Rewriting techniques, I: basics, interpretation, termination

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Exercise 1 :

(a) Given the reduction rules

$$((s\ x) + y) \rhd (s\ (x + y)); \quad (0 + x) \rhd x$$

Can $(s\ ((s\ 0) + 0))$ be *reduced*? Can it be *rewritten*? Provide the substitution, the context and the term t being reduced.

(b) A *string rewrite system* (SRS for short) is a TRS over a signature that contains only unary function symbols. Given the (string) reductions

$$a(b(x)) \rhd b(a(x))$$

Can $a(a(b(x)))$ be reduced? Can $a(b(a(x)))$ be reduced? Can they be rewritten?

(c) Build a reduced string rewrite system that is not terminating.

Exercise 2 :
Given the following term rewriting system (TRS):

$$\begin{array}{ccc} x \times \mathsf{0} \to \mathsf{0} & \qquad & x + \mathsf{0} \to x \\ \mathsf{0} \times x \to \mathsf{0} & & \mathsf{0} + x \to x \\ \mathsf{s}(x) \times y \to (x \times y) + y & & x + \mathsf{s}(y) \to \mathsf{s}(x + y) \\ x \times \mathsf{s}(y) \to (x \times y) + x & & \mathsf{s}(x) + y \to \mathsf{s}(x + y) \end{array}$$

Show the *reduction graph* of $((\mathsf{0} \times \mathsf{0}) + \mathsf{0}) + \mathsf{s}(\mathsf{0})$.

Exercise 3 :
Given the signature $(\{\mathbb{N},\ \mathsf{List}\}, \{\mathsf{0},\ \mathsf{s},\ \epsilon,\ :,\ \mathbin{/\!\!/\!\!\backslash\!\!\backslash},\ \mathsf{sort}\})$ where the set of functions is typed as follows:

$$\mathsf{0} : \mathbb{N}, \qquad \mathsf{s} : \mathbb{N} \to \mathbb{N}, \qquad \epsilon : \mathsf{List}, \qquad (:) : \mathbb{N} \times \mathsf{List} \to \mathsf{List},$$
$$\mathbin{/\!\!/\!\!\backslash\!\!\backslash} : \mathsf{List} \times \mathsf{List} \to \mathsf{List}, \qquad \mathsf{sort} : \mathsf{List} \to \mathsf{List}$$

Define a finite TRS that simulates the *mergesort algorithm*. If needed, you can define auxiliary sorts and function symbols.

A **polynomial interpretation on integers** is the following:

- a subset A of $\mathbb{N}$;
- for every symbol f of arity n, a polynomial $\mathsf{P}_f \in \mathbb{N}[X_1, \ldots, X_n]$;
- for every $a_1, \ldots, a_n \in A$, $\mathsf{P}_f(a_1, \ldots, a_n) \in A$;
- for every $a_1, \ldots, a_i > a_i', \ldots, a_n \in A$, $\mathsf{P}_f(a_1, \ldots, a_i, \ldots, a_n) > \mathsf{P}_f(a_1, \ldots, a_i', \ldots, a_n)$;

Then $(A, (\mathsf{P}_f)_f, >)$ is a well-founded monotone algebra.

Exercise 4 :
Consider the TRS R consisting of the rewrite rules

$$\begin{array}{ccc} \mathsf{0} + y \to y & \mathsf{0} \dot{-} y \to \mathsf{0} & \mathsf{min}(x, y) \to x \dot{-} (x \dot{-} y) \\ \mathsf{s}(x) + y \to \mathsf{s}(x + y) & \mathsf{s}(x) \dot{-} \mathsf{0} \to \mathsf{s}(x) & \mathsf{max}(x, y) \to (x + y) \dot{-} \mathsf{min}(x, y) \\ & \mathsf{s}(x) \dot{-} \mathsf{s}(y) \to x \dot{-} y & \end{array}$$

(a) Show that the following interpretation over $\mathbb{N}$ is indeed a interpretation and that is compatible with R (that is, for each rule $\ell \to r \in R$, we have, for any assignment α, $[\alpha](\ell) >_{\mathbb{N}} [\alpha](r)$)

$$\begin{array}{ccc} \mathsf{0}_{\mathbb{N}} = 0 & +_{\mathbb{N}}(x, y) = 2x + y + 1 & \mathsf{min}(x, y) = 2x + y + 3 \\ \mathsf{s}_{\mathbb{N}}(x) = x + 1 & \dot{-}_{\mathbb{N}}(x, y) = x + y + 1 & \mathsf{max}_{\mathbb{N}}(x, y) = 4x + 2y + 6 \end{array}$$

(b) Find natural numbers a, b, c, d, e and f such that the following interpretation over $\mathbb{N}$ is compatible with R,

$$\begin{array}{lll} \mathsf{0}_{\mathbb{N}} = 0 & +_{\mathbb{N}}(x,y) = 2x + y + 1 & \mathsf{min}_{\mathbb{N}}(x,y) = 3x + by + c \\ \mathsf{s}_{\mathbb{N}}(x) = x + 1 & \dot{-}_{\mathbb{N}}(x,y) = x + 2y + a & \mathsf{max}_{\mathbb{N}}(x,y) = dx + ey + f \end{array}$$

Exercise 5 :
Prove the termination of the following TRS

$$\begin{array}{ll} \mathsf{0} \times x \to \mathsf{0} & x + \mathsf{0} \to x \\ \mathsf{s}(x) \times y \to (x \times y) + y & x + \mathsf{s}(y) \to \mathsf{s}(x + y) \end{array}$$

using the polynomial interpretation on natural numbers:

$$\mathsf{P}_{0} = 2 \qquad \mathsf{P}_{\mathsf{s}}(X) = X + 1 \qquad \mathsf{P}_{+}(X, Y) = X + 2Y \qquad \mathsf{P}_{\times}(X, Y) = (X + Y)^2$$

Is this polynomial interpretation suitable to prove termination of the TRS of Exercise 2?

Exercise 6 :
Let R be a rewrite system on a signature $\mathcal{F}$, and I a model of R, that is, an $\mathcal{F}$-algebra $(A, (f_I)_{f\in\mathcal{F}})$ such that $R \subseteq =_I$ where $t =_I u$ iff for all $\xi : \mathcal{V} \to A$, $t\xi = u\xi$.

Let $\mathcal{F}^I$ be the signature such that $\mathsf{f}_{a_1,\dots,a_n} \in \mathcal{F}^I_n$ iff $\mathsf{f} \in \mathcal{F}_n$ and let $\mathsf{lab}(R) = \{\mathsf{lab}(\ell, \xi) \to \mathsf{lab}(r, \xi) | \ell \to r \in R, \xi : \mathcal{V} \to A\}$, where $\mathsf{lab}(x, \xi) = x$ and $\mathsf{lab}(\mathsf{f}\ t_1\ \dots\ t_n, \xi) = \mathsf{f}_{t_1\xi,\dots,t_n\xi}\ \mathsf{lab}(t_1, \xi)\ \dots\ \mathsf{lab}(t_n, \xi)$, where for any term t, $t\xi \in A$ is the substitution generalised to terms that is, the rewrite system obtained by labeling function symbols by the semantics of their arguments.

1. Prove that $\to_R$ terminates iff $\to_{\mathsf{lab}(R)}$ terminates.
2. Prove that a polynomial interpretation cannot prove the termination of the following system

$$\begin{array}{ll} \mathsf{f}\ (\mathsf{s}\ X) \to \mathsf{f}\ (\mathsf{p}\ (\mathsf{s}\ X)) \diamond (\mathsf{s}\ X) & \mathsf{p}\ (\mathsf{s}\ \mathsf{z}) \to \mathsf{z} \\ \mathsf{p}\ (\mathsf{s}\ (\mathsf{s}\ X)) \to \mathsf{s}\ (\mathsf{p}\ (\mathsf{s}\ X)) & \end{array}$$

3. Prove that this rewrite system can be proved terminating using 1.

A **polynomial interpretation on real numbers** is the following:

- a subset A of $\mathbb{R}^+$;
- a positive real number δ;
- for every symbol f of arity n, a polynomial $\mathsf{P}_f \in \mathbb{R}[X_1, \dots, X_n]$;
- for every $a_1, \dots, a_n \in A$, $\mathsf{P}_f(a_1, \dots, a_n) \in A$;
- for every $a_1, \dots, a_i >_\delta a'_i, \dots, a_n \in A$, $\mathsf{P}_f(a_1, \dots, a_i, \dots, a_n) >_\delta \mathsf{P}_f(a_1, \dots, a'_i, \dots, a_n)$ where $x >_\delta y$ iff $x > y + \delta$.

Then $(A, (\mathsf{P}_f)_f, >_\delta)$ is a well-founded monotone algebra.

Exercise 7 :
Consider the following two TRS:

$$\begin{aligned} R_1 =\{ & \mathsf{l}(\mathsf{p}(x)) \to \mathsf{p}(\mathsf{p}(\mathsf{l}(x))),\ \mathsf{p}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{p}(x))),\ \mathsf{p}(x) \to \mathsf{a}(x, x), \\ & \mathsf{s}(x) \to \mathsf{a}(x, \mathsf{0}),\ \mathsf{s}(x) \to \mathsf{a}(\mathsf{0}, x)\ \} \\ R_2 =\{ & \mathsf{r}(\mathsf{r}(\mathsf{r}(x))) \to \mathsf{a}(\mathsf{r}(x), \mathsf{r}(x)),\ \mathsf{s}(\mathsf{a}(\mathsf{r}(x), \mathsf{r}(x))) \to \mathsf{r}(\mathsf{r}(\mathsf{r}(x)))\ \} \end{aligned}$$

1. Prove that $R_1 \cup R_2$ terminates using the following polynomial interpretation on real numbers: $\delta = 1$, $\mathsf{P}_0(X) = 0$, $\mathsf{P}_{\mathsf{l}}(X) = X^2$, $\mathsf{P}_{\mathsf{s}}(X) = X + 4$, $\mathsf{P}_{\mathsf{p}}(X) = 3X + 5$, $\mathsf{P}_{\mathsf{a}}(X, Y) = X + Y$ and $\mathsf{P}_{\mathsf{r}}(X) = \sqrt{2}X + 1$.
2. Prove that in any polynomial interpretation on natural numbers proving the termination of R_1 it must hold that $\mathsf{P}_{\mathsf{s}}(X)$ is of the form $X + s_0$ and $\mathsf{P}_{\mathsf{a}}(X, Y)$ is of the form $X + Y + a_0$, with $s_0 > a_0$.
hint: look at the dominant terms of the polynomials computed from the rewrite rules.
3. Deduce that the termination of $R_1 \cup R_2$ cannot be proved using a polynomial interpretation of integers.