Notes on the Expanding Box Model (EBM) version of the MHD3D Code

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1 Basic equations

Conservation form of the MHD equation is used

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) \tag{1a}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} = -\nabla \cdot \left[\rho \mathbf{u}\mathbf{u} + \left(p + \frac{1}{2} B^2 \right) \mathbf{I} - \mathbf{B}\mathbf{B} \right] \tag{1b}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (-\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{1c}$$

$$\frac{\partial e}{\partial t} = -\nabla \cdot \left[\left(e + p + \frac{1}{2} B^2 \right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B} \right] \tag{1d}$$

where

$$e = \frac{p}{\kappa - 1} + \frac{1}{2} \rho u^2 + \frac{1}{2} B^2 \tag{2}$$

is the energy density.

2 Iteration of the code

The code operates in Fourier space and the conserved quantity $\mathbf{f} = (\rho, \rho\mathbf{u}, \mathbf{B}, e)$ is evolved. After each time step (sub-step in the 3rd order Runge-Kutta method), $\mathbf{f}$ is transformed back to real space and the primitive quantities $(\mathbf{u}, p)$ are calculated in real space. Then the 18 fluxes

$$\mathbf{F} = \left[\rho \mathbf{u}, \quad \rho \mathbf{u}\mathbf{u} + \left(p + \frac{1}{2} B^2 \right) \mathbf{I} - \mathbf{B}\mathbf{B}, \quad -\mathbf{u} \times \mathbf{B}, \quad \left(e + p + \frac{1}{2} B^2 \right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B} \right] \tag{3}$$

are calculated in real space and transformed to Fourier space for the time integration of $\mathbf{f}$ in Fourier space. Dealiasing is done after each update of $\mathbf{f}$ in Fourier space.

3 Implicit resistive term

The resistive term $\eta\nabla^2 B$ can be easily integrated implicitly:

$$\frac{\mathbf{B}^{n+1}-\mathbf{B}^n}{\Delta t}=\mathbf{F}-\eta k^2\mathbf{B}^{n+1} \tag{4}$$

so

$$\mathbf{B}^{n+1}=\frac{1}{1+\eta k^2\Delta t}\left(\mathbf{B}^n+\mathbf{F}\Delta t\right) \tag{5}$$

Here Δt on the denominator can be the time step of each sub-step inside Runge-Kutta method and $\mathbf{F}\Delta t$ can be of more complicated form used in RK method.

4 Expanding Box Model

Applying EBM to the MHD equation in conservation form needs much care. The mass conservation equation and the induction equation are straightforward:

$$\frac{\partial\rho}{\partial t}=-\nabla\cdot(\rho\mathbf{u})-\frac{2}{\tau}\rho \tag{6}$$

$$\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times(-\mathbf{u}\times\mathbf{B})+\eta\nabla^2\mathbf{B}-\frac{1}{\tau}\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}\mathbf{B} \tag{7}$$

Then we should be careful of the momentum equation: notice that

$$\frac{\partial(\rho\mathbf{u})}{\partial t}+\nabla\cdot(\rho\mathbf{u}\mathbf{u})=\left[\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}\right]+\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})\right]\mathbf{u} \tag{8}$$

Normally the term inside the second bracket on the RHS vanishes. However, in the EBM it doesn't because of Eq (6). That is to say, we need to include the expansion effect of the density in the momentum equation:

$$\begin{aligned}\frac{\partial(\rho\mathbf{u})}{\partial t}&=-\nabla\cdot\left[\rho\mathbf{u}\mathbf{u}+\left(p+\frac{1}{2}B^2\right)\mathbf{I}-\mathbf{B}\mathbf{B}\right]-\frac{2}{\tau}\rho\mathbf{u}-\frac{1}{\tau}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\rho\mathbf{u}\\&=-\nabla\cdot\left[\rho\mathbf{u}\mathbf{u}+\left(p+\frac{1}{2}B^2\right)\mathbf{I}-\mathbf{B}\mathbf{B}\right]-\frac{1}{\tau}\begin{pmatrix}2&0&0\\0&3&0\\0&0&3\end{pmatrix}\rho\mathbf{u}\end{aligned} \tag{9}$$

Similarly, if we expand the energy equation, we will find that the expansion term should be

$$\begin{aligned}
E_p &= -2\frac{\kappa}{\kappa-1}\frac{p}{\tau} + \frac{1}{2}u^2 \times \left(-\frac{2\rho}{\tau}\right) - \frac{1}{\tau}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\rho\mathbf{u}\right]\cdot\mathbf{u} - \frac{1}{\tau}\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{B}\right]\cdot\mathbf{B} \\
&= -2\frac{\kappa}{\kappa-1}\frac{p}{\tau} - \frac{\rho}{\tau}\left(u_x^2 + 2u_y^2 + 2u_z^2\right) - \frac{1}{\tau}\left(2B_x^2 + B_y^2 + B_z^2\right)
\end{aligned} \tag{10}$$

Note that E_p must be calculated in real space while other expansion terms can be added in Fourier space because they are linear to $\mathbf{f}$.

5 Corotating EBM

In order to study the stream-interaction region, we need to rotate the expanding coordinates with an angle α along z axis such that

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \tag{11}$$

where $\tilde{\mathbf{x}}$ is the normal expanding coordinates and $\mathbf{x}'$ is the corotating expanding coordinates. In the code, all quantities are defined on the coordinates $\mathbf{x}'$ and $\mathbf{k}'$ but the vectors are still along $\tilde{\mathbf{x}}$. The only thing we need to do is modifying the wave vector before each derivative. For example, the y derivative will be

$$\begin{aligned}
\frac{\partial}{\partial y} &= \frac{R_0}{R}\frac{\partial}{\partial \tilde{y}} \\
&= \frac{R_0}{R}\left(\frac{\partial x'}{\partial \tilde{y}}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial \tilde{y}}\frac{\partial}{\partial y'}\right) \\
&= \frac{R_0}{R}\left(-ik_x'\sin\alpha + ik_y'\cos\alpha\right)
\end{aligned} \tag{12}$$

Similarly we have

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial \tilde{x}} \\
&= \frac{\partial x'}{\partial \tilde{x}}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial \tilde{x}}\frac{\partial}{\partial y'} \\
&= ik_x'\cos\alpha + ik_y'\sin\alpha
\end{aligned} \tag{13}$$

$$\begin{aligned}
\frac{\partial}{\partial z} &= \frac{R_0}{R}\frac{\partial}{\partial z'} \\
&= \frac{R_0}{R}ik_z'
\end{aligned} \tag{14}$$

For second order derivatives, we get

$$\begin{aligned}
\partial_{xx} &= \partial_{\tilde{x}\tilde{x}} \\
&= \left(\frac{\partial x'}{\partial \tilde{x}}\right)^2\partial_{x'x'} + 2\frac{\partial x'}{\partial \tilde{x}}\frac{\partial y'}{\partial \tilde{x}}\partial_{x'y'} + \left(\frac{\partial y'}{\partial \tilde{x}}\right)^2\partial_{y'y'} \\
&= -\left({k_x'}^2\cos^2\alpha + 2\cos\alpha\sin\alpha k_x'k_y' + {k_y'}^2\sin^2\alpha\right)
\end{aligned} \tag{15a}$$

$$
\begin{aligned}
\partial_{yy} &= \left(\frac{R_0}{R}\right)^2 \partial_{\tilde{y}\tilde{y}} \\
&= \left(\frac{R_0}{R}\right)^2 \left[\left(\frac{\partial x'}{\partial \tilde{y}}\right)^2 \partial_{x'x'} + 2\frac{\partial x'}{\partial \tilde{y}}\frac{\partial y'}{\partial \tilde{y}}\partial_{x'y'} + \left(\frac{\partial y'}{\partial \tilde{y}}\right)^2 \partial_{y'y'}\right] \\
&= -\left(\frac{R_0}{R}\right)^2 \left({k'_x}^2 \sin^2\alpha - 2\cos\alpha\sin\alpha k'_x k'_y + {k'_y}^2 \cos^2\alpha\right)
\end{aligned} \tag{15b}
$$

$$
\begin{aligned}
\partial_{zz} &= \left(\frac{R_0}{R}\right)^2 \partial_{\tilde{z}\tilde{z}} \\
&= -\left(\frac{R_0}{R}\right)^2 {k'_z}^2
\end{aligned} \tag{15c}
$$

The only place that needs the second order derivatives is the resistive term which needs the Laplacian:

$$
\begin{aligned}
\nabla^2 &= \partial_{xx} + \partial_{yy} + \partial_{zz} \\
&= -{k'_x}^2 \left[\cos^2\alpha + \left(\frac{R_0}{R}\right)^2 \sin^2\alpha\right] - {k'_y}^2 \left[\sin^2\alpha + \left(\frac{R_0}{R}\right)^2 \cos^2\alpha\right] \\
&\quad - 2k'_x k'_y \cos\alpha\sin\alpha \left[1 - \left(\frac{R_0}{R}\right)^2\right] - {k'_z}^2 \left(\frac{R_0}{R}\right)^2
\end{aligned} \tag{16}
$$

6 Hall term

Adding the Hall term requires the calculation of the current density. In each time step (sub-step), before calculating the flux, we first calculate the current density in Fourier space, which is simply:

$$
\hat{\mathbf{J}} = i\mathbf{k} \times \hat{\mathbf{B}} \tag{17}
$$

Then we do inverse Fourier transform to get $\mathbf{J}$ in real space. The next step is simply add the following term

$$
\mathbf{E}_h = d_i \frac{\mathbf{J} \times \mathbf{B}}{\rho} \tag{18}
$$

to the electric field when calculating the flux functions. We also need to modify the time-step estimate by adding a Hall-related speed:

$$
c_h \sim \frac{d_i\,|B|}{\rho\Delta x} \tag{19}
$$

which is very large in the case of small Δx.