Linear Algebra + Stats + Probability Notes

Bhargav *github.com/brpy/ml-notes*

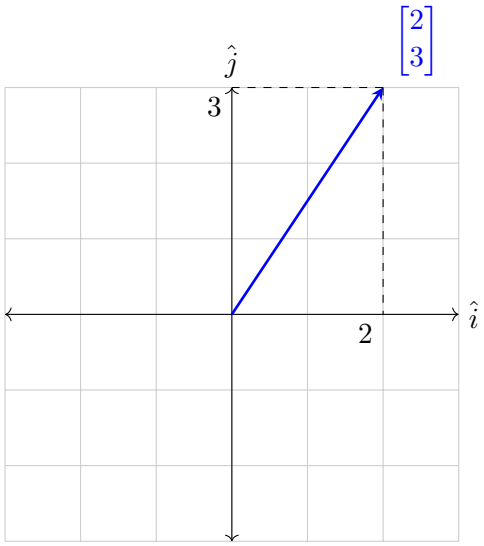
May 15, 2021

1 Linear Algebra

The following notes is primarily made for my revision. I might have skipped some topics that seemed obvious to me. I do not guarantee factual correctness of the notes. If you feel there are any errors, open a github issue/pr. Notes material is collected from various sources. Image credits are given in tex document and in src.csv

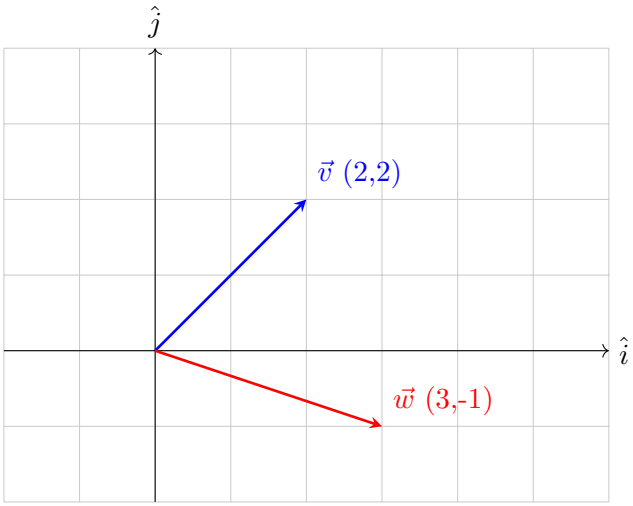
---

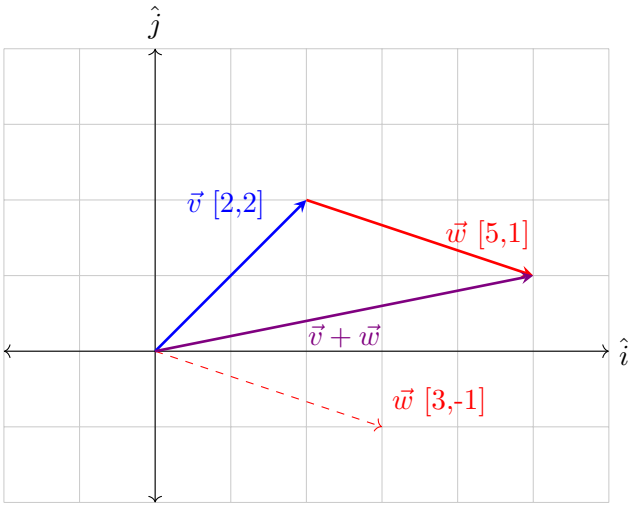
Vectors:



- This vector is represented as $2\hat{i}+3\hat{j}$ where $\hat{i}$ and $\hat{j}$ are unit vectors perpendicular to each other also known as basis vectors (in 2d).
- It is also represented as a column matrix $\begin{bmatrix}2\\3\end{bmatrix}$
- A vector is represented by its length and direction wrto. basis vectors $\hat{i}$, $\hat{j}$.
- A vector can be freely moved without changing its length and the direction it is pointing to.
- So any vector in space can be represented using a linear combination of $\hat{i}$, $\hat{j}$ by moving the starting point to origin.

Adding 2 Vectors:





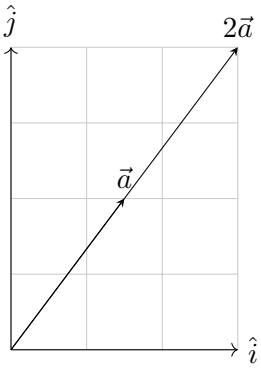
This is also known as Triangular law of vector addition.

- Can also be interpreted as,

$$\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix}$$

- The vector $\vec{w}$ is moved along $\vec{v}$ such that the starting point of $\vec{w}$ meets the end point of $\vec{v}$
- Both $\vec{v}$ and $\vec{w}$ direction and lengths remain unchanged.

Scaling:



- 2 *scales* the vector $\vec{a}$. So, 2 is a Scalar.
- Similarly basis vectors $\hat{i}$ and $\hat{j}$ can be scaled to represent any vector in 2d plane.

Span:

- $a\vec{v} + b\vec{w} \implies$ Linear combination of $\vec{v}$, $\vec{w}$
- Set of all vectors of linear combination of $\vec{v}$, $\vec{w}$; $a\vec{v} + b\vec{w}$ is called **span**.
- For most vectors span consists of all points on the plane.
- If $\vec{v}$, $\vec{w}$ lie on same line, span is a line passin through origin.
- If both $\vec{v}$, $\vec{w}$ are zero, span is zero.

Linear (in)dependent:

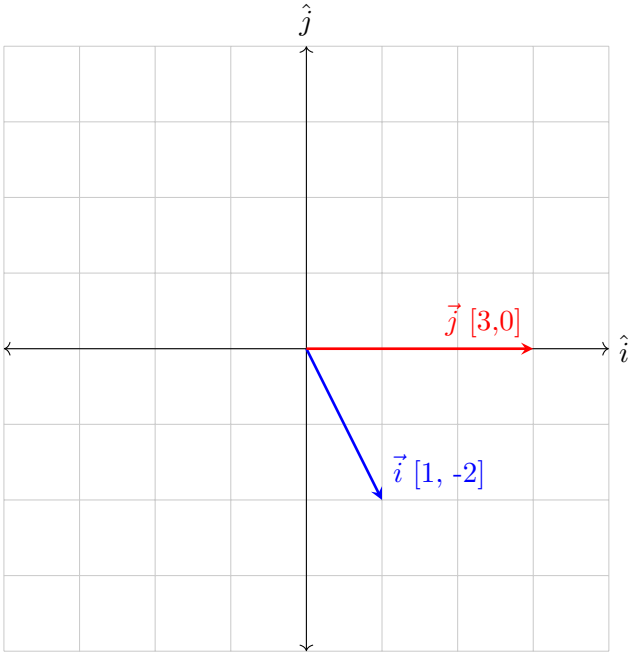
- If one vector can be represented as a linear combination of other, then the vectors are linearly dependent.
- A linearly dependent vector doesn't add to span of a vector.
- If one vector adds a dimension to a span of a vector, they are linearly independent.

Transformation:

- Takes a vector and gives a output vector.

Linear Transformation:
The conditions for the tranformation to be linear are:

- All lines must remain lines.
- Origin must be remain inplace.
- This makes all equidistant parallel lines remain equidistant and parallel.
- Matrices = Transformation of space.
- For example, the linear transform $\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ transforms the 2d plane such a way that the new $\hat{i}$ lands at $\begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix}$ and $\hat{j}$ lands at $\begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix}$



The new basis vectors are blue $\vec{i}$ and red $\vec{j}$.

Now in this new transformed space, every vector has to be represented as a linear combination of these two new basis vectors.

The vector $\vec{v}$ = $-1\hat{i} + 2\hat{j}$ after transformation lands at the point $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ in the new vector space.

Even after the transformation, the linear combination doesn't change. So, $\hat{i}$ and $\hat{j}$ are replaced by $\begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix}$ and $\begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix}$ respectively.

So the new transformed $\vec{v}$ becomes,

$$\vec{v}_{new} = -1\hat{i}_{new} + 2\hat{j}_{new} \tag{1}$$

$$\vec{v}_{new} = -1 \begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix} + 2 \begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix} \tag{2}$$

$$\vec{v}_{new} = \begin{bmatrix} -1\hat{i} \\ 2\hat{j} \end{bmatrix} + \begin{bmatrix} 6\hat{i} \\ 0\hat{j} \end{bmatrix} \tag{3}$$

$$\vec{v}_{new} = \begin{bmatrix} 5\hat{i} \\ 2\hat{j} \end{bmatrix} \tag{4}$$

For any vector $x\hat{i} + y\hat{j}$, after applying the transformation $\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ becomes,

$$\begin{bmatrix} x\hat{i} \\ y\hat{j} \end{bmatrix} \overset{lr.\ transform}{\Longrightarrow} x \begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix} + y \begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix} = \begin{bmatrix} 1x + 3y\hat{i} \\ -2x + 0y\hat{j} \end{bmatrix} \tag{5}$$

Removing $\hat{i}$, $\hat{j}$ for legibility,

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{lr.\ transform}{\Longrightarrow} x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix} \tag{6}$$

This is the basis for Matrix multiplication.

- $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is where $\hat{i}$ lands and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is where $\hat{j}$ (basis vectors) lands after the transformation.
- So the 2X2 matrix $\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ itself can represent the transformation.
- This explains the *rules* for multiplication and why matrix multiplication is not commutative.
- This also explains why the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix. Since, this *transform* actually does nothing, $\hat{i}$ and $\hat{j}$ remain unchanged.

For any transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{Transformation} \overbrace{\begin{bmatrix} x \\ y \end{bmatrix}}^{old\ vector} = x \overbrace{\begin{bmatrix} a \\ c \end{bmatrix}}^{where\ \hat{i}\ lands} + y \underbrace{\begin{bmatrix} b \\ d \end{bmatrix}}_{where\ \hat{j}\ lands} = \overbrace{\begin{bmatrix} ax + by \\ cx + by \end{bmatrix}}^{Transformed\ matrix} \tag{7}$$

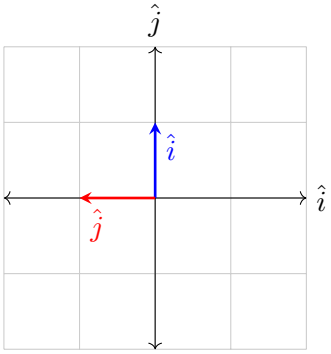
- $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are the new basis vectors; where old $\hat{i}$ and $\hat{j}$ land. These are the new transformed basis vectors.
- Now these new basis vectors have to be used to represent all the vectors in it's span. In other words the linear combination of these two basis vectors.
- From equation 7 it can be observed that the scalars x and y scale the corresponding new basis vectors.
- For no transformation (or) Identity transform / Multiplication,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{8}$$

Which makes sense !!

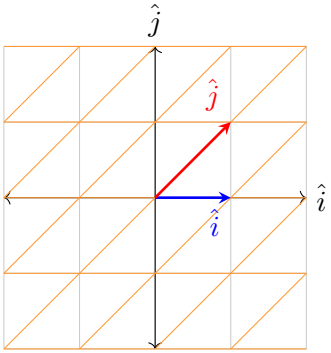
Counterclock Transform:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



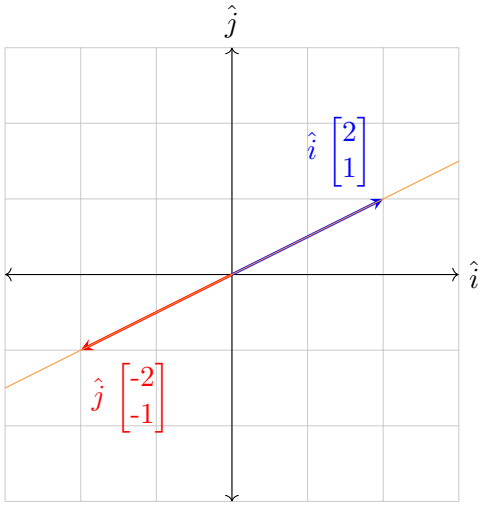
Shear Transform:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Few notable points:

- In transformations, order matters. $M_1 M_2 \neq M_2 M_1$ because $f(g(x)) \neq g(f(x))$
- The associative property $(AB)C = A(BC)$ holds true because the order is C, B then A regardless.
- For a $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$ transformation all 2d space is squished into a line.



These two vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}-2\\-1\end{bmatrix}$ are linearly dependent vectors.

- Two transforms for ex. shear and rotation can be composed into a single transform. This composition transform is nothing but a multiplication of shear and rotation matrix.

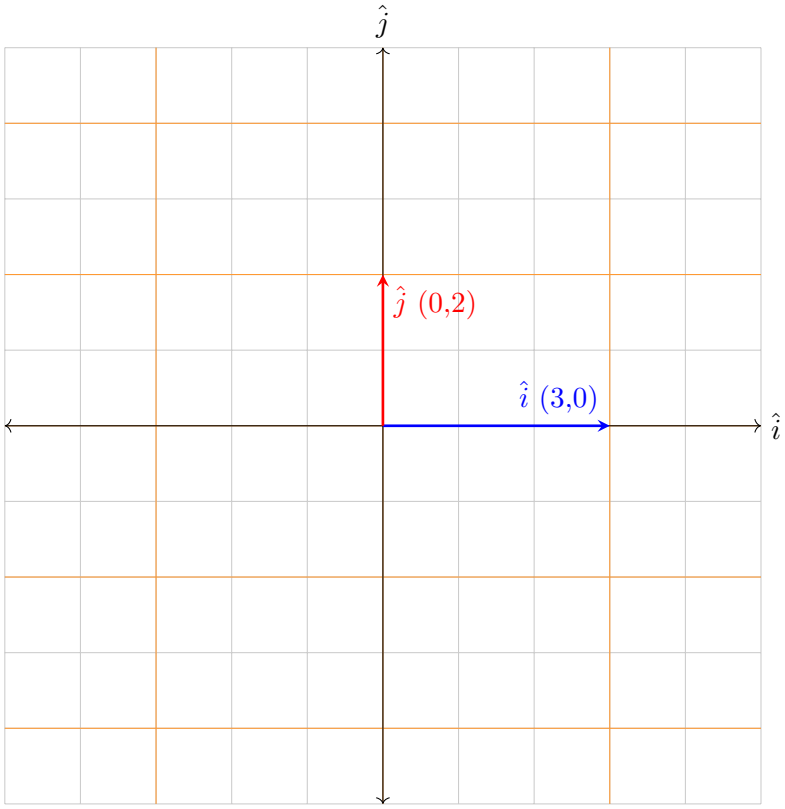
$$\underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}_{Shear}\overbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}^{Rotation}\begin{bmatrix}x\\y\end{bmatrix} = \underbrace{\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}}_{Compostion}\begin{bmatrix}x\\y\end{bmatrix} \tag{9}$$

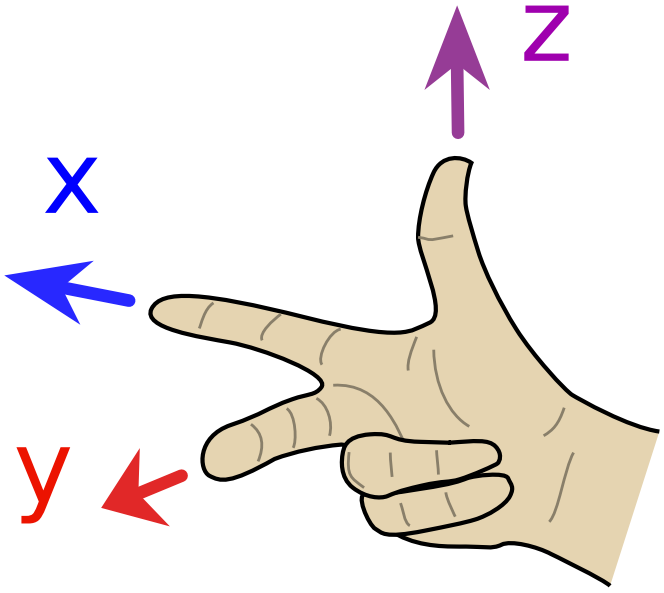
- Product or composition of two matrices/ transforms,

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}e & f\\g & h\end{bmatrix} = \begin{bmatrix}ae+bg & af+bh\\ce+dg & cf+dh\end{bmatrix} \tag{10}$$

Determinant:

- Transformation $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$ scales $\hat{i}$ by factor 3 $\hat{j}$ by a factor 2.





- We observe that the unit square in transformed space is scaled from 1 sq. unit to 6 sq. units. So the determinant of the transform/matrix $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$ is 6. In case of 3d, volume is scaled.
- For a matrix/transform $\begin{bmatrix}a & b\\c & d\end{bmatrix}$, the determinat is $ad - bc$.
- The determinant can be $-ve$ if the axis cross each other during the transformation. It has the effect of *flipping* or *inverting* the space.
- This can be checked using right hand rule to check if the axis still lie in the same orientation after transformation. If not, the determinant is $-ve$.
- For 3d space, determinant is measured by unit volume instead of area.
- Determinant can be zero if the space is squished. For ex. if transformed to a line or point in 2d or to a plane, line or a point in 3d. This has the effect of reduction in number of dimensions.

$$det(M_1 M_2) = det(M_1) det(M_2) \tag{11}$$

System of equations:

$A.\vec{X} = \vec{V}$

$3x + 1y + 4z = 1$
$3x + 9y + 2z = 6$
$3x + 3y + 3z = 8$

- Solving these system of equations imply, finding the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that if applied the transformation $\begin{bmatrix} 3 & 1 & 4 \\ 3 & 9 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, would land on the vector $\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$.
- This can only be solved if A^{-1} exists. Since space cannot be unpacked since there is/are lost dimension(s) if $det(A) = 0$. So $det(A) \neq 0$ has to be true for the system of equations to be solved using,

$$\vec{X} = A^{-1}\vec{V} \tag{12}$$

Rank:

- Rank is the number of dimensions of the transformed space.
- The Rank of a matrix ; output space.

  Rank : 1 $\implies$ output transformation : Line

  Rank : 2 $\implies$ output transformation : Plane

  Rank : 0 $\implies$ output transformation : Point

Column space:

- Set of all possible linear combinations or span of column vectors of A. Or,
- Set of all possible $A\vec{V}$

Null space:

- Set of vectors that get squished into origin after transformation.

Dot Product:

- $\vec{a}$ and $\vec{b}$ are two vectors and angle between them is θ.

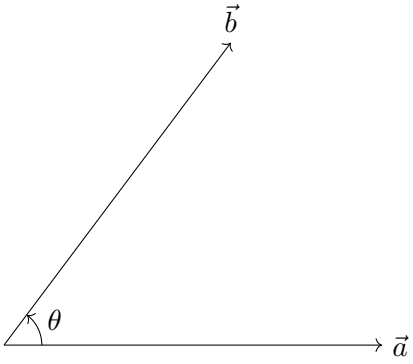
$$\vec{a} = [a_1, a_2, ...a_n] \tag{13}$$

$$\vec{b} = [b_1, b_2, ...b_n] \tag{14}$$

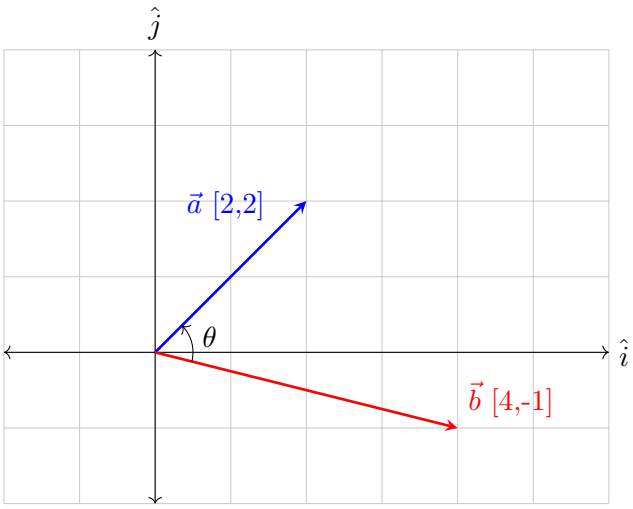
$$\vec{a}.\vec{b} = (a_1)*(b_1)+(a_2)*(b_2)+...+(a_n)*(b_n) \tag{15}$$

- Geometric representation:

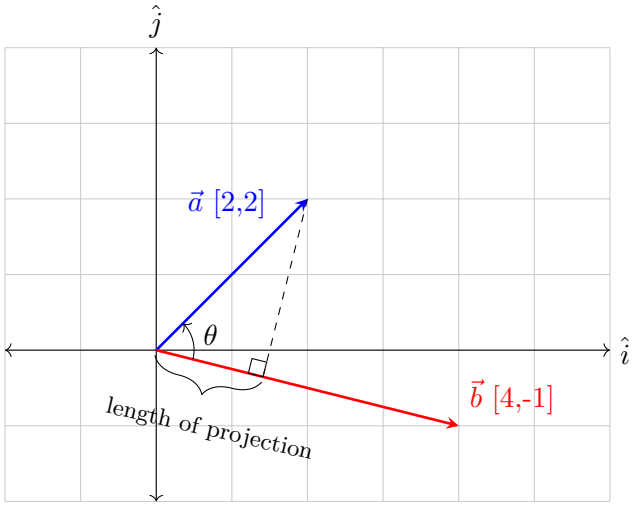
$$\vec{a}.\vec{b} = ||a|| * ||b|| * cos\theta \tag{16}$$



dot product of two vectors = (length of any vector) x (length of projection made on the vector)



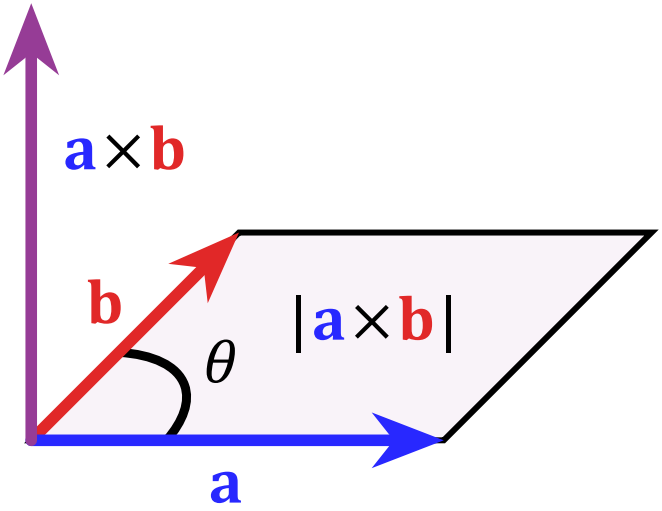
Projecting $\vec{a}$ on $\vec{b}$:



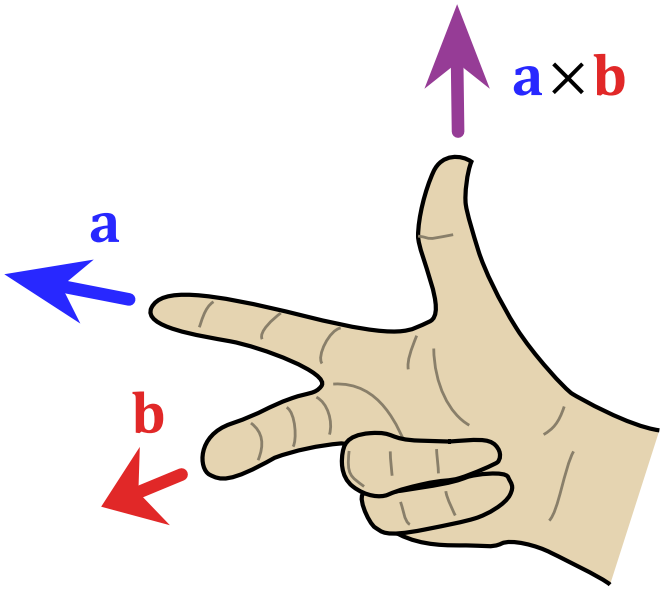
- Length of projection = $||a||cos\theta$
- Dot product = $||b||.||a||cos\theta$
- The length of projection does not depend on the length of $\vec{b}$. It only depends on length of $\vec{a}$ and θ.
- Projecting $\vec{b}$ on $\vec{a}$ will result in the same dot product result. The order does not matter.
- If projection does not lie betweeon origin and the end point of the other vector, you can extend the other vector since length of projection is not affected by it.
- Dot product is -ve if projection is on -ve side.

Cross Product:

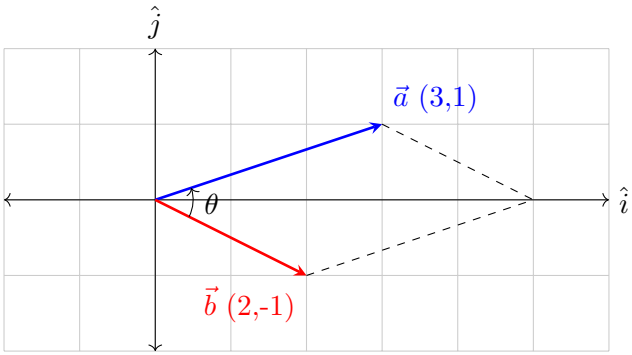
- Directed area product.



- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$



- Length of $\vec{a} \times \vec{b} = det(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix})$
- The cross product vector is normal to the area.



- The area of this parallelogram is $det(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix})$
- The direction of the cross product is normal to the area, i.e, in the direction of $\hat{k}$
- The magnitude of this vector is the area of this parallelogram i.e, determinant.

$$\vec{a} \times \vec{b} = ||a||.||b||sin\theta \tag{17}$$

Eigen:

- For a few transformations, vectors only scale. i.e, only stretch or compress.
- Such vectors are called eigen vectors of the transformation.
- Such vectors are transformed into it's own span.

- The scaled value of such vectors (scalar) is the eigen value. It can be +ve or -ve.

$$A\vec{V} = \lambda\vec{V} \tag{18}$$

where:

A is the transformation,

$\vec{V}$ is the eigen vector and,

λ is the eigen value.

Vector from origin:

- As discussed previously, any vector can be moved to origin for reference, without changing it's direction it is pointing to and it's length.
- To make a vector starting from $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and ending at $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ as a reference vector, we can apply triangular law of addition to get a vector which starts from origin and is equivalent with the vector starting and ending at the above two points.
- We need to remember that the above vectors themselves start at origin. So, we can get the vector by doing $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Matrices:

Diagonal Matrix:

- A matrix with all 0's except diagonal elements.

Identity Matrix:

- A Matrix with all diagonal elements as 1.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scalar Matrix:

- A Matrix with only equal diagonal elements.
- A scalar matrix can be represented by a single scalar.

$$K.I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Upper Triangular Matrix:

- A Matrix with all elements below diagonal are 0

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

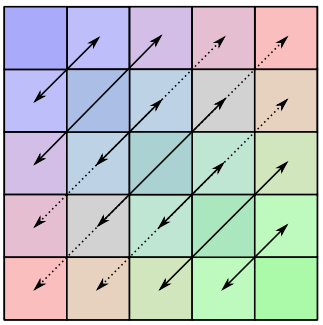
Lower Triangular Matrix:

- A Matrix with all elements above diagonal are 0

$$\begin{bmatrix} 3 & 0 & 0 \\ 7 & 2 & 0 \\ 5 & 3 & 1 \end{bmatrix}$$

Symmetric Matrix:

- A square matrix which is equal to its transpose.
- $A = A^T$
- $a_{ij} = a_{ji}$ for every i and j.



Skew Symmetric Matrix:

- A square matrix which is equal to negative of its transpose.
- $A = -A^T$
- $a_{ij} = -a_{ji}$ for every i and j.

Orthogonal Matrix:

- A square matrix whose column vectors and row vectors are orthonormal.
- $AA^T = A^TA = I$ or,
- $A^T = A^{-1}$
- $a_{ij} = -a_{ji}$ for every i and j.

Lines, Planes and Hyperplanes:

- **Line** - A line has 2 degrees of freedom. You need a point and a slope or two points to uniquely identify a line.
- Equation of a line
  1. $y = mx + c$; slope = m ; y - intercept = c ; passes through $(0, c)$
  2. $ax + by + c = 0$ is a simple linear form with 2 unknowns
  3. $y - y_1 = m(x - x_1) + c$; slope = m ; passes through x_1 and y_1
- Slope is the *tan* of angle x axis makes with the line in anti-clock wise direction.
- Any 2 points on a line can uniquely identify the line.
- An object on a line can move in 1 dimension.

- **Plane** - A plane is a flat 2d surface.
- A xy plane is an example of a plane.
- To represent a plane you need 3 non collinear points or a point and a normal.
- A normal is a vector that is perperndicular to all vectors on the surface.
  1. So, by definition a plane is set all (x, y, z) points that satisfy,

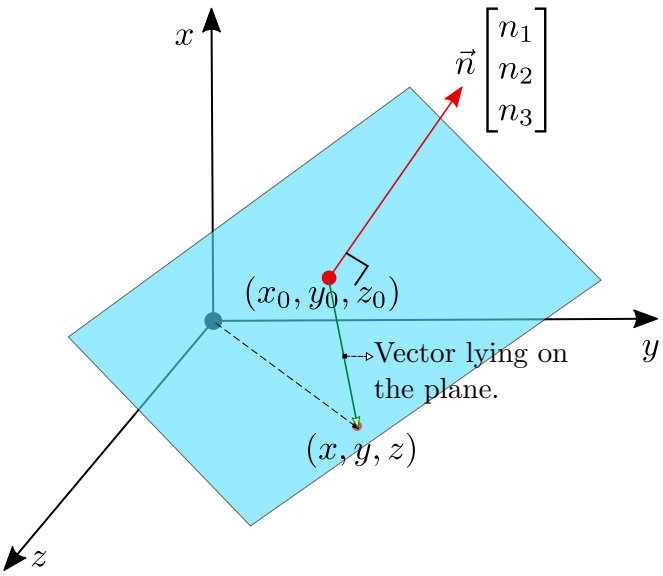
$$dot(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \vec{n}) = 0 \qquad (19)$$

  Where:

  $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ is the normal to the plane and starts at point $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ which is located on the plane.

- The equation subsequently solves to,

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \quad (20)$$



- **Hyperplane** - Is an extension to Line and a plane in higher dimensions.

- A **Linear regression** model can successfully represent a line, plane or a hyperplane.

Miscellaneous:

- Distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Length of a vector $a\hat{i} + b\hat{j} + c\hat{k}$ is

$$\sqrt{a^2 + b^2 + c^2}$$

  also known as L2 Norm. (Check knn notes for info on norms.)

- Distance from a plane $ax + by + cz = 0$ to a point (x_1, y_1) is

$$\frac{|ax_1 + by_1 + cz_1|}{\sqrt{a^2 + b^2 + c^2}}$$

Credits:

- Most of this notes is taken from 3Blue1Brown's Essence of Linear Algebra playlist. Some images are taken from wikipedia.

2 Statistics

Population and Sample:

- (Statistical) population is a set of all the items or events which is of interest for a question or experiment.
- (Statistical) sample is a subset of population. The aim of sampling is that our sample represents the population.
  The advantage is that it is faster and cheaper to collect data than for the entire population.

Figure 1: *A random sampling process*

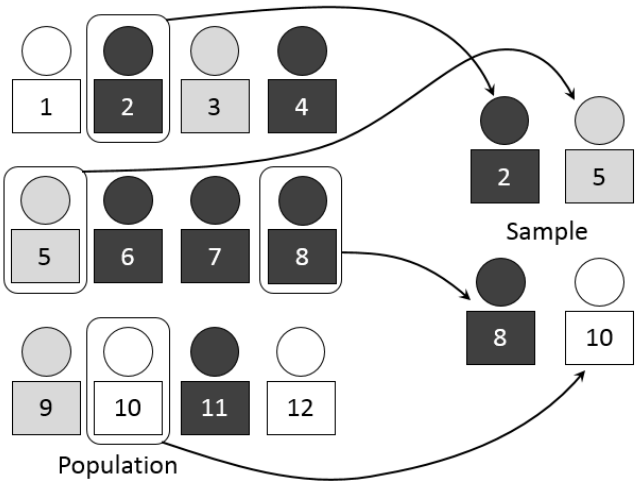
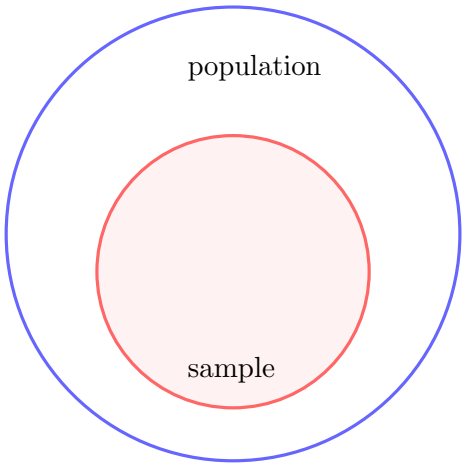


Figure 2: *A sample is a subset of population*



Random variable: (X)

- A Random variable is a measurable function defined on probability space that maps from the sample space to real numbers.
- It is called a **Discrete Random variable** if the range of the function is countable.
- It is called a **Continous Random variable** if the range of the function is uncountably infinite or an interval.
- Both these types of Random variable has a distribution.
  - If X is a **Discrete Random variable**, it's distribution is a discrete probability distribution i.e. can be described by a probability mass function (pmf) that assigns a probability to each value in the range of X.
  - If X is a **Continous Random variable**, and *absolutely continuous*, its distribution can be described by a probability density function (pdf).
    Pdf assigns probailities to an interval and since the range is absolutely continuous, each individual point must have a probability of 0.
- Not all continuous random variables are absolutely continuous.
  A mixture distribution is one such counterexample; such random variables cannot be described by a probability density or a probability mass function.
- But all Random variable can be described by a Cumulative distribution function (cdf).
- Ex. for a discrete rv is dice rolls and for continuous rv is height or incomes of people.

Probability Distribution:

- A Probability Distribution is the mathematical function that gives probabilities of occurences of different possible outcomes for an experiment.
- It is a mathematical description of a random phenomenon in terms of its sample space and the probabilities of events (subsets of the sample space).

Probability Density Function : (pdf)

- Pdf is a function used to specify the random variable falling within a particular range of values, as opposed to taking any particular value(which is 0, since there are infinite number of values).
- Probablity is the integral or area under the curve over an interval.
- It's value at any given point can be interpreted as providing the relative likelyhood that the value of the random variable would equal that sample.
- In other words, while the absolute likelihood for a continuous random variable to take on any particular value is 0 (since there is an infinite set of possible values to begin with), the value of the PDF at two different samples can be used to infer, in any particular draw of the random variable, how much more likely it is that the random variable would equal one sample compared to the other sample.

Properties of pdf:

- Area under the curve or AUC(pdf) $= 1$
- Value at any point of pdf curve has to be ≥ 0
- x-axis : Random variable X
  y-axis : Relative likelyhood/ probability density.
- For a continuous rv, it's pdf has to also be continuous.
- A pdf doesn't have to be differentiable. For ex. Laplace distribution is not differentiable.

$$\frac{1}{2}e^{-|x|}$$

- But a CDF has to be differentiable.
- A point on pdf provides relative likelyhood. Whereas, area under an interval gives probability.
- Few random variables may not have pdf. They are called Singular distributions.

Gaussian/ Normal Distribution: (N)

$$P(X = x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \frac{-(x-\mu)^2}{2\sigma^2} \tag{21}$$

where:
μ - Mean or expectation
σ - Standard deviation
σ^2 - Variance

- The significance of Normal distribution comes from various factors like,
  - Central limit theorem
  - Many natural properties follow a roughly Normal distribution
  - For a fixed mean and std. deviation, Normal disitribution has maximum entropy.
- Though Normal distribution is commonly referred to as a bell curve, there are also other distributions in the bell curve family.

Figure 3: *pdf of a normal distribution*

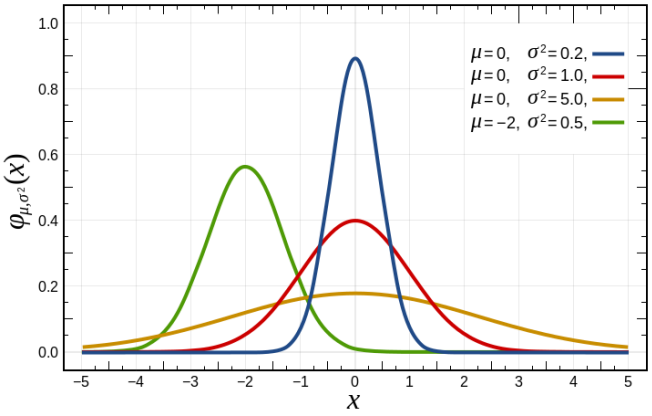
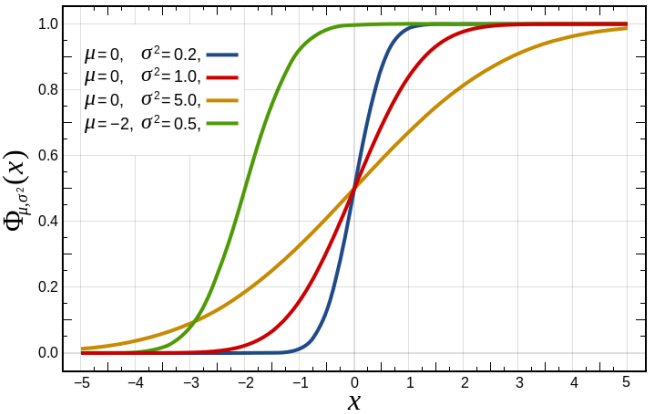


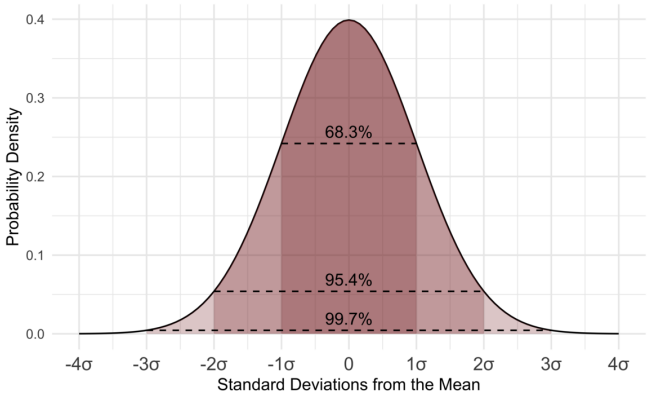
Figure 4: *cdf of a normal distribution*



68-95-99.7 Rule:

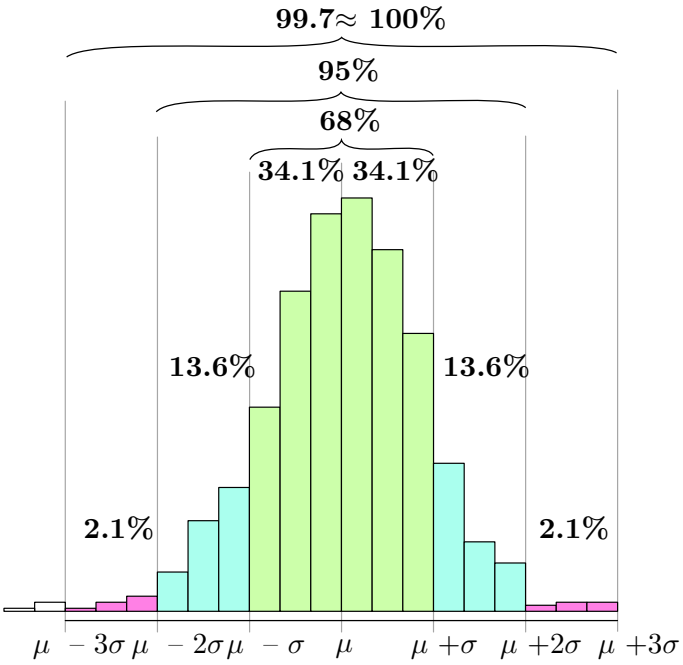
- It is an empirical rule that a percentage of values lie within a band around the mean for a normal distribution.

Figure 5: *pdf of a normal distribution with $\mu = 0$*



- The percentage of values are,
  - 68% of values lie within μ-σ and $\mu + \sigma$
  - 95% of values lie within μ-2σ and μ+2σ
  - 99.7% of values lie within μ-3σ and μ+3σ
- Is valid for any value of μ and σ

Figure 6: *pdf of a normal distribution with mean μ and std-dev. σ*



Cumulative distribution function: (CDF)

- The probability that rv X will take a value less than x. $P(X < x)$.
- Fig 4 on page 10, represents cdf of some normal distributions.
- CDF of a distribution has to differentiable and non decreasing.
- x-axis : Random variable X
  y-axis : $P(X < x)$
- Slope at a point on cdf gives pdf.

$$f(x) = \frac{dF(x)}{dx} \tag{22}$$

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{23}$$

where:

$F(x)$ is cdf.

$f(x)$ is pdf.

- All random variable have cdf.
- On observing CDFs it is easy to get an idea on percentiles.

Probability mass function: (PMF)

- Probability mass function (PMF) is a function that gives the probability that a discrete random variable is exactly equal to some value.
- It is associated with discrete random variable.
- Unlike pdf's there's no need to integrate to get probabilities.
- Some examples are Bernoulli distribution and Binomial distribution.
- Sum of all probabilities has to equal 1, and no values can be negative.

Figure 7: *pmf of a fair dice*

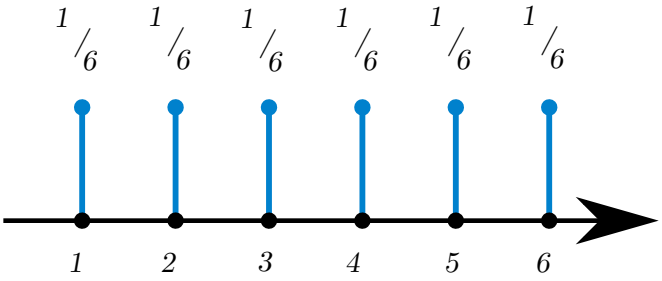
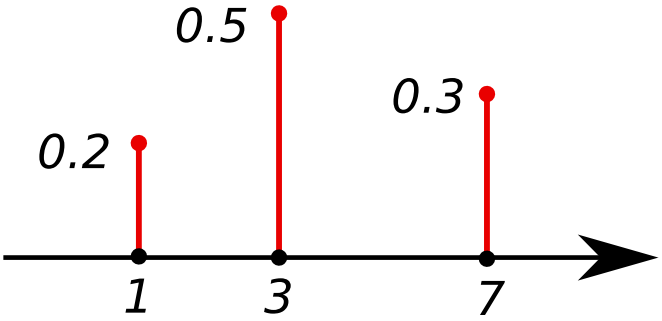


Figure 8: *A discrete probaility distribution function*



Mean, Median and Mode:

- Mean (μ) :

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

where:

μ : mean of the population

n : size of the population

x_i : value from population

- Median :
Central most observation after sorting. If two values are in middle, median is the average of the two.
- Mode: Most frequent observation.
- These are the measures of central tendency.

Variance: (σ^2)

- Measures how far a set of numbers are spread.

$$\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{n} \tag{24}$$

where:

σ^2 : population variance

n : size of population

μ : mean of population

- It is the expectation of the squared deviation of a random variable from its mean.
- The problem with variance as a measure of spread about mean is that, it is not standardised. And the units of variance are $unit^2$

Standard deviation: (σ)

- Measure of the amount of variation or dispersion of a set of values.

$$\sigma = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{n}} \tag{25}$$

- Measures how far a set of numbers are spread.
- Has some useful properties compared to variance as discussed above.
- Why use square?
  - Squaring helps with penalising $-ve$ and $+ve$ deviations equally and sum will not be zero.
  - Squaring can emphasize larger differences.
  - Low std. deviation indicates values are closer to mean.
  - It also helps with algebra due to it being continuously differentiable and is the true L2 distance (norm).

Expected value (or) Expectation:

- Expectation is numerically the average value or mean of a Random variable X.
- It is the weighted average of a rv with weights as it's frequencies or probabilities.
- For the set of values $\{1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3\}$ the expected value is nerically same as the mean or weighted average (weights are probabilities).
- It holds true for large number of picks or realizations which are independent.
- It usually refers to single event or experiment.

Bessel's correction, Sample variance and Sample std. deviation:

- Same formula of variance and std. deviation can be used to find sample variance and sample std. distribution.
- But In statistics often sample variance (s^2) and sample std. deviation (s) are used to represent population variance (σ^2) and population std. deviation (σ).
- This is due to finding population variance and std. deviation might not be feasible. So, a sample is drawn from population to find s^2 and s which approximate to σ and σ^2.
- This approximation (estimate) is biased. The formula overestimates. Check https://en.wikipedia.org/wiki/Bessel's_correction
- So, to remove this bias a correction factor is applied.
- Uncorrected: (using same formula as population) (biased estimate of population mean)

$$s^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n} \tag{26}$$

$$s = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}} \tag{27}$$

- Corrected using bessel's: (unbiased estimate of population mean)

$$s^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1} \tag{28}$$

$$s = \sqrt{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}} \tag{29}$$

where:

x_i : i^{th} sample

s^2 : sample variance

s : sample std. deviation

n : size of sample

$\bar{x}$: sample mean

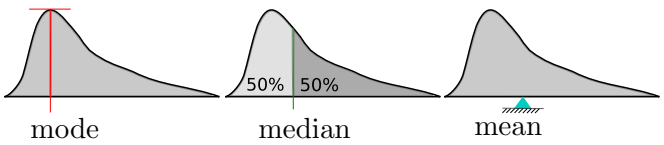
- One can understand Bessel's correction as the degrees of freedom in the residuals vector (residuals, not errors, because the population mean is unknown):
- Bessel's correction is only necessary when the population mean is unknown, and one is estimating both population mean and population variance from a given sample, using the sample mean to estimate the population mean.
- In that case there are n degrees of freedom in a sample of n points, and simultaneous estimation of mean and variance means one degree of freedom goes to the sample mean and the remaining n − 1 degrees of freedom (the residuals) go to the sample variance.
- However, if the population mean is known, then the deviations of the observations from the population mean have n degrees of freedom (because the mean is not being estimated – the deviations are not residuals but errors) and Bessel's correction is not applicable.

Symmetric distribution:

- A symmetric distribution is one where it's pdf is symmetric about a vertical line.
- There should exist a value x_0 such that $f(x_0 + \delta) = f(x_0 - \delta)$ for all δ.
- Properties:
  - The mean and median both occur at the symmteric point x_0
  - If distribution is unimodal (has only 1 mode), mode coincides with mean and median.
  - Skewness is zero.

Mean, Median, Mode from PDF point of view:

Figure 9: *Mean, Median, Mode of an arbitrary distribution*



- We observe that mode is the peak value of pdf. i.e, most frequent/probable.
- Median is the middle most value that splits the area into 2 equal parts.

$$\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$$

or

$$\int_{m}^{\infty} f(x)dx = \frac{1}{2}$$

where m is the median.

- The *mass* of the pdf is balanced at the mean point. or

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

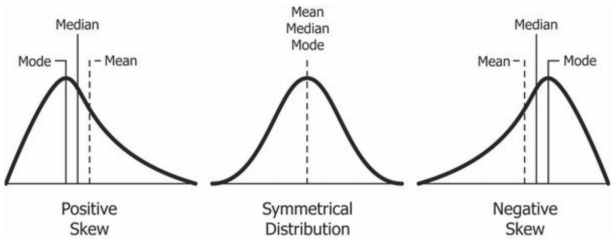
where $E[X]$ is the expected value or average of random variable X.

- The distinction between mean and median is, median is not affected by the *distribution* of area as long as the area towards left and right remain equal to each other.
- Whereas, mean is affected by it. Adding more *mass* at the extremes severly shifts the mean point toward that extreme.
- This is aligned with the conventional definitions of median and mode discussed previously.

Skewness:

- The word *skewness* means *asymmetry.*
- In statistics, skewness is a **measure of asymmetry** of a pdf about it's mean.
- Skewness value can be 0, +ve, -ve or undefined.
- For unimodal distributions, (which have only one mode),

Figure 10: *Skewness General rule of thumb for unimodal distributions*



- However this may not always be true. If one tail is long and thin and other tail is fat and short skewness may still be zero.
- Negative skew : Mass is concentrated on the right; Left skewed, left tailed despite the fact that the curve appears leaning to the right.
- Right skew : Mass is concentrated on the left; Right skewed, right tailed despite the fact that the curve appears leaning to the left.
- Note that this is only a general rule for unimodal distributions. (Not always holds true)
- Pearson's moment coefficient of skewness:

$$\bar{\mu_3} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

Kurtosis:

- Is a **measure of tailedness**.
- Describes the extent of *outliers* or *extreme* values.
- Does not describe peakedness or data configuration near mean.
- Kurtosis of Gaussian/ Normal distrubution N is 3.
- It is common to compare kurtosis of other distributions with normal. The value is called excess kurtosis.
- Excess kurtosis = kurtosis - 3
- K < 3 : Platykurtic distribution. Fewer no. and less extreme outliers than Gaussian (N).
  Tails approach zero faster than gaussian. Thin, short tails.
- K > 3 : Leptokurtic distribution. More no. and high extreme outliers than Gaussian (N).
  Tails approach zero slower than gaussian. Fat, long tails.
- Pearson moments:

$$\bar{\mu_4} = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]$$

Standardization:

- Transforming any distribution into a distribution with $\mu = 0$ and $\sigma = \sigma^2 = 1$.
- The distribution and it's shape remains the same but the scale changes.
- By this process, we can bring all distributions into one frame of reference.

- Commonly Mean centering scaling.

$$Z = \frac{x_i - \mu}{\sigma}$$

- Less affected by outliers than Normalization.
- Rescales data to have mean $\mu = 0$ and std. deviation $\sigma = 1$
- Gives features comparative scaling.
- Standardization is recommended for distance based models and faster convergence in neural networks.

Normalization:

- Scales all values into the range [0,1]
- Also called 0-1 scaling.

$$X_i = \frac{x_i - min}{max - min}$$

- The minimum data point becomes zero and maximum data point becomes 1.
- More affected by outliers than Standardization because min and max values are directly used in calculation Normalized values.
- Standardization is preferred over Normalization most of the times.
- Give features the same scale.

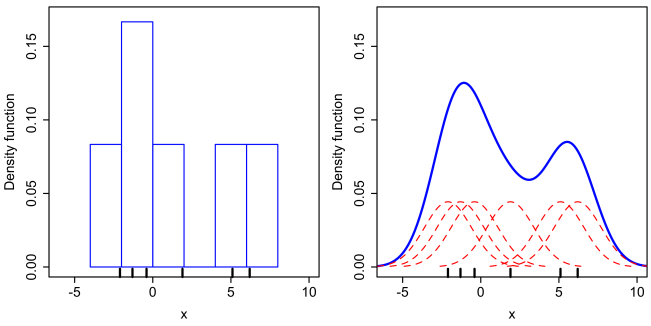
Standard Normal variate:

- A standard normal variate is an outcome from a Normal distribution N with mean $\mu = 0$ and $\sigma = \sigma^2 = 1$. $N(0, 1)$

KDE (Kernel density estimation):

- KDE is a non parametric way to estimate the probability distribution function (pdf) of a random variable.
- Kernel density estimation is a fundamental data smoothing problem where inferences about the population are made, based on a finite data sample.
- Bandwidth h is a free parameter. When h tends to 0 there is no smoothing. When h tends to ∞ the estimate is the shape of the kernel used rather than distribution.

Figure 11: *histogram (left) and kernel density estimate (right) constructed using the same data with six individual kernels*



- Generally Gaussian kernels are used.

Sampling Distribution:

- Sampling distribution of a statistic is when a statistic is sampled from a distribution.
- Depending on the sampling procedure, sample size and statistic the sampling distribution may vary.
- If the statistic is mean, the distribution of these means, or averages, is called the "sampling distribution of the sample mean"

Central Limit theorem:

- For many distributions, the sampling distribution of the sample mean tends to follow a *Normal distribution*.
- The distribution is $N(\mu, \frac{\sigma^2}{n})$ where variance is reduced by a factor of n where n is the sample size. The mean remains the same.
- A general rule of thumb is to have a sample size $n > 30$.
- It is important to note that the sampling statitstic used here is Mean.

Q-Q Plot: (quantile-quantile plot)

- Is a graphical method for comparing two probability distributions.
- It involves plotting the two distributions quantiles against each other.

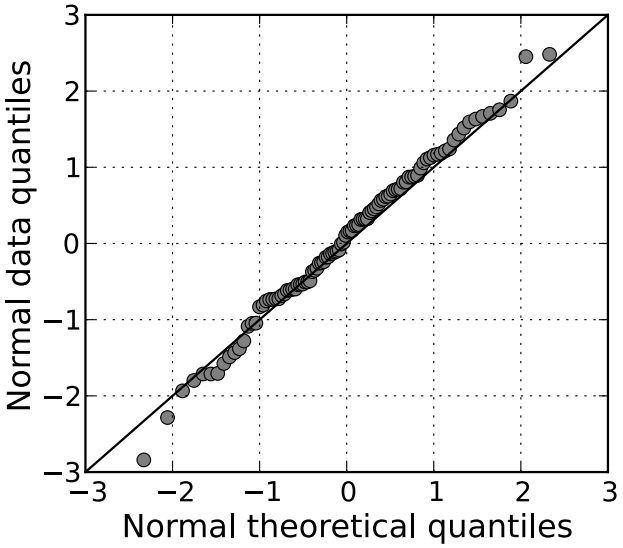
- If D^1 and D^2 are the two distributions to compare, pairs of points (D_i^1, D_i^2) are plotted.

  where:

  D_i is the i^{th} percentile of of distribution D

- If both distributions are identical, x coordinate and y coordinate will be the same and the qq plot will look like a line of equation $y = x$.

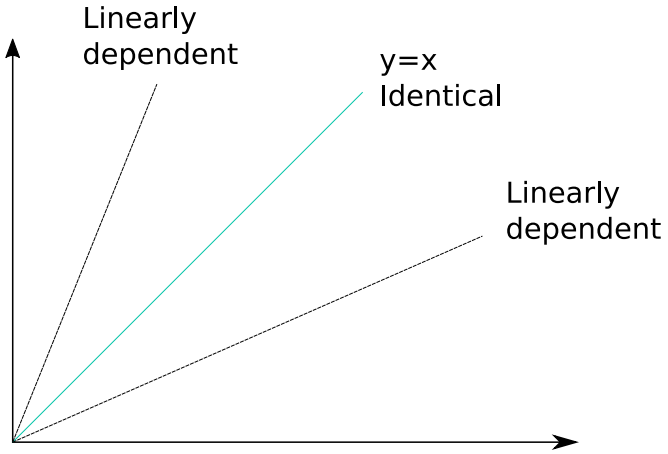
Figure 12: *qq plot of 2 normal distributions*



- If the distributions are same but different scales (pdf of same shapes but different sizes), qq plot will result in a strainght line.

- The slope of this linear relationship represents the scale difference.

Figure 13: *qq plot comparision*



- Through qq plot, experimental data can be compared to theoretical distributions.

Chebyshev's Inequality:

- For a wide range of distributions, no more than a certain fraction of values can be more than a certain distance from mean.

- In other words, there's an upper limit on the number of *extreme* values.

- When distribution is not known, but μ and σ are known and finite and σ is non zero,

$$P(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

or,

$$P(\mu - k\sigma < X < \mu + k\sigma) > 1 - \frac{1}{k^2}$$

- This is a weaker inequality than 68-95-99.7 rule of normal distribution.

Uniform Distribution:

- Equiprobable; Equal probabilities for all outcomes.

Figure 14: *uniform pmf*

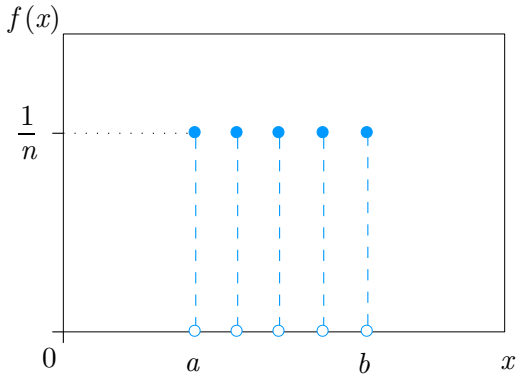
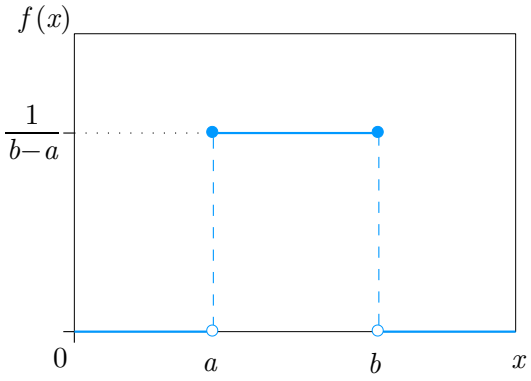


Figure 15: *uniform pdf*



Bernouli Distribution:

- Discrete probability distribution which takes value 1 with probbility p and 0 with probability $1-p$
- Yes or no outcomes; Coin toss.
- Probailities of two outcomes are success: p and failure: $q = 1 - p$

Binomial Distribution:

- Discrete probability distribution of n Bernouli trials with success probability as p
- Each Bernouli trial $\implies$ Yes or no outcome; Coin toss. Performed once (n (no. of trials) = 1).
- Pmf = $Pr(k; n, p) = {}^nC_k \quad p^k(1-p)^{n-k}$

  where, $k = 0, 1, 2, ..., n$

  no. of succeses = k and no. of trials = n

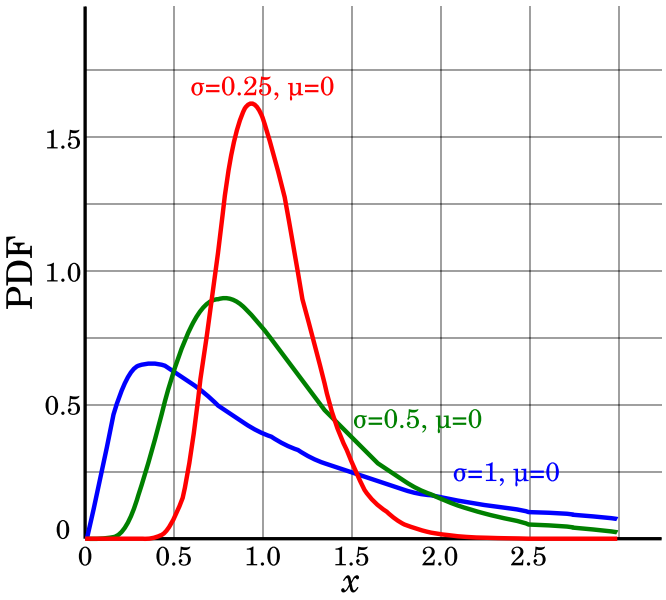
Multinomial Distribution:

- There is also *Multinomial Distribution* that is a generalization of Binomial distribution.
- A Multinomial Dicstribution can have more than 2 outcomes.
- A Binomial distribution is a Multinomial Distribution with no. of outcomes = 2 and n(trials) > 1.

Log-normal distribution:

- If X is log normal, then $\log_e(X)$ is normally distributed.

Figure 16: *log normal distributions with same $\mu = 0$ and different σ*



- High σ leads to fatter tail
- Some occurences
  - Length of comments
  - Time spent on online articles
  - Income of 97% - 99% population

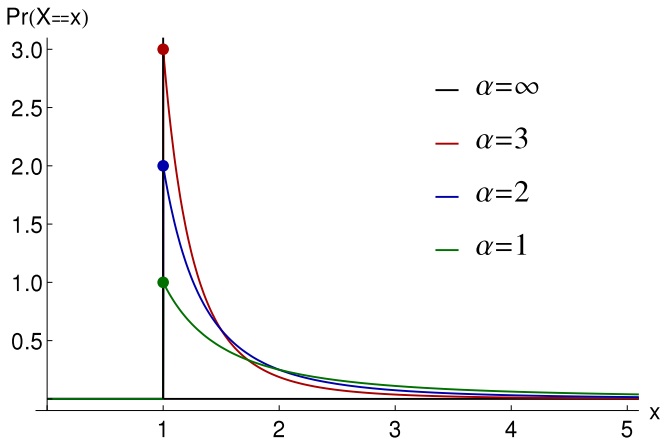
Power law:

- In statistics, a power law is a functional relationship between two quantities, where a relative change in one quantity results in a proportional relative change in the other quantity, independent of the initial size of those quantities.
- one quantity varies as a power of another.
- The area of a square in terms of the length of its side, if the length is doubled, the area is multiplied by a factor of four.

Pareto distribution:

- Is a power law distribution
- Originally invented to represent wealth distribution.
- The pareto principle or 80-20 Rule states that 80% of outcomes are due to 20% causes. Only Pareto distributions with shape value (α) of $\log_4 5 \approx 1.16$ precisely reflect it.
- The parameters are scale param x_m and shape param α

Figure 17: *Pareto Type I probability density functions for various α with $x_m = 1$.*



$$\overline{F}(x) = \Pr(X > x) = \begin{cases} \left(\frac{x_{\mathrm{m}}}{x}\right)^{\alpha} & x \geq x_{\mathrm{m}}, \\ 1 & x < x_{\mathrm{m}}, \end{cases}$$

- Low σ leads to fatter tail.

Power Transform: (Box - Cox Transform)

- Many machine learning algorithms perform better when the distribution of variables is Gaussian. (Linear regression, Naive Bayes)
- More statistical tests can be performed if dstribution follows Normal/ Gaussian.
- Power transforms are a technique for transforming numerical input or output variables to have a Gaussian or more-Gaussian-like probability distribution.
- *Box - Cox* is one of the Power Transforms. There are other transforms like Yeo-Johnson Transform.

$$y_i^{(\lambda)} = \begin{cases} \dfrac{y_i^{\lambda} - 1}{\lambda} & \text{if } \lambda \neq 0, \\ \ln y_i & \text{if } \lambda = 0, \end{cases}$$

  where, $y_1, y_2, ..., y_n$ are data points.
- The parameter λ is estimated using the profile likelihood function and using goodness-of-fit tests.

Co-Variance:

- Compares two Random variables.
- Represents the direction of variability between two RV.

$$Cov(X, Y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y) \quad (30)$$

  Where:

  μ_x : Expectation of x

  μ_y : Expectation of y
- Covariance (X, X) = Var (X)
- Cov(X, Y) = +ve when X↑ $\implies$ Y ↑
- Cov(X, Y) = +ve when X↑ $\implies$ Y ↓ , or
- Cov(X, Y) = +ve when X↓ $\implies$ Y ↑
- Co variance is susceptible by units and scale of data, similar to variance.
- The magnitude of covariance in itself cannot represent much. Only the sign indicates the direction of variation.

Figure 18: *Covariance*

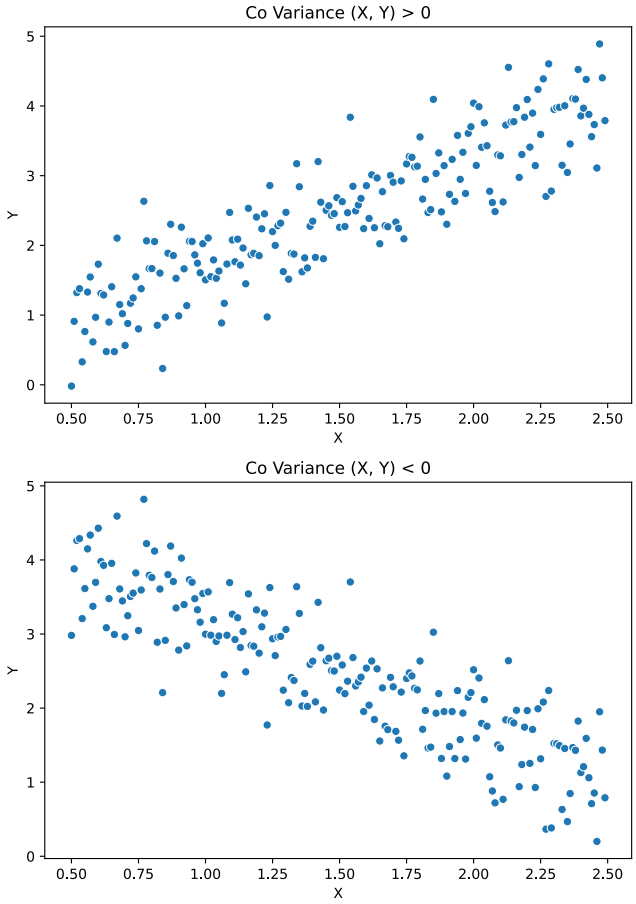


Figure 19: *Pearson Correlation ex.1*

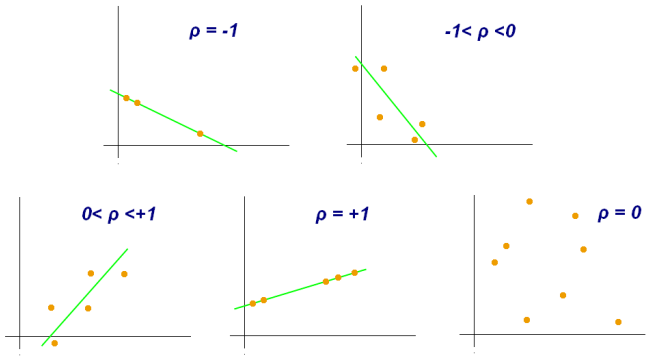
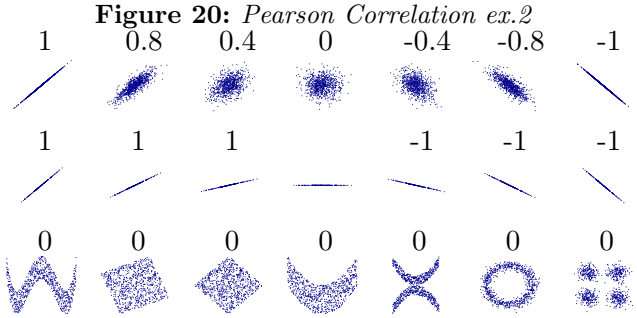


Figure 20: *Pearson Correlation ex.2*



Correlation:

- Correlation also tells us how strong the relationship is between two rv's.
- The magnitude of Correlation can represent the strength of linear dependence/ association between the variables.
- It is important before we draw any conclusions to remember that *Correlation does not imply causation.*

Pearson Correlation Coefficient: (ρ)

- It is the covariance of two variables, divided by the product of their standard deviations.
- It is a normalised version of covariance.

$$\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} \tag{31}$$

Spearman Rank Correlation: (r)

- Involves Pearson Correlation of ranks of RV.

$$r_s = \rho_{\text{r}_X,\text{r}_Y} = \frac{\text{cov}(\text{r}_X,\text{r}_Y)}{\sigma_{\text{r}_X}\sigma_{\text{r}_Y}}, \tag{32}$$

where:

r_s : Spearman rank correlation

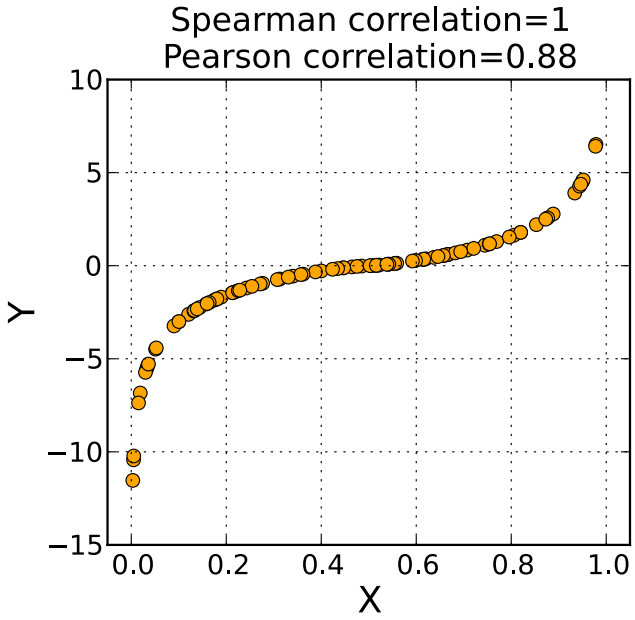
r_X : Rank of rv X

r_Y : Rank of rv Y

- Pearson's correlation assesses *linear relationships* while,
- Spearman's correlation assesses *monotonic relationships.* (Doesn't have to be linear)
- r = 1 for a strictly monotonically increasing function.

Figure 21: *Spearnman Correlation ex.1*

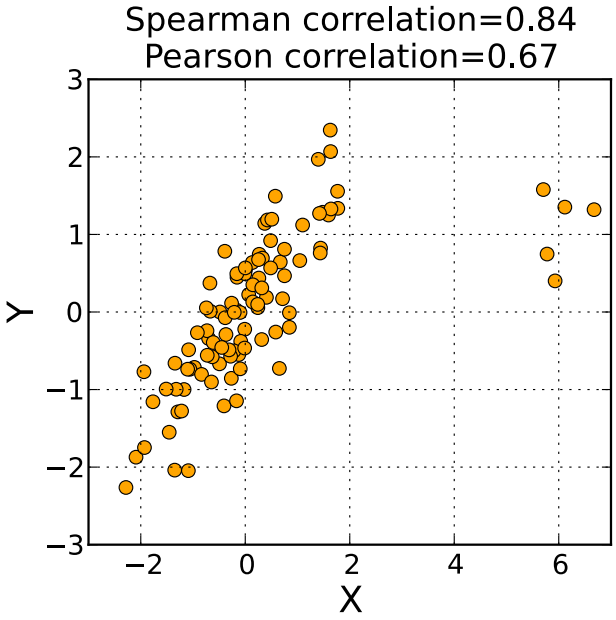
r = 1 because it is monotonically increasing.



- Spearman correlation is less sensitive to outliers because the max value it can have is its rank; Unlike in Pearson.

Figure 22: *Spearnman Correlation ex.2*

r is less sensitive than the Pearson correlation to strong outliers



- r = −1 for a strictly monotonically decreasing function.
- Ranks for each rv are given *starting from 1*, in *ascending* fashion.

	X	Y	r_X	r_Y
sample1	160	52	4	3
sample2	150	66	2	4
sample3	170	68	5	5
sample4	140	46	1	1
sample5	158	51	3	2

where:

r_X : Rank of rv X

r_Y : Rank of rv Y

Confidence Interval:

- Point estimate is 100% certainity of a value.
- C.I. tells range and probability
- Let $X \quad N(\mu, \sigma)$ and $\mu = 168$ cm $\sigma = 5$cm
  (μ-2σ,μ+2σ) - 95%
  [158, 178] with 95% probability ?!

Confidence Interval for mean μ of a r.v.: