Euler inversion

$$f_i(\bar{d}, \bar{p}) = (x_i - x_0)\partial_x h_i + (y_i - y_0)\partial_y h_i + (z_i - z_0)\partial_z h_i + \eta(h_i - b) = 0$$

$$\bar{d} = \begin{bmatrix} h_1 \\ \vdots \\ h_L \\ \partial_x h_1 \\ \vdots \\ \partial_x h_L \\ \partial_y h_1 \\ \vdots \\ \partial_y h_L \\ \partial_z h_1 \\ \vdots \\ \partial_z h_L \end{bmatrix} \qquad \bar{p} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ b \end{bmatrix}$$

$N = 3L \times 1$ $\qquad$ $M = 4 \times 1$

$$\Gamma(\bar{d}, \bar{p}, \bar{\lambda}) = \underbrace{[\bar{d}^o - \bar{d}]^T \bar{\bar{W}}_d [\bar{d}^o - \bar{d}]}_{\phi(\bar{d})} + \underbrace{\mu\, \bar{p}^T \bar{\bar{W}}_p \bar{p}}_{\psi(\bar{p})} + \bar{\lambda}^T \bar{f}(\bar{d}, \bar{p})$$

$\phi(\bar{d})$ → Data misfit

$\psi(\bar{p})$ → Damping

Jacobians, Hessians, gradients

$$\bar{\bar{H}}_p = \bar{\bar{\nabla}}_p \psi = 2\mu \bar{\bar{W}}_p \qquad \bar{\bar{H}}_d = \bar{\bar{\nabla}}_d \phi = 2 \bar{\bar{W}}_d$$

$$\bar{\nabla}_p \psi = 2\mu \bar{\bar{W}}_p \bar{p} \qquad \bar{\nabla}_d \phi = -2 \bar{\bar{W}}_d [\bar{d}^o - \bar{d}]$$

$$\bar{\bar{A}} = \begin{bmatrix} -\partial_x h_1 & -\partial_y h_1 & -\partial_z h_1 & -\eta \\ \vdots & \vdots & \vdots & \vdots \\ -\partial_x h_L & -\partial_y h_L & -\partial_z h_L & -\eta \end{bmatrix} \Longrightarrow \text{predicted (not observed)}$$

$L \times M$

$$\bar{\bar{B}} = \begin{bmatrix} \eta & 0 & \cdots & 0 & (x_1 - x_0) & 0 & \cdots & 0 & (y_1 - y_0) & 0 & \cdots & 0 & (z_1 - z_0) & 0 & \cdots & 0 \\ 0 & \eta & \cdots & 0 & 0 & (x_2 - x_0) & \cdots & 0 & 0 & (y_2 - y_0) & \cdots & 0 & 0 & (z_2 - z_0) & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \eta & 0 & 0 & \cdots & (x_L - x_0) & 0 & 0 & \cdots & (y_L - y_0) & 0 & 0 & \cdots & (z_L - z_0) \end{bmatrix}$$

$L \times N$

Iterations

Start from $\Rightarrow$

$\bar{p}_0 \Rightarrow$ mid-point of survey

$\bar{d}_0 \Rightarrow$ Null or $\bar{d}^o$ (?)

$$\bar{\Delta p} = \bar{p}_{k+1} - \bar{p}_k \qquad \bar{\Delta d} = \bar{d}_{k+1} - \bar{d}_k$$

$$\left[\bar{\bar{A}}_k^T \underbrace{\left[\bar{\bar{B}}_k \bar{\bar{W}}_d^{-1} \bar{\bar{B}}_k^T\right]^{-1}} \bar{\bar{A}}_k + \mu \bar{\bar{W}}_p\right] \bar{\Delta p} = \bar{\bar{A}}_k^T \left[\bar{\bar{B}}_k \bar{\bar{W}}_d^{-1} \bar{\bar{B}}_k^T\right]^{-1} \left[-\bar{\bar{B}}_k \bar{\bar{W}}_d^{-1} \bar{\bar{W}}_d [\bar{d}^o - \bar{d}_k] - \bar{f}_k\right] - \mu \bar{\bar{W}}_p \bar{p}_k$$

$$\bar{\bar{Q}}_k = \left[\bar{\bar{B}}_k \bar{\bar{W}}_d^{-1} \bar{\bar{B}}_k^T\right]^{-1}$$

$$\boxed{\left[\bar{\bar{A}}_k^T \bar{\bar{Q}}_k \bar{\bar{A}} + \mu \bar{\bar{W}}_p\right] \bar{\Delta p} = -\bar{\bar{A}}_k^T \bar{\bar{Q}}_k \left[\bar{f}_k + \bar{\bar{B}}_k [\bar{d}^o - \bar{d}_k]\right] - \mu \bar{\bar{W}}_p \bar{p}_k}$$

$$\bar{\Delta d} = -\bar{\bar{W}}_d^{-1} \left[\bar{\bar{B}}_k^T \bar{\bar{Q}}_k \bar{\bar{B}}_k \bar{\bar{W}}_d^{-1} - \bar{\bar{I}}\right] \bar{\bar{W}}_d [\bar{d}^o - \bar{d}_k] - \bar{\bar{W}}_d^{-1} \bar{\bar{B}}_k^T \bar{\bar{Q}}_k [\bar{\bar{A}}_k \bar{\Delta p} + \bar{f}_k]$$

$$\boxed{\bar{\Delta d} = \left[\bar{\bar{I}} - \bar{\bar{W}}_d^{-1} \bar{\bar{B}}_k^T \bar{\bar{Q}}_k \bar{\bar{B}}_k\right] [\bar{d}^o - \bar{d}_k] - \bar{\bar{W}}_d^{-1} \bar{\bar{B}}_k^T \bar{\bar{Q}}_k [\bar{f}_k + \bar{\bar{A}}_k \bar{\Delta p}]}$$

Provides a solution for $\bar{p}$ and predicted data $\bar{d}$ that satisfies Euler's equation.

Control the step size

$$\bar{\bar{Q}}_k = \left[\bar{\bar{B}}_k \left[(1+\alpha) \bar{\bar{W}}_d\right]^{-1} \bar{\bar{B}}_k^T\right]^{-1}$$

$$\left[\bar{\bar{A}}_k \bar{\bar{Q}}_k \bar{\bar{A}} + (\mu+\alpha) \bar{\bar{W}}_p\right] \bar{\Delta p} = -\bar{\bar{A}}_k^T \bar{\bar{Q}}_k \left[\bar{f}_k + \bar{\bar{B}}_k [\bar{d}^o - \bar{d}_k]\right] - \mu \bar{\bar{W}}_p \bar{p}_k$$

$$\bar{\Delta d} = \left[\bar{\bar{I}} - \left[(1+\alpha) \bar{\bar{W}}_d\right]^{-1} \bar{\bar{B}}_k^T \bar{\bar{Q}}_k \bar{\bar{B}}_k\right] [\bar{d}^o - \bar{d}_k] - \left[(1+\alpha) \bar{\bar{W}}_d\right]^{-1} \bar{\bar{B}}_k \bar{\bar{Q}}_k \left[\bar{f}_k + \bar{\bar{A}}_k \bar{\Delta p}\right]$$