MATH 113 - Matrix Theory; Midterm Review

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1 Notes on 3.F: Duality

Definition. *A linear functional on V is a linear map from V to $\mathbf{F}$, i.e. an element of $\mathcal{L}(V, \mathbf{F})$.*

Definition. *The dual space of V, denoted V' is the vector space of all linear functions on V. In other words, $V' = \mathcal{L}(V, \mathbf{F})$.*

Note that $\dim V' = \dim V$. This follows from 3.61, which states that $\dim \mathcal{L}(V, W) = \dim(V) \dim(W)$.
Definition. *If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list $\phi_1, \ldots, \phi_n$ of elements of V' where each ϕ_j is the linear functional on V such that*

$$\phi_j(v_k) = \begin{cases} 1; & \text{if } k = j \\ 0; & \text{if } k \neq j. \end{cases}$$

2 Review

- Let $T \in L(V, W)$. Then T is injective if and only if $\operatorname{null} T = \{0\}$.
- Fundamental theorem of linear maps. Let T be a linear map from V to W. Then

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

- Matrix of a linear map. Suppose $T \in L(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T w.r.t. these bases is the $m \times n$ matrix whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

(Look at picture on pg. 71).

- A linear functional on V is a linear map from V to F.
- The dual space of V, denoted V', is the vector space of all linear functionals on V, with $V' = L(V, F)$. Note that $\dim V' = \dim V$.
- If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is the list $\phi_1, \ldots, \phi_n$ of elements of V', where each ϕ_j satisfies $\phi_j(v_k) = 1$ if $k = j$, else 0.
- Suppose V is finite dimensional and U is s subspace of V. Then

$$\dim U + \dim U^0 = \dim V.$$

- (3.5) Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

- Volume functions. A function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a volume function if it satisfies bilinearity and alternation.
  - Bilinearity: $f(u, av + bw) = af(u, v) + bf(u, w)$ and $f(au + bv, w) = af(u, w) + bf(v, w)$.
  - Alternation, $f(u, u) = 0$.
- Note that volume functions are a vector space.
- In general, can write volume functions in general for multilinear functions.
- Swapping two vectors changes the sign of the volume.
- For any permutation, define $inv(\sigma)$ to be the number of inversions. The number of times that $i < j$, but $\sigma(i) > \sigma(j)$.
- For any permutation, define the sign as $(-1)^{inv(\sigma)}$.
- If A_{ij} is an $n \times n$ over a field k, then $\det A$ is

$$\det A = \sum_{\sigma \in \Sigma_n} (\sigma) A_{\sigma(1),1} \ldots A_{\sigma(n),n}$$

- Q1a. If T is injective, then $Tv_1 = Tv_2$ implies $v_1 = v_2$. To show linear independence, we claim that

$$a_1 Tv_1 + a_2 Tv_2 + \cdots + a_n Tv_n = 0$$

implies $v_i = 0$ for all i.

- Q2.