Optimization

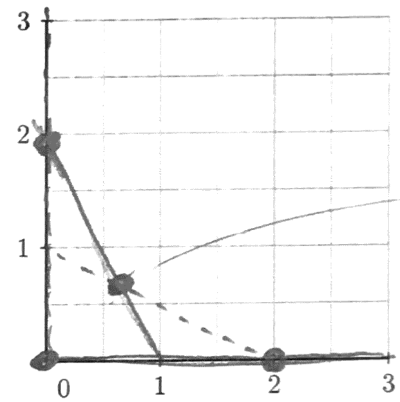
1. Given the function,

$$f(x, y) = 2xy - x^2y - y^2x.$$

(a) Find all the critical points. The graph below might be helpful.

$\nabla f = \langle 2y - 2xy - y^2,$

$2x - x^2 - 2xy \rangle$



$x = y$

$x = 2 - 2x$

$x = \frac{2}{3}$

$2y - 2xy - y^2 = 0$

$y(y + 2x - 2) = 0$

$y = 0,\ 2 - 2x$

$2x - x^2 - 2xy = 0$

$x = 0,\ 2 - 2y$

$$\boxed{\begin{matrix}(0,0)\\(0,2)\\(2,0)\\(\frac{2}{3},\frac{2}{3})\end{matrix}}$$

$$f(x, y) = 2xy - x^2y - y^2x$$

(b) Classify each critical point using the Hessian matrix.

$$\frac{\partial^2 f}{\partial x^2} = -2y \qquad \frac{\partial^2 f}{\partial y^2} = -2x \qquad \frac{\partial^2 f}{\partial x \partial y} = 2 - 2x - 2y$$

$$H = \begin{vmatrix} -2y & 2-2x-2y \\ 2-2x-2y & -2x \end{vmatrix} = 4xy - 4(x+y-1)^2$$

$H(0,0) = -4 \longrightarrow$ max

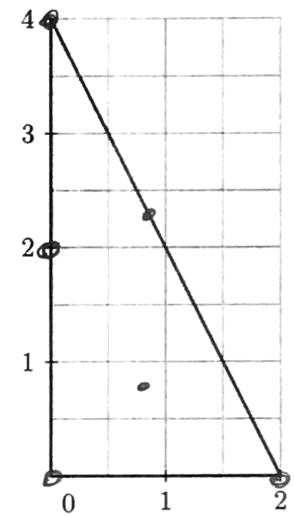
$H(0,2) = -4 \longrightarrow$ max

$H(2,0) = -4 \longrightarrow$ max

$H(\frac{2}{3}, \frac{2}{3}) = \frac{4}{3} \longrightarrow$ min

$$f(x,y) = 2xy - x^2y - y^2x$$

(c) Find the maximum and minimum values of the function in the region within the triangle shown below, including the edges of the triangle.



Find extrema along $y = 4-2x$:

$f(x, 4-2x) = 2x(4-2x) - x^2(4-2x) - (4-2x)^2x$

$f'(x, 4-2x) = 8 - 8x - 8x + 6x^2 - 12x^2 + 32x - 16$

$= -8 + 16x - 6x^2 = 0$

$x = \frac{8 \pm \sqrt{64-48}}{6} = 2, \frac{2}{3}$

$f(0,0) = 0$

$f(0,2) = 0$

$f(2,0) = 0$

$f(4,0) = 0$

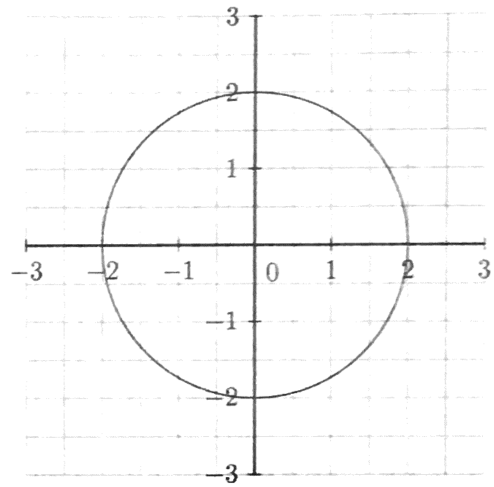
$f(\frac{2}{3}, \frac{2}{3}) = \frac{8}{9} - \frac{16}{27} = \frac{8}{27}$

$f(\frac{2}{3}, \frac{8}{3}) = -\frac{64}{27}$

Max @ $f(\frac{2}{3}, \frac{2}{3}) = \frac{8}{27}$

Min @ $f(\frac{2}{3}, \frac{8}{3}) = -\frac{64}{27}$

2. Given the function $f(x, y) = 5x^2 - 6xy + 5y^2$ with constraint $x^2 + y^2 = 4$, shown below.



(a) Find all points where the gradient of f is parallel (or anti-parallel) to the gradient of $g(x, y) = x^2 + y^2$.

$\vec{\nabla f} = \langle 10x - 6y, 10y - 6x \rangle$

$\vec{\nabla g} = \langle 2x, 2y \rangle$

$\vec{\nabla f} = \lambda \vec{\nabla g}$

$\begin{cases} 10x - 6y = \lambda 2x \\ 10y - 6x = \lambda 2y \end{cases}$

$(5-\lambda)x = 6y$

$(5-\lambda)y = 6x$

$\boxed{\pm x = y}$

(b) Find all points on the constraint $x^2 + y^2 = 4$ where the gradient of f is parallel (or anti-parallel) to the gradient of $g(x, y) = x^2 + y^2$.

$\begin{cases} x = \pm y \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{array}{l} x = \pm\sqrt{2} \\ y = \pm\sqrt{2} \end{array}$

(c) Find the maximum and minimum values of $f(x, y)$ under the constraint $x^2 + y^2 = 4$.

$f(-\sqrt{2}, -\sqrt{2}) = 8$

$f(-\sqrt{2}, \sqrt{2}) = 32$

$f(\sqrt{2}, -\sqrt{2}) = 32$

$f(\sqrt{2}, \sqrt{2}) = 8$

Max value : 32
min value : 8