

Change of Variable

1. Find the area between the ellipses $x^2 + 4y^2 = 4$ and $x^2 + 4y^2 = 16$. [Hint: the change of variables $\underbrace{u = x, v = 2y}$ might be helpful.]

$$\begin{aligned} u^2 + v^2 &= 4 \\ u^2 + v^2 &= 16 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\text{let} \\ u &= r \cos \theta \\ v &= r \sin \theta \end{aligned}$$

$$A = \int_0^{2\pi} \int_2^4 \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(r,\theta)} dr d\theta$$

$$= \int_0^{2\pi} \int_2^4 \frac{1}{2} r dr d\theta$$

$$= \boxed{16\pi}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$$= \frac{1}{\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}}$$

$$= \frac{1}{2}$$

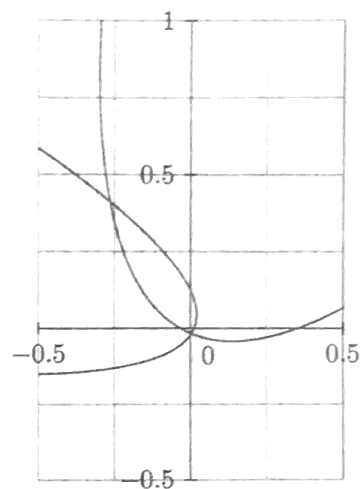
2. Calculate the area in the plane bounded by the rotated parabolas $y - 2x = (3y + x)^2$ and $(3y + x) = (y - 2x)^2$ shown below.

$$\text{Let } u = y - 2x$$

$$v = 3y + x$$

$$\Rightarrow u = v^2$$

$$v = u^2$$



$$A = \int_0^1 \int_{u^2}^{\sqrt{u}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv du$$

$$= \int_0^1 \int_{u^2}^{\sqrt{u}} \frac{1}{7} dv du$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix}}$$

$$= \frac{1}{7} \left[\frac{2u^{3/2}}{3} - \frac{u^3}{3} \right]_0^1$$

$$= \frac{1}{7} \left(\frac{2}{3} - \frac{1}{3} \right) = \boxed{\frac{1}{21}}$$

3. Let T be the solid bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If the density is given by $\delta(x, y, z) = z^2$ find M , the mass of the solid.

Let $u = \frac{x}{a}$

$v = \frac{y}{b}$

$w = \frac{z}{c}$

$\delta = c^2 w^2$

$\frac{\partial(x, y, z)}{\partial(u, v, w)} = abc$

$$M = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} abc^2 w^2 dw dv du$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^1 abc^2 \rho^2 \cos^2 \phi \cdot \rho^2 \sin \phi d\rho d\theta d\phi$$

$$= 2\pi abc^3 \int_0^\pi \frac{1}{5} \cos^2 \phi \sin \phi d\phi$$

$$= 2\pi abc^3 \left[-\frac{1}{15} \cos^3 \phi \right]_0^\pi$$

$$= 2\pi abc^3 \left(\frac{1}{15} + \frac{1}{15} \right)$$

$$= \boxed{\frac{4\pi abc^3}{15}}$$

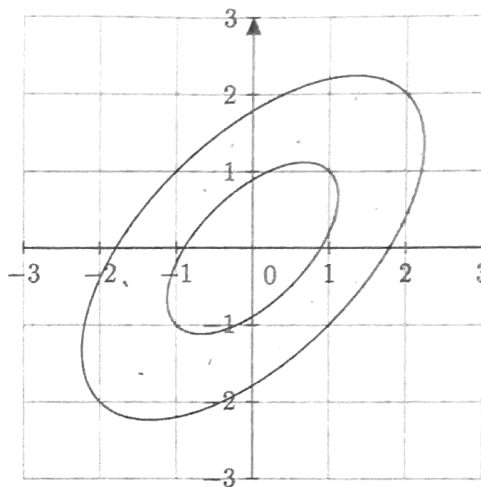
4. Find the area between the rotated ellipses $5x^2 - 6xy + 5y^2 = 4$ and $5x^2 - 6xy + 5y^2 = 16$. [Hint: try the substitutions $x = u + v, y = u - v$.]

$$5x^2 - 6xy + 5y^2 = 4$$

$$5(u+v)^2 - 6(u+v)(u-v) + 5(u-v)^2 = 4$$

$$4u^2 + 16v^2 = 4$$

$$\begin{cases} u^2 + 4v^2 = 1 \\ u^2 + 4v^2 = 4 \end{cases}$$



$$A = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot (\text{Area in } u,v \text{ space})$$

$$= 2 \left(2\pi - \frac{1}{2}\pi \right)$$

$$= \boxed{3\pi}$$