - \cdots \\ \cos \mathbf{X} &= \mathbf{I} - \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^4}{4!} - \frac{\mathbf{X}^6}{6!} + \cdots \end{aligned}$$

结合式 (4) 可得

$$\begin{aligned} \frac{\partial \text{tr}(e^{\mathbf{X}})}{\partial \mathbf{X}} &= \frac{\partial}{\partial \mathbf{X}} \text{tr} \left(\mathbf{I} + \mathbf{X} + \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^3}{3!} + \cdots \right) \\ &= \frac{\partial \text{tr}(\mathbf{I})}{\partial \mathbf{X}} + \frac{\partial \text{tr}(\mathbf{X})}{\partial \mathbf{X}} + \frac{1}{2!} \frac{\partial \text{tr}(\mathbf{X}^2)}{\partial \mathbf{X}} + \frac{1}{3!} \frac{\partial \text{tr}(\mathbf{X}^3)}{\partial \mathbf{X}} + \cdots \\ &= \mathbf{I} + \mathbf{X} + \frac{\mathbf{X}^2}{2!} + \cdots = e^{\mathbf{X}} \end{aligned}$$

以及

$$\begin{aligned} \frac{\partial \text{tr}(\sin \mathbf{X})}{\partial \mathbf{X}} &= \frac{\partial}{\partial \mathbf{X}} \text{tr} \left(\mathbf{X} - \frac{\mathbf{X}^3}{3!} + \frac{\mathbf{X}^5}{5!} - \cdots \right) \\ &= \frac{1}{1!} \frac{\partial \text{tr}(\mathbf{X})}{\partial \mathbf{X}} - \frac{1}{3!} \frac{\partial \text{tr}(\mathbf{X}^3)}{\partial \mathbf{X}} + \frac{1}{5!} \frac{\partial \text{tr}(\mathbf{X}^5)}{\partial \mathbf{X}} - \cdots \\ &= \mathbf{I} - \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^4}{4!} - \cdots = \cos \mathbf{X} \\ \frac{\partial \text{tr}(\cos \mathbf{X})}{\partial \mathbf{X}} &= \frac{\partial}{\partial \mathbf{X}} \text{tr} \left(\mathbf{I} - \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^4}{4!} - \frac{\mathbf{X}^6}{6!} + \cdots \right) \\ &= \frac{\partial \text{tr}(\mathbf{I})}{\partial \mathbf{X}} - \frac{1}{2!} \frac{\partial \text{tr}(\mathbf{X}^2)}{\partial \mathbf{X}} + \frac{1}{4!} \frac{\partial \text{tr}(\mathbf{X}^4)}{\partial \mathbf{X}} - \frac{1}{6!} \frac{\partial \text{tr}(\mathbf{X}^6)}{\partial \mathbf{X}} + \cdots \\ &= -\mathbf{X} + \frac{\mathbf{X}^3}{3!} - \frac{\mathbf{X}^5}{5!} + \cdots = -\sin \mathbf{X} \end{aligned}$$

均与单变量的求导公式一样。

- 取 $\mathbf{U} = \mathbf{I}$, $\mathbf{V} = \mathbf{A} \otimes \mathbf{X}$, 则

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{A} \otimes \mathbf{X})}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\frac{\partial \mathbf{A} \otimes \mathbf{X}}{\partial x_{ji}} \right) = \text{tr} \left(\mathbf{A} \otimes \frac{\partial \mathbf{X}}{\partial x_{ji}} \right) = \text{tr}(\mathbf{A} \otimes \mathbf{E}_{ji}) = \text{tr}(\mathbf{A}) \delta_{ij} \\ &\implies \frac{\partial \text{tr}(\mathbf{A} \otimes \mathbf{X})}{\partial \mathbf{X}} = \text{tr}(\mathbf{A}) \mathbf{I} \end{aligned}$$

- 取 $\mathbf{U} = \mathbf{I}$, $\mathbf{V} = \mathbf{X} \otimes \mathbf{X}$, 则

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{X} \otimes \mathbf{X})}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\frac{\partial \mathbf{X} \otimes \mathbf{X}}{\partial x_{ji}} \right) = \text{tr} \left(\frac{\partial \mathbf{X}}{\partial x_{ji}} \otimes \mathbf{X} + \mathbf{X} \otimes \frac{\partial \mathbf{X}}{\partial x_{ji}} \right) \\ &= \text{tr}(\mathbf{E}_{ji} \otimes \mathbf{X}) + \text{tr}(\mathbf{X} \otimes \mathbf{E}_{ji}) = 2\text{tr}(\mathbf{X}) \delta_{ij} \\ &\implies \frac{\partial \text{tr}(\mathbf{X} \otimes \mathbf{X})}{\partial \mathbf{X}} = 2\text{tr}(\mathbf{X}) \mathbf{I} \end{aligned}$$

6.2 行列式对矩阵求导

设 $\mathbf{X} \in \mathbb{R}^{m \times n}$ 、 $\mathbf{A} \in \mathbb{R}^{l \times m}$ 、 $\mathbf{B} \in \mathbb{R}^{n \times l}$ 、 $\mathbf{Y} = \mathbf{AXB} \in \mathbb{R}^{l \times l}$ ， \mathbf{A} 、 \mathbf{B} 与 \mathbf{X} 无关，结合式 (1) 易知

$$\left[\frac{\partial |\mathbf{AXB}|}{\partial \mathbf{X}} \right]_{ij} = \frac{\partial |\mathbf{Y}|}{\partial x_{ji}} = \sum_p \sum_q \frac{\partial |\mathbf{Y}|}{\partial y_{pq}} \frac{\partial y_{pq}}{\partial x_{ji}} = \text{tr} \left(\frac{\partial |\mathbf{Y}|}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial x_{ji}} \right)$$

其中第二项

$$\frac{\partial \mathbf{Y}}{\partial x_{ji}} = \frac{\partial (\mathbf{AXB})}{\partial x_{ji}} = \mathbf{A} \frac{\partial \mathbf{X}}{\partial x_{ji}} \mathbf{B} = \mathbf{A} \mathbf{E}_{ji} \mathbf{B}$$

记 y_{ji} 有一个微小增量 ϵ 后的矩阵为 $\mathbf{Y}(y_{ji} + \epsilon)$ ，根据第 j 行拉普拉斯展开易知

$$|\mathbf{Y}(y_{ji} + \epsilon)| - |\mathbf{Y}| = \epsilon C_{ji}$$

其中 C_{ji} 是关于 y_{ji} 的代数余子式，因此

$$\left[\frac{\partial |\mathbf{Y}|}{\partial \mathbf{Y}} \right]_{ij} = \frac{\partial |\mathbf{Y}|}{\partial y_{ji}} = \lim_{\epsilon \rightarrow 0} \frac{|\mathbf{Y}(y_{ji} + \epsilon)| - |\mathbf{Y}|}{\epsilon} = C_{ji}$$

故第一项

$$\frac{\partial |\mathbf{Y}|}{\partial \mathbf{Y}} = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \mathbf{Y}^*$$

代入可得

$$\begin{aligned} \left[\frac{\partial |\mathbf{AXB}|}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\frac{\partial |\mathbf{Y}|}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial x_{ji}} \right) = \text{tr}(\mathbf{Y}^* \mathbf{A} \mathbf{E}_{ji} \mathbf{B}) = [\mathbf{BY}^* \mathbf{A}]_{ij} \\ &\implies \frac{\partial |\mathbf{AXB}|}{\partial \mathbf{X}} = \mathbf{B}(\mathbf{AXB})^* \mathbf{A} \end{aligned}$$

若 \mathbf{X} 、 \mathbf{A} 、 \mathbf{B} 均为可逆方阵，则 $\mathbf{Y} = \mathbf{AXB}$ 亦为可逆方阵，于是

$$\frac{\partial |\mathbf{AXB}|}{\partial \mathbf{X}} = \mathbf{B}(\mathbf{AXB})^* \mathbf{A} = \mathbf{B}|\mathbf{AXB}|(\mathbf{AXB})^{-1} \mathbf{A} = |\mathbf{AXB}| \mathbf{X}^{-1} \quad (6)$$

进一步若 $\mathbf{A} = \mathbf{B} = \mathbf{I}$ ，则

$$\frac{\partial |\mathbf{X}|}{\partial \mathbf{X}} = \mathbf{X}^* = |\mathbf{X}| \mathbf{X}^{-1}$$

由此可得

$$\frac{\partial |\mathbf{X}^n|}{\partial \mathbf{X}} = \frac{\partial |\mathbf{X}|^n}{\partial \mathbf{X}} = n|\mathbf{X}|^{n-1} \mathbf{X}^* = n|\mathbf{X}|^n \mathbf{X}^{-1} = n|\mathbf{X}^n| \mathbf{X}^{-1}$$

若 a 与 \mathbf{X} 无关，则

$$\frac{\partial \ln |a\mathbf{X}|}{\partial \mathbf{X}} = \frac{\partial \ln a^m |\mathbf{X}|}{\partial \mathbf{X}} = \frac{\partial \ln a^m}{\partial \mathbf{X}} + \frac{\partial \ln |\mathbf{X}|}{\partial \mathbf{X}} = \frac{1}{|\mathbf{X}|} \frac{\partial |\mathbf{X}|}{\partial \mathbf{X}} = \frac{\mathbf{X}^*}{|\mathbf{X}|} = \mathbf{X}^{-1}$$

设 $\mathbf{X} \in \mathbb{R}^{m \times n}$ 、 $\mathbf{A} \in \mathbb{R}^{m \times m}$ 、 $\mathbf{Y} = \mathbf{X}^\top \mathbf{A} \mathbf{X} \in \mathbb{R}^{n \times n}$ 可逆， \mathbf{A} 与 \mathbf{X} 无关，易知有

$$\begin{aligned} \left[\frac{\partial |\mathbf{X}^\top \mathbf{A} \mathbf{X}|}{\partial \mathbf{X}} \right]_{ij} &= \text{tr} \left(\mathbf{Y}^* \frac{\partial \mathbf{X}^\top \mathbf{A} \mathbf{X}}{\partial x_{ji}} \right) = \text{tr} \left(\mathbf{Y}^* \frac{\partial \mathbf{X}^\top}{\partial x_{ji}} \mathbf{A} \mathbf{X} \right) + \text{tr} \left(\mathbf{Y}^* \mathbf{X}^\top \mathbf{A} \frac{\partial \mathbf{X}}{\partial x_{ji}} \right) \\ &= \text{tr}(\mathbf{Y}^* \mathbf{E}_{ij} \mathbf{A} \mathbf{X}) + \text{tr}(\mathbf{Y}^* \mathbf{X}^\top \mathbf{A} \mathbf{E}_{ji}) = [\mathbf{A} \mathbf{X} \mathbf{Y}^*]_{ji} + [\mathbf{Y}^* \mathbf{X}^\top \mathbf{A}]_{ij} \end{aligned}$$

于是

$$\begin{aligned} \frac{\partial |\mathbf{X}^\top \mathbf{A} \mathbf{X}|}{\partial \mathbf{X}} &= (\mathbf{A} \mathbf{X} \mathbf{Y}^*)^\top + \mathbf{Y}^* \mathbf{X}^\top \mathbf{A} = (\mathbf{A} \mathbf{X} |\mathbf{X}^\top \mathbf{A} \mathbf{X}| (\mathbf{X}^\top \mathbf{A} \mathbf{X})^{-1})^\top + |\mathbf{X}^\top \mathbf{A} \mathbf{X}| (\mathbf{X}^\top \mathbf{A} \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{A} \\ &= |\mathbf{X}^\top \mathbf{A} \mathbf{X}| (\mathbf{X}^\top \mathbf{A}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{A}^\top + |\mathbf{X}^\top \mathbf{A} \mathbf{X}| (\mathbf{X}^\top \mathbf{A} \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{A} \\ &= |\mathbf{X}^\top \mathbf{A} \mathbf{X}| ((\mathbf{X}^\top \mathbf{A}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{A}^\top + (\mathbf{X}^\top \mathbf{A} \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{A}) \end{aligned}$$

若 \mathbf{A} 对称，则

$$\frac{\partial |\mathbf{X}^\top \mathbf{A} \mathbf{X}|}{\partial \mathbf{X}} = 2 |\mathbf{X}^\top \mathbf{A} \mathbf{X}| (\mathbf{X}^\top \mathbf{A} \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{A}$$

- 若 \mathbf{X} 、 \mathbf{A} 是方阵，则其均可逆，于是

$$\frac{\partial |\mathbf{X}^\top \mathbf{A} \mathbf{X}|}{\partial \mathbf{X}} = 2 |\mathbf{X}^\top| |\mathbf{A}| |\mathbf{X}| \mathbf{X}^{-1} \mathbf{A}^{-1} \mathbf{X}^{-\top} \mathbf{X}^\top \mathbf{A} = 2 |\mathbf{X}|^2 |\mathbf{A}| \mathbf{X}^{-1}$$

- 若 $\mathbf{A} = \mathbf{I}$ ，则

$$\frac{\partial |\mathbf{X}^\top \mathbf{X}|}{\partial \mathbf{X}} = 2 |\mathbf{X}^\top \mathbf{X}| (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top = 2 |\mathbf{X}^\top \mathbf{X}| \mathbf{X}^\dagger$$

以及

$$\frac{\partial \ln |\mathbf{X}^\top \mathbf{X}|}{\partial \mathbf{X}} = \frac{1}{|\mathbf{X}^\top \mathbf{X}|} \frac{\partial |\mathbf{X}^\top \mathbf{X}|}{\partial \mathbf{X}} = 2 \mathbf{X}^\dagger$$