Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$, topologized by the family of seminorms

$$\mathrm{p_x}(\mathrm{f}) = |\mathrm{f}(\mathrm{x})| \quad (0 \le \mathrm{x} \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{\mathrm{f_n}\}$ in X such that (a) $\{\mathrm{f_n}\}$ converges to 0 as $\mathrm{n} \to \infty$, but (b) if $\{\gamma_\mathrm{n}\}$ is any sequence of scalars such that $\gamma_\mathrm{n} \to \infty$ then $\{\gamma_\mathrm{n}\mathrm{f_n}\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0,1]$.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms $\mathrm{p_x}$ is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$\mathrm{V}^{((\mathrm{x},\mathrm{k})} \triangleq \{\mathrm{p_x} < 2^{\text{-}\mathrm{k}}\} \quad (\mathrm{x} \in [0,1], \mathrm{k} \in \mathbf{N}) \tag{1.1}$$

is then a local base for a topology τ on X. Given $\{\mathrm{f_n} : \mathrm{n} = 1,2,3,\dots\}$, we set

$$\mathtt{off}(\mathrm{U}) \triangleq \sum_{\mathrm{n}=1}^{\infty} [\mathrm{f_n} \notin \mathrm{U}] \quad (\mathrm{U} \in \tau), \tag{1.2}$$

with the convention $\mathtt{off}(\mathrm{U}) = \infty$ whether the sum has no finite support. So,

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{U}^{(\mathrm{i})}) = \sum_{\mathrm{n}=1}^{\infty}\sum_{\mathrm{i}=1}^{\mathrm{m}} [\mathrm{f_n} \notin \mathrm{U}^{(\mathrm{i})}] \ge \mathtt{off}(\mathrm{U}^{(1)} \cap \dots \cap \mathrm{U}^{(\mathrm{m})}) \tag{1.3}$$

We first assume that $\{\mathrm{f_n}\}$ τ-converges to some f in X, *i.e.*

$$\mathtt{off}(\mathrm{f} + \mathrm{V}) < \infty \quad (\mathrm{V} \in \mathscr{B}). \tag{1.4}$$

The special cases $\mathrm{V} = \mathrm{V}^{(\mathrm{x},\mathrm{k})}$ mean the pointwise convergence of $\{\mathrm{f_n}\}$. Conversely, assume that $\{\mathrm{f_n}\}$ does not τ-converges to any g in X, *i.e.*

$$\forall \mathrm{g} \in \mathrm{X}, \exists \mathrm{V}^{(\mathrm{g})} \in \mathscr{B} : \mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{g})}) = \infty. \tag{1.5}$$

Given g, $\mathrm{V}^{(\mathrm{g})}$ is then an intersection $\mathrm{V}^{(\mathrm{x}^{(1)},\mathrm{k}^{(1)})} \cap \dots \cap \mathrm{V}^{(\mathrm{x}^{(\mathrm{m})},\mathrm{k}^{(\mathrm{m})})}$. Thus

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{x}^{(\mathrm{i})},\mathrm{k}^{(\mathrm{i})})}) \overset{(1.3)}{\ge} \mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{g})}) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{x}^{(\mathrm{i})},\mathrm{k}^{(\mathrm{i})})})$ must then be ∞. This implies that convergence of $\mathrm{f_n}$ to g fails at point $\mathrm{x_i}$. g being arbitrary, we so conclude that $\mathrm{f_n}$ does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on, *k*, *n* and *p* run on $\mathbf{N}_+$. Let $\mathtt{dyadic}(\mathrm{x})$ be the usual dyadic expansion of a real number x, so that $\mathtt{dyadic}(\mathrm{x})$ is an aperiodic binary sequence **iff** x is irrational. Define

$$\mathrm{f_n}(\mathrm{x}) \triangleq \begin{cases} 2^{-\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathtt{dyadic}(\mathrm{x})_{-\mathrm{k}}} & (\mathrm{x} \in [0,1] \setminus \mathbf{Q}) \\ 0 & (\mathrm{x} \in [0,1] \cap \mathbf{Q}) \end{cases} \tag{1.7}$$

so that $f_n(x) \underset{n\to\infty}{\longrightarrow} 0$ and take scalars γ_n such that $\underset{n\to\infty}{\longrightarrow} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose $n^{(p)}$ among those *almost all* n that are large enough to satisfy

$$n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)} \tag{1.8}$$

(start with $n^{(-1)} = n^{(0)} = 0$) and so obtain

$$2^p < \gamma_{n^{(p)}} : 0 < n^{(p)} - n^{(p-1)} \underset{p\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator χ of $\{n^{(1)}, n^{(2)}, \dots\}$ is then aperiodic, *i.e.*

$$x^{(\gamma)} \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \tag{1.10}$$

is irrational. Consequently,

$$\texttt{dyadic}(x^{(\gamma)})_{-k} = \chi_k. \tag{1.11}$$

We now easily see that

$$\chi_1 + \cdots + \chi_{n^{(p)}} = p, \tag{1.12}$$

which, combined with (1.7), yields

$$f_{n^{(p)}}(x^{(\gamma)}) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n^{(p)}} f_{n^{(p)}}(x^{(\gamma)}) > 1. \tag{1.14}$$

We have so established that the subsequence $\{\gamma_{n^{(p)}} f_{n^{(p)}}\}$ does not tend pointwise to 0, hence neither does the whole sequence $\{\gamma_n f_n\}$. In other words, (b) holds, which is in violent contrast with [1.28]: X is then not metrizable. So ends the proof. $\square$