MATH 116 - Complex Analysis

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1 9-24-18: Introduction

We can build up complex numbers with a few basic axioms.

1. $(1, 0)$ - unit.
2. $(0, 1)^2 = -(1, 0)$.

3. Bi-linear in z_1, z_2 (i.e. linear with respect to each argument).

Suppose $z = x + iy$. We define the *conjugation* operator as $\overline{z} = x - iy$, such that

$$z\overline{z} = x^2 + y^2 = |z|^2.$$

We can also express z in polar coordinates, so that

$$z = x + iy = r(\cos\phi + i\sin\phi).$$

We can extend the Taylor series of the exponential function on the real line to the complex plane by defining:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \ldots.$$

It is easy to check that this definition satisfies the usual properties:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}; \qquad e^{x+iy} = e^x e^{iy}.$$

We can similarly define

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} + \ldots.$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots.$$

We can combine these formulae to obtain $e^{iy} = \cos y + i\sin y$ (Euler).

Combining this with the previous definition, we can write

$$re^{i\phi} = r(\cos\phi + i\sin\phi).$$

Now, if $z = re^{i\phi}$, we can write $z^{-1} = \frac{1}{r}e^{-i\phi}$. This gives you a very natural geometric interpretation of inversion (conjugation + scaling).

Note that it is straightforward to derive trigonometric identities from Euler's formula; for example it is easy to see that

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi.$$

Linear functions. Suppose we have a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$. We can write this as

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

And furthermore, the following axioms must be satisfied:

- $F(z_1 + z_2) = F(z_1) + F(z_2)$.
- $F(\lambda z) = \lambda F(z)$.

One question: we could have either $\lambda \in \mathbb{R}$ (termed a real linear map) or $\lambda \in \mathbb{C}$ (termed a complex valued linear map).

If F is a complex linear map, we must have $F(iz) = iF(z)$ (i.e. the matrix has to commute). Furthermore, we must have $F(z) = F(z \cdot 1) = zF(1) = c$, where $c = a + ib$. So

$$F(z) = (a+ib)(x+iy) = (ax - by) + (ay + bx)i.$$

Also,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix}.$$

It follows that F is complex if and only if $a = d$ and $b = -c$.

If $z = x + iy$, we can write $x = \frac{1}{2}(z + \overline{z})$ and $y = -\frac{i}{2}(z - \overline{z})$. Now, set $A = a + ic$, $B = b + id$, so we can write.

$$\frac{1}{2}(A - iB)z\frac{1}{2}(A + iB)\overline{z} = \alpha z + \beta \overline{z}.$$

Importantly, αz is complex linear while $\beta\overline{z}$ is complex antilinear (which means $F(\lambda z) = \overline{\lambda} F(z)$.

This proves that any real linear map can be written as a sum of a complex linear map and a complex antilinear map.

2 Differential 1-forms

Here, $\mathbb{R}^2_z$ denotes the space $\mathbb{R}^2$ with the origin shifted to the point z. A differential 1-form is a function of arguments of 2 kinds: of a point $z \in U$ and a vector $h \in \mathbb{R}^2_z$. It depends linearly on h and arbitrarily (but usually continuously and even differentiably) on z.

We will need only 1-forms on domains in $\mathbb{R}^2$. A differential 1-form λ on a domain $U \subset \mathbb{R}^2$ is a field of linear functions $\lambda_z = \mathbb{R}^2_z \to \mathbb{R}$. Thus a 1-form is a function of arguments of 2 kinds: of a point $z \in U$ and a vector $h \in \mathbb{R}^2_z$.

Given a real valued function $f : U \to \mathbb{R}^2$ on U, its differential df is an example of a differential form: $d_z(f)(h) = \frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2$. In particular, differentials dx and dy of the coordinate functions x, y are differential 1-forms. Any other differential form can be written as a linear combination of dx and dy:

$$\lambda = Pdx + Qdy,$$

where $P, Q : U \to \mathbb{R}$ are functions on the domain U.

A differential 1-form λ is exact if $\lambda = df$. The function f is called the primitive of the 1-form λ. The necessary condition for exactness if that λ is closed which by definition means $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

3 Complex projective line, or Riemann sphere

Consider the space $\mathbb{C}^n$. Similar to the real case, one can *projectivise* $\mathbb{C}^n$. $\mathbb{C}P^n$ is defined as the space of all complex lines through the origin. For us, the one-dimensional complex projective space is most relevant, the complex projective line ($\mathbb{C}P^1$).

Any vector $z = (z_1, z_2) \in \mathbb{C}^2$ generates the 1-dimensional complex subspace (complex line) denoted as

$$l_z = (z) = \{\lambda z : \lambda \in \mathbb{C}\}.$$

This line l_z can be viewed as a point of $\mathbb{C}P^1$. Any proportional vector $\overline{z} = \mu z$ generates the same line. Fix an affine line $L_1 = \{z_2 = 1\} \subset \mathbb{C}^2$. Any line from $\mathbb{C}P^1$ except $\{z_2 = 0\}$

4 Riemann surfaces

A Riemann surface is a 1-dimensional complex manifold. A set S is called a Riemann surface if there exist subsets $U_\lambda \subset X, \lambda \in \Delta$, where Δ is a finite of countable set of indices, and for every $\lambda \in \Delta$ a map $\Phi_\lambda : U_\lambda \to$ such that

- $S = \bigcup_{\lambda \in \Delta}$
- The image $G_\lambda = \Phi_\lambda(U_\lambda)$ is an open set in $\mathbb{C}$.

- The map Φ_λ viewed as a map $U_\lambda \to G_\lambda$ is one to one.
- For any two sets $U_\lambda, U_\mu, \lambda, \mu \in \Delta$, the images $\Phi_\lambda(U_\lambda \cap U_\mu), \Psi_\mu(U_\lambda \cap U_\mu) \subset$ are open and the map

$$h_{\lambda,\mu} = \Phi_\mu \circ \Phi_\lambda^{-1} : \Phi_\lambda(U_\lambda \cap U_\mu) \to \Phi_\mu(U_\lambda \cap U_\mu) \subset^n$$

5 Key ideas

5.1 Basic facts

1. $e^z = \sum_{n=0}^\infty \frac{z^n}{n!}$.
2. $\sin z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$
3. $\cos z = \sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!}$.

5.2 Main results

Cauchy's integral formulas.

Suppose f is holomorphic on an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{z - \zeta}\, d\zeta.$$

Cauchy's integral formulas for derivatives.

Let f be holomorphic on an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Cauchy-Riemann. f is analytic iff $u_x = v_y$, $u_y = -v_x$.

C^1 **class.** Every holomorphic function is a domain U is of class C^1, i.e. its derivative continuously depends on the point of U.

Cauchy's theorem.

Liouville's theorem. If f is entire and bounded, then f is constant.

Singularities and poles. A point singularity (or isolated singularities) of f is a $z_0 \in \mathbb{C}$ such that f is defined in a neighborhood of z_0 but not at the point z_0 itself. A zero for the holomorphic function f is z_0 such that $f(z_0) = 0$. By analtic continuation, the zeros of a non-trivial holomorphic function are isolated. A function F defined in a deleted neighborhood of z_0 has a pole at z_0 if the function $\frac{1}{f}$, defined to be to be zero at z_0, is holomorphic in a full neighborhood of z_0.

Pole power series representation. If f has a pole of order n at z_0, then

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z),$$

where G is a holomorphic function in a neighborhood of z_0.

Residue at a pole. The residue of f at that pole is defined as the coefficient a_{-1}, so that $\operatorname{res}_{z_0} f = a_{-1}$. In particular, if f has a pole of order n at z_0, then

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z).$$

Residue formula, and corollary. Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then

$$\int_C f(z)\, dz = 2\pi i \operatorname{res}_{z_0} f.$$

Suppose f is holomorphic on an open set containing a circle C and its interior, except for poles at the points $z_1, \cdots, z_N$ inside C. Then

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k} f.$$

Conformal map. A bijective holomorphic function $f : U \to V$ is called a conformal map or biholomorphism.

Riemann mapping theorem. Suppose Ω is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that

$$F(z_0) = 0; \qquad F'(z_0) > 0.$$

Corollary (3.2) Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.

Mantel's theorem.

6 Midterm review sheet

6.1 Cauchy-Riemann equations

f is holomorphic iff $u_x = v_y; u_y = -v_x$.

Differential operators w.r.t. z and $\overline{z}$.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

6.2 Cauchy integral formula + applications

Suppose f is holomorphic on an open set that contains the closure of a disc D. If C is the boundary circle, then for any $z \in D$:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

n-th derivative. If f is holomorphic in an open se TΩ, then f has infinitely many complex derivatives in Ω. If $C \subset \Omega$ is a cricle whose interior is only contained in Ω, then for all z in the interior of C:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\, zeta.$$

Cauchy inequality + quick proof. If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then

$$|f^{(n)}(z_0)| \leq \frac{n!||f||_C}{R^n}.$$

Proof. By the Cauchy integral formula, we obtain

$$\begin{aligned}
|f^{(n)}(z_0)| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right| \\
&= \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}} Rie^{i\theta} d\theta\right| \\
&\leq \frac{n!}{2\pi}\frac{||f||_C}{R^n} 2\pi.
\end{aligned}$$

□

Liouville's theorem. If f is entire and bounded, then f is constant.

Proof. By Cauchy inequality, we obtain

$$|f'(z_0)| \leq \frac{B}{R},$$

where B is some bound for f. Taking $R \to \infty$, we obtain the desired result. ☐

Quick proof of FTA. Suppose P has no roots. Then $\frac{1}{P(z)}$ is bounded and entire. But then $\frac{1}{P(z)}$ is constant, which is a contradiction.

Schwarz reflection principle. Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in all of Ω such that $F = f$ on Ω^+.

Proof. For $z \in \Omega^-$, define $F(z)$ by

$$F(z) = \overline{f(\overline{z})},$$

look at power series expansions, and invoke the symmetry principle. ☐

6.3 Power series

Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0 :

$$f(Z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n = \frac{f^{(n)}(z_0)}{n!}$.

Analytic continuation. Suppose f and F are analytic in regions Ω, Ω' with $\Omega \subset \Omega'$. If the two functions agree on the smaller set Ω, then F is an analytic continuation of f into the region Ω', and is uniquely determined by f.

In particular, suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for all z in some non-empty open subset of Ω. Then $f(z) = g(z)$ throughout Ω.

6.4 Exponential function and logarithm

Complex logarithm. Write

$$\log z = \log r + i\theta;$$

principal branch when $|\theta| < \pi$. Constructively, we can write

$$\log_\Omega(z) = F(z) = \int_\gamma f(w)\, dw,$$

where γ is any curve connecting 1 to z. Standard path of integration is to take $1 \to r \in \mathbb{R}$ and then $r \to z$, so that

$$\begin{aligned}\log z &= \int_1^r \frac{dx}{x} + \int_\eta \frac{dw}{w} \\ &= \log r + \int_0^\theta \frac{ire^{it}}{re^{it}}\, dt \\ &= \log r + i\theta.\end{aligned}$$

Note that in general

$$\log(z_1 z_2) \neq \log z_1 + \log z_2.$$

Taylor expansion for $\log(1+x)$. For the principal branch, we can write

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots.$$

6.5 Meromorphic functions

Definition of a meromorphic function. A function f on an open set Ω is meromorphic if there exists a sequence of points $z_0, z_1, \dots$ that has no limit points in Ω and such that

- f is holomorphic in $\Omega \setminus \{z_0, z_1, \dots\}$
- f has poles at the points $\{z_0, z_1, \dots\}$.

Casorati-Weierstrass. Suppose f is holomorphic in the punctured disc $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity at z_0. Then the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.

6.6 Argument principle and Rouche's theorem

Argument principle. Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = Z - P,$$

where Z is the number of zeros inside C, and P is the number of poles inside C.

Rouche's theorem. Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| < |g(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside C.

Proof. Let $f_t(z) = f(z) + tg(z); t \in [0,1]$. Argue that

$$n_t = \frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)}\, dz$$

is constant; and in particular that $n_0 = n_1$. $\square$

Open mapping theorem. If f and holomorphic and nonconstant in a region Ω, then f is open.

Maximum modulus principle. If f is a nonconstant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.

Proof. Immediate from open mapping theorem. $\square$

6.7 Computation of integrals using residues

Residue limit identity. If f has a pole of order n at z_0, then

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z - z_0)^n f(z).$$

Residue theorem. Suppose that f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at the points z_i inside γ. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k} f.$$

Integrals to know.

- $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$.
- $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin \pi a}; 0 < a < 1$.
- $\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x}\, dx = \frac{1}{\cosh \pi \xi}$.

6.8 Harmonic functions and harmonic conjugates

Definition. A real of complex valued C^2-smooth function f on a domain $U \subset \mathbb{C}$ is harmonic if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Unique determination. let $f, g : U \to \mathbb{R}$ be two harmonic functions which extend continuously to the boundary ∂U. Suppose that $f = g$ on ∂U. Then $f = g$ on U.

Proof. Suppose for $a \in U$ we have $f(a) > g(a)$; then consider $f - g$ and apply maximum modulus principle; contradiction. □

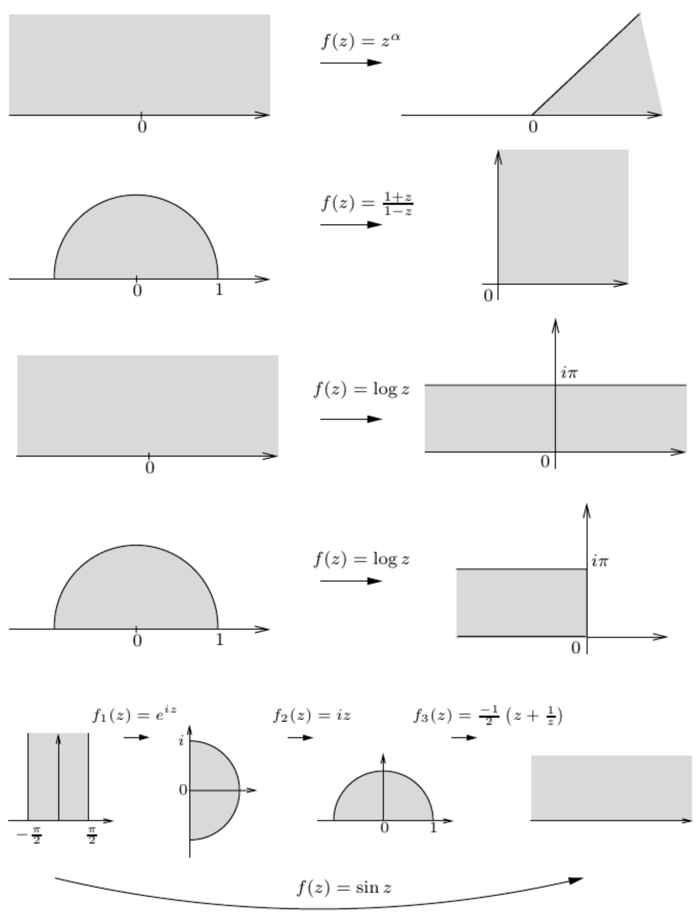
Log-composition. If f is a holomorphic function then $h(z) = \ln |f(z)|$ is harmonic.

Harmonic conjugate. The harmonic conjugate to a function $u(x, y)$ is a function $v(x, y)$ such that $u + iv$ is analytic.

Example. The harmonic conjugate of $u(x, y) = e^x \sin y$ is $-e^x \cos y + C$.

6.9 Elementary conformal mappings

Examples to know:



6.10 Properties of fractional linear transformations

Fractional linear transformations are mappings of the form

$$z \mapsto \frac{az + b}{cz + d}.$$

They always map circles and lines to circles and lines.