Machine Learning: Yoshinari Fujinuma

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LECTURE 8

Slides adapted from Chenhao Tan, Jordan Boyd-Graber, Chris Ketelsen

Logistics

- HW1 deadline is today
- HW2 will be available on Github today

Learning objectives

- Introduce logistic regression
- Introduce Naïve Bayes
- (Bonus) Understand generative models vs. discriminative models

Outline

Probabilistic classification

Logistic regression

Naïve Bayes

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Recap

Perceptron

- Learn weights $\boldsymbol{w}$ and b via the perceptron algorithm
- Predict $\hat{y}$ via $\hat{y} = \text{sign}(\boldsymbol{w} \cdot \boldsymbol{x} + b)$
- $\hat{y} = \{-1, +1\}$

Recap

Do we want a prediction

- “human” or
- “It’s human, but human for 70%, and zombie for 30%”

Outline

Probabilistic classification

Logistic regression

Naïve Bayes

What are we talking about?

- Probabilistic classification: $P(Y|X)$
- Classification uses: ad placement, spam detection
- Building block of other machine learning methods

Logistic Regression: Definition

- Weight vector β_i
- Feature x_i
- "Bias" β_0

$$P(Y = 0|X) = \frac{1}{1 + \exp\left[\beta_0 + \sum_i \beta_i x_i\right]} \tag{1}$$

$$P(Y = 1|X) = 1 - P(Y = 0|X) \tag{2}$$

Logistic Regression: Definition

- Weight β_i
- Feature x_i
- For shorthand, we'll say that

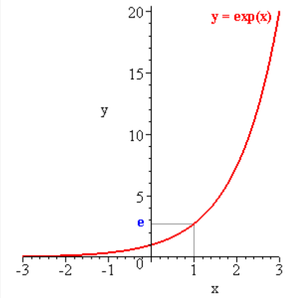
$$P(Y = 1|X) = \sigma(\beta_0 + \sum_i \beta_i x_i) \tag{3}$$

$$P(Y = 0|X) = 1 - \sigma(\beta_0 + \sum_i \beta_i x_i) \tag{4}$$

- Where $\sigma(z) = \frac{1}{1+exp[-z]}$

What's this "exp" doing?

Exponential function



- $\exp[x]$ is shorthand for e^x
- e is a special number, about 2.71828
  - It's the function whose derivative is itself

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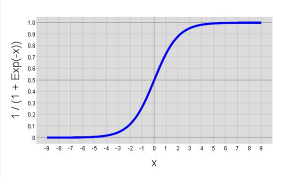
Logistic function



- $\exp[x]$ is shorthand for e^x
- e is a special number, about 2.71828
  - It's the function whose derivative is itself
- The "logistic" function is $\sigma(z) = \frac{1}{1+e^{-z}}$
- Always between 0 and 1.
  - Allows us to model probabilities
- it's "smooth"

Logistic Regression Example: Spam Classification

feature	coefficient	weight
bias	β_0	0.1
"viagra"	β_1	2.0
"mother"	β_2	-1.0
"work"	β_3	-0.5
"nigeria"	β_4	3.0

- What does $Y = 1$ mean?

Example 1: Empty Document?

$X = \{\}$

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- $Y = 1$: spam

Example 1: Empty Document?

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- $P(Y=0) = \frac{1}{1+\exp[0.1]} =$
- $P(Y=1) = \frac{\exp[0.1]}{1+\exp[0.1]} =$

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Example 1: Empty Document?

$X = \{\}$

- $P(Y = 0) = \frac{1}{1+\exp[0.1]} = 0.48$
- $P(Y = 1) = \frac{\exp[0.1]}{1+\exp[0.1]} = 0.52$
- Bias β_0 encodes the prior probability of a class

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Example 2

$X = \{\text{Mother}, \text{Nigeria}\}$

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- $P(Y=0) = \frac{1}{1+\exp[0.1-1.0+3.0]} =$
- $P(Y=1) = \frac{\exp[0.1-1.0+3.0]}{1+\exp[0.1-1.0+3.0]} =$
- Include bias, and sum the other weights

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Example 2

$X = \{\text{Mother}, \text{Nigeria}\}$

- $P(Y=0) = \frac{1}{1+\exp[0.1-1.0+3.0]} = 0.11$
- $P(Y=1) = \frac{\exp[0.1-1.0+3.0]}{1+\exp[0.1-1.0+3.0]} = 0.89$
- Include bias, and sum the other weights

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Example 3

$X = \{$Mother, Work, Viagra, Mother$\}$

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- Multiply feature presence by weight

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Example 3

$X = \{$Mother, Work, Viagra, Mother$\}$

- $P(Y=0) = \frac{1}{1+\exp[0.1-1.0-0.5+2.0-1.0]} = 0.60$
- $P(Y=1) = \frac{\exp[0.1-1.0-0.5+2.0-1.0]}{1+\exp[0.1-1.0-0.5+2.0-1.0]} = 0.40$
- Multiply feature presence by weight

How is Logistic Regression Used?

- Given a set of weights $\vec{\beta}$, we know how to compute the conditional likelihood $P(y|\vec{\beta}, x)$
- Find the set of weights $\vec{\beta}$ that maximize the conditional likelihood on training data (next lecture)
- **Intuition**: higher weights mean that this feature implies that this feature is a good feature for the positive class

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Logistic regression

Naïve Bayes

Spam Classification

HAM	SPAM	SPAM	SPAM	HAM
work	nigeria	fly	money	fly
buy	opportunity	buy	buy	home
money	viagra	nigeria	fly	nigeria

Goal: Estimate $P(Y|X)$

Bayesian Classifiers

What is different from logistic regression?
We model $P(Y|X)$ using **Bayes Rule** i.e.,

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

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HAM	SPAM	SPAM	SPAM	HAM
work buy money	nigeria opportunity viagra	fly buy nigeria	money buy fly	fly home nigeria

- $P(Y = SPAM) = \frac{3}{5}$
- $P(Y = HAM) = \frac{2}{5}$
- The fraction of spams in the training data

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When classifying SPAM vs. HAM, then

$$\frac{P(X|Y = \text{SPAM})P(Y = \text{SPAM})}{P(X)} \text{ vs } \frac{P(X|Y = \text{HAM})P(Y = \text{HAM})}{P(X)}$$

The denominator does not affect the decision of the estimated class

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- Given assumptions about the nature of SPAM/HAM emails, the probability that we observe this particular email

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How do we estimate $P(X|Y)$?

The Naïve Bayes Assumption

We make the following assumption on $P(X|Y)$:

$$P(X|Y) = \prod_{j=1}^{N} P(x_j|Y)$$

i.e., features X are **independent** given class Y

- x_j: each word in a document
- N: Number of words in a document

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In reality, this is not true e.g.,
$P(X = \{\text{peanut, butter}\}|SPAM)$

Do the words “peanut” and “butter” occur independent to each other? i.e.,
$P(X = \{\text{peanut, butter}\}|SPAM) = P(X = \{\text{peanut}\}|SPAM)P(X = \{\text{butter}\}|SPAM)$

Classifying Unseen Examples using Naïve Bayes Classifier

Training Data:

HAM	SPAM	SPAM	SPAM	HAM
work	nigeria	fly	money	fly
buy	opportunity	buy	buy	home
money	viagra	nigeria	fly	nigeria

Unseen Example: $X =$ work, nigeria

$$
\begin{aligned}
P(Y = HAM|X) &\propto P(X|Y = HAM)P(Y = HAM) \\
&= P(\text{work}|Y = HAM)P(\text{nigeria}|Y = HAM) \cdot \frac{2}{5} \\
&= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{2}{5}
\end{aligned}
$$

Naive Bayes Classifier: More examples

What about this case:

- want to identify the type of fruit given a set of features: color, shape and size
- color: red, green, yellow or orange (discrete)
- shape: round, oval or long+skinny (discrete)
- size: diameter in inches (continuous)

Naive Bayes Classifier: More examples

Conditioned on type of fruit, these features are not necessarily independent:

Given category “apple,” the color “green” has a higher probability given “size < 2”:

$$P(green \mid size < 2,\ apple) > P(green \mid apple)$$

Generative vs. Discriminative Models

Discriminative

Model only conditional probability $p(Y|X)$, excluding the data X.

Logistic regression

- Logistic: A special mathematical function it uses
- Regression: Combines a weight vector with observations to create an answer

Generative

Model joint probability $p(X, Y)$ including the data X.

Naïve Bayes

- Uses Bayes rule to reverse conditioning $p(X|Y) \to p(Y|X)$
- Naïve because it ignores joint probabilities within the data distribution

Contrasting Naïve Bayes and Logistic Regression

- Naïve Bayes is easier for learning
- Naïve Bayes works better on smaller datasets
- Logistic regression works better on medium-sized datasets
- On huge datasets, both algorithms perform about the same (data always win)
- The Naïve Bayes assumption

- Next Monday, we will cover it more in depth