The Heat Equation

1. Given the heat equation

$$\begin{aligned} u_t &= ku_{xx}, \quad 0 < x < 2, \quad t > 0 \\ u(0,t) &= u(2,t) = 0, \quad t > 0 \\ u(x,0) &= 1, \quad 0 < x < 2 \end{aligned}$$

   (a) We will look for all solutions of the form $u(x,t) = X(x)T(t)$. Plug this into the heat equation to find an eigenvalue equation in x.

   (b) Why do the boundary conditions on u, i.e. $u(0,t) = u(2,t) = 0$ imply that $X(0) = X(2) = 0$?

   (c) Solve the eigenvalue equation in x.

(d) Solve for $T(t)$.

(e) Why can we write $u(x,t) = \sum_n T_n(t)X_n(x)$, where $X_n(x) = \sin\left(\frac{n\pi x}{2}\right)$?

(f) How do we match the solution up to $u(x, 0) = 1$?

2. Solve the heat equation with Dirichlet boundary conditions

$$\begin{aligned} u_t &= ku_{xx}, \quad 0 < x < 2, \quad t > 0 \\ u_x(0,t) &= u_x(2,t) = 0, \quad t > 0 \\ u(x,0) &= f(x), \quad 0 < x < 2 \end{aligned}$$