Chapter 12 Review

1. Given the points $A(1, 2, 3)$ and $B(5, 4, -2)$.

   (a) Find both the parametric and symmetric equations of the straight line connecting A and B.

   $$\vec{v} = B - A = \langle 4, 2, -5 \rangle$$

   $$\begin{cases} x = 4t + 1 \\ y = 2t + 2 \\ z = -5t + 3 \end{cases} \qquad \frac{x-1}{4} = \frac{y-2}{2} = \frac{z-3}{-5}$$

   (b) Find the plane perpendicular to this line that goes through the point $(1, 2, 4)$.

   $$\vec{n} = \vec{v}$$

   $$4x + 2y - 5z = 4(1) + 2(2) - 5(4)$$

   $$\boxed{4x + 2y - 5z = -12}$$

   (c) Find a plane that this line does not intersect.

   $\vec{n} \perp \vec{v} \Rightarrow$ choose $\vec{n} = \langle 1, -2, 0 \rangle$ $\quad (\vec{n} \cdot \vec{v} = 0)$

   $$\boxed{x - 2y + 0z = 0}$$

   $\hookrightarrow$ check line does not intersect

   $x = 4t + 1$

   $y = 2t + 2$

   $(\cancel{4t} + 1) - 2(\cancel{2t} + 2) = 0$

   $-3 = 0$

   ✓ $\longrightarrow$ doesn't intersect

2. Give examples of the following, or explain why no such example exists.

(a) An equation for a cylinder so that the point $(1, 2, 3)$ is on its surface.

See week 2

$x^2+y^2=5$

(b) A paraboloid that opens downwards in the z direction and intersects the xy-plane in the ellipse $4x^2 + 9y^2 = 36$.

See week 2

$z = 36 - 4x^2 - 9y^2$

(c) Two parallel lines L_1 and L_2 in the parallel planes $x + 2y + 2z = 3$ and $x + 2y + 2z = 6$ so that the distance between L_1 and L_2 is 3.

See week 2

(d) Two skew lines L_1 and L_2 that sit in parallel planes $2x + 2y + z = 5$ and $2x + 2y + z = -4$ so that the minimum distance between L_1 and L_2 is 9.

See week 2

3. Given the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, 3t \rangle$.

(a) Find the unit tangent vector at time t.

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -\sin t, \cos t, 3 \rangle}{\sqrt{1+9}} = \boxed{\frac{1}{\sqrt{10}} \langle -\sin t, \cos t, 3 \rangle}$$

(b) Find an equation for the line tangent to the curve at $t = \pi$.

$\vec{r}\,' = \vec{v}\,' t' + \vec{r}_0\,'$ (remember $y = mx + y_0$)

can use $\vec{v} = \vec{T}$ if preferred

$\vec{v}\,' = \vec{v}(\pi) = \langle -\sin \pi, \cos \pi, 3 \rangle = \langle 0, -1, 3 \rangle$

$\vec{r}_0\,' = \vec{r}(\pi) = \langle \cos \pi, \sin \pi, 3\pi \rangle = \langle -1, 0, 3\pi \rangle$

$\vec{r}\,' = \langle 0, -1, 3 \rangle t + \langle -1, 0, 3\pi \rangle \longleftrightarrow \begin{cases} x = -1 \\ y = -t \\ z = 3t + 3\pi \end{cases}$

(c) Calculate the curvature κ of the curve at $t = \pi$.

$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$ $\quad ds = \|\vec{v}\|\, dt$

$= \left\| \frac{d\left(\frac{\vec{v}}{\|\vec{v}\|}\right)}{\|\vec{v}\|\, dt} \right\|$

$= \frac{1}{\|\vec{v}\|} \left\| \frac{d}{dt} \frac{\vec{v}}{\|\vec{v}\|} \right\|$

$= \frac{1}{\sqrt{10}} \left\| \frac{d}{dt} \left(\frac{1}{\sqrt{10}} \langle -\sin t, \cos t, 3 \rangle \right) \right\|$

$= \frac{1}{10} \sqrt{1}$

$= \boxed{\frac{1}{10}}$

Alt:

$\kappa = \frac{\| \vec{r}\,' \times \vec{r}\,'' \|}{\| \vec{r}\,' \|^3}$

$= \frac{\| \langle 0, -1, 3 \rangle \times \langle 1, 0, 0 \rangle \|}{\sqrt{10}^3}$

$= \frac{\| \langle 0, 3, 1 \rangle \|}{\sqrt{10}^3} = \boxed{\frac{1}{10}}$

(d) Where does this line intersect the plane $z = 0$.

$z = 3t = 0$

$t = 0$

$\boxed{\vec{r}(0) = \langle 1, 0, 0 \rangle}$

4. For the following parameterization $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

(a) Find the velocity and acceleration vectors at time $t = 1$, $\mathbf{v}(1) = \mathbf{r}'(1)$ and $\mathbf{a}(1) = \mathbf{r}''(1)$.

$$\vec{v}(1) = \vec{r}'(1) = \langle 1, 2t, 3t^2 \rangle \Big|_{t=1} = \boxed{\langle 1, 2, 3 \rangle}$$

$$\vec{a}(1) = \vec{r}''(1) = \langle 0, 2, 6t \rangle \Big|_{t=1} = \boxed{\langle 0, 2, 6 \rangle}$$

(b) Find the unit tangent vector $\mathbf{T}$ and the unit normal vector $\mathbf{N}$ at time $t = 1$. [Hint: you can find the normal direction by taking $\mathbf{v} \times \mathbf{a} \times \mathbf{v}$.]

$$\hat{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{1+4+9}} = \boxed{\frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle}$$

$$\vec{N} = \langle 1, 2, 3 \rangle \times \langle 0, 2, 6 \rangle \times \langle 1, 2, 3 \rangle = \langle 6, -6, 2 \rangle \times \langle 1, 2, 3 \rangle$$

$$= \langle -22, -16, 18 \rangle$$

$$\hat{N} = \frac{\vec{N}}{\|\vec{N}\|} = \frac{\langle -11, -8, 9 \rangle}{\sqrt{121+64+81}}$$

$$= \boxed{\frac{1}{\sqrt{266}} \langle -11, -8, 9 \rangle}$$

(c) Find the curvature κ at time $t = 1$.

$$\kappa = \left\| \frac{d\hat{T}}{ds} \right\|$$

$$= \frac{1}{\|\vec{v}\|} \left\| \frac{d\hat{T}}{dt} \right\|$$

$$= \frac{1}{\sqrt{14}} \left\| \frac{\vec{r}' \times \vec{r}''}{\|r'\|^2} \right\|$$

$$= \frac{1}{\sqrt{14}^3} \| \langle 6, -6, 2 \rangle \|$$

$$= \frac{\sqrt{38}}{\sqrt{7}} \cdot \frac{1}{14}$$

The Gradient

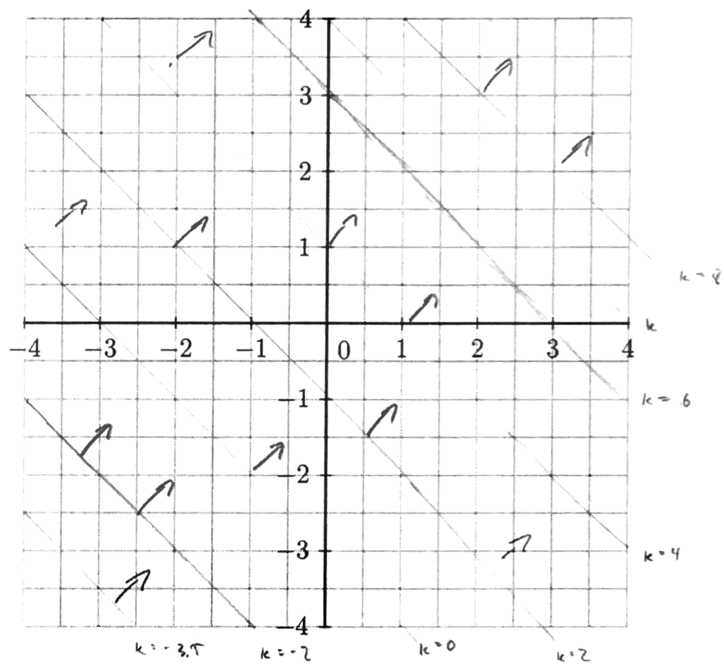
5. Given each of the following equations,

   (a) On the axes on the right, draw level curves of the function $f(x, y)$.

   (b) Find the gradient of f, $\langle f_x, f_y \rangle$, and draw some gradient vectors on the graph of the level curves. What is true about the gradient and level curves?

   - $f(x, y) = x + y + 3$ = k

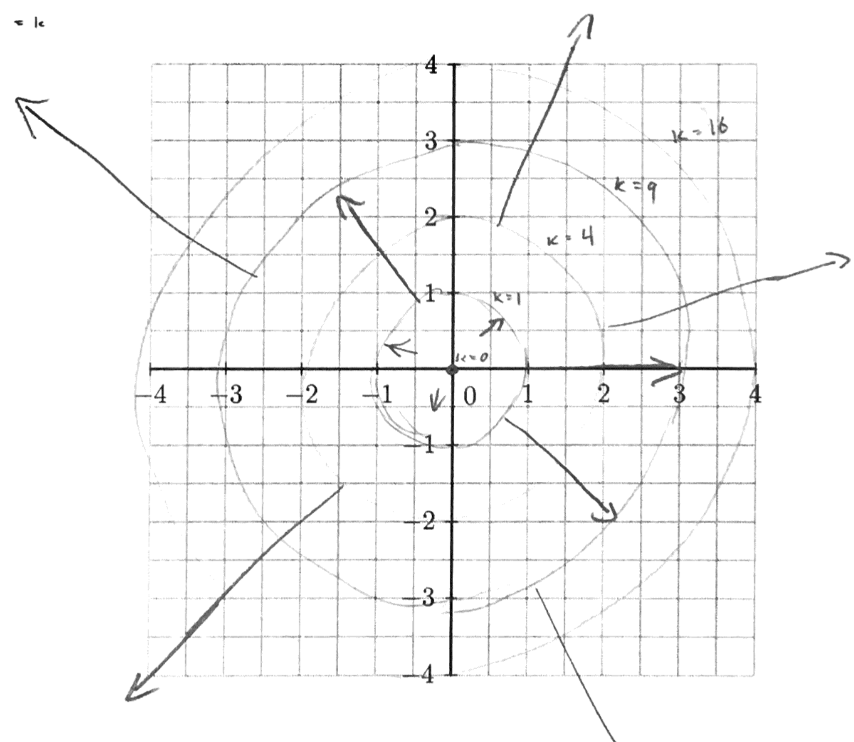
     $\vec{\nabla} f = \langle 1, 1 \rangle$



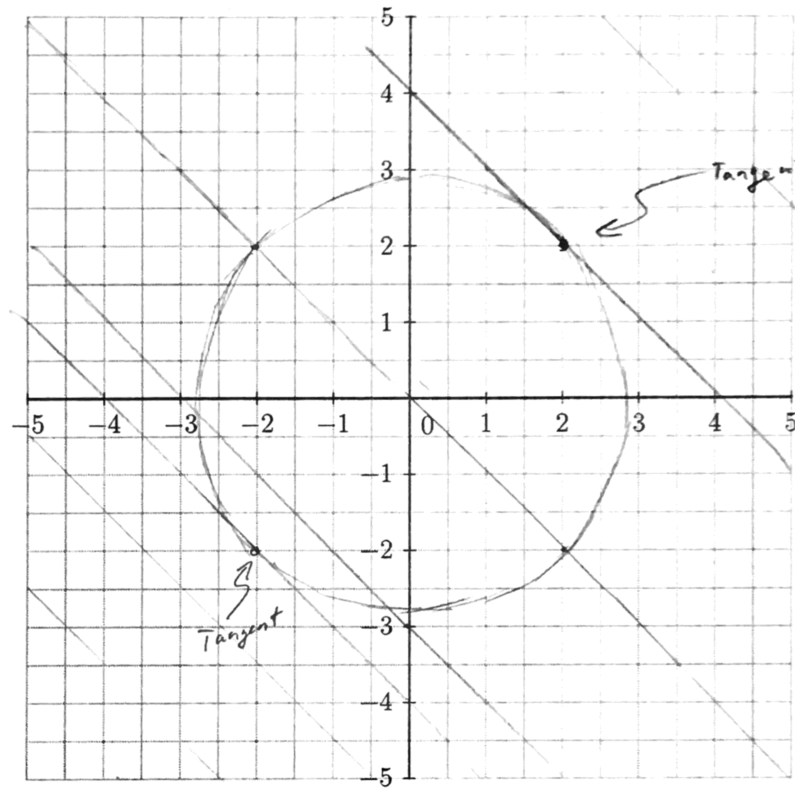
   - $f(x, y) = x^2 + y^2$ = k

     $\vec{\nabla} f = \langle 2x, 2y \rangle$

     $\vec{\nabla} f \perp$ level sets



(c) Show on the graph below that the points where the level curves of $f(x, y) = x + y + 3$ are tangent to the curve $x^2 + y^2 = 8$ are $(2, 2)$ and $(-2, -2)$.



(d) Show that the gradient of $f(x, y) = x+y+3$ and the gradient of $g(x, y) = x^2+y^2$ are parallel (or antiparallel) at (2,2) and (-2,-2).

$\vec{\nabla f} = \langle 1, 1 \rangle$

$\vec{\nabla g} = \langle 2x, 2y \rangle$

$= 2 \langle x, y \rangle$

@ $(2,2)$

$\vec{\nabla f} = \langle 1, 1 \rangle$

$\vec{\nabla g} = \langle 4, 4 \rangle$

$\Downarrow$

$\vec{\nabla f} = \frac{1}{4} \vec{\nabla g}$

$\Downarrow$

$\vec{\nabla f} \parallel \vec{\nabla g}$

@ $(-2,-2)$

$\vec{\nabla f} = \langle 1, 1 \rangle$

$\vec{\nabla g} = \langle -4, -4 \rangle$

$\vec{\nabla f} = -\frac{1}{4} \vec{\nabla g}$

$\Downarrow$

$\vec{\nabla f} \parallel \vec{\nabla g}$

(e) Are these two points the maximum and minimum value of $f(x, y) = x + y + 3$ under the constraint $g(x, y) = x^2 + y^2 = 8$?

Yes.