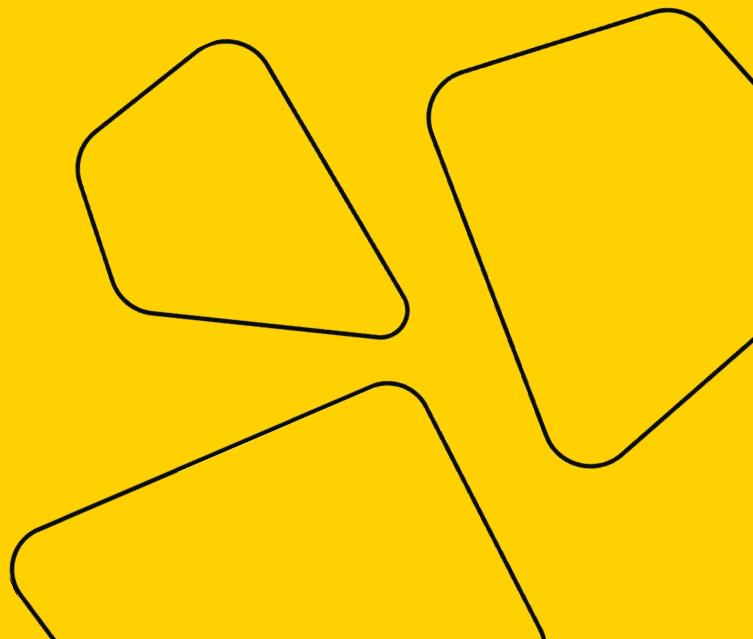


# Unsupervised Learning

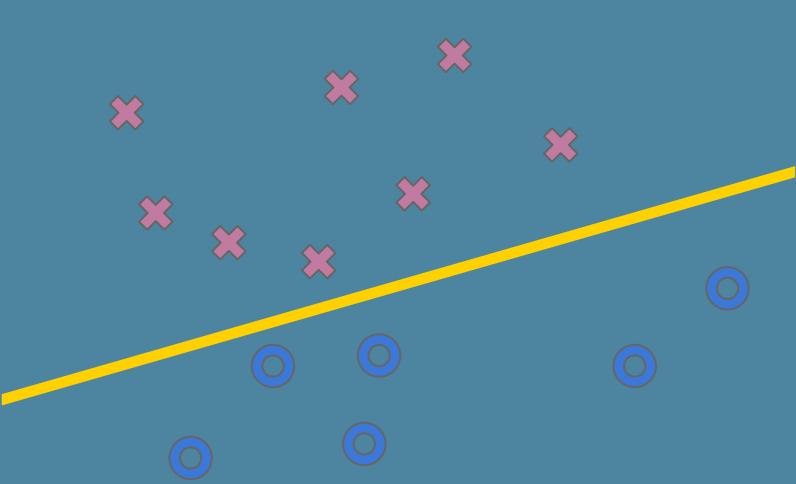
Vladislav Goncharenko  
MIPT, 2021



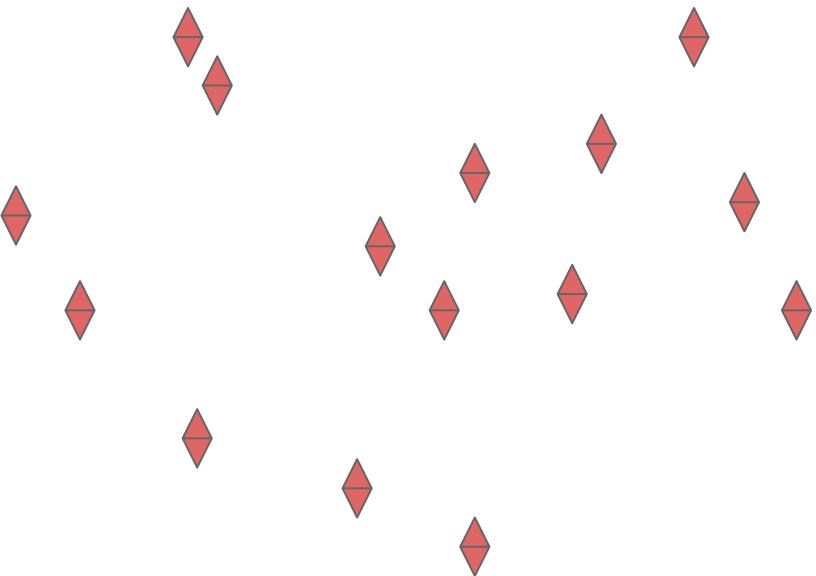
# Outline

- Geometrical machine learning
  - Dimensionality curse
  - Manifold assumption
- Dimensionality reduction
  - Feature selection
  - Multidimensional Scaling (MDS)
  - Isomap
  - Locally linear embedding (LLE)
  - t-SNE
- Clustering
  - k-means
  - DBSCAN
  - Hierarchical clustering
  - metrics
- Density estimation
  - Kernel density estimation

# Supervised learning



# Unsupervised learning



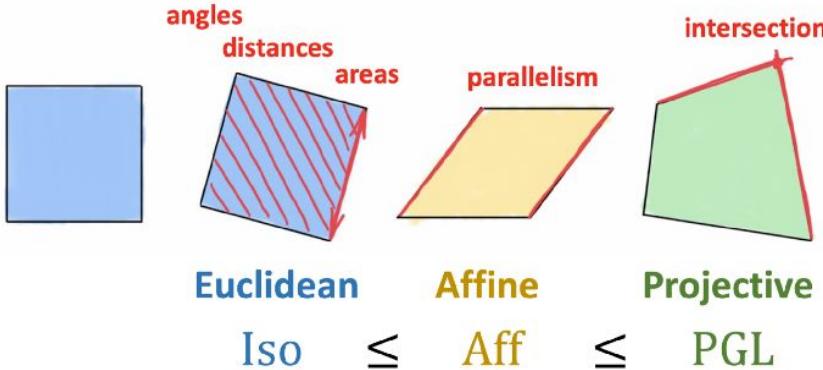
# Geometrical machine learning

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girafe  
ai

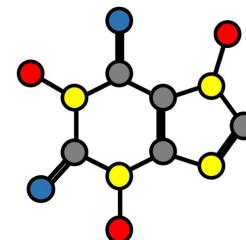
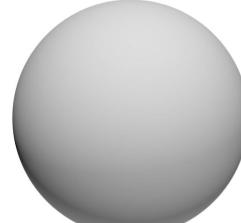
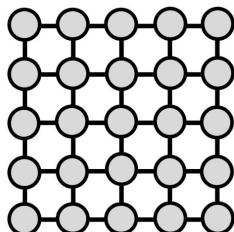
01

# Geometrical machine learning



The breakthrough insight of Klein was to approach the definition of geometry as the **study of invariants**, or in other words, structures that are preserved under a certain type of transformations (symmetries)

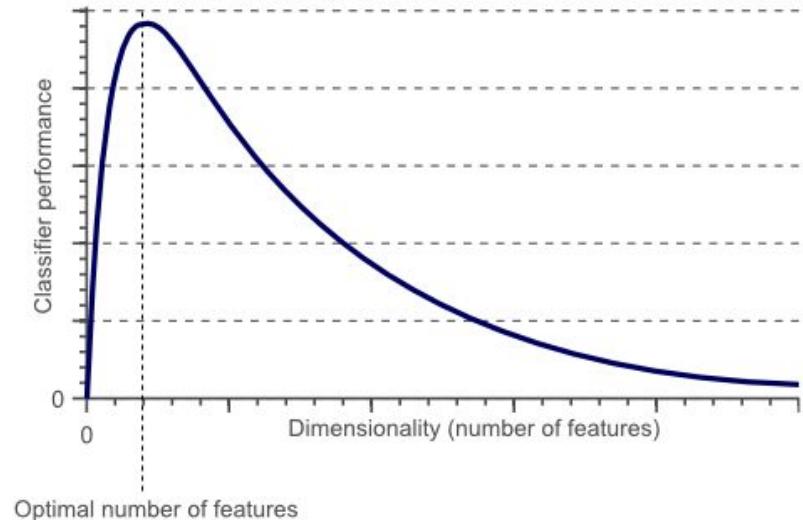
[Article introducing a book on Geometric Deep Learning](#)



# Dimensionality curse

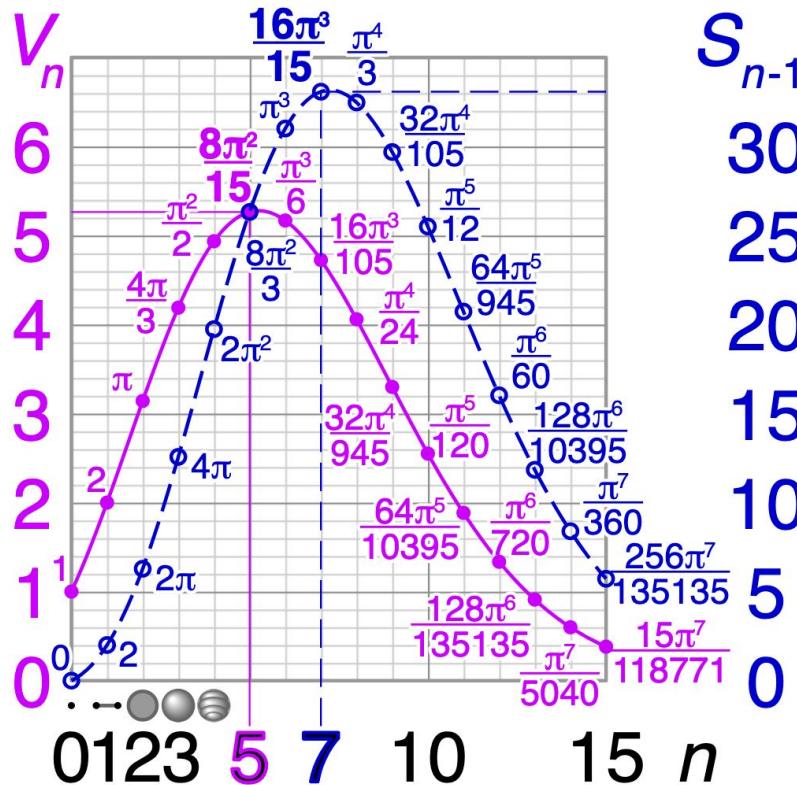


Certain behaviours or effects that appear when analysing data in high dimensions, that do not occur in low-dimensional spaces





# Sphere volume decrease

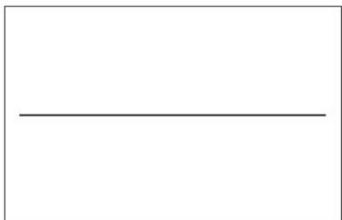


[image source](#)

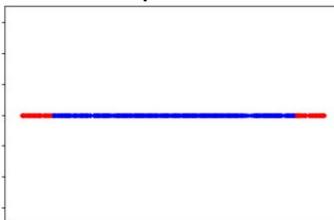
# Distance in high dimensional space



Line of length 1



Line with 500 randomly generated points



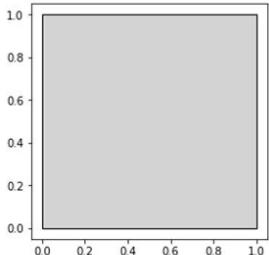
Inside points

Points that fall within 10% of the distance to the edges

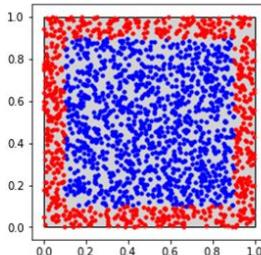
Ratio of inside points to total points = 80%

Average Distance between 2 points = 0.34

Square of side 1



2000 randomly generated points



Inside points

Points that fall within 10% of the distance to the edges

Ratio of inside points to total points = 63%

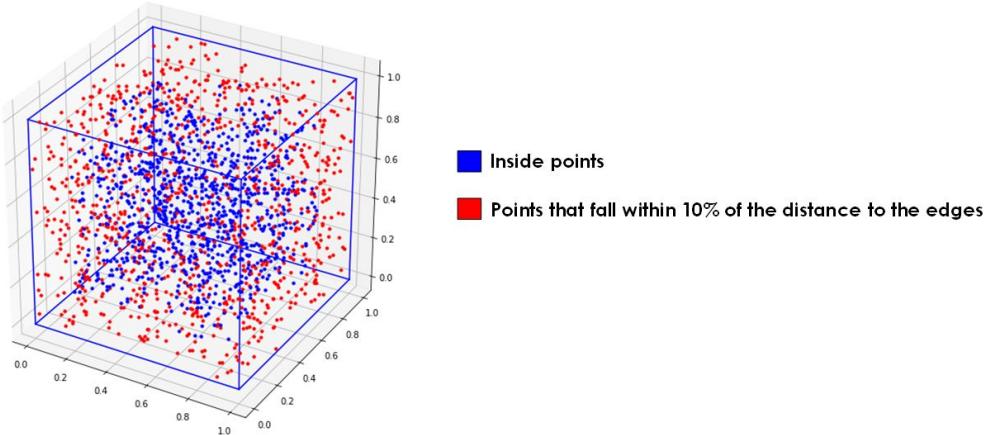
Average Distance between 2 points = 0.52

[image source](#)

# Distance in high dimensional space



Cube of side 1



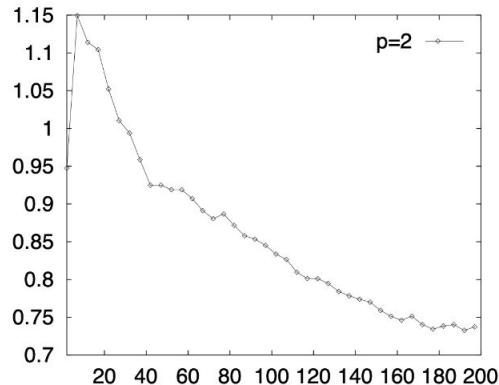
Ratio of inside points to total points = 51%  
Average Distance between 2 points = 0.65

Nº of dimensions	% Outside Points	Average distance (A,B)
1	20%	0.34
2	37%	0.52
3	49%	0.65

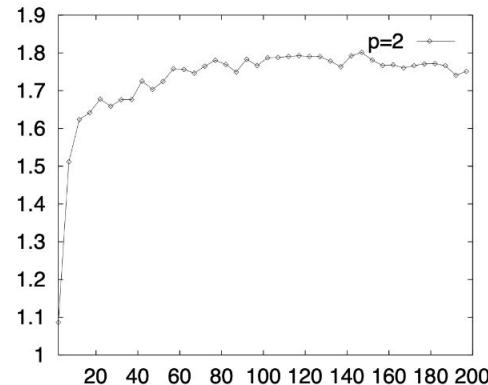
# Distance relative contrast



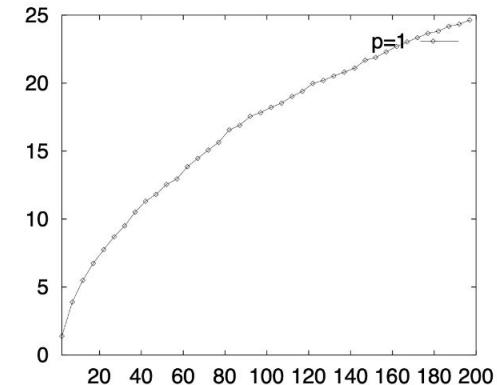
Take random points uniformly distributed in D dimensional cube and calculate distance to the farthest point and to the closest point. Plot their difference depending on D for different Minkowski metrics



L3



L2



L1

On the Surprising Behavior of Distance Metric in High-Dimensional Space,  
Aggarwal et al., 2002

# Conclusions



- Distance loses its meaning - closest and farthest points are equally far
- Proximity concept becomes ill defined
- Lower powers of Minkowski metrics are more sustainable to dimensionality curse

Nº of dimensions	% Outside Points	Average distance (A,B)
1	20%	0.34
2	37%	0.52
3	49%	0.65

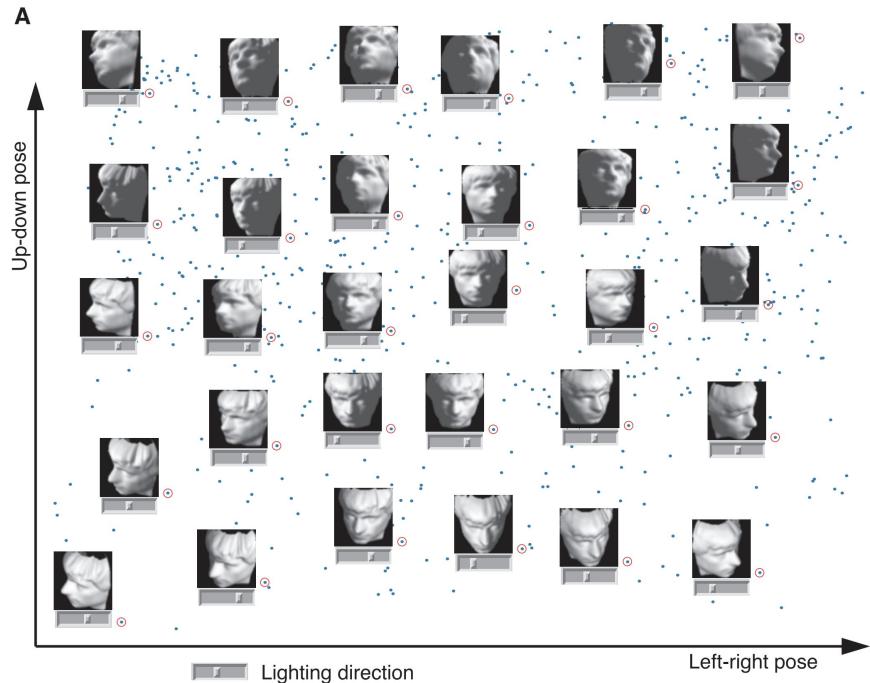
# Manifold assumption



The data lie approximately on a surface (called manifold) of usually much lower dimension than the input space

So problem dimensionality could be (non-)linearly reduced or other tasks solved

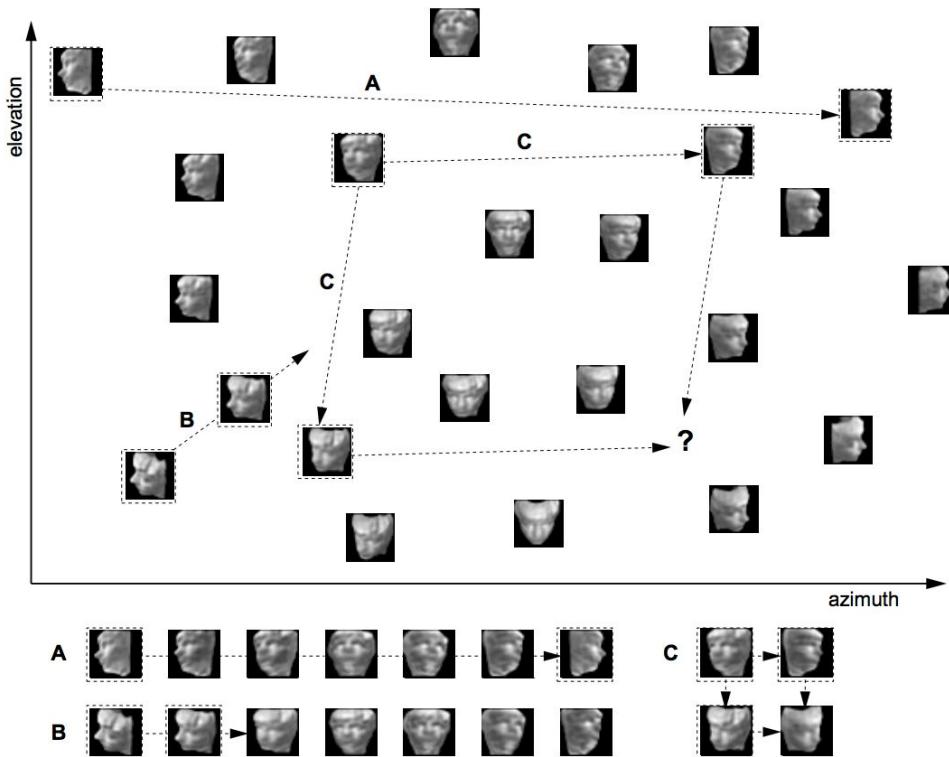
Sometimes dimensionality of manifold is referred as [intrinsic dimension](#) (see [this article](#))



[Tenenbaum, de Silva, Langford](#)  
A Global Geometric Framework for Nonlinear Dimensionality Reduction



# Latent space



Latent (embedding) space describes data in coordinates more relevant to humans' reason and often allows useful linear operations:

- Interpolation (A)
- Extrapolation (B)
- Analogy (C)

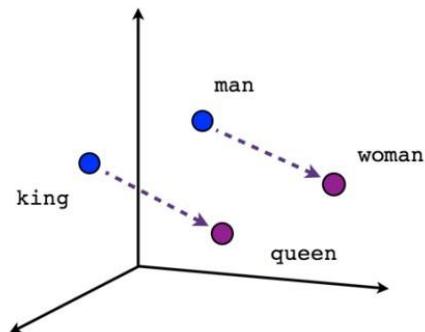
This process is also called embedding space 'walking'



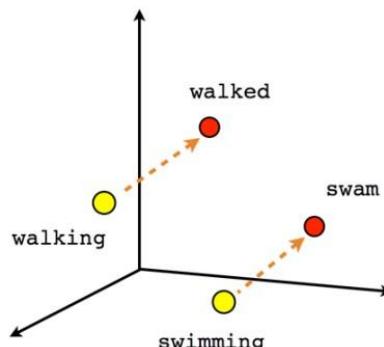
# Latent space example

Word2vec is a method to embed words from text corpus into linear space

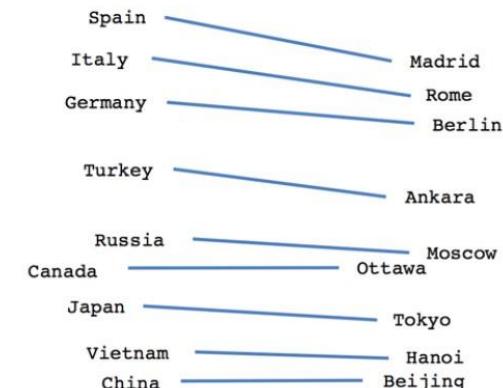
Read more: [manifold assumption](#), [assessing assumption](#)



Male-Female



Verb tense



Country-Capital

# Dimensionality reduction

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girafe  
ai

02

# Поставить задачу!



# Feature selection

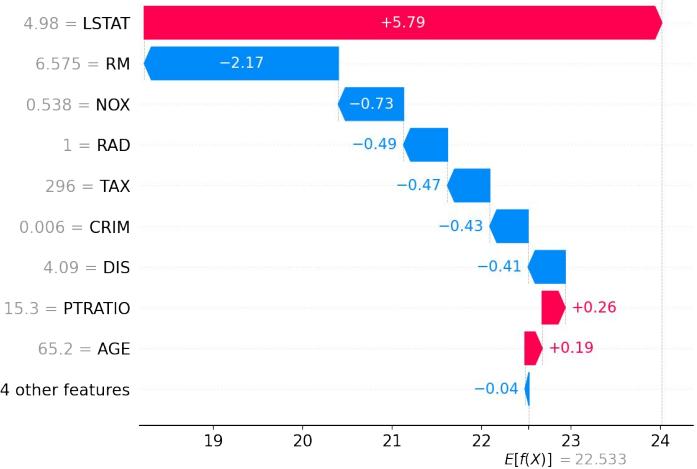
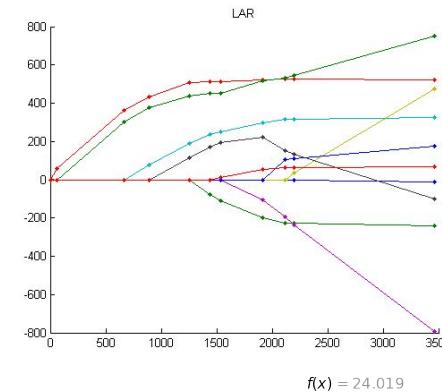


Select subset of existing features to use in further modelling

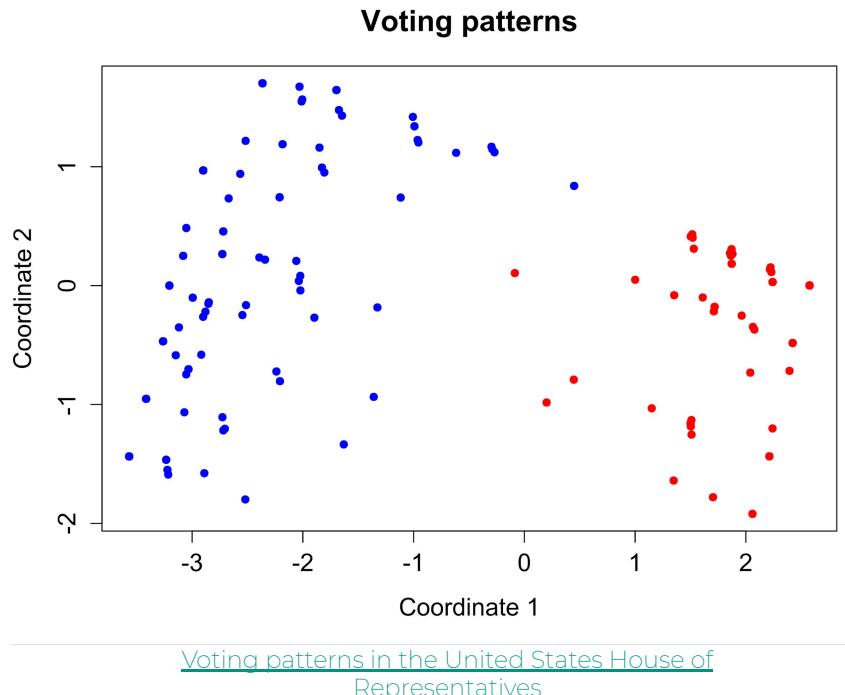
Usually is based on some supervised method

Examples:

- stepwise regression
- LARS
- SHAP values
- etc...



# Multidimensional Scaling (MDS)



Goal:

Linearly embed to given lower space

Solution:

PCA

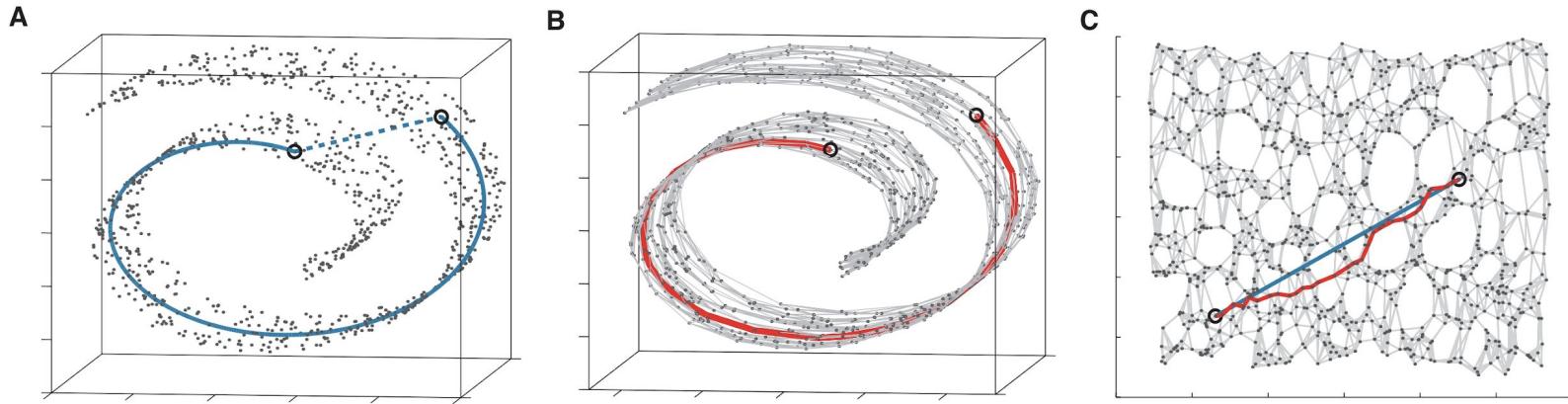
$$L = \|D_x - D_y\|_2 \rightarrow \min_{y = Ax}$$

$$y = \Lambda^{1/2} V^T$$

Params: p - target dimensionality

Also could be non-linear

# Isomap



Now make distancies geodesic!

And measure distances on the produced  
graph

Params:

n - number of neighbours to connect

p - dimensionality of manifold

[A Global Geometric Framework for Nonlinear Dimensionality Reduction, Tenenbaum et al., Science, 2002.](#)



# Isomap algorithm

## Step

1 Construct neighborhood graph

Define the graph  $G$  over all data points by connecting points  $i$  and  $j$  if [as measured by  $d_x(i,j)$ ] they are closer than  $\epsilon$  ( $\epsilon$ -Isomap), or if  $i$  is one of the  $K$  nearest neighbors of  $j$  ( $K$ -Isomap). Set edge lengths equal to  $d_x(i,j)$ .

2 Compute shortest paths

Initialize  $d_G(i,j) = d_x(i,j)$  if  $i,j$  are linked by an edge;  $d_G(i,j) = \infty$  otherwise. Then for each value of  $k = 1, 2, \dots, N$  in turn, replace all entries  $d_G(i,j)$  by  $\min\{d_G(i,j), d_G(i,k) + d_G(k,j)\}$ . The matrix of final values  $D_G = \{d_G(i,j)\}$  will contain the shortest path distances between all pairs of points in  $G$  (16, 19).

3 Construct  $d$ -dimensional embedding

Let  $\lambda_p$  be the  $p$ -th eigenvalue (in decreasing order) of the matrix  $\tau(D_G)$  (17), and  $v_p^i$  be the  $i$ -th component of the  $p$ -th eigenvector. Then set the  $p$ -th component of the  $d$ -dimensional coordinate vector  $\mathbf{y}_i$  equal to  $\sqrt{\lambda_p} v_p^i$ .

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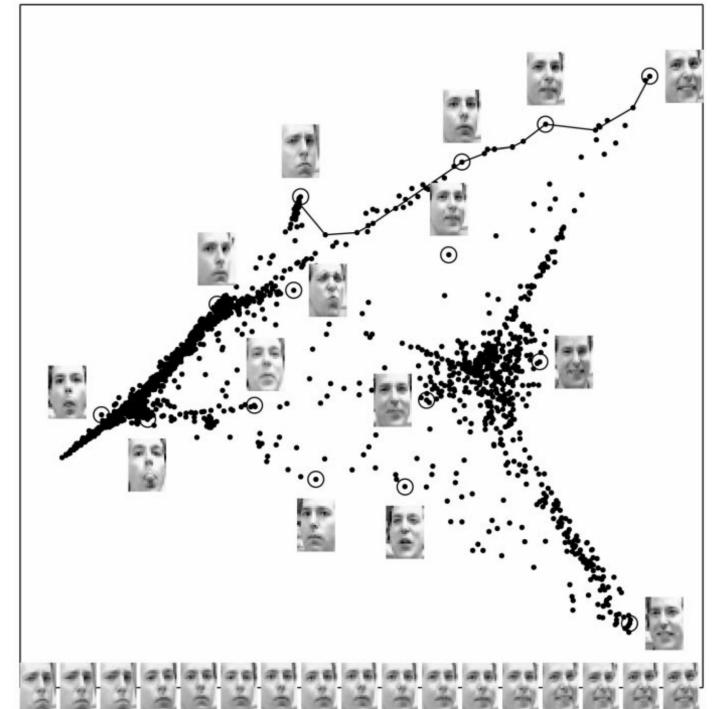
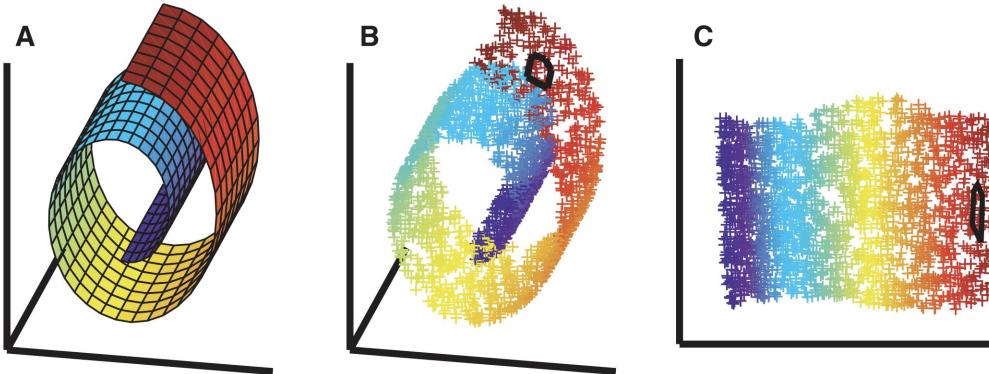
17. The operator  $\tau$  is defined by  $\tau(D) = -HSH/2$ , where  $S$  is the matrix of squared distances  $\{S_{ij} = D_{ij}^2\}$ , and  $H$  is the "centering matrix"  $\{H_{ij} = \delta_{ij} - 1/N\}$  (13).

# Locally linear embedding (LLE)



Smooth manifold locally approximated with hyperplane. Linear pieces are stitched together.

Nonlinear Dimensionality Reduction by Locally Linear Embedding, Roweis et al., Science, 2000



# LLE algorithm



1. estimate point by its K neighbours

$$\varepsilon(W) = \sum_{i=1}^n \left\| x_i - \sum_{j=1}^K W_{ij} x_j \right\|^2$$

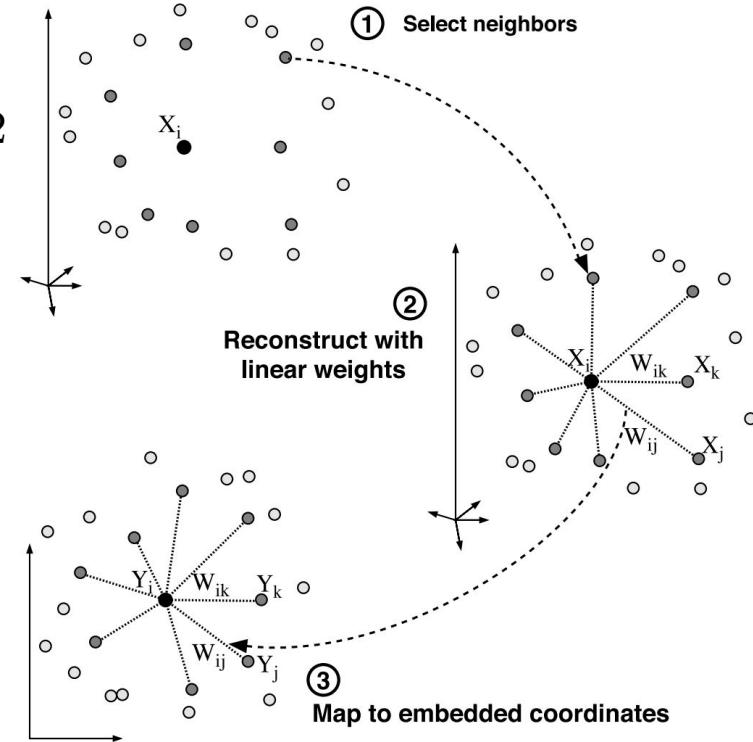
2. Estimate new points based on known relations

$$\Phi(Y) = \sum_{i=1}^n \left\| y_i - \sum_{j=1}^n W_{ij} y_j \right\|^2$$

Params:

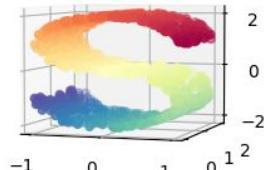
n - number of neighbours to connect

p - dimensionality of manifold

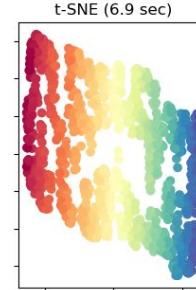
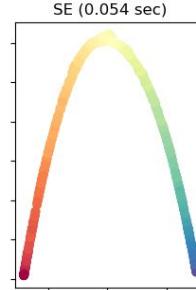
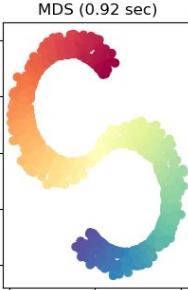
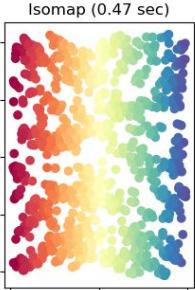
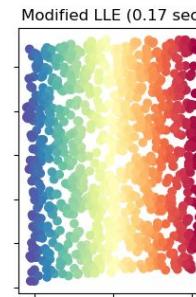
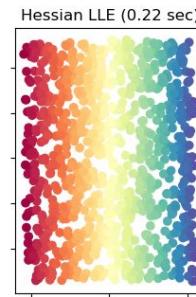
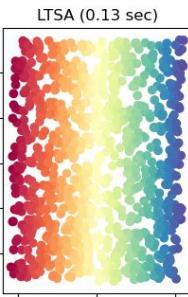
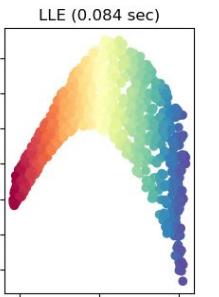




# Many more



Manifold Learning with 1000 points, 10 neighbors



- Hessian Eigenmapping
- Spectral Embedding
- Local Tangent Space Alignment
- Riemannian Geometry
- UMAP
- .....

Read more:

[sklearn manifold methods](#)

[sklearn signals decomposition](#)

# t-SNE



## t-distributed Stochastic Neighbor Embedding

SNE

Stochastic Neighbor Embedding, Hinton et al., NIPS, 2002

# Stochastic Neighbor Embedding



Convert pairwise distances to probabilities, preserve probabilities through the spaces

$$p_{j|i} = \frac{\exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma_i^2}\right)}{\sum_{k \neq i} \exp\left(-\frac{\|x_i - x_k\|^2}{2\sigma_i^2}\right)}$$

asymmetric probability  
of object  $i$  chooses  $j$  as its neighbour

$$q_{j|i} = \frac{\exp(-\|y_i - y_j\|^2)}{\sum_{k \neq i} \exp(-\|y_i - y_k\|^2)}$$

the same in target space

Let's construct embedding s.t. these distributions are close.

*What are close distributions?*

# Kullback–Leibler divergence



$$D_{KL}(P \parallel Q) = \sum_{i,j} p_{j|i} \log \frac{p_{j|i}}{q_{j|i}}$$



Suspiciously similar to Shannon entropy

[Learn more](#)



# SNE problem

$$p_{j|i} = \frac{\exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma_i^2}\right)}{\sum_{k \neq i} \exp\left(-\frac{\|x_i - x_k\|^2}{2\sigma_i^2}\right)}$$

$$q_{j|i} = \frac{\exp(-\|y_i - y_j\|^2)}{\sum_{k \neq i} \exp(-\|y_i - y_k\|^2)}$$

$$D_{KL}(P \parallel Q) \rightarrow \min_Y$$

# t-distributed SNE



Patches over SNE:

1. choose common variance
2. make distributions symmetric

$$p_{ij} = \frac{\exp(-\|x_i - x_j\|^2/2\sigma^2)}{\sum_{k \neq l} \exp(-\|x_k - x_l\|^2/2\sigma^2)}$$

$$p_{ij}^s = \frac{p_{ij} + p_{ji}}{2N}$$

Visualizing Data using t-SNE,  
Maaten, Hinton, 2008, JMLR

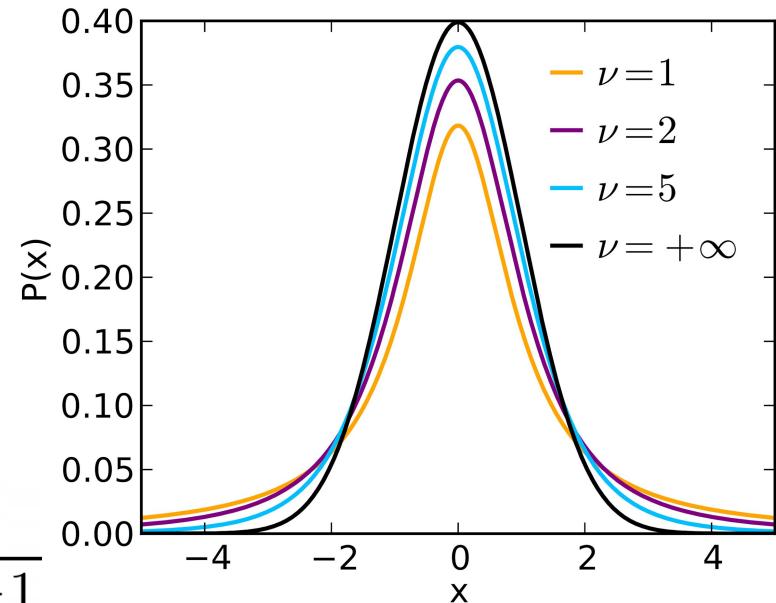
# t-distributed SNE



Patches over SNE:

1. choose common variance
2. make distributions symmetric
3. make it decrease faster than Gaussian  
(use [Student's t-distribution](#))

$$q_{ij} = \frac{(1 + \|y_i - y_j\|^2)^{-1}}{\sum_{k \neq l} (1 + \|y_k - y_l\|^2)^{-1}}$$





# t-SNE problem

$$p_{ij} = \frac{\exp(-\|x_i - x_j\|^2 / 2\sigma^2)}{\sum_{k \neq l} \exp(-\|x_k - x_l\|^2 / 2\sigma^2)} \quad p_{ij}^s = \frac{p_{ij} + p_{ji}}{2N}$$

$$q_{ij} = \frac{(1 + \|y_i - y_j\|^2)^{-1}}{\sum_{k \neq l} (1 + \|y_k - y_l\|^2)^{-1}}$$

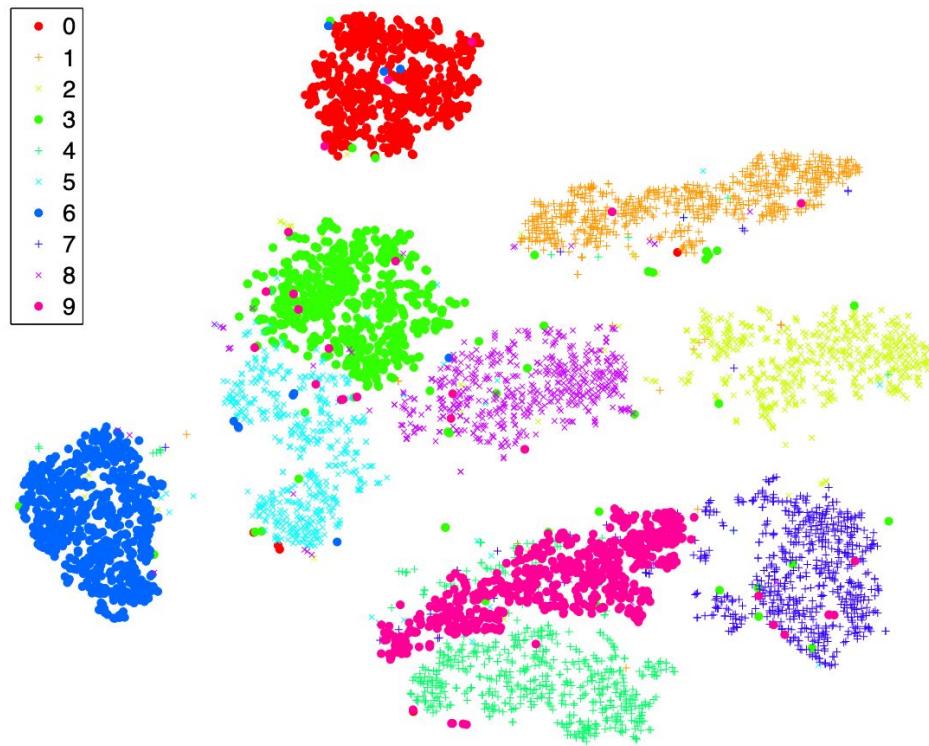
$$D_{KL}(P \parallel Q) \rightarrow \min_Y$$

# Result: nice and light visualizations

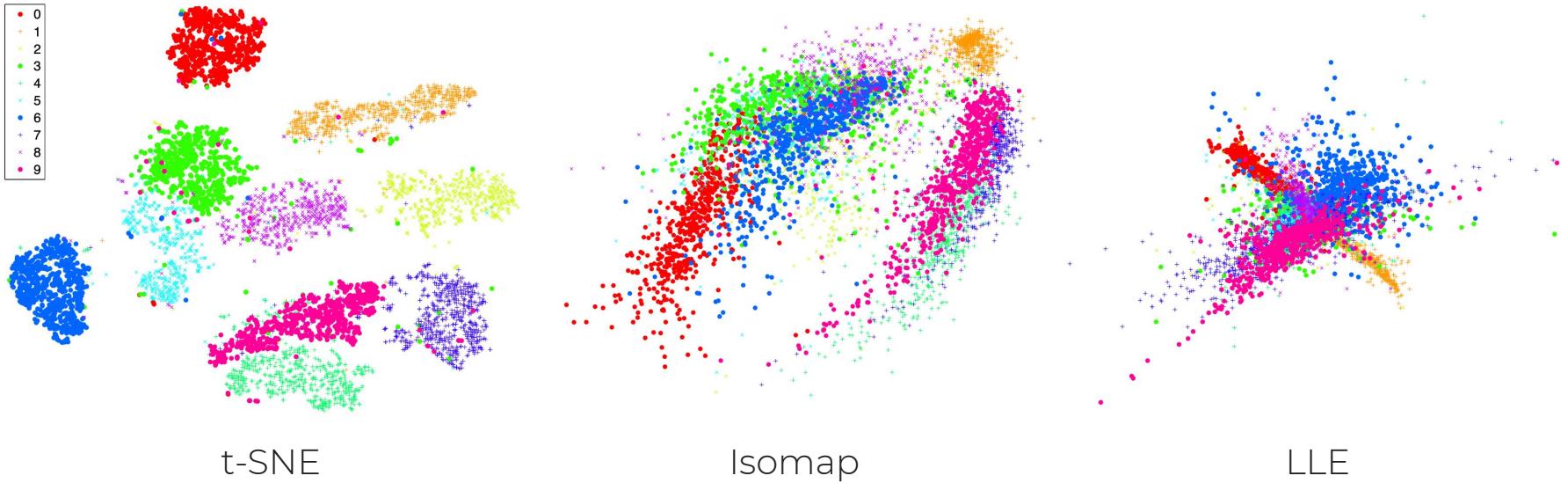


t-SNE on [MNIST dataset](#)

0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9	9	9	9	9	9



# Comparison



# Real life example



MDS (PCA) on faces  
embeddings



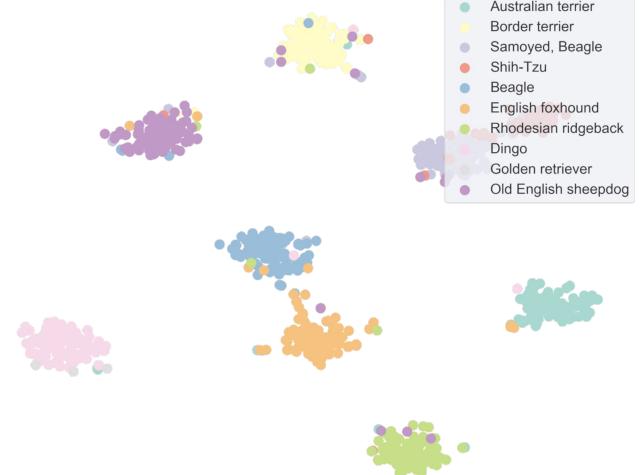
[image source](#)

# Real life example



t-SNE on ArcFace

t-SNE on CosFace



# Clustering

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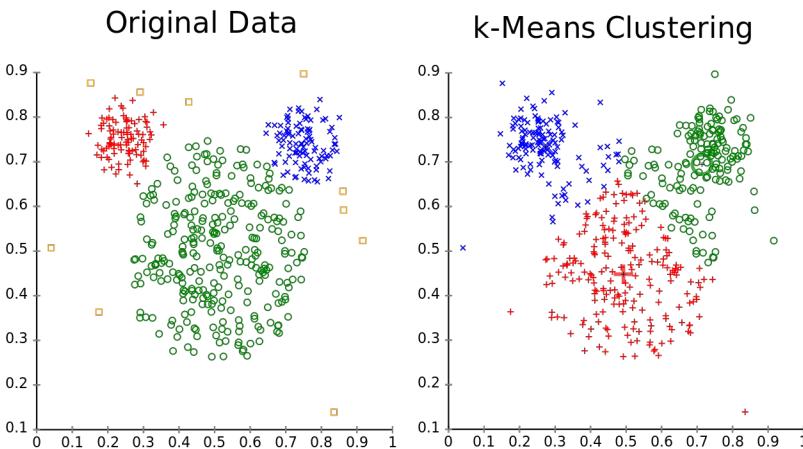
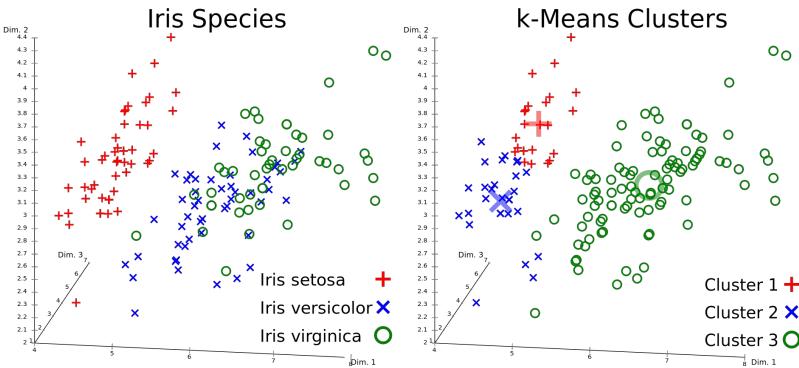
girafe  
ai

03

# Поставить задачу!



# k-means



Based on K nearest neighbours  
algorithm

1. Init clusters centers randomly
2. Define current cluster of an object as a nearest center
3. Calculate new cluster center as a mean of all objects in cluster
4. Repeat from p. 2 until convergence

# k-means

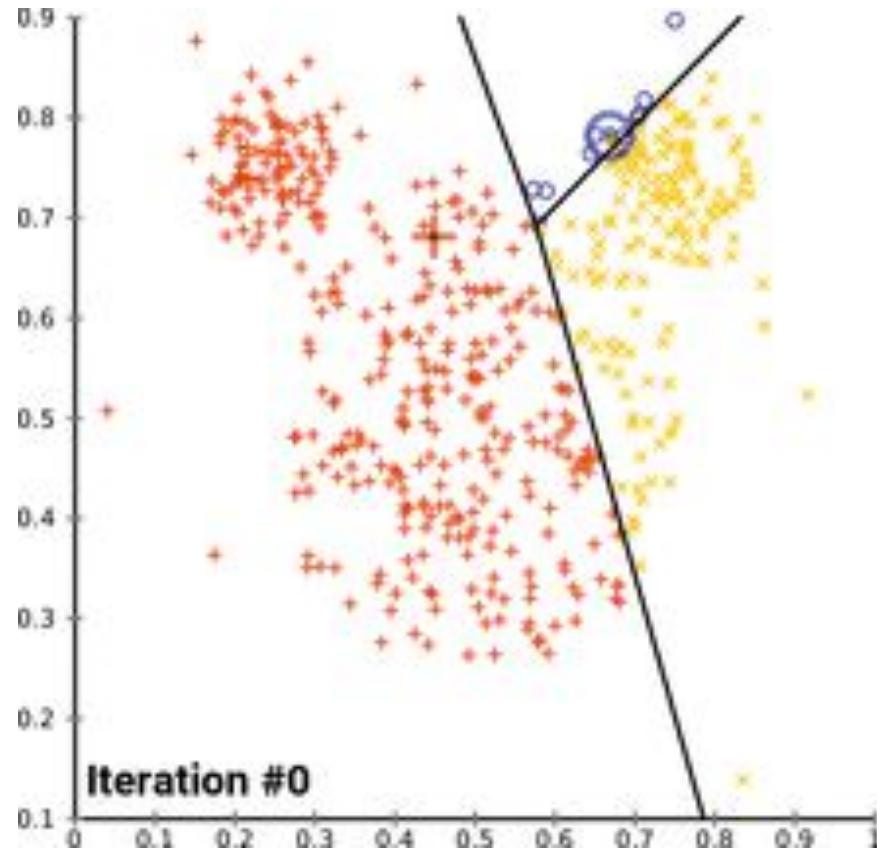


Params:

k - number of clusters

Advanced version: k-means++

The Advantages of Careful Seeding,  
Arthur, Vassilvitskii, 2007, ACM-SIAM  
SODA



# DBSCAN

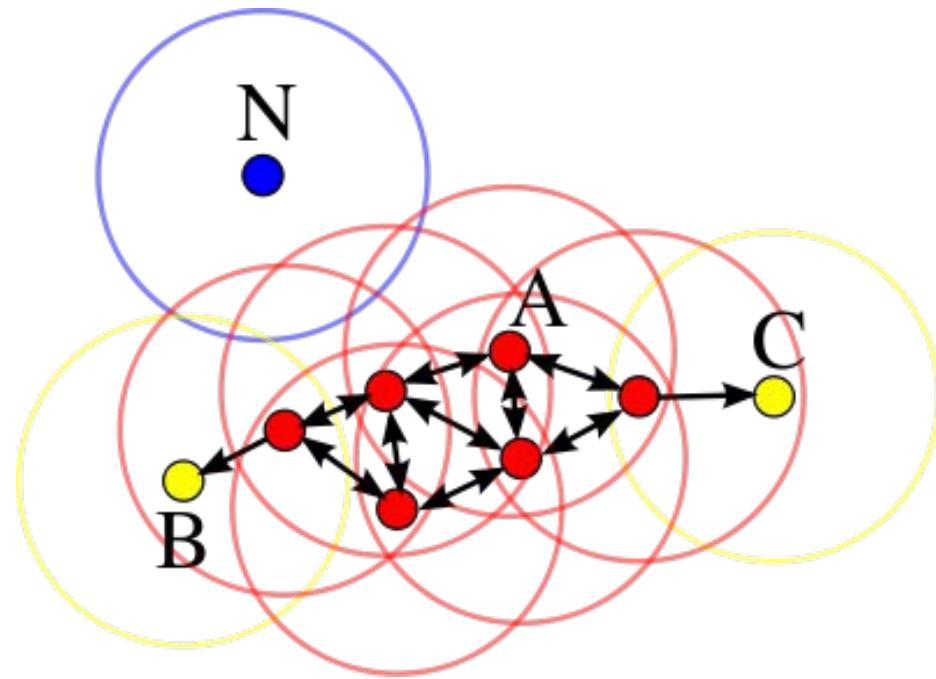


Density-Based Spatial Clustering of Applications with Noise

Split all data points into 3 groups:

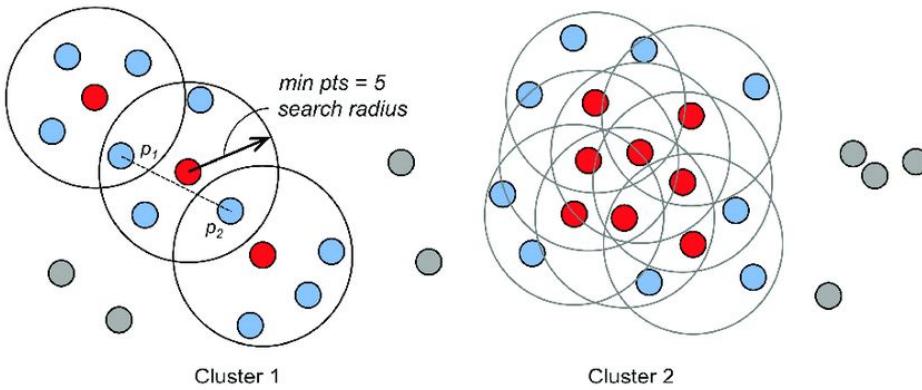
1. Core (red)
2. Border (yellow)
3. Noise (blue)

Core point has at least  $k$  other points in  $\epsilon$ -neighbourhood



A density-based algorithm for discovering clusters in large spatial databases with noise, Ester et al., 1996, KDD-96

# DBSCAN



Any two core or border points in  $\epsilon$ -neighbourhood noted as connected points

Two points both connected to a common point also defined connected (transitivity)

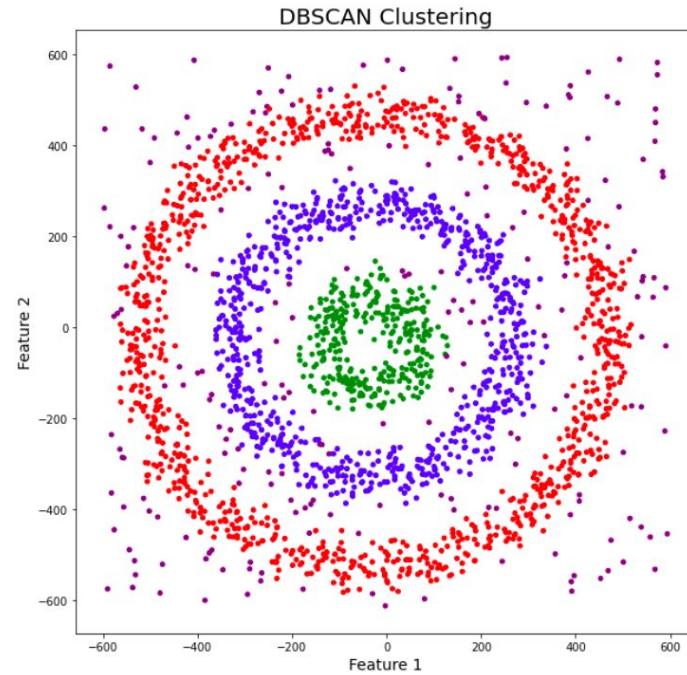
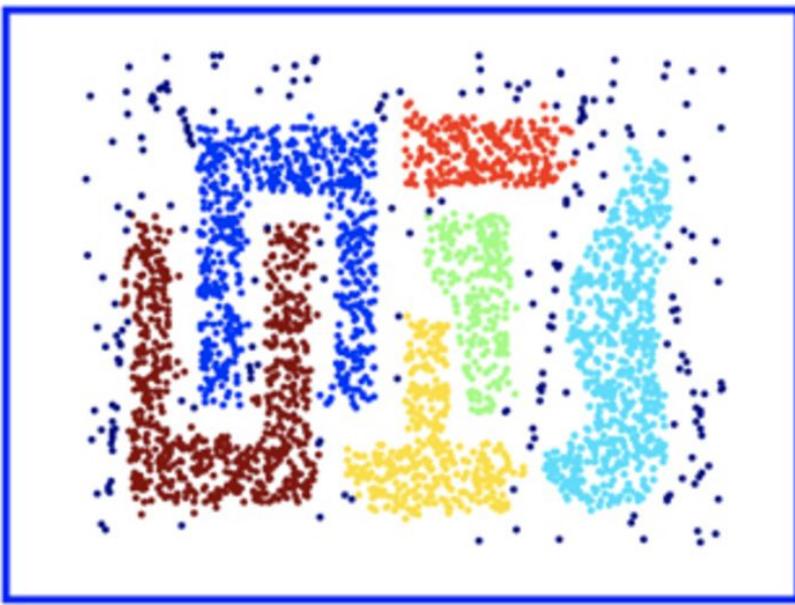
Cluster is defined as maximum connected set of points

Params:

$\epsilon$  - radius of neighbourhood

k - minimal number of neighbours of core point

# DBSCAN examples

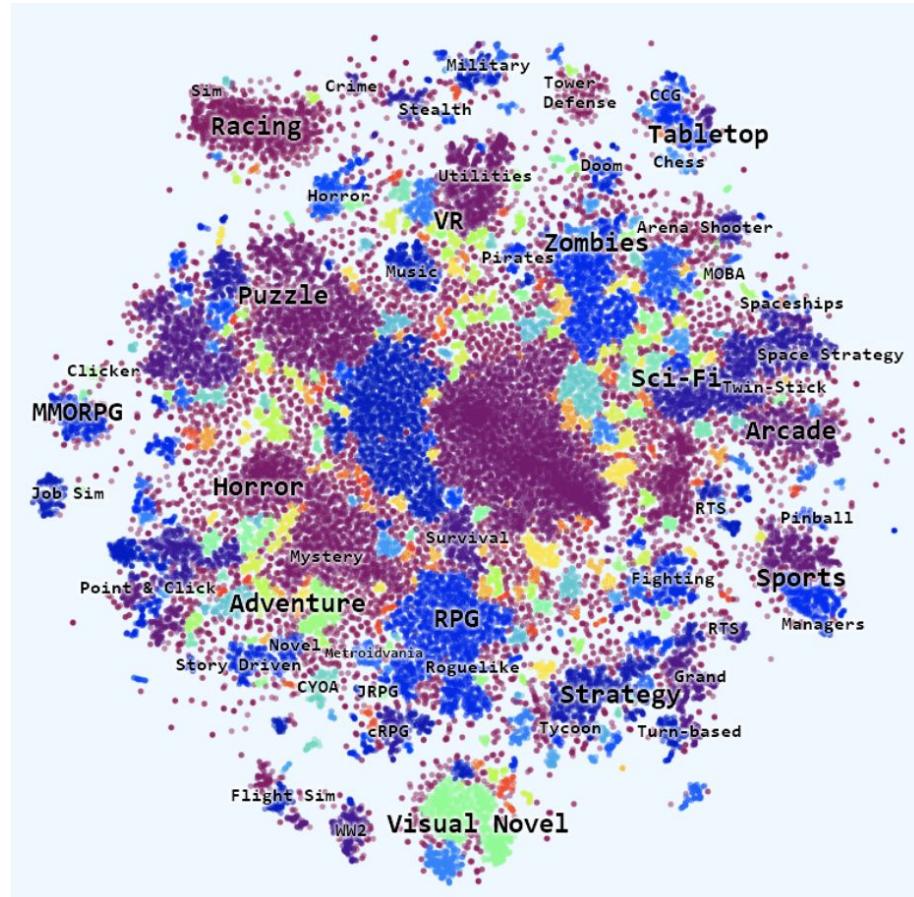


# DBSCAN examples

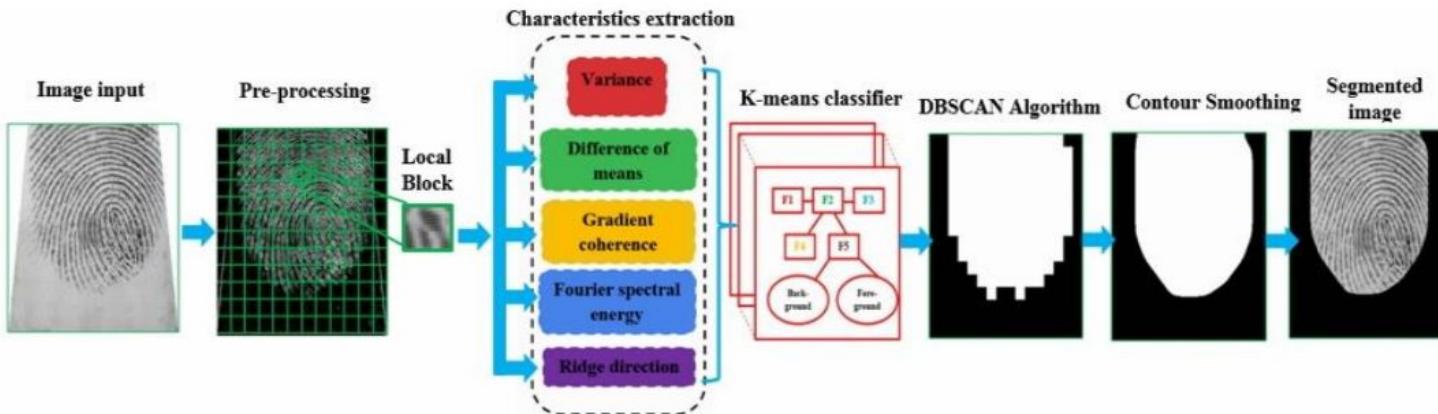


DBSCAN on t-SNE output  
to analyze embeddings (Doc2vec)  
of video games

[image source](#)



# DBSCAN examples



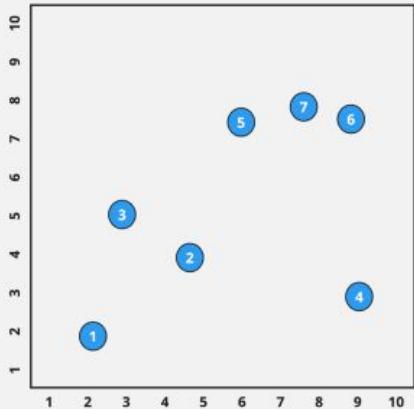
a - original, b - k-means, c - DBSCAN

Improving of Fingerprint Segmentation Images Based on K-MEANS and DBSCAN Clustering, Cherrat et al., 2019, IJECE

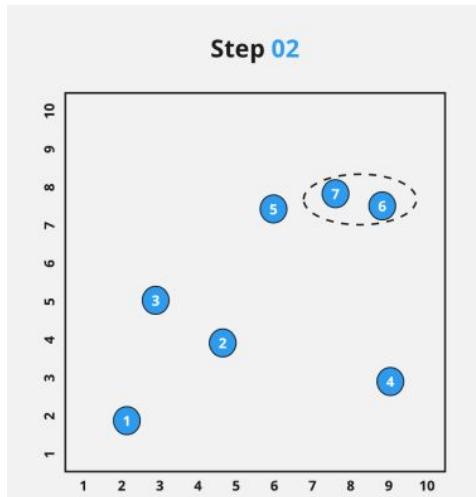
# Hierarchical clustering



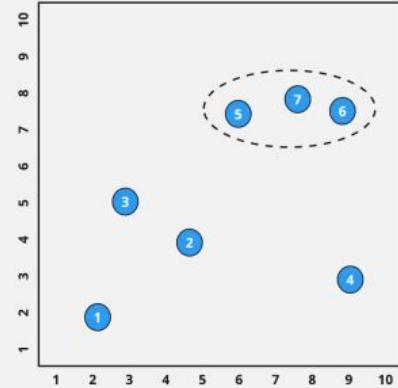
Step 01



Step 02



Step 03

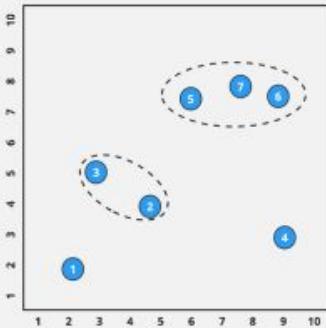


[image source](#)

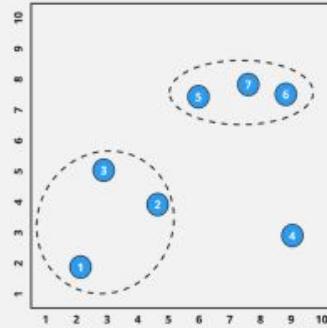
# Hierarchical clustering



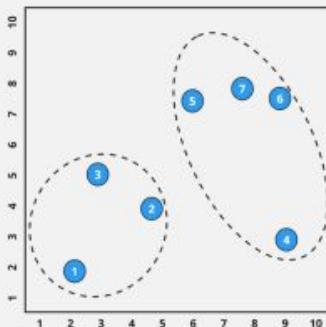
Step 04



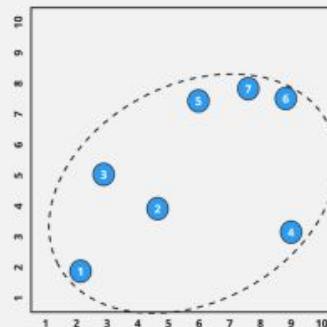
Step 05



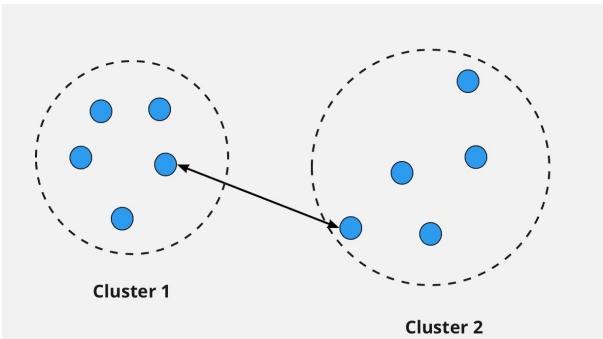
Step 06



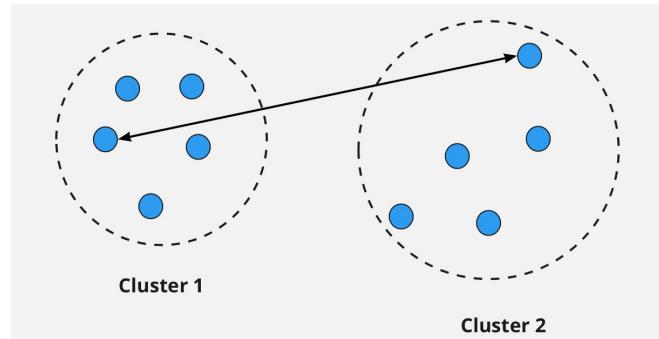
Step 07



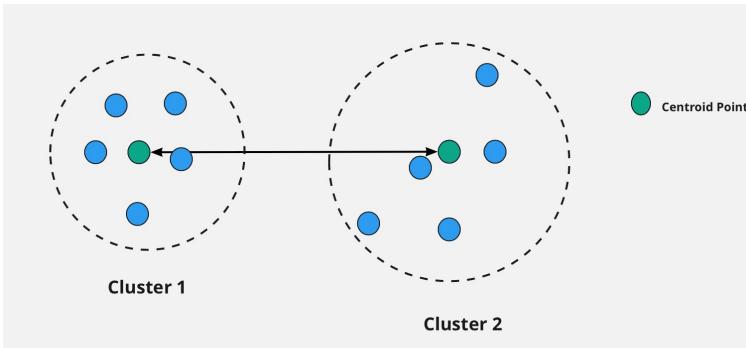
# Distance between clusters



Closest point

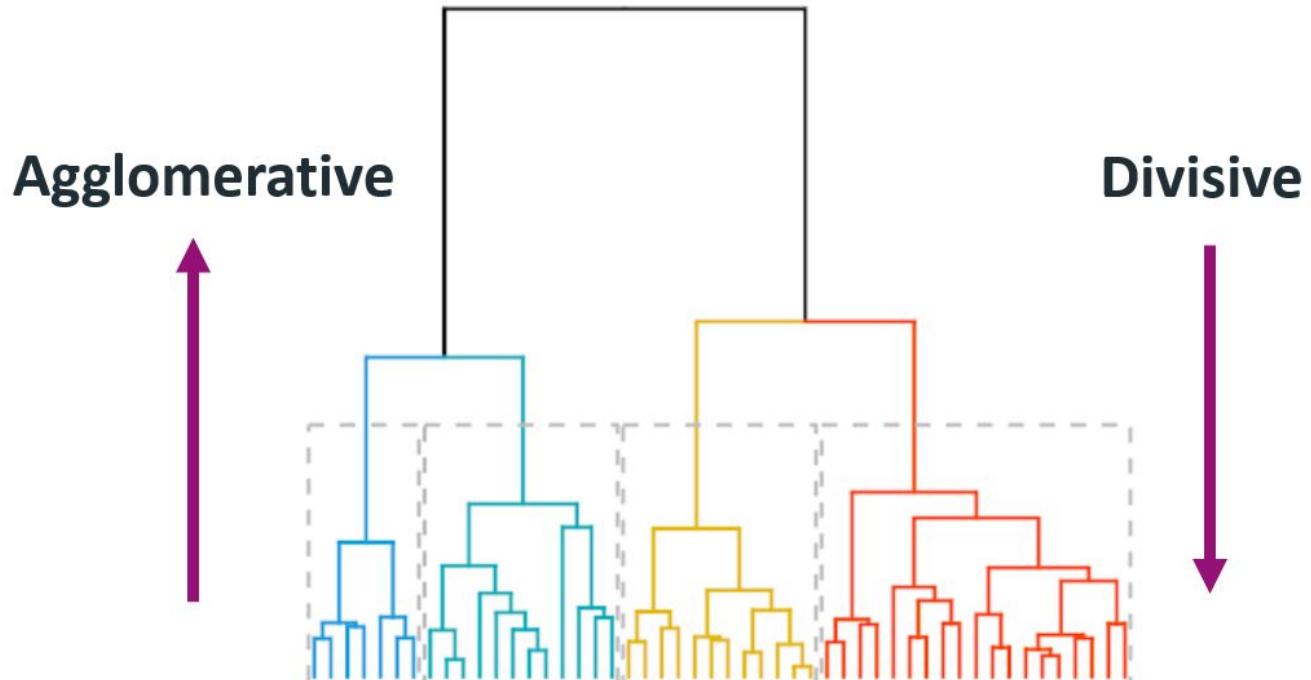


Farthest point



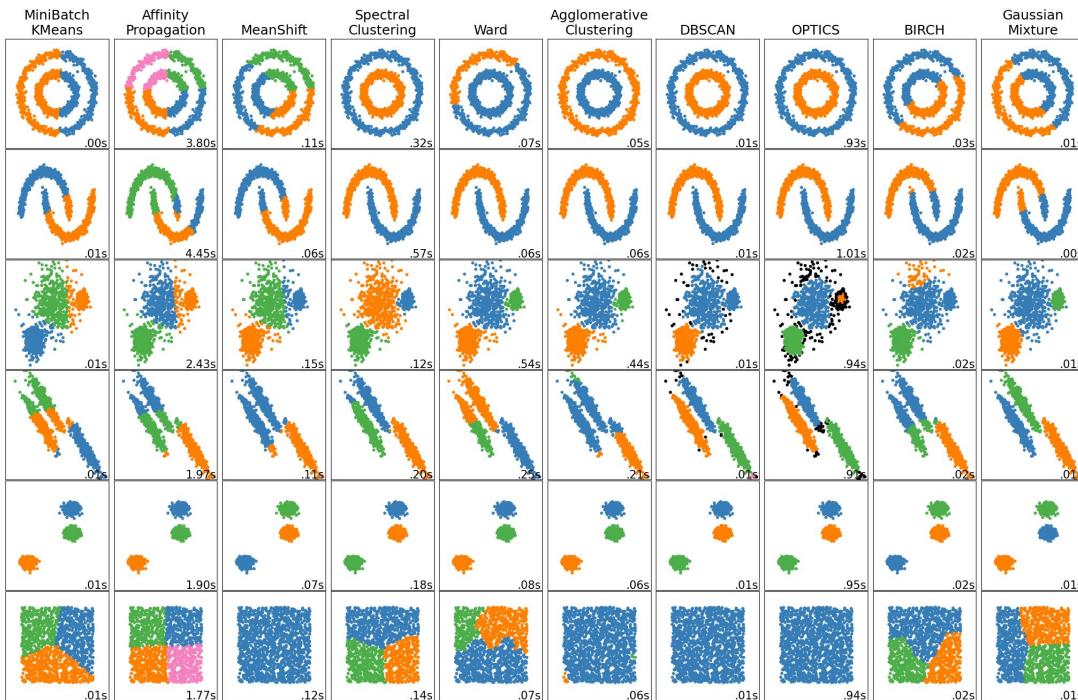
Centroid

# Dendrogram





# Many more



Read more:

[sklearn huge overview](#)

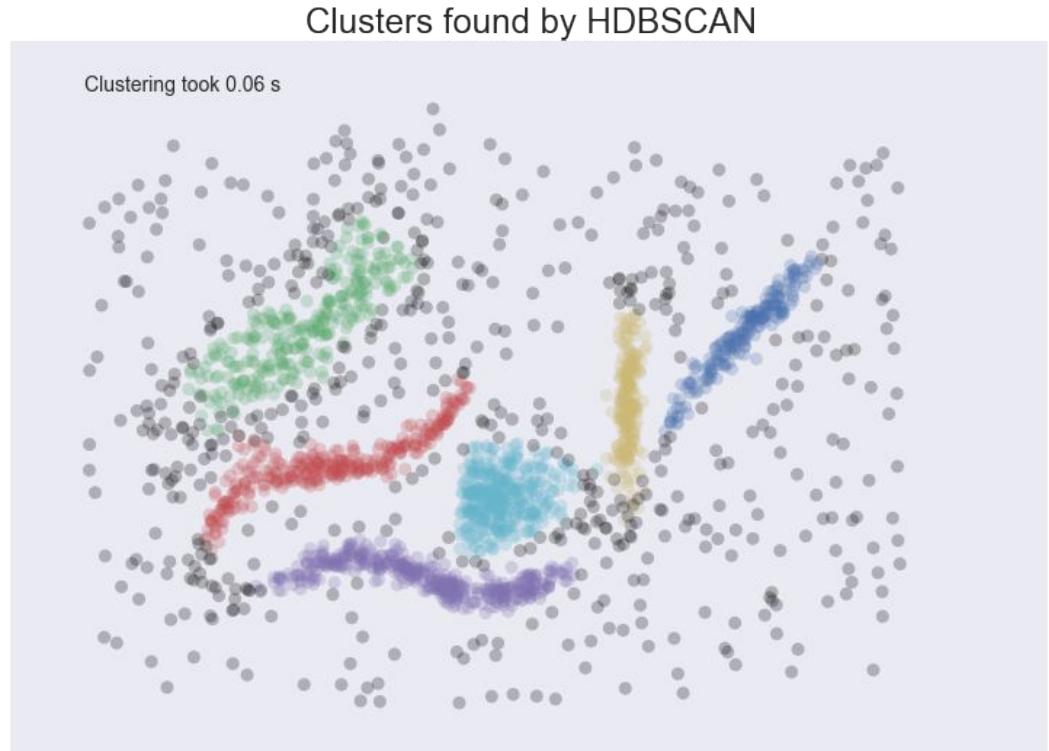
[HDBSCAN \(hierarchy + DBSCAN\)](#)

[timeseries clusterization \(ru\)](#)

# Clustering metrics



- Label based
  - Rand index
  - Mutual Information
  - Homogeneity
  - Completeness
  - V-measure
  - ...
- Label free
  - Silhouette Coefficient
  - Calinski-Harabasz Index
  - Davies-Bouldin Index
  - ...



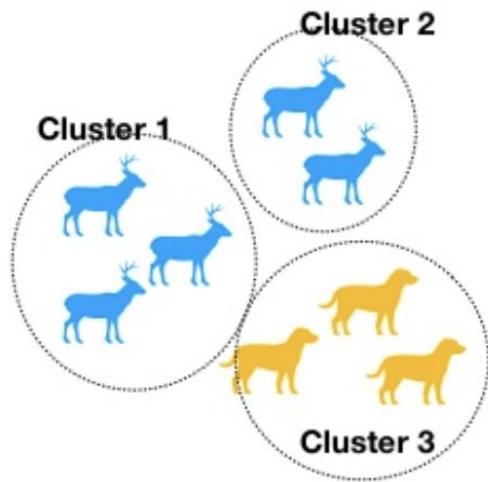
[Detailed explanations \(sklearn docs\)](#)

[image source](#)

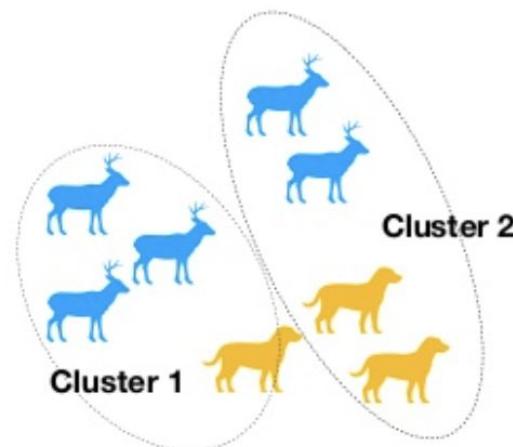


# Homogeneity

Each cluster contains only members of a single class



Good



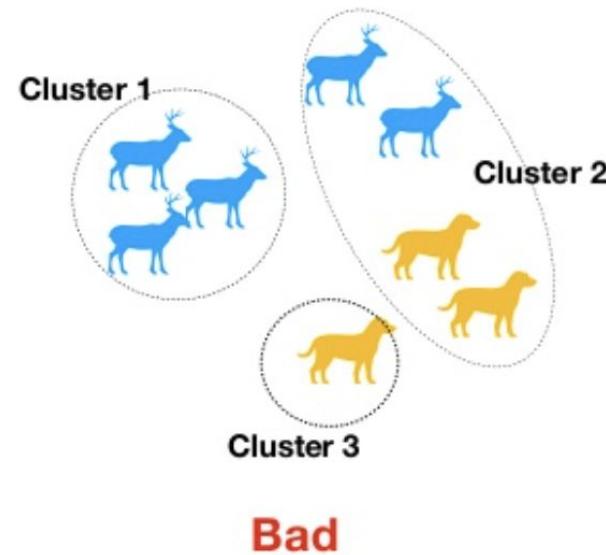
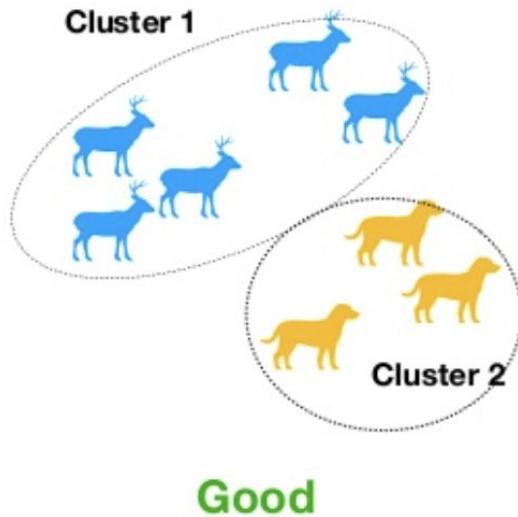
Bad

[image source](#)  
and great slides on topic

# Completeness



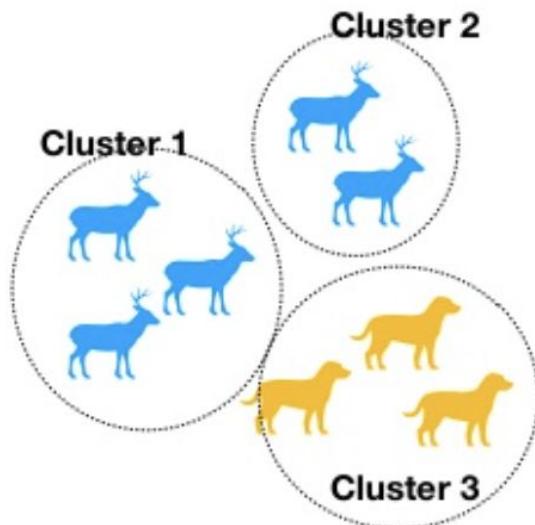
all members of a given class are assigned to the same cluster



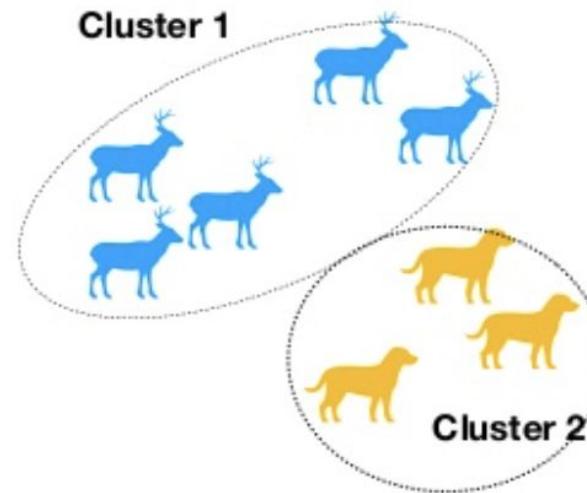


# V-measure

geometric mean of Homogeneity and Completeness



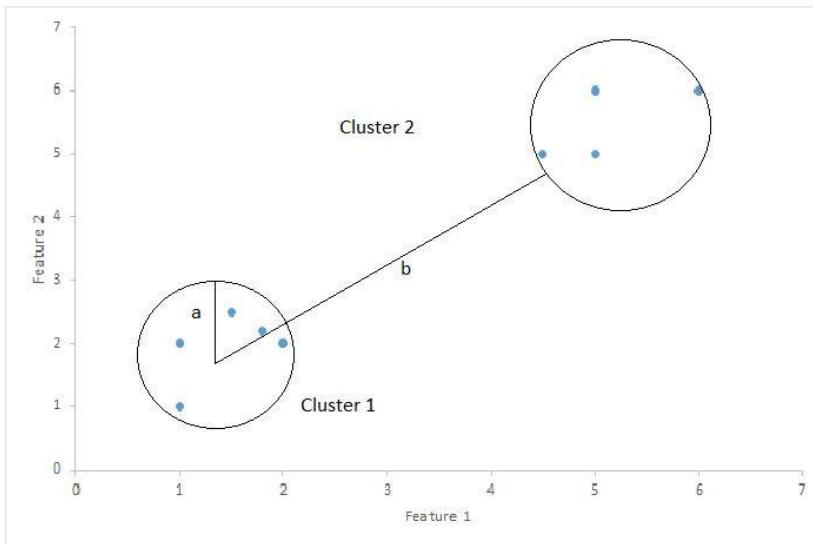
homogeneity score = 1



completeness score = 1



# Silhouette coefficient



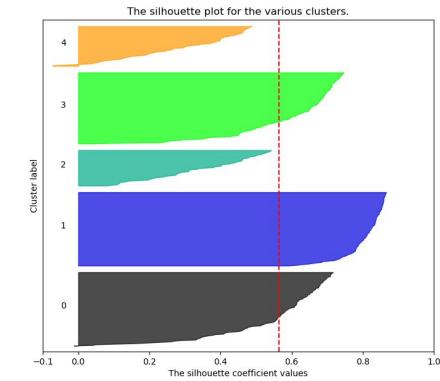
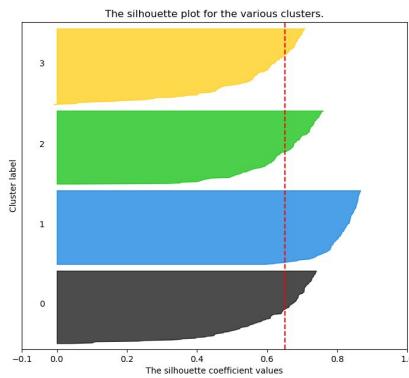
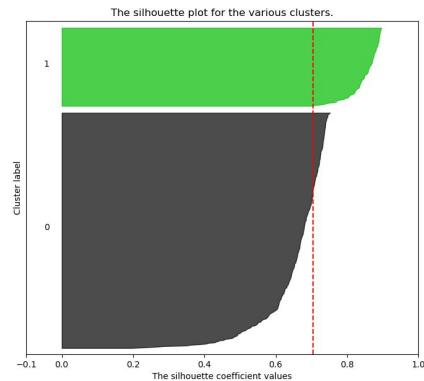
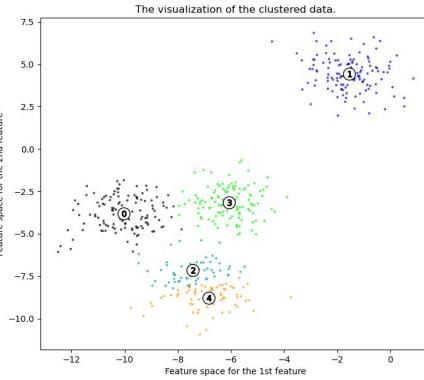
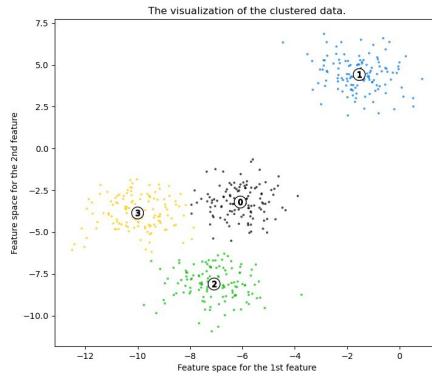
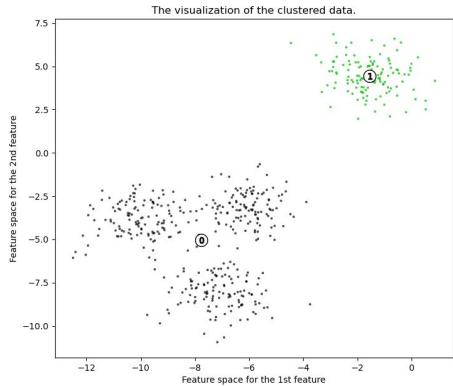
**a:** mean distance to points  
in the same class

**b:** mean distance to points  
in the *next nearest cluster*

$$s_i = \frac{b_i - a_i}{\max(a_i, b_i)}$$

[image source](#)

# Silhouette analysis



[image source](#)

# Density estimation

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girafe  
ai

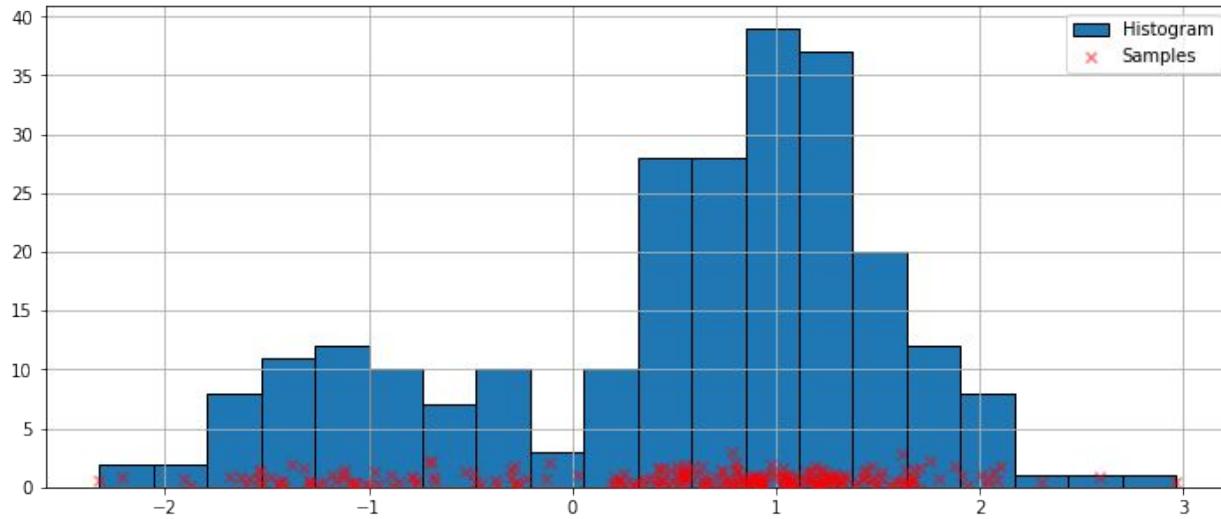
04

# Поставить задачу!





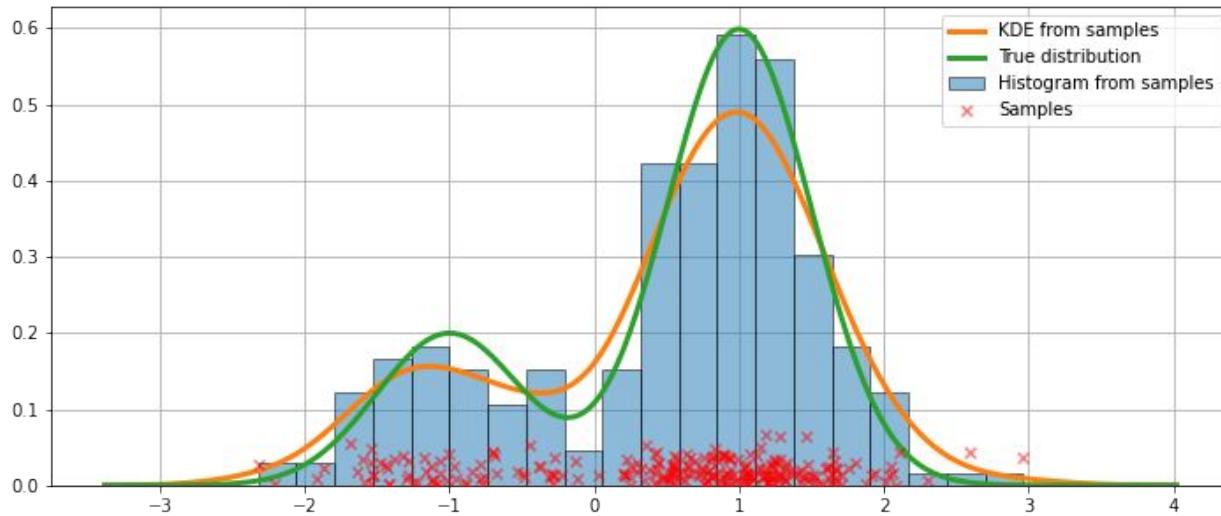
# Kernel density estimation



[statsmodels documentation example](#)

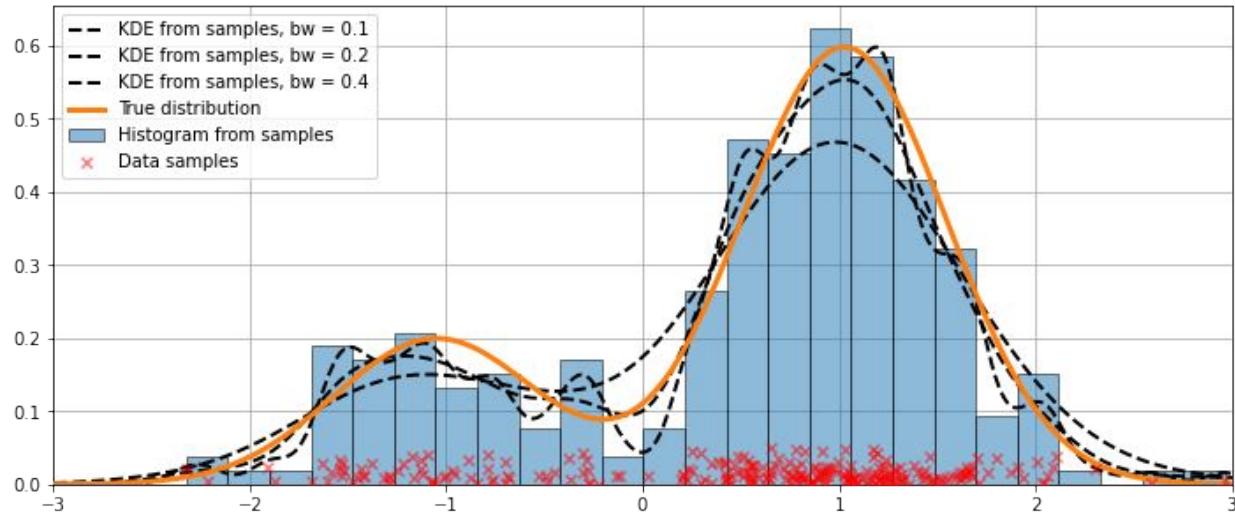


# Kernel density estimation

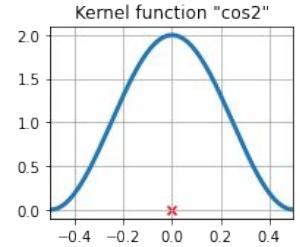
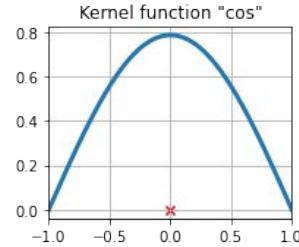
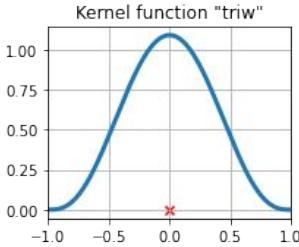
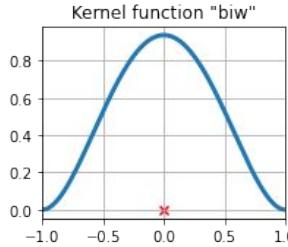
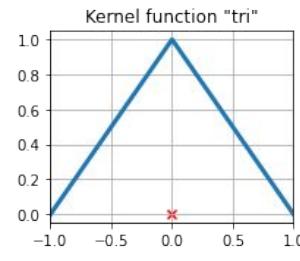
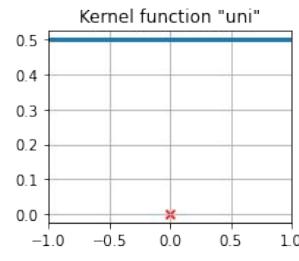
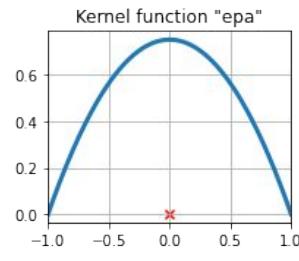
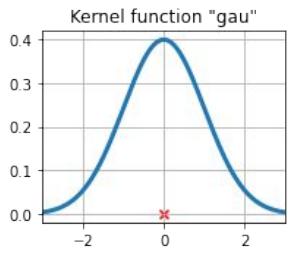




# Window size



# Kernel types



# Revise



- Geometrical machine learning
  - Dimensionality curse
  - Manifold assumption
- Dimensionality reduction
  - Feature selection
  - Multidimensional Scaling (MDS)
  - Isomap
  - Locally linear embedding (LLE)
  - t-SNE
- Clustering
  - k-means
  - DBSCAN
  - Hierarchical clustering
  - metrics
- Density estimation
  - Kernel density estimation

# Thanks for attention!

Questions?

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girafe  
ai





# Notable links

1. [Good lecture on MDS, Isomap, LLE](#)
2. [Lecture on t-SNE](#) (this one is good too)
3. [Slides about clusterization](#)
4. [Metrics in clusterization](#)
5. [Slides about ICA](#)
6. [More clustering methods](#) (in Russian)