Solutions to some exercises from Walter Rudin's
Functional Analysis

gitcordier

September 7, 2020

Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \to \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \to \infty$ then $\{\gamma_n f_n\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0,1]$.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms p_x is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$V_{x,k} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0,1], k \in \mathbf{N}) \tag{1.1}$$

is therefore a local base for a topology τ on X. Given $\{f_n : n = 1, 2, 3, \dots\}$, we put

$$\texttt{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \tag{1.2}$$

with the convention "$\Sigma = \infty$" whether the sum has no finite support. So,

$$\sum_{i=1}^{m} \texttt{off}(U_i) = \sum_{n=1}^{\infty} \sum_{i=1}^{m} [f_n \notin U_i] \ge \texttt{off}(U_1 \cap \cdots \cap U_m). \tag{1.3}$$

We first assume that $\{f_n\}$ τ-converges to some f in X, *i.e.*

$$\texttt{off}(f + V) < \infty \quad (V \in \mathscr{B}). \tag{1.4}$$

The special cases $V_{x,1}, V_{x,2}, \dots$, mean the pointwise convergence $f_n(x) \xrightarrow{n\infty} f(x)$. Conversely, assume that $\{f_n\}$ does not τ-converges to any g in X, *i.e.*

$$\forall g \in X, \exists W \in \mathscr{B} : \texttt{off}(g + W) = \infty. \tag{1.5}$$

Given g, such W is, by definition, a finite intersection $V_{x_1,k_1} \cap \cdots \cap V_{x_m,k_m}$. Thus,

$$\sum_{i=1}^{m} \texttt{off}(g + V_{x_i,k_i}) \overset{(1.3)}{\ge} \texttt{off}(g + W) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\texttt{off}(g + V_{x_i,k_i})$ must then be ∞. In other words, there exists a point x_i for which $\{f_n(x_i)\}$ does not converge to $g(x_i)$. g being arbitrary, we so conclude that f_n does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now prove the second part. From now on, we let *k*, *n* and *p* run on $\mathbf{N}_+$, as $\texttt{dyadic}(x)$ denotes the usual dyadic expansion of x, so that $\texttt{dyadic}(x)$ is an aperiodic binary sequence **iff** x is irrational. Define

$$f_n(x) \triangleq \begin{cases} \exp_2\left(-\sum_{k=1}^{n} \texttt{dyadic}(x)_{-k}\right) & (x \in [0,1] \setminus \mathbf{Q}) \\ 0 & (x \in [0,1] \cap \mathbf{Q}), \end{cases} \tag{1.7}$$

so that $f_n(x) \xrightarrow{n\infty} 0$, and take $\gamma_n \xrightarrow{n\infty} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose n_p among those *almost all* n that are large enough to satisfy

$$n_{p-1} - n_{p-2} < n_p - n_{p-1} \tag{1.8}$$

(start with $n_{-1} = n_0 = 0$) and so obtain

$$2^p < \gamma_{n_p} : 0 < n_p - n_{p-1} \xrightarrow[p \to \infty]{} \infty. \tag{1.9}$$

The indicator χ of $\{n_1, n_2, \dots\}$ in $\mathbf{Z}$ is then aperiodic, *i.e.*

$$\alpha_\gamma \triangleq \sum_{k=1}^\infty \chi_k 2^{-k} \in [0,1] \setminus \mathbf{Q}. \tag{1.10}$$

Hence, χ is not a the infinite-support expansion of a rational number; which forces

$$\texttt{dyadic}(\alpha_\gamma)_{-k} = \chi_k. \tag{1.11}$$

The key ingredient is that

$$\chi_1 + \cdots + \chi_{n_p} = p. \tag{1.12}$$

Combined with (1.7), it yields

$$f_{n_p}(\alpha_\gamma) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n_p} f_{n_p}(\alpha_\gamma) > 1. \tag{1.14}$$

There so exists $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. In other words, (b) holds, which is in violent contrast with 1.28 of [3]: X is therefore not metrizable. So ends the proof. $\qquad\square$

1.2   Exercise 9. Quotient map

Suppose

(a) X *and* Y *are topological vector spaces,*

(b) $\Lambda : X \to Y$ *is linear.*

(c) N *is a closed subspace of* X,

(d) $\pi : X \to X/N$ *is the quotient map, and*

(e) $\Lambda x = 0$ *for every* $x \in N$.

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$, *that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$. *Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \tag{1.15}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'. \tag{1.16}$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$f(\pi x) = 0 \overset{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \tag{1.17}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subset V. \tag{1.18}$$

Since π is open (1.41 (a) of [3]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \tag{1.19}$$

for some neighborhood V of 0_X. So,

$$f(W) = f(\pi(V)) = \Lambda(V). \tag{1.20}$$

As a consequence, if Λ is open, then f(W) is a neighborhood of 0_Y. So ends the proof. $\square$

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \to Y$ is linear, and $\Lambda(X) = Y$.

   (*a*) *Prove that Λ is an open mapping.*

   (*b*) *Assume, in addition, that the null space of Λ is closed, and prove that Λ is continuous.*

Proof. We discard the trivial case $\dim Y = 0$ then henceforth assume that $\dim Y$ has positive dimension n.

Let e range over a base of Y: For each e, there exists x_e in X such that $\Lambda(x_e) = e$, since Λ is onto. So,

$$y = \sum_e y_e \Lambda x_e \quad (y \in Y). \tag{1.21}$$

The sequence $\{x_e\}$ is finite; therefore it is bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$x_e \in sV \text{ for all } x_e. \tag{1.22}$$

Combining this with (1.21) shows that

$$y \in \sum_e \Lambda(V) \quad (y \in Y : |y_e| < s^{-1}), \tag{1.23}$$

which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, with $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$\dim X/N = n. \tag{1.24}$$

f is then an homeomorphism of $X/N \equiv \mathbf{C}^n$ onto Y; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = f \circ \pi$. $\qquad\square$

1.4   Exercise 14. $\mathscr{D}_K$ equipped with other seminorms

Put K $= [0, 1]$ and define $\mathscr{D}_K$ as in Section 1.46. Show that the following three families of seminorms (where n $= 0, 1, 2, \dots$) define the same topology on $\mathscr{D}_K$. If D $= \mathrm{d}/\mathrm{dx}$:

(a) $\|D^n f\|_\infty = \sup\{|D^n f(x)| : \infty < x < \infty\}$

(b) $\|D^n f\|_1 = \int_0^1 |D^n f(x)|\, \mathrm{d}x$

(c) $\|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 \mathrm{d}x \right\}^{1/2}$.

Proof. First, remark that

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty < \infty \tag{1.25}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of $D^n f$ lies in K; which yields

$$|D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \le \int_0^x \left| D^{n+1} f \right| \le \|D^{n+1} f\|_1. \tag{1.26}$$

So,

$$\|D^n f\|_\infty \le \|D^{n+1} f\|_1. \tag{1.27}$$

We now combine (1.25) with (1.27) and so obtain

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty \le \|D^{n+1} f\|_1 \le \cdots \quad (n = 0, 1, 2, \dots). \tag{1.28}$$

Put

$$V_n^{(i)} \triangleq \{f \in \mathscr{D}_K : \|f\|_i < 2^{-n}\} \quad (i = 1, 2, \infty) \tag{1.29}$$

$$\mathscr{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\}, \tag{1.30}$$

so that (1.28) is mirrored in terms of neighborhood inclusions, as follows,

$$V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \cdots . \tag{1.31}$$

Since $V_n^{(i)} \supset V_{n+1}^{(i)}$, $\mathscr{B}^{(i)}$ is a local base of a topology τ_i. But the chain (1.31) forces

$$\tau_1 = \tau_2 = \tau_\infty. \tag{1.32}$$

To see that, choose a set S that is τ_1-open at f, *i.e.* $V_n^{(1)} \subset S - f$ for some n. Next, concatenate this with $V_n^{(2)} \subset V_n^{(1)}$ (see (1.31)) and so obtain $V_n^{(2)} \subset S - f$; which implies that S is τ_2-open at f. Similarly, we deduce, still from (1.31), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.33}$$

So ends the proof. $\square$

1.5   Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Comment   This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms p_n, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1   Let X be a topological space with a countable local base $\{V_n : n = 1, 2, 3, \dots \}$. If $\tilde{V}_n = V_1 \cap \cdots \cap V_n$, then every subsequence $\{\tilde{V}_{\rho(n)}\}$ is a decreasing (*i.e.* $\tilde{V}_{\rho(n)} \supset \tilde{V}_{\rho(n+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $V_n \supset \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X. Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supset \tilde{V}_{\rho(n)}$. $\square$

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$)   Under the same notations of Lemma 1, if $\{V_n\}$ is a decreasing local base, then so is $\{V_{\rho(n)}\}$.

Corollary 2   If $\{Q_n\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{\rho(n)}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $C(\Omega)$'s (respectively $C^\infty(\Omega)$'s) topology of the seminorms p_n, as defined in section 1.44 (respectively 1.46), then the seminorms $p_{\rho(n)}$ define the same topology τ_Q.

Proof. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X. Moreover,

$$Q_{\rho(n)} \subset \mathring{Q}_{\rho(n)+1} \subset Q_{\rho(n)+1} \subset Q_{\rho(n+1)}. \tag{1.34}$$

Thus,

$$Q_{\rho(n)} \subset \mathring{Q}_{\rho(n+1)}. \tag{1.35}$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(n)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(n)}\}$ is a local base. So, $\tau_{Q_\rho} \subset \tau_Q$. Conversely, the above corollary asserts that $\{V_{\rho(n)}\}$ is a local base of τ_Q, which yields $\tau_Q \subset \tau_{Q_\rho}$. $\square$

Lemma 2   If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in allmost all Q_n°, *i.e.* there exists m such that

$$K \subset \mathring{Q}_m \subset \mathring{Q}_{m+1} \subset \mathring{Q}_{m+2} \subset \cdots . \tag{1.36}$$

Proof. The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \quad (n = 1, 2, 3, \dots) \tag{1.37}$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset Q_m^\circ$, for some m. Finally,

$$K \subset \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots . \tag{1.38}$$

$\square$

We are now in a fair position to establish the following:

Theorem   The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subset L_{m_i}^\circ \subset L_{m_i+1}^\circ \subset \cdots$ for some m_i. In particular, the special case $\varkappa_i = m_i + i$ is

$$K_i \subset \overset{\circ}{L}_{\varkappa_i}. \tag{1.39}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \tag{1.40}$$

Combine (1.39) with (1.40) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\varkappa_1} \subset L_{\varkappa_1} \subset \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subset K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots , \tag{1.41}$$

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L. \tag{1.42}$$

So ends the proof $\square$

---

[1] See (b) of 2.5 of [2].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [2].

1.6 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$, *for every multi-index* α.

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{ \phi \in C^\infty(\Omega) : \max\{|D^\beta \phi(x)| : |\beta| \le N, x \in K_N\} < 1/N \right\}. \tag{1.43}$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^\gamma g(x)| : |\gamma| \le N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}. \tag{1.44}$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^\beta D^\alpha g(x)| : |\beta| \le N, x \in K_N\} < \frac{1}{N}. \tag{1.45}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad i.e. \quad D^\alpha\left(V_{N+|\alpha|}\right) \subset V_N, \tag{1.46}$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^\alpha|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^\alpha|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) = D^\alpha\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) \tag{1.47}$$

$$\subset D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha\left(\mathscr{D}_K\right) \tag{1.48}$$

$$\subset V_N \cap \mathscr{D}_K \quad (\text{see } (1.46)) \tag{1.49}$$

So ends the proof. $\square$

Chapter 2

Completeness

2.1 Exercise 3. An equicontinous sequence of measures

Put K $= [-1, 1]$; define $\mathscr{D}_K$ as in section 1.46 (with $\mathbf{R}$ in place of $\mathbf{R}^n$). Supose $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\Lambda\phi = \lim\limits_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ exists for every $\phi \in \mathscr{D}_K$. Show that Λ is a continuous linear functional on $\mathscr{D}_K$. Show that there is a positive integer p and a number $M < \infty$ such that

$$\left| \int_{-1}^{1} f_n(t)\phi(t)dt \right| \leq M \| D^p \|_\infty$$

for all n. For example, if $f_n(t) = n^3 t$ on $[-1/n, 1/n]$ and 0 elsewhere, show that this can be done with $p = 1$. Construct an example where it can be done with $p = 2$ but not with $p = 1$.

We will also consider the case $p = 0$. Since all supports of $\phi, \phi', \phi'', \ldots,$ are in K, we make a specialization of the mean value theorem:

Lemma   If $\phi \in \mathscr{D}_{[a,b]}$, then

$$\| D^\alpha \phi \|_\infty \leq \| D^p \phi \|_\infty \left(\frac{\lambda}{2} \right)^{p-\alpha} \qquad (\alpha = 0, 1, \ldots, p) \tag{2.1}$$

at every order $p = 0, 1, 2, \ldots$; where λ is the length $|b - a|$.

Proof. Let x_0 be in (a, b). We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 $(a < x_1 < x_0)$, such that

$$\phi(x_0) = \phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a). \tag{2.2}$$

Since every $D^p\phi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$\phi(x_0) = D^0\phi(x_0) \tag{2.3}$$

$$= D^1\phi(x_1)(x_0 - a) \tag{2.4}$$

$$= \cdots$$

$$= D^p\phi(x_p)(x_0 - a)\cdots(x_{p-1} - a), \tag{2.5}$$

for all p. More compactly,

$$D^\alpha \phi(x_0) = D^p \phi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a); \qquad (2.6)$$

which yields,

$$|D^\alpha \phi(x)| \le \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (x \in [a, c]) \qquad (2.7)$$

The case $x_0 \ge c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^\alpha \phi(x)| \le \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (2.8)$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\|D^\alpha \phi\|_\infty \le \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}. \qquad (2.9)$$

$\square$

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\phi\rangle \triangleq \int_{\mathbf{R}} u\phi \quad (\phi \in C_0(\mathbf{R})). \qquad (2.10)$$

The following inequalities

$$|\langle u|\phi\rangle| \le \int_{\mathbf{R}} |u\phi| \le \|u\|_{L^1} \quad (\|\phi\|_\infty \le 1) \qquad (2.11)$$

imply that every linear functional

$$\langle u| : C_0(\mathbf{R}) \to \mathbf{C} \qquad (2.12)$$
$$\phi \mapsto \langle u|\phi\rangle$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u|$, since u is determined (a.e) by the integrals $\langle u|\phi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\sup\{|\langle u|\phi\rangle| : \|\phi\|_\infty = 1\} = \|u\|_{L^1}. \qquad (2.13)$$

Note that, in the latter equality, $\le \|u\|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$g_n : \mathbf{R} \to \mathbf{R} \qquad (2.14)$$
$$x \mapsto \begin{cases} n^3 x & \left(x \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right) \\ 0 & \left(x \notin \left[-\frac{1}{n}, \frac{1}{n}\right]\right). \end{cases}$$

---

[1] see 1.32, 4.1 of [3]
[2] See 7.26 of [1].

First, remark that $g_n(x) \xrightarrow[n \to \infty]{} 0$ $(x \in \mathbf{R})$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$\langle g_n | \phi \rangle \xrightarrow[n \to \infty]{} \Lambda \phi, \tag{2.15}$$

where ϕ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\phi_1^+, \ldots, \phi_j^+, \ldots$, from $C_K^\infty(\mathbf{R})$. More specifically,

   (i) $\phi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

   (ii) $\phi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

   (iii) $0 \leq \phi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those ϕ_j^+. Let $\phi_1^-, \ldots, \phi_j^-, \ldots$, mirror the ϕ_j^+, in the sense that $\phi_j^-(x) = \phi_j^+(\text{-x})$, so that

   (iv) $\phi_j \triangleq \phi_j^+ - \phi_j^-$ is odd, as g_n is;

   (v) every ϕ_j is in $C_K^\infty(\mathbf{R})$;

   (vi) The sequence $\{\phi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\|\phi_j\|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n | \phi_j \rangle = 2 \int_0^1 g_n(t) \phi_j^+(t) dt \xrightarrow[j \to \infty]{} 2 \int_0^1 g_n(t) dt = \| g_n \|_{L^1} = n. \tag{2.16}$$

Finally,

$$\| g_n \|_{L^1} \overset{(2.16)}{\leq} \sup\{ |\langle g_n | \phi \rangle| : \| \phi \|_\infty = 1 \} \overset{(2.13)}{\leq} \| g_n \|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$|\langle g_n | \phi \rangle| \leq C_n \quad (\| \phi \|_\infty = 1); \tag{2.18}$$

see (2.11). Furthermore, $\| g_n \|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n | \phi_{\rho(n)} \rangle\}$ such that $\langle g_n | \phi_{\rho(n)} \rangle$ is greater than, say, $n - 0.01$, as $\| \phi_{\rho(n)} \|_\infty = 1$. Consequently, there is no bound M such that

$$|\langle g_n | \phi \rangle| \leq M \quad (\| \phi \|_\infty = 1; n = 1, 2, 3, \ldots). \tag{2.19}$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n | \phi \rangle \xrightarrow[n \to \infty]{} \Lambda \phi \quad (\phi \in C_0(\mathbf{R})) \tag{2.20}$$

---

[3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that ϕ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$\Lambda\phi = \lim_{n\to\infty} \int_K f_n\phi \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.21}$$

for some linear form Λ. Since ϕ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$\Lambda\phi = \lim_{n\to\infty} \langle f_n|\phi\rangle \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.22}$$

Let $K_1, K_2, \ldots$, be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\phi) = \| \phi \|_N \triangleq \max\{|D^\alpha\phi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\phi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\phi \in \mathscr{D}_K : \| \phi \|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\phi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\| f_n \|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [3].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\phi\rangle \xrightarrow[n\to\infty]{} \Lambda\phi$ for all ϕ, *i.e.* $\Lambda - \langle f_n| \xrightarrow[n\to\infty]{} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$|\Lambda\phi| < 1/2 \quad \text{if } \phi \in V_i, \tag{2.23}$$

$$|\Lambda\phi - \langle f_n|\phi\rangle| < 1/2 \quad \text{if } \phi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\phi\rangle| \leq |\Lambda\phi - \langle f_n|\phi\rangle| + |\Lambda\phi| < 1 \quad \text{if } \phi \in V_p. \tag{2.25}$$

Now remark that every $\psi = \psi[\mu, \phi]$, where

$$\psi[\mu, \phi] \triangleq \begin{cases} (1/\mu \cdot 2^p \| \phi \|_p)\phi & (\phi \neq 0, \mu > 1) \\ 0 & (\phi = 0, \mu > 1), \end{cases} \tag{2.26}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\psi\rangle| < 1 \tag{2.27}$$

$$|\langle f_n|\phi\rangle| < 2^p\|\phi\|_p \cdot \mu \tag{2.28}$$

$$|\langle f_n|\phi\rangle| \leq 2^p\|\phi\|_p \tag{2.29}$$

$$|\langle f_n|\phi\rangle| \leq 2^p\{\|D^0\phi\|_\infty + \cdots + \|D^p\phi\|_\infty\}. \tag{2.30}$$

Finally, with the help of (2.1),

$$|\langle f_n|\phi\rangle| \leq 2^p(p+1)\|D^p\phi\|_\infty. \tag{2.31}$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\phi = \lim\limits_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ exists for every $\phi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\phi\rangle| \leq M\|D^p\phi\|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let ϕ keep running on $\mathscr{D}_K$ and so define

$$B_n(\phi) \triangleq \max\{|\phi(x)| : x \in [-1/n, 1/n]\}, \tag{2.32}$$

$$\Delta_n(\phi) \triangleq \max\{|\phi(x) - \phi(0)| : x \in [-1/n, 1/n]\}. \tag{2.33}$$

The mean value asserts that

$$|\phi(1/n) - \phi(-1/n)| \leq B_n(\phi')\,|1/n - (-1/n)| = \frac{2}{n}B_n(\phi'). \tag{2.34}$$

Independently, an integration by parts shows that

$$\langle f_n|\phi\rangle = \left[\frac{n^3t^2}{2}\phi(t)\right]_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)dt \tag{2.35}$$

$$= \frac{n}{2}\left(\phi(1/n) - \phi(-1/n)\right) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)dt. \tag{2.36}$$

Combine (2.34) with (2.36) and so obtain

$$|\langle f_n|\phi\rangle| \leq \frac{n}{2}\,|\phi(1/n) - \phi(-1/n)| + \frac{n^3}{2}\int_{-1/n}^{1/n} t^2|\phi'(t)|dt \tag{2.37}$$

$$\leq B_n(\phi') + \frac{n^3}{2}B_n(\phi')\int_{-1/n}^{1/n} t^2dt \tag{2.38}$$

$$\leq \frac{4}{3}B_n(\phi') \tag{2.39}$$

$$\leq \frac{4}{3}\|\phi'\|_\infty. \tag{2.40}$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\phi')$ tends to $\left|\phi'(0)\right|$, we may expect that f_n tends to $\frac{4}{3}\phi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0) \overset{(2.36)}{=} \frac{\phi(1/n) - \phi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \phi'(0) - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n}(\phi' - \phi'(0))t^2 dt. \qquad (2.41)$$

So,

$$\left|\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0)\right| \leq \left|\frac{\phi(1/n) - \phi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \phi'(0)\right| + \frac{1}{3}\Delta_n(\phi') \underset{n\to\infty}{\longrightarrow} 0. \qquad (2.42)$$

We have just proved that

$$\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\phi'(0) \quad (\phi \in \mathscr{D}_K). \qquad (2.43)$$

In other words,

$$\langle f_n| \underset{n\to\infty}{\longrightarrow} \text{-}\frac{4}{3}\delta', \qquad (2.44)$$

where δ is the *Dirac measure* and $\delta', \delta'', \dots$, its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $\text{-}\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$|\langle f_n|\phi\rangle| \leq \frac{4}{3}\|\phi'\|_\infty \quad (n = 1, 2, 3, \dots) \qquad (2.45)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n|\phi_{\rho(n)}\rangle : \|\phi_{\rho(n)}\|_\infty = 1; n = 1, 2, 3, \dots\} \qquad (2.46)$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle f_n|\phi\rangle| \leq M\|\phi\|_\infty \quad (\phi \in C_K^\infty(\mathbf{R})). \qquad (2.47)$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*[4] $f_n{}'$, as follows

$$f_n{}' : \mathscr{D}_K \to \mathbf{C} \qquad (2.48)$$
$$\phi \mapsto \text{-}\langle f_n|\phi'\rangle.$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$D : \mathscr{D}_K \to \mathscr{D}_K \qquad (2.49)$$
$$\phi \mapsto \phi';$$

---

[4] See 6.1 of [3] for a further discussion.

see Exercise 1.17. f_n' is therefore continuous. Now apply (2.43) with ϕ' and so obtain

$$-\langle f_n|\phi'\rangle \xrightarrow[n\to\infty]{} \frac{4}{3}\phi''(0) \quad (\phi \in \mathscr{D}_K),$$

i.e.

$$f_n' \xrightarrow[n\to\infty]{} \frac{4}{3}\delta''. \tag{2.50}$$

It follows from (2.40) that,

$$|\langle f_n|\phi'\rangle| \le \frac{4}{3}\|\phi''\|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.51}$$

It is therefore possible to uniformly bound f_n' with respect to a norm $\|D^p \cdot \|_\infty$, namely $\|D^2 \cdot \|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle f_n|\phi'\rangle| \le M\|\phi'\|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.52}$$

Define

$$\Phi_j(x) = \int_{-1}^{x} \phi_j. \tag{2.53}$$

The oddness of ϕ_j forces Φ_j to vanish outside [-1, 1]: ϕ_j is therefore in $\mathscr{D}_K$. So, under our assumption,

$$|\langle f_n|\Phi_j'\rangle| \le M\|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots); \tag{2.54}$$

which is

$$|\langle f_n|\phi_j\rangle| \le M \quad (n = 1, 2, 3, \dots). \tag{2.55}$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n|\phi_{\rho(n)}\rangle\}$) and so conclude that there is no constant M such that

$$|\langle|f_n\phi'\rangle| \le M\|\phi'\|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.56}$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$|\langle f_n|\phi'\rangle| \le M\|\phi\|_\infty. \tag{2.57}$$

The mean value theorem (see (2.1)) asserts that

$$|\langle f_n|\phi'\rangle| \le M\|\phi\|_\infty \le M\|\phi'\|_\infty; \tag{2.58}$$

which is, again, a desired contradiction. So ends the proof.

$\square$

2.2 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ *of a function* $f \in L^2(T)$ *(T is the unit circle) by*

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

for all $n \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f$ *exists} is a dense subspace of* $L^2(T)$ *of the first category.*

Proof. Let $f(\theta)$ stand for $f(e^{i\theta})$, so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta. \tag{2.59}$$

We believe it is customary to write

$$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n). \tag{2.60}$$

Moreover, a well known (and easy to prove) result is

$$(e_n, e_{n'}) = [n = n'], \textit{i.e.} \ \{e_n : n \in \mathbf{Z}\} \text{ is an orthormal subset of } L^2(T). \tag{2.61}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

$$\| \Lambda_n \|^2 \stackrel{(2.60)}{=} \| e_{-n} + \cdots + e_n \|^2 \stackrel{(2.61)}{=} \| e_{-n} \|^2 + \cdots + \| e_n \|^2 \stackrel{(2.61)}{=} 2n + 1. \tag{2.62}$$

We now assume, to reach a contradiction, that

$$B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \ldots\} < \infty\} \tag{2.63}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$\| \Lambda_n \| \stackrel{(2.62)}{=} \sqrt{2n + 1} \underset{n\to\infty}{\longrightarrow} \infty. \tag{2.64}$$

We have just established that B is actually of the first category; and so is its subset $L = \{f \in L^2(T) : \lim_{n\longrightarrow\infty} \Lambda_n f$ exists$\}$. We now prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be span$\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$P \subset L \subset L^2(T). \tag{2.65}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.65), that

$$L^2(T) = \overline{P} = \overline{L}. \tag{2.66}$$

So ends the proof   □

2.3 Exercise 9. Boundedness without closedness

Suppose X, Y, Z *are Banach spaces and*

$$B : X \times Y \to Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ *such that*

$$\| B(x, y) \| \leq M \| x \| \| y \| (x \in X, y \in Y).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Let (x, y) range over $X \times Y$. B is continous at the origin; thus, there exists a positive r such that

$$\| B(x, y) \| < 1   (\max\{\| x \|, \| y \|\} < r). \tag{2.67}$$

Given (x, y), we choose two scalars α, β such that $r\alpha > \| x \|$ and $r\beta > \| y \|$. Thus,

$$\| B(x, y) \| = \alpha\beta \| B\left(\alpha^{-1}x, \beta^{-1}y\right) \| \tag{2.68}$$
$$< \alpha\beta. \tag{2.69}$$

We now conclude that

$$B(x, y) \leq r^{-2} \| x \| \| y \|. \tag{2.70}$$

So ends the proof.

As a concrete example, choose $X = Y = Z = C_c(\mathbf{R})$, topologized by the supremum norm. $C_c(\mathbf{R})$ is not complete[5], nevertheless the bilinear product

$$B(f, g) = f \times g   ((f, g) \in C_c(\mathbf{R})^2) \tag{2.71}$$

is bounded, (since $\| B(f, g) \|_\infty = \| f \|_\infty \| g \|_\infty$) and continuous. To see that, pick (u, v) in $C_c(\mathbf{R})^2$: Given any positive scalar ε, there exists another positive scalar r such that $r(r + \| u \| + \| v \|) < \varepsilon$. So, under the following assumption

$$\max\{\| f - u \|_\infty, \| g - v \|_\infty\} < r, \tag{2.72}$$

we reach

$$\| fg - uv \|_\infty \leq \| f - u \|_\infty \cdot \| g \|_\infty + \| u \|_\infty \cdot \| g - v \|_\infty \tag{2.73}$$
$$< r(r + \| v \|) + \| u \| r \tag{2.74}$$
$$< r(r + \| u \| + \| v \|) \tag{2.75}$$
$$< \varepsilon; \tag{2.76}$$

which establishes the continuousness of B. $\square$

---

[5] See 5.4.4 [4]

2.4 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let (X_1, X_2, Z) be topological spaces and B a bilinear mapping

$$B : X_1 \times X_2 \to Z \tag{2.77}$$

From now on, $x = (x_1, x_2)$ denotes an arbitrary element of $X_1 \times X_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $X_1 \times X_2$, *i.e.* given an arbitrary balanced open subset W of Z, there exists in X_i $(i = 1, 2)$ a balanced open subset U_i such that

$$B(U_1 \times U_2) \subset W. \tag{2.78}$$

Let $\nu_i(x)$ denote any scalar that is greater than $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$. So,

$$B(x_1, x_2) = \nu_1(x)\nu_2(x) \cdot B\left(\nu_1(x)^{-1}x_1, \nu_2(x)^{-1}x_2\right) \tag{2.79}$$
$$\in \nu_1(x)\nu_2(x) \cdot B(U_1 \times U_2) \tag{2.80}$$
$$\subset \nu_1(x)\nu_2(x) \cdot W. \tag{2.81}$$

Now pick $p = (p_1, p_2)$ in $X_1 \times X_2$: It directly follows from (2.81) that

$$B(p_1, p_2) - B(x_1, x_2) = B(p_1, p_2 - x_2) + B(p_1 - x_1, x_2 - p_2) + B(p_1 - x_1, p_2) \tag{2.82}$$
$$\in \nu_1(p)\nu_2(p - x) \cdot W + \nu_1(p - x)\nu_2(x - p) \cdot W + \nu_1(p - x)\nu_2(p) \cdot W. \tag{2.83}$$

Let us henceforth assume that

$$p_i - x_i \in [\mu_1(p) + \mu_2(p) + 2]^{-1} \cdot U_i; \tag{2.84}$$

which yields

$$\mu_i(p_i - x_i) \leq [\mu_1(p) + \mu_2(p) + 2]^{-1}. \tag{2.85}$$

Finally, combine the special case

$$\nu_i(p - x) = [\mu_1(p) + \mu_2(p) + 1]^{-1}, \tag{2.86}$$
$$\nu_i(p) = \mu_1(p) + \mu_2(p) + 1 \tag{2.87}$$

with (2.83) and so obtain

$$B(p_1, p_2) - B(x_1, x_2) \in W + W + W. \tag{2.88}$$

W being arbitrary, we have so established the continuousness of B at (p_1, p_2). Since (p_1, p_2) is also arbitrary, the proof is complete. $\qquad\square$

2.5   Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\| f \| = \int_0^1 |f(t)| \ dt.$$

Put $B(f, g) = \int_0^1 f(t)g(t)dt$, *and show that B is a bilinear continuous functional on* $X \times X$ *which is separately but not continuous.*

Proof. Let f denote the first variable, g the second one. Remark that

$$|B(f, g)| < \| f \| \cdot \max_{[0,1]} |g| \ ; \tag{2.89}$$

which is sufficient (1.18 of [3]) to assert that any $f \mapsto B(f, g)$ is continuous. The continuity of all $g \mapsto B(f, g)$ follows (Put $C(g, f) = B(f, g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$|B(f, g)| \le M\| f \|\| g \|. \tag{2.90}$$

Put

$$f_n(X) \triangleq 2\sqrt{n} \cdot X^n \in \mathbf{R}[X] \quad (n = 1, 2, 3, \dots), \tag{2.91}$$

so that

$$\| f_n \| = \frac{2\sqrt{n}}{n + 1} \xrightarrow[n \to \infty]{} 0. \tag{2.92}$$

On the other hand,

$$B(f_n, f_n) = \frac{4n}{2n + 1} > 1. \tag{2.93}$$

Finally, we combine (2.92) and (2.93) with (2.90) and so obtain

$$1 < B(f_n, f_n) \le M\| f_n \|^2 \xrightarrow[n \to \infty]{} 0. \tag{2.94}$$

Our continuousness assumption is then contradicted. So ends the proof.   □

2.6 Exercise 15. Baire cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* Y = X. Hint: Y *must intersect* x + Y *for every* x ∈ X.

Proof. Assume Y is a subgroup of X. Under our assumptions, there exists a sequence $\{E_n : n = 1, 2, 3, \dots\}$ of X such that

(i) $(\overline{E_n})^{\circ} = \emptyset$;

(ii) $X \setminus Y = \bigcup_{n=1}^{\infty} E_n$.

By (i), the complement V_n of $\overline{E_n}$ is a dense open set. Since X is an F-space, it follows from the Baire's theorem that the intersection S of the V_n's is dense in X: So is x + S (x ∈ X). To see that, remark that

$$X = x + \overline{S} \subset \overline{x + S} \tag{2.95}$$

follows from 1.3 (b) of [3]. Since S and x + S are both dense open subsets of X, the Baire's theorem asserts that

$$\overline{(x + S) \cap S} = X. \tag{2.96}$$

Thus,

$$(x + S) \cap S \neq \emptyset. \tag{2.97}$$

Moreover, it follows from (ii) that $X \setminus Y \subset \bigcup_n \overline{E_n}$, *i.e.* Y ⊃ S. Combined with (2.97), this shows that x + Y cuts Y. Therefore, our arbitrary x is an element of the subgroup Y. We have thus established that X ⊂ Y, which achieves the proof. □

2.7   Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* f : X → K *is a closed subset of* X × K. *Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

Proof. Choose a sequence $\{x_n : n = 1, 2, 3, \dots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of X × K. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$f : [0, \infty) \to [0, \infty) \tag{2.98}$$

$$x \mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases}$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}. \tag{2.99}$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b). To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on R_+. This establishes that $(a, b) \in G_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. □

Chapter 3

Convexity

3.1 Exercise 11. Meagerness of the polar

Let X be an infinite-dimensional Fréchet space. Prove that X^*, with its weak*-topology, is of the first category in itself.

This is actually a consequence of the below lemma, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. If X is an infinite dimensional topological vector space whose dual X^* separates points on X, then the polar

$$K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\} \tag{3.1}$$

of any absorbing subset A is a weak*-closed set that has empty interior.

Proof. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous; see 3.14 of [3]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see 3.11 of [3]). Put

$$W \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\}, \tag{3.2}$$

where r_x runs on $\mathbf{R}_+$, as F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$M \triangleq \text{span}(F) \tag{3.3}$$

is finite dimensional. Assume, to reach a contradiction, that $A \subset M$: Each x is then in $tM = M$ for some $t_x > 0$, since A is absorbing. So, $X = M$ is finite dimensional, which is a desired contradiction. We have just established that $A \not\subset M$: Now pick a in $A \setminus M$ and so conclude that

$$b \triangleq \frac{a}{t_a} \in A \tag{3.4}$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M is closed (see 1.21 (b) of [3]): By the Hahn-Banach theorem 3.5 of [3], there exists Λ_a in X^* such that

$$\Lambda_a b > 2 \tag{3.5}$$

$$\Lambda_a(M) = \{0\}. \tag{3.6}$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$|\Lambda_a b + \Lambda_A b| \geq 2 - |\Lambda b| > 1. \tag{3.7}$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

$$\Lambda + W \not\subset K_A. \tag{3.8}$$

Since W and Λ are both arbitrary, this achieves the proof. $\square$

We now give a proof of the original statement:

Corollary.   If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$B_n \triangleq \{x \in X : d(0,x) < 1/n\} \quad (n = 1, 2, 3, \dots) \tag{3.9}$$

form a local base of X. If Λ is in X^*, then

$$|\Lambda| \leq m \text{ on } B_n \tag{3.10}$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$; see 1.18 of [3]. Hence, X^* is the countable union of all

$$m \cdot K_n \quad (m, n = 1, 2, 3, \dots), \tag{3.11}$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [3]. As a consequence, the above lemma implies

$$\left(\overline{K_n}\right)^{\circ} = (K_n)^{\circ} = \emptyset. \tag{3.12}$$

Since the multiplication by m is an homeomorphism (see 1.7 of [3]), this is equivalent to

$$\left(\overline{m \cdot K_n}\right)^{\circ} = m \cdot (K_n)^{\circ} = \emptyset. \tag{3.13}$$

So ends the proof. $\square$

Bibliography

[1] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 1976.

[2] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, 1986.

[3] Walter Rudin. *Functional Analysis*. McGraw-Hill, 1991.

[4] Laurent Schwartz. *Analyse*, volume III (in French). Hermann, 1997.