MATH 171 - Real Analysis

Instructor: George Schaeffer; Notes: Adithya Ganesh

November 16, 2018

Contents

1 9-24-18: Everything is a set

Administrivia:

- Book: Johnsonbaugh and Pfaffenberger
- (Supplement) Rudin's Principles of Mathematical Analysis
- Exam: likely week 5.

1.1 On sets

One motivation for analysis is a problem identified in 1901: Russell's Paradox. Consider

$$R = \{x : x \notin X\} = \text{ the set of all sets that do not contain themselves}$$

Problem: does the set contain itself? Either $R \in R$ or $R \notin R$, but neither is possible.

Rules for what is isn't a set: Zermelo-Frankel axioms.

In particular, under ZF: Can't build $\{x : x \text{ has property } P\}$. You must say

$$\{x \in S : x \text{ has property } P \text{ where } S \text{ is already a set. }\}$$

But going further: the collection of all sets is itself not a set.

Axioms of choice (the Cartesian product of a collection of non-empty sets is non-empty).

We can define the natural numbers in the framework of sets. If x is a set we can define its successor as $S(x) = x \cup \{x\}$.

- 0 = $\emptyset$
- 1 = $\{\emptyset\}$
- 2 = $\{\{\emptyset\}, \emptyset\}$.
- 3 = $\{\{\emptyset\}, \emptyset, \{\{\{\emptyset\}, \emptyset\}\}. = \{0, 1, 2\}$.

1.2 On functions (and cartesian products)

Cartesian Product. Let X and Y be sets. Then we can write

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

How do we define ordered pairs? $(x, y) \neq \{x, y\} = \{y, x\}$ doesn't work, since order matters.

Instead, we want to say

$$(x, y) = \{x, \{x, y\}\}.$$

What is a function? We can write $f : X \to Y$, where X is the domain(f) and Y is the codomain(f). A function $f : X \to Y$ is a subset of $X \times Y$ satisfying the following:

- $\forall x \in X, \exists y \in Y : (x, y) \in f$.
- $\forall x \in X, \forall y, y' \in Y : (x, y) \wedge (x, y') \in f \implies y = y'$.

As a set, for example, $\sin \subseteq \mathbb{R} \times \mathbb{R}$.

1.3 On natural numbers

The set $\mathbb{N}$ is equipped with a sucessor function $S : \mathbb{N} \to \mathbb{N} : x \mapsto S(x)$. There are a few rules attached to this, namely the Peano axioms:

- $\forall x \in \mathbb{N} : S(x) \neq 0$
- S is "injective": If $S(x) = S(y) \implies x = y$.
- Axiom of induction: If $K \subseteq \mathbb{N}$ satisfying
  - $0 \in K$
  - $\forall x \in K, S(x) \in K$.

$\mathbb{N}$ has two binary operations, $+, \cdot$, addition and multiplication.

A binary operation on X is a function $X \times X \to X$.

- + (a, b) = a+b

- $\cdot(a, b) = ab$

$\forall a, a + 0 = a$. $\forall a, b; a + S(b) = S(a + b)$.

2 10-1-18: Suprema and infima

Theorems of R.

- $\mathbb{R}$ is an ordered field.
- Tere are lots of ordered fields: $\mathbb{Q}$.
- Least upper bound axiom: If $S \subseteq \mathbb{R}$ is nonempty and bounded above, then S has a least bound $\in \mathbb{R}$.

Definition. *Let $S \subseteq \mathbb{R}, M \in \mathbb{R}$. We say that M is an upper bound on S is $\forall x \in S : x \leq M$. M is the least upper bound (or the supremeum) of S is $\forall M' < M$, M' is not an upper bound on S.*

Furthermore: $M = \sup(S)$ if

- M is an upper bound on S:
- $\forall M' < M : M'$ is not an upper bound on S.

$$\neg[\forall x \in S : x \leq M']$$
$$\exists x \in S : \neg[x \leq M']$$
$$\exists x : S : x > M'$$
$$\boxed{\forall \epsilon > 0, \exists x \in S : x > M - \epsilon}$$

Easy two step process for proving $M = \sup(S)$.

The "greatest lower bound" axiom is equivalent to the "least upper bound" axiom. Note that for convenience $\sup(\text{unbounded above } S) = +\infty$ and $\sup(\emptyset) = -\infty$.

Consequences of the axioms in $\mathbb{R}$.

Archimedean Property. $\forall a, b \in \mathbb{R}, a, b > 0$, then $\exists n \in \mathbb{N}$ such that $na > b$.

Proof. Let $S = \{n \in \mathbb{N} : na \leq b\}$; which implies $n \leq \frac{b}{a}$. S is nonempty because $0 \in S$. S is bounded above by $\frac{b}{a}$. By LUBA: S has a supremum $m = \sup(S)$. $m+1 \notin S$ and $m+1 \in \mathbb{N}$ (left as an easy exercise).

Why is $m+1 \notin S$? Otherwise $m+1 \leq m$. So $\neg[(m+1)a \leq b]$, i.e. $(m+1)a > b$. □

Note that the Archimedean principle is true in $\mathbb{Q}$ as well. It inherits AP from R. There is also an independent proof just using the construction of $\mathbb{Q}$ as fractions.[1]

Theorem. *The rational numbers form a dense subset of* $\mathbb{R}$.

We start by explaining the definition: $\forall a, b \in \mathbb{R} : a < b \implies [\exists r \in \mathbb{Q} : a < r < b]$. We now mention a lemma that will help us prove the theorem.

Lemma. *If* $a < b \in \mathbb{R}$ *and* $b - a > 1$*, then* $\exists n \in \mathbb{Z} : a < n < b$.

Proof. Let $S = \{n \in \mathbb{Z} : n \leq a\}$.

By LUBA, we let $m = \sup(S)$. Note that $m \in \mathbb{Z}$. $m+1 \notin S$. We can easily verify that $a < m+1 < b$. The first inequality follows from $m+1 \notin S$, and for the second, note that:

$$\begin{aligned} m+1 &< m + (b-a) \\ &\leq a + (b-a) = b. \end{aligned}$$

Here, we have used the fact that $m \in S$, so $m \leq a$. □

Proof. (Main Theorem.) We know $a < b$. By the Archimedean Principle, since $b - a > 0$ and $1 > 0$, there must exist $n \in \mathbb{N}$ such that $n(b-a) > 1$. This implies that $nb - na > 1$. Also $na < nb$.

By the lemma, there exists an integer $k \in \mathbb{N}$ with $na < k < nb$. Dividing by n, we obtain the fraction $\frac{k}{n}$ which satisfies $a < \frac{k}{n} < b$. □

The irrationals are also dense in the reals - just take the rationals and add $\sqrt{2}$, and follow a similar argument.

[1]On HW2: An example of an ordered field, for which the AP fails.

3 Continuity - 10-19

If (X, τ) is a topological space and $S \subseteq X$, we can give S a topology

$$\tau_S = \{U \cap S : \text{ where } U \in \tau_X\}.$$

This is a subspace / induced / inherited / relative topology on S.

Note: $[0, 1]$ w/ the subspace topology from $\mathbb{R}$.

If (X, d_X) is ametric space, $S \subseteq X$, then S is also a metric space, where $d_S = d_X|_{S \times S}$ (restricted for $S \times S$).

If X is a metric space and $S \subseteq X$, then the topoloy from d is the subspace topology.

If U is open / closed in S, it need not be closed in X.

However, if K is compact in S, then K is compact in X.

Self explanatory (?) We say that a topological space X is compact if it is a compact set in its own topology.

Continuous functions. Let (X, d_X) and (Y, d_Y) be metric spaces, let $f : X \to Y$, let $p \in X$. We say that f is continuous at p

1. In the analytic sense if

$$\forall \epsilon > 0, \exists > 0, \forall x \in X : d_X(x, p) < \implies d_Y(f(x), f(p)) < \epsilon.$$

2. In the sequential sense if $\forall$ sequences $(x_n)_n \to p \in X$, the sequence

$$(f(x_n))_n \to f(p).$$

3. In the topological sense if $\forall$ open $V \subseteq Y$, if $f(p) \in V$, then there exists an open $U \subseteq X$ such that $p \in U$ and $f(U) \subseteq V$.

We will show that all of these are equivalent.

Notes on these definitions.

- For the topological sense: can switch "open" for "closed."
- Also, (iii) is the definition of "continuous' at p" for general topological spaces. It doesn't require a metric on X or Y.
- In (i), δ can depend on both p an ϵ.

Outline of proof of equivalence.

- $1 \implies 2$ (easy; definition pushing).

- $2 \implies 1$ is slightly harder. We will prove this by contrapositive. We'll show: if f is not analytically continuous at p, then it's not sequentially cts.

$$\forall \varepsilon > 0, \forall \delta > 0; \exists x \in X : d_X(x,p) < p \text{ and } d_Y(f(x), f(p)) \geq \varepsilon.$$

  In particular, we can define $(x_n)_{n=1}^{\infty}$ so that

$$d_X(x_n, p) < \frac{1}{n} \text{ and } d_Y(f(x_n), f(p)) \geq \varepsilon.$$

  This means that $x_n \to p$, but $f(x_n) \not\to f(p)$.

- $3 \implies 1$. Let $\varepsilon > 0$. We know f is topologically continuous, so let $V = B_\varepsilon(f(p))$ is open. Then there exists an open $U \ni p$ such that $f(U) \subseteq V$. Because U is open, p is interior to U, so $\exists \delta$ such that $B_\delta(p) \subseteq U$.

  So, $x \in B_\delta(p) \implies x \in U \implies f(x) \in V \implies f(x) \in B_\varepsilon(f(p))$.

- $1 \implies 3$ is similar to the previous, just reverse all the statements.

Definition. *The function $f : X \to Y$ is continuous if it is continuous at all $p \in X$.*

Translated definitions to "everywhere continuous."

- Analytic continuity is easy.
- Sequential continuity: $\forall$ convergent sequences $(x_n)_n$, $(f(x_n))_n$ is convergent and $\lim f(x_n) = f(\lim x_n)$.
- Topological continuity: f is continuous if for all open $V \subseteq Y$, $f^{-1}(V)$ is open in U.

For example, consider $f(x) = x^2$. Note that $f\left((-1,1)\right)) = [0,1)$. If f is continuous and U is open $f(U)$ may not be open.

Also, look at $g(x) = \frac{1}{x}$ on $(0, \infty)$. Consider $C = \mathbb{Z}_{>0}$, and then look at the set $g(C) = \left\{\frac{1}{n} : n \geq 1\right\}$ is not closed.

Continuous functions don't preserve openness and they don't preserve closedness, but they preserve compactness.

Theorem. *If $f : X \to Y$ is continuous, (for (X, Y) topological spaces) and $K \subseteq X$ is compact in X, then $f(K)$ is compact in Y.*

Proof. Need to show that every open cover of A has a finite subcover. Let H be an open cover of $f(K)$. Let $G = \{f^{-1}(V) : V \in H\}$. Now, G covers K. Since f is continuous, it is an open cover[2] □

[2] Recall that a topological space X is called compact if each of its open covers has a finite subcover. That is, X is compact if for every collection C of open subsets of X such that

$$X = \bigcup_{x \in C} x,$$

Corollary. (Extreme value theorem). If $f : X \to \mathbb{R}$ (X is a compact topological space, f is continuous). Then f achieves a maximum and minimum on X. Meaning, there exists $p, q \in X$ such that $\forall x \in X, f(p) \leq f(x) \leq f(q)$.

Proof. $f(X)$ is a compact $\subseteq \mathbb{R}$, so $f(X)$ is closed and bounded (Heine-Borel). Provided $X \neq \emptyset, f(X) \neq \emptyset$. Now,

- $m = \inf(f(X)) \in f(X)$ and $M = \sup(f(X)) \in f(X)$
- By definition, since $m, M \in f(X)$, $\exists p, q \in X : f(p) = m$ and $f(q) = M$.

. □

The topological proof is surprisingly fast. Indeed, you can prove this using the sequential definition of continuity using the Bolzano-Weierstrass; but it is tricky.

Some remarmks on Heine-Borel:[3].

Next week: we'll discuss the notion of "connectedness."

Take home exam: goes out after class on Wed, have until Friday to finish.

4 10-22: Connectedness

Exam: released on Wednesday after class. You'll have 3 hours + extra time to submit. Can start at any time until midnight - ϵ. Open textbooks (J&P, Rudin), + notes.
Definition. *Let X be a topological space. X is called disconneted if there are nonempty, disjoint open sets U and U' of X such that $X = U \cup U'$. If X is not disconnected, it's called connected.*

Definition. *If X is a topological space and $S \subseteq X$, then S is "connected" if S is connected as a topological space (with respect to the subspace topology).*

Definition (Alternative)**.** *A topological space is connected if the only clopen sets are X and $\emptyset$.*

We now ask: what are the connected subsets of $\mathbb{R}$?

Lemma. *$S \subseteq \mathbb{R}$ is connected iff S is an interval. $\forall x, y \in S, \forall z \in \mathbb{R}, x < z < y \implies z \in S$.*

---

there is a finite subset F of C such that

$$X = \bigcup_{x \in F} x.$$

[3] $f(X)$ is compact $\subseteq \mathbb{R}$, so $f(X)$ is closed and bounded (true in any metric space). The other direction requires being a subset of $\mathbb{R}^n$, which requires Heine-Borel

Proof. We start with the forward direction. Assume S is not an interval. Then there exists some $z \in \mathbb{R}$ so that $x < z < y$ but $z \notin S$. Let $U = (-\infty, z)$ and let $U' = (z, \infty)$. Then it is easy to check that $(S \cap U) \cup (S \cap U')$ is a disconnection of S.

Need to check:

- $S \cap U$ is open in S, because U and U' are open in $\mathbb{R}$.
- The sets $S \cap U$, $S \cap U'$ are disjointed, because U, U' are disjoint.
- $S = (S \cap U) \cup (S \cap U')$.

If (X, τ) is a topological space and $S \subseteq X$, the subspace topology on S is

$$\tau_S = \{S \cap U : U \in \tau\}.$$

Also, τ_S is the coarsest topology such that $S \to X$ inclusion is continuous.

Now, we move on to the reverse direction. Let S be an interval, so that

$$S = (S \cap U) \cup (S \cap U'),$$

where U and U' are open in $\mathbb{R}$, $S \cap U$ and $S \cap U'$ are nonempty. Want to show that they are not disjoint. Let $V = S \cap U, V' = S \cap U'$.

Let $x \in V$ and $y \in V'$. Without loss of generality, assume $x < y$. Also, $\frac{x+y}{2} \in S$, since S is an interval.

We will construct sequences $(x_n)_n$ and $(y_n)_n$ as follows.

- $x_0 = x$ and $y_0 = y$.
- Let $\alpha_{n+1} = \frac{x_n + y_n}{2}$. If $\alpha_{n+1} \in V$, then $x_{n+1} = \alpha_{n+1}$ and $y_{n+1} = y_n$. Otherwise $\alpha_{n+1} \in V'$ and $x_{n+1} = x_n$ and $y_{n+1} = \alpha_{n+1}$.
- $(x_n)_n$ is an increasing sequence in V, and $(y_n)_n$ is a decreasing sequence in V. They are also bounded, since they are termwise bounded by each other.
- $(x_n)_n$ and $(y_n)_n$ both convergence, and since

  $$|x_n - y_n| \leq 2^{-n}|x - y|,$$

  both sequences convergence to the same limit L; $x < L < y \implies L \in S$. Therefore, $L \in V$ or $L \in V'$.
- Suppose $L \in V$. Then $L \in U$. L is an interior to U (since U is open). In particular, $\exists \epsilon > 0$ such that $B_\epsilon(L) \subseteq U$. By the convergence of $(y_n)_n \to L$, $\exists N$ such that

  $$|y_n - L| < \epsilon; \forall n \geq N$$

  In particular, $y_N \in B_\epsilon(L) \subseteq U \implies y_n \in V$. So since $y_N \in V'$, $V \cap V' \neq \emptyset$.

□

Example. *A disconnected set in $\mathbb{R}$: $\mathbb{Q}$ is disconnected.*

Consider a dramatic example: the Cantor set.

- In homework, proved that every open ball is closed in an ultrametric space.
- The Cantor set is "totally disconnected[4]" Turns out that every ultrametric space is topologically equivalent to the Cantor set.

Last time: Let $f : X \to Y$ be a continuous function of topological spaces. If X is compact, then $f(X)$ is compact. This implies the Extreme Value Theorem.

This time: Let $f : X \to Y$ be a continuous function of topological spaces. If X is connected, then $f(X)$ is connected.

Proof. Suppose $f(X)$ is disconnected. then

$$f(X) = (f(X) \cap V) \cup (f(X) \cap V').$$

Let $W = f(X) \cap V$, $W' = f(X) \cap V'$. Let $U = f^{-1}(W)$ and $U' = f^{-1}(W')$.

- $U \cap U' = X$ (by definition of preimage).
- U and U' are open, because $f^{-1}(W) = f^{-1}(f(X) \cap V) = f^{-1}(V)$. Further, $f^{-1}(V)$ is open because f is continuous and V is open.
- They're nonempty because W, W' are nonempty $\subseteq f(X)$.
- They're disjoint because if $x \in U \cap U'$, then $f(x) \in W \cap W'$; but W and W' are disjoint.

□

Corollary. (Intermediate value theorem.) Let X be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous real-valed function. If there are $p, q \in X$ and $c \in \mathbb{R}$ such that $f(p) < c < f(q)$, $\exists \xi \in X$ such that $f(\xi) = c$.

Proof. $f(X)$ is an interval. □

Example (Incomplete topologist's sine curve)**.** *Consider the graph of* $\sin\left(\frac{1}{x}\right)$ *for* $x \in (0, 1)$. *Note that* $\sin\left(\frac{1}{x}\right)$ *is continuous on this interval. This is connected.*
Example (Midcomplete TSC)**.** *Consider* $\{$*Incomplete TSC*$\} \cup \{(0, 0)\}$. *This will be connected, still. But, it is not path connected. Consider a point* (x, y)*; there is no path between* (x, y) *and* $(0, 0)$.

Interestingly, this is a converse to the Intermediate Value Theorem. Has the intermediate value property, but it is not continuous at 0.

[4] a lot of open sets are closed

5 10-24: Uniform continuity

Exam will be ready at 12:30pm. Have 3 hours + 30 extra minutes to scan + upload.

Today: just one proof, and then Q&A time / review.
Theorem. *Let $f : X \to Y$ be a continuous function with X and Y metric spaces. If X is compact then f is uniformly continuous.*

1. Recall that $f : X \to Y$ is continuous if $\forall p \in X, [\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X$
$$d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon.$$
2. $\forall p \in X, \forall (x_n)_n \to p$ if $f(x_n)_n$ converges $\to f(p)$.
3. $\forall$ open $U \subseteq V f^{-1}(V)$ is open in X. "preimage of an open set is open."

Importantly, 1, 2, 3, work in metric spaces, and 3 works in topological space.

Continuity.

In general, when we talk about continuity, we are discussing conditions of the form $\forall p \in X, \forall \varepsilon > 0, \exists > 0$ such that $\forall x \in X[\dots]$.

We say that $f : X \to Y$ (metric spaces) is uniformly continuous if

$$\forall \varepsilon > 0, \exists > 0, \forall x, p \in X : d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \epsilon.$$

The salient difference here is that δ only depends on ε, and no longer depends on p.

Obvious implication. If f is uniformly continuous, it is continuous. The converse, however is false.

Example. Let $f(x) = \frac{1}{x}$ on $(0, +\infty)$. This is a continuous function (on the interval). It is not uniformly continuous.

Consider some point $(p, f(p))$. Suppose we have a range $(f(p) - \epsilon, f(p) + \epsilon)$. Need a delta such that whenever $x \in (p - \delta, p + \delta)$, $f(x) \in (f(p) - \epsilon, f(p) + \epsilon)$. As $p \to 0$, δ stops working (since it will contain the asymptote at 0).

Example. Let $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0, +\infty)$. This function is continuous, but not uniformly so. No matter how small we make δ, there is some point p close to 0 so that $f((p - \delta, p + \delta)) = [-1, 1]$.

Aside: the difference between Lipschitz continuity and uniform continuity. [5]

Proof of theorem. Suppose that $f : X \to Y$ is continuous but not uniformly continuous (where X is compact). Then $\exists \varepsilon > 0, \forall > 0, \exists x, p \in X$ such that $d_X(x, p) <$ and $d_Y(f(x), f(p)) \geq \epsilon$. We want to use the above to build two sequences $(x_n)_n$ and $(p_n)_n$ such that $\forall n \geq 1$,

[5] You can show that $\sqrt{x}$ is uniformly continuous, but not Lipschitz continuous.

$$d_X(x_n, p_n) < \frac{1}{n}; \qquad d_Y(f(x_n), f(p_n)) \geq \varepsilon$$

We have not used compactness yet. By sequential compactness: some subsequence of (x_n) converges. Some subsequence of $(x_n)_n$ converges to $L \in X$, so that $(x_{n_i})_i \to L$. Now, consider (p_{n_i}); we also have $(p_{n_i})_i \to L$.

Therefore

$$\lim_{i\to\infty} f(x_{n_i}) = f(\lim(x_{n_i})_i) = f(L) = f(\lim_{i\to\infty}(p_{n_i})_i) = \lim_{i\to\infty} f(p_{n_i}).$$

(for the first equality we have used continuity). But this is a contradiction, since our sequences are actually far apart.

Recall the theorem that states that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is integrable. The proof of this theorem relies on the notion of uniform continuity.

5.1 Review

We now discuss the sequential version of uniform continuity.

Theorem. *Let $f : X \to Y$ be continuous metric spaces. Then the following are equivalent:*

- *Let f is uniformly continuous.*
- *If $(x_n)_n$ is a Cauchy sequence, then $f(x_n)_n$ is also Cauchy.*

Theorem. *If X is a metric space and $K \subseteq X$ is compact and $(x_n)_n$ is a sequence in K, it has a convergent subsequence whose limit is in K.*

Dense sets. Suppose X is a metric space $S \subseteq X$, S is dense if

- $\overline{S} = X$
- Every nonempty open U overlaps with S.
- For all $x \in X$ and $\forall \epsilon > 0$, $\exists y \in S$ such that $d_X(x, y) < \epsilon$.

For example, $\mathbb{Q}$ is dense in $\mathbb{R}$ (that is, all real numbers can be arbitrarily well approximated by rational numbers).

If X is a metric space with a countable dense set, then we call X separable. This is good because this means that computers can deal with such sets quite well (e.g. floating point).

Theorem. In the space of continuous functions $[0, 1] \to \mathbb{R}$, the rational coefficient polynomials are dense.

6 11-05-18: Integrability and FTCs

Recall the Riemann-Darboux integral. Suppose $f : [a,b] \to \mathbb{R}$ is bounded with $a < b$. Then we can write

$$\int_{\underline{a}}^{b} = \sup\left\{\int_a^b \varphi : \varphi \text{ a step fn }; \varphi \leq f\right\}$$

or

$$\int_a^{\overline{b}} = \inf\left\{\int_a^b \psi : \psi \text{ a step fn and } \psi \geq f \text{ on } [a,b]\right\}.$$

And if they match, f is integrable and $\int_f = \int_{\underline{a}}^b f = \int_a^{\overline{b}} f$.

Proposition (Sequential criterion for integrability)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is bounded, $L \in \mathbb{R}$.*

Subtext: Wts

$$\int_a^b = L.$$

Then $\int_a^b f = L$ if and only if $\exists (\psi_n)_n, (_n)_n \in$ Step$([a,b])$ for all k, $\psi_k \leq f \leq \varphi_k$; and

$$\int_a^b \psi_n \to L; \qquad \int_a^b \varphi_n \to L.$$

Sufficient conditions for integrability.

- If $f : [a,b] \to \mathbb{R}$ is continuous, it is integrable.
- Piecewise continuous.
- Monotonic functions.
- "Piecewise monotone and/or continuous…"
- Thomae's function shows that the converse is not true.

Proposition (Cauchy criterion for integrability.)**.** *If $f : [a,b] \to \mathbb{R}$ is bounded, it is integrable iff $\forall \varepsilon > 0$, there exists $\psi, \varphi \in$ Step$([a,b])$ such that $\varphi \leq f\psi$ and $\int_a^b(\varphi - \psi) = \int_a^b \varphi - \int_a^b \psi < \varepsilon$.*

(Proof follows from the definition of integrability.)

Theorem. *If $f : [a,b] \to \mathbb{R}$ is continuous, then it is integrable.*

Proof. Strategy: for any interval I we want $\varphi(I) - \psi(I)$ to be small.

Let $\varepsilon > 0$. Pick $\delta > 0$ such that $\forall x, y \in [a,b]$, we have

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Pick a partition of $[a, b]$ into disjoint intervals $\{I_k\}_{k=1}^n$ with $|I_k| < \delta$. For each I_k, $f : \overline{I_k} \to \mathbb{R}$ achieves its min and max at $p_k, q_k \in \overline{I}_k$ respectively with

$$f(p_k) \leq f(x) \leq f(q_k)$$

for all $x \in I_k$.

Now, let

$$\varphi = \sum_{k=1}^{n} f(p_k)\mathbf{1}_{I_k}; \qquad \psi = \sum_{k=1}^{n} f(q_k)\mathbf{1}_{I_k}.$$

Fill in details from notes.

□

Theorem (Lebesgue's Riemann integrability condition). *A function f is integrable if and only if*

$$\lambda\left(\{p \in [a, b] : f \text{ is discontinuous at } p\}\right) = 0,$$

that is the set of discontinuities has Lebesgue measure 0 (alternatively, f is almost everywhere continuous).

[6] This theorem implies:

- f is continuous implies it is integrable.
- Monotone functions are continuous.

Let $f : [a, b] \to \mathbb{R}$ be bounded, we call $F : [a, b] \to \mathbb{R}$ a [continuous] antiderivative of f if it is continuous on $[a, b]$, differentiable on (a, b), and $\forall p \in (a, b) : F'(p) = f(p)$.

Theorem (The fundamental theorem of calculus). *Let $f : [a, b] \to \mathbb{R}$ be an integrable function $F : [a, b] \to \mathbb{R} : x \mapsto \int_a^x f$. Then:*

1. *F is Lipschitz continuous.*
2. *If f is continuous at $p \in (a, b)$, then F is differentiable at p and $F'(p) = f(p)$.*
3. *If f is continuous on all of $[a, b]$, then F is an antiderivative of f.*

Proof. The first statement is proven by the corresponding theorem for the upper / lower integrals. Statement (3) follows from statement (2).

Hence, it suffices to prove (2).

If f is continuous at p, then

$$\lim_{x \to p^+} \frac{F(x) - F(p)}{x - p} = f(p).$$

[6] Cannot use this on homework unless you prove it.

In these notes we will prove half of it (since the case $x \to p^-$ is similar.) Make the change of variables $x = p + h$. Then the limit above is equivalent to

$$\lim_{h\to 0^+} \frac{F(p+h) - F(p)}{h} = \lim_{h\to 0^+} \left[\frac{1}{h}\int_p^{p+h} f\right],$$

where

$$F(x) = \int_a^x f.$$

Let $\delta > 0$ such that $\forall x \in [a, b] : |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon$. If $|x - p| < \delta$ then

$$f(p) - \varepsilon < f(x) < f(p) + \varepsilon.$$

So, for $h < \delta$, we have

$$\begin{aligned}\int_p^{p+h} [f(p) - \varepsilon] &\leq \int_p^{p+h} f(x) \\ &\leq \int_p^{p+h} [f(p) + \varepsilon].\end{aligned}$$

This implies that for $h < \delta$,

$$h(f(p) - \varepsilon) \leq \int_p^{p+h} f \leq h(f(p) + \varepsilon);$$

that is

$$f(p) - \varepsilon \leq \frac{1}{h}\int_p^{p+h} f \leq f(p) + \varepsilon.$$

The punch line is that

$$\left|\lim_{h\to 0^+} \left[\frac{1}{h}\int_p^{p+h} f\right] - f(p)\right| < \varepsilon.$$

In particular, the limit above is equal to $f(p)$. This proves the FTC for the derivative of the integral; Wednesday we will do the integral of the derivative. $\square$

7 Sequences and series of functions

7.1 Pointwise vs. uniform convergence

8 Key ideas

Definition of a metric space.

A metric space is a set X together with a distance $d : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies

- $d(x, x) = 0$.
- $d(x, y) = d(y, x)$.
- $d(x, y) + d(y, z) = d(x, z)$.

These three results together imply $d(x, y) \geq 0$.

Theorem: Cauchy-Schwarz.

Let $a_1, \cdots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Then

$$\left[\sum_{k=1}^{n} a_k b_k\right]^2 \leq \left[\sum_{k=1}^{n} a_k^2\right] \left[\sum_{k=1}^{n} b_k^2\right].$$

To recall inequality, just recall that $||u||||v|| \cos\theta = u \cdot v$.

Convergence in a metric space.

A sequence $(x_n)_n$ is convergent to L if $\forall \varepsilon > 0, \exists N$ such that $n > N \implies |x_n - L| < \varepsilon$.

Cauchy-ness in a metric space.

$\forall \varepsilon > 0, \exists N$ s.t. $\forall m, n \geq N$, $d(x_n, x_m) < \varepsilon$.

Complete metric space, and an example.

A metric space is called complete if Cauchy $\implies$ convergent. $\mathbb{R}$ is a complete metric space.

Convergence in $\mathbb{R}^n$.

$\overline{a}^k \to \overline{a}$ iff $\overline{a}_j^k \to a_j$ for all j.

Similar proof for Cauchy in $\mathbb{R}^n$.

Topological space.

A topological space is a (X, τ) where X is a set and $\tau \subseteq P(x)$ satisfies

- $\emptyset, X \in \tau$
- If $\mathcal{F} \subseteq \tau$, then $\bigcup_{S \in \mathcal{F}} S \in \tau$ (an arbitrary union of sets in τ is in τ).
- If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$ (the intersection of any sets in τ are in τ).

Intuition `https://math.stackexchange.com/a/523794`. Broadly: a topology defines a notion of nearness on a set.

Interior / adherent sets.

Let (X, d) be a metric space, $x \in X, S \in X$. Then X is interior to S if $\exists \varepsilon > 0, B_\varepsilon \subseteq S$.

X is adherent to S if $\forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$.

Limit point / isolated point.

x is a limit point of S if $\forall \epsilon > 0$, $\exists y \neq x$ such that $y \in B_\varepsilon(x) \cap S$.

x is an isolated point of S if $\exists \varepsilon > 0$, s.t. $B_\varepsilon(x) \cap S = \{x\}$.

Open / closed sets.

S is open if every $x \in S$ is interior to S.

S is closed if it contains all its adherent (or limit) points.

Perfect / bounded / dense sets.

S is perfect if it closed and contains no isolated points.

S is bounded if $\exists p \in X$ and $M \geq 0$ so that $\forall x \in S, d(x, p) \leq M$.

S is dense if $\forall x \in X$, x is adherent to S.

Cover of a set.

A cover of a set is a set of subsets whose union equals the original set. If $C = \{U_\alpha; \alpha \in A\}$ is an indexed family of sets U_α then C is a cover of X if

$$X \subseteq \bigcup_{\alpha in A} U_\alpha$$

Compactness.

A subset $K \subseteq X$ of a topological space is called compact if $\forall \mathcal{G} \subseteq T$, with

$$K \subseteq \bigcup \mathcal{G},$$

$\exists$ a finite $\mathcal{G}'$ such that $K \subseteq \bigcup \mathcal{G}'$.

That is, each cover of K has a finite subcover.

Prove that compact sets are closed.

Bolzano-Weierstrass Theorem.

Heine-Borel Theorem.

Inverse powers.

If $\alpha > 0$ and $p \geq 1$ is an integer, there exists a unique $\beta > 0$ such tha $\beta^p = \alpha$.

Proof. Uniqueness is easy once existence is shown. Assume $0 < \alpha < 1$, (since $\alpha = 1$ is easy, and for $\alpha > 1$, just take $\left(\frac{1}{\beta}\right)^n = \frac{1}{\alpha}$. check this detail

Consider two sequences $(a_n)_n$ and $(s_n)_n$ with

$$a_n = \max\left\{k \in \mathbb{Z} : \left(\frac{k}{2^n}\right)^p \leq \alpha\right\}.$$

and $s_n = \frac{a_n}{2^n}$. We define these because $(s_n)_n$ consists of better binary approximations to $\alpha^{1/p}$. Need to check that a_n is well defined, but that is not too difficult.

Claim: $(s_n)_n$ is increasing and bounded above. It is bounded above by 1, since $s_n > 1$ would imply $s_n^p = (a_n/2^n)^p > 1 > \alpha$, which contradicts the definition of a_n. To see that is (s_n) is increasing is straightforward. Thus, $(s_n)_n$ converges to some limit β.

Finally, we will show that $(s_n^p)_n \to \alpha$. This will show that β satisfies $\beta^p = \alpha$. Note that

$$\left(\frac{a_n}{2^n}\right)^p \leq \alpha < \left(\frac{a_n + 1}{2^n}\right)^p$$

Thus, it suffices to check that the difference between the left and right hand sides approaches zero as $n \to \infty$. Note that

$$(a_n + 1)^p - a_n^p = \sum_{k=0}^{p-1} a_n^k,$$

so since $a_n \leq 2^n$, we have $a_n^k \leq 2^{nk} \leq 2^{n(p-1)}$, when $k = 0, \ldots, p-!$.

In particular,

$$\left(\frac{a_n + 1}{2^n}\right)^p - \left(\frac{a_n}{2^n}\right)^p \leq \frac{p \cdot 2^{n(p-1)}}{2^{np}} = \frac{p}{2^n}.$$

Thus, the left hand quantity $\to 0$ as $n \to \infty$. $\square$