

Solutions to some exercises from Walter Rudin's
Functional Analysis

gitcordier

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Notations and Conventions

0.1 Logic

1. **Halmos' iff.** **iff** is a short for “if and only if”.
2. **Definitions (of values) with \triangleq .** Given variables a and b , $a \triangleq b$ means that a is defined as equal to b .
3. **\equiv .** $a \equiv b$ means that there exists a “natural” bijection \rightarrow that maps a to b ; which let us identify a with b . In a metric space context, $a \equiv b$ means that \rightarrow is isometric.
4. **Definitions (formulæ).** Definitions use the **iff** format. In other words, every definition has a “only if”.
5. **Iverson notation.** Given a boolean expression Φ , $[\Phi]$ returns the truth value of Φ , encoded as follows,

$$[\Phi] \triangleq \begin{cases} 0 & \text{if } \Phi \text{ is false;} \\ 1 & \text{if } \Phi \text{ is true.} \end{cases}$$

For example, $[1 > 0] = 1$ but $[\sqrt{2} \in \mathbf{Q}] = 0$.

0.2 Topological vector spaces

1. **Product space**
2. **Scalar field.** The usual (complete) scalar field is \mathbf{C} . A property, *e.g.* linearity, that is true on \mathbf{C} is also true on \mathbf{R} . The complex case is then a *special case* of the real one. Sometimes, this specialization is not purely formal. For example, theorem 12.7 of [3] asserts that, in a Hilbert space H equipped with the inner product $\langle \cdot | \cdot \rangle$, every nonzero linear continuous operator T “breaks orthogonality”, in the sense that there always exists $x = x(T)$ in H that satisfies $\langle Tx | x \rangle \neq 0$. The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exist. To see that, consider the 90° rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitly mentioned*, the extension to the real case will always be obvious. So, taking \mathbf{C} as the scalar field shall mean “*Instead of letting the scalar field undefined, we choose \mathbf{C} for the sake of expressivity. But considering \mathbf{R} instead of \mathbf{C} would actually make no difference here*”.
3. **Finite dimensional spaces.** Let Y be a finite dimensional space. If $\dim Y = 0$, *i.e.* Y is a group of order 1, then $\{\emptyset, Y\}$ is the only possible topology for Y . For instance, in a quotient space X/N , the zero is N and $\{N\}$ is zero-dimensional in X/N .

Assume henceforth that $\dim Y > 0$, *i.e.* Y has a base $F_n = \{f_i : i = 1, \dots, n\}$ for some positive n . The cartesian power $\mathbf{C}^n = \prod_{j=1}^n \mathbf{C}$ is the vector space of all lists (z_1, \dots, z_n) , where $z_j \in \mathbf{C}$ (identify \mathbf{C}^1 with \mathbf{C}). The subset $E_n = \{e_j : j = 1, \dots, n\}$ is the *standard base* of \mathbf{C}^n , *i.e.* $e_j = 1_{\{j\}}$. So, $\dim \mathbf{C}^n = n$. Let $u : \mathbf{C}^n \rightarrow Y$ be the only linear mapping that verifies all $u(e_j) = f_j$. Since u is encoded as the identity matrix, both u and $v = u^{-1}$ exist as isomorphisms. Additionally, \mathbf{C}^n can be equipped with various norms, *e.g.* the *p-norms* $\|\cdot\|_p$ (where $\|(z_1, \dots, z_n)\|_p^p = |z_1|^p + \dots + |z_n|^p$; $p \geq 1$) or $\|\cdot\|_\infty$ (where $\|(z_1, \dots, z_n)\|_\infty = \max |z_j|$). Note that Y inherits any norm $\|\cdot\|$ of \mathbf{C}^n , with $\|u(z_1, \dots, z_n)\| = \|(z_1, \dots, z_n)\|$; which turns u into a isometry of \mathbf{C}^n onto Y . Let $\tau_{\|\cdot\|}$ denote the topology of a norm $\|\cdot\|$. We now go back to the proof of 1.21 of [3] and so equip Y with a its own norm $\|\cdot\|_2$; which turns u into a isometric isomorphism of \mathbf{C}^n onto Y . Y can now be seen as a topological vector space, in at least one fashion; namely, the space $(Y, \tau_{\|\cdot\|_2})$. Let $\tau = \tau_Y$ stand for any arbitrary topology of Y . Hence the following commutative diagram

$$\begin{array}{ccc}
 & (\mathbf{C}^n, \tau_{\|\cdot\|_2}) & \\
 u \swarrow & & \nwarrow v \\
 (Y, \tau_{\|\cdot\|_2}) & & (Y, \tau) \\
 v \searrow & & \nearrow u \\
 & (\mathbf{C}^n, \tau_{\|\cdot\|_2}) &
 \end{array}$$

(1)

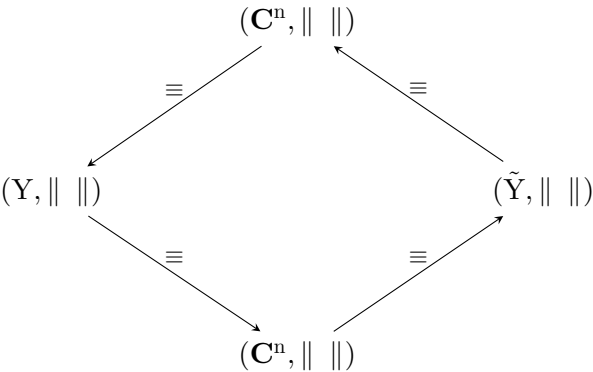
It is now clear that the *identity mapping* $u \circ v$ is an homeomorphism of Y onto Y , which implies that $\tau = \tau_{\|\cdot\|_2}$. In other words, there is only one topology τ for Y , as a topological vector space. This topology is normable, since $\tau = \tau_{\|\cdot\|_2}$. Let $\|\cdot\|_Y$ stand for any norm of Y . The special case $Y = \mathbf{C}^n, F_n = E_n, u = i$ is of considerable interest. TOTO. Now take \tilde{Y} of dimension n then similarly define (obvious notations) \tilde{u}, \tilde{v} and $\tilde{\tau}$.

$$\begin{array}{ccc}
 & (\mathbf{C}^n, \tau_{\mathbf{C}^n}) & \\
 u \swarrow & & \nwarrow \tilde{v} \\
 (Y, \tau) & & (\tilde{Y}, \tilde{\tau}) \\
 v \searrow & & \nearrow \tilde{u} \\
 & (\mathbf{C}^n, \tau_{\mathbf{C}^n}) &
 \end{array}$$

(2)

The homeomorphism between Y and \tilde{Y} leads to *the equivalence of norms at fixed dimension* n , as follows $A\|y\|_Y \leq \|\tilde{u} \circ v(y)\|_{\tilde{Y}} \leq B\|y\|_Y$ ($y \in Y$) for some positive

A,B. Equip Y and \tilde{Y} with the inherited norm $\|\cdot\|$. Y and \tilde{Y} are homeomorphically isomorphic (\equiv) to $\mathbf{C}^n, \|\cdot\|$.



(3)

From now the default norm will be $\|\cdot\|_\infty$.

Chapter 1

Topological Vector Spaces

1.1 Exercise 1. Basic results

Suppose X is a vector space. All sets mentioned below are understood to be subsets of X . Prove the following statements from the axioms as given as in section 1.4.

- (a) If $x, y \in X$ there is a unique $z \in X$ such that $x + z = y$.
- (b) $0 \cdot x = 0 = \alpha \cdot 0$ ($\alpha \in \mathbf{C}, x \in X$).
- (c) $2A \subset A + A$.
- (d) A is convex if and only if $(s + t)A = sA + tA$ for all positive scalars s and t .
- (e) Every union (and intersection) of balanced sets is balanced.
- (f) Every intersection of convex sets is convex.
- (g) If Γ is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of Γ is convex.
- (h) If A and B are convex, so is $A + B$.
- (i) If A and B are balanced, so is $A + B$.
- (j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

Proof. 1. Such property only depends on the group structure of X : Each x in X has an opposite $-x$. Let x' be any opposite of x , so that $x - x = 0 = x + x'$. Thus, $-x + x - x = -x + x + x'$, which is equivalent to $-x = x'$. So is established the uniqueness of $-x$. It is now clear that $x + z = y$ **iff** $z = -x + y$, which asserts both the existence and the uniqueness of z .

2. Remark that

$$(1.1) \quad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$

$$(1.2) \quad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that $0 \cdot x = 0$. So,

$$(1.3) \quad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x.$$

Finally,

$$(1.4) \quad \alpha \cdot 0 \stackrel{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

3. Remark that

$$(1.5) \quad 2x = (1 + 1)x = x + x$$

for every x in X , and so conclude that

$$(1.6) \quad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subset \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets A of X ; which proves (c).

4. If A is convex, then

$$(1.7) \quad A \subset \frac{s}{s+t}A + \frac{t}{s+t}A \subset A;$$

which is

$$(1.8) \quad sA + tA = (s+t)A.$$

Conversely, the special case $s + t = 1$ is

$$(1.9) \quad sA + (1-s)A = A.$$

The latter extends to $s = 0$, since

$$(1.10) \quad 0A + A \stackrel{(b)}{=} \{0\} + A = A.$$

The extension to $s = 1$ is analogously established (or simply use the fact that $+$ is commutative!). So ends the proof.

5. Let A range over B a collection of balanced subsets, so that

$$(1.11) \quad \alpha \bigcap B \subset \alpha A \subset A \subset \bigcup B$$

for all scalars α of magnitude ≤ 1 . The inclusion $\alpha \bigcap B \subset A$ establishes the first part. Now remark that

$$(1.12) \quad \alpha A \subset \bigcup B$$

implies

$$(1.13) \quad \alpha \bigcup B \subset \bigcup B;$$

which achieves the proof.

6. Let A range over C a collection of convex subsets, so that

$$(1.14) \quad (s+t) \bigcap C \subset s \bigcap C + t \bigcap C \subset sA + tA \stackrel{(d)}{=} (s+t)A$$

for all positive scalars s, t . Thus,

$$(1.15) \quad (s+t) \bigcap C \subset s \bigcap C + t \bigcap C \subset (s+t) \bigcap C.$$

We now conclude from (d) that the intersection of C is convex. So ends the proof.

7. Pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i ($i = 1, 2$) lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2 . So, x_1, x_2 are now elements of the convex set C_2 . Every convex combination of our x_1 's is then in $C_2 \subset \bigcup \Gamma$, hence (g).

8. Simply remark that

$$(1.16) \quad s(A + B) + t(A + B) = sA + tA + sB + tB = (s + t)(A + B)$$

for all positive scalars s and t , then conclude from (d) that $A + B$ is convex.

9. Given any α from the closed unit disc,

$$(1.17) \quad \alpha(A + B) = \alpha A + \alpha B \subset A + B.$$

There is no more to prove.

10. Our proof will be based on the following lemma,

If $S \subset X$, then any assertion

- (i) S is a vector subspace of X ;*
 - (ii) S is convex balanced such that $S + S = S$;*
 - (iii) S is convex balanced such that $\lambda S = S \quad (\lambda > 0)$*
- implies the other ones.*

To prove the lemma, let S range over the subsets of X . First, assume that (i) holds: Clearly, every S is convex balanced. Moreover, $S + S \subset S$. Conversely, $S = S + \{0\} \subset S + S$; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \quad nS = (n - 1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). The special case $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$ ($\lambda > 0$) yields

$$(1.19) \quad nS \stackrel{(1.18)}{\subset} S \subset n\lambda S \subset n^2 S,$$

since S is balanced and that $1 < n\lambda < n^2$. Dividing the latter inclusions by n shows that

$$(1.20) \quad S \subset \lambda S \subset nS \stackrel{(1.18)}{\subset} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$(1.21) \quad \alpha S + \beta S = |\alpha|S + |\beta|S \stackrel{(d)}{=} (|\alpha| + |\beta|)S \stackrel{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as S is balanced. Moreover (under the sole assumption that S is balanced), this extends to $|\alpha| + |\beta| = 0$, as follows,

$$(1.22) \quad \alpha S + \beta S = 0S + 0S \stackrel{(b)}{=} \{0\} \stackrel{(b)}{=} 0S \subset S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let V be a collection of vector spaces of X , of intersection I and union U . First, remark that every member of V is convex balanced: So is I (combine (e) with (f)). Next, let Y range over V , so that

$$(1.23) \quad I + I \subset Y + Y \subset Y;$$

which yields

$$(1.24) \quad I + I \subset I.$$

Conversely,

$$(1.25) \quad I = I + \{0\} \subset I + I.$$

It now follows from the lemma's (ii) \Rightarrow (i) that I is a vector subspace of X . Now temporarily assume that S is totally ordered by set inclusion: Combining (e) with (g) establishes that U is convex balanced. To show that U is more specifically a vector subspace, we first remark that such total order implies that either $Z \subset Y$ or $Y \subset Z$, as Z ranges over V . A straightforward consequence is that

$$(1.26) \quad Y \subset Y + Z \subset Y \cup Z.$$

Another one is that $Y \cup Z$ ranges over V as well. Combined with the latter inclusions, this leads to

$$(1.27) \quad U \subset U + U \subset U.$$

It then follows from the lemma's (ii) \Rightarrow (i) that U is a vector subspace of X . Finally, let A, B run through the vector subspaces of X : Combining (h) with (i) proves that $A + B$ is convex balanced as well. Furthermore,

$$(1.28) \quad A + B \stackrel{(i) \Rightarrow (ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as X is an abelian group. We now conclude from (ii) that any $A + B$ is a vector subspace of X . So ends the proof. \square

1.2 Exercise 2. Convex hull

The convex hull of a set A in a vector space X is the set of all convex combinations of members of A , that is the set of all sums $t_1x_1 + \cdots + t_nx_n$ in which $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$; n is arbitrary. Prove that the convex hull of a set A is convex and that is the intersection of all convex sets that contain A .

Proof. The convex hull of a set S will be denoted by $\text{co}(S)$. Remark that $S \subset \text{co}(S)$ (to see that, take $t_1 = 1$ for each x_1 in S) and that $\text{co}(A) \subset \text{co}(B)$ where $A \subset B$ (obvious). Our proof will directly derive from the following lemma,

Let S be a subset of a vector space X : Its convex hull $\text{co}(S)$ is convex and the following statements

- (i) S is convex;
 - (ii) $s_1S + \cdots + s_nS = (s_1 + \cdots + s_n)S$ for all positive scalar variables s_1, \dots, s_n ;
 - (iii) $t_1S + \cdots + t_nS = S$ for all positive scalar variables s_1, \dots, s_n such that $s_1 + \cdots + s_n = 1$;
 - (iv) $\text{co}(S) = S$
- are equivalent.*

More specifically, our proof of the second part will only depend on (i) \Rightarrow (iv).

From now on, we skip the trivial case $S = \emptyset$ then only consider nonempty sets. To prove the first part, let a, b run through the convex combination(s) of S , so that $a = t_1x_1 + \cdots + t_nx_n$ and $b = t_{n+1}x_{n+1} + \cdots + t_{n+p}x_{n+p}$ for some (t_i, x_i) . Every sum $sa + (1-s)b$ ($0 \leq s \leq 1$) is then a convex combination of x_1, \dots, x_{n+p} , since

$$(1.29) \quad sa + (1-s)b = \sum_{i=1}^n st_i x_i + \sum_{i=n+1}^{n+p} (1-s)t_i x_i$$

and

$$(1.30) \quad \sum_{i=1}^n st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s \sum_{i=1}^n t_i + (1-s) \sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets S , this reads

$$(1.31) \quad s \text{co}(S) + (1-s) \text{co}(S) \subset \text{co}(S);$$

which was our first goal. We now aim at the equivalence (i) $\Rightarrow \cdots \Rightarrow$ (iv) \Rightarrow (i): An easy proof by induction makes the implication (i) \Rightarrow (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii) \Rightarrow (iv) derives from the definition of the convex hull. We now close the chain with (iv) \Rightarrow (i), by remarking that S is convex whether $S = \text{co}(S)$. The lemma being proved, let us establish the second part. To do so, start from $F \ni \text{co}(A)$ then possibly enrich F the following way:

$$(1.32) \quad B \in F \Rightarrow B \text{ is convex and contains } A.$$

Note that our definition of F is weaker than the primary assumption “[F only encompasses] all convex sets that contain A ”, which is the special case

$$(1.33) \quad B \in F \Leftrightarrow B \text{ is convex and contains } A.$$

In any case, the key ingredient is that $\text{co}(A) \in \mathcal{F}$ implies

$$(1.34) \quad \text{co}(A) \supset \bigcap \mathcal{F}.$$

Conversely, the following formula

$$(1.35) \quad \text{co}(A) \subset \text{co}(B) \stackrel{(i) \Rightarrow (iv)}{=} B \quad (B \in \mathcal{F})$$

is valid and implies

$$(1.36) \quad \text{co}(A) \subset \bigcap_{B \in \mathcal{F}} B.$$

So ends the proof

□

1.3 Exercise 3. Other basic results

1.4 Exercise 4. A nonempty set whose interior is not

1.5 Exercise 5. A first restatement of boundedness

1.6 Exercise 6. A second restatement of boundedness

1.7 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0, 1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \rightarrow \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \rightarrow \infty$ then $\{\gamma_n f_n\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0, 1]$.) This shows that metrizability cannot be omitted in (b) of Theorem 1.28.

Proof. The family of the seminorms p_x is separating: By 1.37 of [3], the collection \mathcal{B} of all finite intersections of the sets

$$(1.37) \quad V(x, k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k = 1, 2, 3, \dots)$$

is therefore a local base for a topology τ on X . So,

$$(1.38) \quad \sum_{n=1}^{\infty} [f_n \notin \cap_{i=1}^m U_i] \leq \sum_{n=1}^{\infty} \sum_{i=1}^m [f_n \notin U_i] = \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n \notin U_i] \quad (f_n \in X, U_i \in \tau).$$

Now assume that $\{f_n\}$ τ -converges to some f , *i.e.*

$$(1.39) \quad \sum_{n=1}^{\infty} [f_n \notin f + W] < \infty \quad (W \in \mathcal{B}).$$

The special case $W = V(x, k)$ means that, given k , $|f_n(x) - f(x)| < 2^{-k}$ for almost all n , *i.e.* $\{f_n(x)\}$ converges to $f(x)$. Conversely, assume that $\{f_n\}$ does not τ -converges in X , *i.e.*

$$(1.40) \quad \forall f \in X, \exists W \in \mathcal{B} : \sum_{n=1}^{\infty} [f_n \notin f + W] = \infty.$$

W is now the (nonempty) intersection of finitely many $V(x, k)$, say $V(x_1, k_1), \dots, V(x_m, k_m)$. Thus,

$$(1.41) \quad \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)] \stackrel{(1.38)}{\geq} \sum_{n=1}^{\infty} [f_n \notin f + W] \stackrel{(1.40)}{=} \infty.$$

We can now conclude that, for some index i ,

$$(1.42) \quad \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)] = \infty.$$

In other word, $\{f_n(x_i)\}$ fails to converge to $f(x_i)$. We have so proved that τ -convergence is a rewording of pointwise convergence. We now establish the second part.

To do so, we split x into two variables: r if x is rational, α otherwise. The proof is based on the following well-known result: Each α has a *unique* binary expansion. More precisely,

there exists a bijection $b : [0, 1] \setminus \mathbf{Q} \rightarrow \{\beta \in \{0, 1\}^{\mathbf{N}^+} : \beta \text{ is not eventually periodic}\}$ where $b(\alpha) = (\beta_1, \beta_2, \dots)$ is the only bit stream such that

$$(1.43) \quad \alpha = \sum_{k=1}^{\infty} \beta_k \cdot 2^{-k}.$$

Remark that $b(\alpha)_1 + \dots + b(\alpha)_n \xrightarrow{n \rightarrow \infty} \infty$, since $b(\alpha)$ has infinite support, then fix

$$(1.44) \quad f_n(\alpha) \triangleq \frac{1}{b(\alpha)_1 + \dots + b(\alpha)_n} \xrightarrow{n \rightarrow \infty} 0.$$

The actual values $f_n(r)$ are of no interest, as long as every sequence $\{f_n(r) : n = 1, 2, 3, \dots\}$ converges to 0. For example, put $f_n(r) = r/n$, or just $f_n(r) = 0$. We also take $\gamma_n \rightarrow \infty$, *i.e.* given any counting number p , γ_n is greater than p for almost all n . Next, we choose n_p among those *almost all* n that are large enough to satisfy

$$(1.45) \quad n_p - n_{p-1} > p$$

(start with $n_0 = 0$). So, every list $n_p, n_{p'}, n_{p''}, \dots$ that satisfies $n_{p'} - n_p = n_{p''} - n_{p'} = \dots$ is finite (otherwise, $n_{p'} - n_p \geq n_{p+1} - n_p > p \rightarrow \infty$ would hold; see (1.45)). In other words, *the distribution of n_1, n_2, \dots displays no periodic pattern*. As a consequence, the *characteristic function* $\chi : k \mapsto [k \in \{n_1, n_2, \dots\}]$ is not eventually periodic. Combined with (1.43), this establishes that

$$(1.46) \quad \alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k}$$

is irrational. Conversely, still with (1.43),

$$(1.47) \quad b(\alpha_\gamma)_k = \chi_k.$$

Now remark that

$$(1.48) \quad \chi_1 + \dots + \chi_{n_1} = 1$$

$$(1.49) \quad \chi_1 + \dots + \chi_{n_1} + \dots + \chi_{n_2} = 2$$

$$\vdots$$

$$(1.50) \quad \chi_1 + \dots + \chi_{n_1} + \dots + \chi_{n_2} + \dots + \chi_{n_p} = p.$$

Combined with (1.44), this yields

$$(1.51) \quad \gamma_{n_p} f_{n_p}(\alpha_\gamma) = \frac{\gamma_{n_p}}{p} > 1.$$

There so exists a subsequence $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. Since $\{\gamma_n\}$ was arbitrary, this proves (b). \square

1.9 Exercise 9. Quotient map

Suppose

1. X and Y are topological vector spaces,
2. $\Lambda : X \rightarrow Y$ is linear.
3. N is a closed subspace of X ,
4. $\pi : X \rightarrow X/N$ is the quotient map, and
5. $\Lambda x = 0$ for every $x \in N$.

Prove that there is a unique $f : X/N \rightarrow Y$ which satisfies $\Lambda = f \circ \pi$, that is, $\Lambda x = f(\pi(x))$ for all $x \in X$. Prove that f is linear and that Λ is continuous if and only if f is continuous. Also, Λ is open if and only if f is open.

Proof. Bear in mind that π continuously maps X onto the topological (Hausdorff) space X/N , since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$(1.52) \quad f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y.$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$(1.53) \quad \Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x.$$

It straightforwardly derives from (1.52) that f inherits linearity from π and Λ .

Remark. The special case $N = \{\Lambda = 0\}$, i.e. $\Lambda x = 0$ iff $x \in N$ (cf. (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$(1.54) \quad f(\pi x) = 0 \stackrel{(1.52)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$(1.55) \quad \Lambda(U) = f(\pi(U)) \subset V.$$

Since π is open (1.41 (a) of [3]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$(1.56) \quad W = \pi(V)$$

for some neighborhood V of 0_X . So,

$$(1.57) \quad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y . So ends the proof. \square

1.10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \rightarrow Y$ is linear, and $\Lambda(X) = Y$.

1. Prove that Λ is an open mapping.

2. Assume, in addition, that the null space of Λ is closed, and prove that Λ is continuous.

Proof. Discard the trivial case $\Lambda = 0$ and so assume that $\dim Y = n$ for some positive n . Let e range over a base of B of Y then pick W an arbitrary neighborhood of the origin: There so exists V a balanced neighborhood of the origin such that

$$(1.58) \quad \sum_e V \subset W,$$

since addition is continuous. Moreover, for each e , there exists x_e in X such that $\Lambda(x_e) = e$, simply because Λ is onto. So,

$$(1.59) \quad y = \sum_e y_e \cdot \Lambda x_e \quad (y \in Y).$$

As a finite set, $\{x_e : e \in B\}$ is bounded: There so exists a positive scalar s such that

$$(1.60) \quad \forall e \in B, x_e \in s \cdot V.$$

Combining this with (1.59) shows that

$$(1.61) \quad y \in \sum_e y_e \cdot s \cdot \Lambda(V).$$

We now come back to (1.58) and so conclude that

$$(1.62) \quad y \in \sum_e \Lambda(V) \subset \Lambda(W)$$

whether $|y_e| < 1/s$; which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, $\{\Lambda = 0\}$ playing the role of N . Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y . Consequently,

$$(1.63) \quad \dim X/N = n.$$

f is then an homeomorphism of $X/N \equiv \mathbf{C}^n$ onto Y ; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = f \circ \pi$. \square

1.12 Exercise 12. Topology stays, completeness leaves

1.14 Exercise 14. \mathcal{D}_K equipped with other seminorms

Put $K = [0, 1]$ and define \mathcal{D}_K as in Section 1.46. Show that the following three families of seminorms (where $n = 0, 1, 2, \dots$) define the same topology on \mathcal{D}_K . If $D = d/dx$:

$$1. \|D^n f\|_\infty = \sup\{|D^n f(x)| : 0 < x < 1\}$$

$$2. \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$3. \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

Proof. First, remark that

$$(1.64) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of $D^n f$ lies in K ; which yields

$$(1.65) \quad |D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1.$$

So,

$$(1.66) \quad \|D^n f\|_\infty \leq \|D^{n+1} f\|_1.$$

We now combine (1.64) with (1.66) and so obtain

$$(1.67) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots \quad (n = 0, 1, 2, \dots).$$

Put

$$(1.68) \quad V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\} \quad (i = 1, 2, \infty)$$

$$(1.69) \quad \mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\},$$

so that (1.67) is mirrored in terms of neighborhood inclusions, as follows,

$$(1.70) \quad V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \dots.$$

Since $V_n^{(i)} \supset V_{n+1}^{(i)}$, $\mathcal{B}^{(i)}$ is a local base of a topology τ_i . But the chain (1.70) forces

$$(1.71) \quad \tau_1 = \tau_2 = \tau_\infty.$$

To see that, choose a set S that is τ_1 -open at f , i.e. $V_n^{(1)} \subset S - f$ for some n . Next, concatenate this with $V_n^{(2)} \subset V_n^{(1)}$ (see (1.70)) and so obtain $V_n^{(2)} \subset S - f$; which implies that S is τ_2 -open at f . Similarly, we deduce, still from (1.70), that

$$(1.72) \quad \tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}.$$

So ends the proof. □

1.16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms p_n , then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (e.g. [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1 *Let X be a topological space with a countable local base $\{V_n : n = 1, 2, 3, \dots\}$. If $\tilde{V}_n = V_1 \cap \dots \cap V_n$, then every subsequence $\{\tilde{V}_{\varrho(n)}\}$ is a decreasing (i.e. $\tilde{V}_{\varrho(n)} \supset \tilde{V}_{\varrho(n+1)}$) local base of X .*

Proof. The decreasing property is trivial. Now remark that $V_n \supset \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X . Then so is $\{\tilde{V}_{\varrho(n)}\}$, since $\tilde{V}_n \supset \tilde{V}_{\varrho(n)}$. \square

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$) *Under the same notations of Lemma 1, if $\{V_n\}$ is a decreasing local base, then so is $\{V_{\varrho(n)}\}$.*

Corollary 2 *If $\{Q_n\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{\varrho(n)}\}$ also satisfies these conditions. Furthermore, if τ_Q is the $C(\Omega)$'s (respectively $C^\infty(\Omega)$'s) topology of the seminorms p_n , as defined in section 1.44 (respectively 1.46), then the seminorms $p_{\varrho(n)}$ define the same topology τ_Q .*

Proof. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X . Moreover,

$$(1.73) \quad Q_{\varrho(n)} \subset \overset{\circ}{Q}_{\varrho(n)+1} \subset Q_{\varrho(n)+1} \subset Q_{\varrho(n+1)}.$$

Thus,

$$(1.74) \quad Q_{\varrho(n)} \subset \overset{\circ}{Q}_{\varrho(n+1)}.$$

In other words, $Q_{\varrho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\varrho(n)}\}$ then defines a topology τ_{Q_ϱ} for which $\{V_{\varrho(n)}\}$ is a local base. So, $\tau_{Q_\varrho} \subset \tau_Q$. Conversely, the above corollary asserts that $\{V_{\varrho(n)}\}$ is a local base of τ_Q , which yields $\tau_Q \subset \tau_{Q_\varrho}$. \square

Lemma 2 *If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in almost all Q_n° , i.e. there exists m such that*

$$(1.75) \quad K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \dots$$

Proof. The following definition

$$(1.76) \quad C_n \triangleq K \setminus \overset{\circ}{Q}_n$$

shapes $\{C_n\}$ as a decreasing sequence of compact¹ sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude² that the C_n 's intersection contains a point that is not in any $\overset{\circ}{Q}_n$. On the other hand, the conditions specified in [1.44] force the $\overset{\circ}{Q}_n$'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, i.e. $K \subset \overset{\circ}{Q}_m$, for some m . Finally,

$$(1.77) \quad K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots$$

□

We are now in a fair position to establish the following:

Theorem *The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.*

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\kappa}$ for every subsequence $\{L_{\kappa(n)}\}$. Now apply the above Lemma 2 with K_i ($i = 1, 2, 3, \dots$) and so conclude that $K_i \subset \overset{\circ}{L}_{m_i} \subset \overset{\circ}{L}_{m_i+1} \subset \cdots$ for some m_i . In particular, the special case $\kappa_i = m_i + i$ is

$$(1.78) \quad K_i \subset \overset{\circ}{L}_{\kappa_i}.$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$(1.79) \quad L_j \subset \overset{\circ}{K}_{\lambda_j}$$

Combine (1.78) with (1.79) and so obtain

$$(1.80) \quad K_1 \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{L}_{\kappa_{\lambda_{\kappa_1}}} \subset \cdots,$$

which means that the sequence $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$(1.81) \quad \tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L.$$

So ends the proof

□

¹ See (b) of 2.5 of [2].

² In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of 2.6 of [2].

1.17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ is a continuous mapping of $C^\infty(\Omega)$ into $C^\infty(\Omega)$ and also of \mathcal{D}_K into \mathcal{D}_K , for every multi-index α .

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an arbitrary neighborhood of the origin. There so exists N such that U contains

$$(1.82) \quad V_N = \left\{ \varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}.$$

Now pick g in $V_{N+|\alpha|}$, so that

$$(1.83) \quad \max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}.$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitly used). The special case $\gamma = \beta + \alpha$ yields

$$(1.84) \quad \max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

$$(1.85) \quad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad i.e. \quad D^\alpha(V_{N+|\alpha|}) \subset V_N,$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$, we first remark that the collection of the $V_N \cap \mathcal{D}_K$ is a local base of the subspace topology of \mathcal{D}_K . $V_{N+|\alpha|} \cap \mathcal{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$(1.86) \quad D^\alpha|_{\mathcal{D}_K}(V_{N+|\alpha|} \cap \mathcal{D}_K) = D^\alpha(V_{N+|\alpha|} \cap \mathcal{D}_K)$$

$$(1.87) \quad \subset D^\alpha(V_{N+|\alpha|}) \cap D^\alpha(\mathcal{D}_K)$$

$$(1.88) \quad \subset V_N \cap \mathcal{D}_K \quad (\text{see (1.85)})$$

So ends the proof. □

Chapter 2

Completeness

2.3 Exercise 3. An equicontinuous sequence of measures

Put $K = [-1, 1]$; define \mathcal{D}_K as in section 1.46 (with \mathbf{R} in place of \mathbf{R}^n). Suppose $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\Lambda\varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathcal{D}_K$. Show that Λ is a continuous linear functional on \mathcal{D}_K . Show that there is a positive integer p and a number $M < \infty$ such that

$$\left| \int_{-1}^1 f_n(t)\varphi(t)dt \right| \leq M \|D^p \varphi\|_\infty$$

for all n . For example, if $f_n(t) = n^3 t$ on $[-1/n, 1/n]$ and 0 elsewhere, show that this can be done with $p = 1$. Construct an example where it can be done with $p = 2$ but not with $p = 1$.

We will also consider the case $p = 0$. Since all supports of $\varphi, \varphi', \varphi'', \dots$, are in K , we make a specialization of the mean value theorem:

Lemma If $\varphi \in \mathcal{D}_{[a,b]}$, then

$$(2.1) \quad \|D^\alpha \varphi\|_\infty \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (\alpha = 0, 1, \dots, p)$$

at every order $p = 0, 1, 2, \dots$; where λ is the length $|b - a|$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 ($a < x_1 < x_0$), such that

$$(2.2) \quad \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a).$$

Since every $D^p \varphi$ lies in $\mathcal{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \dots < x_p < \dots < x_0$ such that

$$(2.3) \quad \varphi(x_0) = D^0 \varphi(x_0)$$

$$(2.4) \quad = D^1 \varphi(x_1)(x_0 - a)$$

$$= \dots$$

$$(2.5) \quad = D^p \varphi(x_p)(x_0 - a) \cdots (x_{p-1} - a),$$

for all p . More compactly,

$$(2.6) \quad D^\alpha \varphi(x_0) = D^p \varphi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a);$$

which yields,

$$(2.7) \quad |D^\alpha \varphi(x)| \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c])$$

The case $x_0 \geq c$ outputs a “reversed” result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$(2.8) \quad |D^\alpha \varphi(x)| \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$(2.9) \quad \|D^\alpha \varphi\|_\infty \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}.$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u , we put

$$(2.10) \quad \langle u | \varphi \rangle \triangleq \int_{\mathbf{R}} u \varphi \quad (\varphi \in C_0(\mathbf{R})).$$

The following inequalities

$$(2.11) \quad |\langle u | \varphi \rangle| \leq \int_{\mathbf{R}} |u \varphi| \leq \|u\|_{L^1} \quad (\|\varphi\|_\infty \leq 1)$$

imply that every linear functional

$$(2.12) \quad \begin{aligned} \langle u | : C_0(\mathbf{R}) &\rightarrow \mathbf{C} \\ \varphi &\mapsto \langle u | \varphi \rangle \end{aligned}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u |$, since u is determined (a.e) by the integrals $\langle u | \varphi \rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*¹ operator $\langle u |$, of norm

$$(2.13) \quad \sup\{|\langle u | \varphi \rangle| : \|\varphi\|_\infty = 1\} = \|u\|_{L^1}.$$

Note that, in the latter equality, $\leq \|u\|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem². We now consider the special cases $u = g_n$, where g_n is

$$(2.14) \quad \begin{aligned} g_n : \mathbf{R} &\rightarrow \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) \end{cases} \end{aligned}$$

¹ see 1.32, 4.1 of [3]

² See 7.26 of [1].

First, remark that $g_n(x) \rightarrow 0$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \rightarrow \infty$). Nevertheless, we will show that the $\langle g_n |$ converge pointwise³ on \mathcal{D}_K *i.e.* there exists a τ_K -continuous linear form Λ such that

$$(2.15) \quad \langle g_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \varphi,$$

where φ ranges over \mathcal{D}_K . We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\varphi_1^+, \dots, \varphi_j^+, \dots$, from $C_K^\infty(\mathbf{R})$. More specifically,

$$(i) \quad \varphi_j^+ = 1 \text{ on } [e^{-j}, 1 - e^{-j}];$$

$$(ii) \quad \varphi_j^+ = 0 \text{ on } \mathbf{R} \setminus [-1, 1];$$

$$(iii) \quad 0 \leq \varphi_j^+ \leq 1 \text{ on } \mathbf{R};$$

see [1.46] of [3] for a possible construction of those φ_j^+ . Let $\varphi_1^-, \dots, \varphi_j^-, \dots$, mirror the φ_j^+ , in the sense that $\varphi_j^-(x) = \varphi_j^+(-x)$, so that

$$(iv) \quad \varphi_j \triangleq \varphi_j^+ - \varphi_j^- \text{ is odd, as } g_n \text{ is};$$

$$(v) \quad \text{every } \varphi_j \text{ is in } C_K^\infty(\mathbf{R});$$

$$(vi) \quad \text{The sequence } \{\varphi_j\} \text{ converges (pointwise) to } 1_{[0,1]} - 1_{[-1,0]}, \text{ and } \|\varphi_j\|_\infty = 1.$$

Thus, with the help of the Lebesgue's convergence theorem,

$$(2.16) \quad \langle g_n | \varphi_j \rangle = 2 \int_0^1 g_n(t) \varphi_j^+(t) dt \xrightarrow{j \rightarrow \infty} 2 \int_0^1 g_n(t) dt = \|g_n\|_{L^1} = n.$$

Finally,

$$(2.17) \quad \|g_n\|_{L^1} \stackrel{(2.16)}{\leq} \sup\{|\langle g_n | \varphi \rangle| : \|\varphi\|_\infty = 1\} \stackrel{(2.13)}{\leq} \|g_n\|_{L^1};$$

which is the desired result. So, in terms of boundedness constants: Given n , there exists $C_n < \infty$ such that

$$(2.18) \quad |\langle g_n | \varphi \rangle| \leq C_n \quad (\|\varphi\|_\infty = 1);$$

see (2.11). Furthermore, $\|g_n\|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n ; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n | \varphi_{\rho(n)} \rangle\}$ such that $\langle g_n | \varphi_{\rho(n)} \rangle$ is greater than, say, $n - 0.01$, as $\|\varphi_{\rho(n)}\|_\infty = 1$. Consequently, there is no bound M such that

$$(2.19) \quad |\langle g_n | \varphi \rangle| \leq M \quad (\|\varphi\|_\infty = 1; n = 1, 2, 3, \dots).$$

In other words, the g_n have no *uniform bound* in L^1 , *i.e.* the collection of all continuous linear mappings $\langle g_n |$ is not equicontinuous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n |$ do not converge pointwise (or “vaguely”, in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$(2.20) \quad \langle g_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \varphi \quad (\varphi \in C_0(\mathbf{R}))$$

³ See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n |$ do not converge pointwise, in a relevant space, to some Λ (see (2.15)).

From now on, unless the contrary is explicitly stated, we assume that φ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$(2.21) \quad \Lambda\varphi = \lim_{n \rightarrow \infty} \int_K f_n \varphi \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

for some linear form Λ . Since φ vanishes outside K , we can suppose without loss of generality that the support of f_n lies in K . So, (2.21) can be restated as follows,

$$(2.22) \quad \Lambda\varphi = \lim_{n \rightarrow \infty} \langle f_n | \varphi \rangle \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

Let K_1, K_2, \dots , be compact sets that satisfy the conditions specified in 1.44 of [3]. \mathcal{D}_K is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms p_1, p_2, \dots ; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

- (a) $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha \varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\varphi \in \mathcal{D}_K$;
- (b) The collection of the sets $V_N = \{\varphi \in \mathcal{D}_K : \|\varphi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K , the subspace topology of \mathcal{D}_K ; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\varphi \in V_m$ then conclude that $\langle f_n |$ is bounded by $\|f_n\|_{L^1}$ on V_m : Every linear functional $\langle f_n |$ is therefore τ_K -continuous; see 1.18 of [3].

To sum it up:

- (i) \mathcal{D}_K , equipped the topology τ_K , is a Fréchet space (see section 1.46 of [3]);
- (ii) Every linear functional $\langle f_n |$ is continuous with respect to this topology;
- (iii) $\langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda\varphi$ for all φ , i.e. $\Lambda - \langle f_n | \xrightarrow{n \rightarrow \infty} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n | \}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n | \}$, since addition is continuous. There so exists i, j such that, for all n ,

$$(2.23) \quad |\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i,$$

$$(2.24) \quad |\Lambda\varphi - \langle f_n | \varphi \rangle| < 1/2 \quad \text{if } \varphi \in V_j.$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$(2.25) \quad |\langle f_n | \varphi \rangle| \leq |\Lambda\varphi - \langle f_n | \varphi \rangle| + |\Lambda\varphi| < 1 \quad \text{if } \varphi \in V_p.$$

Now remark that every $\psi = \psi[\mu, \varphi]$, where

$$(2.26) \quad \psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\varphi\|_p) \varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases}$$

keeps in V_p . Finally, it is clear that each below statement implies the following one.

$$(2.27) \quad |\langle f_n | \psi \rangle| < 1$$

$$(2.28) \quad |\langle f_n | \varphi \rangle| < 2^p \|\varphi\|_p \cdot \mu$$

$$(2.29) \quad |\langle f_n | \varphi \rangle| \leq 2^p \|\varphi\|_p$$

$$(2.30) \quad |\langle f_n | \varphi \rangle| \leq 2^p \{\|D^0 \varphi\|_\infty + \cdots + \|D^p \varphi\|_\infty\}.$$

Finally, with the help of (2.1),

$$(2.31) \quad |\langle f_n | \varphi \rangle| \leq 2^p(p+1)\|D^p \varphi\|_\infty.$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3 x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

$$(a) \quad \Lambda \varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t) \varphi(t) dt \text{ exists for every } \varphi \in \mathcal{D}_K;$$

$$(b) \quad \text{A uniform bound } |\langle f_n | \varphi \rangle| \leq M \|D^p \varphi\|_\infty \text{ (} n = 1, 2, 3, \dots \text{) exists for all those } f_n, \text{ with } p = 1 \text{ as the smallest possible } p.$$

Bear in mind that $K \subset K_m$ and shift the K_N 's indices, so that K_{m+1} becomes K_1 , K_{m+2} becomes K_2 , and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let φ keep running on \mathcal{D}_K and so define

$$(2.32) \quad B_n(\varphi) \triangleq \max\{|\varphi(x)| : x \in [-1/n, 1/n]\},$$

$$(2.33) \quad \Delta_n(\varphi) \triangleq \max\{|\varphi(x) - \varphi(0)| : x \in [-1/n, 1/n]\}.$$

The mean value asserts that

$$(2.34) \quad |\varphi(1/n) - \varphi(-1/n)| \leq B_n(\varphi') |1/n - (-1/n)| = \frac{2}{n} B_n(\varphi').$$

Independently, an integration by parts shows that

$$(2.35) \quad \langle f_n | \varphi \rangle = \left[\frac{n^3 t^2}{2} \varphi(t) \right]_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt$$

$$(2.36) \quad = \frac{n}{2} (\varphi(1/n) - \varphi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt.$$

Combine (2.34) with (2.36) and so obtain

$$(2.37) \quad |\langle f_n | \varphi \rangle| \leq \frac{n}{2} |\varphi(1/n) - \varphi(-1/n)| + \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\varphi'(t)| dt$$

$$(2.38) \quad \leq B_n(\varphi') + \frac{n^3}{2} B_n(\varphi') \int_{-1/n}^{1/n} t^2 dt$$

$$(2.39) \quad \leq \frac{4}{3} B_n(\varphi')$$

$$(2.40) \quad \leq \frac{4}{3} \|\varphi'\|_\infty.$$

Futhermore, (2.39) gives a hint about the convergence of f_n : Since $B_n(\varphi')$ tends to $|\varphi'(0)|$, we may expect that f_n tends to $\frac{4}{3}\varphi'(0)$. This is actually true: A straightforward computation shows that

$$(2.41) \quad \langle f_n | \varphi \rangle - \frac{4}{3}\varphi'(0) \stackrel{(2.36)}{=} \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} (\varphi' - \varphi'(0))t^2 dt.$$

So,

$$(2.42) \quad \left| \langle f_n | \varphi \rangle - \frac{4}{3}\varphi'(0) \right| \leq \left| \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0) \right| + \frac{1}{3}\Delta_n(\varphi') \xrightarrow{n \rightarrow \infty} 0.$$

We have just proved that

$$(2.43) \quad \langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3}\varphi'(0) \quad (\varphi \in \mathcal{D}_K).$$

In other words,

$$(2.44) \quad \langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} -\frac{4}{3}\delta',$$

where δ is the *Dirac measure* and δ', δ'', \dots , its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $-\frac{4}{3}\delta'$ is τ_K -continuous, and from (2.40) that

$$(2.45) \quad |\langle f_n | \varphi \rangle| \leq \frac{4}{3} \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$(2.46) \quad \{\langle f_n | \varphi_{p(n)} \rangle : \|\varphi_{p(n)}\|_\infty = 1; n = 1, 2, 3, \dots\}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$(2.47) \quad |\langle f_n | \varphi \rangle| \leq M \|\varphi\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

The previous bound of $\langle f_n |$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*⁴ f_n' , as follows

$$(2.48) \quad \begin{aligned} f_n' : \mathcal{D}_K &\rightarrow \mathbf{C} \\ \varphi &\mapsto -\langle f_n | \varphi' \rangle. \end{aligned}$$

It has been proved that every $\langle f_n |$ is continuous. So is

$$(2.49) \quad \begin{aligned} D : \mathcal{D}_K &\rightarrow \mathcal{D}_K \\ \varphi &\mapsto \varphi'; \end{aligned}$$

⁴ See 6.1 of [3] for a further discussion.

see Exercise 1.17. f_n' is therefore continuous. Now apply (2.43) with φ' and so obtain

$$-\langle f_n | \varphi' \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3} \varphi''(0) \quad (\varphi \in \mathcal{D}_K),$$

i.e.

$$(2.50) \quad f_n' \xrightarrow{n \rightarrow \infty} \frac{4}{3} \delta''.$$

It follows from (2.40) that,

$$(2.51) \quad |\langle f_n | \varphi' \rangle| \leq \frac{4}{3} \|\varphi''\|_\infty \quad (n = 1, 2, 3, \dots).$$

It is therefore possible to uniformly bound f_n' with respect to a norm $\|D^p \cdot\|_\infty$, namely $\|D^2 \cdot\|_\infty$. Then arises a question: Is 2 the smallest p ? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$(2.52) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Define

$$(2.53) \quad \Phi_j(x) = \int_{-1}^x \varphi_j.$$

The oddness of φ_j forces Φ_j to vanish outside $[-1, 1]$: φ_j is therefore in \mathcal{D}_K . So, under our assumption,

$$(2.54) \quad |\langle f_n | \Phi_j' \rangle| \leq M \|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots);$$

which is

$$(2.55) \quad |\langle f_n | \varphi_j \rangle| \leq M \quad (n = 1, 2, 3, \dots).$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n | \varphi_{\rho(n)} \rangle\}$) and so conclude that there is no constant M such that

$$(2.56) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Finally, assume, to reach a contradiction, that there exists a constant M such that

$$(2.57) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi\|_\infty.$$

The mean value theorem (see (2.1)) asserts that

$$(2.58) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi\|_\infty \leq M \|\varphi'\|_\infty;$$

which is, again, a desired contradiction. So ends the proof. □

2.6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ of a function $f \in L^2(T)$ (T is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all $n \in \mathbf{Z}$ (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^n \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$ is a dense subspace of $L^2(T)$ of the first category.

Proof. Let $f(\theta)$ stand for $f(e^{i\theta})$, so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$(2.59) \quad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

$$(2.60) \quad \Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well known (and easy to prove) result is

$$(2.61) \quad (e_n, e_{n'}) = [n = n'], \text{ i.e. } \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric (\equiv) identification $L^2 \equiv (L^2)^*$. So,

$$(2.62) \quad \|\Lambda_n\|^2 \stackrel{(2.60)}{=} \|e_{-n} + \cdots + e_n\|^2 \stackrel{(2.61)}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \stackrel{(2.61)}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

$$(2.63) \quad B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \dots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$(2.64) \quad \|\Lambda_n\| \stackrel{(2.62)}{=} \sqrt{2n+1} \xrightarrow{n \rightarrow \infty} \infty.$$

We have just established that B is actually of the first category; and so is its subset $L = \{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$. We now prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be $\text{span}\{e_k : k \in \mathbf{Z}\}$, the collection of the trigonometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$. Combining (2.60) with (2.61) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n . Thus,

$$(2.65) \quad P \subset L \subset L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.65), that

$$(2.66) \quad L^2(T) = \bar{P} = \bar{L}.$$

So ends the proof □

2.9 Exercise 9. Boundedness without closedness

Suppose X, Y, Z are Banach spaces and

$$B : X \times Y \rightarrow Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ such that

$$\|B(x, y)\| \leq M \|x\| \|y\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Let (x, y) range over $X \times Y$: Since B is continuous at the origin, there exists a positive r such that

$$(2.67) \quad \|x\| + \|y\| < r \Rightarrow \|B(x, y)\| < 1.$$

Now consider all scalars s, t such that $2\|x\| < rs$ and $2\|y\| < rt$: The following bound

$$(2.68) \quad \|B(x, y)\| = st \|B(x/s, y/t)\| \stackrel{(2.67)}{<} st$$

is effective, since $r > \|x\|/s + \|y\|/t$. Finally, remark that s, t have infimum bound $2\|x\|/r, 2\|y\|/r$ then so obtain

$$(2.69) \quad \|B(x, y)\| \leq \frac{4}{r^2} \|x\| \|y\|;$$

which achieves the proof.

As a concrete example, choose $X = Y = Z = C_c(\mathbf{R})$, topologized by the supremum norm. $C_c(\mathbf{R})$ is not complete (see 5.4.4 of [4]), nevertheless the bilinear product

$$\begin{aligned} B : C_c(\mathbf{R})^2 &\rightarrow C_c(\mathbf{R}) \\ (f, g) &\mapsto f \cdot g \end{aligned}$$

is bounded (since $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$), and continuous. To show this, pick a positive scalar ε smaller than 1 then put

$$(2.70) \quad r \triangleq \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty}$$

(bear in mind that $r < 1$). The Stone-Weierstrass theorem (see 7.26 of [1]) asserts the existence of (u, v) in $C_c(\mathbf{R})^2$ such that

$$(2.71) \quad \|f - u\|_\infty + \|g - v\|_\infty < r.$$

Next, remark that $\|u\|_\infty \leq r + \|f\|_\infty$ and so obtain

$$(2.72) \quad \|fg - uv\|_\infty = \|(f - u) \cdot g + u \cdot (g - v)\|_\infty$$

$$(2.73) \quad \leq \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty$$

$$(2.74) \quad < r \cdot \|g\|_\infty + (r + \|f\|_\infty) \cdot r$$

$$(2.75) \quad < r \cdot (r + \|f\|_\infty + \|g\|_\infty)$$

$$(2.76) \quad < \varepsilon.$$

Since ε was arbitrary, it is now established that B continuous. □

2.10 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0, 0)$.

Proof. Let (X_1, X_2, Z) be topological spaces and B a bilinear mapping

$$(2.77) \quad B : X_1 \times X_2 \rightarrow Z$$

From now on, $x = (x_1, x_2)$ denotes an arbitrary element of $X_1 \times X_2$. We henceforth assume that B is continuous at the origin $(0, 0)$ of $X_1 \times X_2$, *i.e.* given an arbitrary balanced open subset W of Z , there exists in X_i ($i = 1, 2$) a balanced open subset U_i such that

$$(2.78) \quad B(U_1 \times U_2) \subset W.$$

Let $\nu_i(x)$ denote any scalar that is greater than $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$. So,

$$(2.79) \quad B(x_1, x_2) = \nu_1(x)\nu_2(x) \cdot B(\nu_1(x)^{-1}x_1, \nu_2(x)^{-1}x_2)$$

$$(2.80) \quad \in \nu_1(x)\nu_2(x) \cdot B(U_1 \times U_2)$$

$$(2.81) \quad \subset \nu_1(x)\nu_2(x) \cdot W.$$

Now pick $p = (p_1, p_2)$ in $X_1 \times X_2$: It directly follows from (2.81) that

$$(2.82) \quad B(p_1, p_2) - B(x_1, x_2) = B(p_1, p_2 - x_2) + B(p_1 - x_1, x_2 - p_2) + B(p_1 - x_1, p_2)$$

$$(2.83) \quad \in \nu_1(p)\nu_2(p - x) \cdot W + \nu_1(p - x)\nu_2(x - p) \cdot W + \nu_1(p - x)\nu_2(p) \cdot W.$$

Let us henceforth assume that

$$(2.84) \quad p_i - x_i \in [\mu_1(p) + \mu_2(p) + 2]^{-1} \cdot U_i;$$

which yields

$$(2.85) \quad \mu_i(p_i - x_i) \leq [\mu_1(p) + \mu_2(p) + 2]^{-1}.$$

Finally, combine the special case

$$(2.86) \quad \nu_i(p - x) = [\mu_1(p) + \mu_2(p) + 1]^{-1},$$

$$(2.87) \quad \nu_i(p) = \mu_1(p) + \mu_2(p) + 1$$

with (2.83) and so obtain

$$(2.88) \quad B(p_1, p_2) - B(x_1, x_2) \in W + W + W.$$

W being arbitrary, we have so established the continuousness of B at (p_1, p_2) . Since (p_1, p_2) is also arbitrary, the proof is complete. \square

2.12 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| \, dt.$$

Put $B(f, g) = \int_0^1 f(t)g(t)dt$, and show that B is a bilinear continuous functional on $X \times X$ which is separately but not continuous.

Proof. Let f denote the first variable, g the second one. Remark that

$$(2.89) \quad |B(f, g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient (1.18 of [3]) to assert that any $f \mapsto B(f, g)$ is continuous. The continuity of all $g \mapsto B(f, g)$ follows (Put $C(g, f) = B(f, g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$(2.90) \quad |B(f, g)| < M\|f\|\|g\|.$$

Put

$$(2.91) \quad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \quad (n = 1, 2, 3, \dots),$$

so that

$$(2.92) \quad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand,

$$(2.93) \quad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.93) and (2.90) with (2.92) and so obtain

$$(2.94) \quad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Our continuousness assumption is then contradicted. So ends the proof. \square

2.15 Exercise 15. Baire cut

Suppose X is an F -space and Y is a subspace of X whose complement is of the first category. Prove that $Y = X$. Hint: Y must intersect $x + Y$ for every $x \in X$.

Proof. Assume Y is a subgroup of X . Under our assumptions, there exists a sequence $\{E_n : n = 1, 2, 3, \dots\}$ of X such that

$$(i) \quad (\overline{E_n})^\circ = \emptyset;$$

$$(ii) \quad X \setminus Y = \bigcup_{n=1}^{\infty} E_n.$$

By (i), the complement V_n of $\overline{E_n}$ is a dense open set. Since X is an F -space, it follows from the Baire's theorem that the intersection S of the V_n 's is dense in X : So is $x + S$ ($x \in X$). To see that, remark that

$$(2.95) \quad X = x + \overline{S} \subset \overline{x + S}$$

follows from 1.3 (b) of [3]. Since S and $x + S$ are both dense open subsets of X , the Baire's theorem asserts that

$$(2.96) \quad \overline{(x + S) \cap S} = X.$$

Thus,

$$(2.97) \quad (x + S) \cap S \neq \emptyset.$$

Moreover, it follows from (ii) that $X \setminus Y \subset \bigcup_n \overline{E_n}$, i.e. $Y \supset S$. Combined with (2.97), this shows that $x + Y$ cuts Y . Therefore, our arbitrary x is an element of the subgroup Y . We have thus established that $X \subset Y$, which achieves the proof. \square

2.16 Exercise 16. An elementary closed graph theorem

Suppose that X and K are metric spaces, that K is compact, and that the graph of $f : X \rightarrow K$ is a closed subset of $X \times K$. Prove that f is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of K cannot be omitted from the hypothesis, even when X is compact.

Proof. Choose a sequence $\{x_n : n = 1, 2, 3, \dots\}$ whose limit is an arbitrary a . By compactness of K , the graph G of f contains a subsequence $\{(x_{p(n)}, f(x_{p(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of $X \times K$. G is closed; therefore, $\{(x_{p(n)}, f(x_{p(n)}))\}$ converges in G . So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.98) \quad \begin{aligned} f : [0, \infty) &\rightarrow [0, \infty) \\ x &\mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \end{aligned}$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$(2.99) \quad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b) . To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on \mathbb{R}_+ . This establishes that $(a, b) \in G_+$, hence the closedness G_+ . Finally, we conclude that G is closed, as a finite union of closed sets. \square

Chapter 3

Convexity

3.3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subset X$ an *internal point* of A if $A - x_0$ is an absorbing set.

- (a) Suppose A and B are disjoint convex sets in X , and A has an internal point. Prove that there is a nonconstant linear functional Λ such that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. (The proof is similar to that of Theorem 3.4)
- (b) Show (with $X = \mathbf{R}^2$, for example) that it may not be possible to have $\Lambda(A)$ and $\Lambda(B)$ disjoint, under the hypotheses of (a).

Proof. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C . Since A and B are disjoint, $b_0 - x_0$ is not in C , hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [3] to establish the existence of a linear functional $\Lambda : X \rightarrow \mathbf{R}$ such that

$$(3.1) \quad \Lambda \leq p$$

and

$$(3.2) \quad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \quad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

$$(3.4) \quad \Lambda a \leq \Lambda b.$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2 . There so exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

$$(3.5) \quad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$(3.6) \quad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad .$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A . This implies

$$(3.7) \quad \Lambda b_1 \stackrel{(3.6)}{<} \Lambda a_3 \stackrel{(3.4)}{\leq} \Lambda b_1 \quad ;$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is \mathbf{R}^2 . Fetch

$$(3.8) \quad S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$

$$(3.9) \quad S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$$

$$(3.10) \quad A \triangleq S_1 \cup S_2,$$

$$(3.11) \quad B \triangleq X \setminus A.$$

Pick (x_i, y_i) in S_i . Let t range over the unit interval, and so obtain

$$(3.12) \quad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subset A.$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A . Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.* A° is nonempty¹: A contains an internal point.

Let L be a vector line of \mathbf{R}^2 . In other words, L is the null space of a linear functional $\Lambda : \mathbf{R}^2 \rightarrow \mathbf{R}$ (to see this, take some nonzero u in L^\perp and set $\Lambda x = (x, u)$ for all x in \mathbf{R}^2). One easily checks that both A and B cut L . Hence

$$(3.13) \quad \Lambda(L) = \{0\} \subset \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad .$$

So ends the proof. □

¹For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [2].

3.11 Exercise 11. Meagerness of the polar

Let X be an infinite-dimensional Fréchet space. Prove that X^* , with its weak*-topology, is of the first category in itself.

This is actually a consequence of the below lemma, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *If X is an infinite dimensional topological vector space whose dual X^* separates points on X , then the polar*

$$(3.14) \quad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\}$$

of any absorbing subset A is a weak-closed set that has empty interior.*

Proof. Let x range over X . The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous; see 3.14 of [3]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*-closed. As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see 3.11 of [3]). Put

$$(3.15) \quad W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \quad (r_x > 0)$$

where F runs through the nonempty finite subsets of X . Clearly, the collection of all such W is a local base of X^* . Pick one of those W and remark that the following subspace

$$(3.16) \quad M \triangleq \text{span}(F)$$

is finite dimensional. Assume, to reach a contradiction, that $A \subset M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M , which is a desired contradiction. We have just established that $A \not\subset M$: Now pick a in $A \setminus M$ and so conclude that

$$(3.17) \quad b \triangleq \frac{a}{t_a} \in A$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M , as a finite dimensional space, is closed (see 1.21 (b) of [3] for a proof): By the Hahn-Banach theorem 3.5 of [3], there exists Λ_a in X^* such that

$$(3.18) \quad \Lambda_a b > 2$$

and

$$(3.19) \quad \Lambda_a(M) = \{0\}.$$

The latter equality implies that Λ_a vanishes on F ; hence Λ_a is an element of W . On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$(3.20) \quad |\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that $\Lambda + \Lambda_a$ is not in K_A . We have thus proved that

$$(3.21) \quad \Lambda + W \not\subset K_A.$$

Since W and Λ are both arbitrary, this achieves the proof. \square

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X , so that the following sets

$$(3.22) \quad B_n \triangleq \{x \in X : d(0, x) < 1/n\} \quad (n = 1, 2, 3, \dots)$$

form a local base of X . If Λ is in X^* , then

$$(3.23) \quad |\Lambda| \leq m \text{ on } B_n$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$; see 1.18 of [3]. Hence, X^* is the countable union of all

$$(3.24) \quad m \cdot K_n \quad (m, n = 1, 2, 3, \dots),$$

where K_n is the polar of B_n . Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [3]. As a consequence, the above lemma implies

$$(3.25) \quad (\overline{K_n})^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by m is an homeomorphism (see 1.7 of [3]), this is equivalent to

$$(3.26) \quad (\overline{m \cdot K_n})^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof. □

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

4.1 Exercise 1. Basic results

Let φ be the embedding of X into X^{**} described in Section 4.5. Let τ be the weak topology of X , and let σ be the weak*-topology of X^{**} - the one induced by X^* .

- (a) Prove that φ is an homeomorphism of (X, τ) onto a dense subspace of (X^{**}, σ) .
- (b) If B is the closed unit ball of X , prove that $\varphi(B)$ is σ -dense in the closed unit ball of X^{**} . (Use the Hahn-Banach separation theorem.)
- (c) Use (a), (b), and the Banach-Alaoglu theorem to prove that X is reflexive if and only if B is weakly compact.
- (d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.
- (e) If X is reflexive and Y is a closed subspace of X , prove that X/Y is reflexive.
- (f) Prove that X is reflexive if and only if X^* is reflexive.
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\varphi(X)$ of X^{**} .

Proof. Let ψ be the isometric embedding of X^* into X^{***} . The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$(4.1) \quad \varphi^{-1} : \varphi(X) \rightarrow X$$

$$(4.2) \quad \varphi(x) \mapsto x$$

is an homeomorphism (with respect to τ and σ). We first consider

$$(4.3) \quad V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}, \psi x^* \rangle| < r\} \quad (x^* \in X^*, r > 0);$$

$$(4.4) \quad U \triangleq \{x \in X : |\langle x, x^* \rangle| < r\} \quad (x^* \in X^*, r > 0).$$

and remark that the so defined V 's (respectively U 's) shape a local subbase \mathcal{S}_σ (respectively \mathcal{S}_τ) of σ (respectively τ). We now observe that

$$(4.5) \quad U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathcal{S}_\sigma, U \in \mathcal{S}_\tau) \quad ,$$

since φ^{-1} is one-to-one. This remains true whether we enrich each subbase \mathcal{S} with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that φ^{-1} and its inverse φ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$(4.6) \quad \overline{\varphi(X)}_{\sigma} \subset X^{**} \quad .$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ -closure of $\varphi(X)$. By [3.5], there so exists y^* in X^* such that

$$(4.7) \quad \langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ;$$

$$(4.8) \quad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces y^* to be a the zero of X^* . The functional ψy^* is then the zero of X^{***} : (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \quad \varphi(B) \subset B^{**} \quad ,$$

since φ is isometric. Hence

$$(4.10) \quad \overline{\varphi(B)}_{\sigma} \subset \overline{(B^{**})}_{\sigma} = B^{**} \quad .$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\varphi(B)}_{\sigma}$ contains a vector z^{**} . By [3.7], there exists y^* in X^* such that

$$(4.11) \quad |\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_{\sigma} \quad ;$$

$$(4.12) \quad \langle z^{**}, \psi y^* \rangle > 1 \quad .$$

It follows from (4.11) that

$$(4.13) \quad |\psi y^*| \leq 1 \quad \text{on } \varphi(B) \quad , \quad \text{i.e.} \quad |y^*| \leq 1 \quad \text{on } B \quad .$$

We have so proved that

$$(4.14) \quad y^* \in B^* \quad .$$

Since z^{**} lies in B^{**} , it is now clear that

$$(4.15) \quad |\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ;$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

$$(4.16) \quad B \text{ is weakly compact if and only if } \varphi(B) \text{ is weak}^*\text{-compact.}$$

If B is weakly compact, then $\varphi(B)$ is weak*-closed. So,

$$(4.17) \quad \varphi(B) = \overline{\varphi(B)}_{\sigma} \stackrel{(b)}{=} B^{**} \quad .$$

φ is therefore onto, *i.e.* X is reflexive.

Conversely, keep φ as onto: one easily checks that $\varphi(B) = B^{**}$. The image $\varphi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X , whose closed unit ball is B . Let Y be a norm-closed subspace of X : Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let \equiv stand for “isometrically isomorphic” and apply twice [4.9] to obtain, first

$$(4.18) \quad (X/Y)^* \equiv Y^\perp \quad ,$$

next,

$$(4.19) \quad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \quad .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z , the weak-topology of Z^* is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\varphi(X)$ is a norm-closed subspace of X^{**} ; *cf.* [4.5]. Hence $\varphi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \varphi(X)$ is weakly compact, *i.e.* weak*-compact (to see this, apply (L) with $Z = X^*$).

By (a), B is therefore weakly compact, *i.e.* X is reflexive; see (c). So ends the proof. \square

4.13 Exercise 13. Operator compactness in a Hilbert space

4.15 Exercise 15. Hilbert-Schmidt operators

Chapter 6

Distributions

6.1 Exercise 1. Test functions are almost polynomial

6.6 Exercise 6. Around the supports of some distributions

6.9 Exercise 9. Convergence in $\mathcal{D}(\Omega)$ vs. convergence in $\mathcal{D}'(\Omega)$

(a) Prove that a set $E \subset \mathcal{D}(\Omega)$ is bounded if and only if

$$\sup\{|\Lambda\varphi| : \varphi \in E\} < \infty$$

for every $\Lambda \in \mathcal{D}'(\Omega)$.

(b) Suppose $\{\varphi_j\}$ is a sequence in $\mathcal{D}(\Omega)$ such that $\{\Lambda\varphi_j\}$ is a bounded sequence of numbers, for every $\Lambda \in \mathcal{D}'(\Omega)$. Prove that some subsequence of $\{\varphi_j\}$ converges, in the topology of $\mathcal{D}(\Omega)$.

(c) Suppose $\{\Lambda_j\}$ is a sequence in $\mathcal{D}'(\Omega)$ such that $\{\Lambda_j\varphi\}$ is bounded, for every $\varphi \in \mathcal{D}(\Omega)$. Prove that some subsequence of $\{\Lambda_j\}$ converges in $\mathcal{D}'(\Omega)$ and that the convergence is uniform on every bounded subset of $\mathcal{D}(\Omega)$. *Hint: By the Banach-Steinhaus theorem, the restrictions of the Λ_j to \mathcal{D}_K are equicontinuous. Apply Ascoli's theorem.*

PROOF. Since $\mathcal{D}(\Omega)$ locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $E = \{\varphi_j : j \in \mathbf{N}\}$ is bounded: so is \overline{E} . By (c) of [6.5], there exists some \mathcal{D}_K that contains \overline{E} . Since \mathcal{D}_K has the Heine-Borel property (see [1.46]), \overline{E} is τ_K -compact. Apply [A4] with the metrizable space \mathcal{D}_K (see [1.46]) to conclude that \overline{E} has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds.

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