

MATH 113 - Matrix Theory

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Contents

1	Lecture 1	1
1.1	Course logistics	1
1.2	Introduction	2
1.3	Fields	2
2	Lecture 2	4
2.1	What it means to read proofs	4
2.2	Vector spaces	4
3	Lecture 4	7
4	Notes on 3.F: Duality	9
5	Key Ideas	9

1 Lecture 1

1.1 Course logistics

- Website: web.stanford.edu/~mkemeny/math113.html
- Grade breakdown
 - 30% homework
 - 30% midterm
 - 40% final
- Office hours: Tues 2pm (382-E).
- Text: Axler, *Linear Algebra Done Right*.

1.2 Introduction

Objective. Solving linear equations in an abstract way.

Linear algebra is a useful fundamental framework to have. In some sense linear algebra is really “all we can do.” And in higher math classes, we can often reduce problems to linear algebra.

Example. In differential geometry, you will take a linear approximation of the object using a tangent plane. And you can approximate your function using a linear mapping between vector spaces. Calculus is a special case of this kind of setting.

Think of this course as somewhere between philosophy and an applied engineering math class. It is key to understand the proofs and think about theory deeply. A good mental exercise: take a theorem that you think you understand - and try to reproduce the steps of the proof. Also - look for patterns; notice when different theorems use similar arguments.

1.3 Fields

A field is a set \mathbb{F} , on which it makes sense to add elements, subtract elements, multiply, and divide. Furthermore, these operations should satisfy the usual rules of arithmetic.

In some sense, there are two operations which are more primary (addition / multiplication), and subtraction / division can be viewed as inverse operations.

Definition. A binary operation f is a function $f : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$. That is, for any two elements $x, y \in \mathbb{F}$, we are given a rule to associate a third element $z = f(x, y)$ (note that order matters).

Examples of things we want to be fields or not

- \mathbb{Q} , the rational numbers. Clearly we have binary operations $+$ and \times . Further, \mathbb{Q} has an additive and multiplicative identity (0 and 1 respectively). We also have an inverse operation to addition. Note: we almost have an inverse operation for multiplication (0 cannot be inverted).
- $\mathbb{N} \cup \{0\}$. This is not a field, because we don't have additive inverses.
- Even if we add additive inverses, \mathbb{Z} is not a field (we don't have multiplicative inverses for nonzero elements).
- Consider $\mathbb{R}^2 := \{(x, y) | x, y \in \mathbb{R}\}$. We can clearly define addition $((x, y), (x', y')) \rightarrow (x + x', y + y')$. There is an additive identity $(0, 0)$, and an additive inverse $(-x, -y)$. We can define multiplication the same way: componentwise. And there is a multiplicative identity $(1, 1)$. But we have a problem: we cannot invert points $(x, 0)$ or $(0, y)$ where $x, y \neq 0$.

We will now formally define fields.

Definition. A field \mathbb{F} is a set together with two binary operations $+$: $\mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ and \cdot : $\mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$, satisfying the following axioms.

- *There exists additive and multiplicative identities. That is, $\exists 0, 1 \in \mathbb{F}$ with $0 \neq 1$ such that*

$$x + 0 = x \forall x \in \mathbb{F},$$

$$x \cdot 1 = x \forall x \in \mathbb{F}.$$

- *Existence of inverses. For all $x \in \mathbb{F}$, there exists $-x$ such that*

$$x + (-x) = 0.$$

And for all $x \neq 0 \in \mathbb{F}$, there exists x^{-1} such that

$$x \cdot x^{-1} = 1.$$

.

- *Commutativity of the operations. We have*

$$x + y = y + x \forall x, y \in \mathbb{F}$$

$$x \cdot y = y \cdot x \forall x, y \in \mathbb{F}$$

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- *Associativity of the operations (this one is somewhat non-obvious). We have*

$$x + (y + z) = (x + y) + z \forall x, y, z \in \mathbb{F}$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \forall x, y, z \in \mathbb{F}$$

- *Distributivity of the operations (this can be viewed as a sort of compatibility between addition and multiplication). In particular,*

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Example. Let's go back to \mathbb{R}^2 . With a different definition of multiplication on \mathbb{R}^2 , we can make it a field. We will leave addition as defined earlier (componentwise addition).

Define $(a, b) \cdot (c, d) := (ac - bd, bc + ad)$. Then $(1, 0)$ is a multiplicative identity, since

$$(a, b) \cdot (1, 0) = (a, b).$$

Using this rule, it's straightforward to see that we have multiplicative inverses. In particular, observe that

$$(a, b)^{-1} = \frac{1}{a^2 + b^2}(a, -b),$$

for $(a, b) \neq (0, 0)$.

Claim. There exists an element $x \in \mathbb{R}^2$ with this operation with the property

$$x \cdot x = -1.$$

Note: $(0, 1)$ satisfies this, since

$$\begin{aligned} (0, 1) \cdot (0, 1) &= (0^2 - 1^2, 0 + 0) \\ &= -(1, 0) = -1. \end{aligned}$$

Defining $i = (0, 1)$, we can get the complex numbers, where we write $x + iy$ for (x, y) .

Next time: we will define vector spaces. In words, a vector space V over a field \mathbb{K} will be a set for which we have two operations, vector addition (rule for adding elements of V) and scalar multiplication. (rule for multiplying elements of V by elements in K). Elements in V are called vectors, elements in \mathbb{K} are called scalars.

2 Lecture 2

2.1 What it means to read proofs

Any proof consists of a sequence of statements. Need to read proofs sequentially - it is critical to understand each statement before you move to the next statement. If you get stuck, think deeply about the statements.

2.2 Vector spaces

Last time, we defined fields as sets equipped with two operations: addition and multiplication. The main fields we saw were \mathbb{R} and \mathbb{C} . We defined the field of complex numbers as giving \mathbb{R}^2 a special multiplication operation. In the homework you will encounter a field \mathbb{F}_p with p elements, where p is a prime.

A vector space V over a field \mathbb{K} is a set V equipped with two operations, “vector addition” and “scalar multiplication.” We call elements $v \in V$ vectors and elements $\lambda \in \mathbb{K}$ scalars. Vector addition will be a rule for adding together two vectors. Scalar multiplication will be a rule for multiplying a scalar by a vector.

Motivating example. Consider the plane in \mathbb{R}^2 . A vector in \mathbb{R}^2 will be a vector space over \mathbb{R} .

To add vectors, consider $\vec{v} = (a, b)$ and $\vec{w} = (a', b')$. Then $\vec{v} + \vec{w} = (a + a', b + b')$.

For scalar multiplication, if $\lambda \in \mathbb{R}$ and $\vec{v} = (a, b)$, we have $\lambda\vec{v} = (\lambda a, \lambda b)$.

Geometrically, in vector addition, we move the head of \vec{w} to the tail of \vec{v} to obtain the sum.

If $\lambda > 0$, then geometrically, multiplying by λ has the effect of scaling \vec{v} by a factor of λ . If $\lambda < 0$, then $\lambda\vec{v} = -|\lambda|\vec{v}$, where we scale by λ , and the minus sign changes the direction. Note, we will soon consider objects that cannot be geometrically analyzed, so one shouldn't be too attached to geometric intuition.

Importantly, this extends to any field \mathbb{K} . We can define

$$\mathbb{K}^2 = \{(a, b) | a, b \in \mathbb{K}\},$$

with the same addition and multiplication operations:

$$\begin{aligned}(a, b) + (a', b') &= (a + a', b + b') \\ \lambda(a, b) &= (\lambda a, \lambda b).\end{aligned}$$

Note that the above equations, the $+$ on the left side is a different operation than the $+$ on the right side. The left $+$ is *vector addition*, while the right $+$ is addition over the field \mathbb{K} . More concretely, we can define $+_v$ as vector addition and \cdot_v as scalar multiplication. We would obtain

$$\begin{aligned}+_v : V \times V &\rightarrow V \\ \cdot_v : \mathbb{K} \times V &\rightarrow V\end{aligned}$$

Aside. The difference between \rightarrow and \mapsto : we can write $f(x) = \cos(x)$ as

$$\begin{aligned}f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \cos x\end{aligned}$$

This allows us to differentiate the function signature from the formula.

We now proceed to the formal definition.

Definition. Let \mathbb{K} be a field. A vector space V over \mathbb{K} is a set V together with two operations.

Vector addition.

$$\begin{aligned}V \times V &\rightarrow V \\ (\vec{v}, \vec{w}) &\mapsto \vec{v} + \vec{w}\end{aligned}$$

Scalar multiplication.

$$\begin{aligned}\mathbb{K} \times V &\rightarrow V \\ (\lambda, \vec{v}) &\mapsto \lambda\vec{v}.\end{aligned}$$

Satisfying the following axioms:

- *Commutativity of vector addition.* For all $\vec{v}, \vec{w} \in V$, we must have $\vec{v} + \vec{w} = \vec{w} + \vec{v}$. (Note: formally, we have not defined operations of $V \times \mathbb{K}$, so we don't have commutativity of scalar multiplication.)
- *Associativity of vector addition / scalar multiplication.* For all $\vec{u}, \vec{v}, \vec{w} \in V$, we have

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

In addition, if we have $\lambda, \mu \in \mathbb{K}$, we have

$$(\lambda\mu)\vec{v} = \lambda(\mu\vec{v})$$

Note that these are different multiplication operators (multiplication in the field, and scalar multiplication in the vector space).

- *The usual axioms for addition / multiplication.*

- *Existence of an additive identity.* There exists

$$0 \in V; \text{ such that } \vec{u} + 0 = \vec{u}; \forall \vec{u} \in V$$

- *Existence of an additive inverse.*

$$\forall \vec{v} \in V, \exists -\vec{v} \in V; \text{ such that } \vec{v} + (-\vec{v}) = 0.$$

- *Existence of a multiplicative identity.* There exists some element $1 \in \mathbb{K}$ such that

$$1\vec{v} = \vec{v}; \forall \vec{v} \in V.$$

- *Distributivity.* We have

$$\lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$$

$$(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$$

for all $\lambda, \mu \in \mathbb{K}, \vec{u}, \vec{v} \in V$.

The bad news: we have six axioms to learn. But the good news: we mostly care about 1 (maybe 2) examples.

Main example. (Finite dimensional vector space) Let $V = \mathbb{K}^n$, the set of ordered n -tuples of elements $(x_1, \dots, x_n) | x_i \in \mathbb{K}, 1 \leq i \leq n$. (If $n \geq 4$, we do not try to visualize this).

Then \mathbb{K}^n is a vector space over \mathbb{K} with addition defined as follows.

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

Similarly, we can define scalar multiplication as follows:

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n).$$

Exercise. Check that the axioms hold.

More general example. (Infinite dimensional vector space) Let \mathbb{K} be a field, and let S be any set. Let V be the set of functions $f : S \rightarrow \mathbb{K}$. We can define addition and scalar multiplication.

- (Addition) Let $f_1, f_2 \in V$, i.e. $f_1 : S \rightarrow \mathbb{K}$, $f_2 : S \rightarrow \mathbb{K}$. We can define $(f_1 + f_2)(s) := f_1(s) + f_2(s)$.

$$\begin{aligned} f_1 + f_2 &: S \rightarrow \mathbb{K} \\ s &\mapsto f_1(s) + f_2(s). \end{aligned}$$

- (Scalar multiplication) For any $\lambda \in \mathbb{K}$, we can define $\lambda f_1 : S \rightarrow \mathbb{K}$ as

$$\lambda f_1(s) = \underbrace{\lambda f_1}_{\text{multiplication in } \mathbb{K}}(s).$$

Remark. Importantly, note that the first example is a special case of the second example. Take $S = \{1, \dots, n\}$. Then you can think of an n -tuple as a function

$$f : \{1, \dots, n\} \rightarrow \mathbb{K}$$

where we can think of an n -tuple as a function:

$$\begin{aligned} (x_1, \dots, x_n) &\in \mathbb{K}^n \\ f(x_i) &\mapsto x_i \end{aligned}$$

One advantage of this formulation is that we can think of an infinite sequence (x_1, x_2, \dots) as a function $f : \mathbb{N} \rightarrow \mathbb{K}$. So the set of sequences $\{(x_1, \dots) | x_i \in \mathbb{K}\}$ forms a vector space. That is, $\{f : \mathbb{N} \rightarrow \mathbb{K}\}$ is also a vector space.

Concretely, consider $k = \mathbb{R}$, and $n = 2$. If we have the vector $(3, 5)$, we can consider this as a function f with $f(1) = 3$ and $f(2) = 5$.

3 Lecture 4

Recall that $U \subset V$ is a vector subspace if

- $0 \in U$
- Closed under addition
- Closed under scalar multiplication

Example. Let $V = k^2$, that is

$$V = \{(a, b) | a, b \in k\}$$

Choose any $v = (a, b)$. Then line

$$L_v = (v) = \{\text{all scalar multiples of } v\}$$

is a subspace.

It is easy to show that the additive identity exists, and that it satisfies closure under addition and scalar multiplication.

Definition. Let V, W be k vector spaces. Let $T : U \rightarrow W$ be a linear map. This means that

- $T(a + b) = T(a) + T(b)$ “linearity”
- $T(\lambda a) = \lambda T(a)$ “homogeneity”

Then we define the kernel of T (or nullspace) as the subset of V such that

$$\text{Ker } T = \{v \in V | T(v) = 0\}.$$

Lemma. Let $T : V \rightarrow W$ be a linear map. Then we have $T(0) = 0$.

Proof. Use the fact that

$$0 = 0 + 0$$

So

$$T(0) = T(0) + T(0)$$

Subtracting, we get $T(0) = 0$. □

Note that if T is a linear map, then T is injective, i.e. if $v, w \in V$ such that $T(v) = T(w)$ then $v = w$, if and only if $\text{Ker } T = \{0\}$.

So intuitively, the kernel is a kind of “measure” of how far T is from being injective.

Proposition. $\text{Ker } T \subset V$ is a subspace.

Proof. We will prove each axiom sequentially.

- Note that $0 \in \text{Ker } T$ since $T(0) = 0$ from the lemma.
- Now, need to check closure under addition. By linearity, note that

$$T(v + w) = T(v) + T(w) = 0 + 0 = 0.$$

- Now, need to check closure under scalar multiplication. By homogeneity, note that

$$T(\lambda v) = \lambda T(v) = 0.$$

□

Example. Consider the set

$$U : \{ f : \mathbb{R} \rightarrow \mathbb{R} \mid f^{(n)} : \mathbb{R} \rightarrow \mathbb{R} \text{ exists and is continuous} \}$$

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4 Notes on 3.F: Duality

Definition. A linear functional on V is a linear map from V to \mathbf{F} , i.e. an element of $\mathcal{L}(V, \mathbf{F})$.

Definition. The dual space of V , denoted V' is the vector space of all linear functions on V . In other words, $V' = \mathcal{L}(V, \mathbf{F})$.

Note that $\dim V' = \dim V$. This follows from 3.61, which states that $\dim \mathcal{L}(V, W) = \dim(V) \dim(W)$.

Definition. If v_1, \dots, v_n is a basis of V , then the dual basis of v_1, \dots, v_n is the list ϕ_1, \dots, ϕ_n of elements of V' where each ϕ_j is the linear functional on V such that

$$\phi_j(v_k) = \begin{cases} 1; & \text{if } k = j \\ 0; & \text{if } k \neq j. \end{cases}$$

5 Key Ideas