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# POPULATION AND QUANTITATIVE GENETICS

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The L<sup>A</sup>T<sub>E</sub>X code and R code for this book are kept here <https://github.com/cooplab/popgen-notes/> and again are  
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# 1

## *Introduction*

EVOLUTION IS CHANGE OVER TIME. Biological evolution is the change over time in the genetic composition of a population.<sup>1</sup> Our population is made up of a set of interbreeding individuals, the genetic composition of which is made up of the genomes that each individual carries. While at first this definition of evolution seems at odds with the common textbook view of the evolution of phenotypes, such as the changing shape of finch beaks over generations, it is genetic changes that underpin these phenotypic changes.

The genetic composition of the population can alter due to the death of individuals or the migration of individuals in or out of the population. If our individuals vary in the number of children they have, this also alters the genetic composition of the population in the next generation. Every new individual born into the population subtly changes the genetic composition of the population. Their genome is a unique combination of their parents' genomes, having been shuffled by segregation and recombination during meioses, and possibly changed by mutation. These individual events seem minor at the level of the population, but it is the accumulation of small changes in aggregate across individuals and generations that is the stuff of evolution. It is the compounding of these small changes over tens, hundreds, and millions of generations that drives the amazing diversity of life that has emerged on this earth.

Population genetics is the study of the genetic composition of natural populations and its evolutionary causes and consequences. Quantitative genetics is the study of the genetic basis of phenotypic variation and how phenotypic changes can evolve. Both fields are closely conceptually aligned as we'll see throughout these notes. They seek to describe how the genetic and phenotypic composition of populations can be changed over time by the forces of mutation, recombination, selection, migration, and genetic drift. To understand how these forces interact, it is helpful to develop simple theoretical models to help our

<sup>1</sup> DOBZHANSKY, T., 1951 *Genetics and the Origin of Species* (3rd Ed. ed.), pp. 16

intuition. In these notes we will work through these models and summarize the major areas of population- and quantitative-genetic theory.

While the models we will develop will seem naïve, and indeed they are, they are nonetheless incredibly useful and powerful. Throughout the course we will see that these simple models often yield accurate predictions, such that much of our understanding of the process of evolution is built on these models. We will also see how these models are incredibly useful for understanding real patterns we see in the evolution of phenotypes and genomes, such that much of our analysis of evolution, in a range of areas from human medical genetics to conservation, is based on these models. Therefore, population and quantitative genetics are key to understanding various applied questions, from how medical genetics identifies the genes involved in disease to how we preserve species from extinction.

Population genetics emerged from early efforts to reconcile Mendelian genetics with Darwinian thought. Part of the power of population genetics comes from the fact that the basic rules of transmission genetics are simple and nearly universal. One of the truly remarkable things about population genetics is that many of the important ideas and mathematical models emerged before the 1940s, long before the mechanistic-basis of inheritance (DNA) was discovered, and yet the usefulness of these models has not diminished. This is a testament to the fact that the models are established on a very solid foundation, building from the basic rules of genetic transmission combined with simple mathematical and statistical models.

Much of this early work traces to the ideas of R.A. Fisher, Sewall Wright, and J.B.S. Haldane, who, along with many others, described the early principals and mathematical models underlying our understanding of the evolution of populations. Building on this conceptual fusion of genetics and evolution, there followed a flourishing of evolutionary thought, the modern evolutionary synthesis, combining these ideas with those from the study of speciation, biodiversity, and paleontology. In total this work showed that both short-term evolutionary change and the long-term evolution of biodiversity could be well understood through the gradual accumulation of evolutionary change within and among populations. This evolutionary synthesis continues to this day, combining new insights from genomics, phylogenetics, ecology, and developmental biology.

Population and quantitative genetics are a necessary but not a sufficient description of evolution; it is only by combining the insights of many fields that a rich and comprehensive picture of evolution emerges. We certainly do not need to know the genes underlying the displays of the birds of paradise to study how the divergence of these displays, due to sexual selection, may drive speciation. Indeed, as we'll

"All models are wrong but some are useful" - Box (1979).

See PROVINE (2001) for a history of early population genetics.

PROVINE, W. B., 2001 *The origins of theoretical population genetics: with a new afterword.* University of Chicago Press

"DOBZHANSKY (1951) once defined evolution as 'a change in the genetic composition of the populations' an epigram that should not be mistaken for the claim that everything worth saying about evolution is contained in statements about genes"

- LEWONTIN

see in our discussion of quantitative genetics, we can predict how populations respond to selection, including sexual selection and assortative mating, without any knowledge of the loci involved. Nor do we need to know the precise selection pressures and the ordering of genetic changes to study the emergence of the tetrapod body plan. We do not necessarily need to know all the genetic details to appreciate the beauty of these, and many other, evolutionary case-studies. However, every student of biology gains from understanding the basics of population and quantitative genetics, allowing them to base their studies and speculations on a solid bedrock of understanding of the processes that underpin all evolutionary change.



# 2

## Allele and Genotype Frequencies

In this chapter we will work through how the basics of Mendelian genetics play out at the population level in sexually reproducing organisms.

Loci and alleles are the basic currency of population genetics—and indeed of genetics. If all individuals in the population carry the same allele, we say that the locus is *monomorphic*; at this locus there is no genetic variability in the population. If there are multiple alleles in the population at a locus, we say that this locus is *polymorphic* (this is sometimes referred to as a segregating site).

Table 2.1 show a small stretch orthologous sequence for the ADH locus from samples from *Drosophila melanogaster*, *D. simulans*, and *D. yakuba*. *D. melanogaster* and *D. simulans* are sister species and *D. yakuba* is a close outgroup to the two. Each column represents a single haplotype from an individual (the individuals are diploid but were inbred so they're homozygous for their haplotype). Only sites that differ among individuals of the three species are shown. Site 834 is an example of a polymorphism; some *D. simulans* individuals carry a *C* allele while others have a *T*. Fixed differences are sites that differ between the species but are monomorphic within the species. Site 781 is an example of a fixed difference between *D. melanogaster* and the other two species.

We can also annotate the alleles and loci in various ways. For example, position 781 is a non-synonymous fixed difference. We call the less common allele at a polymorphism the *minor allele* and the common allele the *major allele*, e.g. at site 1068 the *T* allele is the minor allele in *D. melanogaster*. We call the more evolutionarily recent of the two alleles the *derived allele* and the older of the two the *ancestral allele*. The *T* allele at site 1068 is the derived allele as the *C* is found in both the other species, suggesting that the *T* allele arose via a *C* → *T* mutation.

**Question 1. A)** How many segregating sites does the sample

A *locus* (plural: *loci*) is a specific spot in the genome. A locus may be an entire gene, or a single nucleotide base pair such as A-T. At each locus, there may be multiple genetic variants segregating in the population—these different genetic variants are known as *alleles*.

pos.	con.	a	b	c	d	e	f	g	h	i	j	k	l	a	b	c	d	e	f	g	h	i	j	k	l	NS/S
781	G	T	T	T	T	T	T	T	T	T	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	NS
789	T	-	-	-	-	-	-	-	-	-	-	-	-	C	C	C	C	C	C	C	C	C	C	C	C	S
808	A	-	-	-	-	-	-	-	-	-	T	T	T	G	G	G	G	G	G	G	G	G	G	G	NS	
816	G	T	T	T	T	-	-	-	-	-	-	-	C	C	-	-	-	-	-	-	-	-	-	-	S	
834	T	-	-	-	-	-	-	-	-	-	-	-	C	-	-	-	-	-	-	-	-	-	-	-	S	
859	C	-	-	-	-	-	-	-	-	-	-	-	G	G	G	G	G	G	G	G	G	G	G	G	NS	
867	C	-	-	-	-	-	-	-	-	-	-	-	G	G	G	G	G	A	G	G	G	G	G	G	S	
870	C	T	T	T	T	T	T	T	T	T	-	-	-	-	-	-	-	-	-	-	-	-	-	-	S	
950	G	-	-	-	-	-	-	-	-	-	-	-	A	-	-	-	-	-	-	-	-	-	-	-	S	
974	G	-	-	-	-	-	-	-	-	-	T	-	T	T	T	T	-	-	-	-	-	-	-	-	S	
983	T	-	-	-	-	-	-	-	-	-	-	-	C	C	C	C	C	C	C	C	C	C	C	C	S	
1019	C	-	-	-	-	-	-	-	-	-	-	-	-	-	-	A	-	-	-	-	-	-	-	-	S	
1031	C	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	A	-	-	-	-	-	-	-	S	
1034	T	-	-	-	-	-	-	-	-	-	-	-	C	C	C	C	C	-	C	-	C	C	C	S		
1043	C	-	-	-	-	-	-	-	-	-	-	-	-	-	-	A	-	-	-	-	-	-	-	S		
1068	C	T	T	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	S	
1089	C	-	-	-	-	-	-	-	-	-	A	A	A	A	A	A	-	-	-	-	-	-	-	-	NS	
1101	G	-	-	-	-	-	-	-	-	-	-	-	A	A	A	A	A	A	A	A	A	A	A	A	NS	
1127	T	-	-	-	-	-	-	-	-	-	-	-	C	C	C	C	C	C	C	C	C	C	C	C	S	
1131	C	-	-	-	-	-	-	-	-	-	-	-	-	-	-	T	-	-	-	-	-	-	-	-	S	
1160	T	-	-	-	-	-	-	-	-	-	-	-	C	C	C	C	C	C	C	C	C	C	C	C	S	

from *D. simulans* have in the ADH gene?

**B)** How many fixed differences are there between *D. melanogaster* and *D. yakuba*?

## 2.1 Allele frequencies

Allele frequencies are a central unit of population genetics analysis, but from diploid individuals we only get to observe genotype counts. Our first task then is to calculate allele frequencies from genotype counts. Consider a diploid autosomal locus segregating for two alleles ( $A_1$  and  $A_2$ ). We'll use these arbitrary labels for our alleles, merely to keep this general. Let  $N_{11}$  and  $N_{12}$  be the number of  $A_1A_1$  homozygotes and  $A_1A_2$  heterozygotes, respectively. Moreover, let  $N$  be the total number of diploid individuals in the population. We can then define the relative frequencies of  $A_1A_1$  and  $A_1A_2$  genotypes as  $f_{11} = N_{11}/N$  and  $f_{12} = N_{12}/N$ , respectively. The frequency of allele  $A_1$  in the population is then given by

$$p = \frac{2N_{11} + N_{12}}{2N} = f_{11} + \frac{1}{2}f_{12}. \quad (2.1)$$

Note that this follows directly from how we count alleles given individuals' genotypes, and holds independently of Hardy–Weinberg proportions and equilibrium (discussed below). The frequency of the alternate allele ( $A_2$ ) is then just  $q = 1 - p$ .

### 2.1.1 Measures of genetic variability

**Nucleotide diversity ( $\pi$ )** One common measure of genetic diversity is the average number of single nucleotide differences between haplotypes chosen at random from a sample. This is called nucleotide diversity and is often denoted by  $\pi$ . For example, we can calculate  $\pi$  for our ADH locus from Table 2.1 above: we have 6 sequences from *D. simulans* (a-f), there's a total of 15 ways of pairing these sequences, and

Table 2.1: Variable sites in exons 2 and 3 of the ADH gene in *Drosophila* McDONALD and KREITMAN (1991). The first column (pos.) gives the position in the gene; exon 2 begins at position 778 and we've truncated the dataset at site 1175. The second column gives the consensus nucleotide (con.), i.e. the most common base at that position; individuals with nucleotides that match the consensus are marked with a dash. The first columns of sequence (a-l) are from *D. melanogaster*; the next columns (a-f) give sequences from *D. simulans*, and the final set of columns (a-l) from *D. yakuba*. The last column shows whether the difference is a non-synonymous (N) or synonymous (S) change.

$$\pi = \frac{1}{15} ((2+1+1+1+0)+(3+3+3+2)+(0+0+1)+(0+1)+(1)) = 1.2\bar{6} \quad (2.2)$$

where the first bracketed term gives the pairwise differences between a and b-f, the second bracketed term the differences between b and c-f and so on.

Our  $\pi$  measure will depend on the length of sequence it is calculated for. Therefore,  $\pi$  is usually normalized by the length of sequence, to be a per site (or per base) measure. For example, our ADH sequence covers 397bp of DNA and so  $\pi = 1.2\bar{6}/397 = 0.0032$  per site in *D. simulans* for this region. Note that we could also calculate  $\pi$  per synonymous site (or non-synonymous). For synonymous site  $\pi$ , we would count up number of synonymous differences between our pairs of sequences, and then divide by the total number of sites where a synonymous change could have occurred.<sup>1</sup>

*Number of segregating sites.* Another measure of genetic variability is the total number of sites that are polymorphic (segregating) in our sample. One issue is that the number of segregating sites will grow as we sequence more individuals (unlike  $\pi$ ). Later in the course, we'll talk about how to standardize the number of segregating sites for the number of individuals sequenced (see eqn (3.39)).

*The frequency spectrum.* We also often want to compile information about the frequency of alleles across sites. We call alleles that are found once in a sample *singletons*, alleles that are found twice in a sample *doubletons*, and so on. We count up the number of loci where an allele is found  $i$  times out of  $n$ , e.g. how many singletons are there in the sample, and this is called the *frequency spectrum*. We'll want to do this in some consistent manner, so we often calculate the minor allele frequency spectrum, or the frequency spectrum of derived alleles.

**Question 2.** How many minor-allele singletons are there in *D. simulans* in the ADH region?

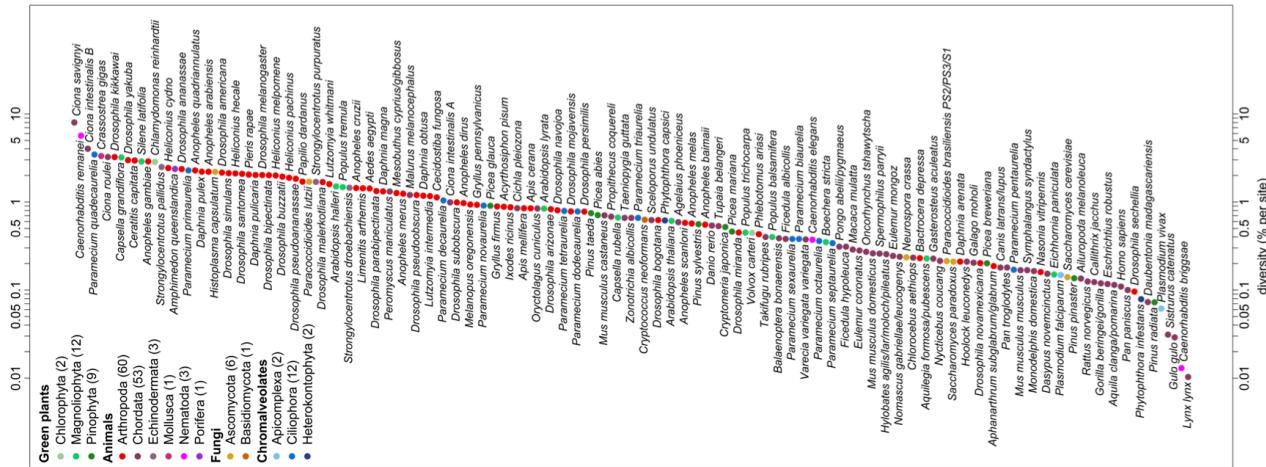
*Levels of genetic variability across species.* Two observations have puzzled population geneticists since the inception of molecular population genetics. The first is the relatively high level of genetic variation observed in most obligately sexual species. This first observation, in part, drove the development of the Neutral theory of molecular evolution, the idea that much of this molecular polymorphism may simply reflect a balance between genetic drift and mutation. The second observation is the relatively narrow range of polymorphism across species

<sup>1</sup> Technically we would need to divide by the total number of possible point mutations that would result in a synonymous change; this is because some mutational changes at a particular nucleotide will result in a non-synonymous or synonymous change depending on the base-pair change.

students were confused whether you are defining 'minor allele' with reference to the species or to the whole group. There are no major allele singletons with reference to the individual species, but I think it's confusing because the only example of a minor allele given in the text by its definition is the 'minor allele in *D. melanogaster*' ... maybe it would be best to clarify in the question or just give an example of the minor allele in the whole sample instead.

with vastly different census sizes. This observation represented a puzzle as Neutral theory predicts that levels of genetic diversity should scale population size. Much effort in theoretical and empirical population genetics has been devoted to trying to reconcile models with these various observations. We'll return to discuss these ideas throughout our course.

The first observations of molecular genetic diversity within natural populations were made from surveys of allozyme data, but we can revisit these general patterns with modern data.



For example, LEFFLER *et al.* (2012) compiled data on levels of within-population, autosomal nucleotide diversity ( $\pi$ ) for 167 species across 14 phyla from non-coding and synonymous sites (Figure 2.2). The species with the lowest levels of  $\pi$  in their survey was Lynx, with  $\pi = 0.01\%$ , i.e. only 1/10000 bases differed between two sequences. In contrast, some of the highest levels of diversity were found in *Ciona savignyi*, Sea Squirts, where a remarkable 1/12 bases differ between pairs of sequences. This 800-fold range of diversity seems impressive, but census population sizes have a much larger range.

### 2.1.2 Hardy–Weinberg proportions

Imagine a population mating at random with respect to genotypes, i.e. no inbreeding, no assortative mating, no population structure, and no sex differences in allele frequencies. The frequency of allele  $A_1$  in the population at the time of reproduction is  $p$ . An  $A_1A_1$  genotype is made by reaching out into our population and independently drawing two  $A_1$  allele gametes to form a zygote. Therefore, the probability that an individual is an  $A_1A_1$  homozygote is  $p^2$ . This probability is also the expected frequencies of the  $A_1A_1$  homozygote in the popula-

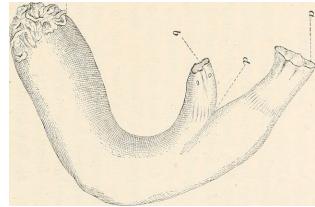


Figure 2.1: Sea Squirt (*Ciona intestinalis*). Einleitung in die vergleichende gehirnphysiologie und Vergleichende psychologie. Loeb, J. 1899.

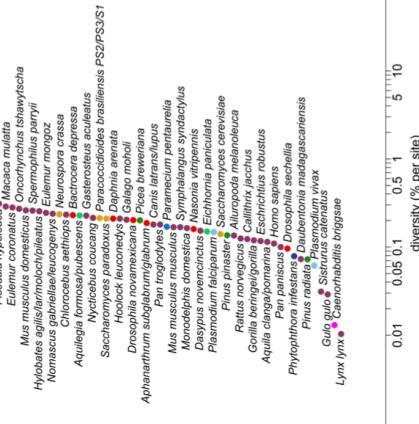


Figure 2.2: Levels of autosomal nucleotide diversity for 167 species across 14 phyla. Figure 1 from LEFFLER *et al.* (2012). Points are ranked by their  $\pi$ , and coloured by their phylum. Note the log-scale.

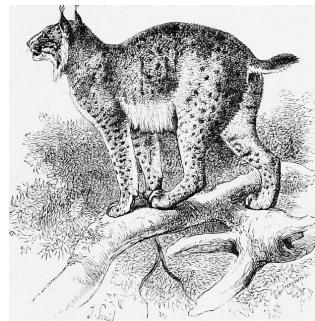


Figure 2.3: Eurasian Lynx (*Lynx lynx*). An introduction to the study of mammals living and extinct. Flower, W.H. and Lydekker, R. 1891.

tion. The expected frequency of the three possible genotypes are

$$\begin{array}{ccc} f_{11} & f_{12} & f_{22} \\ \hline p^2 & 2pq & q^2 \end{array}$$

Note that we only need to assume random mating with respect to our focal allele in order for these expected frequencies to hold in the zygotes forming the next generation. Evolutionary forces, such as selection, change allele frequencies within generations, but do not change this expectation for new zygotes, as long as  $p$  is the frequency of the  $A_1$  allele in the population at the time when gametes fuse.

**Question 3.** On the coastal islands of British Columbia there is a subspecies of black bear (*Ursus americanus kermodei*, Kermode's bear). Many members of this black bear subspecies are white; they're sometimes called spirit bears. These bears aren't hybrids with polar bears, nor are they albinos. They are homozygotes for a recessive change at the MC1R gene. Individuals who are *GG* at this SNP are white while *AA* and *AG* individuals are black.

Below are the genotype counts for the MC1R polymorphism in a sample of bears from British Columbia's island populations from RITLAND *et al.*

<i>AA</i>	<i>AG</i>	<i>GG</i>
42	24	21

What are the expected frequencies of the three genotypes under HWE?

See Figure 2.5 for a nice empirical demonstration of Hardy-Weinberg proportions. The mean frequency of each genotype closely match their HW expectations, and much of the scatter of the dots around the expected line is due to our small sample size ( $\sim 60$  individuals). While HW often seems like a silly model, it often holds remarkably well within populations. This is because individuals don't mate at random, but they do mate at random with respect to their genotype at most of the loci in the genome.

**Question 4.** You are investigating a locus with three alleles, A, B, and C, with allele frequencies  $p_A$ ,  $p_B$ , and  $p_C$ . What fraction of the population is expected to be homozygotes under Hardy-Weinberg?

Microsatellites are regions of the genome where individuals vary for the number of copies of some short DNA repeat that they carry. These regions are often highly variable across individuals, making them a suitable way to identify individuals from a DNA sample. This so-called DNA-fingerprinting has a range of applications from establishing paternity, identifying human remains, to matching individuals to DNA samples from a crime scene. The FBI make use of the CODIS



Figure 2.4: Kermode's bear. Extinct and vanishing mammals of the western hemisphere. 1942. Glover A.



Figure 2.5: Demonstrating Hardy–Weinberg proportions using 10,000 SNPs from the HapMap European (CEU) and African (YRI) populations. Within each of these populations the allele frequency against the frequency of the 3 genotypes; each SNP is represented by 3 different coloured points. The solid lines show the mean genotype frequency. The dashed lines show the predicted genotype frequency from Hardy–Weinberg equilibrium.

database<sup>2</sup>. The CODIS database contains the genotypes of over 13 million people, most of whom have been convicted of a crime. Most of the profiles record genotypes at 13 microsatellite loci that are tetranucleotide repeats (since 2017, 20 sites have been genotyped).

The allele counts for two loci (D16S539 and TH01) are shown in table 2.2 and 2.3 for a sample of 155 people of European ancestry. You can assume these two loci are on different chromosomes.

allele name	80	90	100	110	120	121	130	140	150
allele count	3	34	13	102	97	1	44	13	3

allele name	60	70	80	90	93	100	110
allele counts	84	42	37	67	77	1	2

**Question 5.** You extract a DNA sample from a crime scene. The genotype is 100/80 at the D16S539 locus and 70/93 at TH01.

**A)** You have a suspect in custody. Assuming this suspect is innocent and of European ancestry, what is the probability that their genotype would match this profile by chance (a false-match probability)?

**B)** The FBI uses  $\geq 13$  markers. Why is this higher number necessary to make the match statement convincing evidence in court?

**C)** An early case that triggered debate among forensic geneticists was a crime among the Abenaki, a Native American community in Vermont (see LEWONTIN, 1994, for discussion). There was a DNA sample from the crime scene, and the perpetrator was thought likely

<sup>2</sup> CODIS: Combined DNA Index System

Table 2.2: Data for 155 Europeans at the D16S539 microsatellite from CODIS from ALGEE-HEWITT *et al.*. The top row gives the number of tetranucleotide repeats for each allele, the bottom row gives the sample counts.

Table 2.3: Same as 2.2 but for the TH01 microsatellite.

to be a member of the Abenaki community. Given that allele frequencies vary among populations, why would people be concerned about using data from a non-Abenaki population to compute a false match probability?

## 2.2 Allele sharing among related individuals and Identity by Descent

All of the individuals in a population are related to each other by a giant pedigree (family tree). For most pairs of individuals in a population these relationships are very distant (e.g. distant cousins), while some individuals will be more closely related (e.g. sibling/first cousins). All individuals are related to one another by varying levels of relatedness, or *kinship*. Related individuals can share alleles that have both descended from the shared common ancestor. To be shared, these alleles must be inherited through all meioses connecting the two individuals (e.g. surviving the  $1/2$  probability of segregation each meiosis). As closer relatives are separated by fewer meioses, closer relatives share more alleles. In Figure 2.6 we show the sharing of chromosomal regions between two cousins. As we'll see, many population and quantitative genetic concepts rely on how closely related individuals are, and thus we need some way to quantify the degree of kinship among individuals.



Figure 2.6: First cousins sharing a stretch of chromosome identical by descent. The different grandparental diploid chromosomes are coloured so we can track them and recombinations between them across the generations. Notice that the identity by descent between the cousins persists for a long stretch of chromosome due to the limited number of generations for recombination.

We will define two alleles to be identical by descent (IBD) if they are identical due to transmission from a common ancestor in the past few generations<sup>3</sup>. For the moment, we ignore mutation, and we will be more precise about what we mean by ‘past few generations’ later on. For example, parent and child share exactly one allele identical by descent at a locus, assuming that the two parents of the child are randomly mated individuals from the population. In Figure 2.12, I show a pedigree demonstrating some configurations of IBD.

<sup>3</sup> COTTERMAN, C. W., 1940 A calculus for statistico-genetics. Ph. D. thesis, The Ohio State University; and MALÉCOT, G., 1948 Les mathématiques de l'hérédité

One summary of how related two individuals are is the probability that our pair of individuals share 0, 1, or 2 alleles identical by descent (see Figure 2.7). We denote these probabilities by  $r_0$ ,  $r_1$ , and  $r_2$  respectively. See Table 2.4 for some examples. We can also interpret these probabilities as genome-wide averages. For example, on average, at a quarter of all their autosomal loci full-sibs share zero alleles identical by descent.

One summary of relatedness that will be important is the probability that two alleles picked at random, one from each of the two different individuals  $i$  and  $j$ , are identical by descent. We call this quantity the *coefficient of kinship* of individuals  $i$  and  $j$ , and denote it by  $F_{ij}$ . It is calculated as

$$F_{ij} = 0 \times r_0 + \frac{1}{4}r_1 + \frac{1}{2}r_2. \quad (2.3)$$

The coefficient of kinship will appear multiple times, in both our discussion of inbreeding and in the context of phenotypic resemblance between relatives.

Relationship (i,j)*	$r_0$	$r_1$	$r_2$	$F_{ij}$
parent-child	0	1	0	$1/4$
full siblings	$1/4$	$1/2$	$1/4$	$1/4$
Monzygotic twins	0	0	1	$1/2$
1 <sup>st</sup> cousins	$3/4$	$1/4$	0	$1/16$

**Question 6.** What are  $r_0$ ,  $r_1$ , and  $r_2$  for  $1/2$  sibs? ( $1/2$  sibs share one parent but not the other).

Our  $r$  coefficients are going to have various uses. For example, they allow us to calculate the probability of the genotypes of a pair of relatives. Consider a biallelic locus where allele 1 is at frequency  $p$ , and two individuals who have IBD allele sharing probabilities  $r_0$ ,  $r_1$ ,  $r_2$ . What is the overall probability that these two individuals are both homozygous for allele 1? Well that's

$$\begin{aligned} P(A_1A_1) = & P(A_1A_1|0 \text{ alleles IBD})P(0 \text{ alleles IBD}) \\ & + P(A_1A_1|1 \text{ allele IBD})P(1 \text{ allele IBD}) \\ & + P(A_1A_1|2 \text{ alleles IBD})P(2 \text{ alleles IBD}) \end{aligned} \quad (2.4)$$

Or, in our  $r_0$ ,  $r_1$ ,  $r_2$  notation:

$$\begin{aligned} P(A_1A_1) = & P(A_1A_1|0 \text{ alleles IBD})r_0 \\ & + P(A_1A_1|1 \text{ allele IBD})r_1 \\ & + P(A_1A_1|2 \text{ alleles IBD})r_2 \end{aligned} \quad (2.5)$$

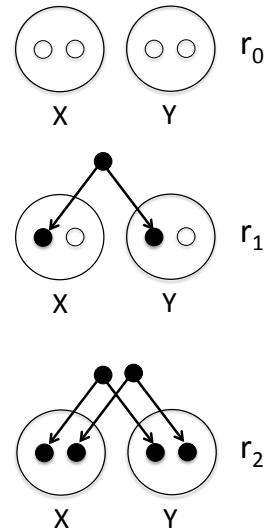


Figure 2.7: A pair of diploid individuals (X and Y) sharing 0, 1, or 2 alleles IBD where lines show the sharing of alleles by descent (e.g. from a shared ancestor).

Table 2.4: Probability that two individuals of a given relationship share 0, 1, or 2 alleles identical by descent on the autosomes. \*Assuming this is the only close relationship the pair shares.

If our pair of relatives share 0 alleles IBD, then the probability that they are both homozygous is  $P(A_1A_1|0 \text{ alleles IBD}) = p^2 \times p^2$ , as all four alleles represent independent draws from the population. If they share 1 allele IBD, then the shared allele is of type  $A_1$  with probability  $p$ , and then the other non-IBD allele, in both relatives, also needs to be  $A_1$  which happens with probability  $p^2$ , so  $P(A_1A_1|1 \text{ alleles IBD}) = p \times p^2$ . Finally, our pair of relatives can share two alleles IBD, in which case  $P(A_1A_1|2 \text{ alleles IBD}) = p^2$ , because if one of our individuals is homozygous for the  $A_1$  allele, both individuals will be. Putting this all together our equation (2.5) becomes

$$P(A_1A_2) = p^4r_0 + p^3r_1 + p^2r_2 \quad (2.6)$$

Note that for specific cases we could also calculate this by summing over all the possible genotypes their shared ancestor(s) had; however, that would be much more involved and not as general as the form we have derived here.

We can write out terms like eq (2.6) for all of the possible configurations of genotype sharing/non-sharing between a pair of individuals. Based on this we can write down the expected number of polymorphic sites where our individuals are observed to share 0, 1, or 2 alleles.

**Question 7.** The genotype of our suspect in Question 5 turns out to be 100/80 for D16S539 and 70/80 at TH01. The suspect is not a match to the DNA from the crime scene; however, they could be a sibling.

Calculate the joint probability of observing the genotype from the crime and our suspect:

- A) Assuming that they share no close relationship.
- B) Assuming that they are full sibs.
- C) Briefly explain your findings.

There's a variety of ways to estimate the relationships among individuals using genetic data. An example of using allele sharing to identify relatives is offered by the work of Nancy Chen (in collaboration with Stepfanie Aguillon, see CHEN *et al.*, 2016; AGUILLO et al., 2017). CHEN *et al.* has collected genotyping data from thousands of Florida Scrub Jays at over ten thousand loci. These Jays live at the Archbold field site, and have been carefully monitored for many decades allowing the pedigree of many of the birds to be known. Using these data she estimates allele frequencies at each locus. Then by equating the observed number of times that a pair of individuals share 0, 1, or 2 alleles to the theoretical expectation, she estimates the probability of  $r_0$ ,  $r_1$ , and  $r_2$  for each pair of birds. A plot of these are shown in Figure 2.9, showing how well the estimates match those known from the pedigree.



Figure 2.8: Florida Scrub-Jays (*Aphelocoma coerulescens*). The birds of America : from drawings made in the United States and their territories. 1880. Audubon J.J.



Figure 2.9: Estimated coefficient of kinship from Florida Scrub Jays. Each point is a pair of individuals, plotted by their estimated IBD ( $r_1$  and  $r_2$ ) from their genetic data. The points are coloured by their known pedigree relationships. Note that most pairs have low kinship, and no recent genealogical relationship, and so appear as black points in the lower left corner. Thanks to Nancy Chen for supplying the data.



Figure 2.10: A simulation of sharing between first cousins. The regions of your grandmother's 22 autosomes that you inherited are coloured red, those that your cousins inherited are coloured blue. In the third panel we show the overlapping genomic regions in purple, these regions will be IBD in you and your cousin. If you are full first cousins, you will also have shared genomic regions from your shared grandfather, not shown here. Details about how we made these simulations here.

*Sharing of genomic blocks among relatives.* We can more directly see the sharing of the genome among close relatives using high-density SNP genotyping arrays. Below we show a simulation of you and your first cousin's genomic material that you both inherited from your shared grandmother. Colored purple are regions where you and your cousin will have matching genomic material, due to having inherited it IBD from your shared grandmother.

You and your first cousin will share at least one allele of your genotype at all of the polymorphic loci in these purple regions. There's a

range of methods to detect such sharing. One way is to look for unusually long stretches of the genome where two individuals are never homozygous for different alleles. By identifying pairs of individuals who share an unusually large number of such putative IBD blocks, we can hope to identify unknown relatives in genotyping datasets. In fact, companies like 23&me and Ancestry.com use signals of IBD to help identify family ties.

As another example, consider the case of third cousins. You share one of eight sets of great-great grandparents with each of your (likely many) third cousins. On average, you and each of your third cousins each inherit one-sixteenth of your genome from each of those two great-great grandparents. This turns out to imply that on average, a little less than one percent of your and your third cousin's genomes ( $2 \times (1/16)^2 = 0.78\%$ ) will be identical by virtue of descent from those shared ancestors. A simulated example where third cousins share blocks of their genome (on chromosome 16 and 2) due to their great, great grandmother is shown in Figure 2.11.



Figure 2.11: A simulation of sharing between third cousins, the details are the same as in Figure 2.10.

Note how if you compare Figure 2.11 and Figure 2.10, individuals inherit less IBD from a shared great, great grandmother than from a shared grandmother, as they inherit from more total ancestors further back. Also notice how the sharing occurs in shorter genomic blocks, as it has passed through more generations of recombination during meiosis. These blocks are still detectable, and so third cousins can be detected using high-density genotyping chips, allowing more distant relatives to be identified than single marker methods alone.<sup>4</sup> More distant relations than third cousins, e.g. fourth cousins, start to have

<sup>4</sup> Indeed the suspect in case of the Golden State Killer was identified through identifying third cousins that genetically matched a DNA sample from an old crime scene (see a [here](#) for more details).

a significant probability of sharing none of their genome IBD. But you have many fourth cousins, so you will share some of your genome IBD with some of them; however, it gets increasingly hard to identify the degree of relatedness from genetic data the deeper in the family tree this sharing goes.

### 2.2.1 Inbreeding

We can define an inbred individual as an individual whose parents are more closely related to each other than two random individuals drawn from some reference population.

When two related individuals produce an offspring, that individual can receive two alleles that are identical by descent, i.e. they can be homozygous by descent (sometimes termed autozygous), due to the fact that they have two copies of an allele through different paths through the pedigree. This increased likelihood of being homozygous relative to an outbred individual is the most obvious effect of inbreeding. It is also the one that will be of most interest to us, as it underlies a lot of our ideas about inbreeding depression and population structure. For example, in Figure 2.12 our offspring of first cousins is homozygous by descent having received the same IBD allele via two different routes around an inbreeding loop.

As the offspring receives a random allele from each parent ( $i$  and  $j$ ), the probability that those two alleles are identical by descent is equal to the kinship coefficient  $F_{ij}$  of the two parents (Eqn. 2.3). This follows from the fact that the genotype of the offspring is made by sampling an allele at random from each of our parents.

$f_{11}$	$f_{12}$	$f_{22}$
$(1 - F)p^2 + Fp$	$(1 - F)2pq$	$(1 - F)q^2 + Fq$

The only way the offspring can be heterozygous ( $A_1A_2$ ) is if their two alleles at a locus are not IBD (otherwise they would necessarily be homozygous). Therefore, the probability that they are heterozygous is

$$(1 - F)2pq, \quad (2.7)$$

where we have dropped the indices  $i$  and  $j$  for simplicity. The offspring can be homozygous for the  $A_1$  allele in two different ways. They can have two non-IBD alleles that are not IBD but happen to be of the allelic type  $A_1$ , or their two alleles can be IBD, such that they inherited allele  $A_1$  by two different routes from the same ancestor. Thus, the probability that an offspring is homozygous for  $A_1$  is

$$(1 - F)p^2 + Fp. \quad (2.8)$$



Figure 2.12: Alleles being transmitted through an inbred pedigree. The two sisters (mum and aunt) share two alleles identical by descent (IBD). The cousins share one allele IBD. The offspring of first cousins is homozygous by descent at this locus.

Table 2.5: **Generalized Hardy–Weinberg**

Therefore, the frequencies of the three possible genotypes can be written as given in Table 2.5, which provides a generalization of the Hardy–Weinberg proportions.

Note that the generalized Hardy–Weinberg proportions completely specify the genotype probabilities, as there are two parameters ( $p$  and  $F$ ) and two degrees of freedom (as  $p$  and  $q$  have to sum to one). Therefore, any combination of genotype frequencies at a biallelic site can be specified by a combination of  $p$  and  $F$ .

**Question 8.** The frequency of the  $A_1$  allele is  $p$  at a biallelic locus. Assume that our population is randomly mating and that the genotype frequencies in the population follow from HW. We select two individuals at random to mate from this population. We then mate the children from this cross. What is the probability that the child from this full sib-mating is homozygous?

*Multiple inbreeding loops in a pedigree.* Up to this point we have assumed that there is at most one inbreeding loop in the recent family history of our individuals, i.e. the parents of our inbred individual have at most one recent genealogical connection. However, an individual who has multiple inbreeding loops in their pedigree can be homozygous by descent thanks to receiving IBD alleles via multiple different different loops. To calculate inbreeding in pedigrees of arbitrary complexity, we can extend beyond our original relatedness coefficients  $r_0$ ,  $r_1$ , and  $r_2$  to account for higher order sharing of alleles IBD among relatives. For example, we can ask, what is the probability that *both* of the alleles in the first individual are shared IBD with one allele in the second individual? There are nine possible relatedness coefficients in total to completely describe kinship between two diploid individuals, and we won't go in to them here as it's a lot to keep track of. However, we will show how we can calculate the inbreeding coefficient of an individual with multiple inbreeding loops more directly.

Let's say the parents of our inbred individual (B and C) have  $K$  shared ancestors, i.e. individuals who appear in both B and C's recent family trees. We denote these shared ancestors by  $A_1, \dots, A_K$ , and we denote by  $n$  the total number of individuals in the chain from B to C via ancestor  $A_i$ , including B, C, and  $A_i$ . For example, if B is C's aunt, then B and C share two ancestors, which are B's parents and, equivalently, C's grandparents. In this case, there are  $n=4$  individuals from B to C through each of these two shared ancestor. In the general case, the kinship coefficient of B and C, i.e. the inbreeding coefficient of their child, is

$$F = \sum_{i=1}^K \frac{1}{2^{n_i}} (1 + f_{A_i}) \quad (2.9)$$

where  $f_{A_i}$  is the inbreeding coefficient of the ancestor  $A_i$ . What's happening here is that we sum over all the mutually-exclusive paths in the pedigree through which B and C can share an allele IBD. With probability  $1/2^{n_i}$ , a pair of alleles picked at random from B and C is descended from the same ancestral allele in individual  $A_i$ , in which case the alleles are IBD.<sup>5</sup> However, even if B inherits the maternal allele and C inherits the paternal allele of shared ancestor  $A_i$ , if  $A_i$  was themselves inbred, with probability  $f_{A_i}$  those two alleles are themselves IBD. Thus a shared *inbred* ancestor further increases the kinship of B and C.



Multiple inbreeding loops increase the probability that a child is homozygous by descent at a locus, which can be calculated simply by plugging in  $F$ , the child's inbreeding coefficient, into our generalized HW equation.

As one extreme example of the impact of multiple inbreeding loops in an individual's pedigree, let's consider king Charles II of Spain, the last of the Spanish Habsburgs. Charles was the son of Philip IV of Spain and Mariana of Austria, who were uncle and niece. If this were the only inbreeding loop, then Charles would have had an inbreeding coefficient of  $1/8$ . Unfortunately for Charles, the Spanish Habsburgs had long kept wealth and power within their family by arranging marriages between close kin. The pedigree of Charles II is shown in Figure 2.13, and multiple inbreeding loops are apparent. For example, Phillip III, Charles II's grandfather and great-grandfather, was himself

<sup>5</sup> For example, in the case of our aunt-nephew case, assuming that the aunt's two parents are their only recent shared ancestors, then  $F = 1/2^4 + 1/2^4 = 1/8$ , in agreement with the answer we would obtain from eqn (2.3).

Figure 2.13: The pedigree of King Charles II of Spain



Figure 2.14: Charles the second of Spain (by Juan Carreño de Miranda, 1685).

a child of an uncle-niece marriage.

ALVAREZ *et al.* (2009) calculated that Charles II had an inbreeding coefficient of 0.254, equivalent to a full-sib mating, thanks to all of the inbreeding loops in his pedigree. Therefore, he is expected to have been homozygous by descent for a full quarter of his genome. As we'll talk about later in these notes, this means that Charles may have been homozygous for a number of recessive disease alleles, and indeed he was a very sickly man who left no descendants due to his infertility.<sup>6</sup> Thus plausibly the end of one of the great European dynasties came about through inbreeding.

### 2.2.2 Calculating inbreeding coefficients from genetic data

If the observed heterozygosity in a population is  $H_O$ , and we assume that the generalized Hardy–Weinberg proportions hold, we can set  $H_O$  equal to  $f_{12}$ , and solve Eq. (2.7) for  $F$  to obtain an estimate of the inbreeding coefficient as

$$\hat{F} = 1 - \frac{f_{12}}{2pq} = \frac{2pq - f_{12}}{2pq}. \quad (2.10)$$

As before,  $p$  is the frequency of allele  $A_1$  in the population. This can be rewritten in terms of the observed heterozygosity ( $H_O$ ) and the heterozygosity expected in the absence of inbreeding,  $H_E = 2pq$ , as

$$\hat{F} = \frac{H_E - H_O}{H_E} = 1 - \frac{H_O}{H_E}. \quad (2.11)$$

Hence,  $\hat{F}$  quantifies the deviation due to inbreeding of the observed heterozygosity from the one expected under random mating, relative to the latter.

**Question 9.** Suppose the following genotype frequencies were observed for an esterase locus in a population of *Drosophila* (A denotes the “fast” allele and B denotes the “slow” allele):

AA	AB	BB
0.6	0.2	0.2

What is the estimate of the inbreeding coefficient at the esterase locus?

If we have multiple loci, we can replace  $H_O$  and  $H_E$  by their means over loci,  $\bar{H}_O$  and  $\bar{H}_E$ , respectively. Note that, in principle, we could also calculate  $F$  for each individual locus first, and then take the average across loci. However, this procedure is more prone to introducing a bias if sample sizes vary across loci, which is not unlikely when we are dealing with real data.

Genetic markers are commonly used to estimate inbreeding for wild and/or captive populations of conservation concern. As an example of

<sup>6</sup> Pedro Gargantilla, who performed Charles' autopsy, stated that his body "did not contain a single drop of blood; his heart was the size of a peppercorn; his lungs corroded; his intestines rotten and gangrenous; he had a single testicle, black as coal, and his head was full of water." While some of this description may refer to actual medical conditions, some of these details seem a little unlikely.

this, consider the case of the Mexican wolf (*Canis lupus baileyi*), also known as the lobo, a sub-species of gray wolf.

They were extirpated in the wild during the mid-1900s due to hunting, and the remaining five lobos in the wild were captured to start a breeding program. vonHOLDT *et al.* (2011) estimated the current-day, average expected heterozygosity to be 0.18, based on allele frequencies at over forty thousand SNPs. However, the average lobo individual was only observed to be heterozygous at 12% of these SNPs. Therefore, the average inbreeding coefficient for the lobo is  $F = 1 - 0.12/0.18$ , i.e.  $\sim 33\%$  of a lobo's genome is homozygous due to recent inbreeding in their pedigree.

*Genomic blocks of homozygosity due to inbreeding.* As we saw above, close relatives are expected to share alleles IBD in large genomic blocks. Thus, when related individuals mate and transmit alleles to an inbred offspring, they transmit these alleles in big blocks through meiosis. An example, lets return to the case of our hypothetical first cousins from Figure 2.6. If this pair of individuals had a child, one possible pattern of genetic transmission is shown in Figure 2.16. The child has inherited the red stretch of chromosome via two different routes through their pedigree from the grandparents. This is an example of an autozygous segment, where the child is homozygous by descent at all of the loci in this red region. The inbreeding coefficient



of the child sets the proportion of their genome that will be in these autozygous segments. For example, a child of first full cousins is expected to have 1/16 of their genome in these segments. The more distant the loop in the pedigree, the more meioses that chromosomes have been through and the shorter individual blocks will be. A child of first cousins will have longer blocks than a child of second cousins, for example.

Individuals with multiple inbreeding loops in their family tree can have a high inbreeding coefficient due to the combined effect of many



Figure 2.15: Grey wolf (*Canis lupus*). Dogs, jackals, wolves, and foxes: a monograph of the Canidae. 1890. y J.G. Keulemans.

Figure 2.16: .

small blocks of autozygosity. For example, Carlos the second had an inbreeding coefficient that is equivalent to that of the child of full-sibs, with a quarter of his genome expected to homozygous by descent, but this would be made up of many shorter blocks.

We can hope to detect these blocks by looking for unusually long genomic runs of homozygosity (ROH) sites in an individual's genome. One way to estimate an individual's inbreeding coefficient is then to total up the proportion of an individual's genome that falls in such ROH regions. This estimate is called  $F_{ROH}$ .

An example of using  $F_{ROH}$  to study inbreeding comes from the work of SAMS and BOYKO (2018), who identified runs of homozygosity in 2,500 dogs, ranging from 500kb up to many megabases. Fig-



Figure 2.17: English bulldog. The dogs of Boytown. 1918. Dyer, W. A.



Figure 2.18: The distribution of  $F_{ROH}$  of individuals from various dog breeds from SAMS and BOYKO (2018).

ure 2.18 shows the distribution of  $F_{ROH}$  of individuals in each dog breed for the X and autosome. In Figure 2.19 this is broken down by the length of ROH segments.

Dog breeds have been subject to intense breeding that has resulted in high levels of inbreeding. Of the population samples examined, Doberman Pinschers have the highest levels of their genome in runs of homozygosity ( $F_{ROH}$ ), somewhat higher than English bulldogs. In 2.19 we can see that English bulldogs have more short ROH than Doberman Pinschers, but that Doberman Pinschers have more of their genome in very large ROH ( $> 16\text{ Mb}$ ). This suggests that English bulldogs have had long history of inbreeding but that Doberman Pinschers have a lot of recent inbreeding in their history.



Figure 2.19: Cumulative density of length of ROH length, measured in megabases (Mb) from SAMS and BOYKO (2018) for various dog breeds. Note that longer lengths of ROH are on the left of the plot.

## 2.3 Summarizing population structure

INDIVIDUALS RARELY MATE COMPLETELY AT RANDOM; your parents weren't two Bilateria plucked at random from the tree of life. Even within species, there's often geographically-restricted mating among individuals. Individuals tend to mate with individuals from the same, or closely related sets of populations. This form of non-random mating is called population structure and can have profound effects on the distribution of genetic variation within and among natural populations.

### 2.3.1 Inbreeding as a summary of population structure.

It turns out that statements about inbreeding represent one natural way to summarize population structure. We defined inbreeding as having parents that are more closely related to each other than two individuals drawn at random from some reference population. The question that naturally arises is: Which reference population should we use? While I might not look inbred in comparison to allele frequencies in the United Kingdom (UK), where I am from, my parents certainly are not two individuals drawn at random from the world-wide population. If we estimated my inbreeding coefficient  $F$  using allele frequencies within the UK, it would be close to zero, but would likely be larger if we used world-wide frequencies. This is because there is a somewhat lower level of expected heterozygosity within the UK than in the human population across the world as a whole.

WRIGHT<sup>7</sup> developed a set of ‘F-statistics’ (also called ‘fixation indices’) that formalize the idea of inbreeding with respect to different levels of population structure. See Figure 2.20 for a schematic diagram. Wright defined  $F_{XY}$  as the correlation between random gametes, drawn from the same level  $X$ , relative to level  $Y$ . We will return to why  $F$ -statistics are statements about correlations between alleles in just a moment. One commonly used  $F$ -statistic is  $F_{IS}$ , which is the inbreeding coefficient between an individual ( $I$ ) and the subpopulation ( $S$ ). Consider a single locus, where in a subpopulation ( $S$ ) a fraction  $H_I = f_{12}$  of individuals are heterozygous. In this subpopulation, let the frequency of allele  $A_1$  be  $p_S$ , such that the expected heterozygosity under random mating is  $H_S = 2p_S(1 - p_S)$ . We will write  $F_{IS}$  as

$$F_{IS} = 1 - \frac{H_I}{H_S} = 1 - \frac{f_{12}}{2p_S q_S}, \quad (2.12)$$

a direct analog of eqn. 2.10. Hence,  $F_{IS}$  is the relative difference between observed and expected heterozygosity due to a deviation from random mating within the subpopulation. We could also compare the observed heterozygosity in individuals ( $H_I$ ) to that expected in the total population,  $H_T$ . If the frequency of allele  $A_1$  in the total population is  $p_T$ , then we can write  $F_{IT}$  as

$$F_{IT} = 1 - \frac{H_I}{H_T} = 1 - \frac{f_{12}}{2p_T q_T}, \quad (2.13)$$

which compares heterozygosity in individuals to that expected in the total population. As a simple extension of this, we could imagine comparing the expected heterozygosity in the subpopulation ( $H_S$ ) to that expected in the total population  $H_T$ , via  $F_{ST}$ :

$$F_{ST} = 1 - \frac{H_S}{H_T} = 1 - \frac{2p_S q_S}{2p_T q_T}. \quad (2.14)$$

We can relate the three  $F$ -statistics to each other as

$$(1 - F_{IT}) = \frac{H_I}{H_S} \frac{H_S}{H_T} = (1 - F_{IS})(1 - F_{ST}). \quad (2.15)$$

Hence, the reduction in heterozygosity within individuals compared to that expected in the total population can be decomposed to the reduction in heterozygosity of individuals compared to the subpopulation, and the reduction in heterozygosity from the total population to that in the subpopulation.

If we want a summary of population structure across multiple subpopulations, we can average  $H_I$  and/or  $H_S$  across populations, and use a  $p_T$  calculated by averaging  $p_S$  across subpopulations (or our samples from sub-populations). For example, the average  $F_{ST}$  across

<sup>7</sup> WRIGHT, S., 1943 Isolation by Distance. *Genetics* 28(2): 114–138; and WRIGHT, S., 1949 The Genetical Structure of Populations. *Annals of Eugenics* 15(1): 323–354

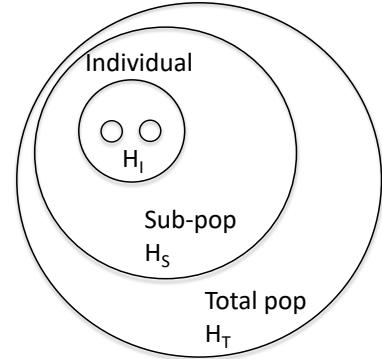


Figure 2.20: The hierarchical nature of F-statistics. The two dots within an individual represent the two alleles at a locus for an individual  $I$ . We can compare the heterozygosity on individuals ( $H_I$ ), to that found by randomly drawing alleles from the sub-population (S), to that found in the total population (T).

$K$  subpopulations (sampled with equal effort) is

$$F_{ST} = 1 - \frac{\bar{H}_S}{H_T}, \quad (2.16)$$

where  $\bar{H}_S = 1/K \sum_{i=1}^K H_S^{(i)}$ , and  $H_S^{(i)} = 2p_i q_i$  is the expected heterozygosity in subpopulation  $i$ . If the total population contains the subpopulation then  $2psqs \leq 2ptqt$ , and so  $F_{IS} \leq F_{IT}$  and  $F_{ST} \geq 0$ . Furthermore, if we have multiple sites, we can replace  $H_I$ ,  $H_S$ , and  $H_T$  with their averages across loci (as above).<sup>8</sup>

As an example of comparing a genome-wide estimate of  $F_{ST}$  to that at individual loci we can look at some data from blue- and golden-winged warblers (*Vermivora cyanoptera* and *V. chrysoptera* 1-2 & 5-6 o, Figure 2.21).

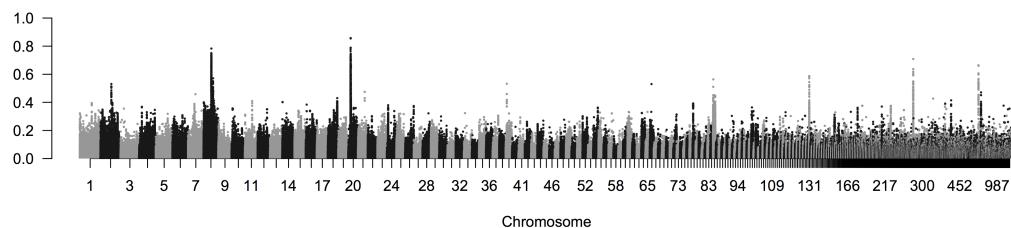
These two species are spread across eastern Northern America, with the golden-winged warbler having a smaller, more northerly range. They're quite different in terms of plumage, but have long been known to have similar songs and ecologies. The two species hybridize readily in the wild; in fact two other previously-recognized species, Brewster's and Lawrence's warbler (4 & 3 in 2.21), are actually found to just be hybrids between these two species. The golden-winged warbler is listed as 'threatened' under the Canadian endangered species act. The golden-winged warbler's habitat is under pressure from human activity and increased hybridization with the blue warbler, which is moving north into its range, also poses a significant issue. TOEWS *et al.* investigated the population genomics of these warblers, sequencing ten golden- and ten blue-winged warblers. They found very low divergence among these species, with a genome-wide  $F_{ST} = 0.0045$ . In Figure 2.22, per SNP  $F_{ST}$  is averaged in 2000bp windows moving along the genome. The average is very low, but some regions of very high  $F_{ST}$

<sup>8</sup> Averaging heterozygosity across loci first, then calculating  $F_{ST}$ , rather than calculating  $F_{ST}$  for each locus individually and then taking the average, has better statistical properties as statistical noise in the denominator is averaged out.



Figure 2.21: The warblers of North America. Chapman, F.M. 1907.

Add ref to Bateson book



stand out. Nearly all of these regions correspond to large allele frequency difference at loci in, or close, to genes known to be involved in plumage colouration difference in other birds. To illustrate these frequency differences TOEWS *et al.* genotyped a SNP in each of these high- $F_{ST}$  regions. Here's their genotyping counts from the SNP in the *Wnt* region, a key regulatory gene involved in feather development:

Figure 2.22:  $F_{ST}$  between blue- and golden-winged warbler population samples at SNPs across the genome. Each dot is a SNP, and SNPs are coloured alternating by scaffold. Thanks to David Toews for the figure.

Species	11	12	22
Blue-winged	2	21	31
Golden-winged	48	12	1

**Question 10.** With reference to the table of *Wnt*-allele counts:

- A) Calculate  $F_{IS}$  in blue-winged warblers.
- B) Calculate  $F_{ST}$  for the sub-population of blue-winged warblers compared to the combined sample.
- C) Calculate mean  $F_{ST}$  across both sub-populations.

*Interpretations of F-statistics* Let us now return to Wright's definition of the  $F$ -statistics as correlations between random gametes, drawn from the same level  $X$ , relative to level  $Y$ . Without loss of generality, we may think about  $X$  as individuals and  $S$  as the subpopulation.

Rewriting  $F_{IS}$  in terms of the observed homozygote frequencies ( $f_{11}$ ,  $f_{22}$ ) and expected homozygosities ( $p_S^2$ ,  $q_S^2$ ) we find

$$F_{IS} = \frac{2p_S q_S - f_{12}}{2p_S q_S} = \frac{f_{11} + f_{22} - p_S^2 - q_S^2}{2p_S q_S}, \quad (2.17)$$

using the fact that  $p^2 + 2pq + q^2 = 1$ , and  $f_{12} = 1 - f_{11} - f_{22}$ . The form of eqn. (2.17) reveals that  $F_{IS}$  is the covariance between pairs of alleles found in an individual, divided by the expected variance under binomial sampling. Thus,  $F$ -statistics can be understood as the correlation between alleles drawn from a population (or an individual) above that expected by chance (i.e. drawing alleles sampled at random from some broader population).

We can also interpret  $F$ -statistics as proportions of variance explained by different levels of population structure. To see this, let us think about  $F_{ST}$  averaged over  $K$  subpopulations, whose frequencies are  $p_1, \dots, p_K$ . The frequency in the total population is  $p_T = \bar{p} = 1/K \sum_{i=1}^K p_i$ . Then, we can write

$$F_{ST} = \frac{2\bar{p}\bar{q} - \frac{1}{K} \sum_{i=1}^K 2p_i q_i}{2\bar{p}\bar{q}} = \frac{\left(\frac{1}{K} \sum_{i=1}^K p_i^2 + \frac{1}{K} \sum_{i=1}^K q_i^2\right) - \bar{p}^2 - \bar{q}^2}{2\bar{p}\bar{q}} = \frac{\text{Var}(p_1, \dots, p_K)}{\text{Var}(\bar{p})}, \quad (2.18)$$

which shows that  $F_{ST}$  is the proportion of the variance explained by the subpopulation labels.

### 2.3.2 Other approaches to population structure

There is a broad spectrum of methods to describe patterns of population structure in population genetic datasets. We'll briefly discuss two broad-classes of methods that appear often in the literature: assignment methods and principal components analysis.

### 2.3.3 Assignment Methods

Here we'll describe a simple probabilistic assignment to find the probability that an individual of unknown population comes from one of  $K$  predefined populations. For example, there are three broad populations of common chimpanzee (*Pan troglodytes*) in Africa: western, central, and eastern. Imagine that we have a chimpanzee, whose population of origin is unknown (e.g. it's from an illegal private collection). If we have genotyped a set of unlinked markers from a panel of individuals representative of these populations, we can calculate the probability that our chimp comes from each of these populations.

We'll then briefly explain how to extend this idea to cluster a set of individuals into  $K$  initially unknown populations. This method is a simplified version of what population genetics clustering algorithms such as STRUCTURE and ADMIXTURE do.<sup>9</sup>

*A simple assignment method* We have genotype data from unlinked  $S$  biallelic loci for  $K$  populations. The allele frequency of allele  $A_1$  at locus  $l$  in population  $k$  is denoted by  $p_{k,l}$ , so that the allele frequencies in population 1 are  $p_{1,1}, \dots, p_{1,L}$  and population 2 are  $p_{2,1}, \dots, p_{2,L}$  and so on.

You genotype a new individual from an unknown population at these  $L$  loci. This individual's genotype at locus  $l$  is  $g_l$ , where  $g_l$  denotes the number of copies of allele  $A_1$  this individual carries at this locus ( $g_l = 0, 1, 2$ ).

The probability of this individual's genotype at locus  $l$  conditional on coming from population  $k$ , i.e. their alleles being a random HW draw from population  $k$ , is

$$P(g_l | \text{pop } k) = \begin{cases} (1 - p_{k,l})^2 & g_l = 0 \\ 2p_{k,l}(1 - p_{k,l}) & g_l = 1 \\ p_{k,l}^2 & g_l = 2 \end{cases} \quad (2.19)$$

Assuming that the loci are independent, the probability of the individual's genotype across all  $S$  loci, conditional on the individual coming from population  $k$ , is

$$P(\text{ind.} | \text{pop } k) = \prod_{l=1}^S P(g_l | \text{pop } k) \quad (2.20)$$

We wish to know the probability that this new individual comes from population  $k$ , i.e.  $P(\text{pop } k | \text{ind.})$ . We can obtain this through Bayes' rule

$$P(\text{pop } k | \text{ind.}) = \frac{P(\text{ind.} | \text{pop } k)P(\text{pop } k)}{P(\text{ind.})} \quad (2.21)$$

<sup>9</sup> PRITCHARD, J. K., M. STEPHENS, and P. DONNELLY, 2000 Inference of population structure using multilocus genotype data. *Genetics* 155(2): 945–959; and ALEXANDER, D. H., J. NOVEMBRE, and K. LANGE, 2009 Fast model-based estimation of ancestry in unrelated individuals. *Genome research* 19(9): 1655–1664

where

$$P(\text{ind.}) = \sum_{k=1}^K P(\text{ind.}|\text{pop } k)P(\text{pop } k) \quad (2.22)$$

is the normalizing constant. We interpret  $P(\text{pop } k)$  as the prior probability of the individual coming from population  $k$ , and unless we have some other prior knowledge we will assume that the new individual has an equal probability of coming from each population  $P(\text{pop } k) = 1/K$ .

We interpret

$$P(\text{pop } k|\text{ind.}) \quad (2.23)$$

as the posterior probability that our new individual comes from each of our  $1, \dots, K$  populations.

More sophisticated versions of this are now used to allow for hybrids, e.g., we can have a proportion  $q_k$  of our individual's genome come from population  $k$  and estimate the set of  $q_k$ 's.

### Question 11.

Returning to our chimp example, imagine that we have genotyped a set of individuals from the Western and Eastern populations at two SNPs (we'll ignore the central population to keep things simpler). The frequency of the capital allele at two SNPs ( $A/a$  and  $B/b$ ) is given by

Population	locus A	locus B
Western	0.1	0.85
Eastern	0.95	0.2

**A)** Our individual, whose origin is unknown, has the genotype  $AA$  at the first locus and  $bb$  at the second. What is the posterior probability that our individual comes from the Western population versus Eastern chimp population?

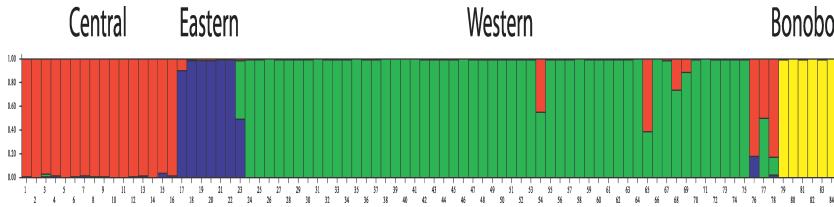
**B)** Let's assume that our individual is a hybrid. At each locus, with probability  $q_W$  our individual draws an allele from the Western population and with probability  $q_C = 1 - q_W$  they draw an allele from the Eastern population. What is the probability of our individual's genotype given  $q_C$ ?

**Optional** You could plot this probability as a function of  $q_W$ . How does your plot change if our individual is heterozygous at both loci?

*Clustering based on assignment methods* While it is great to be able to assign our individuals to a particular population, these ideas can be pushed to learn about how best to describe our genotype data in terms of discrete populations without assigning any of our individuals to populations *a priori*. We wish to cluster our individuals into  $K$  unknown populations. We begin by assigning our individuals at random to these  $K$  populations.

1. Given these assignments we estimate the allele frequencies at all of our loci in each population.
2. Given these allele frequencies we chose to reassign each individual to a population  $k$  with a probability given by eqn. (2.20).

We iterate steps 1 and 2 for many iterations (technically, this approach is known as *Gibbs Sampling*). If the data is sufficiently informative, the assignments and allele frequencies will quickly converge on a set of likely population assignments and allele frequencies for these populations.



To do this in a full Bayesian scheme we need to place priors on the allele frequencies (for example, one could use a beta distribution prior). Technically we are using the joint posterior of our allele frequencies and assignments. Programs like STRUCTURE, use this type of algorithm to cluster the individuals in an “unsupervised” manner (i.e. they work out how to assign individuals to an unknown set of populations). See Figure 2.23 for an example of Becquet *et al* using STRUCTURE to determine the population structure of chimpanzees.

STRUCTURE-like methods have proven incredible popular and useful in examining population structure within species. However, the results of these methods are open to misinterpretation, see LAWSON *et al.* (2018) for a recent discussion. Two common mistakes are 1) taking the results of STRUCTURE-like approaches for some particular value of  $K$  and taking this to represent the best way to describe population-genetic variation. 2) Thinking that these clusters represent ‘pure’ ancestral populations.

There is no right choice of  $K$ , the number of clusters to partition into. There are methods of judging the ‘best’  $K$  by some statistical measure given some particular dataset, but that is not the same as saying this is the most meaningful level on which to summarize population structure in data. For example, running STRUCTURE on world-wide human populations for low value of  $K$  will result in population clusters that roughly align with continental populations (ROSENBERG *et al.*, 2002). However, that does not tell us that assigning ancestral at the level of continents is a particularly meaningful way of partitioning individuals. Running the same data for higher value of  $K$ ,

Figure 2.23: BECQUET *et al.* (2007) genotyped 78 common chimpanzee and 6 bonobo at over 300 polymorphic markers (in this case microsatellites). They ran STRUCTURE to cluster the individuals using these data into  $K = 4$  populations. In the above figure they show each individual as a vertical bar divided into four colours depicting the estimate of the fraction of ancestry that each individual draws from each of the four estimated populations. We can see that these four colours/populations correspond to: Red, central; blue, eastern; green, western; yellow, bonobo.

or within continental regions, will result in much finer-scale partitioning of continental groups (ROSENBERG *et al.*, 2002; LI *et al.*, 2008). No one of these layers of population structure identified is privileged as being more meaningful than another.

It is tempting to think of these clusters as representing ancestral populations, which themselves are not the result of admixture. However, that is not the case, for example, running STRUCTURE on world-wide human data identifies a cluster that contains many European individuals, however, on the basis of ancient DNA we know that modern Europeans are a mixture of distinct ancestral groups.

### 2.3.4 Principal components analysis

Principal component analysis (PCA) is a common statistical approach to visualize high dimensional data, and used by many fields. The idea of PCA is to give a location to each individual data-point on each of a small number principal component axes. These PC axes are chosen to reflect major axes of variation in the data, with the first PC being that which explains largest variance, the second the second most, and so on. The use of PCA in population genetics was pioneered by Cavalli-Sforza and colleagues and now with large genotyping datasets, PCA has made come back.<sup>10</sup>

Consider a dataset consisting of  $N$  individuals at  $S$  biallelic SNPs. The  $i^{th}$  individual's genotype data at locus  $\ell$  takes a value  $g_{i,\ell} = 0, 1$ , or  $2$  (corresponding to the number of copies of allele  $A_1$  an individual carries at this SNP). We can think of this as a  $N \times S$  matrix (where usually  $N \ll S$ ).

Denoting the sample mean allele frequency at SNP  $\ell$  by  $p_\ell$ , it's common to standardize the genotype in the following way

$$\frac{g_{i,\ell} - 2p_\ell}{\sqrt{2p_\ell(1 - p_\ell)}} \quad (2.24)$$

i.e. at each SNP we center the genotypes by subtracting the mean genotype ( $2p_\ell$ ) and divide through by the square root of the expected variance assuming that alleles are sampled binomially from the mean frequency ( $\sqrt{2p_\ell(1 - p_\ell)}$ ). Doing this to all of our genotypes, we form a data matrix (of dimension  $N \times S$ ). We can then perform principal components analysis of this data matrix to uncover the major axes of genotype variance in our sample. Figure 2.24 shows a PCA from BECQUET *et al.* (2007) using the same chimpanzee data as in Figure 2.23.

It is worth taking a moment to delve further into what we are doing here. There's a number of equivalent ways of thinking about what PCA is doing. One of these ways is to think that when we do PCA we are building the individual by individual covariance matrix and per-

<sup>10</sup> MENOZZI, P., A. PIAZZA, and L. CAVALLI-SFORZA, 1978 Synthetic maps of human gene frequencies in Europeans. *Science* 201(4358): 786–792; and PATTERSON, N., A. L. PRICE, and D. REICH, 2006 Population structure and eigenanalysis. *PLoS genetics* 2(12): e190



forming an eigenvalue decomposition of this matrix (with the eigenvectors being the PCs). This individual by individual covariance matrix has entries the  $[i, j]$  given by

$$\frac{1}{S-1} \sum_{\ell=1}^S \frac{(g_{i,\ell} - 2p_\ell)(g_{j,\ell} - 2p_\ell)}{2p_\ell(1-p_\ell)} \quad (2.25)$$

Note that this is the covariance, and is very similar to those we encountered in discussing  $F$ -statistics as correlations (equation (2.17)), except now we are asking about the covariance between two individuals above that expected if they were both drawn from the total sample at random (rather than the covariance of alleles within a single individual). So by performing PCA on the data we are learning about the major (orthogonal) axes of the kinship matrix.

As an example of the application of PCA, let's consider the case of the putative ring species in the Greenish warbler (*Phylloscopus trochiloides*) species complex. This set of subspecies exists in a ring around the edge of the Himalayan plateau. ALCAIDE *et al.* (2014) collected 95 greenish warbler samples from 22 sites around the ring, and the sampling locations are shown in figure 2.25.

Figure 2.24: Principal Component Analysis by BECQUET *et al.* (2007) using the same chimpanzee data as in Figure 2.23. Here they plot the location of each individual on the first two principal components (called eigenvectors) in the left panel, and on the second and third principal components (eigenvectors) in the right panel. PCA, The individuals identified as all of one ancestry by STRUCTURE cluster together by population (solid circles). While the nine individuals identified by STRUCTURE as hybrids (open circles) are for the most part fall at intermediate locations in the PCA. There are two individuals (red open circles) reported as being of a particular population but that appear to be hybrids.

Figure 2.25: The sampling locations of 22 populations of Greenish warblers from ALCAIDE *et al.* (2014). The samples are coloured by the GENETICS 35 subspecies



It is thought that these warblers spread from the south, northward in two different directions around the inhospitable Himalayan plateau, establishing populations along the western edge (green and blue populations) and the eastern edge (yellow and red populations). When they came into secondary contact in Siberia, they were reproductive isolated from one another, having evolved different songs and accumulated other reproductive barriers from each other as they spread independently north around the plateau, such that *P. t. viridanus* (blue) and *P. t. plumbeitarsus* (red) populations presently form a stable hybrid zone.

ALCAIDE *et al.* (2014) obtained sequence data for their samples at 2,334 snps. In Figure 2.27 you can see the matrix of kinship coefficients, using (2.25), between all pairs of samples. You can already see a lot about population structure in this matrix. Note how the red and yellow samples, thought to be derived from the Eastern route around the Himalayas, have higher kinship with each other, and blue and the (majority) of the green samples, from the Western route, form a similarly close group in terms of their higher kinship.

We can then perform PCA on this kinship matrix to identify the major axes of variation in the dataset. Figure 2.28 shows the samples plotted on the first two PCs. The two major routes of expansion clearly occupy different parts of PC space. The first principal component distinguishes populations running North to South along the western route of expansion, while the second principal component distinguishes among populations running North to South along the Eastern route of expansion. Thus genetic data supports the hypothesis that the Greenish warblers speciated as they moved around the Himalayan plateau. However, as noted by ALCAIDE *et al.* (2014), it also suggests additional complications to the traditional view of these



Figure 2.26: Greenish warbler, subsp. *viridanus* (*Phylloscopus trochiloides viridanus*). Coloured figures of the birds of the British Islands. 1885. Lilford T. L. P.. Greenish warblers are rare visitors to the UK.



Figure 2.27: The matrix of kinship coefficients calculated for the 95 samples of Greenish warblers. Each cell in the matrix gives the pairwise kinship coefficient calculated for a particular pair. Hotter colours indicating higher kinship. The x and y labels of individuals are the population labels from Figure 2.25, and coloured by subspecies label as in that figure. The rows and columns have been organized to cluster individuals with high kinship.

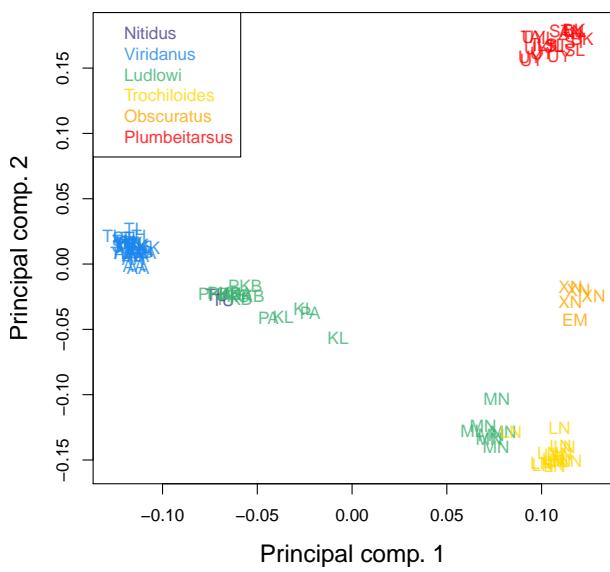


Figure 2.28: The 95 greenish warbler samples plotted on their locations on the first two principal components. The labels of individuals are the population labels from Figure 2.25, and coloured by subspecies label as in that figure.

warblers as an unbroken ring species, a case of speciation by continuous geographic isolation. The *Ludlowi* subspecies shows a significant genetic break, with the southern most MN samples clustering with the *Trochiloides* subspecies, in both the PCA and kinship matrix (Figures 2.28 and 2.27), despite being much more geographically close to the other *Ludlowi* samples. This suggests that genetic isolation is not just a result of geographic distance, and other biogeographic barriers must be considered in the case of this broken ring species.

Finally, while PCA is a wonderful tool for visualizing genetic data, care must be taken in its interpretation. The U-like shape in the case of the Greenish warbler PC might be consistent with some low level of gene flow between the red and the blue populations, pulling them genetically closer together and helping to form a genetic ring as well as a geographic ring. However, U-like shapes are expected to appear in PCAs even if our populations are just arrayed along a line, and more complex geometric arrangements of populations in PC space can result under simple geographic models (NOVEMBRE and STEPHENS, 2008). Inferring the geographical and population-genetic history of species requires the application of a range of tools; see ALCAIDE *et al.* (2014) and BRADBURD *et al.* (2016) for more discussion of the Greenish warblers.

### 2.3.5 Correlations between loci, linkage disequilibrium, and recombination

Up to now we have been interested in correlations between alleles at the same locus, e.g. correlations within individuals (inbreeding) or between individuals (relatedness). We have seen how relatedness between parents affects the extent to which their offspring is inbred. We now turn to correlations between alleles at different loci.

*Recombination* To understand correlations between loci we need to understand recombination a bit more carefully. Let us consider a heterozygous individual, containing  $AB$  and  $ab$  haplotypes. If no recombination occurs between our two loci in this individual, then these two haplotypes will be transmitted intact to the next generation. While if a recombination (i.e. an odd number of crossing over events) occurs between the two parental haplotypes, then  $1/2$  the time the child receives an  $Ab$  haplotype and  $1/2$  the time the child receives an  $aB$  haplotype. Effectively, recombination breaks up the association between loci. We'll define the recombination fraction ( $r$ ) to be the probability of an odd number of crossing over events between our loci in a single meiosis. In practice we'll often be interested in relatively short regions such that recombination is relatively rare, and so we might think that  $r = r_{BP}L \ll \frac{1}{2}$ , where  $r_{BP}$  is the average recombination rate (in Morgans) per base pair (typically  $\sim 10^{-8}$ ) and  $L$  is the number of base pairs separating our two loci.

*Linkage disequilibrium* The (horrible) phrase linkage disequilibrium (LD) refers to the statistical non-independence (i.e. a correlation) of alleles in a population at different loci. It's an awful name for a fantastically useful concept; LD is key to our understanding of diverse topics, from sexual selection and speciation to the limits of genome-wide association studies.

Our two biallelic loci, which segregate alleles  $A/a$  and  $B/b$ , have allele frequencies of  $p_A$  and  $p_B$  respectively. The frequency of the two locus haplotype  $AB$  is  $p_{AB}$ , and likewise for our other three combinations. If our loci were statistically independent then  $p_{AB} = p_A p_B$ , otherwise  $p_{AB} \neq p_A p_B$ . We can define a covariance between the  $A$  and  $B$  alleles at our two loci as

$$D_{AB} = p_{AB} - p_A p_B \quad (2.26)$$

and likewise for our other combinations at our two loci ( $D_{Ab}$ ,  $D_{aB}$ ,  $D_{ab}$ ). Gametes with two similar case alleles (e.g. A and B, or a and b) are known as *coupling* gametes, and those with different case alleles are known as *repulsion* gametes (e.g. a and B, or A and b). Then,

we can think of  $D$  as measuring the *excess* of coupling to repulsion gametes. These  $D$  statistics are all closely related to each other as  $D_{AB} = -D_{Ab}$  and so on. Thus we only need to specify one  $D_{AB}$  to know them all, so we'll drop the subscript and just refer to  $D$ . Also a handy result is that we can rewrite our haplotype frequency  $p_{AB}$  as

$$p_{AB} = p_A p_B + D. \quad (2.27)$$

If  $D = 0$  we'll say the two loci are in linkage equilibrium, while if  $D > 0$  or  $D < 0$  we'll say that the loci are in linkage disequilibrium (we'll perhaps want to test whether  $D$  is statistically different from 0 before making this choice). You should be careful to keep the concepts of linkage and linkage disequilibrium separate in your mind. Genetic linkage refers to the linkage of multiple loci due to the fact that they are transmitted through meiosis together (most often because the loci are on the same chromosome). Linkage disequilibrium merely refers to the covariance between the alleles at different loci; this may in part be due to the genetic linkage of these loci but does not necessarily imply this (e.g. genetically unlinked loci can be in LD due to population structure).

**Question 12.** You genotype 2 bi-allelic loci (A & B) segregating in two mouse subspecies (1 & 2) which mate randomly among themselves, but have not historically interbreed since they speciated. On the basis of previous work you estimate that the two loci are separated by a recombination fraction of 0.1. The frequencies of haplotypes in each population are:

Pop	$p_{AB}$	$p_{Ab}$	$p_{aB}$	$p_{ab}$
1	.02	.18	.08	.72
2	.72	.18	.08	.02

- A) How much LD is there within species? (i.e. estimate  $D$ )  
 B) If we mixed individuals from the two species together in equal proportions, we could form a new population with  $p_{AB}$  equal to the average frequency of  $p_{AB}$  across species 1 and 2. What value would  $D$  take in this new population before any mating has had the chance to occur?

Our linkage disequilibrium statistic  $D$  depends strongly on the allele frequencies of the two loci involved. One common way to partially remove this dependence, and make it more comparable across loci, is to divide  $D$  through by its the maximum possible value given the frequency of the loci. This normalized statistic is called  $D'$  and varies between +1 and -1. In Figure 2.29 there's an example of LD across the TAP2 region in human and chimp. Notice how physically close SNPs, i.e. those close to the diagonal, have higher absolute values of  $D'$  as



Figure 2.29: LD across the TAP2 gene region in a sample of Humans and Chimps. The rows and columns are consecutive SNPs, with each cell giving the absolute  $D'$  value between a pair of SNPs. Note that these are different sets of SNPs in the two species, as shared polymorphisms are very rare.

closely linked alleles are separated by recombination less often allowing high levels of LD to accumulate. Over large physical distances, away from the diagonal, there is lower  $D'$ . This is especially notable in humans as there is an intense, human-specific recombination hotspot in this region, which is breaking down LD between opposite sides of this region.

Another common statistic for summarizing LD is  $r^2$  which we write as

$$r^2 = \frac{D^2}{p_A(1-p_A)p_B(1-p_B)} \quad (2.28)$$

As  $D$  is a covariance, and  $p_A(1-p_A)$  is the variance of an allele drawn at random from locus  $A$ ,  $r^2$  is the squared correlation coefficient. Note that this  $r$  in  $r^2$  is NOT the recombination fraction.

Figure 2.31 shows  $r^2$  for pairs of SNPs at various physical distances in two population samples of *Mus musculus domesticus*. Again LD is highest between physically close markers as LD is being generated faster than it can decay via recombination; more distant markers have much lower LD as here recombination is winning out. Note the decay of LD is much slower in the advanced-generation cross population than in the natural wild-caught population. This persistence of LD across megabases is due to the limited number of generations for recombination since the cross was created.



Figure 2.30: *Mus musculus*. A history of British quadrupeds, including the Cetacea. 1874. Bell T., Tomes, R. F.m Alston E. R.

Figure 2.31: The decay of LD for autosomal SNP in *Mus musculus domesticus*, as measured by  $r^2$ , in a wild-caught mouse population from Arizona and a set of advanced-generation crosses between inbred lines of lab mice. Each dot gives the  $r^2$  for a pair of SNPs a given physical distance apart, for a total of  $\sim 3000$  SNPs. The solid black line gives the mean, the jagged the 95<sup>th</sup> percentile, and the flat red line a cutoff for significant LD. From LAURIE *et al.*

*The generation of LD.* Various population genetic forces can generate LD. Selection can generate LD by favouring particular combinations of alleles. Genetic drift will also generate LD, not because particular combinations of alleles are favoured, but simply because at random particular haplotypes can by chance drift up in frequency. Mixing between divergent populations can also generate LD, as we saw in the mouse question above.

*The decay of LD due to recombination* We will now examine what happens to LD over the generations if we only allow recombination to occur in a very large population (i.e. no genetic drift, i.e. the frequencies of our loci follow their expectations). To do so, consider the frequency of our  $AB$  haplotype in the next generation,  $p'_{AB}$ . We lose a fraction  $r$  of our  $AB$  haplotypes to recombination ripping our alleles apart but gain a fraction  $rp_{APB}$  per generation from other haplotypes recombining together to form  $AB$  haplotypes. Thus in the next generation

$$p'_{AB} = (1 - r)p_{AB} + rp_{APB} \quad (2.29)$$

The last term above, in eqn 2.29, is  $r(p_{AB} + p_{Ab})(p_{AB} + p_{aB})$  simplified, which is the probability of recombination in the different diploid genotypes that could generate a  $p_{AB}$  haplotype.

We can then write the change in the frequency of the  $p_{AB}$  haplotype as

$$\Delta p_{AB} = p'_{AB} - p_{AB} = -rp_{AB} + rp_{APB} = -rD \quad (2.30)$$

So recombination will cause a decrease in the frequency of  $p_{AB}$  if there is an excess of  $AB$  haplotypes within the population ( $D > 0$ ), and an increase if there is a deficit of  $AB$  haplotypes within the population ( $D < 0$ ). Our LD in the next generation is

$$\begin{aligned} D' &= p'_{AB} - p'_{APB} \\ &= (p_{AB} + \Delta p_{AB}) - (p_A + \Delta p_A)(p_B + \Delta p_B) \\ &= p_{AB} + \Delta p_{AB} - p_{APB} \\ &= (1 - r)D \end{aligned} \quad (2.31)$$

where we can cancel out  $\Delta p_A$  and  $\Delta p_B$  above because recombination only changes haplotype, not allele, frequencies. So if the level of LD in generation 0 is  $D_0$ , the level  $t$  generations later ( $D_t$ ) is

$$D_t = (1 - r)^t D_0 \quad (2.32)$$

Recombination is acting to decrease LD, and it does so geometrically at a rate given by  $(1 - r)$ . If  $r \ll 1$  then we can approximate this by an exponential and say that

$$D_t \approx D_0 e^{-rt} \quad (2.33)$$

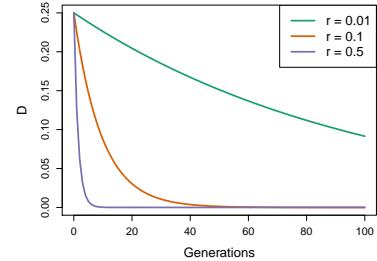


Figure 2.32: The decay of LD from an initial value of  $D_0 = 0.25$  over time (Generations) for a pair of loci a recombination fraction  $r$  apart.

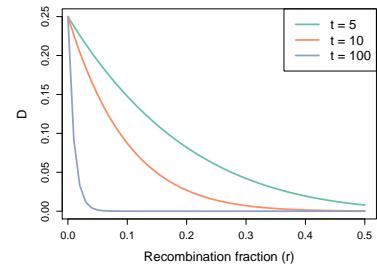


Figure 2.33: The decay of LD from an initial value of  $D_0 = 0.25$  due to recombination over  $t$  generations, plotted across possible recombination fractions ( $r$ ) between our pair of loci.

**Question 13.** You find a hybrid population between the two mouse subspecies described in the question above, which appears to be comprised of equal proportions of ancestry from the two subspecies. You estimate LD between the two markers to be 0.0723. Assuming that this hybrid population is large and was formed by a single mixture event, can you estimate how long ago this population formed?

A particularly striking example of the decay of LD generated by the mixing of populations is offered by the LD created by the interbreeding between humans and Neanderthals. Neanderthals and modern Humans diverged from each other likely over half a million years ago, allowing time for allele frequency differences to accumulate between the Neanderthal and modern human populations. The two populations spread back into secondary contact when humans moved out of Africa over the past hundred thousand years or so. One of the most exciting findings from the sequencing of the Neanderthal genome was that modern-day people with Eurasian ancestry carry a few percent of their genome derived from the Neanderthal genome, via interbreeding during this secondary contact. To date the timing of this interbreeding, SANKARARAMAN *et al.* looked at the LD in modern humans between pairs of alleles found to be derived from the Neanderthal genome (and nearly absent from African populations). In Figure 2.35 we show the average LD between these loci as a function of the genetic distance ( $r$ ) between them, from the work of SANKARARAMAN *et al.*.



Assuming a recombination rate  $r$ , we can fit the exponential decay of LD predicted by eqn. (2.33) to the data points in this figure; the fit is shown as a red line. Doing this we estimate  $t = 1200$  generations, or about 35 thousand years (using a human generation time of 29 years). Thus the LD in modern Eurasians, between alleles derived from the interbreeding with Neanderthals, represents over thirty thousand years of recombination slowly breaking down these old associations.

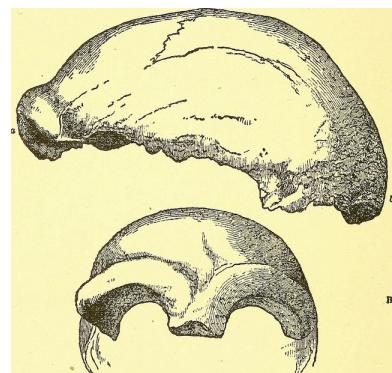


Figure 2.34: The earliest discovered fossil of a Neanderthal, fragments of skull found in a cave in the Neander Valley in Germany. Man's place in nature. 1890. Huxley, T. H.

Figure 2.35: The LD between putative-Neanderthal alleles in a modern European population (the CEU sample from the 1000 Genomes Project). Each point represents the average D statistic between a pair of alleles at a given genetic distance apart (as given on the x-axis and measured in centiMorgans (cM)). The putative Neanderthal alleles are alleles where the Neanderthal genome has a derived allele that is at very low frequency in a modern-human West African population sample (thought to have little admixture from Neanderthals). The red line is the fit of an exponential decay of LD, using non-linear least squared (nls in R).

The calculation done by SANKARARAMAN *et al.* is actually a bit more involved as they account for inhomogeneity in recombination rates and arrive at a date of 47,334 – 63,146 years.

# 3

## *Genetic Drift and Neutral Diversity*

RANDOMNESS IS INHERENT TO EVOLUTION, from the lucky birds blown of course to colonize some new oceanic island, to which mutations arise first in the HIV strain infecting an individual taking anti-retroviral drugs. One major source of stochasticity in evolutionary biology is genetic drift. Genetic drift occurs because more or less copies of an allele by chance can be transmitted to the next generation. This can occur because, by chance, the individuals carrying a particular allele can leave more or less offspring in the next generation. In a sexual population, genetic drift also occurs because Mendelian transmission means that only one of the two alleles in an individual, chosen at random at a locus, is transmitted to the offspring.

Genetic drift can play a role in the dynamics of all alleles in all populations, but it will play the biggest role for neutral alleles. A neutral polymorphism occurs when the segregating alleles at a polymorphic site have no discernible differences in their effect on fitness. We'll make clear what we mean by "discernible" later, but for the moment think of this as "no effect" on fitness.

*The neutral theory of molecular evolution.* The role of genetic drift in molecular evolution has been hotly debated since the 60s when the Neutral theory of molecular evolution was proposed (see OHTA and GILLESPIE, 1996, for a history).<sup>1</sup> The central premise of Neutral theory is that patterns of molecular polymorphism within species and substitution between species can be well understood by supposing that the vast majority of these molecular polymorphisms and substitutions were neutral alleles, whose dynamics were just subject to the vagaries of genetic drift and mutation. Early proponents of this view suggested that the vast majority of new mutations are either neutral or highly deleterious (e.g. mutations that disrupt important protein functions). This latter class of mutations are too deleterious to contribute much to common polymorphisms or substitutions be-

<sup>1</sup> KIMURA, M., 1968 Evolutionary rate at the molecular level. *Nature* *217*(5129): 624–626; KING, J. L. and T. H. JUKES, 1969 Non-darwinian evolution. *Science* *164*(3881): 788–798; and KIMURA, M., 1983 *The neutral theory of molecular evolution*. Cambridge University Press

tween species, because they are quickly weeded out of the population by selection.

Neutral theory can sound strange given that much of the time our first brush with evolution often focuses of adaptation and phenotypic evolution. However, proponents of this world-view didn't deny the existence of advantageous mutations, they simply thought that beneficial mutations are rare enough that their contribution to the bulk of polymorphism or divergence can be largely ignored. They also often thought that much of phenotypic evolution may well be adaptive, but again the loci responsible for these phenotypes are a small fraction of all the molecular change that occur. The original neutral theory of molecular evolution was original proposed to explain protein polymorphism. However, we can apply it more broadly to think about neutral evolution genome-wide. With that in mind, what types of molecular changes could be neutral? Perhaps:

1. Changes in non-coding DNA that don't disrupt regulatory sequences. For example, in the human genome only about 2% of the genome codes for proteins. The rest is mostly made up of old transposable element and retrovirus insertions, repeats, pseudo-genes, and general genomic clutter. Current estimates suggesting that, even counting conserved, functional, non-coding regions that < 10% of our genome is subject to evolutionary constraint (RANDS *et al.*, 2014).
2. Synonymous changes in coding regions, i.e. those that don't change the amino-acid encoded by a codon.
3. Non-synonymous changes that don't have a strong effect on the functional properties of the amino acid encoded, e.g. changes that don't change the size, charge, or hydrophobic properties of the amino acid too much.
4. An amino-acid change with phenotypic consequences, but little relevance to fitness, e.g. a mutation that causes your ears to be a slightly different shape, or that prevents an organism from living past 50 in a species where most individuals reproduce and die by their 20s.

There are counter examples to all of these ideas, e.g. synonymous changes can affect the translation speed and accuracy of proteins and so are subject to selection. However, the list above hopefully convinces you that the general thinking that some portion of molecular change may not be subject to selection isn't as daft as it may have initially sounded.

Various features of molecular polymorphism and divergence have been viewed as consistent with the neutral theory of molecular evo-

lution. The two we'll focus on in this chapter are the high level of molecular polymorphism in many species, see for example Figure 2.2, and the molecular clock. We'll see that various aspects of the original neutral theory have merit in describing some features and types of molecular change, but we'll also see that it is demonstrably wrong in some cases. We'll also see the primary utility of the neutral theory isn't whether it is right or wrong, but that it serves as a simple null model that can be tested and in some cases rejected, and subsequently built on. The broader debate currently in the field of molecular evolution is the balance of neutral, adaptive, and deleterious changes that drive different types of evolutionary change.

### 3.1 Loss of heterozygosity due to drift.

Genetic drift will, in the absence of new mutations, slowly purge our population of neutral genetic diversity, as alleles slowly drift to high or low frequencies and are lost or fixed over time.

Imagine a randomly mating population of a constant size  $N$  diploid individuals, and that we are examining a locus segregating for two alleles that are neutral with respect to each other. This population is randomly mating with respect to the alleles at this locus. See Figures 3.1 and 3.2 to see how genetic drift proceeds, by tracking alleles within a small population.

In generation  $t$  our current level of heterozygosity is  $H_t$ , i.e. the probability that two randomly sampled alleles in generation  $t$  are non-identical is  $H_t$ . Assuming that the mutation rate is zero (or vanishingly small), what is our level of heterozygosity in generation  $t + 1$ ?



Figure 3.1: Loss of heterozygosity over time, in the absence of new mutations. A diploid population of 5 individuals over the generations, with lines showing transmission. In the first generation every individual is a heterozygote.

In the next generation ( $t + 1$ ) we are looking at the alleles in the offspring of generation  $t$ . If we randomly sample two alleles in generation  $t + 1$  which had different parental alleles in generation  $t$ , that is just like drawing two random alleles from generation  $t$ . So the probability that these two alleles in generation  $t + 1$ , that have different parental



Figure 3.2: Loss of heterozygosity over time, in the absence of new mutations. A diploid population of 5 individuals. In the first generation I colour every allele a different colour so we can track their descendants.

alleles in generation  $t$ , are non-identical is  $H_t$ .

Conversely, if the two alleles in our pair had the same parental allele in the proceeding generation (i.e. the alleles are identical by descent one generation back) then these two alleles must be identical (as we are not allowing for any mutation).

In a diploid population of size  $N$  individuals there are  $2N$  alleles. The probability that our two alleles have the same parental allele in the proceeding generation is  $1/(2N)$  and the probability that they have different parental alleles is  $1 - 1/(2N)$ . So by the above argument, the expected heterozygosity in generation  $t + 1$  is

$$H_{t+1} = \frac{1}{2N} \times 0 + \left(1 - \frac{1}{2N}\right) H_t \quad (3.1)$$

Thus, if the heterozygosity in generation 0 is  $H_0$ , our expected heterozygosity in generation  $t$  is

$$H_t = \left(1 - \frac{1}{2N}\right)^t H_0 \quad (3.2)$$

i.e. the expected heterozygosity within our population is decaying geometrically with each passing generation. If we assume that  $1/(2N) \ll 1$  then we can approximate this geometric decay by an exponential decay (see Question 2 below), such that

$$H_t = H_0 e^{-t/(2N)} \quad (3.3)$$

i.e. heterozygosity decays exponentially at a rate  $1/(2N)$ .

In Figure 3.3 we show trajectories through time for 40 independently simulated loci drifting in a population of 50 individuals. Each population was started from a frequency of 30% some drift up and some drift down eventually being lost or fixed from the population, but on average, across simulations, the allele frequency doesn't change. We also track heterozygosity, you can see that heterozygosity sometimes goes up, and sometimes goes down, but on average we are losing heterozygosity, and this rate of loss is well predicted by eqn. (3.2).

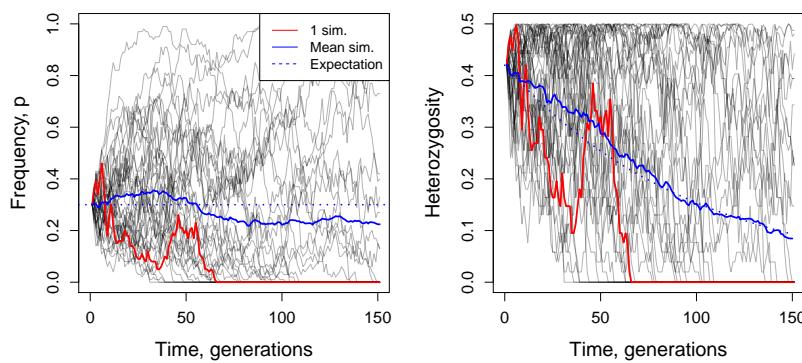


Figure 3.3: Change in allele frequency and loss of heterozygosity over time for 40 replicates. Simulations of genetic drift in a diploid population of 50 individuals, in the absence of new mutations. We start 40 independent, biallelic loci each with an initial allele at 30% frequency. The left panel shows the allele frequency over time and the right panel shows the heterozygosity over time, with the mean decay matching eqn. (3.2).

**Question 1.** You are in charge of maintaining a population of delta smelt in the Sacramento river delta. Using a large set of microsatellites you estimate that the mean level of heterozygosity in this population is 0.005. You set yourself a goal of maintaining a level of heterozygosity of at least 0.0049 for the next two hundred years. Assuming that the smelt have a generation time of 3 years, and that only genetic drift affects these loci, what is the smallest fully outbreeding population that you would need to maintain to meet this goal?

Note how this picture of decreasing heterozygosity stands in contrast to the consistency of Hardy-Weinberg equilibrium from the previous chapter. However, our Hardy-Weinberg *proportions* still hold in forming each new generation. As the offsprings' genotypes in the next generation ( $t + 1$ ) represent a random draw from the previous generation ( $t$ ), if the parental frequency is  $p_t$ , we *expect* a proportion  $2p_t(1 - p_t)$  of our offspring to be heterozygotes (and HW proportions for our homozygotes). However, because population size is finite, the observed genotype frequencies in the offspring will (likely) not match exactly with our expectations. As our genotype frequencies likely change slightly due to sampling, biologically this reflects random variation in family size and Mendelian segregation, the allele frequency will change. Therefore, while each generation represents a sample from Hardy-Weinberg proportions based on the generation before, our genotype proportions are not at an equilibrium (an unchanging state) as the underlying allele frequency changes over the generations. We'll develop some mathematical models for these allele frequency changes later on. For now, we'll simply note that under our simple model of drift (formally the Wright Fisher model), our allele count in the  $t + 1^{th}$  generation represents a binomial sample (of size  $2N$ ) from the popu-

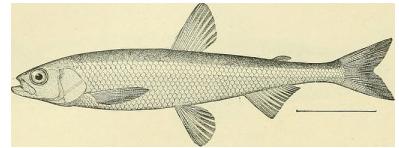


Figure 3.4: Pond smelt (*Hypomesus olidus*), a close relative of delta smelt. Bulletin of the United States Fish Commission. 1906.

lation frequency  $p_t$  in the previous generation. If you've read to here, please email Prof Coop a picture of JBS Haldane in a striped suit with the title "I'm reading the chapter 3 notes". (It's well worth googling JBS Haldane and to read more about his life; he's a true character and one of the last great polymaths. )



Figure 3.6: Loss of heterozygosity in the Black-footed Ferrets in their declining population. Data from WISELY *et al.* (2002).



Figure 3.5: The black-footed ferret (*M. nigripes*). Wild animals of North America, The National geographical society, 1918. BHL

To see how a decline in population size can affect levels of heterozygosity, let's consider the case of black-footed ferrets (*Mustela nigripes*). The black-footed ferret population has declined dramatically through the twentieth century due to destruction of their habitat. In 1979, when the last known black-footed ferret died in captivity, they were thought to be extinct. In 1981, a very small wild population was rediscovered (40 individuals), but in 1985 this population suffered a number of disease outbreaks. All of the 18 remaining wild individuals were brought into captivity, 7 of which reproduced. Thanks to intense captive breeding efforts and conservation work, a wild population of over 300 individuals has been established since. However, because all of these individuals are descended from those 7 individuals who survived the bottleneck, diversity levels remain low. WISELY *et al.* measured heterozygosity at a number of microsatellites in individuals from museum collections, showing the sharp drop in diversity as population sizes crashed (see Figure 3.6).

**Question 2.** In mathematical population genetics, a commonly used approximation is  $(1 - x) \approx e^{-x}$  for  $x \ll 1$  (formally, this

follows from the Taylor series expansion of  $\exp(-x)$ , ignoring second order and higher terms of  $x$ ). This approximation is especially useful for approximating a geometric decay process by an exponential decay process, e.g.  $(1 - x)^t \approx e^{-xt}$ . Using your calculator, or R, check how good of an approximation this is compared to the exact expression for two values of  $x$ ,  $x = 0.1$ , and  $0.01$ , across two different values of  $t$ ,  $t = 5$  and  $t = 50$ . I.e. calculate both expressions for these values, hand in your answers and briefly comment on your results.

### 3.1.1 Levels of diversity maintained by a balance between mutation and drift

Next we're going to consider the amount of neutral polymorphism that can be maintained in a population as a balance between genetic drift removing variation and mutation introducing new neutral variation, see Figure 3.7 for an example. Note in our example, how no-one allele is maintained at a stable equilibrium, rather an equilibrium level of polymorphism is maintained by a constantly shifting case of alleles.



Figure 3.7: Mutation-drift balance. A diploid population of 5 individuals. In the first generation everyone has the same allele (black). Each generation the transmitted allele can mutate and we generate a new colour. In the bottom plot, I trace the frequency of alleles in our population over time. The mutation rate we use is very high, simply to maintain diversity in this small population.

*The neutral mutation rate.* We'll first want to consider the rate at which neutral mutations arise in the population. Thinking back to our discussion of the neutral theory of molecular evolution, let's suppose that there are only two classes of mutation that can arise in our genomic region of interest: neutral mutations and highly deleterious mutations. The total mutation rate at our locus is  $\mu_T$  per generation, i.e. per transmission from parent to child. A fraction  $C$  of our mutations are new alleles that are highly deleterious and so quickly removed from the population. We'll call this  $C$  parameter the constraint, and it will differ according to the genomic region we consider. The remaining fraction  $(1 - C)$  are our neutral mutations, such that our neutral mutation rate is

$$\mu = (1 - C)\mu_T \quad (3.4)$$

This is the per generation rate.

**Question 3.** It's worth taking a minute to get familiar with both how rare, and how common, mutation is. The per base pair mutation rate in humans is around  $1.5 \times 10^{-8}$  per generation. That means, on average, we have to monitor a site for  $\sim 66.6$  million transmissions from parent to child to see a mutation. Yet populations and genomes are big places, so mutations are common at these levels.

**A)** Your autosomal genome is  $\sim 3$  billion base pairs long ( $3 \times 10^9$ ). You have two copies, the one you received from your mum and one from your dad. What is the average (i.e. the expected) number of mutations that occurred in the transmission from your mum and your dad to you?

**B)** The current human population size is  $\sim 7$  billion individuals. How many times, at the level of the entire human population, is a single base-pair mutated in the transmission from one generation to the next?

*Levels of heterozygosity maintained as a balance between mutation and selection.* Looking backwards in time from one generation to the previous generation, we are going to say that two alleles which have the same parental allele (i.e. find their common ancestor) in the preceding generation have *coalesced*, and refer to this event as a *coalescent event*.

The probability that our pair of randomly sampled alleles have coalesced in the preceding generation is  $1/(2N)$ , the probability that our pair of alleles fail to coalesce is  $1 - 1/(2N)$ .

The probability that a mutation changes the identity of the transmitted allele is  $\mu$  per generation. So the probability of no mutation occurring is  $(1 - \mu)$ . We'll assume that when a mutation occurs it creates some new allelic type which is not present in the population. This assumption (commonly called the infinitely-many-alleles model) makes the math slightly cleaner, and also is not too bad an assumption biologically. See Figure 3.7 for a depiction of mutation-drift balance in this model over the generations.

This model lets us calculate when our two alleles last shared a common ancestor and whether these alleles are identical as a result of failing to mutate since this shared ancestor. For example, we can work out the probability that our two randomly sampled alleles coalesce 2 generations in the past (i.e. they fail to coalesce in generation 1 and then coalesce in generation 2), and that they are identical as

$$\left(1 - \frac{1}{2N}\right) \frac{1}{2N} (1 - \mu)^4 \quad (3.5)$$

Note the power of 4 is because our two alleles have to have failed to

mutate through 2 meioses each.

More generally, the probability that our alleles coalesce in generation  $t + 1$  (counting backwards in time) and are identical due to no mutation to either allele in the subsequent generations is

$$P(\text{coal. in } t+1 \& \text{ no mutations}) = \frac{1}{2N} \left(1 - \frac{1}{2N}\right)^t (1 - \mu)^{2(t+1)} \quad (3.6)$$

To make this slightly easier on ourselves let's further assume that  $t \approx t + 1$  and so rewrite this as:

$$P(\text{coal. in } t+1 \& \text{ no mutations}) \approx \frac{1}{2N} \left(1 - \frac{1}{2N}\right)^t (1 - \mu)^{2t} \quad (3.7)$$

This gives us the approximate probability that two alleles will coalesce in the  $(t + 1)^{\text{th}}$  generation. In general, we may not know when two alleles may coalesce: they could coalesce in generation  $t = 1, t = 2, \dots$ , and so on. Thus, to calculate the probability that two alleles coalesce in *any* generation before mutating, we can write:

$$\begin{aligned} P(\text{coal. in any generation \& no mutations}) &\approx P(\text{coal. in } t = 1 \& \text{ no mutations}) + \\ &\quad P(\text{coal. in } t = 2 \& \text{ no mutations}) + \dots \\ &= \sum_{t=1}^{\infty} P(\text{coal. in } t \text{ generations \& no mutation}) \end{aligned}$$

which follows from basic probability and the fact that coalescing in a particular generation is mutually exclusive with coalescing in a different generation.

While we could calculate a value for this sum given  $N$  and  $\mu$ , it's difficult to get a sense of what's going on with such a complicated expression. Here, we turn to a common approximation in population genetics (and all applied mathematics), where we assume that  $1/(2N) \ll 1$  and  $\mu \ll 1$ . This allows us to approximate the geometric decay as an exponential decay. Then, the probability two alleles coalesce in generation  $t + 1$  and don't mutate can be written as:

$$P(\text{coal. in } t+1 \& \text{ no mutations}) \approx \frac{1}{2N} \left(1 - \frac{1}{2N}\right)^t (1 - \mu)^{2t} \quad (3.8)$$

$$\approx \frac{1}{2N} e^{-t/(2N)} e^{-2\mu t} \quad (3.9)$$

$$= \frac{1}{2N} e^{-t(2\mu+1/(2N))} \quad (3.10)$$

Then we can approximate the summation by an integral, giving us:

$$\frac{1}{2N} \int_0^\infty e^{-t(2\mu+1/(2N))} dt = \frac{1/(2N)}{1/(2N) + 2\mu} = \frac{1}{1 + 4N\mu} \quad (3.11)$$

The equation above gives us the probability that our two alleles coalesce at some point in time, and do not mutate before reaching their common ancestor. Equivalently, this can be thought of as the probability our two alleles coalesce *before* mutating, i.e. that they are homozygous.

Then, the complementary probability that our pair of alleles are non-identical (or heterozygous) is simply one minus this. The following equation gives the equilibrium heterozygosity in a population at equilibrium between mutation and drift:

$$H = \frac{4N\mu}{1 + 4N\mu} \quad (3.12)$$

compound parameter  $4N\mu$ , the population-scaled mutation rate, will come up a number of times so we'll give it its own name:

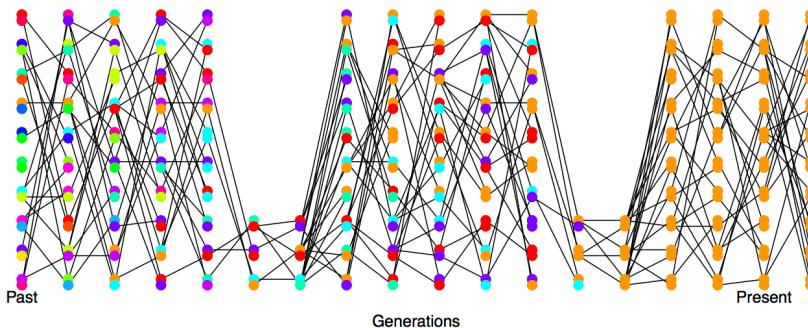
$$\theta = 4N\mu \quad (3.13)$$

So all else being equal, species with larger population sizes should have proportionally higher levels of neutral polymorphism.

**Question 4.** The sequence-level heterozygosity in *Capsella grandiflora* (grand shepherd's purse) is  $\sim 2\%$  per base. Assuming a mutation rate of  $10^{-9} bp^{-1}$  per generation, what is your estimate of the population size of *C. grandiflora*?

### 3.1.2 The effective population size

In practice, populations rarely conform to our assumptions of being constant in size with low variance in reproductive success. Real populations experience dramatic fluctuations in size, and there is often high variance in reproductive success. Thus rates of drift in natural populations are often a lot higher than the census population size would imply. See Figure 3.8 for a depiction of a repeatedly bottlenecked population losing diversity at a fast rate.



This result was derived by KIMURA and CROW (1964) and MALÉCOT (1948) (see MALÉCOT, 1969, for an English translation, the lack of earlier translation meant this result was missed). Technically we're assuming that every new mutation creates a new allele, the so-called "infinitely many alleles" model, otherwise our pair of sequences could be identical due to repeat or back mutation. See this GENETICS blog post and EWENS (2016) for a nice discussion of the history.

the effective population size ( $N_e$ ) is the population size that would result in the same rate of drift in an idealized population of constant size (following our modeling assumptions) as that observed in our true population .

Figure 3.8: Loss of heterozygosity over time in a bottlenecking population. A diploid population of 10 individuals, that bottlenecks down to three individuals repeatedly. In the first generation, I colour every allele a different colour so we can track their descendants. There are no new mutations.

To cope with this discrepancy, population geneticists often invoke the concept of an *effective population size* ( $N_e$ ). In many situations (but not all), departures from model assumptions can be captured by substituting  $N_e$  for  $N$ .

If population sizes vary rapidly in size, we can (if certain conditions are met) replace our population size by the harmonic mean population size. Consider a diploid population of variable size, whose size is  $N_t$   $t$  generations into the past. The probability our pairs of alleles have not coalesced by generation  $t$  is given by

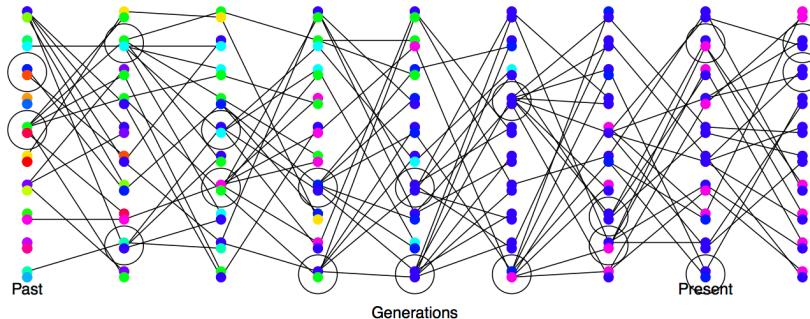
$$\prod_{i=1}^t \left(1 - \frac{1}{2N_i}\right) \quad (3.14)$$

Note that this simply collapses to our original expression  $(1 - \frac{1}{2N})^t$  if  $N_i$  is constant. Under this model, the rate of loss of heterozygosity in this population is equivalent to a population of effective size

$$N_e = \frac{1}{\frac{1}{t} \sum_{i=1}^t \frac{1}{N_i}}. \quad (3.15)$$

This is the harmonic mean of the varying population size.<sup>2</sup>

Thus our effective population size, the size of an idealized constant population which matches the rate of genetic drift, is the harmonic mean true population size over time. The harmonic mean is very strongly affected by small values, such that if our population size is one million 99% of the time but drops to 1000 every hundred or so generations,  $N_e$  will be much closer to 1000 than a million.



Variance in reproductive success will also affect our effective population size. Even if our population has a large constant size  $N$  individuals, if only small proportion of them get to reproduce, then the rate of drift will reflect this much smaller number of reproducing individuals. See Figure 3.9 for a depiction of the higher rate of drift in a population where there is high variance in reproductive success.

To see one example of this, consider the case where  $N_F$  of females get to reproduce and  $N_M$  males get reproduce. While every individual

<sup>2</sup> To see this, note that if  $1/(N_i)$  is small, then we can approximate (3.14) using the exponential approximation:

$$\prod_{i=1}^t \exp\left(-\frac{1}{2N_i}\right) = \exp\left(-\sum_{i=1}^t \frac{1}{2N_i}\right). \quad (3.16)$$

When we put the product inside the exponent, it becomes a sum. We can also write the probability of not coalescing by generation  $t$  in a population of constant size ( $N_e$ ) as an exponential, so that it takes the same form as the expression above on the right. Comparing the exponent in the two cases, we see

$$\frac{t}{2N_e} = \sum_{i=1}^t \frac{1}{2N_i} \quad (3.17)$$

So that if we want a constant effective population size ( $N_e$ ) that has the same rate of loss of heterozygosity as our variable population, we need to rearrange and solve this equation to give (3.15).

Figure 3.9: High variance on reproductive success increases the rate of genetic drift. A diploid population of 10 individuals, where the circled individuals have much higher reproductive success. In the first generation I colour every allele a different colour so we can track their descendants, there are no new mutations.

has a mother and a father, not every individual gets to be a parent. In practice, in many animal species far more females get to reproduce than males, i.e.  $N_M < N_F$ , as a few males get many mating opportunities and many males get no/few mating opportunities (see JANICKE *et al.*, 2016, for a broad analysis, and note that there are certainly many exceptions to this general pattern). When our two alleles pick an ancestor, 25% of the time our alleles were both in a female ancestor, in which case they are IBD with probability  $1/(2N_F)$ , and 25% of the time they are both in a male ancestor, in which case they coalesce with probability  $1/(2N_M)$ . The remaining 50% of the time, our alleles trace back to two individuals of different sexes in the prior generation and so cannot coalesce. Therefore, our probability of coalescence in the preceding generation is

$$\frac{1}{4} \left( \frac{1}{2N_M} \right) + \frac{1}{4} \left( \frac{1}{2N_F} \right) \quad (3.18)$$

i.e. the rate of coalescence is the harmonic mean of the two sexes' population sizes, equating this to  $\frac{1}{2N_e}$  we find

$$N_e = \frac{4N_F N_M}{N_F + N_M} \quad (3.19)$$

Thus if reproductive success is very skewed in one sex (e.g.  $N_M \ll N/2$ ), our effective population size will be much reduced as a result. For more on how different evolutionary forces affect the rate of genetic drift, and their impact on the effective population size, see CHARLESWORTH (2009).

**Question 5.** You are studying a population of 500 males and 500 females Hamadryas baboons. Assume that all of the females but only 1/10 of the males get to mate: **A)** What is the effective population size for the autosome?

**B)** Do you expect the *ratio* of X-chromosome to autosomal diversity to be higher or lower in this species compared to a species where the sexes have more similar variance in reproductive success? Explain the intuition behind your answer.

### 3.2 The Coalescent and patterns of neutral diversity

"Life can only be understood backwards; but it must be lived forwards." – Kierkegaard

*Pairwise Coalescent time distribution and the number of pairwise differences.* Thinking back to our calculations we made about the loss of neutral heterozygosity and equilibrium levels of diversity (in Sections 3.1 and 3.1.1), you'll note that we could first specify which



Figure 3.10: Male Hamadryas baboons. Brehm's Tierleben. Brehm, A.E. 1893. Up to ten females live in a harem with a single male.

generation a pair of sequences coalesce in, and then calculate some properties of heterozygosity based on that. That's because neutral mutations do not affect the probability that an individual transmits an allele, and so don't affect the way in which we can trace ancestral lineages back through the generations.

As such, it will often be helpful to consider the time to the common ancestor of a pair of sequences, and then think of the impact of that time to coalescence on patterns of diversity. See Figure 3.11 for an example of this.



Figure 3.11: A simple simulation of the coalescent process. The simulation consists of a diploid population of 10 individuals (20 alleles). In each generation, each individual is equally likely to be the parent of an offspring (and the allele transmitted is indicated by a light grey line). We track a pair of alleles, chosen in the present day, back 14 generations until they find a common ancestor.

The probability that a pair of alleles have failed to coalesce in  $t$  generations and then coalesce in the  $t + 1$  generation back is

$$P(T_2 = t + 1) = \frac{1}{2N} \left(1 - \frac{1}{2N}\right)^t \quad (3.20)$$

Thus the coalescent time of our pair of alleles is a Geometrically distributed random variable, where the probability of success is  $1/(2N)$ ; we denote this by  $T_2 \sim \text{Geo}(1/(2N))$ . The expected (i.e. the mean over many replicates) coalescent time of a pair of alleles is then

$$\mathbb{E}(T_2) = 2N \quad (3.21)$$

generations.

Conditional on a pair of alleles coalescing  $t$  generations ago, there are  $2t$  generations in which a mutation could occur. If the per generation mutation rate is  $\mu$ , then the expected number of mutations between a pair of alleles coalescing  $t$  generations ago is  $2t\mu$  (the alleles have gone through a total of  $2t$  meioses since they last shared a common ancestor). So we can write the expected number of mutations

Blurring our eyes a little, we can see that 3.20 is

$$\approx \frac{1}{2N} e^{-t/(2N)} \quad (3.22)$$

and so think of a continuous random variable, i.e. we could say that the coalescent time of a pair of sequences ( $T_2$ ) is approximately exponentially distributed with a rate  $1/(2N)$ , i.e.  $T_2 \sim \text{Exp}(1/(2N))$ . Formally we can do this by taking the limit of the discrete process more carefully.

$(S_2)$  separating two alleles drawn at random from the population as

$$\begin{aligned}\mathbb{E}(S_2) &= \sum_{t=0}^{\infty} \mathbb{E}(S_2|T_2 = t)P(T_2 = t) \\ &= \sum_{t=0}^{\infty} 2\mu t P(T_2 = t) \\ &= 2\mu \mathbb{E}(T_2) \\ &= 4\mu N\end{aligned}\tag{3.23}$$

We'll assume that mutation is rare enough that it never happens at the same basepair twice, i.e. no multiple hits, such that we get to see all of the mutation events that separate our pair of sequences.<sup>3</sup> Thus the number of mutations between a pair of sites is the observed number of differences between a pair of sequences. In the previous chapter we denote the observed number of pairwise differences at putatively neutral sites separating a pair of sequences as  $\pi$  (we usually average this over a number of pairs of sequences for a region). Therefore, under our simple, neutral, constant population-size model we expect

$$\mathbb{E}(\pi) = 4N\mu = \theta\tag{3.24}$$

So we can get an empirical estimate of  $\theta$  from  $\pi$ , let's call this  $\hat{\theta}_\pi$ , by setting  $\hat{\theta}_\pi = \pi$ , i.e. our observed level of pairwise genetic diversity. If we have an independent estimate of  $\mu$ , then from setting  $\pi = \hat{\theta}_\pi = 4N\mu$  we can furthermore obtain an estimate of the population size  $N$  that is consistent with our levels of neutral polymorphism. If we estimate the population size this way, we should call it the effective coalescent population size ( $N_e$ ). It's best to think about  $N_e$  estimated from neutral diversity as a long-term, effective population size for the species, but there's a boat load of caveats that come along with that assumption. For example, past bottlenecks and population expansions are all subsumed into a single number and so this estimated  $N_e$  may not be very representative of the population size at any time. That said, it's not a bad place to start when thinking about the rate of genetic drift for neutral diversity in our population over long time-periods.<sup>4</sup>

Lets take a moment to distinguish our expected heterozygosity (eqn. 3.12) from our expected number of pairwise differences ( $\pi$ ). Our expected heterozygosity is the probability that two alleles at a locus, sampled from a population at random, are different from each other. If one or more mutations have occurred since a pair of alleles last shared a common ancestor, then our sequences will be different from each other. On the other hand, our  $\pi$  measure keeps track of the average total number of differences between our loci. As such,  $\pi$  is often a more useful measure, as it records the number of differences between

<sup>3</sup> This is called the infinitely-many-sites assumption, which should be fine if  $N\mu_{BP} \ll 1$ , where  $\mu_{BP}$  is the mutation rate per base pair.

<sup>4</sup> Up to this point we've been describing only neutral processes, however, selection can also alter levels of polymorphism. For example, if some synonymous sites directly experience selection, then even if we use  $\pi$  calculated for on synonymous changes we may underestimate the coalescent effective population size. As we'll see later in the notes, selection at linked sites can also impact neutral diversity. As such, if we can, we may want to use genomic sites subject to the weakest selective constraints, and also far from gene-dense or otherwise very constrained regions of the genome, to estimate  $N_e$  from  $\pi$ . But even then caution is warranted.

the sequences, not just whether they are different from each other (however, for certain types of loci, e.g. microsatellites, heterozygosity is often used as we cannot usually count up the minimum number of mutations in a sensible way). In the case where our locus is a single basepair, the two measures will usually be close to one another, as  $H \approx \theta$  for small values of  $\theta$ . For example, comparing two sequences at random in humans,  $\pi \approx 1/1000$  per basepair, and the probability that a specific base pair differs between two sequences is  $\approx 1/1000$ . However, these two quantities start to differ from each other when we consider regions with higher mutation rates. For example, if we consider a 10kb region, our mutation rate will 10,000 times larger than a single base pair. For this length of sequence the probability that two randomly chosen haplotypes differ is quite different from the number of mutational differences between them. (Try a mutation rate of  $10^{-8}$  per base and a population size of 10, 000 in our calculations of  $\mathbb{E}[\pi]$  and  $H$  to see this.)

**Question 6.** ROBINSON *et al.* (2016) found that the endangered Californian Channel Island fox on San Nicolas had very low levels of diversity ( $\pi = 0.000014\text{bp}^{-1}$ ) compared to its close relative the California mainland gray fox ( $0.0012\text{bp}^{-1}$ ).

**A)** Assuming a mutation rate of  $2 \times 10^{-8}$  per bp, what effective population sizes do you estimate for these two populations?

**B)** Why is the effective population size of the Channel Island fox so low? [Hint: quickly google Channel island foxes to read up on their history, also to see how ridiculously cute they are.]

**Question 7.** In your own words describe why the coalescent time of a pair of lineages scales linearly with the (effective) population size.

*More details on the pairwise coalescent and the randomness of mutation.* We've derived the expected number of differences between a pair of sequences and talked about how variable the coalescent time is for a pair of sequences. The mutation process is also very variable; even if two sequences coalesce in the very distant past by chance, they may still be identical in the present if there was no mutation during that time.

Conditional on the coalescent time  $t$ , the probability that our pair of alleles are separated by  $S_2$  mutations since they last shared a common ancestor is

$$P(S_2|T_2 = t) = \binom{2t}{j} \mu^j (1 - \mu)^{2t-j} \quad (3.25)$$

i.e. mutations happen in  $j$  generations and do not happen in  $2t - j$  generations (with  $\binom{2t}{j}$  ways this combination of events can possibly



Figure 3.12: Gray Fox, *Urocyon cinereoargenteus*. Pearson and Warren. Diseases and enemies of poultry. (1897) BHL.

happen). Assuming that  $\mu \ll 1$  and that  $2t - j \approx 2t$ , then we can approximate the probability that we have  $S_2$  mutations as a Poisson distribution:

$$P(S_2|T_2 = t) = \frac{(2\mu t)^j e^{-2\mu t}}{j!} \quad (3.26)$$

i.e. a Poisson with mean  $2\mu t$ . We'll not make much use of this result, but it is very useful in thinking about how to simulate the process of mutation.

### 3.3 The coalescent process of a sample of alleles.

Usually we are not just interested in pairs of alleles, or the average pairwise diversity. Generally we are interested in the properties of diversity in samples of a number of alleles drawn from the population. Instead of just following a pair of lineages back until they coalesce, we can follow the history of a sample of alleles back through the population.

Consider first sampling three alleles at random from the population. The probability that all three alleles choose exactly the same ancestral allele one generation back is  $1/(2N)^2$ . If  $N$  is reasonably large, then this is a very small probability. As such, it is very unlikely that our three alleles coalesce all at once, and in a moment we'll see that it is safe to ignore such unlikely events.



Figure 3.13: A simple simulation of the coalescent process for three lineages. We track the ancestry of three modern-day alleles, the first pair (blue and purple) coalesce four generations back, after which there are only two independent lineages we are tracking. This pair then coalesces twelve generations in the past. Note that different random realizations of this process will differ from each other a lot.

The probability that a specific pair of alleles find a common ancestor in the preceding generation is still  $1/(2N)$ . There are three possible pairs of alleles, so the probability that no pair finds a common ances-

tor in the preceding generation is

$$\left(1 - \frac{1}{2N}\right)^3 \approx \left(1 - \frac{3}{2N}\right) \quad (3.27)$$

In making this approximation we are multiplying out the right hand-side and ignoring terms of  $1/N^2$  and higher. See Figure 3.13 for a random realization of this process.

More generally, when we sample  $i$  alleles there are  $\binom{i}{2}$  pairs,<sup>5</sup> i.e.  $i(i - 1)/2$  pairs. Thus, the probability that no pair of alleles in a sample of size  $i$  coalesces in the preceding generation is

$$\left(1 - \frac{1}{(2N)}\right)^{\binom{i}{2}} \approx \left(1 - \frac{\binom{i}{2}}{2N}\right) \quad (3.28)$$

while the probability any pair coalesces is  $\approx 2N/\binom{i}{2}$ .

We can ignore the possibility that more than pairs of alleles (e.g. tripletons) simultaneously coalesce at once as terms of  $1/N^2$  and higher can be ignored as they are vanishingly rare. Obviously in reasonable sample sizes there are many more triples ( $\binom{i}{3}$ ) and higher order combinations than there are pairs ( $\binom{i}{2}$ ), but if  $i \ll N$  then we are safe to ignore these terms.

When there are  $i$  alleles, the probability that we wait until the  $t + 1$  generation before any pair of alleles coalesces is

$$P(T_i = t + 1) = \frac{\binom{i}{2}}{2N} \left(1 - \frac{\binom{i}{2}}{2N}\right)^t \quad (3.29)$$

Thus the waiting time to the first coalescent event while there are  $i$  lineages is a geometrically distributed random variable with probability of success  $\binom{i}{2}/2N$ , which we denote by

$$T_i \sim \text{Geo}\left(\binom{i}{2}/2N\right). \quad (3.30)$$

The mean waiting time till any of pair within our sample coalesces is

$$\mathbb{E}(T_i) = \frac{2N}{\binom{i}{2}} \quad (3.31)$$

After a pair of alleles first finds a common ancestral allele some number of generations back in the past, we only have to keep track of that common ancestral allele for the pair when looking further into the past. Thus when a pair of alleles in our sample of  $i$  alleles coalesces, we then switch to having to follow  $i - 1$  alleles back in time. Then when a pair of these  $i - 1$  alleles coalesce, we then only have to follow  $i - 2$  alleles back. This process continues until we coalesce back to a sample of two, and from there to a single most recent common ancestor (MRCA).

<sup>5</sup> said as “i choose 2”

To see the continuous time version of this, note that (3.29) is

$$\approx \frac{\binom{i}{2}}{2N} \exp\left(-\frac{\binom{i}{2}}{2N} t\right) \quad (3.32)$$

The waiting time  $T_i$  to the first coalescent event in a sample of  $i$  alleles is thus exponentially distributed with rate  $\binom{i}{2}/2N$ , i.e.  $T_i \sim \text{Exp}\left(\binom{i}{2}/2N\right)$ .

*Simulating a coalescent genealogy* To simulate a coalescent genealogy at a locus for a sample of  $n$  alleles we therefore simply follow the following algorithm:

1. Set  $i = n$ .
2. Simulate a random variable to be the time  $T_i$  to the next coalescent event from  $T_i \sim \text{Exp}((\frac{i}{2})/2N)$
3. Choose a pair of alleles to coalesce at random from all possible pairs.
4. Set  $i = i - 1$
5. Continue looping steps 1-3 until  $i = 1$ , i.e. the most recent common ancestor of the sample is found.

By following this algorithm we are generating realizations of the genealogy of our sample.

### 3.3.1 Expected properties of coalescent genealogies and mutations.

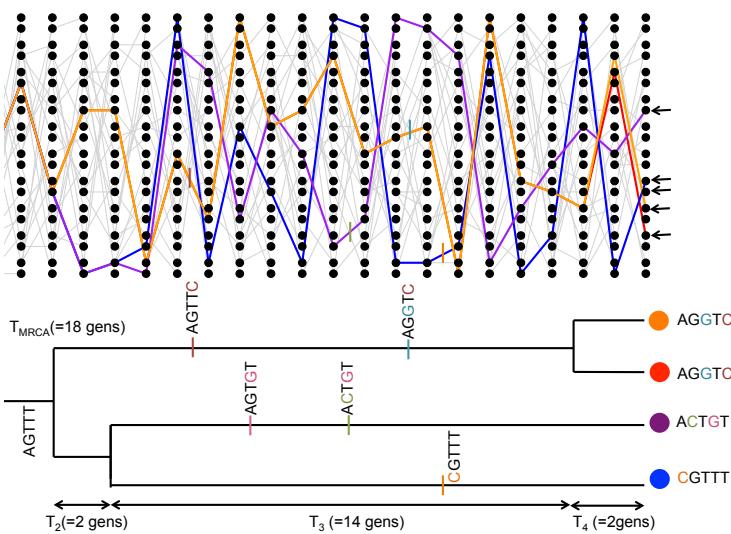


Figure 3.14: A simple coalescent tree from a single coalescent simulation, tracing the genealogy of 4 alleles with mutational changes marked with dashes showing transitions away from the MRCA sequence (AGTTT). The  $T_{MRCA}$  is the sum of  $T_4$ ,  $T_3$ , and  $T_2$ . The total time in the tree is  $T_{tot} = 4T_4 + 2T_3 + 2T_2 = 54$  generations.

*The expected time to the most recent common ancestor.* We will first consider the time to the most recent common ancestor of the entire sample ( $T_{MRCA}$ ). This is

$$T_{MRCA} = \sum_{i=n}^2 T_i \quad (3.33)$$

generations back, where we are summing from  $i = n$  alleles counting backwards to  $i = 2$  alleles (see Figure 3.14 for example). As our coalescent times for different  $i$  are independent, the expected time to the most recent common ancestor is

$$\mathbb{E}(T_{MRCA}) = \sum_{i=n}^2 \mathbb{E}(T_i) = \sum_{i=n}^2 2N / \binom{i}{2} \quad (3.34)$$

Using the fact that  $\frac{1}{i(i-1)} = \frac{1}{i-1} - \frac{1}{i}$  and a bit of rearrangement, we can rewrite this as

$$\mathbb{E}(T_{MRCA}) = 4N \left( 1 - \frac{1}{n} \right) \quad (3.35)$$

So the average  $T_{MRCA}$  scales linearly with population size  $N$ . Interestingly, as we move to larger and larger samples (i.e.  $n \gg 1$ ), the average time to the most recent common ancestor converges on  $4N$ . What's happening here is that in large samples our lineages typically coalesce rapidly at the start and very soon coalesce down to a much smaller number of lineages.

**Question 8.** Assume an autosomal effective population of 10,000 individuals (roughly the long-term human estimate) and a generation time of 30 years. What is the expected time to the most recent common ancestor of a sample of 20 people? What is this time for a sample of 500 people?

*The expected total time in a genealogy and the number of segregating sites.* Mutations fall on specific lineages of the coalescent genealogy and are transmitted to all descendants of their lineage. Furthermore, under the infinitely-many-sites assumption, each mutation creates a new segregating site. The mutation process is a *Poisson process*, and the longer a particular lineage, i.e. the more generations of meioses it represents, the more mutations that can accumulate on it. The total number of segregating sites in a sample is thus a function of the *total* amount of time in the genealogy of the sample, or the sum of all the branch lengths on the genealogical tree,  $T_{tot}$ . Our total amount of time in the genealogy is

$$T_{tot} = \sum_{i=n}^2 iT_i \quad (3.36)$$

as when there are  $i$  lineages, each contributes a time  $T_i$  to the total time (see Figure 3.14 for an example). Taking the expectation of the total time in the genealogy,

$$\mathbb{E}(T_{tot}) = \sum_{i=n}^2 i \frac{2N}{\binom{i}{2}} = \sum_{i=n}^2 \frac{4N}{i-1} = \sum_{i=n-1}^1 \frac{4N}{i} \quad (3.37)$$

we see that our expected total amount of time in the genealogy scales linearly with our population size  $N$ . Our expected total amount of time is also increasing with sample size  $n$ , but is doing so very slowly.

This again follows from the fact that in large samples, the initial coalescence usually happens very rapidly, so that extra samples add little to the total amount of time in the genealogical tree.

We saw above that the number of mutational differences between a pair of alleles that coalescence  $T_2$  generations ago was Poisson with a mean of  $2\mu T_2$ , where  $2T_2$  is the total branch length in this simple 2-sample genealogical tree. A mutation that occurs on any branch of our genealogy will cause a segregating polymorphism in the sample (meeting our infinitely-many-sites assumption). Thus, if the total time in the genealogy is  $T_{tot}$ , there are  $T_{tot}$  generations for mutations. So the total number of mutations segregating in our sample ( $S$ ) is Poisson with mean  $\mu T_{tot}$ . Thus the expected number of segregating sites in a sample of size  $n$  is

$$\mathbb{E}(S) = \mu \mathbb{E}(T_{tot}) = \sum_{i=n-1}^1 \frac{4N\mu}{i} = \theta \sum_{i=n-1}^1 \frac{1}{i} \quad (3.38)$$

Note that this is growing with the sample size  $n$ , albeit very slowly (roughly at the rate of the log of the sample size). We can use this formula to derive another estimate of the population scaled mutation rate  $\theta$ , by setting our observed number of segregating sites in a sample ( $S$ ) equal to this expectation. We'll call this estimator  $\hat{\theta}_W$ :

$$\hat{\theta}_W = \frac{S}{\sum_{i=n-1}^1 1/i} \quad (3.39)$$

This estimator of  $\theta$  was devised by WATTERSON (1975), hence the  $W$ .

*The neutral site-frequency spectrum.* We can use our coalescent process to find the expected number of derived alleles present  $i$  times out of a sample size  $n$ , e.g. how many singletons ( $i = 1$ ) do we expect to find in our sample? For example, in Figure 3.14 in our sample of four sequences, there are 3 singletons and 2 doubletons. The number of sites with these different allele frequencies depends on the lengths of specific genealogical branches. A mutation that falls on a branch with  $i$  descendants will create a derived allele with frequency  $i$ . For example, in our example tree in Figure 3.14, the total number of generations where a mutation could arise and be a doubleton is  $T_3 + 2T_2$ , the total length of the branch ancestral to just the orange and red allele ( $T_3 + T_2$ ) plus the branch ancestral to just the blue and purple allele ( $T_2$ ).

To get a better sense of how  $T_{tot}$  grows with the sample size, we can approximate the sum 3.37 by an integral, which will work for large  $n$ . The result is  $\int_1^{n-1} \frac{4N}{i} di = 4N \log(n - 1)$ .

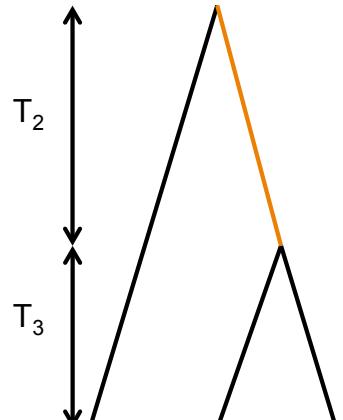


Figure 3.15: A tree for three samples; note that this is the only possible tree shape (treating the tips as unlabeled, i.e. I don't care which pair of sequences carry a doubleton, just that

To see how we could go about working this out, lets start by considering the simple coalescent tree, shown in Figure 3.15, for sample of 3 alleles drawn from a population. Mutations that fall on the branches coloured in black will be derived singletons, while mutations that fall along the orange branch will be doubletons in the sample. The total number of generations where a singleton mutation could arise is  $3T_3 + T_2$ . Note that we only count the time where there are two lineages ( $T_2$ ) once. So our expected number of singletons, using eqn (3.31), is

$$\mathbb{E}(S_i) = \mu (3\mathbb{E}(T_3) + \mathbb{E}(T_2)) = \mu \left( 3 \frac{2N}{3} + 2N \right) = \theta \quad (3.40)$$

By similar logic, the time where doubletons could arise is  $T_2$  and our expected number of doubletons is  $\mathbb{E}(S_i) = \theta/2$ . Thus, there are on average half as many doubletons as singletons.

Extending this logic to larger samples might be doable, but is tedious (I mean really tedious: for 10 alleles there are thousands of possible tree shapes and the task quickly gets impossible even computationally). A nice, relatively simple proof of the neutral site frequency spectrum is given by HUDSON, but we won't give this here. The general form is:

$$\mathbb{E}(S_i) = \frac{\theta}{i} \quad (3.41)$$

i.e. there are twice as many singletons as doubletons, three times as many singletons as tripletons, and so on. The other thing that will be helpful for us to know is that neutral alleles at intermediate frequency tend to be old, and those that are rare in the sample are young. We expect to see a lot more rare alleles in our sample than common alleles.

**Question 9.** There are two possible tree shapes that could relate four samples. Draw both of them and separately colour (or otherwise mark) the branches by where singletons, doubletons, and tripleton derived alleles could arise.

We can also ask the probability of observing a derived allele segregating at frequency  $i/n$  given that the site is polymorphic in our sample of size  $n$  (i.e. given that  $0 < i < n$ ). We can obtain this probability by dividing the expected number of sites segregating for an allele at frequency  $i$  by the expected number segregating at all of the possible allele frequencies for polymorphisms in our sample

$$P(i|0 < i < n) = \frac{\mathbb{E}(S_i)}{\sum_{j=1}^{n-1} \mathbb{E}(S_j)} = \frac{1/i}{\sum_{j=1}^{n-1} 1/j}. \quad (3.42)$$

We can interpret this probability as the fraction of polymorphic sites we expect to find at a frequency  $i/n$ .

*tests based on the site frequency spectrum* Population geneticists have proposed a variety of ways to test whether an observed site frequency spectrum conforms to its neutral, constant-population expectations. These tests are useful for detecting population size changes using data across many loci, or for detecting the signal of selection at individual loci. One of the first tests was proposed by TAJIMA, and is called Tajima's  $D$ . Tajima's  $D$  is

$$D = \frac{\theta_\pi - \theta_W}{C} \quad (3.43)$$

where the numerator is the difference between the estimate of  $\theta$  based on pairwise differences and that based on segregating sites. As these two estimators both have expectation  $\theta$  under the neutral, constant-population model, the expectation of  $D$  is zero. The denominator  $C$  is a positive constant; it's the square-root of an estimator of the variance of this difference under the constant population size, neutral model. This constant was chosen for  $D$  to have mean zero and variance 1 under the null model, so we can test for departures from this simple null model.

An excess of rare alleles compared to the constant-population, neutral model will result in a negative Tajima's  $D$ , because each additional rare allele increases the number of segregating sites by 1, but only has a small effect on the number of pairwise differences between samples. In contrast, a positive Tajima's  $D$  reflects an excess of intermediate frequency alleles relative to the constant-population, neutral expectation. Alleles at intermediate-frequency increase pairwise diversity more per segregating site than typical, thus increasing  $\theta_\pi$  more than  $\theta_W$ .

### 3.3.2 Demography and the coalescent

We've already seen how changes in population size can change the rate at which heterozygosity is lost from the population (see the discussion around eqn. (3.14)). If the population size in generation  $i$  is  $N_i$ , the probability that a pair of lineages coalesce is  $1/2N_i$ ; this conforms to our intuition that if the population size is small, the rate at which pairs of lineages find their common ancestor is faster. We can potentially accommodate rapid random fluctuations in population size by simply using the effective population size  $N_e$  in place of  $N$ . However, longer term more systematic changes in population size will distort the coalescent genealogies, and hence patterns of diversity, in more systematic ways.

We can see how demography potentially distorts the observed frequency spectrum away from the neutral expectation in a very large sample of humans shown in Figure 3.20. For comparison, the neu-

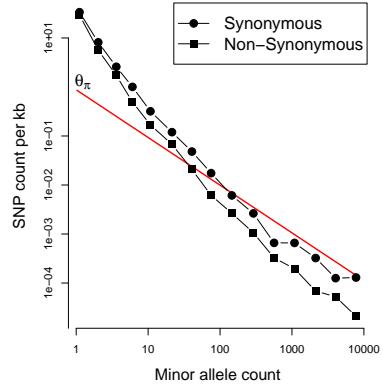


Figure 3.16: Data from 202 genes from 14002 people of European ancestry (28004 alleles). Note the double log-scale. Redrawn from NELSON *et al.*. The red line gives the neutral, constant population size estimate of the site frequency spectrum, our equation (3.41), using a  $\theta$  estimated from  $\pi$ . Note how the non-synonymous changes are even more skewed towards rare alleles, that's likely due to selection against non-synonymous alleles acting to push them towards rare frequency.

tral frequency spectrum, eqn (3.41), is shown as a red line. There are vastly more rare alleles than expected under our neutral, constant-population-size model, but the neutral prediction and reality agree somewhat more for alleles that are more common.



Why is this? Well, these patterns are likely the result of the very recent explosive growth in human populations. If the population has grown rapidly, then the pairwise-coalescent rate in the past may be much higher than the coalescent rate closer to the present. (see Figure 3.17).

One consequence of a recent population expansion is that there is much less genetic diversity in the population than you'd predict using the census population size. Humans are one example of this effect; there are 7 billion of us alive today, but this is due to very rapid population growth over the past thousand to tens of thousands of years. Our level of genetic diversity is very much lower than you'd predict given our census size, reflecting our much smaller ancestral population. A second consequence of recent population expansion is that the deeper coalescent branches are much more squished together in time, compared to those in a constant population. Mutations on deeper branches are the source of alleles at more intermediate frequencies, and so there are even fewer intermediate-frequency alleles in growing populations. That's why there are so many rare alleles, especially singletons, in this large sample of Europeans.

Another common demographic scenario is a population bottleneck. In a bottleneck, the population size crashes dramatically, and sub-

Figure 3.17: A realization of the coalescent process in a growing population. The population underwent a period of doubling every generation. The initial population size of just two individuals, maintained for a number of generations, is obviously highly unrealistic but serves our purpose.

sequently recovers. For example, our population may have had size  $N_{\text{Big}}$  and crashed down to  $N_{\text{Small}}$ . One example of a bottleneck is shown in Figure 3.18. Looking at a sample of lineages drawn from the



population today, if the bottleneck was somewhat recent ( $\ll N_{\text{Big}}$  generations in the past) many of our lineages will not have coalesced before reaching the bottleneck, moving backward in time. But during the bottleneck our lineages coalesce at a much higher rate, such that many of our lineages will coalesce if the bottleneck lasts long enough ( $\sim N_{\text{Small}}$  generations). If the bottleneck is very strong, then all of our lineages will coalesce during the bottleneck, and the resulting site frequency spectrum may look very much like our population growth model (i.e. an excess of rare alleles). However, if some pairs of lineages escape coalescing during the bottleneck, they will coalesce much more deeply in time (e.g. the blue and orange ancestral lineages in 3.18).



An example of this is shown Figure 3.19, data from BRANDVAIN *et al.*. *Mimulus nasutus* is a selfing species that arose recently from an out-crossing progenitor *M. guttatus*, and experienced a strong bottleneck. *M. guttatus* has a very high levels of genetic diversity ( $\pi = 4\%$  at synonymous sites), but *M. nasutus* has lost much of this diversity

Figure 3.18: A realization of the coalescent process in a bottlenecked population. Our population underwent a bottleneck eight generations in the past.

Figure 3.19: Diversity along the *Mimulus* genome. Black dots give  $\pi$  in 1kb windows between chromosomes sampled from two individuals, the red line is a moving average (data from BRANDVAIN *et al.*). Pairwise coalescent times ( $t$ ) estimated assuming  $t = \pi/2\mu$  using  $\mu_{BP} = 10^{-9}$ .



Figure 3.20: *M. guttatus* by Pierre-Joseph Redouté.

( $\pi = 1\%$ ). Looking along the genome, between a pair of *M. guttatus* chromosomes, levels of diversity are fairly uniformly high.

But in comparing two *M. nasutus* chromosomes, diversity is low because the pair of lineages generally coalesce recently. Yet in a few places we see levels of diversity comparable to *M. guttatus*; these regions correspond to genomic sites where our pair of lineages fail to coalesce during the bottleneck and subsequently coalesce much more deeply in the ancestral *M. guttatus* population.



Figure 3.21: Data for polymorphism from Maize and Teosinte: 774 genes redrawn from WRIGHT *et al.* (2005). **Left)** Genetic diversity levels in maize and Teosinte samples at each of these genes. Note how diversity levels are lower in maize than teosinte, i.e. most points are below the red  $x = y$  line. **Right)** The distribution of Tajima's D in maize and teosinte, see how the maize distribution is shifted towards positive values.

Mutations that arise on deeper lineages will be at intermediate frequency in our sample, and so mild bottlenecks can lead to an excess of intermediate frequency alleles compared to the standard constant-population model. This can skew Tajima's D, see eqn 3.43, towards positive values and away from its expectation of zero. One example of this skew is shown in Figure 3.21. Maize (*Zea mays* subsp.*mays*) was domesticated from its wild progenitor teosinte (*Zea mays* subsp.*parviglumis*) roughly ten thousand years ago. We can see how the bottleneck associated with domestication has resulted in a loss of genetic diversity in maize, compared to teosinte, and the polymorphism that remains is somewhat skewed towards intermediate frequencies resulting in more positive values of Tajima's D.

**Question 10.** VOIGHT *et al.* (2005) sequenced 40 autosomal regions from 15 diploid samples of Hausa people from Yaounde, Cameroon. The average length of locus they sequenced for each region was 2365bp. They found that the average number of segregating sites per locus was  $S = 11.1$  and the average  $\pi = 0.0011$  per base over the loci. Is Tajima's D positive or negative?

### 3.4 Molecular Evolution and the fixation of neutral alleles

"history is just one damn thing after another" -Arnold Toynbee



Figure 3.22: Maize (*Zea mays*). Prof. Dr. Thomé's Flora von Deutschland. 1886. Thomé, O. W.

It is very unlikely that a rare neutral allele accidentally drifts up to fixation; more likely, such an allele will be eventually lost from the population. However, populations experience a large and constant influx of rare alleles due to mutation, so even if it is very unlikely that an individual allele fixes within the population, some neutral alleles will fix by chance.



Figure 3.23: Each allele initially present in a small diploid population is given a different colour so we can track their descendants over time. By the 9th generation, all of the alleles present in the population can trace their ancestry back to the orange allele.

*Probability of the eventual fixation of a neutral allele* An allele which reaches fixation within a population is an ancestor to the entire population. In a particular generation there can only be a single allele that all other alleles at the locus in a later generation can claim as an ancestor (See Figure 3.23). At a neutral locus, the actual allele does not affect the number of descendants that the allele has (this follows from the definition of neutrality: neutral alleles don't leave more or less descendants on average than other neutral alleles). An equivalent way to state this is that the allele labels don't affect anything; thus the alleles are *exchangeable*. As a consequence of being exchangeable, any allele is equally likely to be the ancestor of the entire population. In a diploid population of size  $N$ , there are  $2N$  alleles, all of which are equally likely to be the ancestor of the entire population at some later time point. So if our allele is present in a single copy, the chance that it is the ancestor to the entire population in some future generation is  $1/(2N)$ , i.e. the chance our neutral allele is eventually fixed is  $1/(2N)$ . In Figure 3.23, our orange allele in the first generation is one of 10 differently coloured alleles, and so has a  $1/10$  chance of being the ancestor of the entire population at some later time point (and in this simulation it does become the common ancestor, by the 9th generation).

More generally, if our neutral allele is present in  $i$  copies in the population, of  $2N$  alleles, the probability that this allele becomes fixed

is  $i/(2N)$ , i.e. the probability that a neutral allele is eventually fixed is simply given by its frequency ( $p$ ) in the population. (We can also derive this result by letting  $Ns \rightarrow 0$  in eqn. (7.12), a result we'll encounter later.)

A newly arisen mutation only becomes a fixed difference if it is lucky enough to be the ancestor of the entire population. As we saw above, this occurs with probability  $1/(2N)$ .

How long does it take on average for such an allele to fix within our population? Well, in developing equation (3.35) we've seen that it takes  $4N$  generations for a large sample of alleles to all trace their ancestry back to a single most recent common ancestral allele. Any single-base pair change which arose as a single mutation at a locus, and fixed in the population, must have been present in the sequence transmitted by the most recent common ancestor of the population at that locus. Thus it must take roughly  $4N$  generations for a neutral allele present in a single copy within the population to the ancestor of all alleles within our population. This argument can be made more precise, but in general we would still find that it takes  $\approx 4N$  generations for a neutral allele to go from its introduction to fixation within the population.

*Rate of substitution of neutral alleles* A substitution between populations that do not exchange gene flow is simply a fixation event within one population. The rate of substitution is therefore the rate at which new alleles fix in the population, so that the long-term substitution rate is the rate at which mutations arise that will eventually become fixed within our population.

Lets assume, based on our discussion of the neutral theory of molecular evolution, that there are only two classes of mutational changes that can occur with a region, highly deleterious mutations and neutral mutations. A fraction  $C$  of all mutational changes are highly deleterious, and cannot possibly contribute to substitution nor polymorphism. The other  $1 - C$  fraction of mutations are neutral. If our mutation rate is  $\mu$  per transmitted allele per generation, then a total of  $2N\mu(1 - C)$  neutral mutations enter our population each generation.

Each of these neutral mutations has a  $1/(2N)$  probability chance of eventually becoming fixed in the population. Therefore, the rate at which neutral mutations arise that eventually become fixed within our population is

$$2N\mu(1 - C)\frac{1}{2N} = \mu(1 - C) \quad (3.44)$$

Thus the rate of substitution, under a model where newly arising alleles are either highly deleterious or neutral, is simply given by the mutation rate of neutral alleles, i.e.  $\mu(1 - C)$ .

Consider a pair of species that have diverged for  $T$  generations, i.e. orthologous sequences shared between the species last shared a common ancestor  $T$  generations ago. If these species have maintained a constant  $\mu$  over that time, they will have accumulated an average of

$$2\mu(1 - C)T \quad (3.45)$$

neutral substitutions. This assumes that  $T$  is a lot longer than the time it takes to fix a neutral allele, such that the total number of alleles introduced into the population that will eventually fix is the total number of substitutions.

This is a really pretty result as the population size has completely canceled out of the neutral substitution rate. However, there is another way to see this in a more straight forward way. If I look at a sequence in me compared to, say, a particular chimp, I'm looking at the mutations that have occurred in both of our germlines since they parted ways  $T$  generations ago. Since neutral alleles do not alter the probability of their transmission to the next generation, we are simply looking at the mutations that have occurred in  $2T$  generations worth of transmissions. Thus the average number of neutral mutational differences separating our pair of species is simply  $2\mu(1 - C)T$ .

A number of observations follow under this model, from equation (3.45), the first is that a primary determinant of patterns of molecular evolution in a genomic region is the level of constraint ( $C$ ). This pattern generally seems to hold empirically: non-coding regions often evolve more rapidly than coding regions; synonymous substitutions accumulate faster than nonsynonymous; nonsynonymous changes faster in less vital proteins than ones that are absolutely necessary for early development. Note that this is not a unique prediction of the neutral model, e.g. lower pleiotropy means that less constrained regions may be better able to evolve adaptively. However, it is a fantastically useful general insight, e.g. it allows us to spot putatively functional non-coding regions by looking for genomic regions that have very low levels of divergence among distantly related species.

The second important insight, and critical for the development of the neutral theory, is that equation (3.45) is seemingly consistent with ZUCKERKANDL and PAULING (1965)'s hypothesis of a surprisingly constant, protein molecular clock. See Figure 3.24 for an example using the data of DICKERSON (1971). If we double the amount of time separating a pair of species  $T$ , we double the number of substitutions predicted. Note that for this to be true  $T$  must be measured in generations. To explain a protein molecular clock between species that clearly differed dramatically in generation time it was hypothesized that the mutation rate actually scaled with generation time, i.e. short-lived organisms introduced less mutations per generation, e.g. as they

"functionally less important molecules or parts of a molecule evolve faster than more important ones."

— KIMURA and OHTA (1974)



Figure 3.24: The numbers of substitutions between various pairs of groups, for three proteins, plotted against the time these groups shared a common ancestor in the fossil record. Data from DICKERSON (1971). The number of observed amino-acid differences is corrected for multiple hits to obtain the corrected number of changes estimated to occur. The lines give the linear regression, constrained to pass through the origin, for each protein. The slope of the regression is given next to the protein name.

had fewer rounds of mitosis. This generation-time assumption meant that the mutation rate per year could be constant, such that  $\mu T$  would be a constant for pairs of species that had diverged for similar geological times, which are measured in years, even if the organisms differed in generation time. This assumption would allow neutral theory to be consistent with a protein molecular clock measured in years. We now know that this critical generation time assumption is false, organisms with shorter generation times have somewhat higher mutation rates per year, and so a strict neutral model is inconsistent with the protein molecular clock. We'll return to these ideas when we discuss the fate of very weakly selected mutations in Chapter 7 and OHTA (1973)'s Nearly Neutral theory. If you are still reading this send Graham a picture of Tomoko Ohta receiving the Crafoord Prize, an analog of the Nobel prize for biology, for her contributions to molecular evolution.

*The contribution of ancestral polymorphism to divergence.* If we are considering  $T$  to represent the divergence between long-separated species, then we can think of  $T$  as the time that the species split. However, for more recently diverged populations and species, we need to include the fact that the sorting of ancestral polymorphism contributes to divergence among species. In Figure 3.25, we see our two populations split  $T_s$  generations ago. However, the coalescence of our A and B lineage is necessarily deeper in time than  $T_s$ . The top mutation was polymorphic in the ancestral population but now contributes to the divergence between A and B. Assuming that our ancestral population had effective size  $N_A$  individuals, and that our populations

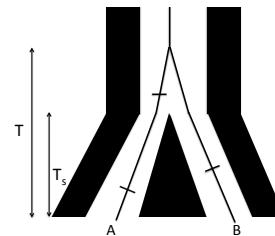


Figure 3.25: The genealogy of two alleles one sampled from population A and B. Mutations on the lineages are shown as dashes. The pair of alleles coalesce in the ancestral population of A and B. The two populations split  $T_s$  generations ago, with no subsequent gene flow, but the two lineages must coalesce deeper in time.

split cleanly with no subsequent gene flow, then

$$T = T_s + 2N_A. \quad (3.46)$$

If our species split time is very large compared to  $2N$  then we can think of  $T$  as the split time.

**Question 11.** For this, and the next question, assume that humans and chimp diverged around  $5.5 \times 10^6$  years ago, have a generation time 20 years, that the speciation occurred instantaneously in allopatry with no subsequent gene flow, and the ancestral effective population size of the human and chimp common ancestor population was 10,000 individuals.

Nachman and Crowell sequenced 12 pseudogenes in human and chimp and found substitutions at 1.3% of sites.

**A)** What is the mutation rate per site per generation at these genes?

**B)** All of the pseudogenes they sequenced are on the autosomes.

What would your prediction be for pseudogenes on the X and Y chromosomes, given that there are fewer rounds of replication in the female germline than in the male germline.

### 3.5 Tests of molecular evolution.

#### 3.5.1 Comparing the rates of non-synonymous to synonymous substitutions $d_N/d_S$

One common tool in molecular evolution is to compare the estimated number (or rates) of substitutions in different classes of genomic sites, for example the ratio of the number of non-synonymous to synonymous substitutions in a given gene. The simplest way to calculate  $d_N$  is to count up the non-synonymous changes and divide by the total number of positions in the gene where a non-synonymous point mutation could occur. We can do likewise for synonymous changes  $d_S$ , and then take the ratio  $d_N/d_S$ . This is a helpful conceptual way to think about what  $d_N/d_S$  represents, however, this ignores the fact that particular changes are more likely to occur by mutation and also does not account for multiple hits. Therefore, in practice the ratio  $d_N/d_S$  is more usually calculated by model-based likelihood and bayesian methods that can account for these features.

For the vast majority of protein-coding genes in the genome we see that  $d_N/d_S < 1$ , this is consistent with the view that non-synonymous sites are much more constrained than synonymous, i.e. that most non-synonymous mutations are deleterious and quickly removed from the population. If we are willing to make the assumption that all

synonymous changes are neutral,  $d_S = 2T\mu$ , then we can estimate the degree of constraint. (Note that synonymous changes can sometimes be subject to both positive and negative selection, but this assumption is a useful starting place.)

Assuming that a fraction  $C$  of non-synonymous changes are too deleterious to contribute to polymorphism then, if  $T$  generations of divergence have elapsed between the two populations we expect  $d_N$  neutral non-synonymous substitutions, where

$$d_N = 2T(1 - C)\mu \quad (3.47)$$

Then

$$d_N/d_S = (1 - C) \quad (3.48)$$

therefore, if we assume that non-synonymous mutations can only be strongly deleterious or neutral, we estimate the fraction of mutational changes that are constrained by negative selection as  $C = 1 - d_N/d_S$ . This has the interpretation of being the fraction of non-synonymous mutations that are quickly weeded out of the population by selection, and so do not contribute to divergence among species.

We can test whether our gene is evolving in a constrained way at the protein level by estimating  $d_N/d_S$  and testing whether this is significantly less than 1. This provides evolutionary evidence that a stretch of DNA proposed to be protein coding is subject to selective constraint, and so likely encodes for a functional protein. We can also perform this test on specific branches of a phylogeny for a gene, to test which branches the gene is subject to constraint, or to test for changes in the level of constraint.

*Loss of constraint at pseudogenes.* While most protein genes evolve under constraint, we can find examples of genes that are evolving in a less constrained manner. The simplest example of this is where the gene has lost function, e.g. has recently become pseudogenized. When a gene completely loses function there will be no selection against non-synonymous changes and so they are just as free to accumulate as synonymous changes, and so  $d_N/d_S = 1$ . Pseudogenes are a wonderful example of the extension of Darwin's ideas about vestigial traits ('Rudimentary organs') to the DNA level, we can still recognize a once useful word (gene) whose spelling is slowly degrading. Genes can lose function because of inactivating mutations that stop them being transcribed or translated into functional proteins, such genes are called pseudogenes. Our genomes are filled with old pseudogenes whose meaning are slowly being eroded through the accumulation of neutral substitutions. One nice example of a gene that has been repeatedly lost function, i.e. become repeatedly pseudogenized, is the Enamlin gene from the study of MEREDITH *et al.* (2009).

"Rudimentary organs may be compared with the letters in a word, still retained in the spelling, but become useless in the pronunciation, but which serve as a clue .. for its derivation." – DARWIN (1859) pg. 455

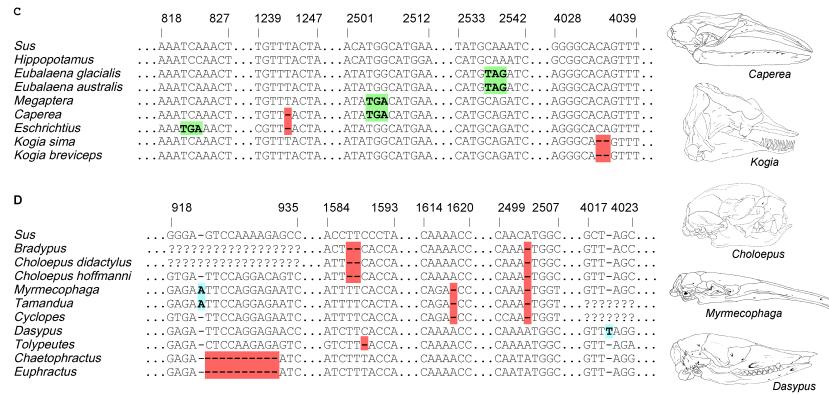


Figure 3.26: Examples of frameshift mutations (insertions blue, deletions red) and premature stop codons in Enamlin in Cetacea and Xenarthra. Figure from MEREDITH *et al.*

The protein Enamlin is a key structural protein involved in the outer cap of enamel on teeth. Various mammals have secondarily evolved diets that do not require hard teeth, and so greatly reduced the selection pressure for hard enamel or teeth at all. For example, two-toed sloths (*Choloepus*); Pygmy sperm whales (*Kogia*); aardvark (*Orycteropus*) lack enamel on teeth. While others have lost their teeth entirely e.g. giant anteaters (*Myrmecophaga*) and Baleen whales. Due to this relaxation of constraint of the phenotype the Enamlin gene has accumulated pseudogenizing substitutions, premature stop codons, and frameshift mutations (see Figure 3.26 for examples). MEREDITH *et al.* sequenced Enamlin across a range of species and found that none of the species with Enamel have frameshift mutations in Enamlin, while 17/20 of species that lack Enamel or teeth have frameshifts in Enamlin and all of them carry premature stop codons (Figure 3.28).

The branches of the Enamlin phylogeny with a functional Enamlin gene (black) had an estimated  $d_N/d_S = 0.51$  consistent with the protein evolving in a constrained manner. While the branches with a pseudogenized Enamlin had  $d_N/d_S = 1.02$  consistent with the gene evolving an unconstrained way. The branches where the gene was likely transitioning from a functional to non-function state (pseudogenization and mixed, blue and purple) had an intermediate values of  $d_N/d_S = 0.83 - 0.98$  consistent with them transitioning from a constrained to unconstrained mode of protein evolution somewhere along these branches of the phylogeny.

*Adaptive evolution and  $d_N/d_S$ .* Clearly genes are not only subject to neutral and deleterious mutations, beneficial mutations must also arise and fix from time to time. Lets assume that a fraction  $B$  of non-synonymous mutations that arise are beneficial such that  $2N\mu B$  beneficial mutations arise per generation. Now newly arisen beneficial



Figure 3.27: Two-toed sloth (*Choloepus hoffmanni*). An introduction to the study of mammals, living and extinct. 1891. Flower W. H. and Lydekker R.

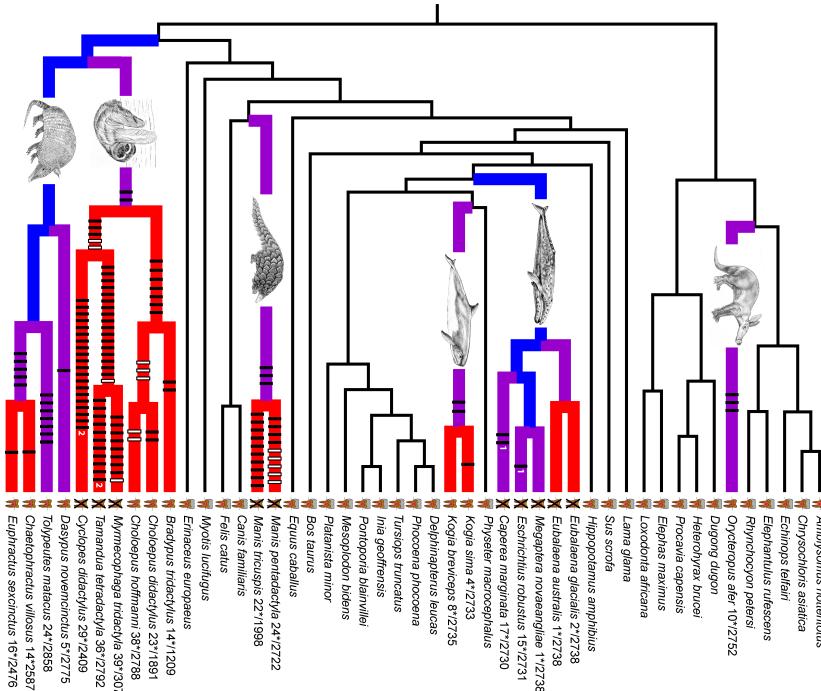


Figure 3.28: The tooth symbol next to each taxon shows whether they have teeth, lack enamel, or lack teeth. Branches of the phylogeny are coloured by whether their Enamlin is functional (black), pre-mutation (blue), mixed (purple), and pseudo-genic (red). The black and white vertical bars on branches show frameshift mutations. The Numbers after taxon names indicate minimum number of stop codons in the sequence / the length of the sequence. Figure from MEREDITH *et al.*

alleles are not destined to fix in the population, as they may be lost to genetic drift when they are rare in the population (we'll discuss how to calculate the fixation probability for beneficial alleles in Chapter 7). A newly arisen beneficial allele reach fixation in the population with probability  $f_B$  from its initial frequency of  $1/2N$ . This fixation probability may be much higher than that of neutral mutations, but still much less than 1. If  $2T$  generations of divergence have elapsed between the two populations then a total of

$$dN = 2T(1 - C - B)\mu + 2T \times (2N\mu B) \times f_B \quad (3.49)$$

non-synonymous substitutions will have accumulated. Then

$$d_N/d_S = (1 - C - B) + 2NBf_B \quad (3.50)$$

assuming again that all synonymous mutations are neutral. Note that this means that our estimates of  $C$  using  $1 - d_N/d_S$  will be a lower bound on the true constraint if even a small fraction of mutations are beneficial. Those cases where the gene is evolving more rapidly at the protein level than at synonymous sites, i.e.  $d_N/d_S > 1$ , are potentially strong candidates for positive selection rapidly driving change at the protein level. We can identify genes that have  $d_N/d_S$  significantly greater than one, either on the complete gene phylogeny, or on particular branches. Note that is is a very conservative test that

few genes in the genome meet, as many genes that are fixing adaptive non-synonymous substitutions will have  $d_N/d_S < 1$ . This is because they may still evolve in a constrained way (i.e.  $d_N/d_S < 1$ ) even if adaptive substitutions are common at a gene if they are outweighed by the loss of non-synonymous mutations to negative selection.

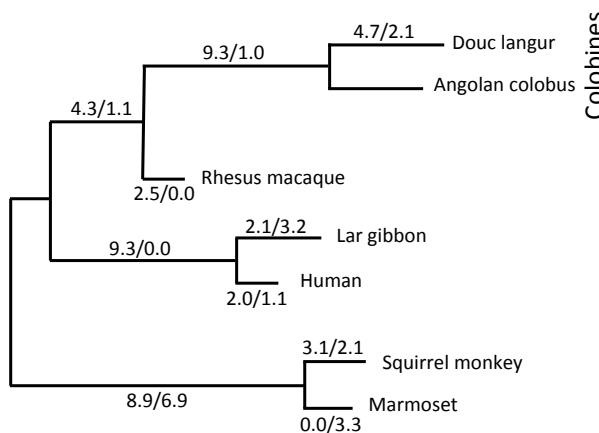


Figure 3.29: A phylogram for the primate lysozyme gene redrawn from YANG. For each branch the numbers give the estimated average number of non-synonymous to synonymous changes in the lysozyme protein.

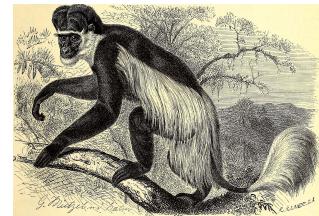


Figure 3.30: Abyssinian black-and-white colobus (*Colobus guereza*). Brehm's Tierleben, Brehm, A.E. 1893. A member of the leaf-eating Colobines.



Figure 3.31: (hoatzin (*Opisthocomus hoazin*). A history of birds Pycraft, W.P. 1910. A leaf-eating bird.

molecular evolution at a gene based on this intuition (building on the conceptually similar HKA test HUDSON *et al.*, 1987). McDONALD and KREITMAN took the case where we have polymorphism data at a gene for one species and divergence to a closely related species. They partitioned polymorphism and fixed differences in a sample into nonsynonymous and synonymous changes:

	Poly.	Fixed
Non-Syn.	$P_N$	$D_N$
Syn.	$P_S$	$D_S$
Ratio	$P_N/P_S$	$D_N/D_S$

Under neutral theory we expect a smaller number of non-synonymous to synonymous fixed differences ( $P_N/P_S < 1$ ) but exactly the same expectation holds for polymorphism ( $P_N/P_S$ ). Lets consider a gene with  $L_S$  and  $L_N$  sites where a synonymous and non-synonymous mutations could arise respectively. We can think of the underlying gene genealogy at our gene, see Figure 3.32, with the total time on the coalescent genealogy within the species as  $T_{tot}$  and the total time for fixed differences between our species  $T'_{div}$ . Then under neutrality we expect  $\mu L_N(1 - C)T_{tot}$  non-synonymous polymorphisms (i.e. our number of segregating sites), and  $\mu L_N(1 - C)T'_{div}$  fixed differences. We can then fill out the rest of our table as follows:

	Poly.	Fixed
Non-Syn.	$\mu L_N(1 - C)T_{tot}$	$\mu L_N(1 - C)T'_{div}$
Syn.	$\mu L_N T_{tot}$	$\mu L_S T'_{div}$
Ratio	$L_N(1 - C)/(L_S)$	$L_N(1 - C)/(L_S)$

Therefore, we expect the ratio of non-synonymous to synonymous changes to be the same for polymorphism and divergence under a strict neutral model. We can test this expectation of equal ratios via the standard G-test of a  $2 \times 2$  table. If the ratio of  $N/S$  is significantly higher divergence than polymorphism we have evidence that non-synonymous substitutions are accumulating more rapidly than we would predict given levels of constraint alone.

### 3.6 Neutral diversity and population structure

Up to now we have assumed that our alleles that we have modelled in the coalescent setting are drawn from a randomly mating population such that any pair of lineages is equally likely to coalesce with each other. However, when there is population structure this assumption is violated.

We have previously written the measure of population structure  $F_{ST}$  as

$$F_{ST} = \frac{H_T - H_S}{H_T} \quad (3.51)$$

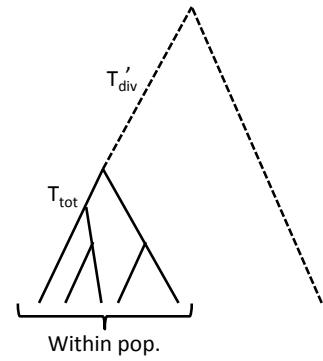


Figure 3.32:

where  $H_S$  is the probability that two alleles sampled at random from a subpopulation differ, and  $H_T$  is the probability that two alleles sampled at random from the total population differ.

*A simple population split model* Imagine a population of constant size of  $N_e$  diploid individuals that  $T$  generations in the past split into two daughter populations (sub-populations) each of size  $N_e$  individuals, who do not subsequently exchange migrants. In the current day we sample an equal number of alleles from both subpopulations.

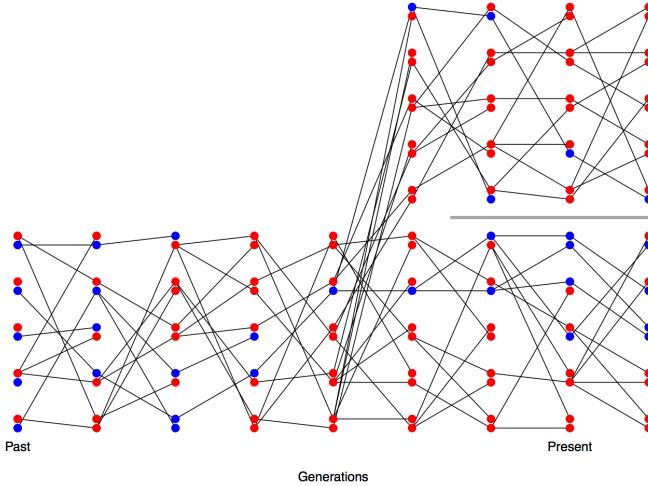


Figure 3.33: Change in allele frequencies following a population split.

Consider a pair of alleles sampled within one of our sub-populations, and think about their per site heterozygosity. They have experienced a population of size  $N_e$  and so the probability that they differ is  $H_S \approx 4N_e\mu$  (assuming that  $N_e\mu \ll 1$ , using our equation 3.12 for heterozygosity within a population ).

The heterozygosity in our total population is a little more tricky to calculate. Assuming that we equally sample both sub-populations, when we draw two alleles from our total sample, 50% of the time they are drawn from the same subpopulation and 50% of the time they are drawn from different subpopulations. Therefore, our total heterozygosity is given by

$$H_T = \frac{1}{2}H_S + \frac{1}{2}H_B \quad (3.52)$$

where  $H_B$  is the probability that a pair of alleles drawn from our two different sub-populations differ from each other. Our pair of alleles can not find a common ancestor with each other for at least  $T$  generations into the past as they are in distinct populations (not connected by migration). Once our alleles find themselves back in the combined ancestral population it takes them on average  $2N$  generations. So the

total opportunity for mutation between our pair of alleles sampled from different populations is  $2(T + 2N)$  generation, such that the probability that our pairs of alleles is different is

$$H_B \approx 2\mu(T + 2N) \quad (3.53)$$

We can plug this into our expression for  $H_T$ , and then that in turn into  $F_{ST}$ . Doing so we find that

$$F_{ST} \approx \frac{\mu T}{\mu T + 4N_e \mu} = \frac{T}{T + 4N_e} \quad (3.54)$$

note that  $\mu$  cancels out of this. In this simple toy model  $F_{ST}$  is increasing because the amount of between population diversity increases with the divergence time of the two populations (initially linearly with  $T$ ). It does so at a rate give by  $T/(4N_e)$  so that differentiation will be higher between populations separated by long divergence times or with small effective population sizes.

**Question 12.** The genome-wide  $F_{ST}$  between Bornean and Sumatran orang-utan species samples (*Pongo pygmaeus* and *Pongo abelii*) is  $\approx 0.37$  (LOCKE *et al.*, 2011), representing a deep population split between the species (potentially with little subsequent gene flow). Within the populations the genome-wide average Watterson's estimate of  $\theta_W = 1.4\text{kb}^{-1}$ , estimated from the number of segregating sites. Assume a generation time of 20 years, and a mutation rate of  $2 \times 10^{-8}$  per base per generation. How far in the past did the two populations diverge?

*A simple model of migration between an island and the mainland.* We can also use the coalescent to think about patterns of differentiation under a simple model of migration drift equilibrium. Lets consider a small island population that is relatively isolated from a large mainland population, and that both of these populations are constant in size. We'll assume that the expected heterozygosity for a pair of alleles sampled on the mainland is  $H_M$ .

Our island has a population size  $N_I$  that is very small compared to our mainland population. Each generation some low fraction  $m$  of our individuals on the island have migrant parents from the mainland the generation before. Our island may also send migrants back to the mainland, but these are a drop in the ocean compared to the large population size on the mainland and their effect can be ignored.

If we sample an allele on the island back and trace its ancestral lineage backward in time, each generation our ancestral allele have a low probability  $m$  of being descended from the mainland in the proceeding generation (if we go far enough the allele eventually has to

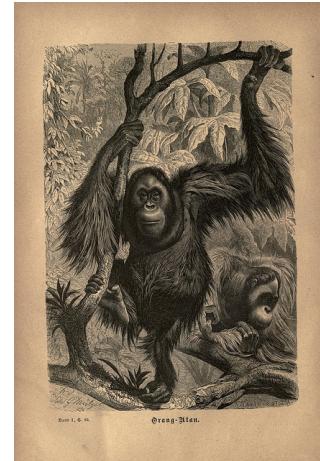


Figure 3.34: Orangutan (*Pongo*). Brehms thierleben, allgemeine kunde des thierreichs. Brehm, A. E.

be descended from an allele on the mainland). The probability that a pair of alleles sampled on the island are descended from a shared recent common ancestral allele on the island, is the probability that our pair of alleles coalesce before either lineage migrates. For example, the probability that our pair of alleles coalesce  $t + 1$  generations back is

$$\frac{1}{2N_I} (1-m)^{2(t+1)} \left(1 - \frac{1}{2N_I}\right)^t \approx \frac{1}{2N_I} \exp\left(-t\left(\frac{1}{2N_I} + 2m\right)\right), \quad (3.55)$$

with the approximation following from assuming that  $m \ll 1$  &  $\frac{1}{(2N_I)} \ll 1$  (note that this is very similar to our derivation of heterozygosity above). The probability that our alleles coalesce before either one of them migrates off the island, irrespective of the time, is

$$\int_0^\infty \frac{1}{2N_I} \exp\left(-t\left(\frac{1}{2N_I} + 2m\right)\right) dt = \frac{1/(2N_I)}{1/(2N_I) + 2m}. \quad (3.56)$$

Lets assume that the mutation rate is very low such as it is very unlikely that the pair of alleles mutate before they coalesce on the island. Therefore, the only way that the alleles can be different from each other is if one or other of them migrates to the mainland, which happens with probability

$$1 - \frac{1/(2N_I)}{1/(2N_I) + 2m} \quad (3.57)$$

Conditional on one or other of our alleles migrating to the mainland both of our alleles represent independent draws from the mainland and so differ from each other with probability  $H_M$ . Therefore, the level of heterozygosity on the island is given by

$$H_I = \left(1 - \frac{1/(2N_I)}{1/(2N_I) + 2m}\right) H_M \quad (3.58)$$

So the reduction of heterozygosity on the island compared to the mainland is

$$F_{IM} = 1 - \frac{H_I}{H_M} = \frac{1/(2N_I)}{1/(2N_I) + 2m} = \frac{1}{1 + 4N_I m}. \quad (3.59)$$

The level of inbreeding on the island compared to the mainland will be high in the migration rate is low and the effective population size of the island is low, as allele frequencies on the island are drifting and diversity is not being replenished on the island by migration. The key parameter here is the number individuals on the island replaced by immigrants from the mainland each generation ( $N_I m$ ).

We have framed this as being about the reduction in genetic diversity on the island compared to the mainland. However, if we consider collecting individuals on the island and mainland in proportion to

population sizes the total level of heterozygosity would be  $H_T = H_M$ , as samples from our mainland would greatly outnumber those from our island. Therefore, considering our island our sub-population we have derived another simple model of  $F_{ST}$ .

**Question 13.** You are investigating a small river population of sticklebacks, which receives infrequent migrants from a very large marine population. At a set of (putatively neutral biallelic markers the freshwater population has frequencies:

0.2, 0.7, 0.8

at the same markers the marine population has frequencies:

0.4, 0.5 and 0.7.

From studying patterns of heterozygosity at a large collection of markers, you have estimated the long term effective size of your freshwater population is 2000 individuals.

What is your estimate of the migration rate from the marine populations into the river?

*Incomplete lineage sorting* Because it can take a long time for an polymorphism to drift up or down in frequency, multiple population splits may occur during the transit of an allele. This can lead to incongruence between the overall population tree and the information about relationships present at individual loci. In Figure 3.35 and 3.36 we show a simulations of three populations where the bottom population splits off from the other two first, followed by the subsequent splitting of the top and the middle population. We start both simulations with a newly introduced red allele being polymorphic in the combined ancestral polymorphism. The most likely fate of this allele is that it is quickly lost from the population, but sometimes the allele can drift up in frequency and be polymorphic when the populations split, as the alleles in our two figures have done. If the allele is lost/fixed in the descendent populations before the next population split our allele configuration will agree with the population tree, as it does in Figure 3.35, and so the gene tree will agree with population tree (as shown in the left side of Figure 3.37). However, if the allele persists as a polymorphism in the ancestral population till the top and the middle populations split, then the allele can fix in one of these populations and not the other. Such an event can lead to a substitution pattern that disagrees with the population tree, as in Figure 3.36. If we were construct a phylogeny using the variation we would see a disagreement between the gene tree and population tree. In Figure 3.36 an allele drawn from the top and the bottom populations are necessarily more closely related to each other than either is to an allele drawn from population 2, as tracing our allelic lineages from top and bottom pop-

ulations back through time they must coalesce with each other before we reach the point where the red mutation arose; while a lineage from the middle population has not yet to coalesce with either. An example of this in terms of the underlying tree is shown on the right side of Figure 3.37 .

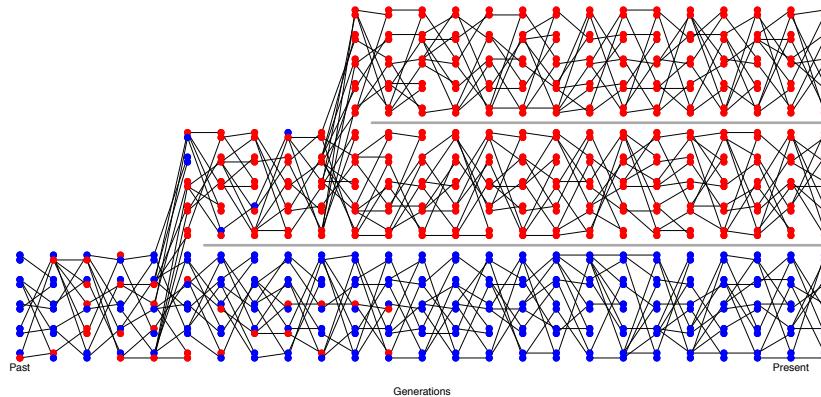


Figure 3.35:

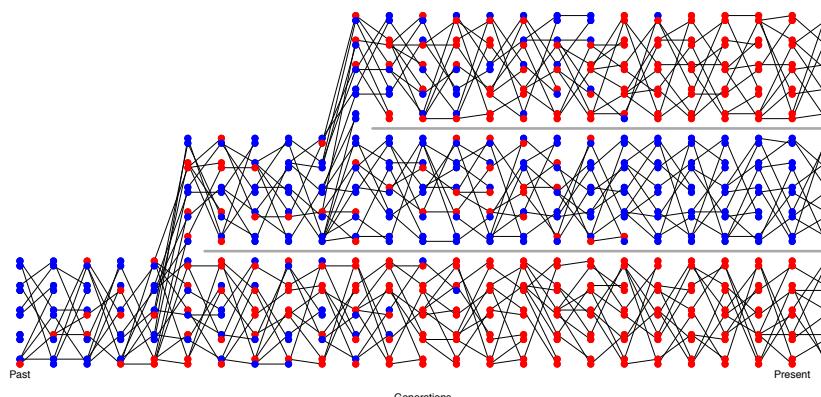


Figure 3.36:



Figure 3.37: The population tree of three populations ((A, B), C) is shown blocked out with black shapes. Two different coalescent trees are relating a single allele drawn from A, B, and C are shown with thinner lines.

A natural, pedigree analogy to this is the fact that while two biological siblings are more closely related to each other genealogically than either is to their cousin, at any given locus one of the

siblings can share an allele IBD with their cousin that they do not share with their own sibling, due to the randomness of mendelian segregating down their pedigree. The average relatedness of the individuals/population disagrees with the patterns of relatedness at a particular locus.

As an empirical example of incomplete lineage sorting lets consider the work of JENNINGS and EDWARDS who sequenced a single allele from three different species of Australian grass finches (*Poephila*); two sister species of long-tailed finches (*Poephila acuticauda* and *P. hecki*) and the black-throated finch (*Poephila cincta*, see Figure 3.38). They collected sequence data for 30 genes, and constructed phylogenetic gene trees at each of these loci, resulting in 28 well resolved gene trees. 16 gene trees of the gene trees showed *P. acuticauda* and *P. hecki* as sisters with *P. cincta* (the tree ((A,H),C) ). While for twelve genes the gene tree was discordant with the population tree: for seven of their genes *P. hecki* fell as an outgroup to the other two; and at five *P. acuticauda* fell as an outgroup (the trees ((A,C),H) and ((H,C),A) respectively).

Lets use the coalescent to understand this discordance between gene trees and species trees. Lets assume that two sister populations (A & B) split  $t_1$  generations in the past, with a deeper split from a third outgroup population (C)  $t_2$  generations in the past. We'll assume that there's no gene flow among our populations after each split. We can trace back the ancestral lineages of our three alleles. The first opportunity for the A & B lineages to coalesce is  $t_1$  generations. If they coalesce with each other in their shared ancestral population before  $t_2$  in the past, left side of Figure 3.37, their gene tree will definitely agree with the population tree. So the only way for the gene tree to disagree with the population tree is for the A & B lineages to fail to coalesce in their shared ancestral population, this happens with probability  $(1 - 1/2N)^{t_2 - t_1}$ . We'll get a discordant gene tree, if A & B make it back to the shared ancestral population with C without coalescing, and then one or other of them coalesces with the C lineage first. This happens with probability 2/3, as at the first pairwise-coalescent event there are three possible pairs two of which (A & C and B & C) result in a discordant tree. So the probability that we get a coalescent tree that is discordant with the population tree is

$$\frac{2}{3} (1 - 1/2N)^{t_2 - t_1}. \quad (3.60)$$

Thus we should expect gene-tree population-tree discordance when population split in rapid succession, and/or population sizes are large.

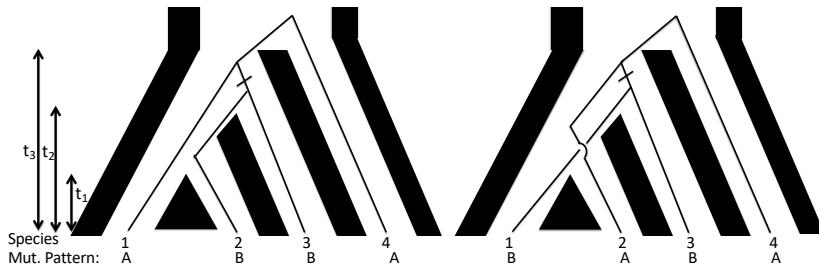
**Question 14.** Lets return to JENNINGS and EDWARDS's Australian grass finches example, they estimated that the ancestral pop-



Figure 3.38: Banded Grass Finch (*P. cincta*). Illustration by Elizabeth Gould. Birds of Australia Gould J. 1840.

ulation size of our two long-tailed finches was four hundred thousand. What is your best estimate of the inter-speciation time, i.e.  $t_2 - t_1$ ?

*Testing for gene flow.* We often want to test whether gene flow has occurred between populations. For example, we might want to establish a case that interbreeding between humans and Neanderthals occurred or demonstrate that gene flow occurred after two populations began to speciate. A broad range of methods have been designed to test for gene flow, and to estimate gene flow rates, based on neutral expectations. Here'll we briefly just discuss one method based on some simple coalescent ideas. Above we assumed that gene-tree population-tree discordance was due to incomplete lineage sorting due to populations rapidly splitting. However, gene flow among populations can also lead to gene-tree discordance. While both ILS and gene flow can lead to discordance, under simplifying assumptions ILS implies more symmetry in how these discordancies manifest themselves.



Take a look at Figure 3.39. In both cases the lineages from A and B fail to coalesce in their initial shared ancestral population, and one or other of them coalesces with the lineage from C first. Each option is equally likely, therefore the mutational patterns ABBA and BABA are equally likely to occur under ILS.<sup>6</sup>

However, if gene flow occurs from population C into population B the lineage from B can more recently coalesce with lineage C, and so we should see more ABBAAs than BABAs. To test for this we can sample four sequences from our populations and count up the number of sites that show the two mutational patterns consistent with the gene-tree discordance  $n_{ABBA}$  and  $n_{BABA}$  and take

$$\frac{n_{ABBA} - n_{BABA}}{n_{ABBA} + n_{BABA}} \quad (3.61)$$

this statistic will have expectation zero if the gene-tree discordance is due to ILS, and be skewed negative if gene flow occurred from C into B (positive if gene flow occurred from C into A).

Figure 3.39: In both the left and right trees ILS has occurred between our single lineages sampled from populations A, B, and C. Imagine that population D, is an somewhat distant outgroup such that the lineages from A-C (nearly) always coalesce with each other, before coalescing with D. The small dash on the branch indicates the mutation A→B occurring, giving rise to the ABBA or BABA mutational pattern shown at the bottom.

<sup>6</sup> here we have to assume no structure in the ancestral population.

# 4

## *Phenotypic variation and resemblance between relatives.*

All that we mean when we speak of a gene [allele] for pink eyes is, a gene which differentiates a pink eyed fly from a normal one — not a gene [allele] which produces pink eyes per se, for the character pink eyes is dependent on the action of many other genes.

- STURTEVANT (1915)

THE DISTINCTION BETWEEN GENOTYPE AND PHENOTYPE is one of the most useful ideas in Biology.<sup>1</sup> The genotype of an individual (the genome), for most purposes, is decided when the sperm fertilizes egg. The phenotype of an individual represents any measurable aspect of an organism. Your height, to the amount of RNA transcribed from a given gene, to what you ate last Tuesday; all of these are phenotypes. Nearly any phenotype we can choose to measure about an organism represents the outcome of the instructions encoded by that genome played out through an incredibly complicated developmental, and physiological or behavioural processes which interact with a myriad of environmental and stochastic factors. Honestly it boggles the mind how organisms work as well as they do, let alone that you managed to eat lunch last Tuesday.

The simplest way to measure the genotype-phenotype relationship is to calculate the phenotypic mean for each genotype at a locus. For example, WANG *et al.* (2018)<sup>2</sup> explored the genetic basis of budset time in European aspen (*Populus tremula*), the effect of a specific SNP on the phenotype is shown in Figure 4.2. Budset timing is a key trait underlying local adaptation to varying growing season length. The SNP falls in the gene (*PtFT2*) that is known to play a strong role in flowering time regulation in other plants.

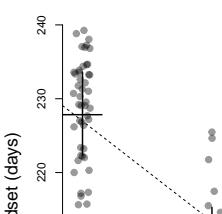
One way for us to assess the relationship between genotype phenotype is to fit a linear regression, i.e. the best fitting linear line through the data, of phenotype on genotype. The slope of this line has the

<sup>1</sup> JOHANNSEN, W., 1911 The Genotype Conception of Heredity. *The American Naturalist* 45(531): 129–159



Figure 4.1: *P. tremula*. Der baum. H. Schacht. 1860. BHL

<sup>2</sup> WANG, J., J. DING, B. TAN, K. M. ROBINSON, I. H. MICHELSON, A. JOHANSSON, B. NYSTEDT, D. G. SCOFIELD, O. NILSSON, S. JANSSON, and OTHERS, 2018 A major locus controls local adaptation and adaptive life history variation in a perennial plant. *Genome biology* 19(1): 72



interpretation of being the average effect of substituting a copy of allele 2 for a copy of allele 1. In our Aspen example the slope is  $-13.6$ , i.e. swapping a single  $T$  for a  $G$  allele moves the budset forward by  $13.6$  days, with the  $GG$  homozygote predicted to set buds  $27.2$  days earlier than the  $TT$  homozgote. As a measure of significance of this relationship we can calculate the p-value of this regression. To try and identify loci that are associated with our trait genome-wide we can conduct this regression at each SNP in the genome. One common way to display these results is to plot the logarithm of the p-value for each SNP along genome (a so-called Manhattan plot). Here's one from WANG *et al.* (2018) for their the Aspen budset phenotype

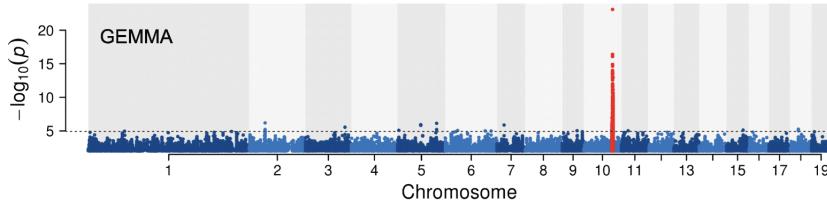


Figure 4.3: Manhattan plot of the p-value of the linear association between genotype and budset in Aspen. Each dot represents the test at a single SNP, plotted at physical coordinate on the genome. Different chromosomes are plotted in alternating colours. The SNPs surrounding the PtFT2 gene are shown in red.

The SNP with the most significant p-value is the PtFT2 SNP. Note that other SNPs in the surrounding region also light up as showing a significant association with budset timing. This is because loci that are in LD with a functional locus may in turn show an association, not because they directly affect the phenotype but simply because the genotypes at the two loci are themselves are non-randomly associated. Here is a zoomed in version (Figure 2 in WANG *et al.* (2018)) with SNPs coloured by the strength of their LD with the putatively functional SNP. Note how SNPs with strong LD with the functional allele

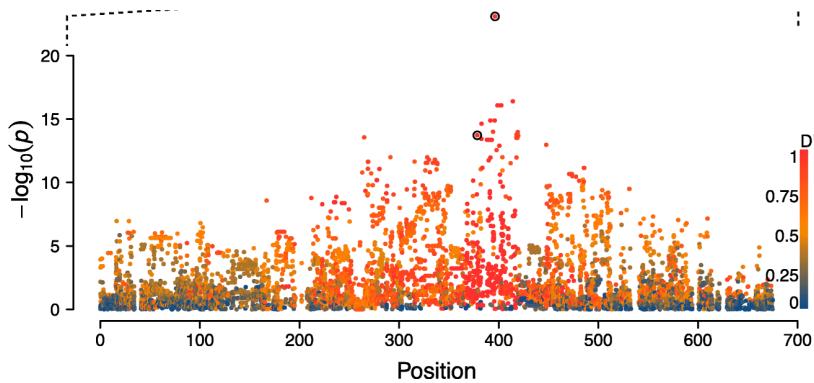


Figure 4.4: The Manhattan plot zoomed in on the top-hit (red SNPs from Figure 4.3). SNPs are now coloured by their  $D_f$  value with the most significant SNP.  $D_f$  is the LD covariance between a pair of loci ( $D$ , eqn XXX) normalized by the largest value it can take given the allele frequencies. From Figure 2 of

(redder points) have more significant p-values.

Variation in some traits seems to have a relatively simple genetic basis. In our Aspen example there is one clear large-effect locus, which

explains 62% of the variation in budget. However, many phenotypes are likely much more genetically complex in their variation involving the functional effect of many alleles at many polymorphic loci. Such genetically complex traits are called polygenic traits.

We can use our understanding of the sharing of alleles between relatives to understand the phenotypic resemblance between relatives in quantitative phenotypes. We can then use this to understand the evolutionary change in quantitative phenotypes in response to selection.

#### 4.0.1 A simple additive model of a trait

Let's imagine that the genetic component of the variation in our trait is controlled by  $L$  autosomal loci that act in an additive manner. The frequency of allele 1 at locus  $l$  is  $p_l$ , with each copy of allele 1 at this locus increasing your trait value by  $a_l$  above the population mean.

The phenotype of an individual, let's call her  $i$ , is  $X_i$ . Her genotype at SNP  $l$ , is  $G_{i,l}$ . Here  $G_{i,l} = 0, 1$ , or  $2$  represents the number of copies of allele 1 she has at this SNP. Her expected phenotype, given her genotype, is then

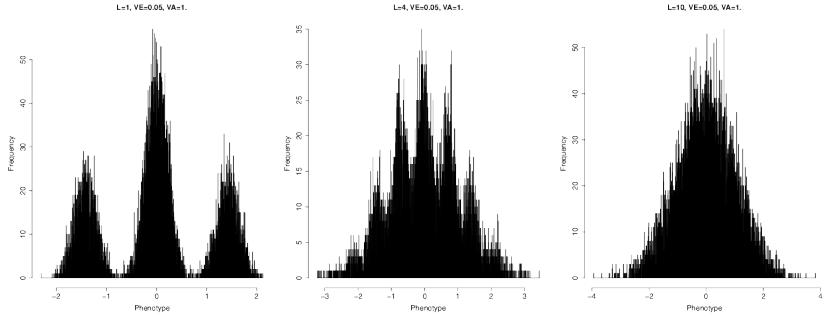
$$\mathbb{E}(X_i|G_{i,1}, \dots, G_{i,L}) = \mu + X_{A,i} = \mu + \sum_{l=1}^L G_{i,l}a_l \quad (4.1)$$

where  $\mu$  is the mean phenotype in our population, and  $X_{A,i}$  is the deviation away from the mean phenotype due to her genotype. Now in reality the genetic phenotype is a function of the expression of those alleles in a particular environment. Therefore, we can think of this expected phenotype as being an average across a set of environments that occur in the population.

When we measure our individual's observed phenotype we see

$$X_i = \mu + X_{A,i} + X_{E,i} \quad (4.2)$$

where  $X_E$  is the deviation from the mean phenotype due to the environment. This  $X_E$  included the systematic effects of the environment our individual finds herself in and all of the noise during development, growth, and the various random insults that life throws at our individual. If a reasonable number of loci contribute to variation in our trait then we can approximate the distribution of  $X_{A,i}$  by a normal distribution due to the central limit theorem (see Figure 4.5). Thus if we can approximate the distribution of the effect of environmental variation on our trait ( $X_{E,i}$ ) also by a normal distribution, which is reasonable as there are many small environmental effects, then the distribution of phenotypes within the population ( $X_i$ ) will be normally distributed (see Figure 4.5).



Note that as this is an additive model we can decompose eqn. 4.2 into the effects of the two alleles at each locus, in particular we can rewrite it as

$$X_i = \mu + X_{iM} + X_{iP} + X_{iE} \quad (4.3)$$

where  $X_{iM}$  and  $X_{iP}$  are the contribution to the phenotype of the allele that our individual received from her mother (maternal alleles) and father (paternal alleles) respectively. This will come in handy in just a moment when we start thinking about the phenotype covariance of relatives.

Now obviously this model seems silly at first sight as alleles don't only act in an additive manner, as they interact with alleles at the same loci (dominance) and at different loci (epistasis). Later we'll relax this assumption, however, we'll find that if we are interested in evolutionary change over short time-scales it is actually only the "additive component" of genetic variation that will (usually) concern us. We will define this more formally later on, but for the moment we can offer the intuition that parents only get to pass on a single allele at each locus on to the next generation. As such, it is the effect of these transmitted alleles, averaged over possible matings, that is an individual's average contribution to the next generation (i.e. the additive effect of the alleles that their genotype consists of).

#### 4.0.2 Additive genetic variance and heritability

As we are talking about an additive genetic model we'll talk about the additive genetic variance ( $V_A$ ), the variance due to the additive effects of segregating genetic variation. This is a subset of the total genetic variance if we allow for non-additive effects.

The variance of our phenotype across individuals ( $V$ ) can write this as

$$V = \text{Var}(X_A) + \text{Var}(X_E) = V_A + V_E \quad (4.4)$$

in writing this we are assuming that there is no covariance between

**Figure 4.5:** The convergence of the phenotypic distribution to a normal distribution. Each of the three histograms shows the distribution of the phenotype in a large sample, for increasing large numbers of loci ( $L$ ). I have simulated each individual's phenotype following equation 4.1 and 4.2. Specifically I simulate each individual's biallelic genotype at  $L$  loci, assuming Hardy-Weinberg proportions and that the allele is at 50% frequency. I assume that all of the alleles have equal effects and combine them additively together. I then add an environmental contribution, which is normally distributed with variance 0.05. Note that in the left two pictures you can see peaks corresponding to different genotypes.

$X_{G,i}$  and  $X_{E,i}$  i.e. there is no covariance between genotype and environment.

Our additive genetic variance can be written as

$$V_A = \sum_{l=1}^L Var(G_{i,l}a_l) \quad (4.5)$$

where  $Var(G_{i,l}a_l)$  is the contribution to the additive variance among individuals of the  $l$  locus. Assuming random mating we can write our additive genetic variance as

$$V_A = \sum_{l=1}^L a_l^2 2p_l(1 - p_l) \quad (4.6)$$

where the  $2p_l(1 - p_l)$  term follows the binomial sampling of two alleles per individual at each locus.

*The narrow sense heritability* We would like a way to think about what proportion of the variation in our phenotype across individuals is due to genetic differences as opposed to environmental differences. Such a quantity will be key in helping us think about the evolution of phenotypes. For example, if variation in our phenotype had no genetic basis then no matter how much selection changes the mean phenotype within a generation the trait will not change over generations.

We'll call the proportion of the variance that is genetic the heritability, and denote it by  $h^2$ . We can then write this as

$$h^2 = \frac{Var(X_A)}{V} = \frac{V_A}{V} \quad (4.7)$$

remember that we thinking about a trait where all of the alleles act in a perfectly additive manner. In this case our heritability  $h^2$  is referred to as the narrow sense heritability, the proportion of the variance explained by the additive effect of our loci. When we allow dominance and epistasis into our model we'll also have to define the broad sense heritability (the total proportion of the phenotypic variance attributable to genetic variation).

The narrow sense heritability of a trait is a useful quantity, indeed we'll see shortly that it is exactly what we need to understand the evolutionary response to selection on a quantitative phenotype. We can calculate the narrow sense heritability by using the resemblance between relatives. For example, if our phenotype was totally environmental we should not expect relatives to resemble each other any more than random individuals drawn from the population. Now the obvious caveat here is that relatives also share an environment, so may resemble each other due to shared environmental effects.

#### 4.0.3 The covariance between relatives

So we'll go ahead and calculate the covariance in phenotype between two individuals (1 and 2) who have a phenotype  $X_1$  and  $X_2$  respectively.

$$\text{Cov}(X_1, X_2) = \text{Cov}((X_{1M} + X_{1P} + X_{1E}), ((X_{2M} + X_{2P} + X_{2E})) \quad (4.8)$$

We can expand this out in terms of the covariance between the various components in these sums.

To make our task easier we (and most analyses) will assume two things

1. that we can ignore the covariance of the environments between individuals (i.e.  $\text{Cov}(X_{1E}, X_{2E}) = 0$ )
2. that we can ignore the covariance between the environment variation experienced by an individual and the genetic variation in another individual (i.e.  $\text{Cov}(X_{1E}, (X_{2M} + X_{2P})) = 0$ ).

The failure of these assumptions to hold can severely undermine our estimates of heritability, but we'll return to that later. Moving forward with these assumptions, we can write our phenotypic covariance between our pair of individuals as

$$\text{Cov}(X_1, X_2) = \text{Cov}((X_{1M}, X_{2M}) + \text{Cov}(X_{1M}, X_{2P}) + \text{Cov}(X_{1P}, X_{2M}) + \text{Cov}(X_{1P}, X_{2P}) \quad (4.9)$$

This is saying that under our simple additive model we can see the covariance in phenotypes between individuals as the covariance between the allelic effects in our individuals. We can use our results about the sharing of alleles between relatives to obtain these terms. But before we write down the general case lets quickly work through some examples.

*The covariance between Identical Twins* Lets first consider the case of a pair of identical twins from two unrelated parents. Our pair of twins share their maternal and paternal allele identical by descent ( $X_{1M} = X_{2M}$  and  $X_{1P} = X_{2P}$ ). As their maternal and paternal alleles are not correlated draws from the population, i.e. have no probability of being *IBD* as we've said the parents are unrelated, the covariance between their effects on the phenotype is zero (i.e.  $\text{Cov}(X_{1P}, X_{2M}) = \text{Cov}(X_{1M}, X_{2P}) = 0$ ). In that case eqn. 4.9 is

$$\text{Cov}(X_1, X_2) = \text{Cov}((X_{1M}, X_{2M}) + \text{Cov}(X_{1P}, X_{2P}) = 2\text{Var}(X_{1M}) = V_A \quad (4.10)$$

Now in general identical twins are not going to be super helpful for us in estimating  $h^2$  as under models with non-additive effects identical

twins have higher covariance than we'd expect as they resemble each other also because of the dominance effects as they don't just share alleles they share their entire genotype.

#### *The covariance in phenotype between mother and child .*

If the mother and father are unrelated individuals (i.e. are two random draws from the population) then the mother and a child share one allele IBD at each locus (i.e.  $r_1 = 1$  and  $r_0 = r_2 = 0$ ). Half the time our mother transmits her paternal allele to the child, in which case  $X_{P1} = X_{M2}$  and so  $Cov(X_{P1}, X_{M2}) = Var(X_{P1})$  and all the other covariances in eqn. 4.9 zero, and half the time she transmits her maternal allele to the child  $Cov(X_{M1}, X_{M2}) = Var(X_{M1})$  and all the other terms zero. By this argument  $Cov(X_1, X_2) = \frac{1}{2}Var(X_{M1}) + \frac{1}{2}Var(X_{P1}) = \frac{1}{2}V_A$ .

*The covariance between general pairs of relatives under an additive model* The two examples make clear that to understand the covariance between phenotypes of relatives we simply need to think about the alleles they share IBD. Consider a pair of relatives (1 and 1) with a probability  $r_0$ ,  $r_1$ , and  $r_2$  of sharing zero, one, or two alleles IBD respectively. When they share zero alleles  $Cov((X_{1M} + X_{1P}), (X_{2M} + X_{2P})) = 0$ , when they share one allele  $Cov((X_{1M} + X_{1P}), (X_{2M} + X_{2P})) = Var(X_{1M}) = \frac{1}{2}V_A$ , and when they share two alleles  $Cov((X_{1M} + X_{1P}), (X_{2M} + X_{2P})) = V_A$ . Therefore, the general covariance between two relatives is

$$Cov(X_1, X_2) = r_0 \times 0 + r_1 \frac{1}{2}V_A + r_2 V_A = 2F_{1,2}V_A \quad (4.11)$$

So under a simple additive model of the genetic basis of a phenotype to measure the narrow sense heritability we need to measure the covariance between a set of pairs of relatives (assuming that we can remove the effect of shared environmental noise). From the covariance between relatives we can calculate  $V_A$ , we can then divide this by the total phenotypic variance to get  $h^2$ .

Another way that we can estimate the narrow sense heritability is through the regression of child's phenotype on the parental mid-point phenotype. The parental mid-point phenotype is simple the average of the mum and dad's phenotype. Denoting the child's phenotype by  $X_{kid}$  and mid-point phenotype by  $X_{mid}$  so that if we take the regression  $X_{kid} \sim X_{mid}$  this regression has slope  $\beta = Cov(X_{kid}, X_{mid})/Var(X_{mid})$ . The covariance of  $Cov(X_{kid}, X_{mid}) = \frac{1}{2}V_A$ , and  $Var(X_{mid}) = \frac{1}{2}V$  as by taking the average of the parents we have halved the variance, such that the slope of the regression is

$$\beta_{mid,kid} = \frac{Cov(X_{kid}, X_{mid})}{Var(X_{mid})} = \frac{V_A}{V} = h^2 \quad (4.12)$$



Figure 4.6: *Polemonium viscosum*. Modified from Flowers of Mountain and Plain. New York :H.W. Wilson Co.,1920. BHL.

**Question 1.** Galen explored selection on flower shape in *P. viscosum*. She found that plants with larger corolla flare had more bumblebee visits, which resulted in higher seed set and a 17% increase in corolla flare in the plants contributing to the next generation. Based on the data in Fig. 4.7 what is the expected response in the next generation?

GALEN, C., 1996 Rates of floral evolution: adaptation to bumblebee pollination in an alpine wildflower, *Polemonium viscosum*. Evolution 50(1): 120–125

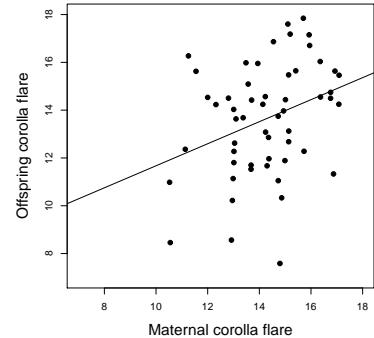


Figure 4.7: The relationship between maternal and offspring corolla flare (flower width) in *P. viscosum*. Redrawn from GALEN.

**Question 2. A)** In polygynous blackbird populations (i.e. males mate with several females), paternal half-sibs can be identified. Suppose that the covariance of tarsus lengths among half-sibs is  $0.25 \text{ cm}^2$  and that the total phenotypic variance is  $4 \text{ cm}^2$ . Use these data to estimate  $h^2$  for tarsus length in this population.

**B)** Why might paternal half-sibs be preferable for measuring heritability than maternal half-sibs?

i.e. the regression of the child's phenotype on the parental midpoint phenotype is an estimate of the narrow sense heritability. This is a common way to estimate heritability, although it doesn't bypass the need to control for environmental correlations between relatives.

Our regression allows us to attempt to predict the phenotype of the child given the phenotypes of the parents; how well we can do this depends on the slope. If the slope is close to zero then the parental phenotypes hold no information about the phenotype of the child, while if the slope is close to one then the parental mid-point is a good guess at the child's phenotype.

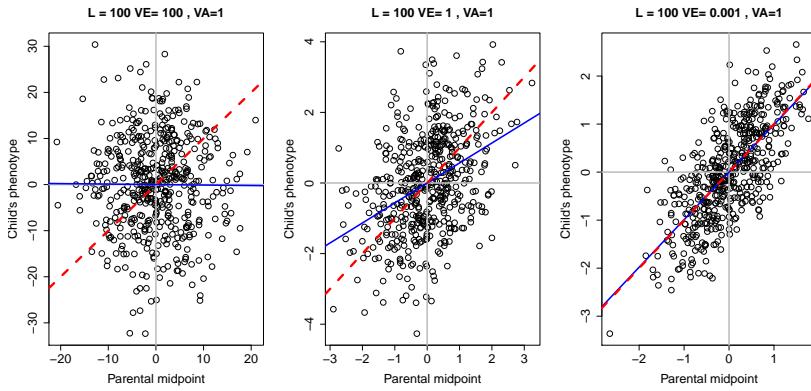


Figure 4.8: Regression of parental mid-point phenotype on child's phenotype. The three panels show decreasing levels of environmental variance ( $V_E$ ) holding the additive genetic variance constant ( $V_A = 1$ ). In these figures we simulate 100 loci, as described in the caption of Figure 4.5. We simulate the genotypes and phenotypes of the two parents, and then simulate the child's genotype following mendelian transmission. The blue line shows  $x = y$  the red line shows the best fitting linear regression line.

More formally the expected phenotype of the child given the parental phenotypes is

$$\mathbb{E}(X_{kid}|X_{mom}, X_{dad}) = \mu + \beta_{mid,kid}(X_{mid} - \mu) = \mu + h^2(X_{mid} - \mu) \quad (4.13)$$

this follows from the definition of linear regression. So to find the child's predicted phenotype we simply take the mean phenotype and add on the difference between our parental mid-point multiplied by our narrow sense heritability.

**Question 3.** Briefly explain Galton's observation of the regression towards mediocrity in light of Mendelian inheritance.

*Estimating additive genetic variance across a variety of different relationships.* In many natural populations we may have access to individuals of a range of different relationships to each other (through monitoring of the paternity of individuals), but relatively few individuals of a given relationship (e.g. sibs). We can try and use this information as fully as possible in a mixed model framework. Considering equation 4.2 we can write an individual's phenotype  $X_i$  as

$$X_i = \mu + X_{A,i} + e_i \quad (4.14)$$

where  $e_i \sim N(0, V_E)$  and  $X_{A,i}$  is normally distributed across individuals with covariance matrix  $V_A A$  where the entries for a pair of individuals  $i$  and  $j$  are  $A_{ij} = 2F_{i,j}$  and  $A_{ii} = 1$ . Given the matrix  $A$  we can estimate  $V_A$ . We can also add fixed effects into this model to account for generation effects, additional mixed effects could also be included to account for shared environments between particular individuals (e.g. a shared nest). This is sometimes called the “animal model”.

#### 4.1 Multiple traits.

Traits often covary with each other, due to both environmentally induced effects (e.g. due to the effects of diet on multiple traits) and due to the expression of underlying genetic covariance between traits. In turn this genetic covariance can reflect pleiotropy, a mechanistic effect of an allele on multiple traits (e.g. variants that effect skin pigmentation often effect hair color) or the genetic linkage of loci independently affecting multiple traits. If we are interested in evolution over short time-scales we can (often) ignore the genetic basis of this correlation.

Consider two traits  $X_{1,i}$  and  $X_{2,i}$  in an individual  $i$ , these could be say the individual’s leg length and nose length. As before we can write these as

$$\begin{aligned} X_{1,i} &= \mu_1 + X_{1,A,i} + X_{1,E,i} \\ X_{2,i} &= \mu_2 + X_{2,A,i} + X_{2,E,i} \end{aligned} \tag{4.15}$$

As before we can talk about the total phenotypic variance ( $V_1, V_2$ ), environmental variance ( $V_{1,E}$  and  $V_{2,E}$ ), and the additive genetic variance in trait one and two ( $V_{1,A}, V_{2,A}$ ). But now we also have to consider the total covariance  $V_{1,2} = Cov(X_1, X_2)$ , the environmentally induced covariance between the traits ( $V_{E,1,2} = Cov(X_{1,E}, X_{2,E})$ ) and the additive genetic covariance ( $V_{A,1,2} = Cov(X_{1,A}, X_{2,A})$ ) between trait one and two.

We can store these values in a matrices

$$\mathbf{V} = \begin{pmatrix} V_1 & V_{1,2} \\ V_{1,2} & V_2 \end{pmatrix} \tag{4.16}$$

and

$$\mathbf{G} = \begin{pmatrix} V_{1,A} & V_{A,1,2} \\ V_{A,1,2} & V_{2,A} \end{pmatrix} \tag{4.17}$$

we can generalize this to an arbitrary number of traits.

We can estimate these quantities, in a similar way to before, by studying the covariance in different traits between relatives:

$$Cov(X_{1,i}, X_{2,j}) = 2F_{i,j}V_{A,1,2} \tag{4.18}$$

A simple example of a genetic covariance is the covariance of male and female phenotypes.

Added genetic correlation, needed for the stalk-eyed flies

add question estimating genetic coar/corr

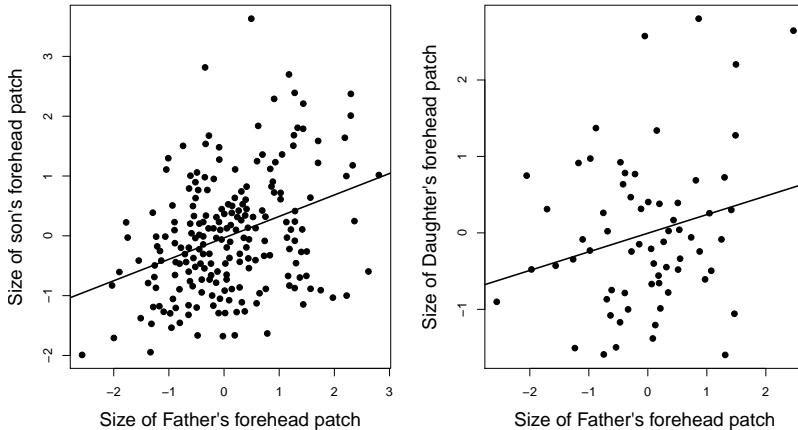


Figure 4.9: Relationship of standardized forehead patch size between fathers and sons and daughters in Pied fly-catchers. Redrawn from POTTI and CANAL.



#### 4.1.1 Non-additive variation.

Up to now we've assumed that our alleles contribute to our phenotype in an additive fashion. However, that does not have to be the case as there may be non-additivity among the alleles present at a locus (*dominance*) or among alleles at different loci (*epistasis*). We can accommodate these complications into our models. We do this by partitioning our total genetic variance into independent variance components.

*Dominance.* To understand the effect of dominance let's consider how the allele that a parent transmits influences their offspring's phenotype. A parent transmits one of their two alleles at a locus to their offspring. Assuming that individuals mate at random, this allele is paired with another allele drawn at random from the population. For example, if your mother transmitted allele 1 to you: with probability  $p$  it would be paired with another allele 1, and you would be a homozygote; and with probability  $q$  it's paired with a 2 allele and you're a heterozygote.

Consider autosomal biallelic locus  $\ell$ , with frequency  $p$  for allele 1, and genotypes 0, 1, and 2 corresponding to how many copies of allele 1 individuals carry. We'll denote the mean phenotype of an individual with genotype 0, 1, and 2 are  $\bar{X}_{\ell,0}$ ,  $\bar{X}_{\ell,1}$ ,  $\bar{X}_{\ell,2}$  respectively. Here this mean is taken over all the environments and genetic backgrounds the

Figure 4.10: *Ficedula hypoleuca*, Pied fly-catcher. Coloured illustrations of British birds, and their eggs. London :G.W. Nickisson, 1842-1850. BHL

alleles are present on. We'll mean center (MC) these phenotypic values setting  $\bar{X}'_{\ell,0} = \bar{X}_{\ell,0} - \mu$ , and likewise for the other genotypes.

We can think about the average (marginal) MC phenotype of an individual who received an allele 1 from their as the average of the MC phenotype for heterozygotes and 11 homozygotes weighted by the probability that an allele 1 is present in these genotypes

$$a_{\ell,1} = p\bar{X}'_{\ell,2} + q\bar{X}'_{\ell,1}, \quad (4.19)$$

Similarly if your parent transmitted a 2 allele to you, your average MC phenotype would be

$$a_{\ell,2} = p\bar{X}'_{\ell,1} + q\bar{X}'_{\ell,0} \quad (4.20)$$

Lets now consider the average phenotype of an offspring of each of our three genotypes

genotype: 0, 1, 2.

additive genetic value:  $a_{\ell,2} + a_{\ell,1}$ ,  $a_{\ell,1} + a_{\ell,2}$ ,  $a_{\ell,1} + a_{\ell,1}$

i.e. mean phenotype of each genotypes' offspring averaged over across all possible matings to other individuals in the population (assuming individuals mate at random). Theses are the additive MC genetic values (breeding values) of our genotypes. Here we are simply adding up the additive contributions of the alleles present in each genotype. These are the genotypic values of the trait that would result taking only the additive effects of the components of the genotype.

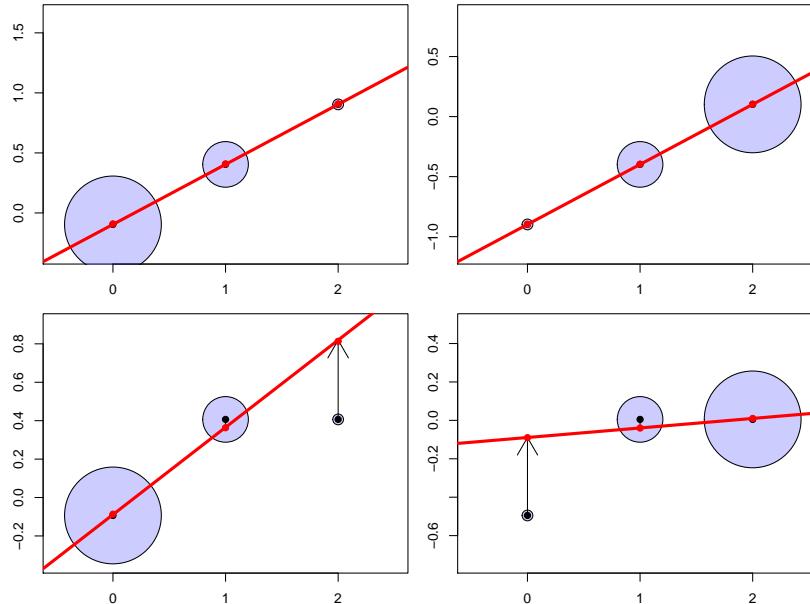


Figure 4.11: The average mean-centered (MC) phenotypes of each genotype. **Top Row:** Additive relationship between genotype and phenotype. **Bottom Row:** Allele 1 is dominant over allele 2, such that the heterozygote has the same phenotype as the 22 genotype (2). The area of each circle is proportion to the fraction of the population in each genotypic class ( $p^2$ ,  $2pq$ , and  $q^2$ ). One the left column  $p = 0.1$  and the right column is  $p = 0.9$ . The additive genetic values of the genotypes are shown as red dots. The regression between phenotype and additive genotype is shown as a red line. The black vertical arrows show the difference between the average MC phenotype and additive genetic value for each genotype.

To illustrate this in Figure 4.11 we plot two different cases of dominance relationship, in the top row an additive polymorphism and in

the second row a fully dominant allele. The additive genetic values of the genotypes are shown as red dots. Note that the additive values of the genotypes line up with the observed MC phenotypic means in the top row when our alleles interact in a completely additive manner. Our additive genetic values always fall along a linear line (the red line in our figure). The additive values are falling best fitting line of linear regression for our population, when phenotype is regressed against the additive genotype (0, 1, 2 copies of allele 1) across all individuals in our population. Note in the dominant case the additive genetic values differ from the observed phenotypic means, and are closer to the observed values for the genotypes that are common in the population.

The difference in the additive effect of the two alleles  $a_{\ell,2} - a_{\ell,1}$  can be interpreted as a average effect of swapping an allele 1 for an allele 2, we'll call this difference  $\alpha_\ell = a_{\ell,2} - a_{\ell,1}$ . Our  $\alpha_\ell$  is also the slope of the regression of genotype against phenotype (the red line in Figure 4.11). Note that the slope of our regression of genotype on phenotype ( $\alpha_\ell$ ) does not depend on allele frequency for our completely additive locus (top row of 4.11). In contrast, when there is dominance, our the slope between genotype on phenotype ( $\alpha_\ell$ ) is a function of allele frequency (bottom row of 4.11). When a dominant allele (1) is rare there is a strong slope of genotype on phenotype, bottom left Figure 4.11. This strong slope is because replacing a single copy of the 2 allele with a 1 allele in an individual has a big effect on average phenotype, as it will most likely move an individual from being a 22 homozygote to being a 12 heterozygote. However, when the dominant allele (1) is common in the population, replacing a 2 allele by a 1 allele in an individual on average has little phenotypic effect. This small effect is because as we are mainly turning heterozygotes into homozygotes (11), who have the same mean phenotype.

As an example of this lets consider the genetics of sexual maturity in Atlantic Salmon.

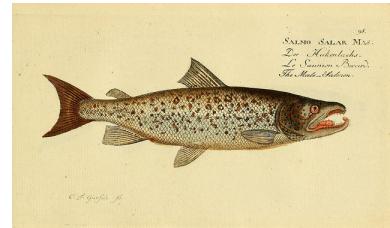
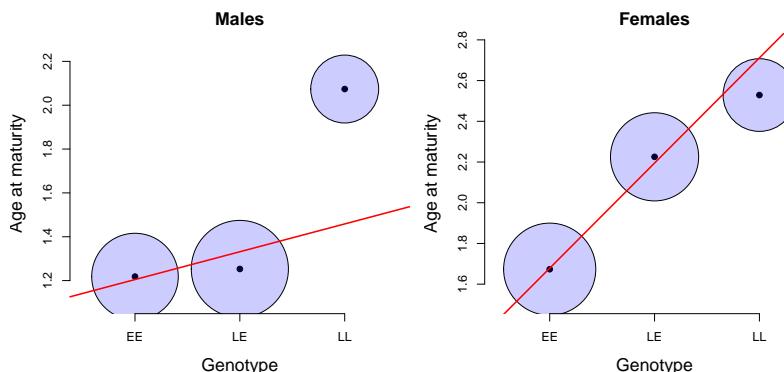


Figure 4.12: Atlantic Salmon (*Salmo salar*). *Histoire naturelle des poissons*. 1796. Bloch, M. E.

Figure 4.13: The average age at sexual maturity for each genotype, broken down by sex. The area of each circle is proportion to the fraction of the population in each genotypic class. The regression between phenotype and additive genotype is shown as a red line. The black vertical arrows show the difference between the average MC phenotype and additive genetic value for each genotype.

The variance in the population phenotype due to these additive breeding values at locus  $\ell$  assuming HW proportions is

$$\begin{aligned} V_{A,\ell} &= p^2(2a_{\ell,2})^2 + 2pq(a_{\ell,1} + a_{\ell,2})^2 + q^2(2a_{\ell,0})^2 \\ &= 2(pa_{\ell,1}^2 + qa_{\ell,2}^2) \\ &= 2pq\alpha_{\ell}^2 \end{aligned} \quad (4.21)$$

The total additive effect can be found by summing this additive genetic variance across loci

$$V_A = \sum_{\ell=1}^L V_{A,\ell} = \sum_{\ell=1}^L 2p_{\ell}q_{\ell}\alpha_{\ell}^2. \quad (4.22)$$

Having assigned the additive genetic variance to be the variance explained by the additive contribution of the alleles at a locus, we define the dominance variance as the population variance among genotypes at a locus due to their deviation from additivity. We can calculate how much each genotypic mean deviates away from its additive prediction at locus  $\ell$  (the length of the arrows in Figure 4.11), for example the heterozygote deviates

$$d_{\ell,1} = \bar{X}'_{\ell,1} - (a_{\ell,1} + a_{\ell,2}) \quad (4.23)$$

away from its additive genetic value, with similar expressions for each of the homozygotes ( $d_{\ell,0}$  and  $d_{\ell,2}$ ). We can then write the dominance variance at our locus as the genotype-frequency weighted sum of our squared dominance deviations

$$V_{D,\ell} = p^2d_{\ell,0}^2 + 2pqd_{\ell,1}^2 + q^2d_{\ell,2}^2. \quad (4.24)$$

Writing our total dominance variance as the sum across loci

$$V_D = \sum_{\ell=1}^L V_{D,\ell}. \quad (4.25)$$

Having now partitioned all of the genetic variance into additive and dominant terms, we can write our total genetic variance as

$$V_G = V_A + V_D. \quad (4.26)$$

We can do this because by construction the covariance between our additive and dominant deviations for the genotypes is zero. We can define the narrow sense heritability as before  $h^2 = V_A/V_P = V_A/(V_G + V_E)$ , which is the proportion of phenotypic variance due to additive genetic variance. We can also define the total proportion of the phenotype variance due to genetic differences among individuals, as the broad-sense heritability  $H^2 = V_G/(V_G + V_E)$ .

Relationship (i,j)*	$Cov(X_i, X_j)$
parent-child	$\frac{1}{2}V_A$
full siblings	$\frac{1}{2}V_A + \frac{1}{4}V_D$
identical (monozygotic) twins	$V_A + V_D$
1 <sup>st</sup> cousins	$\frac{1}{8}V_A$

Table 4.1: Phenotypic covariance between some pairs of relatives, include the dominance variation. \* Assuming this is the only relationship the pair of individuals share (above that expected from randomly sampling individuals from the population).

When dominance is present in the loci influencing our trait ( $V_D > 0$ ), we need to modify our phenotype covariance among relatives to account for this non-additivity. Specifically our equation for the covariance among a general pair of relatives (eqn. 4.11 for additive variation) becomes

$$Cov(X_1, X_2) = 2F_{1,2}V_A + r_2V_D \quad (4.27)$$

where  $r_2$  is the probability that the pair of individuals share 2 alleles identical by descent, making the same assumptions (other than additivity) that we made in deriving eqn. 4.11. In table 4.1 we show the phenotypic covariance for some common pairs of relatives. The regression of offspring phenotype on parental midpoint still has a slope  $V_A/V_P$ .

Full sibs and parent-offspring have the same covariance if there is no dominance variance (as they have the same kinship coefficient  $F_{1,2}$ ). However, when dominance is present ( $V_D > 0$ ), full-sibs resemble each other more than parent-offspring pairs. That's because parents and offspring share precisely one allele, while full-sibs can share both alleles (i.e. the full genotype at a locus) identical by descent. We can attempt to estimate  $V_D$  by comparing different sets of relationships. For example non-identical twins (full sibs born at same time) should have 1/2 the phenotypic covariance of identical twins if  $V_D = 0$ . Therefore, we can attempt to estimate  $V_D$  by looking at whether identical twins have more than twice the phenotypic covariance than non-identical twins.

The most important aspect of this discussion for thinking about evolutionary genetics is that the parent-offspring covariance is still only a function of  $V_A$ . This is because our parent (e.g. the mother) transmits only a single allele, at each locus, to its offspring. The other allele the offspring receives is random (assuming random mating), as it comes from the other unrelated parent (the father). Therefore, the average effect on the offspring phenotype of the allele the offspring receives from the mother, is averaged over the random allele the child receives from the father. Thus we only care about the additive effect of the allele, as parents transmit only alleles not genotypes to their offspring. This means that the short-term response to selection, as described by the breeder equation, depends only on  $V_A$  and the additive effect of alleles. Therefore, if we can estimate the narrow-sense

heritability we can predict the short-term response. However, if alleles display dominance our value of  $V_A$  will change as our loci change frequency, e.g. as dominant alleles become common in the population their contribution to  $V_A$  decreases. Therefore, our value of  $V_A$  will not be a constant across generations.

Up to this point we have only considered dominance and not epistasis. However, we include epistasis in a similar manner (for example among pairs of loci). This gets a little tricky to think about so we will only briefly explain it. We can first estimate the additive effect of the alleles by considering the effect of the alleles averaging over the other alleles they are paired with, just as before. We can then calculate the additive genetic variance from this. We can estimate the dominance variance, by calculating the residual variance among genotypes at a locus unexplained by the additive effect of the loci. We can then estimate the epistatic variance by estimating the residual variance left unexplained among the two locus genotypes from the additive and dominant deviations calculated from each locus separately. In practice these high variance components are hard to estimate, and usually small as much of our variance is assigned to the additive effect. Again we would find that we mostly care about  $V_A$  for predicting short-term evolution, but that the contribution of loci to the additive genetic variance will depend on the epistatic relationships among loci.

**Question 4.** You collect observations of red deer within a generation; recording an individual's number of offspring and phenotypes, which are known to have additive genetic variation, and construct the plots shown in Figure 4.14 (standardizing the phenotypes). Answer the following questions. For each question choose one of the bold options. Briefly justify each of your answers with reference to the breeder's equation and multi-trait breeder's equation. No calculations are required.

**A)** Looking at just figure 4.14 A, in what direction do you expect male antler size to evolve?

**Insufficient information, increase, decrease.**

**B)** Looking just at figures 4.14 B and C, in what direction do you expect male antler size to evolve?

**Insufficient information, increase, decrease.**

**C)** (3 Points) Looking at figures 4.14 A, B, and C in what direction do you expect male antler size to evolve?

**Insufficient information, increase, decrease.**

**Question 5.** How could you use 1/2 sibs vs full-sibs to estimate  $V_D$ ? Why might this be difficult in practice? Why are identical vs non-identical twins better suited for this?

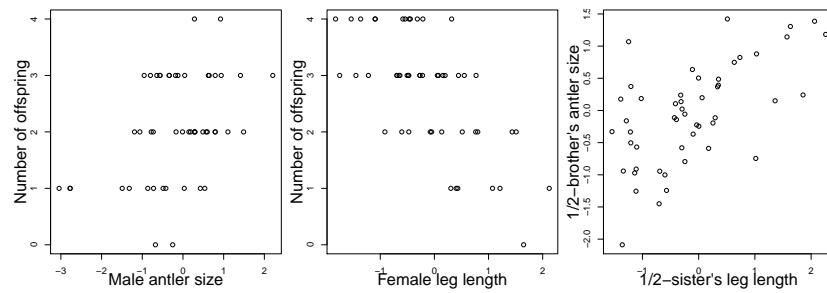


Figure 4.14: Observations of red deer within a generation; recording an individual's number of offspring and phenotypes, which are known to have additive genetic variation. The figures left to right are A-C.

**Question 6.** Can you construct a case where  $V_A = 0$  and  $V_D > 0$ ? You need just describe it qualitatively, you don't need to work out the math. (tricker question).

# 5

## *The response to selection*

Evolution by natural selection requires:

1. Variation in a phenotype
2. That survival is non-random with respect to this phenotypic variation.
3. That this variation is heritable.

Points 1 and 2 encapsulate our idea of Natural Selection, but evolution by natural selection will only occur if the 3rd condition is met. It is the heritable nature of variation that couples change within a generation due to natural selection, to change across generations (evolutionary change).

Lets start by thinking about the change within a generation due to directional selection, where selection acts to change the mean phenotype within a generation. For example, a decrease in mean height within a generation, due to taller organisms having a lower chance of surviving to reproduction than shorter organisms. Specifically, we'll denote our mean phenotype at reproduction by  $\mu_S$ , i.e. after selection has acted, and our mean phenotype before selection acts by  $\mu_{BS}$ . This second quantity may be hard to measure, as obviously selection acts throughout the life-cycle, so it might be easier to think of this as the mean phenotype if selection hadn't acted. So the mean phenotype changes within a generation is  $\mu_S - \mu_{BS} = S$ .

We are interested in predicting the distribution of phenotypes in next generation, in particular we are interested in the mean phenotype in the next generation to understand how directional selection has contributed to evolutionary change. We'll denote the mean phenotype in offspring, i.e. the mean phenotype in the next generation before selection acts, as  $\mu_{NG}$ . The change across generations we'll call the response to selection  $R$  and put this equal to  $\mu_{NG} - \mu_{BS}$ .

The mean phenotype in the next generation is

$$\mu_{NG} = \mathbb{E}(\mathbb{E}(X_{kid}|X_{mom}, X_{dad})) \quad (5.1)$$

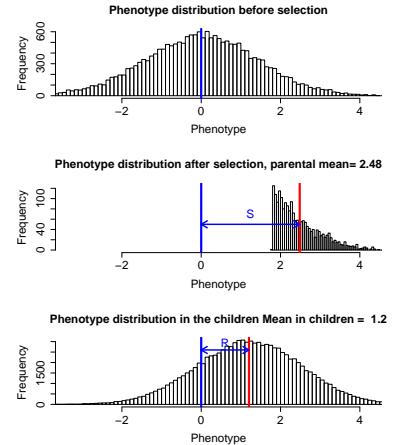


Figure 5.1: **Top.** Distribution of phenotype in parental population prior to selection,  $V_A = V_E = 1$ . **Middle.** Only individuals in the top 10% of the phenotypic distribution are selected to reproduce, the shift in the phenotypic mean is  $S$ . **Bottom.** Phenotypical distribution of children of the selected parents, the shift in the mean phenotype is  $R$ .

where the outer expectation is over the randomly mating of individuals who survive to reproduce. We can use eqn. 4.13 to obtain an expression for this

$$\mu_{NG} = \mu_{BS} + \beta_{mid,kid}(\mathbb{E}(X_{mid}) - \mu_{BS}) \quad (5.2)$$

so to obtain  $\mu_{NG}$  we need to compute  $\mathbb{E}(X_{mid})$  the expected mid-point phenotype of pairs of individuals who survive to reproduce. Well this is just the expected phenotype in the individuals who survived to reproduce ( $\mu_S$ ), so

$$\mu_{NG} = \mu_{BS} + h^2(\mu_S - \mu_{BS}) \quad (5.3)$$

So we can write our response to selection as

$$R = \mu_{NG} - \mu_{BS} = h^2(\mu_S - \mu_{BS}) = h^2S \quad (5.4)$$

So our response to selection is proportional to our selection differential, and the constant of proportionality is the narrow sense heritability. This equation is sometimes termed the Breeders equation. It is a statement that the evolutionary change across generations ( $R$ ) is proportional to the change caused by directional selection within a generation, and the strength of this relationship is determined by the narrow sense heritability.

---

Using the fact that  $h^2 = V_A/V$  we can rewrite this in a different form as

$$R = V_A \frac{S}{V} \quad (5.5)$$

i.e. our response to selection is the additive genetic variance of our trait ( $V_A$ ) multiplied by the change within a generation as a fraction of the total phenotypic variance ( $S/V$ , sometimes called the the selection gradient  $\beta$ ).

*The long-term response to selection* If our selection pressure is sustained over many generations we can use our breeders equation to predict the response. If we are willing to assume that our heritability does not change and we maintain a constant selection gradient, then after  $n$  generations our phenotype mean will have shifted

$$nh^2S \quad (5.6)$$

i.e. our population will keep up a linear response to selection.

*Alternative formulations of the Breeder's equation.* A change in mean phenotype within a generation occurs because of the differential fitness of our organisms. To think more carefully about this change within a

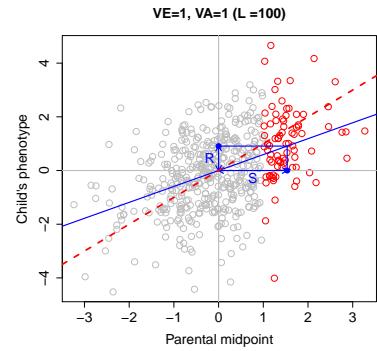


Figure 5.2: A visual representation of the Breeder's equation. Regression of parental mid-point phenotype on child's phenotype ( $V_A = V_E = 1$ ). Under truncation selection only individuals (red) with phenotypes  $> 1$  are bred.

Lost the barnacle question, put it back in.

**Question 1.** A population of red deer were trapped on Jersey (an island off of England) during the last inter-glacial period. From the fossil record we can see that the population rapidly adapted to their new conditions, within 6,000 years they evolved from an estimated mean weight of the population of 200kg to an estimated mean weight of 36kg (a 6 fold reduction)! You estimate that the generation time of red deer is 5 years and, from a current day population, that the narrow sense heritability of the phenotype is 0.5.

A) Estimate the mean change per generation in the mean body weight.

B) Estimate the change in mean body weight caused by selection within a generation. State your assumptions.

C) Assuming we only have fossils from the founding population and the population after 6000 years, should we assume that the calculations accurately reflect what actually occurred within our population?

LISTER, A., 1989 Rapid dwarfing of red deer on Jersey in the last interglacial. Nature 342(6249): 539

generation lets think about a simple fitness model where our phenotype affects the viability of our organisms (i.e. the probability they survive to reproduce). The probability that an individual has a phenotype  $X$  before selection is  $p(X)$ , so that the mean phenotype before selection is

$$\mu_{BS} = \mathbb{E}[X] = \int_{-\infty}^{\infty} xp(x)dx \quad (5.7)$$

The probability that an organism with a phenotype  $X$  survives to reproduce is  $w(X)$ , and we'll think about this as the fitness of our organism. The probability distribution of phenotypes in those who do survive to reproduce is

$$\mathbb{P}(X|\text{survive}) = \frac{p(x)w(x)}{\int_{-\infty}^{\infty} p(x)w(x)dx}. \quad (5.8)$$

where the denominator is a normalization constant which ensures that our phenotypic distribution integrates to one. The denominator also has the interpretation of being the mean fitness of the population, which we'll call  $\bar{w}$ , i.e.

$$\bar{w} = \int_{-\infty}^{\infty} p(x)w(x)dx. \quad (5.9)$$

Therefore, we can write the mean phenotype in those who survive to reproduce as

$$\mu_S = \frac{1}{\bar{w}} \int_{-\infty}^{\infty} xp(x)w(x)dx \quad (5.10)$$

If we mean center our population, i.e. set the phenotype before selection to zero, then

$$S = \frac{1}{\bar{w}} \int_{-\infty}^{\infty} xp(x)w(x)dx \quad (5.11)$$

if  $\mu_S = 0$ . Inspecting this more closely we can see that  $S$  has the form of a covariance between our phenotype  $X$  and our fitness  $w(X)$  ( $Cov(X, w(X))$ ). Thus our change in mean phenotype is directly a measure of the covariance of our phenotype and our fitness. Rewriting our breeder's equation using this observation we see

$$R = \frac{V_A}{V} Cov(X, w(X)) \quad (5.12)$$

we see that the response to selection is due to the fact that our fitness (viability) of our organisms/parents covaries with our phenotype, and that our child's phenotype is correlated with the parent phenotype.

### 5.0.1 The response of multiple traits to selection, the multivariate breeder's equation.

We can generalize these results for multiple traits, to ask how selection on multiple phenotypes plays out over short time intervals. We'll write

our change in the mean our multiple phenotypes within a generation as the vector  $\mathbf{S}$  and our response across multiple generations as the vector  $\mathbf{R}$ . These two quantities are related by

$$\mathbf{R} = \mathbf{G}\mathbf{V}^{-1}\mathbf{S} = \mathbf{G}\beta \quad (5.13)$$

where  $\mathbf{V}$  and  $\mathbf{G}$  are our matrices of the variance-covariance of phenotypes and additive genetic values (eqn. (4.17) (4.16)) and  $\beta$  is a vector of selection gradients (i.e. the change within a generation as a fraction of the total phenotypic variance). <sup>1</sup> To make this a bit more intuitive, consider two traits we are writing

$$\begin{aligned} R_1 &= V_{A,1}\beta_1 + V_{A,1,2}\beta_2 \\ R_2 &= V_{A,2}\beta_2 + V_{A,1,2}\beta_1 \end{aligned} \quad (5.14)$$

where the 1 and 2 index our two different traits. This is a statement that our response in any one phenotype is modified by selection on other traits that covary with that trait. This offers a good way to think about how genetic trade offs play out over evolution over short time-scales.

As an example of correlated responses to selection consider the WILKINSON selection experiment on Stalk-eyed flies (*Cyrtodiopsis dalmanni*). Stalk-eyed flies have evolved amazingly long eye-stalks. In the lab WILKINSON established six populations of wild-caught flies and selected up and down on males eye-stalk to body size ratio for 10 generations (left plot in Figure 5.4). Despite the fact that he did not select on females, he saw a correlated response in the females from each of the lines (right plot), because the genetic correlation between male and female body proportions.

### Question 2.

At the end of ten generation in WILKINSON's experiment (Figure 5.4) the males from the up- and down-selected lines had mean eye-stalk to body ratios of 1.29 and 1.14 respectively. While the females from the up- and down-selected lines had means of 0.9 and 0.82.

**A)** WILKINSON estimated that by selecting the top/bottom 10 males he had on average shifted the mean body ratio by 0.024 within each generation. What is the male heritability of mean eye-stalk to body ratio?

**B)** Assume that the heritability of male and female phenotypes are equal. Assuming that there is no direct selection on female boy-ratios in this experiment, i.e. that all of the response in female is due to correlated selection, can you estimate the genetic correlation?

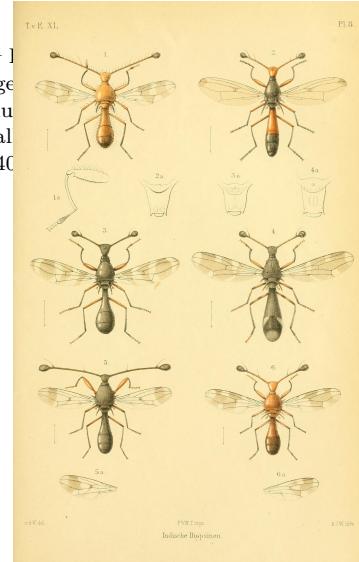


Figure 5.3: Stalk-eyed Flies (*Diopsidae*). Diptera. van der Wulp. 1898. BHL

Need to make sure that genetic corr. is defined and that they have enough info.

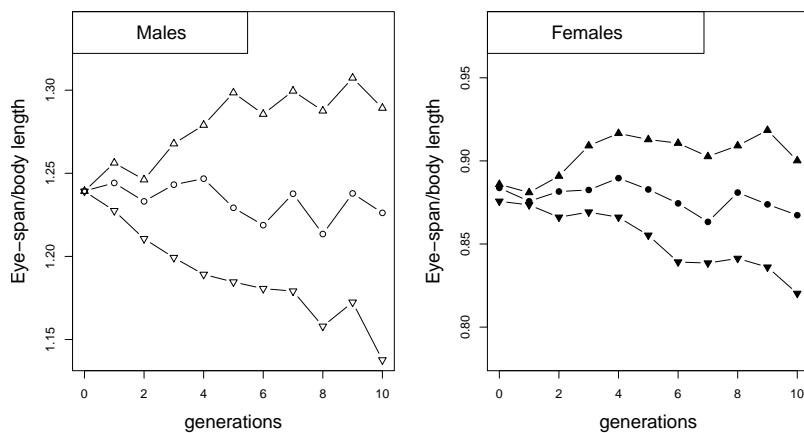


Figure 5.4: WILKINSON selected two of populations for flies for increased and eye-stalk to body length ratio in males (mean shown as up triangles), and two for a decreased ratio (down triangles), by taking the top 10 males with the highest (lowest) ratio out of 50 measures. He also established two control populations (circles). He constructed each generation of females by sampling 10 at random from each population.

### 5.0.2 Some applications of the multivariate trait breeder's equation

#### Hamilton's Rule and the evolution of altruistic and selfish behaviours

Adaptative evolution should push the population towards acting in a way that maximizes their evolutionary fitness, subject to constraints. Yet individuals frequently behave in ways that sacrifice their own fitness for the benefit of others. Hamilton supplied the first general explanation of such altruism. His intuition was that while an individual is losing out of some reproductive output, the alleles underlying an altruistic behaviour can still spread in the population if this cost is outweighed by benefits gained through the transmission of these alleles through a related individual (note that this means that the allele is not acting in an self sacrificing manner, even though individuals may as a result). We can use our quantitative genetics framework to gain some simple intuition for when altruistic behaviours should evolve through kin selection. To do this we can derive a simple version of Hamilton's Rule by thinking about the phenotypes of an individual's kin as genetically correlated phenotype.

To do this we can follow Queller's (1992) treatment, to rederive Hamilton's rule in a quantitative genetics framework (Hamilton's original work did this in a population genetics framework).

So lets imagine that individuals interact in pairs, with our focal individual  $i$  being paired with an individual  $j$ . Imagine that individuals have two possible phenotypes  $X = 1$  or  $0$ , corresponding to providing or withholding some small act of 'Altruism' (we could just as easily flip these labels and call them a unselfish act and a selfish act respectively). Our pairs of individuals interacting could for example be siblings sharing a nest. The altruistic trait could be as simple as grow-

this is work in progress

ing at a slightly slower rates so as to reducing sibling-competition for food from parents, or more complicated acts of altruism such as foregoing their own reproduction so as to help their parents raise their siblings.

Providing the altruistic act has a cost  $C$  to the fitness of our individual, with failing to provide this act has no cost. Receiving this altruistic act confers a fitness benefit  $B$  (not receiving it has no benefit). Hamilton's Rule states that such a trait will spread through the population if the average coefficient of relatedness between the interacting individuals is  $r$ , the average number of alleles that our pairs of individuals share identical by descent at a locus. We can express Hamilton's rule as

$$C < rB \quad (5.15)$$

To sketch a proof of this lets assume that our focal  $i$  individual's relative fitness can be written as

$$W(i, j) = W_0 + W_i + W_j \quad (5.16)$$

where  $W_i$  is the contribution of the fitness of the individual  $i$  due to this phenotype, and  $W_j$  is the contribution to our individual  $i$ 's fitness due to the  $j$ 's behaviour (i.e. phenotype). With the benefit  $B$  and cost  $C$  our  $W(i, j)$  are depicted in Figure 5.5.

We can write the expected change in phenotype is

$$R = \beta_i V_A + \beta_j V_{A,i,j} \quad (5.17)$$

Our altruistic phenotype is increasing in the population if  $R > 0$ , i.e. if

$$\begin{aligned} 0 < & \beta_i V_A + \beta_j V_{A,i,j} \\ -\beta_i < & \beta_j \frac{V_{A,i,j}}{V_A} \end{aligned} \quad (5.18)$$

So what's the average genetic covariance between individual  $i$  and  $j$ 's altruistic phenotype? Well the covariance of the same phenotype between two individual's is just  $2F_{i,j}V_A$  (see (4.11)). our  $F_{i,j}$  is the probability that an allele found in individual  $i$  is identical by descent to an allele drawn from  $j$ ,  $2F_{i,j}$  can be interpreted as the average number of alleles shared between individuals  $i$  and  $j$  our re. So

$$\begin{aligned} -\beta_i < & \beta_j \frac{2F_{i,j}V_A}{V_A} \\ -\beta_i < & \beta_j 2F_{i,j} \\ C < & rB \end{aligned} \quad (5.19)$$

$$(5.20)$$

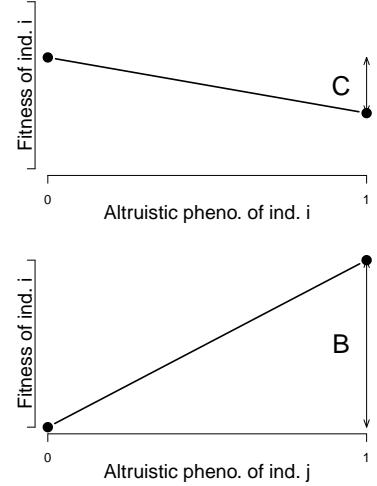


Figure 5.5:

*Sexual selection and the evolution of mate preference by Indirect benefits.* Organisms often put an enormous effort into attracting mates, sometimes at a considerable cost to their chances of survival.

One major reason why individuals are choosy about who they mate with is that it can directly impact their fitness. By choosing a mate with particular characteristics individuals can gain more parental care for their offspring, avoid parasites, or be choosing a mate with higher fertility. However, even in the absence of direct benefits selection can still indirectly favour the evolution of choosiness. These indirect benefits occur because as individuals can have higher fitness offspring by choosing a mate who's phenotype indicates high viability (the so-called good genes hypothesis), or by

We'll denote the display trait, e.g. tail length, in males by  $\sigma$  and the preference trait in females by  $\varphi$ . Our display trait is under direct selection in males, such that its response to selection can be written as

$$R_\sigma = \beta_\sigma V_{A,\sigma} \quad (5.21)$$

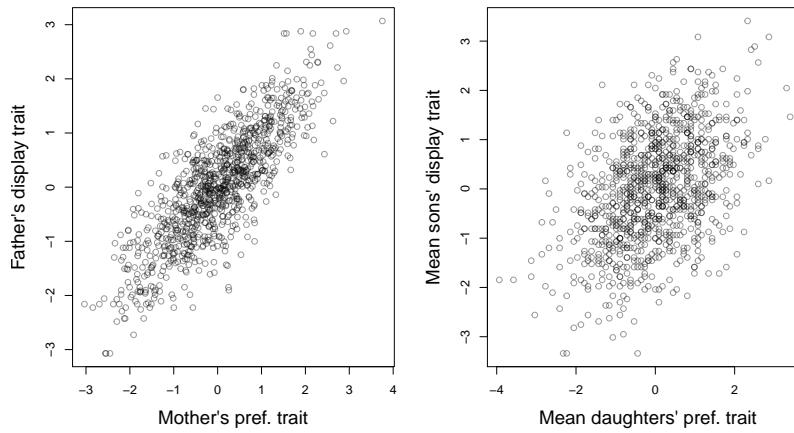
Let's assume that the female preference trait, the degree to which females are attracted to long tails, is not under direct selection  $\beta_\varphi = 0$  then the response to selection of the preference trait can be written as

$$R_\varphi = \beta_\varphi V_{A,\varphi} + \beta_\sigma V_{A,\varphi\sigma} = \beta_\sigma V_{A,\varphi\sigma} \quad (5.22)$$

so the female preference will respond to selection if it is genetically correlated with the male trait, i.e. if  $V_{A,\varphi\sigma} \neq 0$ . There's a number of different ways this genetic correlation could arise. The simplest is that the loci underlying the male trait may have a pleiotropic effect on female preference, however, female preference may often have quite a distinct genetic basis from male display traits.

A more general way in which trait-preference genetic correlations may arise is through assortative mating. As females vary in their tail-length preference, the ones with a preference for longer tails will mate with long-tailed males and the opposite for females with a preference for shorter-tails. Therefore, a genetic correlation between mates display and preference traits will become established, see Figure 5.0.2. The males with the longer tails will also carry the alleles associated with the preference for longer tails, as their long-tailed dads tended to mate with females with a genetic preference for long tails. Similarly the males with shorter tails will carry alleles associated with the preference for shorter tails. Thus if direct selection males with longer tails then the female preference for longer tails will increase as it is genetically correlated via assortative mating.

As an example of how selection on display traits can drive the evolution of preference traits, let's consider some data from guppies. Guppies (*Poecilia reticulata*) are a classic system for studying the interplay of natural and sexual selection. In some populations of guppies, females show a preference for males with more orange colouration. HOUDE established four replicated populations pairs of guppies



(*Poecilia reticulata*) and selected one of each pair for an increase or decreased orange coloration in males; selecting the top/bottom 20 out of 50 males. She randomly chose females from each population to form the next generation, and so did not exert direct selection on females. She measured the response to selection on male colouration and female preference for orange, left and right panels of Figure 5.6 respectively. In the lines that were selected for more orange males.

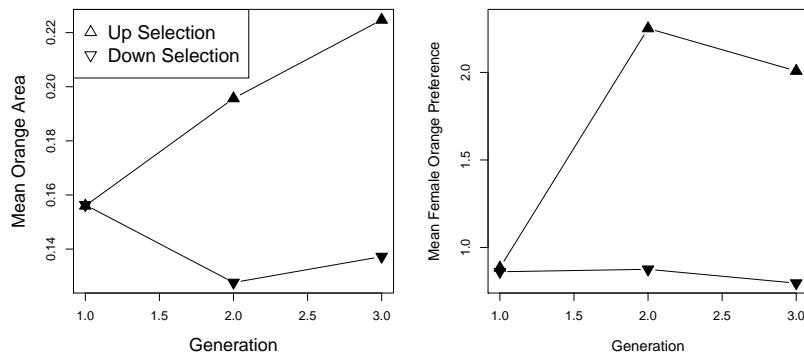


Figure 5.6: Mean phenotypes for the two up- and two down-selected populations of Guppies. Left panel: a response to selection was seen due to the direct selection on male colouration. Right panel: an indirect, correlated response was also seen in female preference. Redrawn from HOODE.

# 6

## *One-locus models of selection*

### *6.0.1 Fitness*

As we have seen, natural selection occurs when there are differences between individuals in fitness. We may define fitness in various ways. Most commonly, it is defined with respect to the contribution of a phenotype or genotype to the next generation. Differences in fitness can arise at any point during the life cycle. For instance, different genotypes or phenotypes may have different survival probabilities from one stage in their life to the stage of reproduction (viability), or they may differ in the number of offspring produced (fertility), or both. Here, we define the absolute fitness of a genotype as the expected number of offspring of an individual of that genotype.

### *6.0.2 Haploid selection model*

We start out by modelling selection in a haploid model, as this is mathematically relatively simple. Let the number of individuals carrying alleles  $A_1$  and  $A_2$  in generation  $t$  be  $P_t$  and  $Q_t$ . Then, the relative frequencies at time  $t$  of alleles  $A_1$  and  $A_2$  are  $p_t = P_t/(P_t + Q_t)$  and  $q_t = Q_t/(P_t + Q_t) = 1 - p_t$ . Further, assume that individuals of type  $A_1$  and  $A_2$  on average produce  $W_1$  and  $W_2$  offspring individuals, respectively. We call  $W_i$  the absolute fitness.

Therefore, in the next generation, the absolute number of carriers of  $A_1$  and  $A_2$  are  $P_{t+1} = W_1 P_t$  and  $Q_{t+1} = W_2 Q_t$ , respectively. The mean absolute fitness of the population at time  $t$  is

$$\overline{W}_t = W_1 \frac{P_t}{P_t + Q_t} + W_2 \frac{Q_t}{P_t + Q_t} = W_1 p_t + W_2 q_t, \quad (6.1)$$

i.e. the sum of the fitness of the two types weighted by their relative frequencies. Note that the mean fitness depends on time, as it is a function of the allele frequencies, which are themselves time dependent.

The frequency of allele  $A_1$  in the next generation is then given by

$$p_{t+1} = \frac{P_{t+1}}{P_{t+1} + Q_{t+1}} = \frac{W_1 P_t}{W_1 P_t + W_2 Q_t} = \frac{W_1 p_t}{W_1 p_t + W_2 q_t} = \frac{W_1}{\bar{W}_t} p_t. \quad (6.2)$$

Importantly, eqn. (6.2) tells us that the change in  $p$  only depends on a ratio of fitnesses. Therefore, we need to specify fitness only up to an arbitrary constant. As long as we multiply all fitnesses by the same value, that constant will cancel out and eqn. (6.2) will hold.

Based on this argument, it is very common to scale absolute fitnesses by the absolute fitness of one of the genotypes, e.g. the most or the least fit genotype, to obtain relative fitnesses. Here, we will use  $w_i$  for the relative fitness of genotype  $i$ . If we choose to scale by the absolute fitness of genotype  $A_1$ , we obtain the relative fitnesses  $w_1 = W_1/W_1 = 1$  and  $w_2 = W_2/W_1$ .

Without loss of generality, we can therefore rewrite eqn. (6.2) as

$$p_{t+1} = \frac{w_1}{\bar{w}} p_t, \quad (6.3)$$

dropping the dependence of the mean fitness on time in our notation, but remembering it. The change in frequency from one generation to the next is then given by

$$\Delta p_t = p_{t+1} - p_t = \frac{w_1 p_t}{\bar{w}} - p_t = \frac{w_1 p_t - \bar{w} p_t}{\bar{w}} = \frac{w_1 p_t - (w_1 p_t + w_2 q_t) p_t}{\bar{w}} = \frac{w_1 - w_2}{\bar{w}} p_t q_t, \quad (6.4)$$

recalling that  $q_t = 1 - p_t$ .

Assuming that the fitnesses of the two alleles are constant over time, the number of the two allelic types  $\tau$  generations after time  $t$  are  $P_{t+\tau} = (W_1)^\tau P_t$  and  $Q_{t+\tau} = (W_2)^\tau Q_t$ , respectively. Therefore, the relative frequency of allele  $A_1$  after  $\tau$  generations past  $t$  is

$$p_{t+\tau} = \frac{(W_1)^\tau P_t}{(W_1)^\tau P_t + (W_2)^\tau Q_t} = \frac{(w_1)^\tau P_t}{(w_1)^\tau P_t + (w_2)^\tau Q_t} = \frac{p_t}{p_t + (w_2/w_1)^\tau q_t}, \quad (6.5)$$

where the last step includes dividing the whole term by  $(w_1)^\tau$  and switching from absolute to relative allele frequencies.

Rearranging eqn. (6.5) and setting  $t = 0$ , we can work out the time  $\tau$  for the frequency of  $A_1$  to change from  $p_0$  to  $p_\tau$ . First, we write

$$p_\tau = \frac{p_0}{p_0 + (w_2/w_1)^\tau q_0} \quad (6.6)$$

and rearrange this to obtain

$$\frac{p_\tau}{q_\tau} = \frac{p_0}{q_0} \left( \frac{w_1}{w_2} \right)^\tau. \quad (6.7)$$

Solving this for  $\tau$  yields

$$\tau = \log \left( \frac{p_\tau q_0}{q_\tau p_0} \right) / \log \left( \frac{w_1}{w_2} \right). \quad (6.8)$$

In practice, it is often helpful to parametrize the relative fitnesses  $w_i$  in a specific way. For example, we may set  $w_1 = 1$  and  $w_2 = 1 - s$ , where  $s$  is called the selection coefficient. Using this parametrization,  $s$  is simply the difference in relative fitnesses between the two alleles. Equation (6.5) becomes

$$p_{t+\tau} = \frac{p_t}{p_t + q_t(1-s)^\tau}, \quad (6.9)$$

as  $w_2/w_1 = 1 - s$ . Then, if  $s \ll 1$ , we can approximate  $(1-s)^\tau$  in the denominator by  $\exp(-s\tau)$  to obtain

$$p_{t+\tau} \approx \frac{p_t}{p_t + q_t e^{-s\tau}}. \quad (6.10)$$

This equation takes the form of a logistic function. That is because we are looking at the relative frequencies of two ‘populations’ (of alleles  $A_1$  and  $A_2$ ) that are growing (or declining) exponentially, under the constraint that  $p$  and  $q$  always sum to 1.

Moreover, eqn. (6.7) for the time  $\tau$  it takes for a certain change in frequency to occur becomes

$$\tau = -\log \left( \frac{p_\tau q_0}{q_\tau p_0} \right) / \log(1-s). \quad (6.11)$$

Assuming again that  $s \ll 1$ , this simplifies to

$$\tau \approx \frac{1}{s} \log \left( \frac{p_\tau q_0}{q_\tau p_0} \right). \quad (6.12)$$

One particular case of interest is the time it takes to go from an absolute frequency of 1 to near fixation in a population of size  $N$ . In this case, we have  $p_0 = 1/N$ , and we may set  $p_\tau = 1 - 1/N$ , which is very close to fixation. Then, plugging these values into eqn. (6.12), we obtain

$$\begin{aligned} tau &= \frac{1}{s} \log \left( \frac{1 - 2/N + 1/N^2}{1/N^2} \right) \\ &\approx \frac{1}{s} (\log(N) + \log(N-2)) \\ &\approx \frac{2}{s} \log(N) \end{aligned} \quad (6.13)$$

where we make the approximations  $N^2 - 2N + 1 \approx N^2 - 2N$  and later  $N - 2 \approx N$ .

**Question 1.** You are studying the frequency of antibiotic-resistant bacteria in a patient. Before administering the antibiotic the frequency of the resistance allele is 1/1000. You administer the antibiotic, alarming just 8 days later you find the frequency of the resistance allele to be 99%. Assume a generation time of 1/4 a day for these bacteria. What is the selection coefficient associated with the resistance to antibiotics?

*Haploid model with fluctuating selection* We can now consider the case where the fitnesses depend on time, and say that  $w_{1,t}$  and  $w_{2,t}$  are the fitnesses of the two types in generation  $t$ . The frequency of allele  $A_1$  in generation  $t + 1$  is

$$p_{t+1} = \frac{w_{1,t}}{\bar{w}_t} p_t, \quad (6.14)$$

which simply follows from eqn. (6.3). The ratio of the frequency of allele  $A_1$  to that of allele  $A_2$  in generation  $t + 1$  is

$$\frac{p_{t+1}}{q_{t+1}} = \frac{w_{1,t} p_t}{w_{2,t} q_t}. \quad (6.15)$$

Therefore, if we think of the two alleles starting in generation  $t$  at frequencies  $p_t$  and  $q_t$ , then  $\tau$  generations later,

$$\frac{p_{t+\tau}}{q_{t+\tau}} = \left( \prod_{i=t}^{\tau-1} \frac{w_{1,i}}{w_{2,i}} \right) \frac{p_t}{q_t}. \quad (6.16)$$

Dempster (1955)

The question of which allele is increasing or decreasing in frequency comes down to whether  $\left( \prod_{i=t}^{\tau-1} \frac{w_{1,i}}{w_{2,i}} \right)$  is  $> 1$  or  $< 1$ . As it is a little hard to think about this ratio, we can instead take the  $\tau^{\text{th}}$  root of it and consider

$$\sqrt[\tau]{\left( \prod_{i=t}^{\tau-1} \frac{w_{1,i}}{w_{2,i}} \right)} = \frac{\sqrt[\tau]{\prod_{i=t}^{\tau-1} w_{1,i}}}{\sqrt[\tau]{\prod_{i=t}^{\tau-1} w_{2,i}}}. \quad (6.17)$$

The term  $\sqrt[\tau]{\prod_{i=t}^{\tau-1} w_{1,i}}$  is the geometric mean fitness of allele  $A_1$  over the  $\tau$  generations past generation  $t$ . Therefore, allele  $A_1$  will only increase in frequency if it has a higher geometric mean fitness than allele  $A_2$  (at least in our simple deterministic model).

### 6.0.3 Diploid model

We will now move on to a diploid model of a single locus with two segregating alleles. We will assume that the difference in fitness between

the three genotypes comes from differences in viability, i.e. differential survival of individuals from the formation of zygotes to reproduction. We denote the absolute fitnesses of genotypes  $A_1A_1$ ,  $A_1A_2$ , and  $A_2A_2$  by  $W_{11}$ ,  $W_{12}$ , and  $W_{22}$ . Specifically,  $W_{ij}$  is the probability that a zygote of genotype  $A_iA_j$  survives to reproduction. Assuming that individuals mate at random, the number of zygotes that are of the three genotypes and form generation  $t$  are

$$Np_t^2, \quad N2p_tq_t, \quad Nq_t^2. \quad (6.18)$$

The mean fitness of the population of zygotes is then

$$\bar{W}_t = W_{11}p_t^2 + W_{12}2p_tq_t + W_{22}q_t^2. \quad (6.19)$$

Again, this is simply the weighted mean of the genotypic fitnesses.

How many zygotes of each of the three genotypes survive to reproduce? An individual of genotype  $A_1A_1$  has a probability of  $W_{11}$  of surviving to reproduce, and similarly for other genotypes. Therefore, the expected number of  $A_1A_1$ ,  $A_1A_2$ , and  $A_2A_2$  individuals who survive to reproduce is

$$NW_{11}p_t^2, \quad NW_{12}2p_tq_t, \quad NW_{22}q_t^2. \quad (6.20)$$

It then follows that the total number of individuals who survive to reproduce is

$$N (W_{11}p_t^2 + W_{12}2p_tq_t + W_{22}q_t^2). \quad (6.21)$$

This is simply the mean fitness of the population multiplied by the population size (i.e.  $N\bar{w}$ ).

The relative frequency of  $A_1A_1$  individuals at reproduction is simply the number of  $A_1A_1$  genotype individuals at reproduction ( $NW_{11}p_t^2$ ) divided by the total number of individuals who survive to reproduce ( $N\bar{w}$ ), and likewise for the other two genotypes. Therefore, the relative frequency of individuals with the three different genotypes at reproduction is

$$\frac{NW_{11}p_t^2}{N\bar{w}}, \quad \frac{NW_{12}2p_tq_t}{N\bar{w}}, \quad \frac{NW_{22}q_t^2}{N\bar{w}} \quad (6.22)$$

(see Table 6.1).

	$A_1A_1$	$A_1A_2$	$A_2A_2$
Absolute no. at birth	$Np_t^2$	$N2p_tq_t$	$Nq_t^2$
Fitnesses	$W_{11}$	$W_{12}$	$W_{22}$
Absolute no. at reproduction	$NW_{11}p_t^2$	$NW_{12}2p_tq_t$	$NW_{22}q_t^2$
Relative freq. at reproduction	$\frac{NW_{11}p_t^2}{N\bar{w}} = \frac{W_{11}}{\bar{w}} p_t^2$	$\frac{NW_{12}2p_tq_t}{N\bar{w}} = \frac{W_{12}}{\bar{w}} 2p_tq_t$	$\frac{NW_{22}q_t^2}{N\bar{w}} = \frac{W_{22}}{\bar{w}} q_t^2$

As there is no difference in the fecundity of the three genotypes, the allele frequencies in the zygotes forming the next generation are simply

Table 6.1: Relative genotype frequencies after one episode of viability selection.

the allele frequency among the reproducing individuals of the previous generation. Hence, the frequency of  $A_1$  in generation  $t + 1$  is

$$p_{t+1} = \frac{W_{11}p_t^2 + W_{12}p_tq_t}{\bar{W}}. \quad (6.23)$$

Note that, again, the absolute value of the fitnesses is irrelevant to the frequency of the allele. Therefore, we can just as easily replace the absolute fitnesses with the relative fitnesses. That is, we may replace  $W_{ij}$  by  $w_{ij} = W_{ij}/W_{11}$ , for instance.

Each of our genotype frequencies is responding to selection in a manner that depends just on its fitness compared to the mean fitness of the population. For example the frequency of the 11 homozygotes increases from birth to adulthood in proportion to  $W_{11}/\bar{W}$ . In fact we can estimate this fitness ratio for each genotype by comparing the frequency at birth compared to adults. As an example of this we'll look at some data from sticklebacks.

Marine threespine stickleback (*Gasterosteus aculeatus*) have independently colonized and adapted to many freshwater lakes as glaciers receded following the last ice age making them a wonderful system for studying the genetics of adaptation. In marine habitats most of the stickleback have armour plates to protect them from predation, but freshwater populations repeatedly evolve the loss of armour plates due to selection on an allele at the Ectodysplasin gene (EDA). This allele is found as a standing variant at very low frequency marine populations; BARRETT *et al.* took advantage of this fact and collected and bred a population of marine individuals carrying both the low- (L) and completely- plated (C) alleles. They introduced the offspring of this cross into four freshwater ponds and monitored genotype frequencies <sup>1</sup> over their life courses:

	CC	LC	LL
Juveniles	0.55	0.23	0.22
Adults	0.21	0.53	0.26
Juv./Adult ( $W_\bullet/\bar{W}$ )	0.4	2.3	1.2
rel. fitness ( $W_\bullet/W_{12}$ )	0.17	1.0	0.54

The heterozygotes have increased in frequency dramatically in the population as their fitness is more than double the mean fitness of the population. We can also calculate the relative fitness of each genotype by dividing through by the fitness of the fittest genotype, the heterozygote in this case (doing this cancels through  $\bar{W}$ ). The relative fitness of the CC is  $\sim 1/5$  of the heterozygote. This calculation does not rely on the genotype frequency being at their HWE in the juveniles.

**Question 2.** A What is the frequency of the low-plated EDA allele (L) at the start of the stickleback experiment?

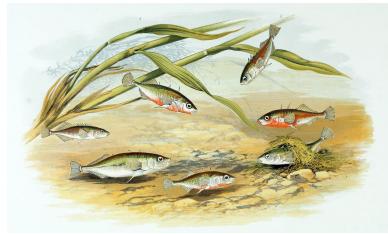


Figure 6.1: Freshwater threespine Stickleback (*G. aculeatus*). British fresh-water fishes. Houghton W 1879. BHL.

<sup>1</sup> The actual dynamics observed by BARRETT *et al.* as in the very young selection reverses direction

**B** What is the frequency in the adults?

**Question 3.** You have been studying an annual wildflower for many generations with two color morphs orange and white. You have discovered that a single bi-allelic locus controls flower color, with the white allele being recessive. The pollinator of these plants is an almost blind bat, so individuals are pollinated at random with respect to flower color. Your population census of 200 individuals showed that the population consisted of 168 orange-flowered individuals, and 32 white-flowered individuals.

Heavy February rainfall creates optimal growing conditions for an exotic herbivorous beetle with a preference for orange-flowered individuals. This year it arrives at your study site with a ravenous appetite. Only 50% of orange-flowered individuals survive its wrath, while 90% of white-flowered individuals survive until the end of the growing season.

**A** What is the initial frequency of the white allele, and what do you have to assume to obtain this?

**B** What is the frequency of the white allele in the seeds forming the next generation?

The change in frequency from generation  $t$  to  $t + 1$  is

$$\Delta p_t = p_{t+1} - p_t = \frac{w_{11}p_t^2 + w_{12}p_tq_t}{\bar{w}} - p_t. \quad (6.24)$$

To simplify this equation, we will first define two variables  $\bar{w}_1$  and  $\bar{w}_2$  as

$$\bar{w}_1 = w_{11}p_t + w_{12}q_t, \quad (6.25)$$

$$\bar{w}_2 = w_{12}p_t + w_{22}q_t. \quad (6.26)$$

These are called the marginal fitnesses of allele  $A_1$  and  $A_2$ , respectively. They are so called as  $\bar{w}_1$  is the average fitness of an allele  $A_1$ , i.e. the fitness of  $A_1$  in a homozygote weighted by the probability it is in a homozygote ( $p_t$ ) plus the fitness of  $A_1$  in a heterozygote weighted by the probability it is in a heterozygote ( $q_t$ ). We further note that the mean relative fitness can be expressed in terms of the marginal fitnesses as

$$\bar{w} = \bar{w}_1p_t + \bar{w}_2q_t, \quad (6.27)$$

where, for notational simplicity, we have omitted the dependence of mean and marginal fitnesses on time.

We can then rewrite eqn. (6.24) using  $\bar{w}_1$  and  $\bar{w}_2$  as

$$\Delta p_t = \frac{(\bar{w}_1 - \bar{w}_2)}{\bar{w}} p_t q_t. \quad (6.28)$$

The sign of  $\Delta p_t$ , i.e. whether allele  $A_1$  increases or decreases in frequency, depends only on the sign of  $(\bar{w}_1 - \bar{w}_2)$ . The frequency of  $A_1$  will keep increasing over the generations so long as its marginal fitness is higher than that of  $A_2$ , i.e.  $\bar{w}_1 > \bar{w}_2$ , while if  $\bar{w}_1 < \bar{w}_2$ , the frequency of  $A_1$  will decrease. Note the similarity between eqn. (6.28) and the respective expression for the haploid model in eqn. (6.4). (We will return to the special case where  $\bar{w}_1 = \bar{w}_2$  shortly).

We can also rewrite (6.24) as

$$\Delta p_t = \frac{1}{2} \frac{p_t q_t}{\bar{w}} \frac{d\bar{w}}{dp}, \quad (6.29)$$

the demonstration of this we leave to the reader. This form shows that the frequency of  $A_1$  will increase ( $\Delta p_t > 0$ ) if the mean fitness is an increasing function of the frequency of  $A_1$  (i.e. if  $\frac{d\bar{w}}{dp} > 0$ ). On the other hand, the frequency of  $A_1$  will decrease ( $\Delta p_t < 0$ ) if the mean fitness is a decreasing function of the frequency of  $A_1$  (i.e. if  $\frac{d\bar{w}}{dp} < 0$ ). Thus, although selection acts on individuals, under this simple model, selection is acting to increase the mean fitness of the population. The rate of this increase is proportional to the variance in allele frequencies within the population ( $p_t q_t$ ).

**Question 4.** Show that eqns. (6.29) and (6.28) are equivalent. (Trickier question.)

So far, our treatment of the diploid model of selection has been in terms of generic fitnesses  $w_{ij}$ . In the following, we will use particular parametrizations to gain insight about two specific modes of selection: directional selection and heterozygote advantage.

#### 6.0.4 Diploid directional selection

Directional selection means that one of the two alleles always has higher marginal fitness than the other one. Let us assume that  $A_1$  is the fitter allele, so that  $w_{11} \geq w_{12} \geq w_{22}$ , and hence  $\bar{w}_1 > \bar{w}_2$ . As we are interested in changes in allele frequencies, we may use relative fitnesses. We parameterize the reduction in relative fitness in terms of a selection coefficient, similar to the one we met in the haploid selection section, as follows:

genotype	$A_1 A_1$	$A_1 A_2$	$A_2 A_2$
absolute fitness	$W_{11}$	$\geq W_{12} \geq$	$W_{22}$
relative fitness (generic)	$w_{11} = W_{11}/W_{11}$	$w_{12} = W_{12}/W_{11}$	$w_{22} = W_{22}/W_{11}$
relative fitness (specific)	1	$1 - sh$	$1 - s$ .

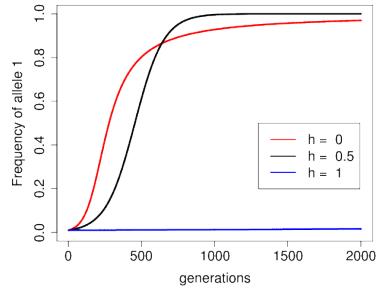


Figure 6.2: The trajectory of the frequency of allele  $A_1$ , starting from  $p_0 = 0.01$ , for a selection coefficient  $s = 0.01$  and three different dominance coefficients. The recessive allele will eventually fix in the population, but it takes a long time.

Here, the selection coefficient  $s$  is the difference in relative fitness between the two homozygotes, and  $h$  is the dominance coefficient. For selection to be directional, we require that  $0 \leq h \leq 1$  holds. The dominance coefficient allows us to move between two extremes. One is when  $h = 0$ , such that allele  $A_1$  is fully dominant and  $A_2$  fully recessive. In this case, the heterozygote  $A_1A_2$  is as fit as the  $A_1A_1$  homozygote genotype. The inverse holds when  $h = 1$ , such that allele  $A_1$  is fully recessive and  $A_2$  fully dominant.

Yellow monkey flowers (*Mimulus guttatus*) have repeatedly adapted to the toxic soils found at copper mines throughout the Californian foothills in the past 150 years. Kevin Wright

We can then rewrite eqn. (6.28) as

$$\Delta p_t = \frac{p_t h s + q_t s(1-h)}{\bar{w}} p_t q_t, \quad (6.30)$$

where

$$\bar{w}_t = 1 - 2p_t q_t s h - q_t^2 s. \quad (6.31)$$

**Question 5.** Comparing the red ( $h = 0$ ) and black ( $h = 0.5$ ) trajectories in Figure 6.2, provide an explanation for why  $A_1$  increases faster initially if  $h = 0$ , but then approaches fixation more slowly compared to the case of  $h = 0.5$ .

As an example of how dominance affects the trajectory of a real polymorphism we'll consider an example from a colour polymorphism in red foxes (*Vulpes vulpes*). There are three colour morphs of red foxes: silver, cross, and red (see Figure 6.3), with this difference primarily controlled by a single polymorphism; with genotypes RR, Rr, rr respectively. The fur pelts of the silver morph fetched three times the price for hunters compared to cross (a smoky red) and red pelts, the latter two being seen as roughly equivalent in worth. Thus the desirability of the pelts acts as a recessive trait, with much stronger selection against the silver homozygotes. As a result of this price difference silver foxes were hunted more intensely and declined as a proportion of the population in Eastern Canada, see Figure 6.4, as documented by ELTON, from 16% to 5% from 1834 to 1937.

HALDANE reanalyzed these data and showed that they were consistent with recessive selection acting against silver morph alone. Note how the heterozygotes (cross) decline somewhat as a result of selection on the silver homozygotes, but overall the R allele is slow to respond to selection as it is 'hidden' from selection in the heterozygote state.

A special case is when  $h = 0.5$ . This case is the case of no dominance, as the interaction among alleles with respect to fitness is



Figure 6.3: Three colour morphs in red fox *V. vulpes*, cross, red, and silver foxes from left to right. From "The larger North American mammals" Nelson, E.W., Fuertes, L.A. 1916.

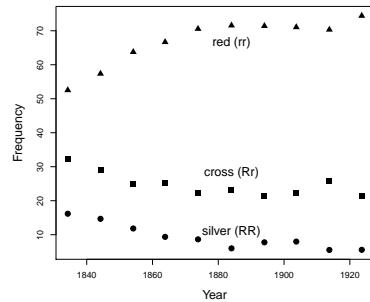


Figure 6.4: Data from ELTON, redrawn from ALLENDORF and HARD.

Add selection lines or get students to do that as an exercise.

strictly additive. Then, eqn. (6.30) simplifies to

$$\Delta p_t = \frac{1}{2} \frac{s}{\bar{w}} p_t q_t. \quad (6.32)$$

If selection is very weak, i.e.  $s \ll 1$ , the denominator ( $\bar{w}$ ) is close to 1 and we have

$$\Delta p_t = \frac{1}{2} s p_t q_t. \quad (6.33)$$

It is instructive to compare eqn. (6.33) to the respective expression under the haploid model. To this purpose, start from the generic term for  $\Delta p_t$  under the haploid model in eqn. (6.4) and set  $w_1 = 1$  and  $w_2 = 1 - s$ . Again, assume that  $s$  is small, so that eqn. (6.4) becomes  $\Delta p_t = s p_t q_t$ . Hence, if  $s$  is small, the diploid model of directional selection without dominance is identical to the haploid model, up to a factor of 1/2. That factor is due to the choice of the parametrisation; we could have set  $w_{11} = 1$ ,  $w_{12} = 1 - s$ , and  $w_{22} = 1 - 2s$  in diploid model instead, in which case the agreement with the haploid model would be perfect.

From this analogy, we can borrow some insight we gained for the haploid model. Specifically, the trajectory of the frequency of allele  $A_1$  in the diploid model without dominance follows a logistic growth curve similar to (6.10). Similarly, eqn. 6.13 for the haploid model suggests that in the diploid model without dominance it takes

$$\tau \approx \frac{4}{s} \log(2N) \quad (6.34)$$

generations for the favourable allele ( $A_1$ ) to transit from its entry into the population ( $p_0 = 1/(2N)$ ) to close to fixation ( $p_\tau = 1 - 1/(2N)$ ). Note again the difference by a factor of 2 due to the choice of parametrization. Also, the total number of alleles is  $2N$  in the diploid model, rather than  $N$ , which explains another factor of 2 in the argument of the logarithm. More generally we can use this correspondance of the trajectory additive diploid model to the haploid model to understand how quickly allele frequencies should change given a selection coefficient between arbitrary frequencies.

### 6.0.5 Heterozygote advantage

What if the heterozygotes are fitter than either of the homozygotes? In this case, it is useful to parameterize the relative fitnesses as follows:

genotype	$A_1 A_1$	$A_1 A_2$	$A_2 A_2$
absolute fitness	$w_{11}$	$< w_{12} >$	$w_{22}$
relative fitness (generic)	$w_{11} = W_{11}/W_{12}$	$w_{12} = W_{12}/W_{11}$	$w_{22} = W_{22}/W_{12}$
relative fitness (specific)	$1 - s_1$	1	$1 - s_2$

Here,  $s_1$  and  $s_2$  are the differences between the relative fitnesses of the two homozygotes and the heterozygote. Note that to obtain relative fitnesses we have divided absolute fitness by the heterozygote fitness. We could use the same parameterization as in the model of directional selection, but the reparameterization we have chosen here makes the math prettier.

In this case, when allele  $A_1$  is rare, it is often found in a heterozygous state and so it increases in frequency. However, when allele  $A_1$  is common, it is often found in the homozygote state, while the allele  $A_2$  is often found in the heterozygote state; it is now  $A_2$  that increases in frequency at the expense of allele 1. Thus, at least in the deterministic model, neither allele can reach fixation and both alleles will be maintained as a balanced polymorphism in the population at an equilibrium frequency.

We can solve for this equilibrium frequency by setting  $\Delta p_t = 0$  in eqn. (6.28), i.e.  $p_t q_t (\bar{w}_1 - \bar{w}_2) = 0$ . Doing so, we find that there are three equilibria, all of which are stable. Two of them are not very interesting ( $p = 0$  or  $q = 0$ ), but the third one is the polymorphic equilibrium, where  $\bar{w}_1 - \bar{w}_2$  holds. Using the parametrization above, we see that the marginal fitnesses of the two alleles are formally equivalent. Insertion of the selection coefficients  $s_1$  and  $s_2$  yields

$$p_e = \frac{s_2}{s_1 + s_2} \quad (6.35)$$

for the equilibrium frequency of interest. This is also the frequency of  $A_1$  at which the mean fitness of the population is maximised.

*Underdominance.* Another case that is of potential interest is the case of fitness underdominance, where the heterozygote is less fit than either of the homozygotes. This can be parametrized as follows:

genotype	$A_1 A_1$	$A_1 A_2$	$A_2 A_2$
absolute fitness	$w_{11}$	$w_{12}$	$w_{22}$
relative fitness (generic)	$w_{11} = W_{11}/W_{12}$	$w_{12} = W_{12}/W_{12}$	$w_{22} = W_{22}/W_{12}$
relative fitness (specific)	$1 + s_1$	1	$1 + s_2$

This case also permits three equilibria,  $p = 0$ ,  $p = 1$ , and a polymorphic equilibrium  $p = p_U$ . However, now only the first two equilibria are stable, while the polymorphic equilibrium is unstable. If  $p < p_U$  then  $\Delta p_t$  is negative and allele  $A_1$  will be lost, while if  $p > p_U$ , allele  $A_1$  will become fixed.

While such alleles might not spread within populations (if  $p_U \gg 0$  and selection is reasonably strong), they are of interest in the study of speciation and hybrid zones. That is because alleles  $A_1$  and  $A_2$  may have arisen in a stepwise fashion, i.e. not by a single mutation, in

separate subpopulations. Now, heterozygote disadvantage will play a potential role in species maintenance, if isolation of the subpopulations is not complete.

**Question 6.** You are studying the polymorphism that affects flight speed in butterflies. The polymorphism does not appear to affect fecundity. Homozygotes for the B allele are slow in flight and so only 40% of them survive to have offspring. Heterozygotes for the polymorphism (Bb) fly quickly and have a 70% probability of surviving to reproduce. The homozygotes for the alternative allele (bb) fly very quickly indeed, but often die of exhaustion, with only 10% of them making it to reproduction.

- A) What is the equilibrium frequency of the B allele?
- B) Calculate the marginal fitnesses of the B and the b allele at the equilibrium frequency.

*Diploid fluctuating fitness* We would like to think about the case where the diploid absolute fitnesses are time-dependent. The three genotypes then have fitnesses  $w_{11,t}$ ,  $w_{12,t}$ , and  $w_{22,t}$  in generation  $t$ . However, this case is much less tractable than the haploid case, as segregation makes it tricky to keep track of the genotype frequencies. We can make some progress and gain some intuition by thinking about how the frequency of allele  $A_1$  changes when it is rare.

Haldane and Jayakar (1963)

Selection pressures clearly fluctuate over time, how does this affect the dynamics of polymorphisms. Two examples of polymorphisms fluctuating in frequency in response to temporally-varying selection are shown in Figure 6.5; thanks to the short lifespan of *Drosophila* we can see seasonally-varying selection. The first is an inversion allele in *Drosophila pseudoobscura*, in *D. pseudoobscura* populations. Throughout western North America two orientations of the chromosome, two inversion alleles, exist: the Chiricahua and Standard alleles. WRIGHT and DOBZHANSKY investigated their frequency over four years at a number of locations, and found that their frequency fluctuated systematically over the seasons in response to selection. Our second example is an insertion-deletion polymorphism in the Insulin-like Receptor gene. PAABY *et al.* tracked the frequency of this allele over a number of seasons, and found it oscillated over the seasons. She and her coauthors also determined that these alleles had large effects on traits such as developmental time and fecundity, which could mediate the maintainance of this polymorphism through life-history trade-offs.

Add figure of het.

need earlier Dob. ref IX

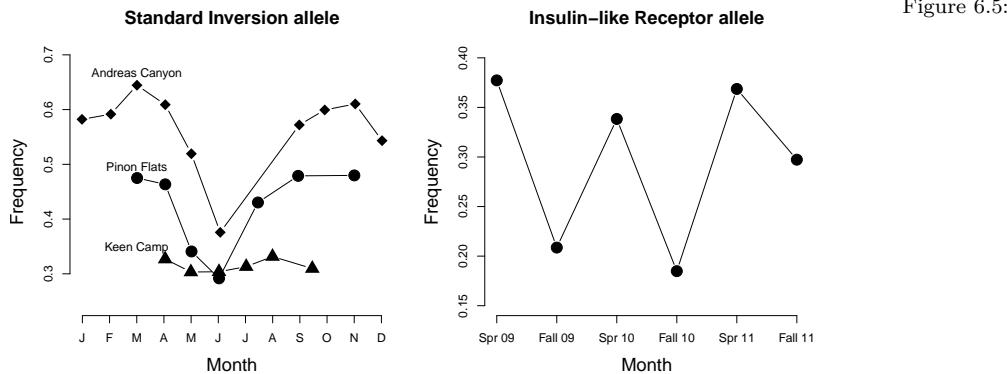


Figure 6.5:

When  $A_1$  is rare, i.e.  $p_t \ll 1$ , its frequency in the next generation (6.23) can be approximated as

$$p_{t+1} \approx \frac{w_{12}}{\bar{w}} p_t. \quad (6.36)$$

To obtain this, we have ignored the  $p_t^2$  term and assumed that  $q_t \approx 1$  in the numerator. Following a similar argument to approximate  $q_{t+1}$ , we can write

$$\frac{p_{t+1}}{q_{t+1}} = \frac{w_{12,t} p_t}{w_{22,t} q_t}. \quad (6.37)$$

Then, starting from out from  $p_0$  and  $q_0$  in generation 0,  $t + 1$  generations later, we have

$$\frac{p_{t+1}}{q_{t+1}} = \left( \prod_{i=0}^{t-1} \frac{w_{12,i}}{w_{22,i}} \right) \frac{p_0}{q_0}. \quad (6.38)$$

From this, we can see, following our haploid argument from above, that the frequency of allele  $A_1$  will increase when rare only if

$$\frac{\sqrt[t]{\prod_{i=0}^{t-1} w_{12,i}}}{\sqrt[t]{\prod_{i=0}^{t-1} w_{22,i}}} > 1, \quad (6.39)$$

i.e. if the heterozygote has higher geometric mean fitness than the  $A_2 A_2$  homozygote.

The question now is, whether allele  $A_1$  will approach fixation in the population, or whether there are cases in which we can obtain a balanced polymorphism. To investigate that, we can simply repeat our analysis for  $q \ll 1$ , and see that in that case

$$\frac{p_{t+1}}{q_{t+1}} = \left( \prod_{i=0}^{t-1} \frac{w_{11,i}}{w_{12,i}} \right) \frac{p_0}{q_0}. \quad (6.40)$$

Now, for allele  $A_1$  to carry on increasing in frequency and to approach fixation, the  $A_1 A_1$  genotype has to be out-competing the heterozygotes. For allele  $A_1$  to approach fixation, we need the geometric mean

of  $w_{11,i}$  to be greater than the geometric mean fitness of heterozygotes ( $w_{12,i}$ ). At the same time, if heterozygotes have higher geometric mean fitness than the  $A_1A_1$  homozygotes, then the  $A_2$  allele will increase in frequency when it is rare. Therefore, a balanced polymorphism can result when the heterozygote has higher geometric fitness than either of the homozygotes.

Intriguingly, we can have a balanced polymorphism even if the heterozygote is never the fittest genotype in any generation, if the heterozygotes has a higher geometric mean fitness than either of the homozygotes. In this case, the polymorphism will remain balanced in the population, despite the fact that the heterozygote is never the fittest genotype.

**Question 7.** Consider a plant population found in one of two different environments each generation. These occur randomly,  $1/3$  of time the population experiences the dry environment and with probability  $2/3$  it experiences the wet environment Y. The Absolute Fitnesses are as follows:

Environment	AA	Aa	aa
Wet.	6.25	5.0	3.75
Dry	3.85	5.0	6.15

**A)** Show whether the equilibrium frequency of A will be 0, 1, or in between.

**B)** If the probabilities of wet and dry environments were equal, i.e., 0.5 how would your conclusion change?

(HINT: Lets write  $w_{AA,dry}$  and  $w_{AA,wet}$  for the fitnesses of the AA homozygote in the two environments. Then if the two environments are equally common  $\prod_{i=0}^t w_{AA,i} \approx w_{AA,dry}^{t/2} w_{AA,wet}^{t/2}$  for large values of  $t$ .)

**Question 8.** OPTIONAL trickier question.

Imagine a randomly-mating population of hermaphrodites. In this population a derived allele (D) segregates that distorts transmission in its favour over the ancestral allele (d) in the production of all the gametes of heterozygotes. The drive leads to a fraction  $r$  of the gametes of heterozygotes (D/d) to carry the D allele ( $r > 0.5$ ). The D allele causes viability problems in the homozygote state such that the relative fitnesses are  $w_{dd} = 1$ ,  $w_{Dd} = 1$ ,  $w_{DD} = 1 - e$ . The allele is currently at frequency  $p$  in the population at birth. Assuming that the population is very large and no mutation occurs:

- A) What is the frequency of the D allele in the next generation, before selection has had a chance to act?
- B) What conditions do you need for a polymorphic equilibrium to be maintained? At what is the equilibrium frequency of this balanced polymorphism?
- C) Imagine the cost of the driver were additive  $w_{dd} = 1$ ,  $w_{Dd} = 1 - e$ ,  $w_{DD} = 1 - 2e$ . Under what conditions can the driver invade the population? Can a polymorphic equilibrium be maintained?

### 6.0.6 Mutation-selection balance

Mutation is constantly introducing new alleles into the population. Therefore, variation can be maintained within a population not only if selection is balancing (e.g. through heterozygote advantage or fluctuating selection over time, as we have seen in the previous section), but also due to a balance between mutation and selection<sup>2</sup>. A case of particular interest is when mutation introduces deleterious alleles and selection acts against these alleles. To study this balance, we return to the model of directional selection, where allele  $A_1$  is advantageous, i.e.

genotype	$A_1A_1$	$A_1A_2$	$A_2A_2$
absolute fitness	$W_{11}$	$\geq W_{12} \geq$	$W_{22}$
relative fitness	$w_{11} = 1$	$w_{12} = 1 - sh$	$w_{22} = 1 - s$ .

For a start, we consider the case where allele  $A_2$  is not completely recessive ( $h > 0$ ), so that the heterozygotes suffer at least some disadvantage. We denote by  $\mu = \mu_{1 \rightarrow 2}$  the mutation rate per generation from  $A_1$  to the deleterious allele  $A_2$ , and assume that there is no reverse mutation ( $\mu_{2 \rightarrow 1} = 0$ ). Let us assume that selection against  $A_2$  is relatively strong compared to the mutation rate, so that it is justified to assume that  $A_2$  is always rare, i.e.  $q_t = 1 - p_t \ll 1$ . Compared to

<sup>2</sup> Haldane, J. B. S., 1937 The Effect of Variation of Fitness. The American Naturalist 71(735): 337–349

previous sections, for mathematical clarity, we also switch from following the frequency  $p_t$  of  $A_1$  to following the frequency  $q_t$  of  $A_2$ . Of course, this is without loss of generality. The change in frequency of  $A_2$  due to selection can be written as

$$\Delta_S q_t = \frac{\bar{w}_2 - \bar{w}_1}{\bar{w}} p_t q_t \approx -hsq_t. \quad (6.41)$$

This approximation can be found by assuming that  $q^2 \approx 0$ ,  $p \approx 1$ , and that  $\bar{w} \approx w_1$ . All of these assumptions make sense if  $q \ll 1$ .

From eqn. (6.41) we see that selection acts to reduce the frequency of  $A_2$  (as both  $h$  and  $s$  are positive), and it does so geometrically across the generations. That is, if the initial frequency of  $A_2$  is  $q_0$ , then its frequency at time  $t$  is approximately

$$q_t = q_0(1 - hs)^t. \quad (6.42)$$

We will now consider the change in frequency induced by mutation. Recalling that  $\mu$  is the mutation rate from  $A_1$  to  $A_2$  per generation, the frequency of  $A_2$  after mutation

$$q' = \mu p_t + q_t = \mu(1 - q_t) + q_t. \quad (6.43)$$

Assuming that  $\mu \ll 1$  and that  $q \ll 1$ , the change in the frequency of allele  $A_2$  due to mutation ( $\Delta_M q_t$ ) can be approximated by

$$\Delta_M q_t = q' - q_t = \mu. \quad (6.44)$$

Hence, when  $A_2$  is rare and the mutation rate is low, mutation acts to linearly increase the frequency of the deleterious allele  $A_2$ .

If selection is to balance deleterious mutation, their combined effect over one generation has to be zero. Therefore, to find the mutation–selection equilibrium, we set

$$\Delta_M q_t + \Delta_S q_t = 0, \quad (6.45)$$

insert eqns. (6.41) and (6.44), and solve for  $q$  to obtain

$$q_e = q_t = \frac{\mu}{hs}. \quad (6.46)$$

We see that the frequency of the deleterious allele  $A_2$  is balanced at the mutation rate ( $\mu$ ) divided by the reduction in relative fitness in the heterozygote ( $hs$ ).

It is worth pointing out that the fitness of the  $A_2A_2$  homozygote has not entered this calculation, as  $A_2$  is so rare that it is hardly ever found in the homozygous state. Therefore, if  $A_2$  has any deleterious effect in a heterozygous state (i.e. if  $h > 0$ ) it is this effect that determines the frequency at which  $A_2$  is maintained in the population. Also, note that by writing the total change in allele frequency as

$\Delta_M q_t + \Delta_S q_t$  we have implicitly assumed that we can ignore terms of order  $\mu \times s$ . That is, we have assumed that there is no interaction between mutation and selection. We can do so as we assumed that both  $\mu$  and  $s$  are small.

If an allele was truly recessive (although few likely are), we have  $h = 0$ , and so eqn. (6.46) is not valid. However, we can make an argument similar to the one above to show that, for truly recessive alleles,

$$q_e = \sqrt{\frac{\mu}{s}}. \quad (6.47)$$

**Question 9.** Oblong-winged katydids (*Amblycorypha oblongifolia*) are usually green. However, some are bright pink, thanks to a erythrism mutation (a nice example of an early Mendelian reasoning in a wonderful titled paper<sup>3</sup>). This condition is thought to be due to a dominant mutation (Cre). Assume that roughly one in ten thousand is bright pink and that the mutation rate at the gene underlying this condition is  $10^{-5}$ . What is the relative fitness of heterozygotes for the pink mutation?

*The genetic load of deleterious alleles* What effect do such deleterious mutations at mutation-selection balance have on the population? It is common to express this effect in terms of a reduction of the mean relative fitness of the population. For a single site at which a deleterious mutation is segregating at  $q_e = \mu/(hs)$ , the mean relative fitness is reduced to

$$\bar{w} = 1 - 2p_e q_e h s - q_e^2 s \approx 1 - 2\mu. \quad (6.48)$$

Somewhat remarkably, the drop in mean fitness due to a site segregating at mutation-selection balance is independent of the selection coefficient against the heterozygote; it depends only the mutation rate. Note that this applies only if the mutation is not totally recessive, i.e. if  $h > 0$ .

A reduction of  $1 - 2\mu$  is very small, given that the mutation rate of a gene is likely  $< 10^{-5}$ . However, if there are many loci segregating at mutation-selection balance, this can accumulate to a substantial so-called genetic load, and a major cause of variation in fitness-related traits among individuals. For example, the human genome contains over twenty thousand genes, and many other functional regions, the vast majority of which will be subject to purifying selection against mutations that disrupt their function. In humans most loss of function (LOF) variants, which severely disrupt a protein-coding gene, are found at low frequencies. However, each human genome typically carries over hundred LOF variants (MACARTHUR *et al.*, 2012; LEK *et al.*, 2016). Not every LOF allele will be deleterious, some could be



Figure 6.6: Oblong-winged katydid.  
Field book of insects. Lutz, F.E. 1918.

<sup>3</sup> WHEELER, W. M., 1907 Pink Insect Mutants. The American Naturalist 41(492): 773–780

advantageous, but the combined load of these LOF must on average lower our fitness, otherwise selection wouldn't be removing them from the population. Each one of us carries a unique set of these LOF alleles.. We differ slightly in how many of these alleles we carry. For example Figure 6.7 shows the distribution of number of LOF alleles carried by 769 individuals of Dutch ancestry. The individuals who carry fewer of these LOF alleles will on average have higher fitness than those individuals with more. We don't know how much lower the fitness of these individuals is, nor how most of these alleles manifest their fitness consequences through disease and other mechanisms.

### 6.0.7 Inbreeding depression

All else being equal, eqn. (6.46) suggests that mutations that have a smaller effect in the heterozygote can segregate at higher frequency under mutation-selection balance. As a consequence, alleles that have strongly deleterious effects in the homozygous state can segregate at low frequencies in the population, as long as they do not have too strong a deleterious effect in heterozygotes. Thus, outbred populations may have many alleles with recessive deleterious effects segregating within them.

**Question 10.** Assume that a deleterious allele has a relative fitness .99 in heterozygotes and a relative fitness 0.2. Assuming that the deleterious allele is at a frequency  $10^{-3}$  at birth the genotype frequencies follow from HWE:

**A)** What is the average fitness of an individual in the population due to this allele?

**B)** What is the fitness of the child of a full-sib mating?

One consequence of this is that inbred individuals from usually outbred populations may have lower fitnesses than outbred individuals. In typically outbred populations, the mean fitness of individuals decreases with the inbreeding coefficient, i.e. this so-called inbreeding depression is a common observation. This is wide-spread observation, dating back to systematic surveys of inbreeding depression by DARWIN. Inbreeding depression is likely as primarily a consequence of being homozygous at many loci for alleles with recessive deleterious effects.

One example of inbreeding depression is shown in Figure 6.9. White campion (*Silene latifolia*) is a dioecious flowering plant, dioecious means that the males and females are separate individuals. RICHARDS performed crosses to create offspring who were outbred, the offspring of half-sibs, full-sibs, and of two generations of full-sib mating. He measured their germination success, which is plotted in Figure 6.9.

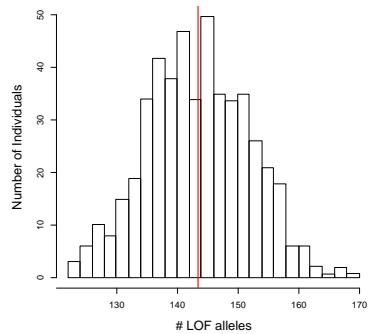


Figure 6.7: The distribution of LOF alleles in 769 individuals from the Genome of the Netherlands project. Redrawn from FRANCIOLI *et al.*. The average individual carries 144 LOF alleles.

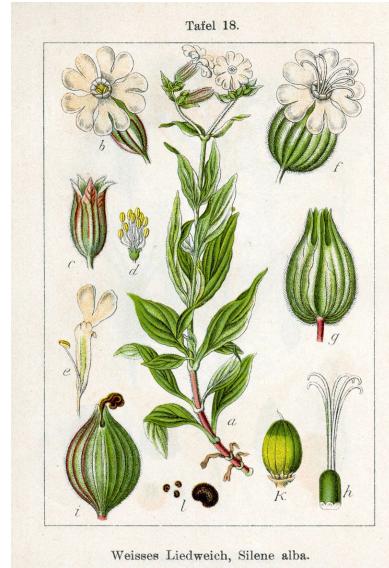


Figure 6.8: White campion (*S. latifolia*)

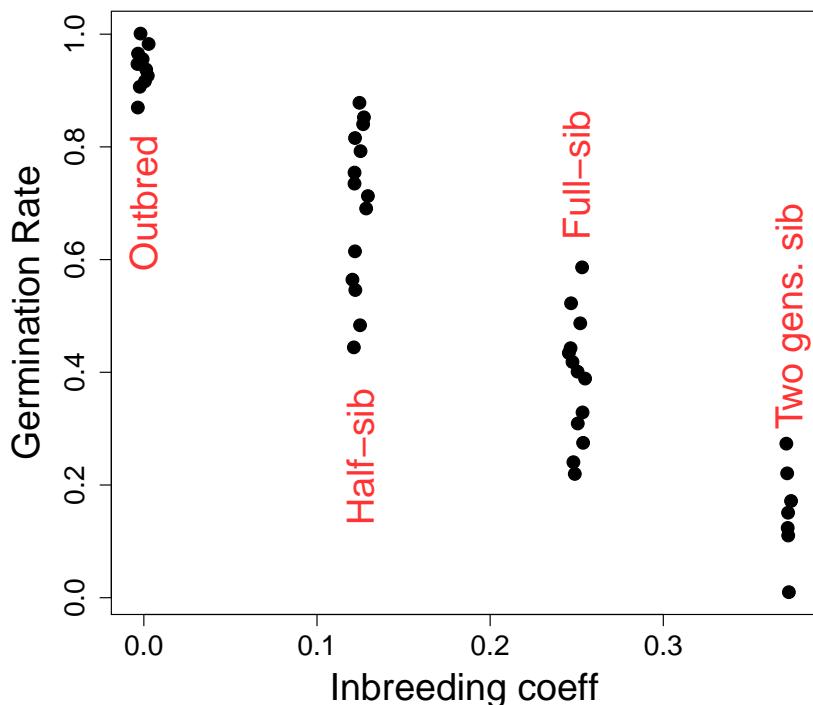


Figure 6.9: Data showing inbreeding depression over different degrees of inbreeding in *S. latifolia*. Each point is the mean seed germination rates for different family crosses. Redrawn from RICHARDS.

Add theory line?

*Purging the inbreeding load.* Populations that regularly inbreed over sustained periods of time are expected to partially purge this load of deleterious alleles. This is because such populations have exposed many of these alleles in a homozygous state, and so selection can more readily remove these alleles from the population.

If the population has sustained inbreeding, such that individuals in the population have an inbreeding coefficient  $F$ , deleterious alleles at each locus will find a new equilibrium frequency. Assuming the model of mutation-selection model, with this inbreeding is, the new equilibrium frequency is

$$q_e = \frac{\mu}{(h(1-f) + f)s} \quad (6.49)$$

Note how the frequency of the deleterious allele is decreased due to the allele now being expressed, and therefore exposed to selection, in homozygotes due to inbreeding.

### 6.0.8 Migration-selection balance

Another reason for the persistence of deleterious alleles in a population is that there is a constant influx of maladaptive alleles from other populations where these alleles are locally adapted. This seems unlikely to be as broad an explanation for the persistence of deleterious alleles genome-wide as mutation-selection balance. However, a brief discussion of such alleles is worthwhile as it helps to inform our ideas about local adaptation.

**Adaptation can occur over a range of geographic scales.**

Adaptation can potentially occur on much finer geographic scales from kilometers down to meters in some species. On such scales dispersal is surely rapidly moving alleles between environments, but the adaptation is maintained by the continued action of selection. An example of adaptation at fine-scales is shown in Figure 6.10.. [Jain and Bradshaw](#) studied the patterns of heavy-metal resistance in plants on mine tailings and in nearby meadows, a set of classic studies of

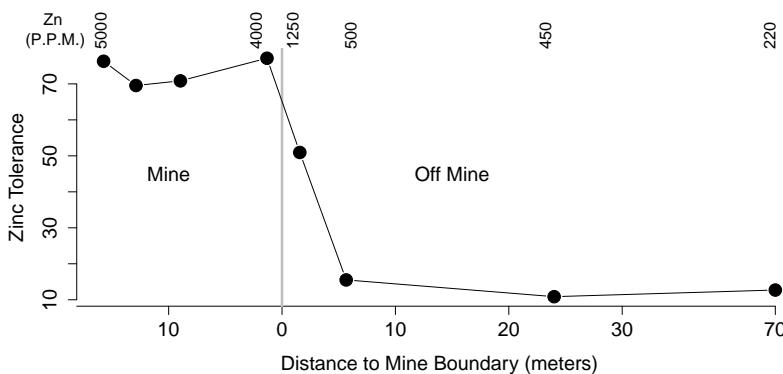


Figure 6.10: Data showing the Zinc tolerance of *Anthoxanthum odoratum* on and off of the Trelogan Mine, Flintshire, North Wales.

As a first pass at this lets consider a haploid two allele model with two different populations, where the relative fitnesses of our alleles are as follows

allele	1	2
population 1	1	1-s
population 2	1-s	1

As a simple model of migration lets suppose within a population a fraction of  $m$  individuals are migrants from the other population, and  $1 - m$  individuals are from the same deme.

To quickly sketch a solution to this, we'll set up a situation analogous to our mutation-selection balance model. To do this let's assume that selection is strong compared to migration ( $s \gg m$ ) then allele 1 will be almost fixed in population 1 and allele 2 will be almost fixed in

population 2. If that is the case, migration changes the frequency of allele 2 in population 1 ( $q_1$ ) by

$$\Delta_{Mig.} q_1 \approx m \quad (6.50)$$

while as noted above  $\Delta_S q_1 = -sq_1$ , so that migration and selection are at an equilibrium when  $0 = \Delta_S q_1 + \Delta_{Mig.} q_1$ , i.e. an equilibrium frequency of allele 2 in population 1 of

$$q_{e,1} = \frac{m}{s} \quad (6.51)$$

so that migration is playing the role of mutation and so migration-selection balance (at least under strong selection) is analogous to mutation selection balance.

We can use this same model by analogy for the case of migration-selection balance in a diploid model, in that case we replace our haploid  $s$  by the cost to heterozygotes  $hs$ .

**Question 11.** You are investigating a small river population of sticklebacks, which receives infrequent migrants from a very large marine population. At a set of (putatively) neutral biallelic markers the freshwater population has frequencies: 0.2, 0.7, 0.8

at the same markers the marine population has frequencies: 0.4, 0.5 and 0.7.

From studying patterns of heterozygosity at a large collection of markers, you have estimated the long term effective size of your freshwater population is 2000 individuals.

A) What is  $F_{ST}$  across these neutral markers in the freshwater population, with respect to the large marine population (i.e. treat the marine population as the total)?

B) You are also studying an unlinked locus involved in the regulation of salt uptake. In the marine population the ancestral allele is at close to fixation, but in your river population the derived allele is at 0.99 frequency. Estimate the selective disadvantage of the ancestral allele in your river population.  
[Hint how can you use neutral differentiation to estimate the migration rate?]

### 6.0.9 Some theory of the spatial distribution of allele frequencies under deterministic models of selection

Imagine a continuous haploid population spread out along a line. Individuals disperse a random distance  $\Delta x$  from its birthplace to the

location where it reproduces, where  $\Delta x$  is drawn from the probability density  $g(\cdot)$ . To make life simple we will assume that  $g(\Delta x)$  is normally distributed with mean zero and standard deviation  $\sigma$ , i.e. migration is unbiased an individuals migrate an average distance of  $\sigma$ .

Our frequency of allele 2 at time  $t$  in the population at spatial location  $x$  is  $q(x, t)$ . Assuming that only dispersal occurs, how does our allele frequency change in the next generation? Our allele frequency in the next generation at location  $x$  reflects the migration from different locations in the proceeding generation. Our population at location  $x$  receives a contribution  $g(\Delta x)q(x + \Delta x, t)$  of allele 2 from the population at location  $x + \Delta x$ , such that the frequency of our allele at  $x$  in the next generation is

$$q(x, t+1) = \int_{-\infty}^{\infty} g(\Delta x)q(x + \Delta x, t)d\Delta x. \quad (6.52)$$

To obtain  $q(x + \Delta x, t)$ , lets take a Taylor series expansion of  $q(x, t)$

$$q(x + \Delta x, t) = q(x, t) + \Delta x \frac{dq(x, t)}{dx} + \frac{1}{2}(\Delta x)^2 \frac{d^2q(x, t)}{dx^2} + \dots \quad (6.53)$$

then

$$q(x, t+1) = q(x, t) + \left( \int_{-\infty}^{\infty} \Delta x g(\Delta x) d\Delta x \right) \frac{dq(x, t)}{dx} + \frac{1}{2} \left( \int_{-\infty}^{\infty} (\Delta x)^2 g(\Delta x) d\Delta x \right) \frac{d^2q(x, t)}{dx^2} + \dots \quad (6.54)$$

$g(\cdot)$  has a mean of zero so  $\int_{-\infty}^{\infty} \Delta x g(\Delta x) d\Delta x = 0$  and has variance  $\sigma^2$  so  $\int_{-\infty}^{\infty} (\Delta x)^2 g(\Delta x) d\Delta x = \sigma^2$  and all higher terms are zero (as all high moments of the normal are zero). Looking at the change in frequency  $\Delta q(x, t) = q(x, t+1) - q(x, t)$  then

$$\Delta q(x, t) = \frac{\sigma^2}{2} \frac{d^2q(x, t)}{dx^2} \quad (6.55)$$

This is a diffusion equation, so that migration is acting to smooth out allele frequency differences with a diffusion constant of  $\frac{\sigma^2}{2}$ . This is exactly analogous to the equation describing how a gas diffuses out to equal density, as both particles in a gas and our individuals of type 2 are performing Brownian motion (blurring our eyes and seeing time as continuous).

We will now introduce fitness differences into our model and set the relative fitnesses of allele 1 and 2 at location  $x$  to be 1 and  $1 + s\gamma(x)$ . To make progress in this model we'll have to assume that selection isn't too strong i.e.  $s\gamma(x) \ll 1$  for all  $x$ . The change in frequency of allele 2 obtained within a generation due to selection is

$$q'(x, t) - q(x, t) \approx s\gamma(x)q(x, t)(1 - q(x, t)) \quad (6.56)$$

i.e. logistic growth of our favoured allele at location  $x$ . Putting our selection and migration terms together we find

$$q(x, t + 1) - q(x, t) = s\gamma(x)q(x, t)(1 - q(x, t)) + \frac{\sigma^2}{2} \frac{d^2q(x, t)}{dx^2} \quad (6.57)$$

in deriving this we have essentially assumed that migration acted upon our original frequencies before selection and in doing so have ignored terms of the order of  $\sigma s$ .

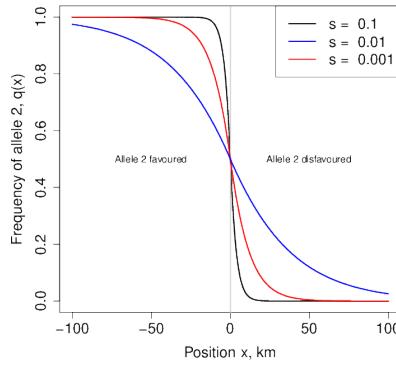


Figure 6.11: An equilibrium cline in allele frequency. Our individuals disperse an average distance of  $\sigma = 1$  km per generation, and our allele 2 has a relative fitness of  $1 + s$  and  $1 - s$  on either side of the environmental change at  $x = 0$ .

*The cline in allele frequency associated with a sharp environmental transition.* To make progress lets consider a simple model of location adaptation where the environment abruptly changes. Specifically we assume that  $\gamma(x) = 1$  for  $x < 0$  and  $\gamma(x) = -1$  for  $x \geq 0$ , i.e. our allele 2 has a selective advantage at locations to the left of zero, while this allele is at a disadvantage to the right of zero. In this case we can get an equilibrium distribution of our two alleles were to the left of zero our allele 2 is at higher frequency, while to the right of zero allele 1 predominates. As we cross from the left to the right side of our range the frequency of our allele 2 decreases in a smooth cline.

Our equilibrium spatial distribution of allele frequencies can be found by setting the LHS of eqn. (6.57) to zero to arrive at

$$s\gamma(x)q(x)(1 - q(x)) = -\frac{\sigma^2}{2} \frac{d^2q(x)}{dx^2} \quad (6.58)$$

We then could solve this differential equation with appropriate boundary conditions ( $q(-\infty) = 1$  and  $q(\infty) = 0$ ) to arrive at the appropriate functional form to our cline. While we won't go into the solution of this equation here, we can note that by dividing our distance  $x$  by  $\ell = \sigma/\sqrt{s}$  we can remove the effect of our parameters from the above equation. This compound parameter  $\ell$  is the characteristic length of

our cline, and it is this parameter which determines over what geographic scale we change from allele 2 predominating to allele 1 predominating as we move across our environmental shift.

The width of our cline, i.e. over what distance do we make this shift from allele 2 predominating to allele 1, can be defined in a number of different ways. One simple way to define the cline width, which is easy to define but perhaps hard to measure accurately, is the slope (i.e. the tangent) of  $q(x)$  at  $x = 0$ . Under this definition the cline width is approximately  $0.6\sigma/\sqrt{s}$ .

*The rate of spatial spread of a beneficial allele.* Consider a beneficial mutation that has arisen in a specific spatial location and has begun to spread geographically. **FINISH THIS.**

# 7

## *The Impact of Genetic Drift on Selected Alleles*

### *7.1 Stochastic loss of strongly selected alleles*

Even strongly selected alleles can be lost from the population when they are sufficiently rare. This is because the number of offspring left by individuals to the next generation is fundamentally stochastic. A selection coefficient of  $s=1\%$  is a strong selection coefficient, which can drive an allele through the population in a few hundred generations once the allele is established. However, if individuals have on average a small number of offspring per generation the first individual to carry our allele who has on average 1% more children could easily have zero offspring, leading to the loss of our allele before it ever gets a chance to spread.

To take a first stab at this problem let's think of a very large haploid population, and in order for this population to stay constant in size we'll assume that individuals without the selected mutation have on average one offspring per generation. While individuals with our selected allele have on average  $1 + s$  offspring per generation. We'll assume that the distribution of offspring number of an individual is Poisson distributed with this mean, i.e. the probability that an individual with the selected allele has  $i$  children is

$$P_i = \frac{(1+s)^i e^{-(1+s)}}{i!} \quad (7.1)$$

Consider starting from a single individual with the selected allele, and ask about the probability of eventual loss of our selected allele starting from this single copy ( $p_L$ ). To derive this we'll make use of a simple argument (derived from branching processes). Our selected allele will be eventually lost from the population if every individual with the allele fails to leave descendants.

1. In our first generation with probability  $P_0$  our individual leaves no copies of itself to the next generation, in which case our allele is lost (Figure 7.1A).

2. Alternatively it could leave one copy of itself to the next generation (with probability  $P_1$ ), in which case with probability  $p_L$  this copy eventually goes extinct (Figure 7.1B).
3. It could leave two copies of itself to the next generation (with probability  $P_2$ ), in which case with probability  $p_L^2$  both of these copies eventually goes extinct (Figure 7.1C).
4. More generally it could leave  $k$  copies ( $k > 0$ ) of itself to the next generation (with probability  $P_k$ ), in which case with probability  $p_L^k$  all of these copies eventually go extinct (e.g. Figure 7.1D).

summing over these probabilities we see that

$$\begin{aligned} p_L &= \sum_{k=0}^{\infty} P_k p_L^k \\ &= \sum_{k=0}^{\infty} \frac{(1+s)^k e^{-(1+s)}}{k!} p_L^k \\ &= e^{-(1+s)} \left( \sum_{k=0}^{\infty} \frac{(p_L(1+s))^k}{k!} \right) \end{aligned} \quad (7.2)$$

well the term in the brackets is itself an exponential expansion, so we can rewrite this as

$$p_L = e^{(1+s)(p_L - 1)} \quad (7.3)$$

solving this would give us our probability of loss for any selection coefficient. Lets rewrite this in terms of the the probability of escaping loss  $p_F = 1 - p_L$ . We can rewrite eqn (7.3) as

$$1 - p_F = e^{-p_F(1+s)} \quad (7.4)$$

to gain an approximation to this lets consider a small selection coefficient  $s \ll 1$  such that  $p_F \ll 1$  and then expanded out the exponential on the right hand side (ignoring terms of higher order than  $s^2$  and  $p_F^2$ ) then

$$1 - p_F \approx 1 - p_F(1 + s) + p_F^2(1 + s)^2/2 \quad (7.5)$$

solving this we find that

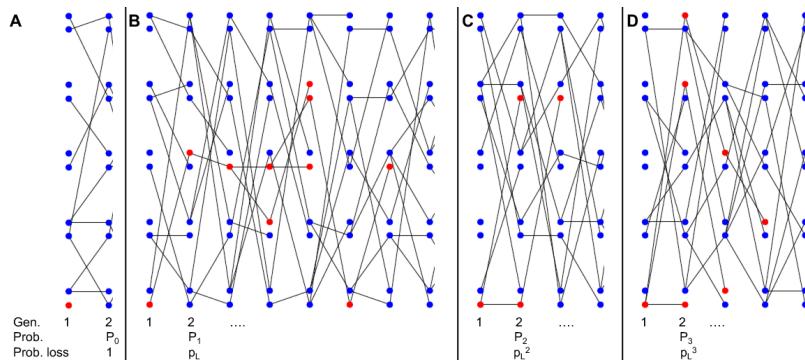
$$p_F = 2s. \quad (7.6)$$

Thus even an allele with a 1% selection coefficient has a 98% probability of being lost when it is first introduced into the population by mutation.

We can also adapt this result to a diploid setting. Assuming that heterozygotes for the 1 allele have  $1 + (1 - h)s$  children, the probability of allele 1 is not lost, starting from a single copy in the population, is

$$p_F = 2(1 - h)s \quad (7.7)$$

for  $h > 0$ . Note this is a slightly different parameterization from our diploid model above.



**Question 1.** Melanic squirrels suffer a higher rate of predation (due to hawks) than normally pigmented squirrels. Melanism is due to a dominant, autosomal mutation. The frequency of melanic squirrels at birth is  $4 \times 10^{-5}$ .

**A)** If the mutation rate to new melanic alleles is  $10^{-6}$ , assuming the melanic allele is at mutation-selection equilibrium, what is the reduction in fitness of the heterozygote?

Suddenly levels of pollution increase dramatically in our population, and predation by hawks now offers an equal (and opposite) advantage to the dark individuals as it once offered to the normally pigmented individuals.

**B)** What is the probability that a single copy of this allele (present just once in the population) is lost?

**C)** If the population size of our squirrels is a million individuals, and is at mutation selection-balance, what is the probability that the population adapts from anyone of these standing pool of melanic alleles?

## 7.2 The interaction between genetic drift and weak selection.

For strongly selected alleles, once the allele has escaped initial loss at low frequencies, their path will be determined deterministically by their selection coefficients. However, if selection is weak the stochasticity of reproduction can play a role in the trajectory an allele takes even when it is common in the population.

To see this lets think of our simple Wright-Fisher model (see R exercise). Each generation we allow a deterministic change in our allele frequency, and then binomially sample two alleles for each of our offspring to construct our next generation.

So the expected change in our allele frequency within a generation

is given just by our deterministic formula. To make things easy on our self lets assume an additive model, i.e.  $h = 1/2$ , and that  $s \ll 1$  so that  $\bar{w} \approx 1$ . This gives us

$$\mathbb{E}(\Delta p) = \frac{s}{2}p(1-p) \quad (7.8)$$

our variance in our allele frequency change is given by

$$Var(p' - p) = Var(p') = \frac{p'(1-p')}{2N} \quad (7.9)$$

this variance in our allele frequency follows from the fact that we are binomially sampling  $2N$  new alleles in the next generation from a frequency  $p'$ . Denoting our count of allele 1 by  $i$  our

$$Var(p' - p) = Var\left(\frac{i}{2N} - p\right) = Var\left(\frac{i}{2N}\right) = \frac{Var(i)}{(2N)^2} \quad (7.10)$$

and from binomial sampling  $Var(i) = 2Np'(1-p')$  and so we arrive at our answer. Assuming that  $s \ll 1$ ,  $p' \approx p$ , then in practice we can use

$$Var(\Delta p) = Var(p' - p) \approx \frac{p(1-p)}{2N}. \quad (7.11)$$

To get our first look at the relative effects of selection vs drift we can simply look at when our change in allele frequency caused selection within a generate is reasonably faithfully passed across the generations. In particular if our expected change in frequency is much great than the variance around this change, genetic drift will play little role in the fate of our selected allele (once the allele is not too rare within the population). When does selected dominant genetic drift? This will happen if  $\mathbb{E}(\Delta p) \gg Var(\Delta p)$  when  $Ns \gg 1$ . Conversely any hope of our selected allele following its deterministic path will be quickly undone if our change in allele frequencies due to selection is much less than the variance induced by drift. So if  $Ns \ll 1$  then drift will dominate the fate of our allele.

To make further progress on understanding the fate of alleles with selection coefficients of the order  $1/N$  requires more careful modeling. However, we can obtain the probability that under our diploid model, with an additive selection coefficient  $s$ , the probability of allele 1 fixing within the population starting from a frequency  $p$  is given by

$$\pi(p) = \frac{1 - e^{-2Ns}}{1 - e^{-2Ns}} \quad (7.12)$$

the proof of this is sketched out below (see Section 7.2.2). A new allele will arrive in the population at frequency  $p = 1/(2N)$ , then its probability of reaching fixation is

$$\pi\left(\frac{1}{2N}\right) = \frac{1 - e^{-s}}{1 - e^{-2Ns}} \quad (7.13)$$

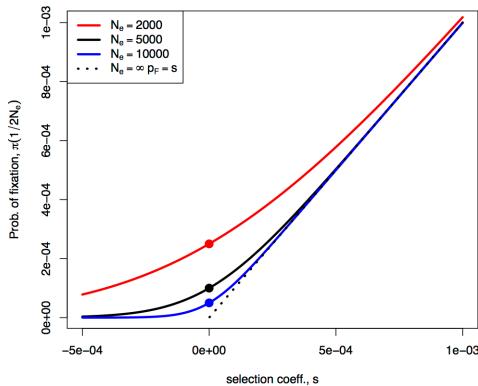


Figure 7.1: The probability of the fixation of a new mutation with selection coefficient  $s$  ( $h = 1/2$ ) in a diploid population of effective size  $N_e$ . The dashed line gives the infinite population solution. The dots give the solution for  $s \rightarrow 0$ , i.e.  $1/(2N_e)$

if  $s \ll 1$  but  $Ns \gg 1$  then  $\pi(\frac{1}{2N}) \approx s$ , which nicely gives us back our result that we obtained above (eqn. (7.7)). Our probability of fixation (eqn. (7.13)) is plotted as a function of  $s$  and  $N$  in Figure 7.1. To recover our neutral result we can take the limit  $s \rightarrow 0$  to obtain our neutral fixation probability  $1/(2N)$ .

In the case where  $Ns$  close to 1 then

$$\pi\left(\frac{1}{2N}\right) \approx \frac{s}{1 - e^{-2Ns}} \quad (7.14)$$

this is greater than our result  $p_F = s$  from the branching process argument (using our additive model of  $h = 1/2$ ), increasingly so for smaller  $N$ . Why is this? The reason why is that  $p_F$  is really the probability of “never being lost” in an infinitely large population. So to persist indefinitely the allele has to escape loss permanently, by never being absorbed by the zero state. When the population size is finite, to fix we only need to reach a size  $2N$  individuals. Weakly beneficial mutations ( $Ns > 1$ ) are slightly more likely to fix than the  $s$  probability, as they only have to reach  $2N$  to never be lost.

### 7.2.1 The fixation of slightly deleterious alleles.

From Figure 7.1 we can see that weakly deleterious alleles can fix, especially in small populations. To understand how likely it is that deleterious alleles accidentally reach fixation by genetic drift, let's assume a diploid model with additive selection (with a selection coefficient of  $-s$  against our allele 2). If  $Ns \gg 1$  then our deleterious allele (allele 2) can not possibly reach fixation. However, if  $Ns$  is not large then

$$\pi\left(\frac{1}{2N}\right) \approx \frac{s}{e^{2Ns} - 1} \quad (7.15)$$

for our deleterious allele. So deleterious alleles can fix within populations (albeit at a low rate) if  $N_s$  is not too large. As above this is because while deleterious mutations will never escape loss in infinite population, but they can become fixed in finite population by reach  $2N$  copies. This is captured by the denominator of the fixation probability under the diffusion model, which that this increases the fixation prob. of alleles with  $|N_s| \geq 1$ . The absorption of alleles at  $2N$  copies can also be modeled in finite individual models (i.e. not the diffusion limit), but we will not go into that here.

**Question 2.** ‘Haldane’s sieve’ is the name for the idea that the mutations that contribute to adaptation are likely to be dominant or at least co-dominant.

- A) Briefly explain this argument with a verbal model relating to the results we’ve developed in the last two chapters.
- B) Haldane’s sieve is thought to be less important for adaptation from previously deleterious standing variation, than adaptation from new mutation. Can you explain the intuition behind of this idea?
- C) Haldane’s sieve is likely to be less important in inbred, e.g. selfing, populations. Why is this?



### 7.2.2 A Sketch Proof of the probability of fixation of weakly selected alleles

We’ll let  $P(\Delta p)$  be the probability that our allele frequency shifts by  $\Delta p$  in the next generation. Using this we can write our probability  $\pi(p)$  in terms of the probability of achieving fixation averaged over the frequency in the next generation

$$\pi(p) = \int \pi(p + \Delta p)P(\Delta p)d(\Delta p) \quad (7.16)$$

This is very similar to the technique that we used deriving our probability of escaping loss in a very large population above.

So we need an expression for  $\pi(p + \Delta p)$ . To obtain this we’ll do a Taylor series expansion of  $\pi(p)$  assuming that  $\Delta p$  is small

$$\pi(p + \Delta p) \approx \pi(p) + \Delta p \frac{d\pi(p)}{dp} + (\Delta p)^2 \frac{d^2\pi(p)}{dp^2}(p) \quad (7.17)$$

ignoring higher order terms.

Taking the expectation over  $\Delta p$  on both sides, as in eqn. 7.16, we obtain

$$\pi(p) = \pi(p) + \mathbb{E}(\Delta p) \frac{d\pi(p)}{dp} + \mathbb{E}((\Delta p)^2) \frac{d^2\pi(p)}{dp^2} \quad (7.18)$$

Well  $\mathbb{E}(\Delta p) = \frac{s}{2}p(1-p)$  and  $Var(\Delta p) = \mathbb{E}((\Delta p)^2) - \mathbb{E}^2(\Delta p)$ , so if  $s \ll 1$  then  $\mathbb{E}^2(\Delta p) \approx 0$ , and  $\mathbb{E}(\Delta p)^2 = \frac{p(1-p)}{2N}$ . This leaves us with

$$0 = \frac{s}{2}p(1-p) \frac{d\pi(p)}{dp} + \frac{p(1-p)}{2N} \frac{d^2\pi(p)}{dp^2} \quad (7.19)$$

and we can specify the boundary conditions to be  $\pi(1) = 1$  and  $\pi(0) = 0$ . Solving this differential equation is somewhat involved process but in doing so we find that

$$\pi(p) = \frac{1 - e^{-2Ns}}{1 - e^{-2Ns}} \quad (7.20)$$

This proof can be extended to alleles with arbitrary dominance, however, this does not lead to a analytically tractable expression so we do not pursue this here.



## *The effect of linked selection on patterns of neutral diversity*

A newly derived allele with an additive selection coefficient  $s$  will take a time  $\tau \approx 4 \log(2N)/s$  generations to reach fixation within our population (see eqn. (6.34)). This short time window offers very little time for recombination between the selected site and linked neutral sites.

First lets imagine examining variation at a locus fully linked to our selected locus, just after our sweep reached fixation. A pair of neutral alleles sampled at this locus must both trace their ancestral lineages back through to the neutral allele on whose background the selected allele initially arose. As that neutral allele, which existed  $\tau$  generations ago is the ancestor of the entire population at this locus. Our individuals who carry the beneficial allele are, from the perspective of these two alleles, exactly like a rapidly expanding population. Therefore, our pair of neutral alleles sampled at our locus will be forced to coalesce  $\approx \tau$  generations ago. This is a very short-time scale compared to the average neutral coalescent tie of  $2N$  generations of a pair of alleles.

If we now allow recombination into our model we can think about a pair of alleles sampled at a neutral locus a recombination distance  $r$  away from our selected site. Our pair of alleles will be forced to coalesce  $\approx \tau$  generations if neither of them reside on haplotypes that the selected allele recombined onto during the sweep. This is equivalent to saying that neither of our neutral alleles recombine off of the beneficial allele's background moving backward in time.

The probability that our lineage fail recombines off our beneficial allele's background and onto the ancestral background in the  $j^{th}$  generation back is

$$r(1 - X(j)) \tag{8.1}$$

so the probability ( $p_{NR}$ ) that our lineage fails to recombine off in the  $\tau$  generations it takes our selected allele to move through the

population is

$$p_{NR} = \prod_{j=1}^{\tau} (1 - r(1 - X(j))) \quad (8.2)$$

assuming that  $r$  is small then  $(1 - r(1 - X(j))) \approx e^{-r(1-X(j))}$ , such that

$$p_{NR} = \prod_{j=1}^{\tau} (1 - r(1 - X(j))) \approx \exp \left( -r \sum_{j=1}^{\tau} 1 - X(j) \right) = \exp \left( -r\tau(1 - \hat{X}) \right) \quad (8.3)$$

where  $\hat{X}$  is the average frequency of the derived allele across the trajectory  $\hat{X} = \frac{1}{\tau} \sum_{j=1}^{\tau} X(j)$ . As our allele is additive its trajectory for frequencies  $< 0.5$  is the mirror image of its trajectory for frequency  $> 0.5$ , therefore its average frequency  $\hat{X} = 0.5$ . So

$$p_{NR} = e^{-r\tau/2}. \quad (8.4)$$

The probability that both of our lineages fail to recombine off the sweep and hence are forced to coalesce is  $p_{NR}^2$ , assuming that they coalesce at a time close to  $\tau$  so that they recombine independently of each other for times  $< \tau$ .

If one or other of our lineages recombine off the sweep it will take them on average  $\approx 2N$  generations to find a common ancestor as we are back our neutral coalescent. Thus the expected time till our pair of lineages find a common ancestor is

$$\mathbb{E}(T_2) = \tau \times p_{NR}^2 + (1 - p_{NR}^2)(\tau + 2N) \approx (1 - p_{NR}^2) 2N \quad (8.5)$$

where this last approximation assumes that  $\tau \ll 2N$ . So the expected pairwise diversity for neutral alleles at a recombination distance  $r$  away from the selected sweep ( $\pi_r$ ) is

$$\mathbb{E}(\pi_r) = 2\mu\mathbb{E}(T_2) \approx \theta (1 - e^{-r\tau}) \quad (8.6)$$

so diversity increases as we move away from the selected site, slowly exponentially plateauing to its neutral expectation  $\theta = 4N\mu$ .

To get a sense of the physical scale over which diversity is reduced consider a region where recombination occurs at a rate  $r_{BP}$  per base pair per generation, and our locus is  $\ell$  base pairs away from the selected site  $r = r_{BP}\ell$  (where  $r_{BP}\ell \ll 1$  so we don't need to worry about more than one recombination event occurring per generation). Typical recombination rates are on the order of  $r_{BP} = 10^{-8}$ , in Figure 8.1 we show the reduction in diversity, given by eqn. (8.6), for two different selection coefficients.

For our expected diversity levels to recover to 50% of its neutral expectation  $\mathbb{E}(\pi_r)/\theta = 0.5$ , requires a physical distance  $\ell^*$  such that  $\log(0.5) = -r_{BP}\ell^*\tau$  as using our expression for  $\tau$  then  $\ell^* = \frac{-\log(0.5)}{r_{BP}\tau}$ .

As  $\tau$  depends inversely on the selection  $s$  (eqn. (6.34)), the width of our trough of reduced diversity depends on  $s/r_{BP}$ . All else being equal we expect stronger sweeps or sweeps in regions of low recombination to have a larger hitchhiking effect. So that a selection coefficient of  $s = 0.1\%$  would reduce diversity over 10's of kb, while a sweep of  $s = 1\%$  would affect  $\sim 100$ kb.

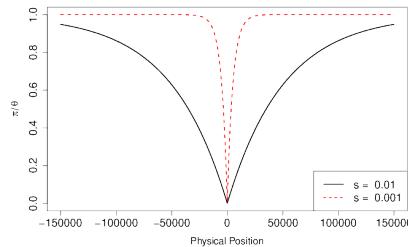


Figure 8.1: Reduction in diversity compared to its neutral expectation as a function of the distance away from a site where a selected allele has just gone to fixation. The recombination rate is  $r_{BP} = 1 \times 10^{-8}$ .

**Question 1.** A recently published study has identified the genetic basis of melanism in the pepper moth. This allele swept to fixation in northern parts of the UK; a classic case of adaptation to industrial pollution (made famous by the work of Kettlewell). The genetic basis of melanism is a transposable element (TE) inserted into a pigmentation gene. The investigators found that diversity is suppressed in a broad region around the TE. Specifically, on the background of the TE, it takes roughly 200 kb in either direction for diversity levels to recover to 50% of genome-wide levels.

Random facts: In all moths and butterflies only males recombine; chromosomes are transmitted without recombination in female. The recombination rate in males is 2.9 cM/Mb. Peppered moths have an effective population size of roughly a hundred thousand individuals. Kettlewell used to eat moths when out collecting them in the field (personal communication, Art. Shapiro).

**A)** Briefly explain how this pattern offers further evidence that the melanic allele was favoured by selection.

**B)** Using this information what is your estimate as to the age of the allele?

**C)** What is your estimate of the selection coefficient favouring this melanic?

### 8.0.1 A simple recurrent model of selective sweeps

We sample a pair of neutral alleles at a locus a genetic distance  $r$  away from a locus where sweeps are initiated within the population at some very low rate  $\nu$  per generation. The waiting time between

sweeps at our locus is exponential  $\sim \text{Exp}(\nu)$ . Each sweep rapidly transits through the population in  $\tau$  generations, such that each sweep is finished long before the next sweep ( $\tau \ll 1/\nu$ ).

As before our chance that our neutral lineage fails to recombine off the sweep is  $p_{NR}$ , such that the probability that our pair of lineages are forced to coalesce by a sweep  $e^{-r\tau}$ . Our lineages therefore have a very low probability

$$\nu e^{-r\tau} \quad (8.7)$$

of being forced to coalesce by a sweep per generation. In addition of lineages can coalesce at a neutral rate of  $1/(2N)$ . Thus the average waiting time till a coalescent event between our neutral pair of lineages due to either a sweep or a neutral coalescent event is

$$\mathbb{E}(T_2) = \frac{1}{\nu e^{-r\tau} + 1/(2N)} \quad (8.8)$$

Now imagine that the sweeps don't occur at a fixed location with respect to our locus of interest, but now occur uniformly at random across our sequence. The sweeps are initiated at a very low rate of  $\nu_{BP}$  per basepair per generation. The rate of coalescent due to sweeps at a locus  $\ell$  basepairs away from our neutral loci is  $\nu_{BP}e^{-r_{BP}\ell\tau}$ . If our neutral locus is in the middle of a chromosome that stretches  $L$  basepairs in either direction the total rate of sweeps per generation that force our pair of lineages to coalesce is

$$2 \int_0^L \nu_{BP} e^{-r_{BP}\ell\tau} d\ell = \frac{2\nu_{BP}}{r_{BP}\tau} (1 - e^{-r_{BP}\tau L}) \quad (8.9)$$

so that if  $L$  is very large ( $r_{BP}\tau L \gg 1$ ) the rate of coalesce per generation due to sweeps is  $\frac{2\nu_{BP}}{r_{BP}\tau}$ . The total rate of coalescence for a pair of lineages per generation is then

$$\frac{2\nu_{BP}}{r_{BP}\tau} + \frac{1}{2N} \quad (8.10)$$

So our average time till a pair of lineages coalesce is

$$\mathbb{E}(T_2) = \frac{1}{\frac{2\nu_{BP}}{r_{BP}\tau} + \frac{1}{2N}} = \frac{r_{BP}2N}{\frac{4N\nu_{BP}}{\tau} + r_{BP}} \quad (8.11)$$

such that our expected pairwise diversity ( $\pi = 2\mu\mathbb{E}(T_2)$ ) in a region of recombination rate  $r_{BP}$  that experiences sweeps at rate  $\nu_{BP}$  is

$$\mathbb{E}(\pi) = \theta \frac{r_{BP}}{\frac{4N\nu_{BP}}{\tau} + r_{BP}} \quad (8.12)$$

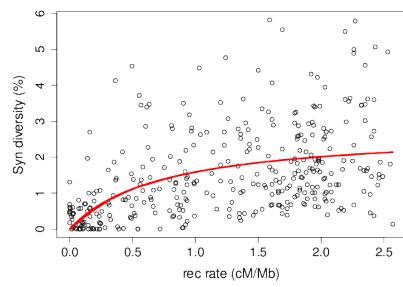


Figure 8.2: The relationship between (sex-averaged) recombination rate and synonymous site pairwise diversity ( $\pi$ ) in *Drosophila melanogaster* using the data of Shapiro et al. 07 (kindly provided by Peter Andolfatto, see Sella et al. 09 for details). The curve is the predicted relationship between  $\pi$  and recombination rate obtained by fitting equation (8.12) to this data using non-linear least squares via the `nls()` function in R.



# 9

## *Interaction of multiple selected loci.*

Consider two biallelic loci segregating for  $A/a$  and  $B/b$ . There are four haplotypes  $AB$ ,  $Ab$ ,  $aB$ ,  $ab$ , which for simplicity we label 1-4. The frequency of our four haplotypes are  $x_1$ ,  $x_2$ ,  $x_3$ , and  $x_4$ . Each individual has a genotype consisting of two haplotypes, we label  $w_{ij}$  the fitness of an individual with the genotype made up of haplotype  $i$  and  $j$  (we assume that  $w_{ij} = w_{ji}$ , i.e. there are no parent of origin effects). Assuming that these fitnesses reflect differences due to viability selection, and that individuals mate at random, we can write the following table of our genotype proportions after selection:

	$AB$	$Ab$	$aB$	$ab$
$AB$	$w_{11}x_1^2$	$w_{12}2x_1x_2$	$w_{13}2x_1x_3$	$w_{14}2x_1x_4$
$Ab$	•	$w_{22}x_2^2$	$w_{23}2x_2x_3$	$w_{24}2x_2x_4$
$aB$	•	•	$w_{33}x_3^2$	$w_{34}2x_3x_4$
$ab$	•	•	•	$w_{44}x_4^2$

This follows from assume that our haplotypes are brought together at random (HWE) then discounted by their fitnesses. Our mean fitness  $\bar{w}$  is the sum of all the entries in the table, which normalized the complete table to sum to one. The frequency of  $AB$  haplotype (1) in the next generation of gametes is

$$x'_1 = \frac{(w_{11}x_1^2 + \frac{1}{2}w_{12}2x_1x_2 + \frac{1}{2}w_{13}2x_1x_3 + \frac{1}{2}(1-r)w_{14}2x_1x_4 + \frac{1}{2}rw_{23}2x_2x_3)}{\bar{w}} \quad (9.1)$$

Here each of the HWE genotype frequencies (e.g.  $2x_1x_2$ ) is weighted by their fitness relative to the mean fitness ( $w_{ij}/\bar{w}$ ), and by their probability of transmitting the  $AB$  haplotype to the next generation, for example  $AB/Ab$  individuals (1/2) transmit the  $AB$  haplotype only half the time. The final two terms include the recombination fraction ( $r$ ). The first term involving recombination refers to the  $AB/ab$  genotype (1/4), who with probability  $(1-r)/2$  transmit a non-recombinant  $AB$  haplotype to the gamete. Similarly the second term refers to a

*Ab/aB* genotype, a proportion  $r/2$  of their gametes carry the recombinant *AB* haplotype.

In the single locus case we defined the marginal fitness of an allele, here it will help us to define the marginal fitness of the  $i^{th}$  haplotype

$$\bar{w}_i = \sum_{j=1}^4 w_{ij} x_j \quad (9.2)$$

this is the fitness of the  $i^{th}$  haplotype averaged over all of the genotypes it could occur in, weighted by their probability under random mating. Using this notation, and with some rearrangement of equation (9.1), we obtain

$$x'_1 = \frac{x_1 \bar{w}_1 - w_{14} r D}{\bar{w}} \quad (9.3)$$

here we have assumed that  $w_{23} = w_{14}$ , i.e. that the fitness of *AB/ab* individuals is the same as *Ab/aB* individuals (i.e. that fitness depends only on the alleles carried by an individual, and not on which chromosome they are carried, this assumption is sometimes called no *cis*-epistasis).

We can then write the change in the frequency of our 1 haplotype as

$$\Delta x_1 = \frac{x_1 (\bar{w}_1 - \bar{w}) - r w_{14} D}{\bar{w}} \quad (9.4)$$

Generalizing from this we write the change in our set of four haplotypes as

$$\Delta x_i = \frac{x_1 (\bar{w}_i - \bar{w}) \pm r w_{14} D}{\bar{w}} \quad (9.5)$$

where the coupling haplotypes 1 and 4 use  $+D$  and repulsion haplotypes 2 and 3 use  $-D$ . Note that the sum of these four  $\Delta x_i$  is zero, as our allele frequencies sum to one.

So the change in the frequency of a haplotype (eg *AB*, haplotype 1) is determined by the interplay of two factors. First the extent to which the marginal fitness of our haplotype is higher (or lower) than the mean fitness of the population (the magnitude and sign of  $(\bar{w}_1 - \bar{w})/\bar{w}$ ). Second whether there is a deficit or any excess of our haplotype compared to linkage equilibrium (the magnitude and sign of  $D$ ) modified by the strength of recombination.

### 9.0.1 Types of interaction between selection and recombination

*Interference.* HIV uses its reverse transcriptase (RT) gene to write itself from an RNA virus into its host's DNA, allowing it to hijack the host's regulatory machinery, a critical part of its life cycle. Efavirenz is an anti-HIV drug, which inhibits HIV's RT protein. Sadly mutations are common in the RT HIV gene, and in the presence of the drug have a profound fitness advantage allowing them to spread.

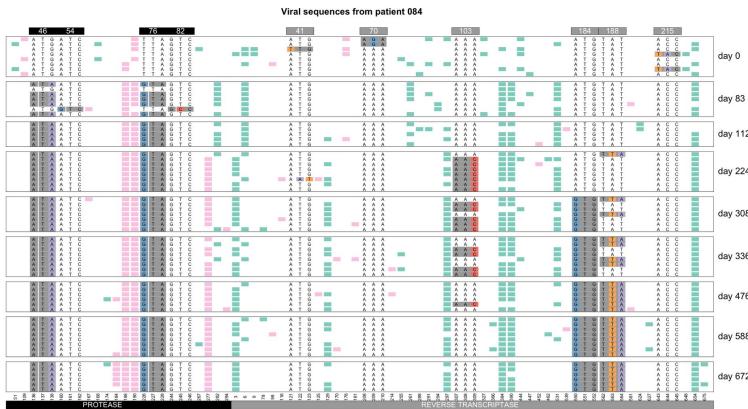


Figure 9.1: Haplotype Figure thanks to Pleuni Pennings.

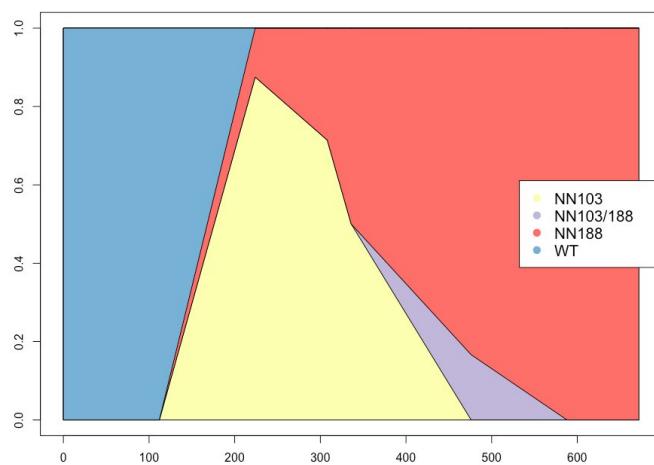


Figure 9.2: Muller plot Figure thanks to Pleuni Pennings.



## Bibliography

, 2013 In North American katydids, green isn't the dominant colour, pink is. Scientific American Blog Network.

AGUILLO, S. M., J. W. FITZPATRICK, R. BOWMAN, S. J. SCHOECH, A. G. CLARK, G. COOP, and N. CHEN, 2017, 08)Deconstructing isolation-by-distance: The genomic consequences of limited dispersal. PLOS Genetics 13(8): 1–27.

ALCAIDE, M., E. S. SCORDATO, T. D. PRICE, and D. E. IRWIN, 2014 Genomic divergence in a ring species complex. Nature 511(7507): 83.

ALEXANDER, D. H., J. NOVEMBRE, and K. LANGE, 2009 Fast model-based estimation of ancestry in unrelated individuals. Genome research 19(9): 1655–1664.

ALGEE-HEWITT, B. F., M. D. EDGE, J. KIM, J. Z. LI, and N. A. ROSENBERG, 2016 Individual identifiability predicts population identifiability in forensic microsatellite markers. Current Biology 26(7): 935–942.

ALLENDORF, F. W. and J. J. HARD, 2009 Human-induced evolution caused by unnatural selection through harvest of wild animals. Proceedings of the National Academy of Sciences 106(Supplement 1): 9987–9994.

ALVAREZ, G., F. C. CEBALLOS, and C. QUINTEIRO, 2009 The role of inbreeding in the extinction of a European royal dynasty. PLoS One 4(4): e5174.

BARRETT, R. D. H., S. M. ROGERS, and D. SCHLUTER, 2008 Natural Selection on a Major Armor Gene in Threespine Stickleback. Science 322(5899): 255–257.

- BECQUET, C., N. PATTERSON, A. C. STONE, M. PRZEWORSKI, and D. REICH, 2007 Genetic structure of chimpanzee populations. *PLoS genetics* 3(4): e66.
- BOX, G. E., 1979 Robustness in the strategy of scientific model building. In *Robustness in statistics*, pp. 201–236. Elsevier.
- BRADBURD, G. S., P. L. RALPH, and G. M. COOP, 2016 A spatial framework for understanding population structure and admixture. *PLoS genetics* 12(1): e1005703.
- BRANDVAIN, Y., A. M. KENNEY, L. FLAGEL, G. COOP, and A. L. SWEIGART, 2014 Speciation and introgression between *Mimulus nasutus* and *Mimulus guttatus*. *PLoS Genetics* 10(6): e1004410.
- CHARLESWORTH, B., 2009 Effective population size and patterns of molecular evolution and variation. *Nature Reviews Genetics* 10(3): 195.
- CHEN, N., E. J. COSGROVE, R. BOWMAN, J. W. FITZPATRICK, and A. G. CLARK, 2016 Genomic Consequences of Population Decline in the Endangered Florida Scrub-Jay. *Current Biology* 26(21): 2974 – 2979.
- COTTERMAN, C. W., 1940 A calculus for statistico-genetics. Ph. D. thesis, The Ohio State University.
- DARWIN, C., 1859 *On the Origin of Species by Means of Natural Selection*. London: Murray. or the Preservation of Favored Races in the Struggle for Life.
- DARWIN, C., 1876 The effect of cross and self fertilization in the vegetable kingdom: Murray. London, UK.
- DICKERSON, R. E., 1971 The structure of cytochrome c and the rates of molecular evolution. *Journal of Molecular Evolution* 1(1): 26–45.
- DOBZHANSKY, T., 1951 *Genetics and the Origin of Species* (3rd Ed. ed.), pp. 16.
- ELTON, C., 1942 *Voles, mice and lemmings. Problems in population dynamics*. Oxford: Clarendon Press.
- EWENS, W. J., 2016 Motoo Kimura and James Crow on the Infinitely Many Alleles Model. *Genetics* 202(4): 1243–1245.

- FRANCIOLI, L. C., A. MENELAOU, S. L. PULIT,  
F. VAN DIJK, P. F. PALAMARA, C. C. ELBERS, P. B.  
NEERINCX, K. YE, V. GURYEV, W. P. KLOOSTERMAN,  
and OTHERS, 2014 Whole-genome sequence variation, population  
structure and demographic history of the Dutch population. *Nature genetics* 46(8): 818.
- GALEN, C., 1996 Rates of floral evolution: adaptation to bumblebee  
pollination in an alpine wildflower, *Polemonium viscosum*. *Evolu-*  
*tion* 50(1): 120–125.
- HALDANE, J., 1942 The selective elimination of silver foxes in east-  
ern Canada. *Journal of Genetics* 44(2-3): 296–304.
- HALDANE, J. B. S., 1937 The Effect of Variation of Fitness. *The*  
*American Naturalist* 71(735): 337–349.
- HOUDE, A. E., 1994 Effect of artificial selection on male colour  
patterns on mating preference of female guppies. *Proc. R. Soc.*  
*Lond. B* 256(1346): 125–130.
- HUDSON, R. R., 2015, 07)A New Proof of the Expected Frequency  
Spectrum under the Standard Neutral Model. *PLOS ONE* 10(7):  
1–5.
- HUDSON, R. R., M. KREITMAN, and M. AGUADÉ, 1987 A test  
of neutral molecular evolution based on nucleotide data. *Genet-*  
*ics* 116(1): 153–159.
- JANICKE, T., I. K. HÄDERER, M. J. LAJEUNESSE, and  
N. ANTHES, 2016 Darwinian sex roles confirmed across the an-  
imal kingdom. *Science advances* 2(2): e1500983.
- JENNINGS, W. B. and S. V. EDWARDS, 2005 Speciational his-  
tory of Australian grass finches (*Poephila*) inferred from thirty gene  
trees. *Evolution* 59(9): 2033–2047.
- JOHANNSEN, W., 1911 The Genotype Conception of Heredity. *The*  
*American Naturalist* 45(531): 129–159.
- KIMURA, M., 1968 Evolutionary rate at the molecular level. *Na-*  
*ture* 217(5129): 624–626.
- KIMURA, M., 1983 *The neutral theory of molecular evolution*. Cam-  
bridge University Press.
- KIMURA, M. and J. F. CROW, 1964 The number of alleles that  
can be maintained in a finite population. *Genetics* 49(4): 725.

- KIMURA, M. and T. OHTA, 1974 On some principles governing molecular evolution. *Proceedings of the National Academy of Sciences* *71*(7): 2848–2852.
- KING, J. L. and T. H. JUKES, 1969 Non-darwinian evolution. *Science* *164*(3881): 788–798.
- KORNEGAY, J. R., J. W. SCHILLING, and A. C. WILSON, 1994 Molecular adaptation of a leaf-eating bird: stomach lysozyme of the hoatzin. *Molecular Biology and Evolution* *11*(6): 921–928.
- LANDE, R., 1979 Quantitative genetic analysis of multivariate evolution, applied to brain: body size allometry. *Evolution* *33*(1Part2): 402–416.
- LAURIE, C. C., D. A. NICKERSON, A. D. ANDERSON, B. S. WEIR, R. J. LIVINGSTON, M. D. DEAN, K. L. SMITH, E. E. SCHADT, and M. W. NACHMAN, 2007, 08)Linkage Disequilibrium in Wild Mice. *PLOS Genetics* *3*(8): 1–9.
- LAWSON, D. J., L. VAN DORP, and D. FALUSH, 2018 A tutorial on how not to over-interpret STRUCTURE and ADMIXTURE bar plots. *Nature communications* *9*(1): 3258.
- LEFFLER, E. M., K. BULLAUGHEY, D. R. MATUTE, W. K. MEYER, L. SEGUREL, A. VENKAT, P. ANDOLFATTO, and M. PRZEWORSKI, 2012 Revisiting an old riddle: what determines genetic diversity levels within species? *PLoS biology* *10*(9): e1001388.
- LEK, M., K. J. KARCZEWSKI, E. V. MINIKEL, K. E. SAMOCHA, E. BANKS, T. FENNELL, A. H. O'DONNELL-LURIA, J. S. WARE, A. J. HILL, B. B. CUMMINGS, and OTHERS, 2016 Analysis of protein-coding genetic variation in 60,706 humans. *Nature* *536*(7616): 285.
- LEWONTIN, R. C., 1994, 05)[DNA Fingerprinting: A Review of the Controversy]: Comment: The Use of DNA Profiles in Forensic Contexts. *Statist. Sci.* *9*(2): 259–262.
- LEWONTIN, R. C., 2001 *Thinking about evolution: historical, philosophical, and political perspectives*, Chapter Natural History and Formalism in Evolutionary Genetics, pp. 7–20. Cambridge University Press.
- LI, J. Z., D. M. ABSHER, H. TANG, A. M. SOUTHWICK, A. M. CASTO, S. RAMACHANDRAN, H. M. CANN, G. S.

- BARSH, M. FELDMAN, L. L. CAVALLI-SFORZA, and OTHERS, 2008 Worldwide human relationships inferred from genome-wide patterns of variation. *science* 319(5866): 1100–1104.
- LISTER, A., 1989 Rapid dwarfing of red deer on Jersey in the last interglacial. *Nature* 342(6249): 539.
- LOCKE, D. P., L. W. HILLIER, W. C. WARREN, K. C. WORLEY, L. V. NAZARETH, D. M. MUZNY, S.-P. YANG, Z. WANG, A. T. CHINWALLA, P. MINX, and OTHERS, 2011 Comparative and demographic analysis of orang-utan genomes. *Nature* 469(7331): 529.
- MACARTHUR, D. G., S. BALASUBRAMANIAN, A. FRANKISH, N. HUANG, J. MORRIS, K. WALTER, L. JOSTINS, L. HABEGGER, J. K. PICKRELL, S. B. MONTGOMERY, and OTHERS, 2012 A systematic survey of loss-of-function variants in human protein-coding genes. *Science* 335(6070): 823–828.
- MALÉCOT, G., 1948 Les mathématiques de l'hérédité.
- MALÉCOT, G., 1969 The Mathematics of Heredity (Revised, edited and translated by Yermanos, DM).
- MCDONALD, J. H. and M. KREITMAN, 1991 Adaptive protein evolution at the Adh locus in *Drosophila*. *Nature* 351(6328): 652.
- MENOZZI, P., A. PIAZZA, and L. CAVALLI-SFORZA, 1978 Synthetic maps of human gene frequencies in Europeans. *Science* 201(4358): 786–792.
- MEREDITH, R. W., J. GATESY, W. J. MURPHY, O. A. RYDER, and M. S. SPRINGER, 2009, 09)Molecular Decay of the Tooth Gene Enamelin (ENAM) Mirrors the Loss of Enamel in the Fossil Record of Placental Mammals. *PLOS Genetics* 5(9): 1–12.
- MESSIER, W. and C.-B. STEWART, 1997 Episodic adaptive evolution of primate lysozymes. *Nature* 385(6612): 151.
- NELSON, M. R., D. WEGMANN, M. G. EHM, D. KEßNER, P. S. JEAN, C. VERZILLI, J. SHEN, Z. TANG, S.-A. BACANU, D. FRASER, and OTHERS, 2012 An abundance of rare functional variants in 202 drug target genes sequenced in 14,002 people. *Science*: 1217876.
- NOVEMBRE, J. and M. STEPHENS, 2008 Interpreting principal component analyses of spatial population genetic variation. *Nature genetics* 40(5): 646.

- OHTA, T., 1973 Slightly deleterious mutant substitutions in evolution. *Nature* *246*(5428): 96.
- OHTA, T. and J. H. GILLESPIE, 1996 Development of neutral and nearly neutral theories. *Theoretical population biology* *49*(2): 128–142.
- PAABY, A. B., A. O. BERGLAND, E. L. BEHRMAN, and P. S. SCHMIDT, 2014 A highly pleiotropic amino acid polymorphism in the *Drosophila* insulin receptor contributes to life-history adaptation. *Evolution* *68*(12): 3395–3409.
- PATTERSON, N., A. L. PRICE, and D. REICH, 2006 Population structure and eigenanalysis. *PLoS genetics* *2*(12): e190.
- POTTI, J. and D. CANAL, 2011 Heritability and genetic correlation between the sexes in a songbird sexual ornament. *Heredity* *106*(6): 945.
- PRITCHARD, J. K., M. STEPHENS, and P. DONNELLY, 2000 Inference of population structure using multilocus genotype data. *Genetics* *155*(2): 945–959.
- PROVINE, W. B., 2001 *The origins of theoretical population genetics: with a new afterword*. University of Chicago Press.
- RANDS, C. M., S. MEADER, C. P. PONTING, and G. LUNTER, 2014 8.2% of the human genome is constrained: variation in rates of turnover across functional element classes in the human lineage. *PLoS genetics* *10*(7): e1004525.
- RICHARDS, C. M., 2000 Inbreeding depression and genetic rescue in a plant metapopulation. *The American Naturalist* *155*(3): 383–394.
- RITLAND, K., C. NEWTON, and H. MARSHALL, 2001 Inheritance and population structure of the white-phased “Kermode” black bear. *Current Biology* *11*(18): 1468 – 1472.
- ROBINSON, J. A., D. ORTEGA-DEL VECCHYO, Z. FAN, B. Y. KIM, C. D. MARSDEN, K. E. LOHMUELLER, R. K. WAYNE, and OTHERS, 2016 Genomic flatlining in the endangered island fox. *Current Biology* *26*(9): 1183–1189.
- ROSENBERG, N. A., J. K. PRITCHARD, J. L. WEBER, H. M. CANN, K. K. KIDD, L. A. ZHIVOTOVSKY, and M. W. FELDMAN, 2002 Genetic structure of human populations. *science* *298*(5602): 2381–2385.

- SAMS, A. J. and A. R. BOYKO, 2018 Fine-scale resolution and analysis of runs of homozygosity in domestic dogs. *bioRxiv*.
- SANKARARAMAN, S., N. PATTERSON, H. LI, S. PÄÄBO, and D. REICH, 2012, 10)The Date of Interbreeding between Neandertals and Modern Humans. *PLOS Genetics* 8(10): 1–9.
- STURTEVANT, A. H., 1915 The behavior of the chromosomes as studied through linkage. *Zeitschrift für induktive Abstammungs-und Vererbungslehre* 13(1): 234–287.
- TAJIMA, F., 1989 Statistical method for testing the neutral mutation hypothesis by DNA polymorphism. *Genetics* 123(3): 585–595.
- TOEWS, D. P., S. A. TAYLOR, R. VALLENDER, A. BRELSFORD, B. G. BUTCHER, P. W. MESSEY, and I. J. LOVETTE, 2016 Plumage Genes and Little Else Distinguish the Genomes of Hybridizing Warblers. *Current Biology* 26(17): 2313 – 2318.
- VOIGHT, B. F., A. M. ADAMS, L. A. FRISSE, Y. QIAN, R. R. HUDSON, and A. DI RIENZO, 2005 Interrogating multiple aspects of variation in a full resequencing data set to infer human population size changes. *Proceedings of the National Academy of Sciences* 102(51): 18508–18513.
- VONHOLDT, B. M., J. P. POLLINGER, D. A. EARL, J. C. KNOWLES, A. R. BOYKO, H. PARKER, E. GEFEN, M. PILOT, W. JEDRZEJEWSKI, B. JEDRZEJEWSKA, V. SIDOROVICH, C. GRECO, E. RANDI, M. MUSIANI, R. KAYS, C. D. BUSTAMANTE, E. A. OSTRANDER, J. NOVEMBRE, and R. K. WAYNE, 2011 A genome-wide perspective on the evolutionary history of enigmatic wolf-like canids. *Genome Research*.
- WANG, J., J. DING, B. TAN, K. M. ROBINSON, I. H. MICHELSON, A. JOHANSSON, B. NYSTEDT, D. G. SCOFIELD, O. NILSSON, S. JANSSON, and OTHERS, 2018 A major locus controls local adaptation and adaptive life history variation in a perennial plant. *Genome biology* 19(1): 72.
- WATTERSON, G., 1975 On the number of segregating sites in genetical models without recombination. *Theoretical population biology* 7(2): 256–276.
- WHEELER, W. M., 1907 Pink Insect Mutants. *The American Naturalist* 41(492): 773–780.

- WILKINSON, G. S., 1993 Artificial sexual selection alters allometry in the stalk-eyed fly *Cyrtodiopsis dalmanni* (Diptera: Diopsidae). *Genetics Research* 62(3): 213–222.
- WISELY, S. M., S. W. BUSKIRK, M. A. FLEMING, D. B. McDONALD, and E. A. OSTRANDER, 2002 Genetic Diversity and Fitness in Black-Footed Ferrets Before and During a Bottleneck. *Journal of Heredity* 93(4): 231–237.
- WRIGHT, S., 1943 Isolation by Distance. *Genetics* 28(2): 114–138.
- WRIGHT, S., 1949 The Genetical Structure of Populations. *Annals of Eugenics* 15(1): 323–354.
- WRIGHT, S. and T. DOBZHANSKY, 1946 Genetics of natural populations. XII. Experimental reproduction of some of the changes caused by natural selection in certain populations of *Drosophila pseudoobscura*. *Genetics* 31(2): 125.
- WRIGHT, S. I., I. V. BI, S. G. SCHROEDER, M. YAMASAKI, J. F. DOEBLEY, M. D. McMULLEN, and B. S. GAUT, 2005 The Effects of Artificial Selection on the Maize Genome. *Science* 308(5726): 1310–1314.
- YANG, Z., 1998 Likelihood ratio tests for detecting positive selection and application to primate lysozyme evolution. *Molecular Biology and Evolution* 15(5): 568–573.
- ZUCKERKANDL, E. and L. PAULING, 1965 Evolutionary divergence and convergence in proteins. In *Evolving genes and proteins*, pp. 97–166. Elsevier.