Optimization

1. Given the function,

$$f(x, y) = 2xy - x^2y - y^2x.$$

(a) Find all the critical points. The graph below might be helpful.

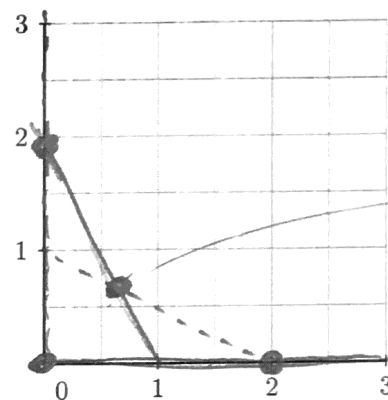$$\nabla f = \langle 2y - 2xy - y^2,$$

$$2x - x^2 - 2xy \rangle$$

$$2y - 2xy - y^2 = 0$$

$$y(y + 2x - 2) = 0$$

$$y = 0, \quad 2 - 2x$$

$$2x - x^2 - 2xy = 0$$

$$x = 0, \quad 2 - 2y$$

$x = y$
$x = 2 - 2x$
$x = \frac{2}{3}$

$(0, 0)$
$(0, 2)$
$(2, 0)$
$\left(\frac{2}{3}, \frac{2}{3}\right)$

$$f(x, y) = 2xy - x^2 y - y^2 x$$

(b) Classify each critical point using the Hessian matrix.

$$\frac{\partial^2 f}{\partial x^2} = -2y \qquad \frac{\partial^2 f}{\partial y^2} = -2x \qquad \frac{\partial^2 f}{\partial x \partial y} = 2 - 2x - 2y$$

$$H = \begin{vmatrix} -2y & 2-2x-2y \\ 2-2x-2y & -2x \end{vmatrix} = 4xy - 4(x+y-1)^2$$

$$H(0,0) = -4 \longrightarrow \text{saddle}$$

$$H(0,2) = -4 \longrightarrow \text{saddle}$$

$$H(2,0) = -4 \longrightarrow \text{saddle}$$

$$H\left(\tfrac{2}{3}, \tfrac{2}{3}\right) = \tfrac{4}{3} \longrightarrow \text{local max}$$

$$f_{xx}\left(\tfrac{2}{3}, \tfrac{2}{3}\right) = -\tfrac{4}{3} \nearrow$$

$$f(x, y) = 2xy - x^2 y - y^2 x$$

(c) Find the maximum and minimum values of the function in the region within the triangle shown below, including the edges of the triangle.
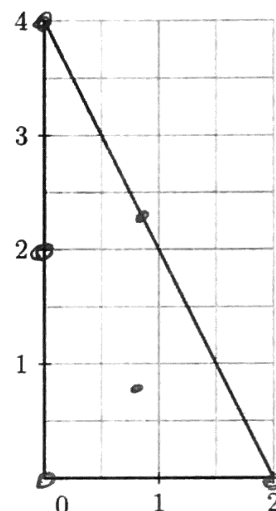
Find extrema along $y = 4 - 2x$:

$$f(x, 4-2x) = 2x(4-2x) - x^2(4-2x) - (4-2x)^2 x$$

$$f'(x, 4-2x) = 8 - 8x - 8x + 6x^2 - 12x^2 + 32x - 16$$

$$= -8 + 16x - 6x^2 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 48}}{6} = 2, \frac{2}{3}$$

$f(0,0) = 0$

$f(0,2) = 0$

$f(2,0) = 0$

$f(4,0) = 0$

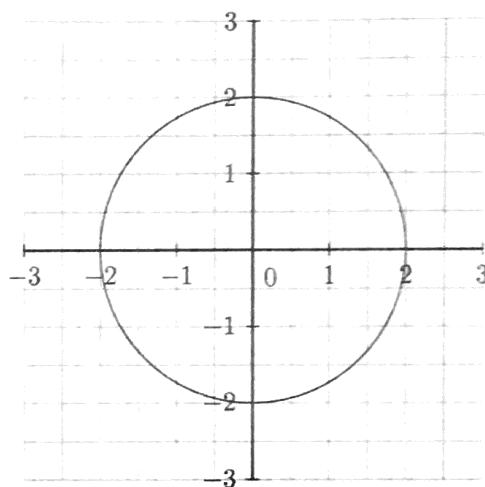$f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{8}{9} - \frac{16}{27} = \boxed{\frac{8}{27}}$

$f\left(\frac{2}{3}, \frac{8}{3}\right) = \boxed{-\frac{64}{27}}$

Max @ $f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{8}{27}$

Min @ $f\left(\frac{2}{3}, \frac{8}{3}\right) = -\frac{64}{27}$

2. Given the function $f(x,y) = 5x^2 - 6xy + 5y^2$ with constraint $x^2 + y^2 = 4$, shown below.

(a) Find all points where the gradient of f is parallel (or anti-parallel) to the gradient of $g(x,y) = x^2 + y^2$.

$\vec{\nabla f} = \langle 10x - 6y, \ 10y - 6x \rangle$

$\vec{\nabla g} = \langle 2x, 2y \rangle$

$\vec{\nabla f} = \lambda \vec{\nabla g}$

$\begin{cases} 10x - 6y = \lambda \, 2x \\ 10y - 6x = \lambda \, 2y \end{cases}$

$(5-\lambda)x = 6y$

$(5-\lambda)y = 6x$

$\boxed{\pm x = y}$

(b) Find all points on the constraint $x^2 + y^2 = 4$ where the gradient of f is parallel (or anti-parallel) to the gradient of $g(x,y) = x^2 + y^2$.

$\begin{cases} x = \pm y \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \ x = \pm\sqrt{2} \\ \qquad\qquad\qquad\qquad y = \pm\sqrt{2}$

(c) Find the maximum and minimum values of $f(x,y)$ under the constraint $x^2 + y^2 = 4$.

$f(-\sqrt{2}, -\sqrt{2}) = 8$

$f(-\sqrt{2}, \sqrt{2}) = 32$

$f(\sqrt{2}, -\sqrt{2}) = 32$

$f(\sqrt{2}, \sqrt{2}) = 8$

$\begin{array}{|l|} \hline \text{Max value : } 32 \\ \text{min value : } 8 \\ \hline \end{array}$

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