Dynamics

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Robotics and Automation course

TAMIU

The Euler Lagrange equations

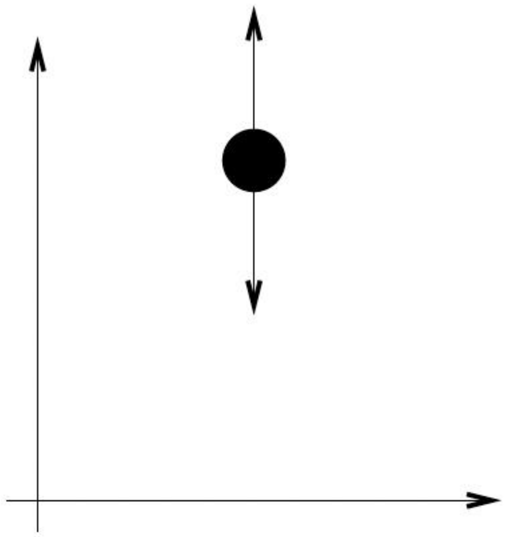


Fig. 6.1 One Degree of Freedom System

$$m\ddot{y} \quad = \quad f - mg$$

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial \boxed{\mathcal{K}}}{\partial \dot{y}}$$

$\mathcal{K} = \frac{1}{2}m\dot{y}^2$ is the **kinetic energy**

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \boxed{\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{y}}}$$

$$mg = \frac{\partial}{\partial y}(mgy) = \boxed{\frac{\partial \mathcal{P}}{\partial y}}$$

$\mathcal{K} = \frac{1}{2}m\dot{y}^2$ is the **kinetic energy**.

$\mathcal{P} = mgy$ is the **potential energy due to gravity**.

Important definitions

We define:

$$\boxed{\mathcal{L} = \mathcal{K} - \mathcal{P}} = \frac{1}{2}m\dot{y}^2 - mgy$$

The Lagrangian

Notice that we can write $m\ddot{y} = f - mg$ as follows:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f} \longrightarrow \text{The Euler Lagrange equation}$$

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2}m\dot{y}^2 - mgy$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{K}}{\partial \dot{y}}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial \mathcal{P}}{\partial y}$$

Example 1

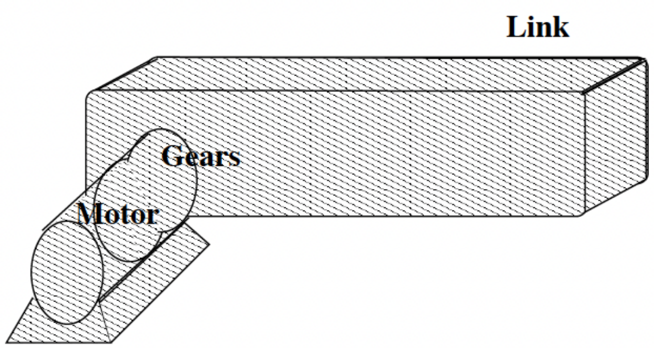
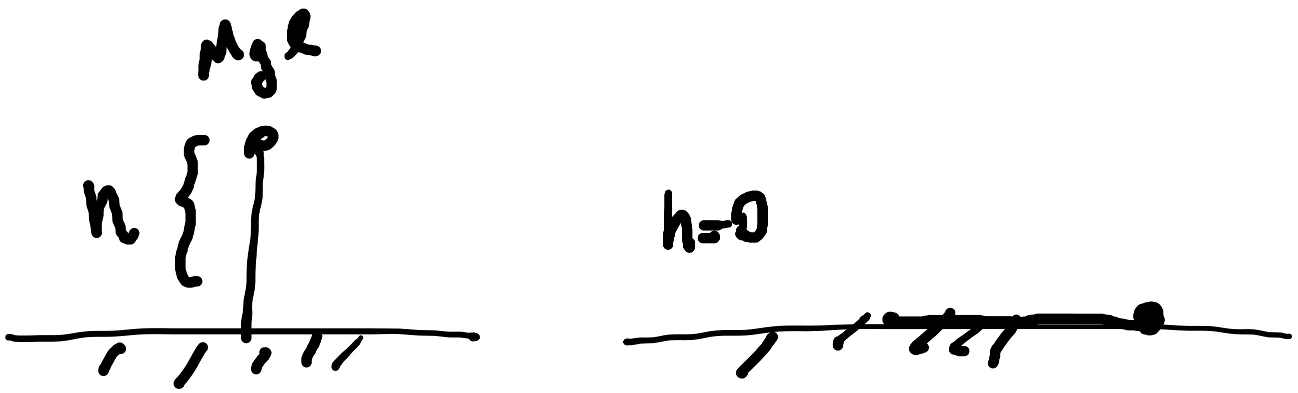


Fig. 6.2 Single-Link Robot.



$h = \ell(1-\cos\theta_\ell)$

Let θ_l and θ_m denote the angles of the link and motor shaft, respectively.
Then, $\theta_m = r\theta_l$, where r: 1 is the gear ratio. J_m, J_l are the rotational inertias.

$$\begin{aligned} K &= \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_\ell\dot{\theta}_\ell^2 \\ &= \frac{1}{2}\underbrace{(r^2J_m + J_\ell)}_{J = r^2J_m + J_\ell}\dot{\theta}_\ell^2 \end{aligned}$$

$$\begin{aligned} P &= Mg\ell(1-\cos\theta_\ell) \\ \mathcal{L} &= \frac{1}{2}J\dot{\theta}_\ell^2 - Mg\ell(1-\cos\theta_\ell) \end{aligned}$$

Example

$$\mathcal{L} = \frac{1}{2} J \dot{\theta}_\ell^2 - Mg\ell(1 - \cos\theta_\ell) \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_\ell} - \frac{\partial \mathcal{L}}{\partial \theta_\ell} = f$$

Substituting this expression into the Euler-Lagrange equations yields the equation of motion

$$\boxed{J\ddot{\theta}_\ell + Mg\ell \sin\theta_\ell = \tau_\ell}$$

Dynamics of the arm!

The generalized force τ_l represents those external forces and torques that are not derivable from a potential function

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In general, for any system of the type considered, an application of the Euler-Lagrange equations leads to a system of n coupled, second order nonlinear ordinary differential equations of the form:

Euler-Lagrange Equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \quad i = 1, \ldots, n$$

Example 2

Let's code this:

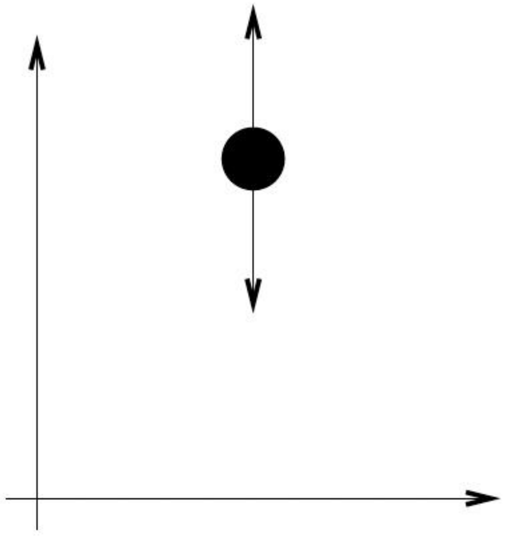

Fig. 6.1 One Degree of Freedom System

$$m\ddot{y} = f - mg$$

Example

dynamics.py > ...

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# System parameters
m = 1.0  # mass (kg)
g = 9.81  # gravitational acceleration (m/s^2)
f = 10  # external force (N) -m*g - 10*y[1]

# Define the differential equation
def system(t, y):
    # y[0] = position (y)
    # y[1] = velocity (v = dy/dt)
    # control: f = -m*g - 10*y[0] - 7*y[1]
    dydt = [y[1], (f - m * g) / m]  # velocity and acceleration
    return dydt
```

Example

```python

# Initial conditions
y0 = [5, 8]  # initial position and initial velocity
t_span = (0, 10)  # time interval (from 0 to 10 seconds)
t_eval = np.linspace(t_span[0], t_span[1], 100)  # points for evaluation

# Solve the differential equation
sol = solve_ivp(system, t_span, y0, t_eval=t_eval)

# Extract results
t = sol.t         # time values
y = sol.y[0]      # position values
v = sol.y[1]      # velocity values

```

Example

```python
# Plot the results
plt.figure(figsize=(10, 5))

# Position plot
plt.subplot(2, 1, 1)
plt.plot(t, y, label='Position')
plt.xlabel('Time (s)')
plt.ylabel('Position (m)')
plt.legend()
plt.grid()

# Velocity plot
plt.subplot(2, 1, 2)
plt.plot(t, v, label='Velocity', color='orange')
plt.xlabel('Time (s)')
plt.ylabel('Velocity (m/s)')
plt.legend()
plt.grid()

plt.tight_layout()
plt.show()
```

Exercise

Create a code for the system of Example 1

General expressions for kinetic and potential energy

The kinetic energy of a rigid object is the sum of two terms:

1. the translational energy obtained by concentrating the entire mass of the object at the center of mass, and
2. the rotational kinetic energy of the body about the center of mass.

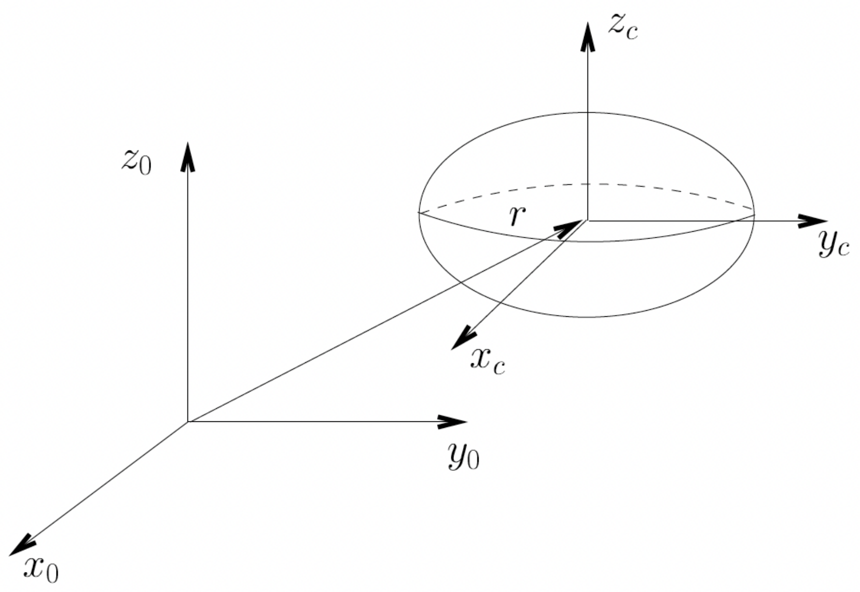

Fig. 6.4 A General Rigid Body

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{I} \omega$$

Inertia tensor expressed in the inertia frame

Similarity transformation for the inertia tensor:

$$\mathcal{I} = R I R^T$$

Inertia tensor expressed in the body frame.

The inertia matrix expressed in the body attached frame is a constant matrix independent of the motion of the object and easily computed.

The inertia tensor

Let the mass density of the object be represented as a function of position $\rho(x, y, z)$

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$\begin{cases} I_{xx} = \int\int\int (y^2 + z^2)\rho(x, y, z) dx\ dy\ dz & I_{xy} = I_{yx} = -\int\int\int xy\rho(x, y, z) dx\ dy\ dz \\ I_{yy} = \int\int\int (x^2 + z^2)\rho(x, y, z) dx\ dy\ dz & I_{xz} = I_{zx} = -\int\int\int xz\rho(x, y, z) dx\ dy\ dz \\ I_{zz} = \int\int\int (x^2 + y^2)\rho(x, y, z) dx\ dy\ dz & I_{yz} = I_{zy} = -\int\int\int yz\rho(x, y, z) dx\ dy\ dz \end{cases}$$

Principal moments of inertia

Cross products of inertia

The integrals in the above expression are computed over the region of space occupied by the rigid body.

Example

Consider the rectangular solid of length, a, width, b, and height, c, shown in the figure and suppose that the density is constant

$$\rho(x, y, z) = \rho$$

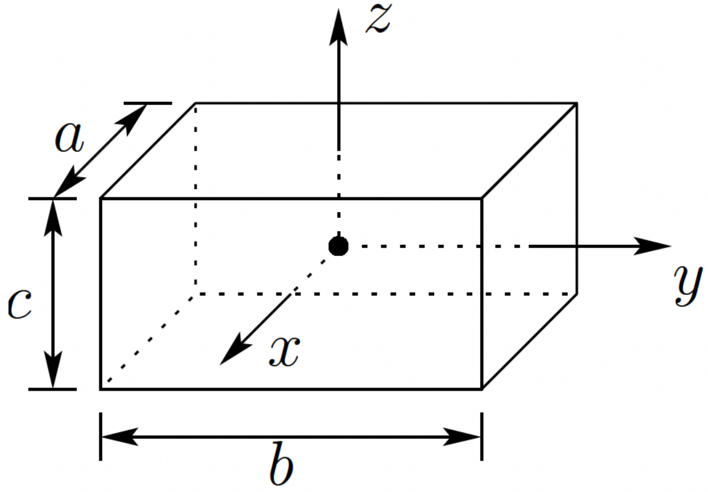

Fig. 6.5 Uniform Rectangular Solid

$$I_{xx} = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2+z^2)\rho(x,y,z)dx\ dy\ dz = \rho\frac{abc}{12}(b^2+c^2)$$

Likewise

$$I_{yy} = \rho\frac{abc}{12}(a^2+c^2) \quad ; \quad I_{zz} = \rho\frac{abc}{12}(a^2+b^2)$$

and the cross products of inertia are zero.

$$\mathcal{I} = RIR^T$$

Kinetic energy of an n-link robot

Now consider a manipulator consisting of n links with Jacobians: $v_i = J_{v_i}(q)\dot{q}, \qquad \omega_i = J_{\omega_i}(q)\dot{q}$

Suppose the mass of link i is m_i and that the inertia matrix of link i, evaluated around a coordinate frame parallel to frame i but whose origin is at the center of mass, equals I_i.

From $\mathcal{K} = \frac{1}{2}mv^Tv + \frac{1}{2}\omega^T\mathcal{I}\omega$ it follows that

$$K = \frac{1}{2}\dot{q}^T \sum_{i=1}^{n} \underbrace{\left[m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q)\right]}_{\text{Matrix } n\times n} \dot{q}$$

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

inertia matrix

$D(q)$ is a symmetric positive definite matrix that is in general configuration dependent.

Potential energy of an n-link robot

In the case of rigid dynamics, the only source of potential energy is gravity. The potential energy of the i-th link can be computed by assuming that the mass of the entire object is concentrated at its center of mass and is given by

$$P_i = g^T r_{ci} m_i$$

where g is vector giving the direction of gravity in the inertial frame and the vector r_{ci} gives the coordinates of the center of mass of link i. The total potential energy is:

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} g^T r_{ci} m_i$$