Solutions to some exercises from Walter Rudin's *Functional Analysis*

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December 2, 2025

Contents

Notations and Conventions

I Logic

i Propositional logic

Given propositional variables p, q, the boolean operators $\neg$, $\vee$, $\wedge$, $\Leftrightarrow$, $\Rightarrow$, $\Leftarrow$, assign boolean *truth values* as follows,

- $\neg$ $\neg \mathrm{p}$ has not the truth value of p.
- $\vee$ The *conjonction* $\mathrm{p} \vee \mathrm{q}$ is true, unless: p false, q false.
- $\wedge$ The *disjunction* is false, unless: p true, q true.
- $\Leftrightarrow$ The *logical equivalence* expresses *tautologies*: $\mathrm{p} \Leftrightarrow \mathrm{q}$ is true, unless: p has not the truth value of q. It is easily checked that $(\mathrm{p} \Leftrightarrow \mathrm{q}) \Leftrightarrow ((\mathrm{p} \Rightarrow \mathrm{q}) \wedge (\mathrm{p} \Leftarrow \mathrm{q}))$; see the below definitions.
- $\Rightarrow$ The logical implication is denoted by $\Rightarrow$: $\mathrm{p} \Rightarrow \mathrm{q}$ means *if (criterion/premise)* p *then (conclusion)* q, or, alternatively, p *implies* q. $\mathrm{p} \Rightarrow \mathrm{q}$ is formally defined as $\neg \mathrm{p} \vee \mathrm{q}$. Remark that the “reasoning” $\mathrm{p} \Rightarrow \mathrm{q}$ is always valid, unless: p true, q false. Moreover, $\mathrm{p} \wedge (\mathrm{p} \Rightarrow \mathrm{q}) \Rightarrow \mathrm{q}$ is always true.
- $\Leftarrow$ $\mathrm{q} \Leftarrow \mathrm{p}$ is $\mathrm{p} \Rightarrow \mathrm{q}$ read backward. A common pronunciation is q *since* p.

For a subtle introduction to proposition logic, see Section 1.3 and Subsection 16.1.3 of [1].

ii Iverson notation

Given a boolean expression ϕ, $[\,\phi\,]$ returns the truth value of ϕ, encoded as follows,

$$[\,\phi\,] \triangleq \begin{cases} 0 & \text{if } \phi \text{ is false;} \\ 1 & \text{if } \phi \text{ is true.} \end{cases}$$

For example, $[\,1 > 0\,] = 1$ but $\left[\,\sqrt{2} \in \mathbf{Q}\,\right] = 0$.

II Special terms

i Halmos’ iff and definitions

iff is a short for “if and only if”. Splitting **iff** into *if-then* clauses shows that it is just a rewording of the logical equivalence $\Leftrightarrow$. All definitions will use the **iff** format; which is consistent with the fact that every definition expresses a tautology.

ii Assigning values

Given variables a and b, $\triangleq$ is a specialization of $=$. We say that $\mathrm{x} \triangleq \mathrm{y}$ **iff** x and y are assumed to be equal. Usually, $\mathrm{x} \triangleq \mathrm{y}$ means that x is assigned the previously known value y (some authors write $\mathrm{x} := \mathrm{y}$) but this is not a limitation. Definitions can be redundant and may overlap. The only restriction is that $\mathrm{x} \triangleq \mathrm{y}$ is inconsistent whether $\mathrm{x} \neq \mathrm{y}$.

iii Equinumerosity

$\mathrm{a} \equiv \mathrm{b}$ means that there exists a bijection $\rightarrow$ that maps a to b; which let us identify a with b. In a metric space context, $\mathrm{a} \equiv \mathrm{b}$ means that $\rightarrow$ is isometric.

III Sets

i Subsets and supersets

Provided a pair (A, X), $\subseteq$ and $\supseteq$ are the regular symbols for set ordering, as follows:

$$\mathrm{A} \subseteq \mathrm{X} \quad \textbf{iff} \quad \mathrm{a} \in \mathrm{A} \Rightarrow \mathrm{a} \in \mathrm{X} \tag{1}$$
$$\mathrm{X} \supseteq \mathrm{A} \quad \textbf{iff} \quad \mathrm{A} \subseteq \mathrm{X}. \tag{2}$$

Note that there is no specific symbol for a strict version: If needed, we will explicitely state that $\mathrm{A} \neq \mathrm{X}$.

IV Topological vector spaces

i Scalar field

The usual (complete) scalar field is **C**. A property, *e.g.* linearity, that is true on **C** is also true on **R**. The complex case is then a *special case* of the real one. Sometimes, this specialization is not harmless. For example, theorem 12.7 of [4] asserts that, in a Hilbert space H equipped with the inner product $\langle \cdot \,|\, \cdot \rangle$, every nonzero linear continuous operator T "breaks orthogonality", in the sense that there always exists $\mathrm{x} = \mathrm{x}(\mathrm{T})$ in H that satisfies $\langle \mathrm{Tx} | \mathrm{x} \rangle \neq 0$. The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exists. To see that, consider the 90° rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitely mentioned*, the exension to the real case will always be obvious. So, taking **C** as the scalar field shall mean

> *Instead of letting the scalar field undefined, we choose* **C** *for the sake of expessivity. But considering* **R** *instead of* **C** *would actually make no difference here.*

ii Vector space bases

Given a vector space X over $\mathbf{C}$ (or, more generally, over a field), a subset B of X is a basis of X **iff** there is a *finite sum*

$$\sum : \mathbf{C}^{\mathrm{B}} \to \mathrm{X} \qquad z \mapsto \sum_{u \in \mathrm{B}} z_u u \tag{3}$$

that bijectively maps all *almost null*

$$z : \mathrm{B} \to \mathbf{K} \qquad u \mapsto z_u \tag{4}$$

onto X. The axiom of choice (AC) forces

(*a*) the existence of such B (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [4] with B playing the role of Λ);

(*b*) all bases to have the same cardinal, which is called the *dimension* of X and is denoted as dim(X).

We now come to the *finite-dimensional* case, *i.e.* dim(X) is a nonnegative integer n. Remark that $n = 0$, *i.e.* $\mathrm{B} = \emptyset$, means that X is a singleton. Our first step consists in studying $\mathbf{C}^n$, which is the standard n-dimensional vector space.

iii Finite-dimensional spaces

From now on, the zero-dimensional case, which is trivial, shall be skipped.

The product topology of $\mathbf{C}^n$

As the n-th power of $\mathbf{C}$, $\mathbf{C}^n$ has a standard base $\{e_k : k = 1, \ldots, n\}$ (where $e_k = 1_{\{k\}}$). Furthermore, it is topologized by the *polydiscs*

$$\prod_{i=1}^{n} D_{r_i} \quad (D_{r_i} \triangleq \{z_i \in C : |z_i| < r_i\}), \tag{5}$$

Equivalently, we may equipp $\mathbf{C}^n$ with the euclidian norm

$$\|z\|_2 \triangleq \sqrt{|z_1|^2 + \cdots + |z_n|^2} \quad (z = (z_1, \ldots, z_n) \in \mathbf{C}^n)\,, \tag{6}$$

whose open balls centered at the origin are all nonempty

$$B_r \triangleq \{z \in \mathbf{C}^n : \|z\|_2 < r\}. \tag{7}$$

To see such equivalence, first pick a positive r then set $r_i = r/\sqrt{n}$, so that

$$\prod_{i=1}^{n} D_{r_i} \subseteq B_r. \tag{8}$$

Next, conversely choosing $r = \min\{r_1, \ldots, r_n\}$ yields

$$B_r \subseteq \prod_{i=1}^{n} D_{r_i}. \tag{9}$$

Topology of a finite-dimensional vector space

It is customary to identify any n-dimensional vector space with $\mathbf{C}^n$ topologized by the euclidian norm; see [iii]. To see this, pick a n-dimensional vector space Y, of basis $\{u_1, \ldots, u_n\}$; see [ii]. Setting $u_k = f(e_k)$ means a special case of [1.20] of [4], where f is an isomorphism of $\mathbf{C}^n$ onto Y. Actually, Y is endowed with the topology $\{f(U) : U \text{ open}\}$, and [1.21] of [4] states that f is an homeomorphism, which implies that

> $\{f(U) : U \text{ open}\}$ *is the only vector space topology for* Y.

As a consequence, Y is necessarily locally convex and bounded; *i.e.* normable; see [1.39] of [4]. Moreover, provided a norm $\|\cdot\|$ on Y, there exists a positive *modulus of continuity* $C = C_f$ such that

$$\|y\| \leq C\|z\|_2 \quad ((z, y) \in f)\,, \tag{10}$$

since f is continuous. Now pick a n-dimensional topological vector space W then repeat the same reasoning, first with $g : \mathbf{C}^n \to W$, next with $h = g \circ f^{-1}$, in the role of f, and so conclude that the homeomorphism h maps Y's topology onto Y's topology and that W is normable. To sum it up,

> $\dim(Y) = \dim(W)$, *i.e.* Y *and* W *are isomorphic each other, means that* Y *and* W *are two normable spaces that are homeomorphic each other.*

We then equip W with a norm $|||\cdot|||$, so that

$$|||w||| \leq C_h\|y\| \quad ((y, w) \in h) \tag{11}$$

for some positive C_h. The special case $g = f$ means that Y's norms are equivalent, in the sense that there exists a positive C_{id} such that

$$|||y||| \leq C_{id}\|y\|. \tag{12}$$

The standard norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$

In all special cases $Y = \mathbf{C}^n$ topologized by the standard norms 1, 2, ∞, the optimal modulus, *i.e.* the smallest $C = C_{i,j}$ such that

$$\|z\|_j \leq C_{i,j}\|z\|_i \quad (z \in \mathbf{C}^n)), \tag{13}$$

is easily derived from definitions - see [1.19] of [4] - with the noticeable exception of $C_{2,1} = \sqrt{n}$, which is usually seen as a special *Cauchy-Schwarz inequality*; see (1) in [12.2] of [4]. The very steps of this classical hack are left to the reader.

Chapter 1

Topological Vector Spaces

1 Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X*. Prove the following statements from the axioms as given as in section 1.4.*

(a) *If* $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ *there is a unique* $\mathrm{z} \in \mathrm{X}$ *such that* $\mathrm{x} + \mathrm{z} = \mathrm{y}$.

(b) $0 \cdot \mathrm{x} = 0 = \alpha \cdot 0 \quad (\alpha \in \mathbf{C}, \mathrm{x} \in \mathrm{X})$.

(c) $2\mathrm{A} \subseteq \mathrm{A} + \mathrm{A}$.

(d) A *is convex if and only if* $(\mathrm{s} + \mathrm{t})\mathrm{A} = \mathrm{sA} + \mathrm{tA}$ *for all positive scalars* s *and* t.

(e) *Every union (and intersection) of balanced sets is balanced.*

(f) *Every intersection of convex sets is convex.*

(g) *If* Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

(h) *If* A *and* B *are convex, so is* $\mathrm{A} + \mathrm{B}$.

(i) *If* A *and* B *are balanced, so is* $\mathrm{A} + \mathrm{B}$.

(j) *Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.*

Proof. (a) Such property only depends on the group structure of X: Each x in X has an opposite -x. Let x' be any opposite of x, so that $\mathrm{x} - \mathrm{x} = 0 = \mathrm{x} + \mathrm{x}'$. Thus, $-\mathrm{x} + \mathrm{x} - \mathrm{x} = -\mathrm{x} + \mathrm{x} + \mathrm{x}'$, which is equivalent to $-\mathrm{x} = \mathrm{x}'$. So is established the uniqueness of -x. It is now clear that $\mathrm{x} + \mathrm{z} = \mathrm{y}$ **iff** $\mathrm{z} = -\mathrm{x} + \mathrm{y}$, which asserts both the existence and the uniqueness of z.

(b) Remark that

$$0 \cdot \mathrm{x} = (0 + 0) \cdot \mathrm{x} = 0 \cdot \mathrm{x} + 0 \cdot \mathrm{x} \tag{1.1}$$
$$= (0 + 0) \cdot \mathrm{x} = 0 + 0 \cdot \mathrm{x} \tag{1.2}$$

then conclude from (a) that $0 \cdot x = 0$. So,

$$0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x. \tag{1.3}$$

Finally,

$$\alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0, \tag{1.4}$$

which proves (b).

(c) Remark that

$$2x = (1 + 1)x = x + x \tag{1.5}$$

for every x in X, and so conclude that

$$2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A \tag{1.6}$$

for all subsets A of X; which proves (c).

(d) If A is convex, then

$$A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A; \tag{1.7}$$

which is

$$sA + tA = (s + t)A. \tag{1.8}$$

Conversely, the special case $s + t = 1$ is

$$sA + (1 - s)A = A. \tag{1.9}$$

The latter extends to $s = 0$, since

$$0A + A \overset{(b)}{=} \{0\} + A = A. \tag{1.10}$$

The extension to $s = 1$ is analogously established (or simply use the fact that + is commutative!). So ends the proof.

(e) Let A range over B a collection of balanced subsets, so that

$$\alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B \tag{1.11}$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap B \subseteq A$ establishes the first part. Now remark that

$$\alpha A \subseteq \bigcup B \tag{1.12}$$

implies

$$\alpha \bigcup B \subseteq \bigcup B; \tag{1.13}$$

which achieves the proof.

(f) Let A range over C a collection of convex subsets, so that

$$(s+t)\bigcap C \subseteq s\bigcap C + t\bigcap C \subseteq sA + tA \overset{(d)}{\subseteq} (s+t)A \tag{1.14}$$

for all positives scalars *s*, *t*. Inclusions at both extremities force

$$s\bigcap C + t\bigcap C = (s+t)\bigcap C. \tag{1.15}$$

We now conclude from (d) that the intersection of C is convex. So ends the proof.

(g) Skip all trivial cases $\Gamma = \emptyset$, $\{\emptyset\}$, $\{\{x\}\}$, $\{\emptyset, \{x\}\}$ then pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i $(i = 1, 2)$ lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_i's is then in $C_2 \subseteq \bigcup \Gamma$. Hence (g).

(h) Simply remark that

$$s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B) \tag{1.16}$$

for all positive scalars *s* and *t*, then conclude from (d) that $A + B$ is convex.

(i) Given any α from the closed unit disc,

$$\alpha(A+B) = \alpha A + \alpha B \subseteq A + B. \tag{1.17}$$

There is no more to prove: $A + B$ is balanced.

(j) Our proof will be based on the following lemma,

If S *is nonempty, then each of the following three properties*
(i) S *is a vector subspace of* X*;*
(ii) S *is convex balanced such that* $S + S = S$*;*
(iii) S *is convex balanced such that* $\lambda S = S \quad (\lambda > 0)$
implies the other two.

To prove the lemma, let *S* run through all nonempty subsets of X. First, assume that (i) holds: Clearly, every S is convex balanced. Moreover, $S + S \subseteq S$. Conversely, $S = S + \{0\} \subseteq S + S$; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$nS = (n-1)S + S = S + S = S \quad (n = 1, 2, 3, \dots) \tag{1.18}$$

with the help of (b) and (d). Pick $\lambda > 0$ then choose n so large that $1 < n\lambda < n^2$. Thus,

$$nS \overset{(1.18)}{\subseteq} S \subseteq n\lambda S \subseteq n^2 S, \tag{1.19}$$

since S is balanced. For instance, set $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$. Dividing the latter inclusions by n shows that

$$S \subseteq \lambda S \subseteq nS \overset{(1.18)}{\subseteq} S, \tag{1.20}$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$\alpha S + \beta S \overset{(a)}{=} |\alpha|S + |\beta|S \overset{(d)}{=} (|\alpha| + |\beta|)S \overset{(iii)}{=} S \quad (|\alpha| + |\beta| > 0); \tag{1.21}$$

where the equality at the left holds as S is balanced. Moreover (under the sole assumption that S is balanced), this extends to $|\alpha| + |\beta| = 0$, as follows,

$$\alpha S + \beta S = 0S + 0S \overset{(b)}{=} \{0\} \overset{(b)}{=} 0S \subseteq S. \tag{1.22}$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let V be a collection of vector spaces of X, of intersection I and union U. First, remark that every member of V is convex balanced: So is I (combine (e) with (f)). Next, let Y range over V, so that

$$I + I \subseteq Y + Y \subseteq Y; \tag{1.23}$$

which yields

$$I + I = I \tag{1.24}$$

(the fact that $I = I + \{0\} \subseteq I + I$ was tacitely used). It now follows from the lemma's (ii) $\Rightarrow$ (i) that I is a vector subspace of X. Now temporarily assume that S is totally ordered by set inclusion: Combining (e) with (g) establishes that U is convex balanced. To show that U is more specifically a vector subspace, we first remark that such total order implies that either $Z \subseteq Y$ or $Y \subseteq Z$, as Z ranges over V. A straightforward consequence is that

$$Y \subseteq Y + Z \subseteq Y \cup Z. \tag{1.25}$$

Another one is that $Y \cup Z$ ranges over V as well. Combined with the latter inclusions, this leads to

$$U \subseteq U + U \subseteq U. \tag{1.26}$$

It then follows from the lemma's (ii) $\Rightarrow$ (i) that U is a vector subspace of X. Finally, let A, B run through all vector subspaces of X: Combining (h) with (i) proves that $A + B$ is convex balanced as well. Furthermore,

$$A + B \overset{(i)\Rightarrow(ii)}{=} (A + A) + (B + B) = (A + B) + (A + B), \tag{1.27}$$

where the equality at the right holds as X is an abelian group. We now conclude from (ii) that any $A + B$ is a vector subspace of X. So ends the proof.

$\square$

2 Exercise 2. Convex hull

The convex hull of a set A *in a vector space* X *is the set of all convex combinations of members of* A*, that is the set of all sums* $t_1x_1 + \cdots + t_nx_n$ *in which* $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$*;* n *is arbitrary. Prove that the convex hull of a set* A *is convex and that is the intersection of all convex sets that contain* A*.*

Proof. The convex hull of a set S will be denoted by co(S). Remark that $S \supseteq \mathrm{eq}\,\mathrm{co}(S)$ (to see that, take $t_1 = 1$ for each x_1 in S) and that $\mathrm{co}(A) \supseteq \mathrm{eq}\,\mathrm{co}(B)$ where $A \supseteq \mathrm{eq}B$ (obvious).

Our proof will directly derive from (i) ⇒ (iv) in the following lemma,

> *Let* S *be a subset of a vector space* X*: Its convex hull* co(S) *is convex and the following statements*
>
> *(i)* S *is convex;*
>
> *(ii)* $s_1S + \cdots + s_nS = (s_1 + \cdots + s_n)S$ *for all positive scalar variables* $s_1, \ldots, s_n$*;*
>
> *(iii)* $t_1S + \cdots + t_nS = S$ *for all positive scalar variables* $s_1, \ldots, s_n$ *such that* $s_1 + \cdots + s_n = 1$*;*
>
> *(iv)* $\mathrm{co}(S) = S$
>
> *are equivalent.*

From now on, we skip the trivial case $S = \emptyset$ then only consider nonempty sets. To prove the first part, let *a*, *b* range over co(S), so that $a = t_1x_1 + \cdots + t_nx_n$ and $b = t_{n+1}x_{n+1} + \cdots + t_{n+p}x_{n+p}$ for some (t_i, x_i). Every sum $sa + (1 - s)b$ $(0 \leq s \leq 1)$ is then in the convex hull of $\{x_1, \ldots, x_{n+p}\}$, since

$$sa + (1 - s)b = \sum_{i=1}^{n} st_ix_i + \sum_{i=n+1}^{n+p} (1 - s)t_ix_i \tag{1.28}$$

and

$$\sum_{i=1}^{n} st_i + \sum_{i=n+1}^{n+p} (1 - s)t_i = s\sum_{i=1}^{n} t_i + (1 - s)\sum_{i=n+1}^{n+p} t_i = 1. \tag{1.29}$$

In terms of sets S, this reads as follows,

$$s\,\mathrm{co}(S) + (1 - s)\,\mathrm{co}(S) \subseteq \mathrm{co}(S); \tag{1.30}$$

which was our fist goal. We now aim at the equivalence (i) ⇒ ⋯ ⇒ (iv) ⇒ (i): An easy proof by induction makes the implication (i) ⇒ (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii) ⇒ (iv) derives from the definition of the convex hull. We now close the chain with (iv) ⇒ (i), by remarking that S is convex whether $S = \mathrm{co}(S)$. The lemma being proved, let us establish the second part.

To do so, we start from the convexity of co(A) then set $\mathrm{F} = \{\mathrm{co(A)}\}$. We may enrich F as follows,

$$\mathrm{B} \in \mathrm{F} \Rightarrow \mathrm{B} \text{ is convex and contains A.} \tag{1.31}$$

Note that our initial predicate "[F only encompasses] *all convex sets that contain A*", is now the special case

$$\mathrm{B} \in \mathrm{F} \Leftrightarrow \mathrm{B} \text{ is convex and contains A.} \tag{1.32}$$

In any case, the key ingredient is that $\mathrm{co(A)} \in \mathrm{F}$ implies

$$\mathrm{co(A)} \supseteq \bigcap_{\mathrm{B}\in\mathrm{F}} \mathrm{B}. \tag{1.33}$$

Conversely, the next formula

$$\mathrm{co(A)} \subseteq \mathrm{co(B)} \overset{\text{(i)}\Rightarrow\text{(iv)}}{=} \mathrm{B} \quad (\mathrm{B} \in \mathrm{F}) \tag{1.34}$$

is valid and implies

$$\mathrm{co(A)} \subseteq \bigcap_{\mathrm{B}\in\mathrm{F}} \mathrm{B}. \tag{1.35}$$

So ends the proof $\square$

3 Exercise 3. Other basic results

Let be X as topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements:

(a) *The convex hull of every open set is open.*

(b) *If X is locally convex then the convex hull of every bounded set is bounded.*

(c) *If A and B are bounded, so is A+B.*

(d) *If A and B are compact, so is A+B.*

(e) *If A is compact and B is closed, then A+B is closed.*

(f) *The sum of two closed sets may fail to be closed.*

Proof. (a) Pick an open set A then let the variables V_i $(i = 1, 2, \dots)$ run through all open subsets of A, so that

$$\mathrm{co}(A) \subseteq \bigcup_{t_i} (t_1 V_1 + \cdots + t_i V_i + \cdots) \subseteq \mathrm{co}(A) \tag{1.36}$$

given all convex combinations $t_1V_1 + \cdots + t_iV_i + \cdots$. We know from Section 1.7 of [4] that those sums are open; which achieves the proof.

(b) Provided a bounded set E, pick V a neighbourhood of 0: By (b) of Section 1.14 in [4], V contains a convex neighbourhood of 0, say W. There so exists a positive scalar s such that

$$E \subseteq tW \subseteq tV \quad (t > s); \tag{1.37}$$

which yields

$$\mathrm{co}(E) \subseteq \mathrm{co}(tW) = t\,\mathrm{co}(W) = tW \subseteq tV. \tag{1.38}$$

So ends the proof.

(c) At fixed V, neighbourhood of the origin, we combine the continuousness of + with Section 1.14 of [4] to conclude that there exists U a balanced neighborhood of the origin such that

$$U + U \subseteq V. \tag{1.39}$$

Moreover, by the very definition of boundedness, $A \subseteq rU$ for some positive scalar r. Similarly, $B \subseteq sU$ for some positive s. Finally,

$$A + B \subseteq rU + sU \subseteq tU + tU \subseteq tV \quad (t > r, s), \tag{1.40}$$

since U is balanced. So ends the proof.

(d) First, A and B are compact: So is $A \times B$. Next, + maps continuously $A \times B$ onto $A + B$. In conclusion, $A + B$ is compact.

(e) From now on, we assume that neither A nor B is empty, since otherwise the result is trivial. Now pick $c \in X$ outside $A + B$: The result will be established by showing that c is not in the closure of $A + B$.

To do so, we let the variable *a* range over A: Every set $a + B$ is closed as well; see Section 1.7 of [4]. Trivially, $a + B \neq c$: By Section 1.10 of [4], there so exists $V = V(a)$ a neighborhood of the origin such that

$$(a + B + V) \cap (c + V) = \emptyset. \tag{1.41}$$

Moreover, there are finitely many $a + V$, say $a_1 + V_1, a_2 + V_2, \ldots$, whose union U contains the compact set A. Therefore,

$$A + B \subseteq U + B. \tag{1.42}$$

Now define

$$W \triangleq V_1 \cap V_2 \cap \cdots , \tag{1.43}$$

so that

$$(a_i + B + V_i) \cap (c + W) \overset{(1.41)}{=} \emptyset \quad (i = 1, 2, \ldots). \tag{1.44}$$

As a conclusion, c is not in the closure of $U + B$. Finally, (1.42) asserts that c is not in $\overline{A + B}$ either; which achieves the proof.

Corollary: If B is the closure of a set S, then

$$A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B \tag{1.45}$$

by (b) of Section 1.13 of [4] (since A is closed; see Section 1.12, from the same source). The special case $A = \{x\}$, $B = X$ will occur in the proof of Exercise 15 in chapter 2.

(f) The last proof will consist in exibhiting a counterexample. To do so, let f be any continuous mapping of the real line such that

(i) $f(x) + f(\text{-}x) \neq 0 \quad (x \in \mathbf{R})$;

(ii) f vanishes at infinity.

For instance, we may combine (ii) with f even and $f > 0$ by setting $f(x) = 2^{-|x|}$, $f(x) = e^{-x^2}$, $f(x) = 1/(1 + |x|)$, ..., and so on.

As a continous function, f has closed graph G; see [2.14] of [4]. Moreover, (i) implies that the origin $(0, 0) \neq (x - x, f(x) + f(\text{-}x))$ is not in $G + G$. On the other hand,

$$\{(0, f(n) + f(\text{-}n)) : n = 1, 2, \ldots\} \subseteq G + G. \tag{1.46}$$

Now the key ingredient is that

$$(0, f(n) + f(\text{-}n)) \xrightarrow[n\to\infty]{(ii)} (0, 0). \tag{1.47}$$

We have so constructed a sequence in $G + G$ that converges outside $G + G$. So ends the proof.

□

4 Exercise 4. A nonempty set whose interior is not

Let be $\mathrm{B} = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$*. Show that* B *is balanced but that its interior is not.*

Proof. It is obvious that the nonempty set B contains the origin $(0, 0)$. Additionally, its interior B° is nonempty as well. Indeed, the following set

$$\{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2] < 1/2\} \subseteq \mathrm{B} \tag{1.48}$$

is a neighborhood of $(1, 2) \in \mathrm{B}$. Moreover, B is balanced, since

$$|\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \quad (|\alpha| \leq 1) \tag{1.49}$$

for all (z_1, z_2) in B. Nevertheless, the nonempty set B° is not balanced, what we now establish by showing that $(0, 0) \notin \mathrm{B}^\circ$. To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$\mathrm{U} \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2] < r\} \subseteq \mathrm{B} \tag{1.50}$$

for some positive r. Clearly, U contains $(r/2, 0)$, and that special case $(r/2, 0) \in \mathrm{B}$ now contradicts the definition of B. So ends the proof. □

5 Exercise 5. A first restatement of boundedness

Consider the definition of "bounded set" given in Section 1.6. Would the content of this definition be altered if it were required merely required that to every neighbourhood V *of 0 corresponds* some $\mathrm{t} > 0$ *such that* $\mathrm{E} \subseteq \mathrm{tV}$*?*

Proof. The answer is: No. To prove it, start from (a) of Section 1.14: V contains W, a balanced neighbourhood of 0. Assume that E is bounded in this weaker sense, *i.e.* there exists a positive t that satisfies

$$\mathrm{E} \subseteq \mathrm{tW}. \tag{1.51}$$

Thus,

$$\mathrm{E} \subseteq \mathrm{tW} \subseteq \mathrm{sW} \subseteq \mathrm{sV} \quad (\mathrm{s} > \mathrm{t}), \tag{1.52}$$

since W is balanced. We so reach the definition given in Section 1.6: The two ones are equivalent. □

6 Exercise 6. A second restatement of boundedness

Prove that a set E *in a topological vector space is bounded if and only if every countable subset of E is bounded.*

Proof. It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighbourhood of the origin: No counting number $\mathrm{n} = 1, 2, \ldots,$ verifies $\mathrm{E} \subseteq \mathrm{nV}$ (see Exercise 1 in Chapter 1). In other words, there exists a sequence $\{\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}, \ldots\} \subseteq \mathrm{E}$ such that

$$\mathrm{x}_\mathrm{n} \notin \mathrm{nV}. \tag{1.53}$$

As a consequence, $\mathrm{x}_\mathrm{n}/\mathrm{n}$ fails to converge to 0 as n tends to ∞. In contrast, $1/\mathrm{n}$ succeeds. It then follows from Section 1.30 that $\{\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}, \ldots\}$ is not bounded. So ends the proof. □

7 Exercise 7. Metrizability & number theory

Let be X *the vector space of all complex functions on the unit interval* $[0,1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \quad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ *in X such that (a)* $\{f_n\}$ *converges to* 0 *as* $n \to \infty$*, but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

1 Justification of the terminology

Proof. The family of the seminorms p_x is separating: The collection $\mathscr{B}$ of all finite intersections of the sets

$$V(x,k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0,1], k = 1,2,3,\dots) \tag{1.54}$$

is therefore a local base for a topology τ on X; see Section 1.37 of [4]. So,

(1.55)
$$\sum_{n=1}^{\infty}\left[f_n \notin \cap_{i=1}^{m} U_i\right] \le \sum_{n=1}^{\infty}\sum_{i=1}^{m}\left[f_n \notin U_i\right] = \sum_{i=1}^{m}\sum_{n=1}^{\infty}\left[f_n \notin U_i\right] \quad (f_n \in X, U_i \in \tau).$$

Now assume that $\{f_n\}$ τ-converges to some f, *i.e.*

$$\sum_{n=1}^{\infty}\left[f_n \notin f + W\right] < \infty \quad (W \in \mathscr{B}). \tag{1.56}$$

The special case $W = V(x,k)$ means that, given k, $|f_n(x) - f(x)| < 2^{-k}$ for almost all n. In other words, $\{f_n(x)\}$ converges to f(x). Conversely, assume that $\{f_n\}$ does not τ-converges in X, *i.e.*

$$\forall f \in X, \exists W \in \mathscr{B} : \sum_{n=1}^{\infty}\left[f_n \notin f + W\right] = \infty. \tag{1.57}$$

W is then the intersection of some $V(x_1,k_1),\dots,V(x_m,k_m)$. Hence

$$\infty \overset{(1.57)}{=} \sum_{n=1}^{\infty}\left[f_n \notin f + W\right] \overset{(1.55)}{\le} \sum_{i=1}^{m}\sum_{n=1}^{\infty}\left[f_n \notin f + V(x_i,k_i)\right]. \tag{1.58}$$

It is now clear that

$$\sum_{n=1}^{\infty}\left[f_n \notin f + V(x_i,k_i)\right] = \infty. \tag{1.59}$$

for some i. In other words, $\{f_n(x_i)\}$ fails to converge to $f(x_i)$. The overall conclusion is that τ-convergence is the X's version of pointwise convergence. □

2 Proof (with the given hint)

We now prove the second part by constructing a specific sequence $\{f_n\}$ that simultaneously satisfies (a) and (b). Indeed, the hint says that there exists a one-to-one and *onto* mapping

$$\begin{aligned} \phi : [0,1] &\to \{\theta : \theta \in \mathbf{R}^{\mathbf{N}_+}, \lim_{\infty} \theta = 0\}. \\ x &\mapsto (\theta_1, \dots, \theta_n, \dots) \end{aligned} \tag{1.60}$$

Proof. We set (under the same notation)

$$f_n(x) \triangleq \theta_n \xrightarrow[n\to\infty]{} 0 \quad \big(x = \phi^{-1}(\theta_1, \dots, \theta_n, \dots)\big) \tag{1.61}$$

so that $x_\gamma = \phi^{-1}\Big(1/\sqrt{1+|\gamma_1|}, \dots, 1/\sqrt{1+|\gamma_n|}, \dots\Big)$ implies

$$\gamma_n f_n(x_\gamma) = \gamma_n/\sqrt{1+|\gamma_n|} \underset{\infty}{\sim} \sqrt{\gamma_n} \xrightarrow[n\to\infty]{} \infty, \tag{1.62}$$

provided $\gamma_n \to \infty$. This proves (b), since $\{\gamma_n f_n(x_\gamma) : n = 1, 2, 3, \dots\}$ diverges. □

3 Proving it the hard way (no hint)

We will use the following simple proposition about binary expansions: each irrational has an eventually aperiodic binary expansion.

Proof. Formally, there exists a one-to-one and *onto* mapping

$$\begin{aligned} \texttt{bin} : [0,1] \setminus \mathbf{Q} &\to \big\{\beta \in \{0,1\}^{\mathbf{N}_+} : \beta \text{ is eventually aperiodic}\big\} \\ v &\mapsto (\beta_1, \dots, \beta_n, \dots) \end{aligned} \tag{1.63}$$

where $(\beta_1, \dots, \beta_n, \dots)$ satisfies

$$v = \beta_1/2 + \cdots + \beta_n/2^n + \cdots . \tag{1.64}$$

First, note that $\beta_1 + \beta_2 + \beta_3 + \cdots = \infty$ then define (under the same notation)

$$f_n(v) \triangleq 1/2^{\beta_1 + \cdots + \beta_n} \xrightarrow[n\to\infty]{} 0. \tag{1.65}$$

Now pick an arbitrary $\gamma_n \to \infty$: Given a positive integer k, $\gamma_n > 4^k$ for almost all n, say $\tilde{n}$. Next, choose $n = n_k$ from $\{\tilde{n}\}$ so large that

$$n_{k+1} - n_k > k + 1 \to \infty. \tag{1.66}$$

This way, $\chi = 1_{\{n_1, n_2, \dots\}}$ *is not eventually periodic on* $\mathbf{N}_+$! Moreover, one easily checks that the specialization $\beta = \chi = \texttt{bin}(v_\gamma)$ yields

$$\beta_1 + \cdots + \beta_{n_1} + \cdots + \beta_{n_k} = k. \tag{1.67}$$

Finally, combining (1.65) with (1.67) shows that

$$\gamma_{n_k} f_{n_k}(v_\gamma) = \gamma_{n_k}/2^k > 4^k/2^k > 2^k \xrightarrow[k\to\infty]{} \infty. \tag{1.68}$$

As a conclusion, every $\gamma_n \to \infty$ contains a subsequence $\{\gamma_{n_k}\}$ that keeps $\gamma_{n_k} f_{n_k}(v)$ away from 0 (for some $v = v_\gamma$) when $k \to \infty$; which is (b). □

9 Exercise 9. Quotient map

Suppose

(a) X *and* Y *are topological vector spaces,*
(b) $\Lambda : X \to Y$ *is linear.*
(c) N *is a closed subspace of* X*,*
(d) $\pi : X \to X/N$ *is the quotient map, and*
(e) $\Lambda x = 0$ *for every* $x \in N$*.*

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [4]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y. \tag{1.69}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'. \tag{1.70}$$

It straightforwardly derives from (1.69) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$f(\pi x) = 0 \overset{(1.69)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \tag{1.71}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [4]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subseteq V. \tag{1.72}$$

Since π is open (1.41 (a) of [4]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [4]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \tag{1.73}$$

for some neighborhood V of 0_X. So,

$$f(W) = f(\pi(V)) = \Lambda(V). \tag{1.74}$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y. So ends the proof. □

10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$, $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$.

(a) *Prove that* Λ *is an open mapping.*

(b) *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. Discard the trivial case $\Lambda = 0$ and assume that $\dim \mathrm{Y} = \mathrm{n}$ for some positive n. Let e range over a basis of B of Y then pick in X an arbitrary neighborhood W of the origin: There so exists V a balanced neighborhood of the origin of X such that

$$\sum_{\mathrm{e}} \mathrm{V} \subseteq \mathrm{W}, \tag{1.75}$$

since addition is continuous. Moreover, for each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, simply because Λ is onto: Given y in Y, of e-component(s) $\mathrm{y_e}$, we now obtain

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \Lambda(\mathrm{x_e}). \tag{1.76}$$

As a finite set, $\{\mathrm{x_e} : \mathrm{e} \in \mathrm{B}\}$ is bounded: There so exists a positive scalar s such that

$$\forall \mathrm{e} \in \mathrm{B}, \mathrm{x_e} \in \mathrm{sV}. \tag{1.77}$$

Combining this with (1.76) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \mathrm{y_e}\, \mathrm{s}\Lambda(\mathrm{V}). \tag{1.78}$$

We now come back to (1.75) and so conclude that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \subseteq \Lambda(\mathrm{W}) \tag{1.79}$$

for if $|\mathrm{y_e}| < 1/\mathrm{s}$; which proves (a) whether B is the standard basis of $\mathrm{Y} = \mathbf{C}^{\mathrm{n}}$ equipped with $\|\cdot\|_\infty$. The general case is now provided for free by [iii].

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. We now conclude with the help of [iii] that f is an homeomorphism of X/N onto Y. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$. □

12 Exercise 12. Topology stays, completeness leaves

Suppose $d_1(x,y) = |x-y|, d_2(x,y) = |\phi(x) - \phi(y)|$, *where* $\phi(x) = x/(1+|x|)$. *Prove that* d_1 *and* d_2 *are metrics on* $\mathbf{R}$ *which induce the same topology, although* d_1 *is complete and* d_2 *is not.*

Proof. First, each d_i $(i = 1, 2)$ induces a topology τ_i whose open balls are all

$$B_i(a, r) \triangleq \{x \in \mathbf{R} : d_i(a, x) < r\} \quad (a \in \mathbf{R}, r > 0). \tag{1.80}$$

Next, remark that the monotonically increasing mapping $\phi : \mathbf{R} \to]\text{-}1, 1[$ is odd and that

$$\phi(x) \xrightarrow[x\to\infty]{} 1. \tag{1.81}$$

ϕ is therefore a τ_1-homeomorphsim of $\mathbf{R}$ onto $]\text{-}1, 1[$. A first consequence is that, at fixed $a \in \mathbf{R}$, given any positive scalar ε, the τ_1-continuousness of ϕ supplies an open ball $B_1(a, \eta)$ on which $|\phi(a) - \phi| < \varepsilon$. In terms of balls B_i, this reads as follows,

$$B_1(a, \eta) \subseteq B_2(a, \varepsilon). \tag{1.82}$$

The second consequence is that the τ_1-continuousness of ϕ^{-1} yields similar inclusions

$$B_2(a, \varepsilon') \subseteq B_1(a, \eta') \tag{1.83}$$

provided $\eta' > 0$. At arbitrary ε, the special case $\eta' = \eta$ is the concatenation

$$B_2(a, \varepsilon') \subseteq B_1(a, \eta) \subseteq B_2(a, \varepsilon); \tag{1.84}$$

which proves that $\tau_1 = \tau_2$. Finally, all inequalities $n < i < j$ over $\mathbf{N}$ together yield

$$d_2(i, j) = |\phi(i) - \phi(j)| \xrightarrow[n\to\infty]{} 0. \tag{1.85}$$

The sequence $n = 0, 1, 2, \ldots$ is therefore τ_2-Cauchy. We will nevertheless establish that it τ_2-diverges. To do so, we start by offering the τ_2-converge to some λ: The triangle inequality immediately dismiss that assumption, as follows,

$$d_2(0, \lambda) \geq d_2(0, n) - d_2(\lambda, n) = \phi(n) - d_2(\lambda, n) \xrightarrow[n\to\infty]{} 1. \tag{1.86}$$

We then conclude that d_2 fails to be complete. □

14 Exercise 14. $\mathscr{D}_K$ equipped with other seminorms

Put $K = [0,1]$ *and define* $\mathscr{D}_K$ *as in Section 1.46. Show that the following three families of seminorms (where* $n = 0, 1, 2, \dots$*) define the same topology on* $\mathscr{D}_K$*. If* $D = d/dx$*:*

(a) $\|D^n f\|_\infty = \sup\{|D^n f(x)| : \infty < x < \infty\}$

(b) $\|D^n f\|_1 = \int_0^1 |D^n f(x)|\, dx$

(c) $\|D^n f\|_2 = \left\{\int_0^1 |D^n f(x)|^2 dx\right\}^{1/2}$.

Proof. Let us equipp $\mathscr{D}_K$ with the inner product $\langle f|g\rangle = \int_0^1 f\,\bar{g}$, so that $\langle f|f\rangle = \|f\|_2^2$. The following

$$\int_0^1 1\,|D^n f| \le \|1\|_2 \|D^n f\|_2 \tag{1.87}$$

is then a Cauchy-Schwarz inequality; see Theorem 12.2 of [4]. We so obtain

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty < \infty \tag{1.88}$$

since K has length 1. Obviously, the support of $D^n f$ lies in K, hence the below equality

$$|D^n f(x)| = \left|\int_0^x D^{n+1} f\right| \le \int_0^x |D^{n+1} f| \le \|D^{n+1} f\|_1. \tag{1.89}$$

Take the supremum over all $|D^n f(x)|$: Combining (1.88) with (1.89) now reads as follows,

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty \le \|D^{n+1} f\|_1 \le \cdots < \infty. \tag{1.90}$$

Finally, put

$$V_n^{(i)} \triangleq \{f \in \mathscr{D}_K : \|f\|_i < 2^{-n}\}, \tag{1.91}$$

$$\mathscr{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\}, \tag{1.92}$$

so that (1.90) is mirrored by neighborhood inclusions, provided $i = 1, 2, \infty$:

$$V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \cdots. \tag{1.93}$$

Their subchains $V_n^{(i)} \supseteq V_{n+1}^{(i)}$ turn $\mathscr{B}^{(i)}$ into a local base of a topology τ_i. The whole chain (1.93) then forces

$$\tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1; \tag{1.94}$$

which achieves the proof. □

16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $\mathrm{p_n}$, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [4] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [3]) about intersection of nonempty compact sets.

Lemma 1 *Let* X *be a topological space with a countable local base* $\{V_n : n = 1, 2, 3, \dots\}$*. If* $\tilde{V}_n = V_1 \cap \dots \cap V_n$*, then every subsequence* $\{\tilde{V}_{\varrho(n)}\}$ *is a decreasing (i.e.* $\tilde{V}_{\varrho(n)} \supseteq \tilde{V}_{\varrho(n+1)}$*) local base of X.*

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_n} \supseteq \tilde{\mathrm{V}}_\mathrm{n}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{n}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\mathrm{\varrho(n)}}\}$, since $\tilde{\mathrm{V}}_\mathrm{n} \supseteq \tilde{\mathrm{V}}_{\mathrm{\varrho(n)}}$. □

The following special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$ is one of the key ingredients:

Corollary 1 (special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$**)** *Under the same notations of Lemma 1, if* $\{V_n\}$ *is a decreasing local base, then so is* $\{V_{\varrho(n)}\}$*.*

Corollary 2 *If* $\{Q_n\}$ *is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence* $\{Q_{\varrho(n)}\}$ *also satisfies theses conditions. Furthermore, if* τ_Q *is the* $C(\Omega)$*'s (respectively* $C^\infty(\Omega)$*'s) topology of the seminorms* p_n*, as defined in section 1.44 (respectively 1.46), then the seminorms* $p_{\varrho(n)}$ *define the same topology* τ_Q*.*

Proof. Let X be $\mathrm{C}(\Omega)$ topologized by the seminorms $\mathrm{p_n}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_n} = \{\mathrm{p_n} < 1/\mathrm{n}\}$, then $\{\mathrm{V_n}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q_{\varrho(n)}} \subseteq \mathring{\mathrm{Q}}_{\mathrm{\varrho(n)+1}} \subseteq \mathrm{Q_{\varrho(n)+1}} \subseteq \mathrm{Q_{\varrho(n+1)}}. \tag{1.95}$$

Thus,

$$\mathrm{Q_{\varrho(n)}} \subseteq \mathring{\mathrm{Q}}_{\mathrm{\varrho(n+1)}}. \tag{1.96}$$

In other words, $\mathrm{Q_{\varrho(n)}}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p_{\varrho(n)}}\}$ then defines a topology $\tau_{\mathrm{Q_\varrho}}$ for which $\{\mathrm{V_{\varrho(n)}}\}$ is a local base. So, $\tau_{\mathrm{Q_\varrho}} \subseteq \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V_{\varrho(n)}}\}$ is a local base of τ_Q, which yields $\tau_\mathrm{Q} \subseteq \tau_{\mathrm{Q_\varrho}}$. □

Lemma 2 *If a sequence of compact sets* $\{Q_n\}$ *satisfies the conditions specified in section 1.44, then every compact set* K *lies in allmost all* Q_n°*, i.e. there exists m such that*

$$K \subseteq \overset{\circ}{Q}_m \subseteq \overset{\circ}{Q}_{m+1} \subseteq \overset{\circ}{Q}_{m+2} \subseteq \cdots . \tag{1.97}$$

Proof. The following definition

$$\mathrm{C_n} \triangleq \mathrm{K} \setminus \overset{\circ}{\mathrm{Q}}_{\mathrm{n}} \tag{1.98}$$

shapes $\{\mathrm{C_n}\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no $\mathrm{C_n}$ is empty and so conclude[2] that the $\mathrm{C_n}$'s intersection contains a point that is not in any $\mathrm{Q_n^\circ}$. On the other hand, the conditions specified in [1.44] force the $\mathrm{Q_n^\circ}$'s collection to be an open cover. This contradiction reveals that $\mathrm{C_m} = \emptyset$, *i.e.* $\mathrm{K} \subseteq \mathrm{Q_m^\circ}$, for some m. Finally,

$$\mathrm{K} \subseteq \overset{\circ}{\mathrm{Q}}_{\mathrm{m}} \subseteq \mathrm{Q_m} \subseteq \overset{\circ}{\mathrm{Q}}_{\mathrm{m+1}} \subseteq \mathrm{Q_{m+1}} \subseteq \overset{\circ}{\mathrm{Q}}_{\mathrm{m+2}} \subseteq \cdots . \tag{1.99}$$

□

We are now in a fair position to establish the following:

Theorem *The topology of* $C(\Omega)$ *does not depend on the particular choice of* $\{K_n\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of* $C^\infty(\Omega)$*, as long as this sequence satisfies the conditions specified in section 1.44.*

Proof. With the second corollary's notations, $\tau_{\mathrm{K}} = \tau_{\mathrm{K}_\lambda}$, for every subsequence $\{\mathrm{K}_{\lambda(\mathrm{n})}\}$. Similarly, let $\{\mathrm{L_n}\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_{\mathrm{L}} = \tau_{\mathrm{L}_\varkappa}$ for every subsequence $\{\mathrm{L}_{\varkappa(\mathrm{n})}\}$. Now apply the above Lemma 2 with $\mathrm{K_i}$ $(\mathrm{i} = 1, 2, 3, \dots)$ and so conclude that $\mathrm{K_i} \subseteq \mathrm{L^\circ_{m_i}} \subseteq \mathrm{L^\circ_{m_i+1}} \subseteq \cdots$ for some $\mathrm{m_i}$. In particular, the special case $\varkappa_{\mathrm{i}} = \mathrm{m_i} + \mathrm{i}$ is

$$\mathrm{K_i} \subseteq \overset{\circ}{\mathrm{L}}_{\varkappa_\mathrm{i}}. \tag{1.100}$$

Let us reiterate the above proof with $\mathrm{K_n}$ and $\mathrm{L_n}$ in exchanged roles then similarly find a subsequence $\{\lambda_{\mathrm{j}} : \mathrm{j} = 1, 2, 3, \dots\}$ such that

$$\mathrm{L_j} \subseteq \overset{\circ}{\mathrm{K}}_{\lambda_\mathrm{j}} \tag{1.101}$$

Combine (1.100) with (1.101) and so obtain

$$\mathrm{K_1} \subseteq \overset{\circ}{\mathrm{L}}_{\varkappa_1} \subseteq \mathrm{L}_{\varkappa_1} \subseteq \overset{\circ}{\mathrm{K}}_{\lambda_{\varkappa_1}} \subseteq \mathrm{K}_{\lambda_{\varkappa_1}} \subseteq \overset{\circ}{\mathrm{L}}_{\varkappa_{\lambda_{\varkappa_1}}} \subseteq \cdots , \tag{1.102}$$

which means that the sequence $\mathrm{Q} = (\mathrm{K_1}, \mathrm{L}_{\varkappa_1}, \mathrm{K}_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_{\mathrm{K}} = \tau_{\mathrm{K}_\lambda} = \tau_{\mathrm{Q}} = \tau_{\mathrm{L}_\varkappa} = \tau_{\mathrm{L}}. \tag{1.103}$$

So ends the proof □

[1] See (b) of 2.5 of [3].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [3].

17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$*, for every multi-index* α.

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{\varphi \in C^\infty(\Omega) : \max\{|D^\beta\varphi(x)| : |\beta| \le N, x \in K_N\} < 1/N\right\}. \tag{1.104}$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^\gamma g(x)| : |\gamma| \le N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}. \tag{1.105}$$

(the fact that $K_N \subseteq K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^\beta D^\alpha g(x)| : |\beta| \le N, x \in K_N\} < \frac{1}{N}. \tag{1.106}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad \textit{i.e.} \quad D^\alpha\left(V_{N+|\alpha|}\right) \subseteq V_N, \tag{1.107}$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^\alpha|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$\begin{aligned}
D^\alpha|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) &= D^\alpha\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) && (1.108)\\
&\subseteq D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha\left(\mathscr{D}_K\right) && (1.109)\\
&\subseteq V_N \cap \mathscr{D}_K \quad \text{(see (1.107))} && (1.110)
\end{aligned}$$

So ends the proof. □

Chapter 2

Completeness

3 Exercise 3. An equicontinous sequence of measures

Put $\mathrm{K} = [-1, 1]$*; define* $\mathscr{D}_{\mathrm{K}}$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^{\mathrm{n}}$*). Supose* $\{\mathrm{f_n}\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\varphi = \lim_{n\to\infty} \int_{-1}^{1} \mathrm{f_n}(\mathrm{t})\varphi(\mathrm{t})\mathrm{dt}$ *exists for every* $\varphi \in \mathscr{D}_{\mathrm{K}}$*. Show that* Λ *is a continuous linear functional on* $\mathscr{D}_{\mathrm{K}}$*. Show that there is a positive integer* p *and a number* $\mathrm{M} < \infty$ *such that*

$$\left| \int_{-1}^{1} \mathrm{f_n}(\mathrm{t})\varphi(\mathrm{t})\mathrm{dt} \right| \leq \mathrm{M}\|\mathrm{D^p}\|_\infty$$

for all n*. For example, if* $\mathrm{f_n}(\mathrm{t}) = \mathrm{n^3 t}$ *on* $[-1/\mathrm{n}, 1/\mathrm{n}]$ *and* 0 *elsewhere, show that this can be done with* $\mathrm{p} = 1$*. Construct an example where it can be done with* $\mathrm{p} = 2$ *but not with* $\mathrm{p} = 1$*.*

We will also consider the case $\mathrm{p} = 0$. Since all supports of $\varphi, \varphi', \varphi'', \dots,$ are in K, we make a specialization of the mean value theorem:

Lemma If $\varphi \in \mathscr{D}_{[\mathrm{a,b}]}$, then

$$\|\mathrm{D}^\alpha\varphi\|_\infty \leq \|\mathrm{D^p}\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{\mathrm{p}-\alpha} \qquad (\alpha = 0, 1, \dots, \mathrm{p}) \tag{2.1}$$

at every order $\mathrm{p} = 0, 1, 2, \dots$; where λ is the length $|\mathrm{b} - \mathrm{a}|$.

Proof. Let $\mathrm{x_0}$ be in $(\mathrm{a, b})$. We first consider the case $\mathrm{x_0} \leq \mathrm{c} = (\mathrm{a} + \mathrm{b})/2$: The mean value theorem asserts that there exists $\mathrm{x_1}$ $(\mathrm{a} < \mathrm{x_1} < \mathrm{x_0})$, such that

$$\varphi(\mathrm{x_0}) = \varphi(\mathrm{x_0}) - \varphi(\mathrm{a}) = \mathrm{D}\varphi(\mathrm{x_1})(\mathrm{x_0} - \mathrm{a}). \tag{2.2}$$

Since every $\mathrm{D}^{\mathrm{p}}\varphi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$\begin{aligned}
\varphi(x_0) &= D^0\varphi(x_0) && (2.3)\\
&= D^1\varphi(x_1)(x_0 - a) && (2.4)\\
&= \cdots \\
&= D^p\varphi(x_p)(x_0 - a)\cdots(x_{p-1} - a), && (2.5)
\end{aligned}$$

for all p. More compactly,

$$D^\alpha\varphi(x_0) = D^p\varphi(x_p)\prod_{k=\alpha}^{p-1}(x_k - a); \tag{2.6}$$

which yields,

$$|D^\alpha\varphi(x)| \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (x \in [a, c]) \tag{2.7}$$

The case $x_0 \geq c$ outputs a “reversed” result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^\alpha\varphi(x)| \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \tag{2.8}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\|D^\alpha\varphi\|_\infty \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}. \tag{2.9}$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\varphi\rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})). \tag{2.10}$$

The following inequalities

$$|\langle u|\varphi\rangle| \leq \int_{\mathbf{R}} |u\varphi| \leq \|u\|_{L^1} \quad (\|\varphi\|_\infty \leq 1) \tag{2.11}$$

imply that every linear functional

$$\begin{aligned} \langle u| : C_0(\mathbf{R}) &\to \mathbf{C} \\ \varphi &\mapsto \langle u|\varphi\rangle \end{aligned} \tag{2.12}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [4]. Conversely, u can be identified with $\langle u|$, since u is determined (a.e) by the integrals $\langle u|\varphi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\sup\{|\langle u|\varphi\rangle| : \|\varphi\|_\infty = 1\} = \|u\|_{L^1}. \tag{2.13}$$

Note that, in the latter equality, $\leq \|u\|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$g_n : \mathbf{R} \to \mathbf{R} \tag{2.14}$$
$$x \mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) . \end{cases}$$

First, remark that $g_n(x) \longrightarrow 0$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$\langle g_n|\varphi\rangle \xrightarrow[n\to\infty]{} \Lambda\varphi, \tag{2.15}$$

where φ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\varphi_1^+, \dots, \varphi_j^+, \dots,$ from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\varphi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;
(ii) $\varphi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;
(iii) $0 \leq \varphi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [4] for a possible construction of those φ_j^+. Let $\varphi_1^-, \dots, \varphi_j^-, \dots,$ mirror the φ_j^+, in the sense that $\varphi_j^-(x) = \varphi_j^+(-x)$, so that

(iv) $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$ is odd, as g_n is;
(v) every φ_j is in $C_K^\infty(\mathbf{R})$;
(vi) The sequence $\{\varphi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\left\|\varphi_j\right\|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n|\varphi_j\rangle = 2\int_0^1 g_n(t)\varphi_j^+(t)dt \xrightarrow[j\to\infty]{} 2\int_0^1 g_n(t)dt = \|g_n\|_{L^1} = n. \tag{2.16}$$

Finally,

$$\|g_n\|_{L^1} \overset{(2.16)}{\leq} \sup\{|\langle g_n|\varphi\rangle| : \|\varphi\|_\infty = 1\} \overset{(2.13)}{\leq} \|g_n\|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$|\langle g_n|\varphi\rangle| \leq C_n \quad (\|\varphi\|_\infty = 1); \tag{2.18}$$

[1] see 1.32, 4.1 of [4]
[2] See 7.26 of [2].
[3] See 3.14 of [4] for a definition of the related topology.

see (2.11). Furthermore, $\|g_n\|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\varphi_{\rho(n)}\rangle\}$ such that $\langle g_n|\varphi_{\rho(n)}\rangle$ is greater than, say, $n - 0.01$, as $\left\|\varphi_{\rho(n)}\right\|_\infty = 1$. Consequently, there is no bound M such that

$$(2.19) \qquad |\langle g_n|\varphi\rangle| \leq M \quad (\|\varphi\|_\infty = 1; n = 1, 2, 3, \ldots).$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [4]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$(2.20) \qquad \langle g_n|\varphi\rangle \xrightarrow[n\to\infty]{} \Lambda\varphi \quad (\varphi \in C_0(\mathbf{R}))$$

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [4]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that φ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$(2.21) \qquad \Lambda\varphi = \lim_{n\to\infty} \int_K f_n\varphi \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

for some linear form Λ. Since φ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$(2.22) \qquad \Lambda\varphi = \lim_{n\to\infty} \langle f_n|\varphi\rangle \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

Let $K_1, K_2, \ldots,$ be compact sets that satisfy the conditions specified in 1.44 of [4]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [4] and Exercise 1.16. We know that $K \subseteq K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha\varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\varphi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\varphi \in \mathscr{D}_K : \|\varphi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [4] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\varphi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|f_n\|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [4].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [4]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\varphi\rangle \xrightarrow[n\to\infty]{} \Lambda\varphi$ for all φ, *i.e.* $\Lambda - \langle f_n| \xrightarrow[n\to\infty]{} 0$.

With the help of [2.6] and [2.8] of [4], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$|\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i, \tag{2.23}$$

$$|\Lambda\varphi - \langle f_n|\varphi\rangle| < 1/2 \quad \text{if } \varphi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\varphi\rangle| \le |\Lambda\varphi - \langle f_n|\varphi\rangle| + |\Lambda\varphi| < 1 \qquad \text{if } \varphi \in V_p. \tag{2.25}$$

Now remark that every $\psi = \psi[\mu, \varphi]$, where

$$\psi[\mu,\varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p\|\varphi\|_p)\varphi & (\varphi \ne 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases} \tag{2.26}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\psi\rangle| < 1 \tag{2.27}$$

$$|\langle f_n|\varphi\rangle| < 2^p\|\varphi\|_p \cdot \mu \tag{2.28}$$

$$|\langle f_n|\varphi\rangle| \le 2^p\|\varphi\|_p \tag{2.29}$$

$$|\langle f_n|\varphi\rangle| \le 2^p\{\|D^0\varphi\|_\infty + \cdots + \|D^p\varphi\|_\infty\}. \tag{2.30}$$

Finally, with the help of (2.1),

$$|\langle f_n|\varphi\rangle| \le 2^p(p+1)\|D^p\varphi\|_\infty. \tag{2.31}$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3x$ on [-1/n, 1/n], 0 elsewhere. Actually, we will prove that

(a) $\Lambda\varphi = \lim\limits_{n\to\infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\varphi\rangle| \le M\|D^p\varphi\|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subseteq K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let φ keep running on $\mathscr{D}_K$ and so define

$$B_n(\varphi) \triangleq \max\{|\varphi(x)| : x \in [-1/n, 1/n]\}, \tag{2.32}$$

$$\Delta_n(\varphi) \triangleq \max\{|\varphi(x) - \varphi(0)| : x \in [-1/n, 1/n]\}. \tag{2.33}$$

The mean value asserts that

$$|\varphi(1/n) - \varphi(\text{-}1/n)| \le B_n(\varphi')|1/n - (\text{-}1/n)| = \frac{2}{n}B_n(\varphi'). \tag{2.34}$$

Independently, an integration by parts shows that

$$\langle f_n|\varphi\rangle = \left[\frac{n^3t^2}{2}\varphi(t)\right]_{\text{-}1/n}^{1/n} - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2\varphi'(t)dt \tag{2.35}$$

$$= \frac{n}{2}\left(\varphi(1/n) - \varphi(\text{-}1/n)\right) - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2\varphi'(t)dt. \tag{2.36}$$

Combine (2.34) with (2.36) and so obtain

$$|\langle f_n|\varphi\rangle| \le \frac{n}{2}|\varphi(1/n) - \varphi(\text{-}1/n)| + \frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2|\varphi'(t)|dt \tag{2.37}$$

$$\le B_n(\varphi') + \frac{n^3}{2}B_n(\varphi')\int_{\text{-}1/n}^{1/n} t^2dt \tag{2.38}$$

$$\le \frac{4}{3}B_n(\varphi') \tag{2.39}$$

$$\le \frac{4}{3}\|\varphi'\|_\infty. \tag{2.40}$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\varphi')$ tends to $|\varphi'(0)|$, we may expect that f_n tends to $\frac{4}{3}\varphi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0) \overset{(2.36)}{=} \frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0) - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n}(\varphi' - \varphi'(0))t^2dt. \tag{2.41}$$

So,

$$\left|\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0)\right| \le \left|\frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0)\right| + \frac{1}{3}\Delta_n(\varphi') \xrightarrow[n\to\infty]{} 0. \tag{2.42}$$

We have just proved that

$$\langle f_n|\varphi\rangle \xrightarrow[n\to\infty]{} \frac{4}{3}\varphi'(0) \quad (\varphi \in \mathscr{D}_K). \tag{2.43}$$

In other words,

$$\langle f_n| \xrightarrow[n\to\infty]{} \text{-}\frac{4}{3}\delta', \tag{2.44}$$

where δ is the *Dirac measure* and $\delta', \delta'', \dots,$ its *derivatives*; see 6.1 and 6.9 of [4].

It follows from the previous part that $\text{-}\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$|\langle f_n|\varphi\rangle| \le \frac{4}{3}\|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots) \tag{2.45}$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$\{\langle \mathrm{f_n}|\varphi_{\rho(\mathrm{n})}\rangle : \left\|\varphi_{\rho(\mathrm{n})}\right\|_\infty = 1; \mathrm{n} = 1, 2, 3, \ldots\} \tag{2.46}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle \mathrm{f_n}|\varphi\rangle| \le \mathrm{M}\|\varphi\|_\infty \quad (\varphi \in \mathrm{C}^\infty_\mathrm{K}(\mathbf{R})). \tag{2.47}$$

The previous bound of $\langle \mathrm{f_n}|$ - see (2.40), is therefore the best possible one, *i.e.* $\mathrm{p} = 1$ is the smallest possible p and, given $\mathrm{p} = 1$, $\mathrm{M} = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give $\mathrm{f_n}$ a *derivative*[4] $\mathrm{f_n}'$, as follows

$$\begin{aligned} \mathrm{f_n}' : \mathscr{D}_\mathrm{K} &\to \mathbf{C} \\ \varphi &\mapsto \text{-} \langle \mathrm{f_n}|\varphi'\rangle \,. \end{aligned} \tag{2.48}$$

It has been proved that every $\langle \mathrm{f_n}|$ is continuous. So is

$$\begin{aligned} \mathrm{D} : \mathscr{D}_\mathrm{K} &\to \mathscr{D}_\mathrm{K} \\ \varphi &\mapsto \varphi'; \end{aligned} \tag{2.49}$$

see Exercise 1.17. $\mathrm{f_n}'$ is therefore continuous. Now apply (2.43) with φ' and so obtain

$$\text{-}\langle \mathrm{f_n}|\varphi'\rangle \xrightarrow[\mathrm{n}\to\infty]{} \frac{4}{3}\varphi''(0) \quad (\varphi \in \mathscr{D}_\mathrm{K}),$$

i.e.

$$\mathrm{f_n}' \xrightarrow[\mathrm{n}\to\infty]{} \frac{4}{3}\delta''. \tag{2.50}$$

It follows from (2.40) that,

$$|\langle \mathrm{f_n}|\varphi'\rangle| \le \frac{4}{3}\|\varphi''\|_\infty \quad (\mathrm{n} = 1, 2, 3, \ldots). \tag{2.51}$$

It is therefore possible to uniformly bound $\mathrm{f_n}'$ with respect to a norm $\|\mathrm{D^p}\cdot\|_\infty$, namely $\|\mathrm{D}^2\cdot\|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle \mathrm{f_n}|\varphi'\rangle| \le \mathrm{M}\|\varphi'\|_\infty \quad (\mathrm{n} = 1, 2, 3, \ldots). \tag{2.52}$$

Define

$$\Phi_\mathrm{j}(\mathrm{x}) = \int_{-1}^{\mathrm{x}} \varphi_\mathrm{j}. \tag{2.53}$$

[4]See 6.1 of [4] for a further discussion.

The oddness of φ_j forces Φ_j to vanish outside [-1, 1]: φ_j is therefore in $\mathscr{D}_K$. So, under our assumption,

$$|\langle f_n|\Phi_j'\rangle| \leq M\|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots); \tag{2.54}$$

which is

$$|\langle f_n|\varphi_j\rangle| \leq M \quad (n = 1, 2, 3, \dots). \tag{2.55}$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n|\varphi_{\rho(n)}\rangle\}$) and so conclude that there is no constant M such that

$$|\langle|f_n\varphi'\rangle| \leq M\|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.56}$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$|\langle f_n|\varphi'\rangle| \leq M\|\varphi\|_\infty. \tag{2.57}$$

The mean value theorem (see (2.1)) asserts that

$$|\langle f_n|\varphi'\rangle| \leq M\|\varphi\|_\infty \leq M\|\varphi'\|_\infty; \tag{2.58}$$

which is, again, a desired contradiction. So ends the proof.

$\square$

6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{\mathrm{f}}(\mathrm{n})$ *of a function* $\mathrm{f} \in \mathrm{L}^2(\mathrm{T})$ *(*T *is the unit circle) by*

$$\hat{\mathrm{f}}(\mathrm{n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{in}\theta}\mathrm{d}\theta$$

for all $\mathrm{n} \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_{\mathrm{n}}\mathrm{f} = \sum_{\mathrm{k}=-\mathrm{n}}^{\mathrm{n}} \hat{\mathrm{f}}(\mathrm{k}).$$

Prove that $\{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \lim_{\mathrm{n}\infty} \Lambda_{\mathrm{n}}\mathrm{f}$ *exists*$\}$ *is a dense subspace of* $\mathrm{L}^2(\mathrm{T})$ *of the first category.*

Proof. Let $\mathrm{f}(\theta)$ stand for $\mathrm{f}(\mathrm{e}^{\mathrm{i}\theta})$, so that $\mathrm{L}^2(\mathrm{T})$ is identified with a closed subset of $\mathrm{L}^2([-\pi, \pi])$, hence the inner product

$$\hat{\mathrm{f}}(\mathrm{n}) = (\mathrm{f}, \mathrm{e}_{\mathrm{n}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(\theta)\mathrm{e}^{-\mathrm{in}\theta}\mathrm{d}\theta. \tag{2.59}$$

We believe it is customary to write

$$\Lambda_{\mathrm{n}}(\mathrm{f}) = (\mathrm{f}, \mathrm{e}_{-\mathrm{n}}) + \cdots + (\mathrm{f}, \mathrm{e}_{\mathrm{n}}). \tag{2.60}$$

Moreover, a well known (and easy to prove) result is

(2.61)

$$(\mathrm{e}_{\mathrm{n}}, \mathrm{e}_{\mathrm{n}'}) = [\mathrm{n} = \mathrm{n}'], \textit{i.e.}\ \{\mathrm{e}_{\mathrm{n}} : \mathrm{n} \in \mathbf{Z}\} \text{ is an orthormal subset of } \mathrm{L}^2(\mathrm{T}).$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $\mathrm{L}^2 \equiv (\mathrm{L}^2)^*$. So,

$$\|\Lambda_{\mathrm{n}}\|^2 \overset{(2.60)}{=} \|\mathrm{e}_{-\mathrm{n}} + \cdots + \mathrm{e}_{\mathrm{n}}\|^2 \overset{(2.61)}{=} \|\mathrm{e}_{-\mathrm{n}}\|^2 + \cdots + \|\mathrm{e}_{\mathrm{n}}\|^2 \overset{(2.61)}{=} 2\mathrm{n} + 1. \tag{2.62}$$

We now assume, to reach a contradiction, that

$$\mathrm{B} \triangleq \{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \sup\{|\Lambda_{\mathrm{n}}\, \mathrm{f}| : \mathrm{n} = 1, 2, 3, \ldots\} < \infty\} \tag{2.63}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [4] asserts that the sequence $\{\Lambda_{\mathrm{n}}\}$ is norm-bounded; which is a desired contradiction, since

$$\|\Lambda_{\mathrm{n}}\| \overset{(2.62)}{=} \sqrt{2\mathrm{n}+1} \xrightarrow[\mathrm{n}\to\infty]{} \infty. \tag{2.64}$$

We have just established that B is actually of the first category; and so is its subset L=$\{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \lim_{\mathrm{n}\longrightarrow\infty} \Lambda_{\mathrm{n}}\mathrm{f}$ exists$\}$. We now prove that L is nevertheless dense in $\mathrm{L}^2(\mathrm{T})$. To do so, we let P be span$\{\mathrm{e}_{\mathrm{k}} : \mathrm{k} \in \mathrm{Z}\}$, the collection of the trignometric polynomials $\mathrm{p}(\theta) = \sum \lambda_{\mathrm{k}}\mathrm{e}^{\mathrm{ik}\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_{\mathrm{n}}(\mathrm{p}) = \sum \lambda_{\mathrm{k}}$ for almost all n. Thus,

$$\mathrm{P} \subseteq \mathrm{L} \subseteq \mathrm{L}^2(\mathrm{T}). \tag{2.65}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $\mathrm{L}^2(\mathrm{T})$. We then conclude, with the help of (2.65), that

$$\mathrm{L}^2(\mathrm{T}) = \overline{\mathrm{P}} = \overline{\mathrm{L}}. \tag{2.66}$$

So ends the proof ☐

9 Exercise 9. Boundedness without closedness

Suppose $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ *are Banach spaces and*

$$\mathrm{B} : \mathrm{X} \times \mathrm{Y} \to \mathrm{Z}$$

is bilinear and continuous. Prove that there exists $\mathrm{M} < \infty$ *such that*

$$\|\mathrm{B}(\mathrm{x}, \mathrm{y})\| \leq \mathrm{M}\|\mathrm{x}\|\|\mathrm{x}\| \quad (\mathrm{x} \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Since B is continous at the origin, there exists a positive r such that

$$\|\mathrm{x}\| + \|\mathrm{y}\| < \mathrm{r} \Rightarrow \|\mathrm{B}(\mathrm{x}, \mathrm{y})\| < 1. \tag{2.67}$$

Given nonzero x, y, let s range over $]0, \mathrm{r}[$, so that the folllowing bound

$$\|\mathrm{B}(\mathrm{x}, \mathrm{y})\| = \frac{4\|\mathrm{x}\|\|\mathrm{y}\|}{\mathrm{s}^2} \left\| \mathrm{B}\left(\frac{\mathrm{s}}{2\|\mathrm{x}\|}\mathrm{x}, \frac{\mathrm{s}}{2\|\mathrm{y}\|}\mathrm{y}\right) \right\| \overset{(2.67)}{<} \frac{4\|\mathrm{x}\|\|\mathrm{y}\|}{\mathrm{s}^2} \tag{2.68}$$

is effective. It is now obvious that

$$\mathrm{B}(\mathrm{x}, \mathrm{y}) \leq \frac{4}{\mathrm{s}^2}\|\mathrm{x}\|\|\mathrm{y}\| \underset{\mathrm{s} \to \mathrm{r}}{\longrightarrow} \frac{4}{\mathrm{r}^2}\|\mathrm{x}\|\|\mathrm{y}\| \quad ((\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y}); \tag{2.69}$$

which achieves the proof.

As a concrete example, choose $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = \mathrm{C_c}(\mathbf{R})$, topologized by the supremum norm. $\mathrm{C_c}(\mathbf{R})$ is not complete (see 5.4.4 of [5]), nevertheless the bilinear product

$$\begin{array}{rccc} \mathrm{B} : & \mathrm{C_c}(\mathbf{R})^2 & \to & \mathrm{C_c}(\mathbf{R}) \\ & (\mathrm{f}, \mathrm{g}) & \mapsto & \mathrm{f} \cdot \mathrm{g} \end{array}$$

is bounded (since $\|\mathrm{f} \cdot \mathrm{g}\|_\infty \leq \|\mathrm{f}\|_\infty \cdot \|\mathrm{g}\|_\infty$), and continuous. To show this, pick a positive scalar ε smaller than 1, provided any (f, g). Next, define

$$\mathrm{r} \triangleq \frac{\varepsilon}{1 + \|\mathrm{f}\|_\infty + \|\mathrm{g}\|_\infty} < 1. \tag{2.70}$$

We now restrict (u, v) to a particular neighborhood of (f, g). More specifically,

$$\|\mathrm{f} - \mathrm{u}\|_\infty + \|\mathrm{g} - \mathrm{v}\|_\infty < \mathrm{r}. \tag{2.71}$$

Next, remark that $\|\mathrm{u}\|_\infty \leq \mathrm{r} + \|\mathrm{f}\|_\infty$ and so obtain (bear in mind that $\mathrm{r} < 1$)

$$\begin{aligned}
\|\mathrm{fg} - \mathrm{uv}\|_\infty &= \|(\mathrm{f} - \mathrm{u}) \cdot \mathrm{g} + \mathrm{u} \cdot (\mathrm{g} - \mathrm{v})\|_\infty && (2.72)\\
&\leq \|\mathrm{f} - \mathrm{u}\|_\infty \cdot \|\mathrm{g}\|_\infty + \|\mathrm{u}\|_\infty \cdot \|\mathrm{g} - \mathrm{v}\|_\infty && (2.73)\\
&< \mathrm{r} \cdot \|\mathrm{g}\|_\infty + (\mathrm{r} + \|\mathrm{f}\|_\infty) \cdot \mathrm{r} && (2.74)\\
&< \mathrm{r} \cdot (\mathrm{r} + \|\mathrm{f}\|_\infty + \|\mathrm{g}\|_\infty) && (2.75)\\
&< \varepsilon. && (2.76)
\end{aligned}$$

Since ε was arbitrary, it is now established that B continuous at every (f, g). $\square$

10 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let $(\mathrm{X}_1, \mathrm{X}_2, \mathrm{Z})$ be topological spaces and B a bilinear mapping

$$\mathrm{B} : \mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{Z}. \tag{2.77}$$

From now on, $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)$ denotes an arbitrary element of $\mathrm{X}_1 \times \mathrm{X}_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $\mathrm{X}_1 \times \mathrm{X}_2$, *i.e.* given an arbitrary **balanced** open subset W of Z, there exists in X_i $(i = 1, 2)$ a **balanced** open subset U_i such that

$$\mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) \subseteq \mathrm{W}. \tag{2.78}$$

In such context, $\lambda_i(\mathrm{x})$ is chosen greater than $\mu_i(\mathrm{x}_i) = \inf\{r > 0 : \mathrm{x}_i \in r \cdot \mathrm{U}_i\}$; see [1.33] of [4] for further reading about the *Minkowski functionals* μ. In other words, x_i lies in $\lambda_i(\mathrm{x})\mathrm{U}_i$, since U_i is balanced. Thus,

$$\begin{aligned}
\mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) &= \lambda_1(\mathrm{x})\lambda_2(\mathrm{x}) \cdot \mathrm{B}(\mathrm{x}_1/\lambda_1(\mathrm{x}), \mathrm{x}_2/\lambda_2(\mathrm{x})) && (2.79)\\
&\in \lambda_1(\mathrm{x})\lambda_2(\mathrm{x}) \cdot \mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) && (2.80)\\
&\subseteq \lambda_1(\mathrm{x})\lambda_2(\mathrm{x}) \cdot \mathrm{W}. && (2.81)
\end{aligned}$$

Pick $\mathrm{p} = (\mathrm{p}_1, \mathrm{p}_2)$ in $\mathrm{X}_1 \times \mathrm{X}_2$, and let $\mathrm{q} = (\mathrm{q}_1, \mathrm{q}_2)$ range over $\mathrm{X} \times \mathrm{Y}$, as a first step: It directly follows from (2.81) that

$$\begin{aligned}
&(2.82)\\
&\mathrm{B}(\mathrm{p}) - \mathrm{B}(\mathrm{q}) = \mathrm{B}(\mathrm{p}_1, \mathrm{p}_2 - \mathrm{q}_2) + \mathrm{B}(\mathrm{p}_1, \mathrm{q}_2) - \mathrm{B}(\mathrm{q}_1, \mathrm{q}_2)\\
&(2.83) \qquad = \mathrm{B}(\mathrm{p}_1, \mathrm{p}_2 - \mathrm{q}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{q}_1, \mathrm{q}_2)\\
&(2.84) \qquad = \mathrm{B}(\mathrm{p}_1, \mathrm{p}_2 - \mathrm{q}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{q}_1, \mathrm{q}_2 - \mathrm{p}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{q}_1, \mathrm{p}_2)\\
&(2.85) \qquad \in \lambda_1(\mathrm{p})\lambda_2(\mathrm{p} - \mathrm{q})\mathrm{W} + \lambda_1(\mathrm{p} - \mathrm{q})\lambda_2(\mathrm{q} - \mathrm{p})\mathrm{W} + \lambda_1(\mathrm{p} - \mathrm{q})\lambda_2(\mathrm{p})\mathrm{W}.
\end{aligned}$$

We now restrict q to a particular neighborhood of p. More specifically,

$$\mathrm{p}_i - \mathrm{q}_i \in \frac{1}{\mu_1(\mathrm{p}_1) + \mu_2(\mathrm{p}_2) + 2}\mathrm{U}_i; \tag{2.86}$$

which implies

$$\mu_i(\mathrm{q}_i - \mathrm{p}_i) = \mu_i(\mathrm{p}_i - \mathrm{q}_i) \leq \frac{1}{\mu_1(\mathrm{p}_1) + \mu_2(\mathrm{p}_2) + 2} \tag{2.87}$$

(the equality at the left is valid, since $\mathrm{U}_i = -\mathrm{U}_i$). The special case

$$\lambda_i(\mathrm{p}) \triangleq \mu_1(\mathrm{p}_1) + \mu_2(\mathrm{p}_2) + 1, \tag{2.88}$$

$$\lambda_i(\mathrm{p} - \mathrm{q}) \triangleq \frac{1}{\mu_1(\mathrm{p}_1) + \mu_2(\mathrm{p}_2) + 1} \triangleq \lambda_i(\mathrm{q} - \mathrm{p}) \tag{2.89}$$

implies that

$$\mathrm{B}(\mathrm{p}) - \mathrm{B}(\mathrm{q}) \in \mathrm{W} + \mathrm{W} + \mathrm{W}, \tag{2.90}$$

since W is balanced. W being arbitrary, we have so established the continuousness of B at arbitrary p; which achieves the proof. □

12 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)|\ dt.$$

Put $B(f,g) = \int_0^1 f(t)g(t)dt$*, and show that* B *is a bilinear continuous functional on* $X \times X$ *which is separately but not continuous.*

Proof. Let f denote the first variable, g the second one. Remark that

$$|B(f,g)| < \|f\| \cdot \max_{[0,1]} |g|; \tag{2.91}$$

which is sufficient (1.18 of [4]) to assert that any $f \mapsto B(f,g)$ is continuous. The continuity of all $g \mapsto B(f,g)$ follows (Put $C(g,f) = B(f,g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$|B(f,g)| < M\|f\|\|g\|. \tag{2.92}$$

Put

$$f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \qquad (n = 1, 2, 3, \dots), \tag{2.93}$$

so that

$$\|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow[n\to\infty]{} 0. \tag{2.94}$$

On the other hand,

$$B(f_n, f_n) = \frac{4n}{2n+1} > 1. \tag{2.95}$$

Finally, we combine (2.95) and (2.92) with (2.94) and so obtain

$$1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow[n\to\infty]{} 0. \tag{2.96}$$

Our continuousness assumption is then contradicted. So ends the proof. □

15 Exercise 15. Baire's cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* $\mathrm{Y}=\mathrm{X}$. Hint: Y *must intersect* $\mathrm{x}+\mathrm{Y}$ *for every* $\mathrm{x}\in\mathrm{X}$.

Proof. Assume that $\mathrm{X}=\mathrm{V_n}\sqcup\overline{\mathrm{E}}_{\mathrm{n}}$ $(\mathrm{n}=1,2,3,\dots)$ such that

(i) $\mathrm{X}=\overline{\mathrm{V}}_{\mathrm{n}}$;
(ii) $\mathrm{X}\setminus\mathrm{Y}=\bigcup_{\mathrm{n}=1}^{\infty}\mathrm{E_n}$.

First, let x range over X, so that $\mathrm{x}+\mathrm{V_n}$ is open and dense as well (because[5] the translation by x is a homeomorphism of X onto X). Next, apply the Baire's theorem twice to establish that

(a) every intersection $\mathrm{W_n}=\mathrm{V_n}\cap[\mathrm{x}+\mathrm{V_n}]$ is dense in X;
(b) so is the (then nonempty) intersection $\bigcap_{\mathrm{n}=1}^{\infty}\mathrm{W_n}$.

Moreover, the intersection $\bigcap_{\mathrm{n}=1}^{\infty}\mathrm{V_n}$ cuts no $\mathrm{E_n}$. To sum up,

$$\mathrm{w}\overset{(b)}{\in}\bigcap_{\mathrm{n}=1}^{\infty}\mathrm{W_n}\overset{(a)}{\subseteq}\bigcap_{\mathrm{n}=1}^{\infty}\mathrm{V_n}\overset{(ii)}{\subseteq}\mathrm{Y} \tag{2.97}$$

for some $\mathrm{w}=\mathrm{w(x)}\in\mathrm{Y}$. Note that w also lies in every $\mathrm{x}+\mathrm{V_n}$, by (a). Hence

$$\mathrm{w}-\mathrm{x}\in\bigcap_{\mathrm{n}=1}^{\infty}\mathrm{V_n}\overset{(2.97)}{\subseteq}\mathrm{Y}. \tag{2.98}$$

Finally, combining (2.97) with (2.98) yields

$$\mathrm{x}=\mathrm{w}-(\mathrm{w}-\mathrm{x})\in\mathrm{Y}-\mathrm{Y}=\mathrm{Y}, \tag{2.99}$$

since Y is a subspace (subgroup) of X. We have so established that

$$\mathrm{X}\subseteq\mathrm{Y}, \tag{2.100}$$

which achieves the proof. ∎

[5] This is also a special case of 1.3 (b) of [4], where: $\mathrm{X}=\mathrm{x}+\mathrm{X}\subseteq\overline{\mathrm{x}+\mathrm{V_n}}$.

16 Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $f : X \to K$ *is a closed subset of* $X \times K$*. Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

Proof. Choose a sequence $\{x_n : n = 1, 2, 3, \dots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of $X \times K$. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$f : [0, \infty) \to [0, \infty) \qquad (2.101)$$
$$x \mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases}$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}. \qquad (2.102)$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b). To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on R_+. This establishes that $(a, b) \in G_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. □

Chapter 3

Convexity

3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subseteq X$ *an internal point of* A *if* $A - x_0$ *is an absorbing set.*

(*a*) *Suppose* A *and* B *are disjoint convex sets in* X*, and* A *has an internal point. Prove that there is a nonconstant linear functional* Λ *such that* $\Lambda(A) \cap \Lambda(B)$ *contains at most one point. (The proof is similar to that of Theorem 3.4)*

(*b*) *Show (with* $X = \mathbf{R}^2$*, for example) that it may not possible to have* $\Lambda(A)$ *and* $\Lambda(B)$ *disjoint, under the hypotheses of (a).*

Proof. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ $(b_0 \in B)$. Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [4] to establish the existence of a linear functional $\Lambda : X \to \mathbf{R}$ such that

$$\Lambda \leq p \tag{3.1}$$

and

$$\Lambda(b_0 - x_0) = 1. \tag{3.2}$$

Then

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B). \tag{3.3}$$

Hence

$$\Lambda a \leq \Lambda b. \tag{3.4}$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There so exists (a_i, b_i) in $A \times B$ $(i = 1, 2)$ such that

$$\Lambda a_i = \Lambda b_i = y_i. \tag{3.5}$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad . \tag{3.6}$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

$$\Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1 \quad ; \tag{3.7}$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\}, \tag{3.8}$$
$$S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}, \tag{3.9}$$
$$A \triangleq S_1 \cup S_2, \tag{3.10}$$
$$B \triangleq X \setminus A. \tag{3.11}$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A. \tag{3.12}$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.* A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a linear functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$\Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad . \tag{3.13}$$

So ends the proof. □

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [3].

11 Exercise 11. Meagerness of the polar

Let X *be an infinite-dimensional Fréchet space. Prove that* X^**, with its weak***-topology, is of the first category in itself.*

This is actually a consequence of the below lemma, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual* X^* *separates points on* X*, then the polar*

$$K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\} \tag{3.14}$$

of any absorbing subset A is a weak**-closed set that has empty interior.*

Proof. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous; see 3.14 of [4]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see 3.11 of [4]). Put

$$W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \qquad (r_x > 0) \tag{3.15}$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$M \triangleq \mathrm{span}(F) \tag{3.16}$$

is finite dimensional. Assume, to reach a contradiction, that $A \subseteq M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subseteq M$: Now pick a in $A \setminus M$ and so conclude that

$$b \triangleq \frac{a}{t_a} \in A \tag{3.17}$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see 1.21 (b) of [4] for a proof): By the Hahn-Banach theorem 3.5 of [4], there exists Λ_a in X^* such that

$$\Lambda_a b > 2 \tag{3.18}$$

and

$$\Lambda_a(M) = \{0\}. \tag{3.19}$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$|\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1. \tag{3.20}$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

$$\Lambda + W \not\subseteq K_A. \tag{3.21}$$

Since W and Λ are both arbitrary, this achieves the proof. □

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$B_n \triangleq \{x \in X : d(0, x) < 1/n\} \qquad (n = 1, 2, 3, \dots) \tag{3.22}$$

form a local base of X. If Λ is in X^*, then

$$|\Lambda| \leq m \text{ on } B_n \tag{3.23}$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$; see 1.18 of [4]. Hence, X^* is the countable union of all

$$m \cdot K_n \qquad (m, n = 1, 2, 3, \dots), \tag{3.24}$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [4]. As a consequence, the above lemma implies

$$\left(\overline{K_n}\right)^\circ = (K_n)^\circ = \emptyset. \tag{3.25}$$

Since the multiplication by m is an homeomorphism (see 1.7 of [4]), this is equivalent to

$$\left(\overline{m \cdot K_n}\right)^\circ = m \cdot (K_n)^\circ = \emptyset. \tag{3.26}$$

So ends the proof. □

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

1 Exercise 1. Basic results

Let φ be the embedding of X *into* X^{**} *decribed in Section 4.5. Let τ be the weak topology of* X*, and let σ be the weak***- topology of* X^{**}*- the one induced by* X^**.*

(*a*) *Prove that φ is an homeomorphism of* (X, τ) *onto a dense subspace of* (X^{**}, σ)*.*

(*b*) *If* B *is the closed unit ball of* X*, prove that* $\varphi(B)$ *is σ-dense in the closed unit ball of* X^{**}*. (Use the Hahn-Banach separation theorem.)*

(*c*) *Use (a), (b), and the Banach-Alaoglu theorem to prove that* X *is reflexive if and only if* B *is weakly compact.*

(*d*) *Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.*

(*e*) *If* X *is reflexive and* Y *is a closed subspace of* X*, prove that* X/Y *is reflexive.*

(*f*) *Prove that* X *is reflexive if and only* X^* *if reflexive.*
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\varphi(X)$ *of* X^{**}*.*

Proof. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$\varphi^{-1} : \varphi(X) \to X \tag{4.1}$$

$$\varphi(x) \mapsto x \tag{4.2}$$

is an homeomorphism (with respect to τ and σ). We first consider

$$V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}|\psi x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0); \tag{4.3}$$

$$U \triangleq \{x \in X : |\langle x|x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0). \tag{4.4}$$

and remark that the so defined V 's (respectively U 's) shape a local subbase $\mathscr{S}_\sigma$ (respectively $\mathscr{S}_\tau$) of σ (respectively τ). We now observe that

$$U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathscr{S}_\sigma,\ U \in \mathscr{S}_\tau) \quad , \tag{4.5}$$

since φ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathscr{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that φ^{-1} and its inverse φ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$\overline{\varphi(X)}_\sigma \subseteq X^{**} \quad . \tag{4.6}$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\varphi(X)$. By [3.5], there so exists y^* in X^* such that

$$\langle \varphi x, \psi y^*\rangle = \langle y^*, \varphi x\rangle = \langle x, y^*\rangle = 0 \quad (x \in X) \quad ; \tag{4.7}$$

$$\langle z^{**}, \psi y^*\rangle = 1 \tag{4.8}$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$\varphi(B) \subseteq B^{**} \quad , \tag{4.9}$$

since φ is isometric. Hence

$$\overline{\varphi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} \quad . \tag{4.10}$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\varphi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$|\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_\sigma \quad ; \tag{4.11}$$

$$\langle z^{**}, \psi y^*\rangle > 1 \quad . \tag{4.12}$$

It follows from (4.11) that

$$|\,\psi y^*\,| \leq 1 \text{ on } \varphi(B)\,,\ \textit{i.e.}\, |\,y^*\,| \leq 1 \text{ on } B \quad . \tag{4.13}$$

We have so proved that

$$y^* \in B^* \quad . \tag{4.14}$$

Since z^{**} lies in B^{**}, it is now clear that

$$|\langle z^{**}, \psi y^{*}\rangle| \leq 1 \quad ; \tag{4.15}$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

(4.16) B is weakly compact if and only if $\varphi(B)$ is weak*-compact.

If B is weakly compact, then $\varphi(B)$ is weak*-closed. So,

$$\varphi(B) = \overline{\varphi(B)}_{\sigma} \overset{(b)}{=} B^{**} \quad . \tag{4.17}$$

φ is therefore onto, *i.e.* X is reflexive.
Conversely, keep φ as onto: one easily checks that $\varphi(B) = B^{**}$. The image $\varphi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(X/Y)^{*} \equiv Y^{\perp} \quad , \tag{4.18}$$

next,

$$(X/Y)^{**} \equiv (Y^{\perp})^{*} \equiv X^{**}/(Y^{\perp})^{\perp} \equiv X/Y \quad . \tag{4.19}$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

> *Given a reflexive Banach space* Z*, the weak***-topology of* Z^* *is its weak one.*

Assume first that X is reflexive. Since B^{*} is weak* compact, by (c) of [4.3], (L) implies that B^{*} is also weakly compact. Then (c) turns X^{*} into a reflexive space.

Conversely, let X^{*} be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\varphi(X)$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\varphi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \varphi(X)$ is weakly compact, *i.e.* weak*-compact (to see this, apply (L) with $Z = X^{*}$).

By (a), B is therefore weakly compact, *i.e.* X is reflexive; see (c). So ends the proof. □

13 Exercise 13. Operator compactness in a Hilbert space

(a) Suppose $T \in \mathscr{B}(X,Y), T_n \in \mathscr{B}(X,Y)$ *for* $n = 1, 2, 3, \ldots,$ *each* T_n *has finite-dimensional range, and* $\lim \|T - T_n\| = 0$*. Prove that* T *is compact.*

(b) Assume Y *is a Hilbert space, and prove the converse of (a): Every compact* $T \in \mathscr{B}(X,Y)$ *can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)*

Proof. Since each T_n is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every T_n as a compact operator. Let $\{x_0^i : i \in \mathbf{N}\}$ be in U the open unit ball of X. Since T_1 is compact, $\{x_0^i\}$ contains a subsequence $\{x_1^i : i \in \mathbf{N}\}$ such that $\{T_1 x_1^i\}$ converges to a point y_1 of Y. The same reasoning can be recursively applied to T_n and $\{x_{n-1}^i\} \subseteq U$ so that $\{T_n x_n^i\}$ tends to some y_n of Y, as $\{x_n^i\}$ is a subsequence of $\{x_{n-1}^i\}$. Then

$$(4.20) \qquad T_n x_p^i \underset{i\to\infty}{\longrightarrow} y_n \quad (p\, n = 1, 2, 3, \cdots) \quad .$$

Applied with $\{x_n^i : (n,i) \in \mathbf{N}^2\}$, a Cantor's diagonal process therefore provides a subsequence $\{\tilde{x}_j : j \in \mathbf{N}\}$ such that

$$(4.21) \qquad T_j \tilde{x}_k \underset{k\to\infty}{\longrightarrow} y_j \quad ;$$

$$(4.22) \qquad T_j \tilde{x}_j \underset{j\to\infty}{\longrightarrow} y_j \quad .$$

We now easily obtain

(4.23)
$$\|T_j \tilde{x}_j - T_k \tilde{x}_k\| \leq \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \underset{k>j\to\infty}{\longrightarrow} 0 \quad .$$

$\{T_j \tilde{x}_j\}$ is then a Cauchy sequence. So is $\{T \tilde{x}_j\}$, since $\|T - T_j\| \to 0$. On the other hand, Y is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix ε as a positive scalar. Since T is compact, Y contains a finite set C such that

$$(4.24) \qquad T(U) \subseteq \bigcup_{c\in C} B(c, \varepsilon) \quad .$$

As a Hilbert space, Y contains a *maximal orthonormal set* (or *Hilbert basis*) M. This implies that $\mathrm{span}(M)$ is dense in Y; *cf.* 4.18 & 4.22 of [3]. The finiteness of C forces M to enclose a finite set S so that

$$(4.25) \qquad \forall c \in C, \exists s(c) \in \mathrm{span}(S) : \|c - s(c)\| < \varepsilon \quad .$$

Let x be in U. It follows from (4.24) that

$$\|\mathrm{T}x - c_x\| < \varepsilon \tag{4.26}$$

for some c_x of C. We now combine (4.25) and (4.26) to obtain

$$\|\mathrm{T}x - s(c_x)\| \leq \|\mathrm{T}x - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon \tag{4.27}$$

As a finite-dimensional subspace, span(S) is closed (see footnote 4, Exercise 1.10). We so obtain

$$\mathrm{Y} = \mathrm{span}(\mathrm{S}) \oplus \mathrm{span}(\mathrm{S})^{\perp} \quad , \tag{4.28}$$

by [12.4]. There so exists a unique projection projection $\pi = \pi(\varepsilon)$ of Y onto itself (see [5.6] for the definition) such that

$$\pi(\mathrm{Y}) = \mathrm{span}(\mathrm{S})\,, \;\; (\mathrm{I} - \pi)(\mathrm{Y}) = \mathrm{span}(\mathrm{S})^{\perp} \quad . \tag{4.29}$$

It is easily checked that π has norm 1. Moreover,

$$\pi s = s \quad (s \in \mathrm{span}(\mathrm{S})) \quad . \tag{4.30}$$

Thus,

$$(\mathrm{I} - \pi)(\mathrm{T}x) = (\mathrm{I} - \pi)(\mathrm{T}x - s(c_x)) \quad (x \in \mathrm{U}) \quad . \tag{4.31}$$

Then,

$$\|(\mathrm{I} - \pi)(\mathrm{T}x)\| \leq \|\mathrm{I} - \pi\|\,\|\mathrm{T}x - s(c_x)\| < 4\varepsilon \quad (x \in \mathrm{U}) \tag{4.32}$$

(the fact that π has norm 1 is hidden in the right side inequality). We have just so proved that

$$\|\mathrm{T} - \pi \circ \mathrm{T}\| \in \underset{\varepsilon \sim 0}{\mathrm{O}}(\varepsilon) \quad . \tag{4.33}$$

That is particularly true whether $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$. Let so T_n be $\pi(\varepsilon_n) \circ \mathrm{T}$ and conclude that these (compact) operators approximate T in the desired fashion, *i.e.*

$$\|\mathrm{T} - \mathrm{T}_n\| \underset{n\to\infty}{\longrightarrow} 0 \quad . \tag{4.34}$$

□

15 Exercise 15. Hilbert-Schmidt operators

Suppose μ *is a finite (or* σ*-finite) positive measure on a measure space* Ω*,* $\mu\times\mu$ *is the corresponding product measure on* $\Omega\times\Omega$*, and* $K\in L^2(\mu\times\mu)$*. Define*

$$(Tf)(s)=\int_\Omega K(s,t)f(t)d\mu(t)\quad [f\in L^2(\mu)].$$

(a) Prove that $T\in\mathscr{B}(L^2(\mu))$ *and that*

$$\|T\|^2\le\int_\Omega\int_\Omega|K(s,t)|^2d\mu(s)d\mu(t).$$

(b) Suppose a_i*,* b_i *are members of* $L^2(\mu)$*, for* $1\le i\le n$*, put* $K_1=\sum a_i(s)b_i(t)$ *and define* T_1 *in terms of* K_1 *a* T *was defined in terms of* K*. Prove that* $\dim\mathscr{R}(T_1)\le n$*.*

(c) Deduce that T *is a compact operator in* $L^2(\mu)$*. Hint: Use exercise 13.*

(d) Suppose $\lambda\in\mathbf{C}$*,* $\lambda\ne0$*. Prove: Either the equation*

$$Tf-\lambda f=g$$

has a unique solution $f\in L^2(\mu)$ *for every* $g\in L^2(\mu)$ *or there are infinitely many solutions for some* g *and none for others. (This is known as the Fredholm alternative.).*

(e) Describe the adjoint of T*.*

Proof. Let X (respectively P) be the Banach space $L^2(\mu)$ (respectively $L^2(\mu\times\mu)$). A consequence of the Radon-Nikodym theorem (*cf.* 6.16 of [3]) is that there exists a group isomorphism $\rho:X\to X^*$, $f\mapsto f^*$ such that

$$\langle u,f^*\rangle=\int_\Omega u\cdot f\mu\quad (u\in X,\ f\in X)\quad . \tag{4.35}$$

Define a.e K_s, $K_t:\Omega\to\mathbf{C}$ by setting

$$K_s(t)\triangleq K_t(s)\triangleq K(s,t)\ \text{ a.e }\quad ((s,t)\in\Omega)\quad . \tag{4.36}$$

T is clearly linear. Moreover,

$$|(Tf)(s)|=|\langle K_s,f^*\rangle|\le\|K_s\|_X\quad (\|f\|_X<1) \tag{4.37}$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with $|K|^2$ to obtain

$$\|Tf\|_X^2\le\int_\Omega\|K_s\|_X^2\,\mu(s)=\|K\|_P^2<\infty\quad (\|f\|_X<1)\quad . \tag{4.38}$$

(a) is then proved.

To show (b), remark that

$$\int_\Omega a_i(s)\cdot b_i\cdot f\,\mu\ \in\mathbf{C}\cdot a_i(s)\ \text{ a.e}\qquad (f\in X,\ s\in\Omega)\quad . \tag{4.39}$$

It is now clear that T maps any f of X into $\mathbf{C} \cdot a_1 + \cdots + \mathbf{C} \cdot a_n$. We so conclude that $\dim R(T_1) \leq n$.

We now aim at (c). The current part refers to Exercise 4.13. X is also a Hilbert space and so contains a Hilbert basis M. Define a.e

$$\begin{aligned} a_b : \Omega &\to \mathbf{C} \\ s &\mapsto (K_s,\, b) \end{aligned} \tag{4.40}$$

whenever b ranges M. Hence,

$$K_s = \sum_{b \in M} a_b(s) \cdot b \quad \text{a.e} \quad (s \in \Omega) \quad . \tag{4.41}$$

Provided any positive scalar ε, there so exists a finite subset $S = S(\varepsilon)$ of M such that

$$\|K_s - \sum_{b \in S} a_b(s) \cdot b\,\|_X < \varepsilon \quad (s \in \Omega) \quad . \tag{4.42}$$

Remark that $\sum_{b \in S} a_b \cdot b$ matches the definition of K_1; *cf.* (b): from now on,

$$K_1 \triangleq \sum_{b \in S} a_b \cdot b \quad . \tag{4.43}$$

It follows from (b) that

$$\dim R(K_1) < \infty \quad . \tag{4.44}$$

Now turn back to (a), with $K - K_1$ playing the role of K, and so obtain

$$\|T - T_1\| < \varepsilon\mu(\Omega) \leq \infty \quad . \tag{4.45}$$

For if μ is finite, use (a) of Exercise 4.13 to conclude that T is compact. Assume henceforth that μ is not (necessarily) finite and pick δ in $\mathbf{R}_+$. The simple functions (with finite measure support) form a dense family of an L^p space ($1 \leq p < \infty$); *cf.* 3.13 of [3]. It then exists a simple function K_δ of $L^2(\mu \times \mu)$ such that

$$(\mu \times \mu)\,(\{K_\delta \neq 0\}) < \infty\,, \;\; \|K - K_\delta\|_P < \delta \quad . \tag{4.46}$$

Define an operator T_δ in terms of K_δ as T was defined in terms of K, and proceed as in (a) with $T - T_\delta$ instead of T. Then

$$\|T - T_\delta\| < \delta \quad . \tag{4.47}$$

The key ingredient is that K_δ can be identified with an element of the finite measure space $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$. What we have attempted to approximate T by T_1 can therefore be reiterated (with K_δ playing the role of K) to achieve an approximation $T_{\delta,1}$ of T_δ so that

$$\|T_\delta - T_{\delta,1}\| < \varepsilon \quad . \tag{4.48}$$

It now follows from (4.47) and (4.48) that

$$\|T - T_{\delta,1}\| \le \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta \quad . \tag{4.49}$$

Since ε and δ were arbitrary, the σ-finite case is proved. We now establish (d).

Provided g of X, let E_g be the following equation on X

$$Tf - \lambda f = g \quad , \tag{4.50}$$

whose solution set is denoted by S_g. Note that S_0 is $\ker(T-\lambda)$ and discard the trivial case $S_0 = X$[1]: each f of X lies in $S_{Tf-\lambda f}$, as some $Tf - \lambda f$'s are nonzero. Some S_g's are then nonempty. Remark that

$$S_g = f + S_0 \quad (f \in S_g) \tag{4.51}$$

in such case. Furthermore, the equality $\beta = \alpha$ of [4.25] yields

$$(T - \lambda I)(X) \ne X, \textit{i.e. } S_0 \ne \{0\} \quad . \tag{4.52}$$

So if $T - \lambda I$ is not onto, not only some S_g's are empty, but also $S_0 \ne \{0\}$. Every nonempty S_g (such sets always exist, see above) is then infinite, by (4.51).
Otherwise, $T - \lambda I$ is bijective and every equation E_g has then a unique solution f. The Fredholm alternative is so proved.

Our last step is the description of T^*. Let $S : X \to X$ be such that

$$(Sf)(t) \triangleq \int_\Omega K_t \cdot f \ \text{a.e} \qquad (f \in X,\ t \in \Omega) \tag{4.53}$$

Proceed as in (a), with S instead of T: S lies in $\mathscr{B}(X)$. Next, we claim that

$$\begin{aligned}
\langle u, T^* f^* \rangle &= \langle Tu, f^* \rangle && (4.54)\\
&= \int_\Omega (Tu) \cdot f \ \mu && (4.55)\\
&= \int_{\Omega^2} K \cdot f \cdot u \ (\mu \times \mu) && (4.56)\\
&= \int_\Omega (Sf) \cdot u \ \mu && (4.57)\\
&= \langle u, (Sf)^* \rangle \quad , && (4.58)
\end{aligned}$$

whenever u and f run through the closed unit ball of X. Since $\|T\|$, $\|T^*\|$ are equal and finite, only exactness of (4.56) is possibly in doubt; the below justification dissipates it. In conclusion,

$$T^* = \rho S \rho^{-1} \quad . \tag{4.59}$$

[1] *e.g.* $X = L^2(\{0\}, \delta)$, so that $I = \lambda^{-1} T$ is compact.

Informally,

$$T^* = S \quad . \tag{4.60}$$

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep u and f as above. Let us introduce

$$A(s) \triangleq \int_\Omega |K_s(t) \cdot u(t)| \; \mu(t) \;\text{ a.e} \quad (s \in \Omega) \quad , \tag{4.61}$$

to make hold the following Cauchy-Schwarz inequality

$$A(s) \leq \|K_s\|_X \quad (s \in \Omega) \quad . \tag{4.62}$$

Thus,

$$\int_{\Omega^2} |K(s, t)\, u(t)\, f(s)| \; \mu(s)\mu(t) = \int_\Omega |f(s)|\, A(s)\, \mu(s) \tag{4.63}$$

$$\leq \int_\Omega |f(s)|\, \|K_s\|_X\, \mu(s) \tag{4.64}$$

$$\leq \left[\int_\Omega \|K_s\|_X^2 \; \mu(s)\right]^{\frac{1}{2}} = \|K\|_P < \infty \quad . \tag{4.65}$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof. □

Chapter 6

Distributions

1 Exercise 1. Test functions are almost polynomial

Suppose f *is a complex continuous function in* $\mathbf{R}^{\mathrm{n}}$*, with compact support. Prove that* $\psi \mathrm{P_j} \to \mathrm{f}$ *uniformly on* $\mathbf{R}^{\mathrm{n}}$*, for some* $\psi \in \mathscr{D}$ *and for some sequence* $\{\mathrm{P_j}\}$ *of polynomials.*

Proof. According to 1.16, Ω is union of a compact sets sequence $\{\mathrm{K_i}\}$ and $\mathrm{supp(f)}$ lies in some $\mathrm{K} = \mathrm{K_i}$, so that f is embedded in $\mathscr{D}(\Omega)$. We can apply [1.10] to ensure that Ω encloses a compact set $\mathrm{S} = \overline{\mathrm{K} + \mathrm{B}(\varepsilon)}$ for sufficiently small $\varepsilon > 0$.

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra $\{\mathrm{g} \in \mathrm{C(S)} : \mathrm{g} \text{ is polynomial}\}$ of $\mathrm{C(S)}$. There so exists a sequence $\{\mathrm{P_j} : \mathrm{j} \in \mathbf{N}\}$ of $\mathbf{R}[\mathrm{X}_1, \dots, \mathrm{X_n}]$ such that

$$\sup_{\mathrm{S}} |\mathrm{f} - \mathrm{P_j}| \xrightarrow[\mathrm{j}\infty]{} 0 \quad . \tag{6.1}$$

By [6.20], the open set $\mathrm{K} + \mathrm{B}(\varepsilon)$ has a local partition of unity $\{\psi_\mathrm{i}\} \subseteq \mathscr{D}(\Omega)$. Moreover, there exists an integer l such that $\psi = \psi_1 + \cdots + \psi_\mathrm{l}$ equals 1 on K. Hence

$$\| \mathrm{f} - \psi \mathrm{P_j} \|_\infty = \| \psi \mathrm{f} - \psi \mathrm{P_j} \|_\infty = \sup_{\mathrm{S}} |\psi \mathrm{f} - \psi \mathrm{P_j}| \tag{6.2}$$

$$= \sup_{\mathrm{S}} |\mathrm{f} - \mathrm{P_j}| \xrightarrow[\mathrm{j}\infty]{(6.1)} 0 \quad . \tag{6.3}$$

□

We will actually prove more by showing that $\mathscr{D}(\Omega)$ is separable for each nonempty open subset Ω of $\mathbf{R}^{\mathrm{n}}$.

Proof. The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the separability of $\mathscr{D}(\Omega)$. To do so, we keep (α, j) in $\mathbf{N}^n \times \mathbf{N}$. Remark that S encloses $\operatorname{supp}(D^\alpha f)$: according to the first part, there exists a sequence $\{P_{\alpha,j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \ldots, X_n]$ such that

$$\|D^\alpha f - \psi P_{\alpha,j}\|_\infty \underset{j\infty}{\longrightarrow} 0 \quad . \tag{6.4}$$

Now let m range over $\{1, 2, 3, \ldots\}$ and set $W_{m,j}$ in $\mathscr{D}(\Omega)$ as follows

$$D^{-\alpha}\varphi \in \mathscr{D}(\Omega) : D^\alpha D^{-\alpha}\varphi = \varphi \quad . \tag{6.5}$$

$$W_{m,j}(x) \triangleq D^{-(m,\ldots,m)}(\psi P_{(m,\ldots,m),j}) \tag{6.6}$$

By (6.4), there exists a natural number k(m) such that

$$\|D^{(m,\ldots,m)}(f - W_{m,j})\|_\infty < 1/m \quad (j \geq k(m)) \quad . \tag{6.7}$$

Assume without loss of generality that S has diameter 1, so that (6.7) yields

$$\|D^\lambda(f - W_{m,k(m)})\|_\infty < 1/m \quad (|\lambda| \leq m) \quad , \tag{6.8}$$

by the mean value theorem. In other words (remark that $\operatorname{supp}(f - W_{m,k(m)}) \subseteq S$),

(6.9)
$$f - W_{m,k(m)} \in U_m \triangleq \{\varphi \in \mathscr{D}_S : \|\varphi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \cdots \quad (m = 1, 2, 3, \ldots) \quad .$$

Pick W in β (see (b) of [6.3]): $W \cap \mathscr{D}_S$ contains a neighbourhood of 0. Hence W contains some U_m, for m sufficiently large. Thus

$$W_{m,k(m)} \underset{m\infty}{\longrightarrow} f \quad (\text{in } \mathscr{D}(\Omega)) \quad . \tag{6.10}$$

We have so established that the $W_{m,k(m)}$'s family is dense in $\mathscr{D}(\Omega)$. We now aim to disclose a countable set $\tilde{W}$ that has the same property.

Choose δ in $\mathbf{R}_+$ and fetch any $W_{m,k(m)}$. Let X be $(X_1, \ldots, X_n)$ and express $P_{(m,\ldots,m),k(m)}$ as

$$P(X) = \sum_{|\gamma| \leq d} p_\gamma \cdot X^\gamma \quad . \tag{6.11}$$

Since $\bar{\mathbf{Q}} = \mathbf{R}$, $\mathbf{Q}[X]$ hosts some $\mathbf{Q}(X) = \sum_{|\gamma|\leq d} q_\gamma \cdot X^\gamma$ such that $|p_\gamma - q_\gamma| < \delta$ for all γ. Thus,

(6.12)
$$|P(x) - Q(x)| \leq \sum_{|\gamma| \leq d} |p_\gamma - q_\gamma|\,|x|^{|\gamma|} \leq \delta \sum_{l \leq d} \binom{l + n - 1}{n - 1} \|x\|_\infty^l \quad (x \in \mathbf{R}^n) \quad .$$

Since S is bounded, we so obtain

$$\|\psi(P - Q)\|_\infty \in O(\delta) \quad . \tag{6.13}$$

Now define $\tilde{W}_m$ in terms of Q as $W_{m,\,k(m)}$ was defined in terms of P, and consider the integrations made in (6.6): each $D^\lambda \tilde{W}_m$ ($|\lambda| \le m$) can be obtained from some of them. So (6.13) yields

$$\|D^\lambda(W_{m,\,k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \quad (|\lambda| \le m) \quad . \tag{6.14}$$

To be more specific, these λ's only exist in finite amount, so the big O can be assumed to be the same for all them. Since δ was arbitrary, combining (6.10) with (6.14) establishes the density of the all $\tilde{W}_m$'s family $\tilde{W}$.

Furthermore, each member of $\tilde{W}$ is only made of two ingredients: ψ and a polynomial of $\mathbf{Q}[X]$. The mapping ψ is attached to some K_i and $\mathbf{Q}[X]$ inherits countableness from $\mathbf{Q}$. Note that the "integrations packs" of (6.6) only exist in countable amount. Our $\tilde{W}$ is then countable. □

6 Exercise 6. Around the supports of some distributions

(a) Suppose that $c_m = \exp\{\text{-}(m!)!\}$, $m = 0, 1, 2, \ldots$. *Does the series*

$$\sum_{m=0}^{\infty} c_m (D^m \varphi)(0)$$

converges for every $\varphi \in C^\infty(R)$*?*

(b) Let Ω *be open in* $\mathbf{R}^n$*, suppose* $\Lambda_i \in \mathscr{D}'(\Omega)$*, and suppose that all* Λ_i *have their supports in some fixed compact* $K \subseteq \Omega$*. Prove that the sequence* $\{\Lambda_i\}$ *cannot converge in* $\mathscr{D}'(\Omega)$ *unless the orders of the* Λ_j *are bounded. Hint: Use the Banach-Steinhaus theorem.*

(c) Can the assumption about the supports be dropped in (b)?

Proof. The answer is: no. Let us establish this assertion. Assume, to reach a contradiction, that the above series converges for every smooth $\varphi : \mathbf{R} \to \mathbf{C}$.

The sequence $\{c_m (D^m\varphi)(0)\}$ so converges to 0. Nevertheless, it is proved in [1.46] that $C^\infty(\Omega)$ is not locally bounded. In other words, it is always possible to excavate a φ for which the magnitude of the m-th derivative at 0 is as large as we please[1], *e.g.* greater than $1/c_m$. A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume $\{\Lambda_j\}$ to converge to some Λ of $\mathscr{D}'(\Omega)$ and we let Q run through the compact sets of Ω. Next, we define

(6.15)
$$S(T, Q) \triangleq \{N \in \mathbf{N}, \exists C \in \mathbf{R}_+ : |T\varphi| \le C\,\|\varphi\|_N \text{ for all } \varphi \text{ of } \mathscr{D}_Q\} \quad (T \in \mathscr{D}(\Omega)) \quad .$$

[1] indeed [1.46] provides sufficient tools for constructive proof of this; see the $\varphi_j - \check{\varphi}_j$ involved in (**??**).

Such subset of $\mathbf{N}$ has a minimum $\omega(T, Q)$. The following value

$$\omega(T) \triangleq \max\{\omega(T, Q) : Q \subseteq \Omega, \ Q \text{ compact } \} \leq \infty \tag{6.16}$$

is then the order of T. Assume, to reach a contradiction, that, after passage to a subsequence,

$$\omega(\Lambda_j, Q_j) = j \quad (j = 1, 2, 3, \dots) \tag{6.17}$$

for some compact $Q = Q_j$. By (a) of [6.24], Q_j cuts $\operatorname{supp}\Lambda_j$, say in p_j. Since K encloses $\operatorname{supp}\Lambda_j$, $\{p_j\}$ tends, after passage to a subsequence, to some p of K. Choose a positive scalar r so that

$$\overline{B}(p, r) \triangleq \{x \in \mathbf{R}^n : |x - p| \leq r\} \subseteq \Omega \quad . \tag{6.18}$$

Such closed ball $\overline{B}(p, r)$ is a compact subset of Ω. By (b) of [6.5] (which refers to [1.46]) $\mathscr{D}_{\overline{B}(p, r)}$ is then a Fréchet space. It now follows from [2.6] that $\{\Lambda_j\}$ is equicontinuous on $\mathscr{D}_{\overline{B}(p, r)}$. There so exists[2] a nonnegative integer N such that

$$|\Lambda\varphi| \leq C\,\|\varphi\|_N \quad (\varphi \in \mathscr{D}_{\overline{B}(p, r)}) \tag{6.19}$$

for some positive constant C. On the other hand, $\overline{B}(p, r)$ contains almost all the p_j's. Hence

$$|\Lambda_N\,\varphi| > C\,\|\varphi\|_N \tag{6.20}$$

for some φ of $\mathscr{D}_{\overline{B}(p,r)}$. (b) is then established.

To prove (c), we introduce a sequence $\{x_m : m \in \mathbf{Z}\}$ of Ω that has no limit point. Let $\{\alpha_m : m \in \mathbf{Z}\}$ be in $\mathbf{N}$ and so define[3]

$$\begin{aligned} \Lambda : \ &\mathscr{D}(\Omega) \to \mathbf{C} \\ &\varphi \mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m}\varphi)(x_m) \end{aligned} \quad . \tag{6.21}$$

Λ belongs to $\mathscr{D}'(\Omega)$, since $\{x_m\}$ has no limit point. Next, we easily check that

$$\begin{aligned} \Lambda_j : \ &\mathscr{D}(\Omega) \to \mathbf{C} \\ &\varphi \mapsto \sum_{|m|\leq j} (D^{\alpha_m}\varphi)(x_m) \end{aligned} \qquad (j \in \mathbf{N}) \tag{6.22}$$

is also a distribution and that $\{\Lambda_j\}$ tends to Λ in $\mathscr{D}'(\Omega)$. Nevertheless, no Λ_j's can have common support because $\{x_m\}$ has no limit point. Our assumption can therefore be dropped. $\square$

[2] For more details, see Exercise 2.3.

[3] As $\Omega = \mathbf{R}$, the case $\alpha_m = m$ is the "counterpart" of the series of (a) and the case $(x_m, \alpha_m) = (m, 0)$ is the *Dirac comb*.

9 Exercise 9. Convergence in $\mathscr{D}(\Omega)$ vs. convergence in $\mathscr{D}'(\Omega)$

(a) Prove that a set $E \subseteq \mathscr{D}(\Omega)$ *is bounded if and only if*

$$\sup\{|\Lambda\varphi| : \varphi \in E\} < \infty$$

for every $\Lambda \in \mathscr{D}(\Omega)$.

(b) Suppose $\{\varphi_j\}$ *is a sequence in* $\mathscr{D}(\Omega)$ *such that* $\{\Lambda\varphi_j\}$ *is a bounded sequence of numbers, for every* $\Lambda \in \mathscr{D}'(\Omega)$. *Prove that some subsequence of* $\{\varphi_j\}$ *converges, in the topology of* $\mathscr{D}(\Omega)$.

(c) Suppose $\{\Lambda_j\}$ *is a sequence in* $\mathscr{D}'(\Omega)$ *such that* $\{\Lambda_j\,\varphi\}$ *is bounded, for every* $\varphi \in \mathscr{D}(\Omega)$. *Prove that some subsequence of* $\{\Lambda_j\}$ *converges in* $\mathscr{D}'(\Omega)$ *and that the convergence is uniform on every bounded subset of* $\mathscr{D}(\Omega)$. *Hint: By the Banach-Steinhaus theorem, the restrictions of the* Λ_j *to* $\mathscr{D}_K$ *are equicontinuous. Apply Ascoli's theorem.*

Proof. Since $\mathscr{D}(\Omega)$ locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $E = \{\varphi_j : j \in \mathbf{N}\}$ is bounded: so is $\overline{E}$. By (c) of [6.5], there exists some $\mathscr{D}_K$ that contains $\overline{E}$. Since $\mathscr{D}_K$ has the Heine-Borel property (see [1.46]), $\overline{E}$ is τ_K-compact. Apply [A4] with the metrizable space $\mathscr{D}_K$ (see [1.46]) to conclude that $\overline{E}$ has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds. □

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