Python for Orbital Mechanics

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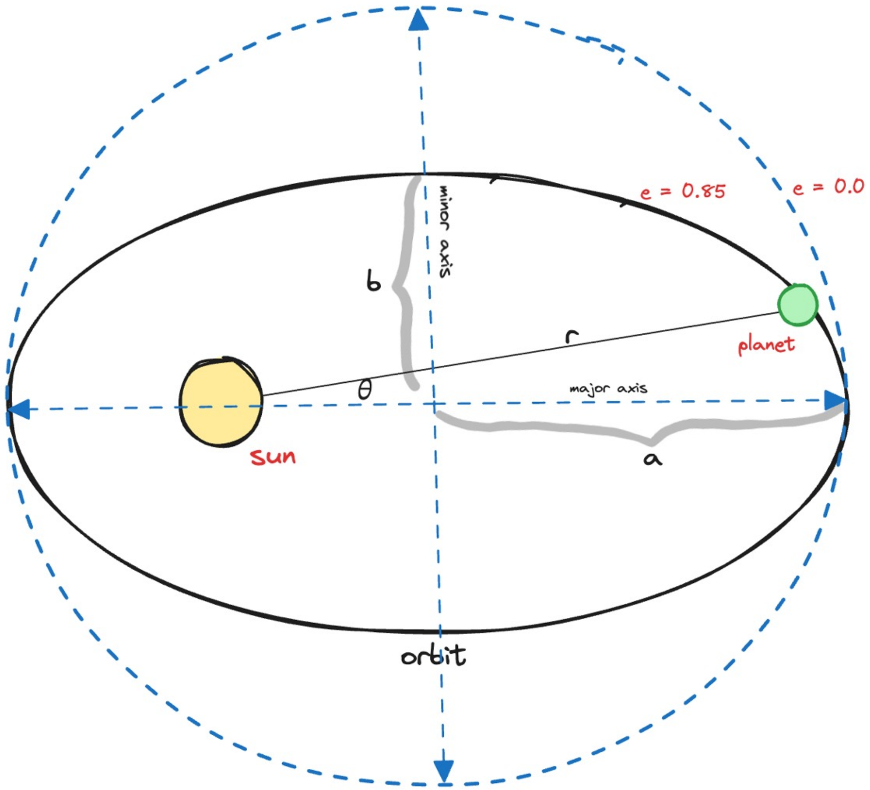
MOVING IN AN ELLIPTICAL ORBIT

DEFINING AN ORBIT



Shape and size

SIZE: Semi-major axis (a) — the sum of the periapsis and apoapsis distances divided by two. For classic two-body orbits, the semi-major axis is the distance between the centers of the bodies, not the distance of the bodies from the center of mass.

$$a = \frac{r_a + r_p}{2}$$

$$T = 2\pi \sqrt{\frac{a^3}{\mu}}$$

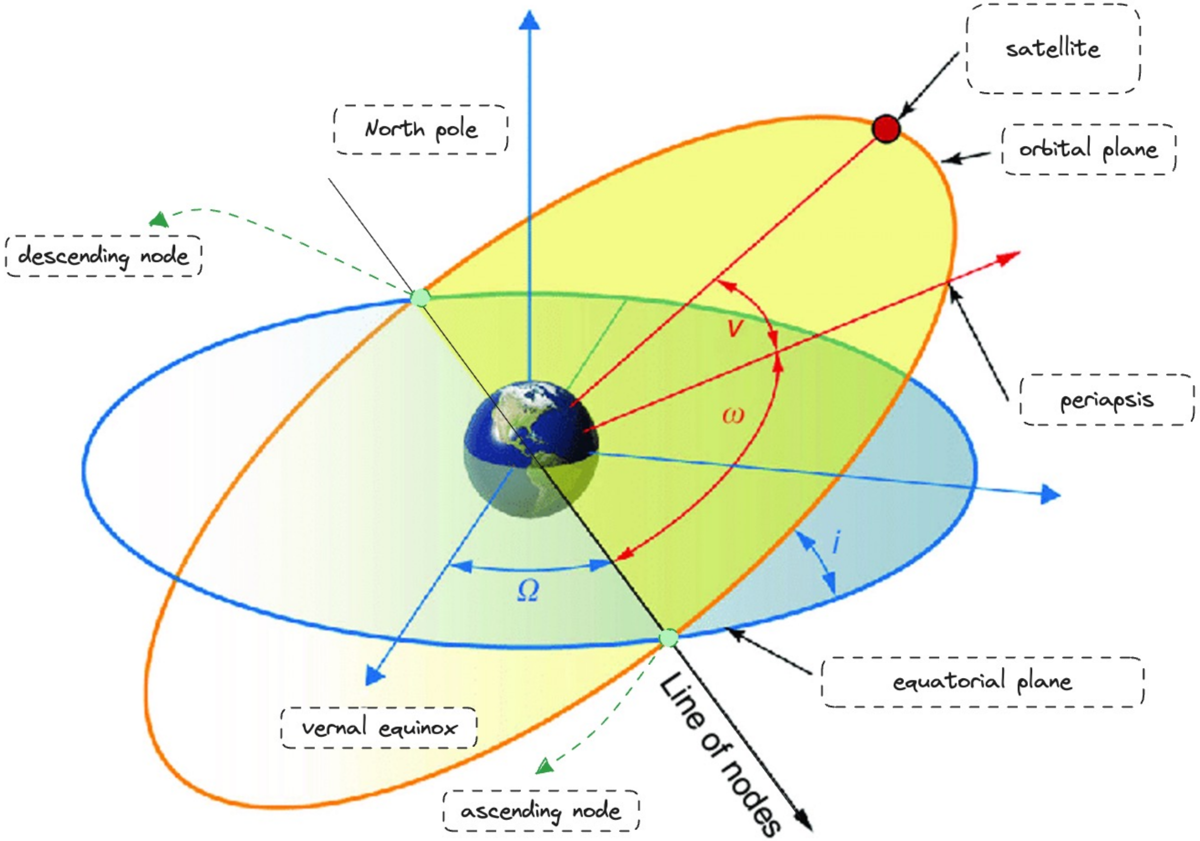
SHAPE: Eccentricity (e)—shape of the ellipse, describing how much it is elongated compared to a circle (not marked in diagram).

$$e = \frac{r_a - r_p}{r_a + r_p}$$

$$0 < e < 1:\ elliptical, \qquad 1 > e:\ hyperbolic$$

$$e = 1 - \frac{r_p}{a}$$

DEFINING AN ORBIT

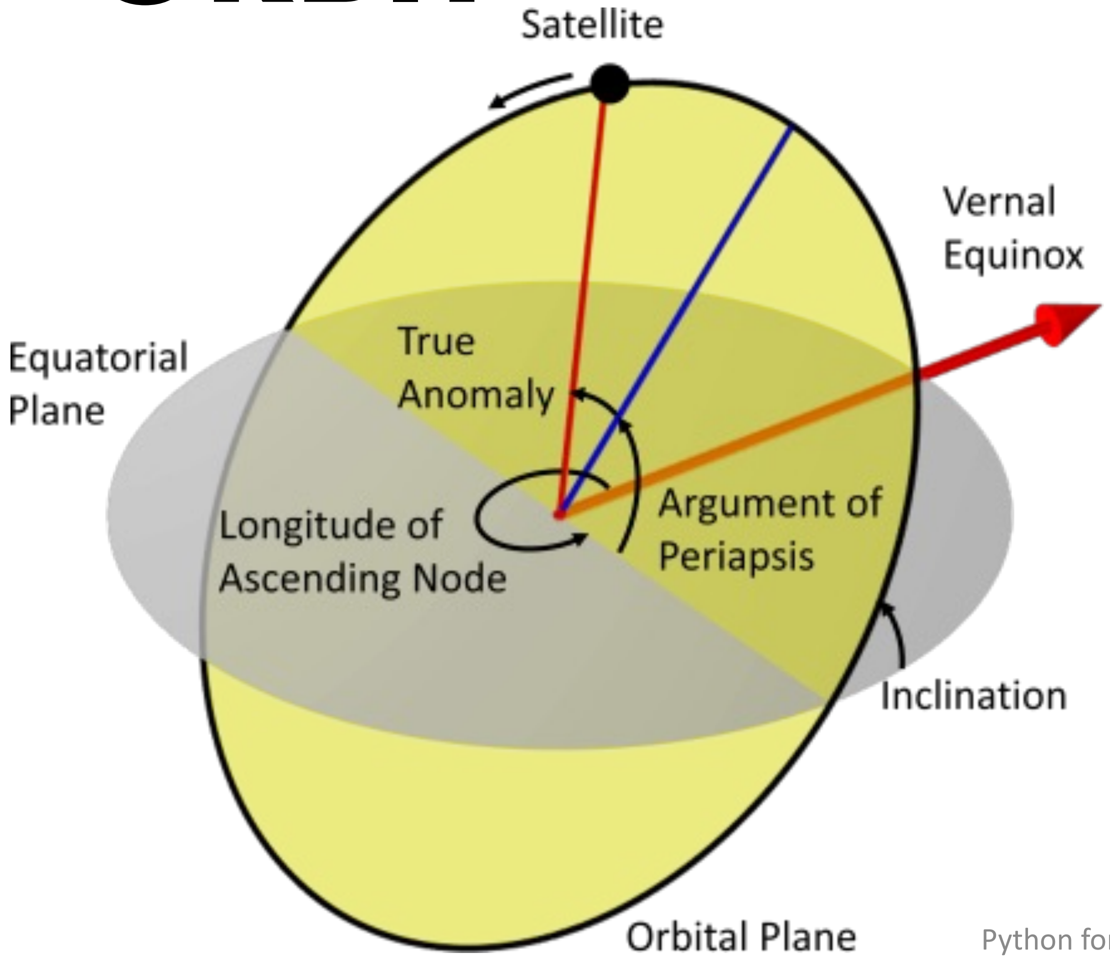


Orientation of orbital plane

TILT: Inclination (i) — vertical tilt of the ellipse with respect to the reference plane, measured at the ascending node (where the orbit passes upward through the reference plane). Tilt angle is measured perpendicular to line of intersection between orbital plane and reference plane.

TWIST: Longitude of the ascending node (Ω) — horizontally orients the ascending node of the ellipse (where the orbit passes from south to north through the reference plane (equatorial) with respect to the reference frame's vernal point. This is measured in the reference plane and is shown as the angle Ω in the diagram.

DEFINING AN ORBIT



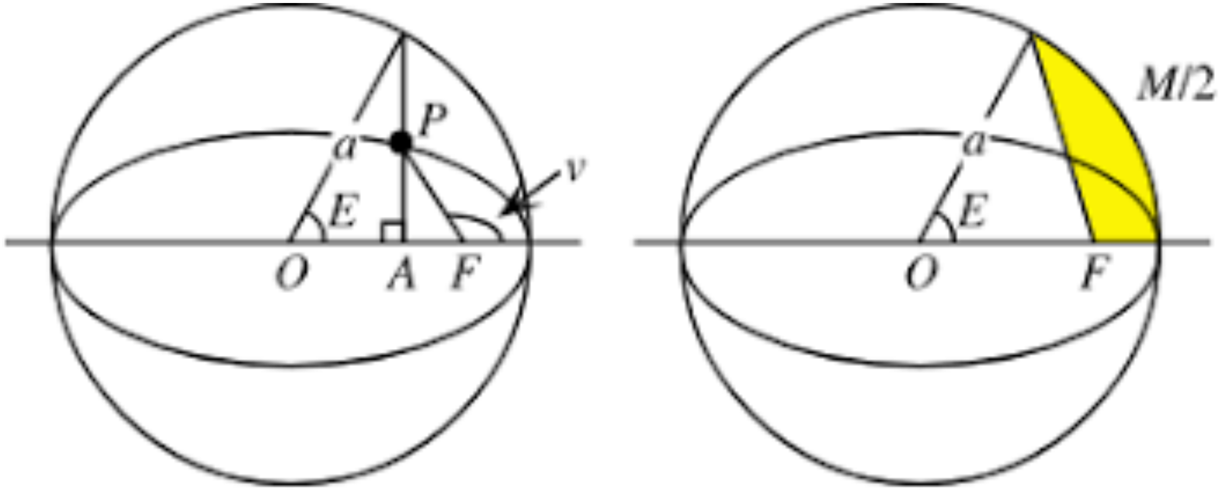
Satellite location and position

Argument of periapsis (ω) defines the orientation of the ellipse in the orbital plane, as an angle measured from the ascending node to the periapsis (the closest point the satellite object comes to the primary object around which it orbits).

True anomaly (ν, θ, or f) at epoch (t0) defines the position of the orbiting body along the ellipse at a specific time (the "epoch").

$$f = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{E}{2}\right)$$

MOVING IN AN ORBIT



<u>Propagating in time</u>

Eccentric anomaly (E) defines the angle obtained by drawing the auxiliary circle of an ellipse with center O and focus F, and drawing a line perpendicular to the semimajor axis and intersecting it at A.

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right)$$

Mean anomaly (M) at epoch (t0) is the angle between the periapsis of an orbit and the position of an imaginary body that orbits in the same period as the real one but at a constant average angular velocity (the mean motion) of the real orbiting body. The angle of mean anomaly is measured from the body being orbited, in the direction of orbital motion. Shown by Kepler's equation below.

$$M = E - e\sin E$$

If P is the orbital period, $\mathrm{n} = \frac{2\pi}{P}$ is the mean motion over a period

$$M(t_2) = M(t_1) + n(t_2 - t_1)$$
$$M = n(t - t_0)$$

FINDING POSITION AND VELOCITY WITH ORBITAL ELEMENTS

Given any orbital elements, we want to solve for the position and velocity.

<u>POLAR EQUATION</u>

$$\mathrm{r} = \frac{a(1-e^2)}{1+ecos(f(t))}\ km$$

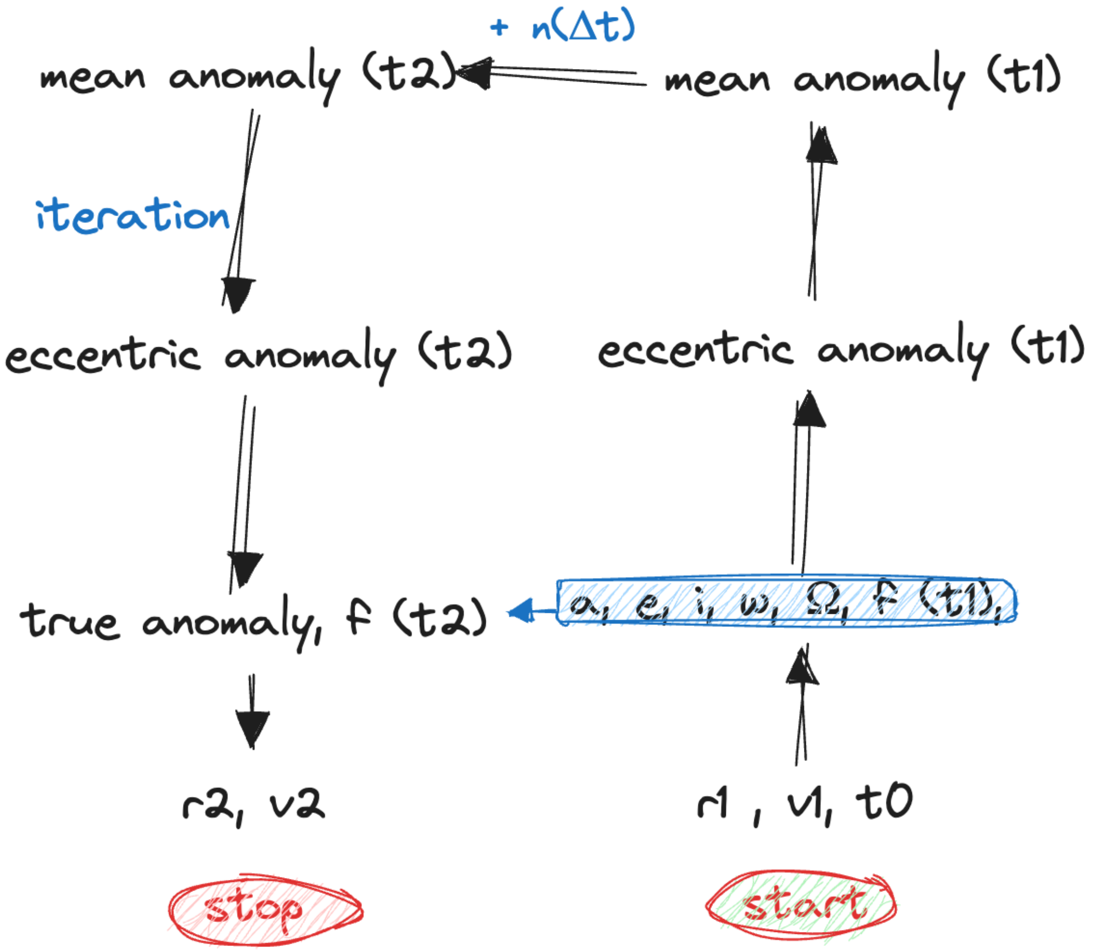
<u>VIS-VIVA EQUATION</u>

$$v = \sqrt{\mu\left(\frac{2}{r}-\frac{1}{a}\right)}\ km/s$$

FINDING POSITION AND VELOCITY WITH ORBITAL ELEMENTS

Given the initial position and velocity vectors, and time of an object in orbit, find the position and velocity of the object at a later time in its orbit.

Step 1. Use true anomaly f(t1), to get eccentric anomaly E(t1)
Step 2. Use E(t1) to find mean anomaly M(t1)
Step 3. Determine mean anomaly M(t2)
Step 4. Use mean anomaly M(t2) to find eccentric anomaly E(t2) by iteration
Step 5. Use eccentric anomaly E(t2) to find true anomaly f(t2)
Step 6. Use the unchanged orbital elements and the true anomaly f(t2) to calculate the new position and velocity vectors.



NEWTON'S ALGORITHM

Given M we want to solve for E with $f(E) = M - E + e \sin E = 0$

General root solving solution

Step 1. Take x_n as initial guess.

Step 2. Approximate $f(x) \cong f(x_n) + f'(x_n)\,(x - x_n)$

Step 3. Solve $f(x_n) + f'(x_n)\,(x - x_n) = 0$, $x_n = x - \frac{f(x_n)}{f'(x_n)}$

Step 4. Update your guess, $x_{n+1} = x - \frac{f(x_n)}{f'(x_n)}$

Step 5. If $f(x_n)$ is sufficiently small, quit.

Step 6. If $f(x_n)$ is not sufficiently small, repeat until it is.

Applied to Kepler's equation

$M = E - e \sin E$

Take $E_1 = M$ as initial guess.

$f(E) = E + e \sin E - M = 0$

$f'(E) = 1 - e \cos E$

$f(E) = f(E_n) + f'(E_n)\,(E - E_n) = 0$

$$E_n = E - \frac{f(E_n)}{f'(E_n)}$$

$$E_n = E - \frac{E + e \sin E - M}{1 - e \cos E}$$

$\|E + e \sin E - M\| > 0.001$

EXAMPLE

A rocket is in a elliptical orbit about the planet Kerbin. Calculate the position of the rocket 15 minutes after periapsis.

$$a = 2{,}160km, \qquad e = 0.832, \qquad Kerbin\ mass = 5.29\ * 10^{22}kg$$

SOLUTION (calculate with meters)

1. Kepler's 3rd law: $T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{a^3}{GM}}$, G = $6.67\ * 10^{-11}$
2. Average angular speed or Mean motion, $\mathrm{n} = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$ rad/sec

3. Mean anomaly M = n * t , t = 15mins = 900secs
4. From $M = E - e\sin E$, solve for E by iteration
5. Find true anomaly with E and e. $f = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{E}{2}\right)$ radians (change to degrees)
6. At t = 15mins after periapsis, rocket is at $\mathrm{r} = \frac{a(1-e^2)}{1+ecos(f(t))}$
7. We can also now find velocity from here, $v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$

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