Linear Algebra + Stats + Probability Notes

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0.1 Linear Algebra:

The following notes is primarily made for my revision. I might have skipped some topics that seemed obvious to me. I do not guarantee factual correctness of the notes. If you feel there are any errors, open a github issue/pr. Notes material is collected from various sources. Image credits are given in tex document and in src.csv

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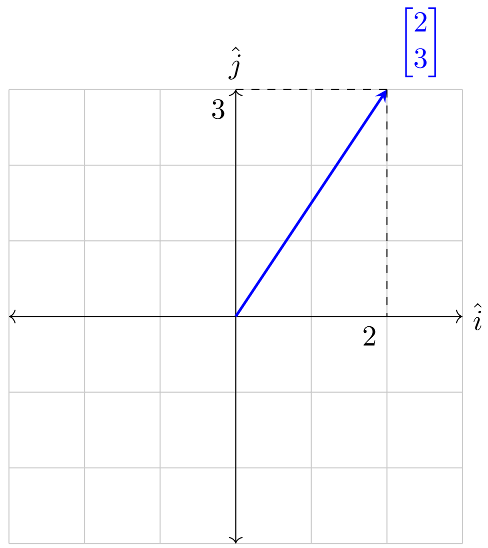
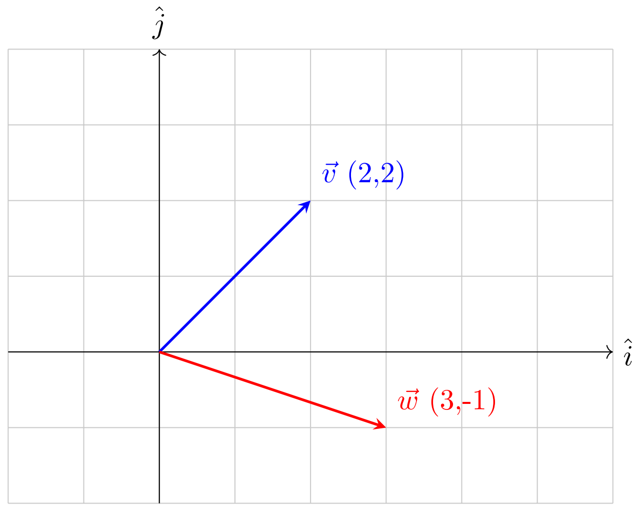
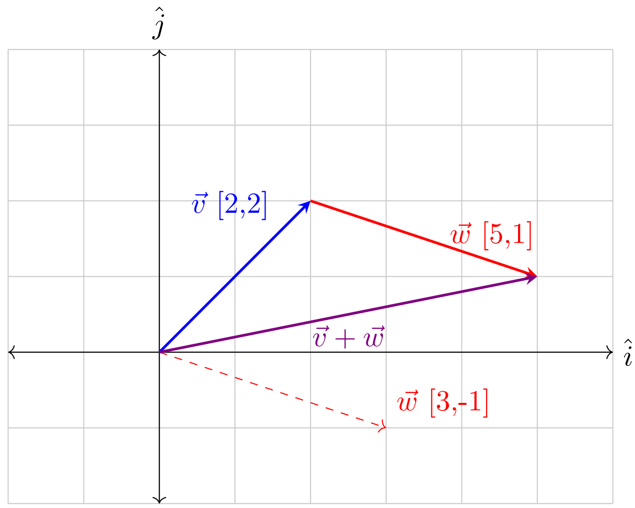
Vectors:



- This vector is represented as $2\hat{\imath}+3\hat{\jmath}$ where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors perpendicular to each other also known as basis vectors (in 2d).
- It is also represented as a column matrix $\begin{bmatrix}2\\3\end{bmatrix}$
- A vector is represented by its length and direction wrto. basis vectors $\hat{\imath}$, $\hat{\jmath}$.
- A vector can be freely moved without changing its length and the direction it is pointing to.
- So any vector in space can be represented using a linear combination of $\hat{\imath}$, $\hat{\jmath}$ by moving the starting point to origin.

Adding 2 Vectors:



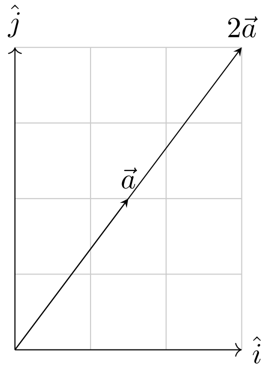


This is also known as Triangular law of vector addition.

- Can also be interpreted as,
  $\begin{bmatrix}5\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix}$
- The vector $\vec{w}$ is moved along $\vec{v}$ such that the starting point of $\vec{w}$ meets the end point of $\vec{v}$

- Both $\vec{v}$ and $\vec{w}$ direction and lengths remain unchanged.

Scaling:



- 2 *scales* the vector $\vec{a}$. So, 2 is a Scalar.
- Similarly basis vectors $\hat{i}$ and $\hat{j}$ can be scaled to represent any vector in 2d plane.

Span:

- $a\vec{v} + b\vec{w} \implies$ Linear combination of $\vec{v}$, $\vec{w}$
- Set of all vectors of linear combination of $\vec{v}$, $\vec{w}$; $a\vec{v} + b\vec{w}$ is called **span**.
- For most vectors span consists of all points on the plane.
- If $\vec{v}$, $\vec{w}$ lie on same line, span is a line passin through origin.
- If both $\vec{v}$, $\vec{w}$ are zero, span is zero.

Linear (in)dependent:

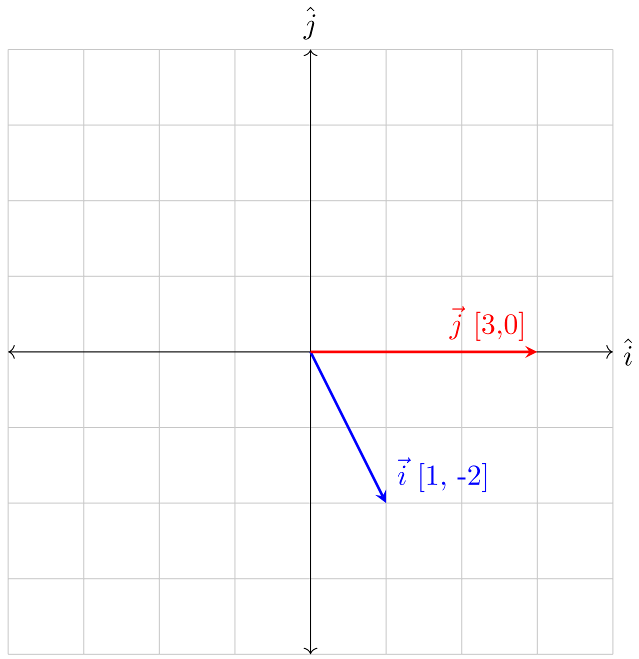
- If one vector can be represented as a linear combination of other, then the vectors are linearly dependent.
- A linearly dependent vector doesn't add to span of a vector.
- If one vector adds a dimension to a span of a vector, they are linearly independent.

Transformation:

- Takes a vector and gives a output vector.

Linear Transformation:
The conditions for the tranformation to be linear are:

- All lines must remain lines.
- Origin must be remain inplace.
- This makes all equidistant parallel lines remain equidistant and parallel.
- Matrices = Transformation of space.
- For example, the linear transform $\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ transforms the 2d plane such a way that the new $\hat{i}$ lands at $\begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix}$ and $\hat{j}$ lands at $\begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix}$



The new basis vectors are blue $\vec{i}$ and red $\vec{j}$.

Now in this new transformed space, every vector has to be represented as a linear combination of these two new basis vectors.

The vector $\vec{v}$ = -1$\hat{i}$ + 2$\hat{j}$ after transformation lands at the point $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ in the new vector space.

Even after the transformation, the linear combination doesn't change. So, $\hat{i}$ and $\hat{j}$ are replaced by $\begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix}$ and $\begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix}$ respectively.

So the new transformed $\vec{v}$ becomes,

$$\vec{v}_{new} = -1\hat{i}_{new} + 2\hat{j}_{new} \tag{1}$$

$$\vec{v}_{new} = -1 \begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix} + 2 \begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix} \tag{2}$$

$$\vec{v}_{new} = \begin{bmatrix} -1\hat{i} \\ 2\hat{j} \end{bmatrix} + \begin{bmatrix} 6\hat{i} \\ 0\hat{j} \end{bmatrix} \tag{3}$$

$$\vec{v}_{new} = \begin{bmatrix} 5\hat{i} \\ 2\hat{j} \end{bmatrix} \tag{4}$$

For any vector $x\hat{i} + y\hat{j}$, after applying the transformation $\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ becomes,

$$\begin{bmatrix} x\hat{i} \\ y\hat{j} \end{bmatrix} \overset{lr.\,transform}{\Longrightarrow} x \begin{bmatrix} 1\hat{i} \\ -2\hat{j} \end{bmatrix} + y \begin{bmatrix} 3\hat{i} \\ 0\hat{j} \end{bmatrix} = \begin{bmatrix} 1x + 3y\hat{i} \\ -2x + 0y\hat{j} \end{bmatrix} \tag{5}$$

Removing $\hat{i}$, $\hat{j}$ for legibility,

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{lr.\,transform}{\Longrightarrow} x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix} \tag{6}$$

This is the basis for Matrix multiplication.

- $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is where $\hat{i}$ lands and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is where $\hat{j}$ (basis vectors) lands after the transformation.
- So the 2X2 matrix $\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$ itself can represent the transformation.
- This explains the *rules* for multiplication and why matrix multiplication is not commutative.
- This also explains why the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix. Since, this *transform* actually does nothing, $\hat{i}$ and $\hat{j}$ remain unchanged.

For any transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{Transformation} \overbrace{\begin{bmatrix} x \\ y \end{bmatrix}}^{old\ vector} = x \overbrace{\begin{bmatrix} a \\ c \end{bmatrix}}^{where\ \hat{i}\ lands} + y \underbrace{\begin{bmatrix} b \\ d \end{bmatrix}}_{where\ \hat{j}\ lands} = \overbrace{\begin{bmatrix} ax + by \\ cx + by \end{bmatrix}}^{Transformed\ matrix} \tag{7}$$

- $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are the new basis vectors; where old $\hat{i}$ and $\hat{j}$ land. These are the new transformed basis vectors.
- Now these new basis vectors have to be used to represent all the vectors in it's span. In other words the linear combination of these two basis vectors.
- From equation 7 it can be observed that the scalars x and y scale the corresponding new basis vectors.

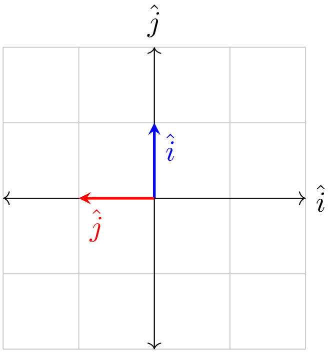
- For no transformation (or) Identity transform / Multiplication,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{x} \\ \text{y} \end{bmatrix} = \begin{bmatrix} \text{x} \\ \text{y} \end{bmatrix} \tag{8}$$

  Which makes sense !!

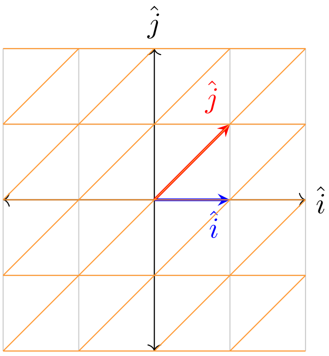
Counterclock Transform:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \text{x} \\ \text{y} \end{bmatrix}$$



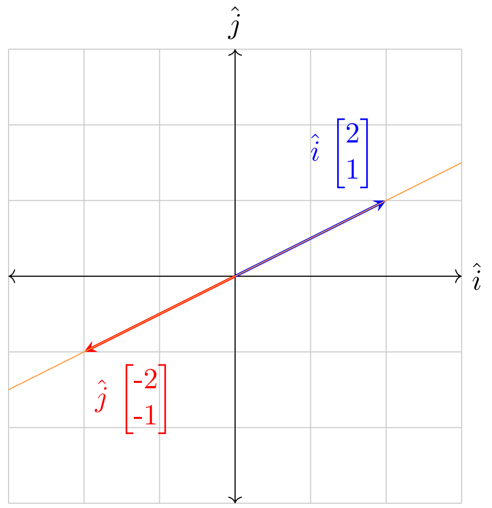
Shear Transform:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{x} \\ \text{y} \end{bmatrix}$$



Few notable points:

- In transformations, order matters. $M_1 M_2 \neq M_2 M_1$ because $f(g(x)) \neq g(f(x))$
- The associative property $(AB)C = A(BC)$ holds true because the order is C, B then A regardless.
- For a $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$ transformation all 2d space is squished into a line.



These two vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}-2\\-1\end{bmatrix}$ are linearly dependent vectors.

- Two transform for ex. shear and rotation can be composed into a single transform. This composition transform is nothing but a multiplication of shear and rotation matrix.

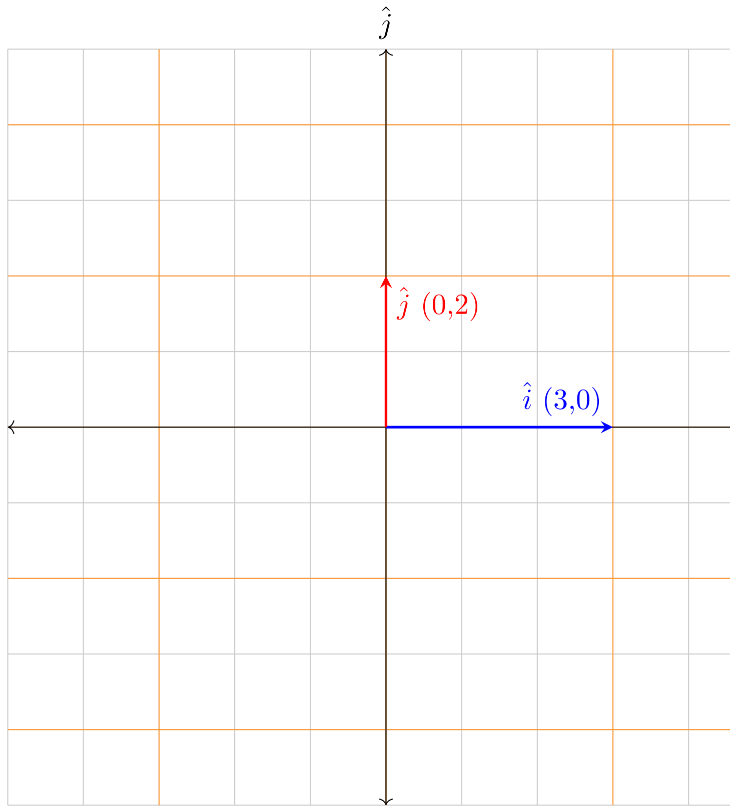
$$\underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}_{\textit{Shear}}\overbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}^{\textit{Rotation}}\begin{bmatrix}x\\y\end{bmatrix}=\underbrace{\begin{bmatrix}1 & -1\\1 & 0\end{bmatrix}}_{\textit{Compostion}}\begin{bmatrix}x\\y\end{bmatrix} \tag{9}$$

- Product or composition of two matrices/ transforms,

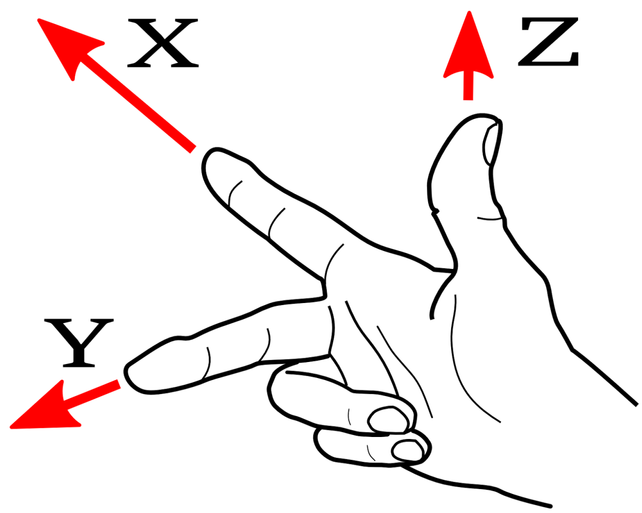
$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}e & f\\g & h\end{bmatrix}=\begin{bmatrix}ae+bg & af+bh\\ce+dg & cf+dh\end{bmatrix} \tag{10}$$

Determinant:

- Transformation $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$ scales $\hat{i}$ by factor 3 $\hat{j}$ by a factor 2.



- We observe that the unit square in transformed space is scaled from 1 sq. unit to 3 sq. units. So the determinant of the transform/matrix $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$ is 3. In case of 3d, volume is scaled.
- For a matrix/transform $\begin{bmatrix}a & b\\c & d\end{bmatrix}$, the determinat is $ad - bc$.
- The determinant can be $-ve$ if the axis cross each other during the transformation. It has the effect of *flipping* or *inverting* the space.



- This can be checked using right hand rule to check if the axis still lie in the same orientation after transformation. If not, the determinant is $-ve$.

- For 3d space, determinant is measured by unit volume instead of area.

- Determinant can be zero if the space is squished. For ex. if transformed to a line or point in 2d or to a plane, line or a point in 3d. This has the effect of reduction in number of dimensions.

$$det(M_1M_2) = det(M_1)det(M_2) \tag{11}$$

System of equations:

$A.\vec{X} = \vec{V}$

$3x + 1y + 4z = 1$
$3x + 9y + 2z = 6$
$3x + 3y + 3z = 8$

- Solving these system of equations imply, finding the vector $\begin{bmatrix} \text{x} \\ \text{y} \\ \text{z} \end{bmatrix}$ that if applied the transformation $\begin{bmatrix} 3 & 1 & 4 \\ 3 & 9 & 2 \\ 3 & 3 & 3 \end{bmatrix}$, would land on the vector $\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$.

- This can only be solved if A^{-1} exists. Since space cannot be unpacked since there is/are lost dimension(s) if $det(A) = 0$. So $det(A) \neq 0$ has to be true for the system of equations to be solved using,

$$\vec{X} = A^{-1}\vec{V} \tag{12}$$

Rank:

- Rank is the number of dimensions of the transformed space.

- The Rank of a matrix ; output space.

  Rank : 1 $\implies$ output transformation : Line

  Rank : 2 $\implies$ output transformation : Plane

  Rank : 0 $\implies$ output transformation : Point

Column space:

- Set of all possible linear combinations or span of column vectors of A. Or,

- Set of all possible $A\vec{V}$

Null space:

- Set of vectors that get squished into origin after transformation.

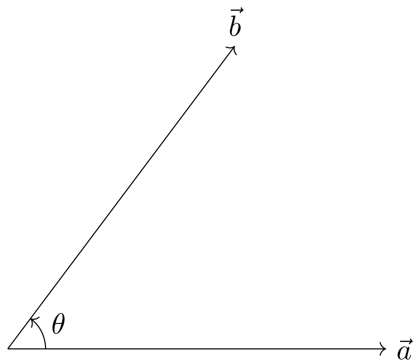
Dot Product:

- $\vec{a}$ and $\vec{b}$ are two vectors and angle between them is θ.

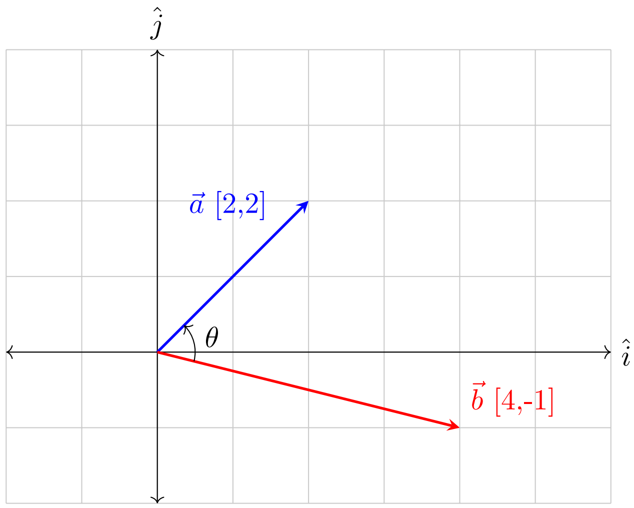
$$\vec{a} = [a_1, a_2, ...a_n] \tag{13}$$
$$\vec{b} = [b_1, b_2, ...b_n] \tag{14}$$
$$\vec{a}.\vec{b} = (a_1)*(b_1)+(a_2)*(b_2)+...+(a_n)*(b_n) \tag{15}$$

- Geometric representation:

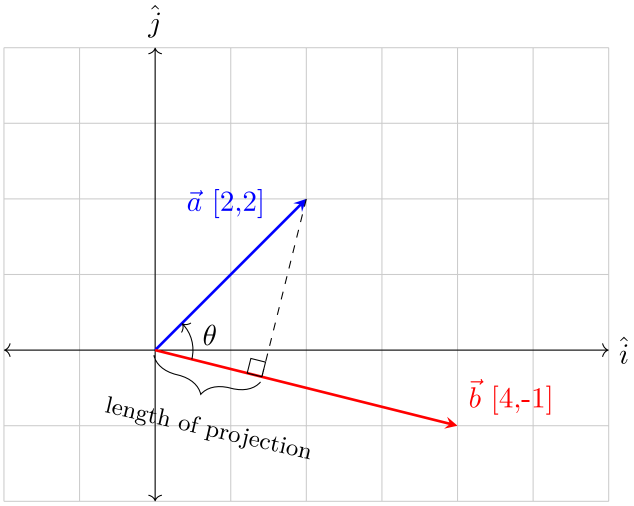
$$\vec{a}.\vec{b} = ||a|| * ||b|| * cos\theta \tag{16}$$



dot product of two vectors = (length of any vector) x (length of projection made on the vector)



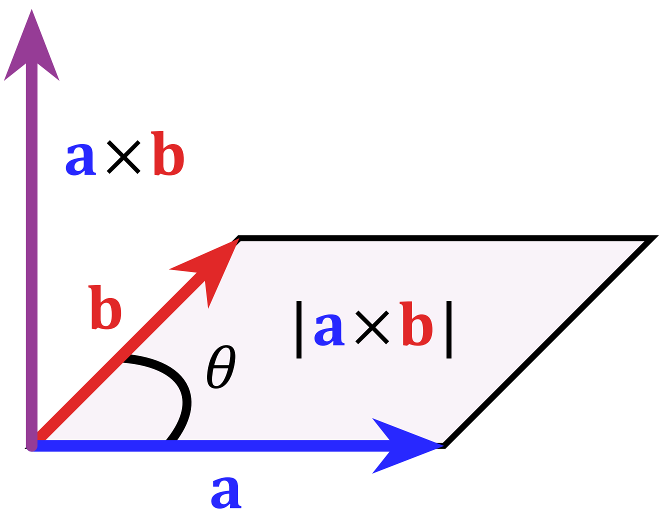
Projecting $\vec{a}$ on $\vec{b}$:



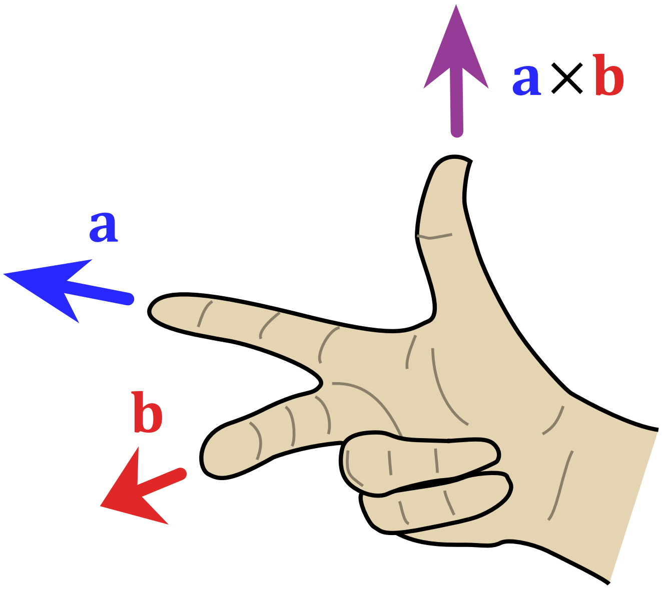
- Length of projection $= a.cos\theta$
- Dot product $= a.cos\theta.b$
- The length of projection does not depend on the length of $\vec{b}$. It only depends on length of $\vec{a}$ and θ.
- Projecting $\vec{b}$ on $\vec{a}$ will result in the same dot product result. The order does not matter.
- If projection does not lie betweeon origin and the end point of the other vector, you can extend the other vector since length of projection is not affected by it.
- Dot product is -ve if projection is on -ve side.

Cross Product:

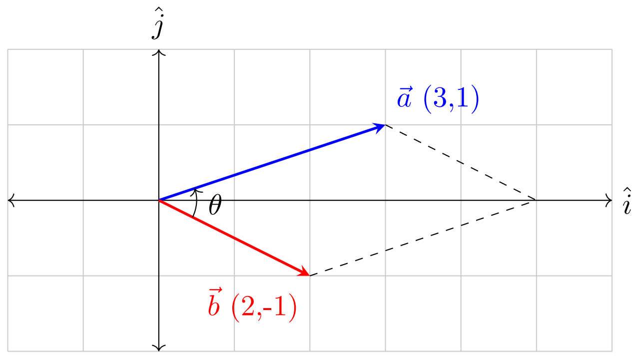
- Directed area product.



- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$



- Length of $\vec{a} \times \vec{b} = det(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix})$
- The cross product vector is normal to the area.



- The area of this parallelogram is $det(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix})$
- The direction of the cross product is normal to the area, i.e, in the direction of $\hat{k}$
- The magnitude of this vector is the area of this parallelogram i.e, determinant.

$$\vec{a} \times \vec{b} = ||a||.||b||sin\theta \tag{17}$$

Eigen:

- For a few transformations, vectors only scale. i.e, only stretch or compress.
- Such vectors are called eigen vectors of the transformation.
- Such vectors are transformed into it's own span.

- The scaled value of such vectors (scalar) is the eigen value. It can be +ve or -ve.

$$A\vec{V} = \lambda\vec{V} \tag{18}$$

where:

A is the transformation,

$\vec{V}$ is the eigen vector and,

λ is the eigen value.

Vector from origin:

- As discussed previously, any vector can be moved to origin for reference, without changing it's direction it is pointing to and it's length.
- To make a vector starting from $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and ending at $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ as a reference vector, we can apply triangular law of addition to get a vector which starts from origin and is equivalent with the vector starting and ending at the above two points.
- We need to remember that the above vectors themselves start at origin. So, we can get the vector by doing $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Matrices:

Diagonal Matrix:

- A matrix with all 0's except diagonal elements.

Identity Matrix:

- A Matrix with all diagonal elements as 1.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scalar Matrix:

- A Matrix with only equal diagonal elements.
- A scalar matrix can be represented by a single scalar.

$$K.I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Upper Triangular Matrix:

- A Matrix with all elements below diagonal are 0

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

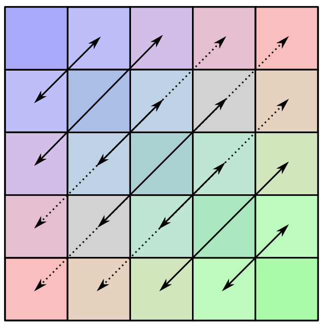
Lower Triangular Matrix:

- A Matrix with all elements above diagonal are 0

$$\begin{bmatrix} 3 & 0 & 0 \\ 7 & 2 & 0 \\ 5 & 3 & 1 \end{bmatrix}$$

Symmetric Matrix:

- A square matrix which is equal to its transpose.
- $A = A^T$
- $a_{ij} = a_{ji}$ for every i and j.



Skew Symmetric Matrix:

- A square matrix which is equal to negative of its transpose.
- $A = -A^T$
- $a_{ij} = -a_{ji}$ for every i and j.

Orthogonal Matrix:

- A square matrix whose column vectors and row vectors are orthonormal.
- $AA^T = A^T A = I$ or,
- $A^T = A^{-1}$
- $a_{ij} = -a_{ji}$ for every i and j.

Lines, Planes and Hyperplanes: