Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Notations and Assumptions

I Special terms

i The iff convention

iff is a shorthand for "if and only if". Splitting **iff** into *if-then* clauses shows that it is a natural language version of the logical equivalence $\Leftrightarrow$. All definitions are understood to be **iff** clauses, which is consistent with the fact that every definition expresses an equivalence.

ii The assignment operator

Given variables a and b, $\mathrm{a} \triangleq \mathrm{b}$ is a special form of $\mathrm{a} = \mathrm{b}$. We say that $\mathrm{a} \triangleq \mathrm{b}$ **iff** a and b are *assumed* to be equal. Typically, $\mathrm{a} \triangleq \mathrm{b}$ is used to indicate that a is assigned the value b (some authors write $\mathrm{a} := \mathrm{b}$) but, in a different context, $\mathrm{a} \triangleq \mathrm{b}$ may also denote $\mathrm{a} =: \mathrm{b}$*, i.e.,* $\mathrm{a} := \mathrm{b}$.

iii Iverson brackets

Given a boolean expression ϕ, the boolean value $[\phi]$ encodes the truth value of ϕ as a `bit`:

$$[\phi] \triangleq \begin{cases} 0 & (\phi \text{ is false}) \\ 1 & (\phi \text{ is true}). \end{cases}$$

For example, $\left[1 > 0\right] = 1$ but $\left[\sqrt{2} \in \mathbf{Q}\right] = 0$. For interpretations of Boolean operators in logic, see [2].

II Sets

i Subsets and supersets

$\subseteq$ and $\supseteq$ are the standard symbols for set ordering. No specific symbol is reserved for strict ordering, every constraint $\mathrm{X} \neq \mathrm{Y}$ will be explicitly stated when a strict subset is intended. Given A and B, $\mathrm{A} \cup \mathrm{B}$ is the union of A and B. More generally, for any collection C, the union U of C is expressed as follows:

$$\mathrm{U} \triangleq \bigcup \mathrm{C} \triangleq \bigcup_{\mathrm{S} \in \mathrm{C}} \mathrm{S}. \tag{1}$$

Similarly, $\cap$ denotes intersection in the same manner.

ii Special mappings

The identity I (or id) is the mapping $\{(\mathrm{x}, \mathrm{x}) : \mathrm{x} \in \mathrm{X}\}$. Similarly, the projection $\pi = \{((\mathrm{x}, \mathrm{y}), \mathrm{x}) : \mathrm{x} \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}\}$ always exists. Observe that I is the diagonal of $\mathrm{X} \times \mathrm{X}$.

iii Equinumerosity

In a metric space context, $\mathrm{X} \equiv \mathrm{Y}$ means that ϕ is a surjective isometry.

III Topological vector spaces

i Scalar field

C extends **R**, which implies that a property, *e.g.,* linearity, that holds over **C** also holds over **R**. The complex case is therefore *stronger* than the real case. This restriction may be significant in some contexts. However, the standard scalar field is **C**. Unless the field is explicitly given, as in [3.1 and 12.7] of [4], we assume that results for **C** apply equally to the real case.

ii Vector space bases

Given a vector space X, a subset B of X is a basis of X **iff** the sum

$$\text{(2)} \qquad \Big\{(z_u)_{u\in B} : z_u \in \mathbf{C}, \{u : z_u \neq 0\} \text{ is finite}\Big\} \to X$$
$$(z_u) \mapsto \sum_{u\in B} z_u u$$

is a bijection from all *finitely supported* families (z_u) onto X. The axiom of choice (AC) forces

(a) the existence of such B (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [4] with B playing the role of Λ);

(b) all bases to have the same cardinality, which is called the *dimension* of X and is denoted as $\dim X$.

We now turn to the finite-dimensional case over the field **C**. The zero-dimensional case is $B = \emptyset$, *i.e.,* $X = \{0\}$. Our first step is to study $\mathbf{C}^n$, the standard n-dimensional vector space, when $n = 1, 2, 3, \ldots$.

iii Finite-dimensional spaces

The product topology of $\mathbf{C}^n$

$\mathbf{C}^n$ has the standard basis $1_{\{1\}}, \ldots, 1_{\{n\}} : \{1, \ldots, n\} \to \{0, 1\}$ so that the scalar z_k is the k-th component of

$$\text{(3)} \qquad (z_1, \ldots, z_k, \ldots, z_n) = z_1 \cdot (\underbrace{1, 0, \ldots}_{1_{\{1\}}}) + \cdots + z_k \cdot (\underbrace{0, \ldots, 1, 0, \ldots}_{1_{\{k\}}}) + \cdots + z_n \cdot (\underbrace{0, \ldots, 1}_{1_{\{n\}}}),$$

as $z = (z_1, \ldots, z_n)$ ranges over $\mathbf{C}^n$. A common notation is to let e_k stand for $1_{\{k\}}$. Moreover, $\mathbf{C}^n$ is endowed with the topology generated by all polydiscs

$$\text{(4)} \qquad \prod_{i=1}^{n} \{\underbrace{z_i \in \mathbf{C} : |z_i| < r_i}_{D_{r_i}}\} \qquad (r_i > 0).$$

Equivalently, we may equip $\mathbf{C}^n$ with the Euclidean norm

$$\text{(5)} \qquad \|z\|_2 \triangleq \sqrt{|z_1|^2 + \cdots + |z_n|^2},$$

whose open balls centered at the origin are all

$$\text{(6)} \qquad B_r \triangleq \Big\{z \in \mathbf{C}^n : \|z\|_2 < r\Big\} \qquad (r > 0).$$

To show the equivalence, first set $r_i = r/\sqrt{n}$. Hence

$$\text{(7)} \qquad \prod_{i=1}^{n} D_{r_i} \subseteq B_r.$$

Conversely, put $r = \min(r_1, \ldots, r_n)$ so that

$$\text{(8)} \qquad B_r \subseteq \prod_{i=1}^{n} D_{r_i}.$$

Topology of a finite-dimensional vector space

It is customary to identify any n-dimensional vector space with $\mathbf{C}^n$ equipped with the Euclidean norm; see [5]. To show this, choose an isomorphism $f : \mathbf{C}^n \to Y$. For instance, let $f(e_k)$ be u_k as in [1.20] of [4] when $\{u_k\}$ is a basis of the n-dimensional Y; see [ii]. It follows from [1.21] of [4] that f is a homeomorphism. A striking consequence is that $\{f(U) : U$ *open in* $\mathbf{C}^n\}$ *is the unique topological vector space topology on* Y. Thus, Y is necessarily locally convex and locally bounded , *i.e.,* normable; see [1.39] of [4]. Note that the formula $\|y\|_Y = \|f^{-1}(y)\|_2$ $(y \in Y)$ defines a norm. Additionally, Y is locally compact, as the closed unit ball of $\mathbf{C}^n$ is compact. Choose an n-dimensional topological vector space W, and repeat the same reasoning with $g : \mathbf{C}^n \to W$, then $h = g \circ f^{-1}$, in place of f. Thus, $h : Y \to W$ is a homeomorphism and W is normable as well. The following assertions are then equivalent in the finite-dimensional context:

(i) $\dim W = \dim Y$,

(ii) W and Y are isomorphic,

(iii) W and Y are homeomorphic and normable.

Furthermore, the norms on W and Y are *equivalent*. That is, for any given norm $\|\cdot\|_Y$ on Y and any given norm $|||\cdot|||_W$ on W, there exists a positive constant $C = C_h$ such that

$$|||w|||_W \le C|||y|||_Y \qquad ((y,w) \in h)\,, \tag{9}$$

as h is continuous. When W = Y, this means that all norms on Y are equivalent, in the sense that

$$|||h(y)|||_Y \le C\|y\|_Y. \tag{10}$$

The standard norms 1, 2, and ∞

When $\mathbf{C}^n$ is equipped with standard norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, the least $C_{i,j}$ such that

$$\|z\|_j \le C_{i,j}\|z\|_i \tag{11}$$

is easily derived from definitions - see [1.19] of [4], except for the case $C_{2,1} = \sqrt{n}$, which requires the Cauchy-Schwarz inequality; see (1) in [12.2] of [4]. The table below records these sharp $C_{i,j}$,

i \ j	1	2	∞
1	1	1	1
2	$\sqrt{n}$	1	1
∞	n	$\sqrt{n}$	1

Table 1. Minimal $C_{i,j}$ for the standard norms $i, j = 1, 2, \infty$.

iv Continuity and boundedness in normed spaces

A linear mapping Λ is said to be *bounded* **iff** $\Lambda(E)$ *is a bounded set for every bounded set* E; see [1.31] of [4]. Linearity implies that $\Lambda : X \to Y$ is bounded **iff**

$$\|\Lambda\| \triangleq \sup\{\|\Lambda x\| : \|x\| < 1\} < \infty, \tag{12}$$

Given normed spaces X and Y, bounded linear maps Λ form a normed space $B(X,Y)$, where norm $\|\Lambda\|$ is given above; see [4.1] of [4]. It is now easy to see that, in the current context, boundedness and continuity coincide. This is a particular case of [2.8] of [4]. When Y is the scalar field, this equivalence also comes from [1.32] of [4]. Furthermore, we observe, that given a collection of bounded mappings $\Lambda \in B(X,Y)$, the property of equicontinuity now reads as

$$\sup_\Lambda \|\Lambda\| < \infty. \tag{13}$$

Thus, *uniform boundedness* and uniform continuity coincide as well; *cf.* [2.4] of [4].

IV Measure theory

i Radon measures

Given a locally compact Hausdorff space X, *e.g.*, $\mathbf{R}^n$, a positive Radon measure Λ is a functional that is *positive* on $C_c(X)$, in the sense that

(i) $\phi \geq 0 \Rightarrow \Lambda\phi \geq 0 \qquad (\phi \in C_c(X))$.

Theorem [2.14] of [3] shows that positivity (i) implies the following *continuity* property (ii):

(ii) For each compact $K \subseteq X$ there exists a "continuity bound" M_K such that

$$|\Lambda\phi| \leq M_K \|\phi\|_\infty \qquad (\phi \in C_c(X) : \operatorname{supp}\phi \subseteq K).$$

Condition (ii) defines Radon measures in a weaker sense; see [5]. Furthermore, Λ is bounded on $C_c(X)$ with respect to the supremum norm **iff** (ii) is strengthened by $\|\Lambda\|_\infty \leq \sup\{M_K : K \subseteq X \text{ compact}\} < \infty$. Such uniformly continuous functionals Λ constitute the space of *bounded* Radon measures. According to [6.19] of [3], each of them can be isometrically identified with a specific regular Borel measure μ. Conversely, when the ambient space X is compact, every Radon measure is $\|\cdot\|_\infty$-bounded. For instance, this case is addressed in Exercise 3 of Chapter 2. In measure theory, by *support* of μ, we mean:

(14) $$\operatorname{supp}\mu \triangleq X \setminus \bigcup V,$$

where V runs through all open sets of measure 0. A very important Radon measure of support $\{0\}$, the *Dirac delta function* is described in [A.3].

ii Lebesgue integration

Theorem [2.14] of [3] states that every positive Radon measure $\Lambda : C_c(X) \to \mathbf{C}$ is identified with a positive and regular Borel measure $\beta : X \to \mathbf{R}$. More precisely,

(15) $$\int_X \phi \, d\beta \triangleq \Lambda\phi \qquad (\phi \in C_c(X)),$$

where the integral on the left-hand side is a Lebesgue integral. The standard example is $X = \mathbf{R}$ with $\beta = \beta_{\mathbf{R}}$ the Lebesgue measure on Borel sets. The regularity property implies that β has total mass

(16) $$\int_X 1 \, d\beta \triangleq \sup\{\Lambda\phi : \phi \in C_c(X), \|\phi\|_\infty \leq 1\} \leq \infty.$$

When we substitute $f\,d\beta$ for $d\beta$, with f X-measurable, (15) takes the form

(17) $$\int_X f\phi \, d\beta = \Lambda\phi.$$

In particular, if f is positive,

(18) $$\int_X f \, d\beta = \sup\{\Lambda\phi : \phi \in C_c(X), \|\phi\|_\infty \leq 1\}.$$

Moreover, given $N = N(\beta)$ denoting $\{f : f = 0 \ \beta\text{-a.e.}\}$, we see that the density f and the measure Λ are identified modulo N in (17). Algebraically speaking:

(19) $$g\beta = f\beta \Leftrightarrow g - f \in N.$$

A central question in calculus is whether the integral of the modulus, namely

(20) $$|f|_1 \triangleq \int_X |f| \, d\beta \qquad (f \text{ X-measurable}),$$

is finite. This motivates the following definitions:

(21) $$\mathscr{L}^1(X,\beta) \triangleq \{f : |f|_1 < \infty\} \subseteq \mathscr{L}^1_{loc}(X,\beta) \triangleq \bigcap \{\mathscr{L}^1(K,\beta) : K \subseteq X \text{ compact.}\}.$$

Standard notation $\mathscr{L}^1(\beta)$ is used to avoid redundancy, or simply $\mathscr{L}^1$ when the context is clear. Key points include:

(a) $\mathcal{L}^1$, equipped with $|\cdot|_1$, is a seminormed space,

(b) N is a closed subspace of the seminormed space $\mathcal{L}^1$.

Together, these two facts imply that $L^1 = \mathcal{L}^1/N$, equipped with the norm

$$\|f + N\|_1 \triangleq |f|_1 \qquad (f \in \mathcal{L}^1), \tag{22}$$

is a Banach space. These definitions and properties extend to L^p spaces ($p > 1$), as follows

$$\mathcal{L}^p \triangleq \{f : |f|^p \in \mathcal{L}^1\}, \tag{23}$$

$$L^p \triangleq \mathcal{L}^p/N, \tag{24}$$

$$\|f + N\|_p \triangleq \left(\int_X |f|^p\right)^{1/p} \qquad (f \in \mathcal{L}^p). \tag{25}$$

Similar *quasi-Banach* L^p spaces exist for $0 < p < 1$, with the notable difference that $\|f + N\|_p$ no longer defines a norm (the triangle inequality is lost). In contrast, $L^\infty = \{f + N : \|f + N\|_\infty < \infty\}$, equipped with the *quotient norm*

$$\|f + N\|_\infty \triangleq \inf\{M : |f| < M \ \beta\text{-a.e.}\} \tag{26}$$

is a Banach space.

Chapter 1

Topological Vector Spaces

1 Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X. *Prove the following statements from the axioms as given in section 1.4.*

(a) *If* $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ *there is a unique* $\mathrm{z} \in \mathrm{X}$ *such that* $\mathrm{x} + \mathrm{z} = \mathrm{y}$.

(b) $0 \cdot \mathrm{x} = 0 = \alpha \cdot 0 \qquad (\alpha \in \mathbf{C}, \mathrm{x} \in \mathrm{X})$.

(c) $2\mathrm{A} \subseteq \mathrm{A} + \mathrm{A}$.

(d) A *is convex if and only if* $(\mathrm{s} + \mathrm{t})\mathrm{A} = \mathrm{sA} + \mathrm{tA}$ *for all positive scalars* s *and* t.

(e) *Every union (and intersection) of balanced sets is balanced.*

(f) *Every intersection of convex sets is convex.*

(g) *If* Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

(h) *If* A *and* B *are convex, so is* $\mathrm{A} + \mathrm{B}$.

(i) *If* A *and* B *are balanced, so is* $\mathrm{A} + \mathrm{B}$.

(j) *Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.*

PROOF. (a) Such a property only depends on the group structure of X: Each x in X has an additive inverse -x. Let x' be any additive inverse of x, so that $\mathrm{x} - \mathrm{x} = 0 = \mathrm{x} + \mathrm{x}'$. Thus, $\text{-x} + \mathrm{x} - \mathrm{x} = \text{-x} + \mathrm{x} + \mathrm{x}'$, which is equivalent to $\text{-x} = \mathrm{x}'$. Therefore, the inverse -x is unique. It is now clear that $\mathrm{x} + \mathrm{z} = \mathrm{y}$ **iff** $\mathrm{z} = \text{-x} + \mathrm{y}$, which asserts both the existence and the uniqueness of z.

(b) Remark that

$$\begin{aligned} 0 \cdot \mathrm{x} &= (0 + 0) \cdot \mathrm{x} = 0 \cdot \mathrm{x} + 0 \cdot \mathrm{x} && (1.1) \\ &= (0 + 0) \cdot \mathrm{x} = 0 + 0 \cdot \mathrm{x} && (1.2) \end{aligned}$$

then conclude from (a) that $0 \cdot \mathrm{x} = 0$. So,

$$0 = 0 \cdot \mathrm{x} = (1 - 1) \cdot \mathrm{x} = \mathrm{x} + (\text{-}1) \cdot \mathrm{x} \Rightarrow \text{-}1 \cdot \mathrm{x} = \text{-x}. \tag{1.3}$$

Finally,

$$\alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (\mathrm{x} + (\text{-}1 \cdot \mathrm{x})) = \alpha \cdot \mathrm{x} + \alpha \cdot (\text{-}1) \cdot \mathrm{x} = (\alpha - \alpha) \cdot \mathrm{x} = 0 \cdot \mathrm{x} = 0, \tag{1.4}$$

which proves (b).

(c) Remark that

$$2x = (1+1)x = x + x \tag{1.5}$$

for every x in X, and so conclude that

$$2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A \tag{1.6}$$

for all subsets A of X; which proves (c).

(d) If A is convex, then

$$A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A; \tag{1.7}$$

which is

$$sA + tA = (s+t)A. \tag{1.8}$$

Conversely, the special case $s + t = 1$ is

$$sA + (1-s)A = A. \tag{1.9}$$

The latter extends to $s = 0$, since

$$0A + A \overset{(b)}{=} \{0\} + A = A. \tag{1.10}$$

The extension to $s = 1$ is analogously established (or simply use the fact that + is commutative!). So ends the proof.

(e) Let A range over B a collection of balanced subsets, so that

$$\alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B \tag{1.11}$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap B \subseteq A$ establishes the first part. Now remark that

$$\alpha A \subseteq \bigcup B \tag{1.12}$$

implies

$$\alpha \bigcup B \subseteq \bigcup B; \tag{1.13}$$

which completes the proof.

(f) Let A range over C a collection of convex subsets, so that

$$(s+t)\bigcap C \subseteq s\bigcap C + t\bigcap C \subseteq sA + tA \overset{(d)}{\subseteq} (s+t)A \tag{1.14}$$

for all positives scalars s, t. Inclusions at both extremities force

$$s\bigcap C + t\bigcap C = (s+t)\bigcap C. \tag{1.15}$$

We then conclude from (d) that the intersection of C is convex. So ends the proof.

(g) We dismiss all trivial cases $\Gamma = \emptyset$, $\{\emptyset\}$, $\{\{x\}\}$, $\{\emptyset, \{x\}\}$ then pick x_1, x_2 in $\bigcup\Gamma$, so that each x_i $(i = 1, 2)$ lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_i's is then in $C_2 \subseteq \bigcup\Gamma$. Hence (g).

(h) Simply remark that

$$s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B) \tag{1.16}$$

for all positive scalars s and t, then conclude from (d) that $A + B$ is convex.

(i) Given any α from the closed unit disc,

$$\alpha(A + B) = \alpha A + \alpha B \subseteq A + B. \tag{1.17}$$

This completes the proof: $A + B$ is balanced.

(j) The proof is based on Lemma [A.1]. Let Γ be a collection of vector subspaces of X. Define $I = \bigcap \Gamma$ and $U = \bigcup \Gamma$. The intersection I is convex and balanced by (e) and (f), because every $Y \in \Gamma$ is convex and balanced. Next, observe that

$$I + I \subseteq Y + Y \subseteq Y \tag{1.18}$$

for all $Y \in \Gamma$. Thus,

$$I + I \subseteq I. \tag{1.19}$$

It now follows from the implication (b) $\Rightarrow$ (a) of Lemma [A.1] that I is a vector subspace of X. We now prove the counterpart of (g) when Γ is totally ordered by set inclusion. Combining (e) with (g) demonstrates that U is convex and balanced. To show that U is a vector subspace, we note that this total ordering of Γ implies

$$Y_1 + Y_2 \subseteq \max(Y_1, Y_2) \tag{1.20}$$

when Y_1 and Y_2 run through Γ. Hence

$$U + U \subseteq U. \tag{1.21}$$

It then follows from the implication (b) $\Rightarrow$ (a) of Lemma [A.1] that U is a vector subspace of X. To prove the counterpart of (h), let each Y_i be a vector subspace of X. Taken together, (h) and (i) imply that $Y_1 + Y_2$ is convex and balanced. Moreover,

$$(Y_1 + Y_2) + (Y_1 + Y_2) = (Y_1 + Y_1) + (Y_2 + Y_2) \subseteq Y_1 + Y_2. \tag{1.22}$$

Finally, we conclude from (b) $\Rightarrow$ (a) of Lemma [A.1] that $Y_1 + Y_2$ is a vector subspace of X.

□

2 Exercise 2. Convex hull

The convex hull of a set A *in a vector space* X *is the set of all convex combinations of members of* A*, that is the set of all sums* $t_1x_1 + \dots + t_nx_n$ *in which* $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$*;* n *is arbitrary. Prove that the convex hull of a set* A *is convex and that is the intersection of all convex sets that contain* A*.*

PROOF. The convex hull of a set S will be denoted by co(S). Remark that $S \supseteq \mathrm{eq}\,\mathrm{co}(S)$ (to see that, take $t_1 = 1$ for each x_1 in S) and that $\mathrm{co}(A) \supseteq \mathrm{eq}\,\mathrm{co}(B)$ where $A \supseteq \mathrm{eq}B$ (obvious).

Our proof will directly derive from (i) $\Rightarrow$ (iv) in the following lemma,

Let S *be a subset of a vector space* X*: Its convex hull* co(S) *is convex and the following statements*

(i) S *is convex;*

(ii) $s_1S + \dots + s_nS = (s_1 + \dots + s_n)S$ *for all positive scalar variables* $s_1, \dots, s_n$*;*

(iii) $t_1S + \dots + t_nS = S$ *for all positive scalar variables* $s_1, \dots, s_n$ *such that* $s_1 + \dots + s_n = 1$*;*

(iv) $\mathrm{co}(S) = S$

are equivalent.

From now on, we skip the trivial case $S = \emptyset$ then only consider nonempty sets. To prove the first part, let a, b range over co(S)so that $a = t_1x_1 + \dots + t_nx_n$ and $b = t_{n+1}x_{n+1} + \dots + t_{n+p}x_{n+p}$ for some (t_i, x_i). Every sum $sa + (1 - s)b$ $(0 \leq s \leq 1)$ is then in the convex hull of $\{x_1, \dots, x_{n+p}\}$, since

$$sa + (1 - s)b = \sum_{i=1}^{n} st_ix_i + \sum_{i=n+1}^{n+p} (1 - s)t_ix_i \tag{1.23}$$

and

$$\sum_{i=1}^{n} st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s\sum_{i=1}^{n} t_i + (1-s)\sum_{i=n+1}^{n+p} t_i = 1. \tag{1.24}$$

In terms of sets S, this reads as follows,

$$s\,co(S) + (1-s)\,co(S) \subseteq co(S); \tag{1.25}$$

which was our first goal. We now prove the equivalence (i) ⇒ ⋯ ⇒ (iv) ⇒ (i): An easy proof by induction makes the implication (i) ⇒ (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii) ⇒ (iv) derives from the definition of the convex hull. We close the chain with (iv) ⇒ (i), by remarking that S is convex whether S = co(S). The lemma being proved, we establish the second part.

To do so, we start from the convexity of co(A) then set $F = \{co(A)\}$. We may enrich F as follows,

$$B \in F \Rightarrow B \text{ is convex and contains } A. \tag{1.26}$$

Note that our initial predicate "[F only encompasses] *all convex sets that contain A*", is now the special case

$$B \in F \Leftrightarrow B \text{ is convex and contains } A. \tag{1.27}$$

In any case, the key ingredient is that $co(A) \in F$ implies

$$co(A) \supseteq \bigcap_{B\in F} B. \tag{1.28}$$

Conversely, the next formula

$$co(A) \subseteq co(B) \overset{(i)\Rightarrow(iv)}{=} B \qquad (B \in F) \tag{1.29}$$

is valid and implies

$$co(A) \subseteq \bigcap_{B\in F} B. \tag{1.30}$$

So ends the proof □

3 Exercise 3. Other basic results

Let be X as topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements:

(a) The convex hull of every open set is open.

(b) If X is locally convex then the convex hull of every bounded set is bounded.

(c) If A and B are bounded, so is A+B.

(d) If A and B are compact, so is A+B.

(e) If A is compact and B is closed, then A+B is closed.

(f) The sum of two closed sets may fail to be closed.

PROOF. (a) Pick an open set A then let the variables V_i $(i = 1, 2, \dots)$ run through all open subsets of Aso that

$$co(A) \subseteq \bigcup_{t_i} (t_1V_1 + \cdots + t_iV_i + \cdots) \subseteq co(A) \tag{1.31}$$

given all convex combinations $t_1V_1 + \cdots + t_iV_i + \cdots$. We know from [1.7] of [4] that those sums are open; which achieves the proof.

(b) Provided a bounded set E, pick V a neighborhood of 0: By (b) of Section 1.14 in [14] of [4], V contains a convex neighborhood of 0, say W. It follows that there exists a positive scalar s such that

$$\mathrm{E} \subseteq \mathrm{tW} \subseteq \mathrm{tV} \qquad (\mathrm{t} > \mathrm{s}). \tag{1.32}$$

Hence

$$\mathrm{co}(\mathrm{E}) \subseteq \mathrm{co}(\mathrm{tW}) = \mathrm{t}\,\mathrm{co}(\mathrm{W}) = \mathrm{tW} \subseteq \mathrm{tV}, \tag{1.33}$$

which completes the proof.

(c) At fixed V, neighborhood of the origin, we combine the continuity of + with [1.14] of [4] to conclude that there exists U a balanced neighborhood of the origin such that

$$\mathrm{U} + \mathrm{U} \subseteq \mathrm{V}. \tag{1.34}$$

Moreover, by the very definition of boundedness, $\mathrm{A} \subseteq \mathrm{rU}$ for some positive scalar r. Similarly, $\mathrm{B} \subseteq \mathrm{sU}$ for some positive s. Finally,

$$\mathrm{A} + \mathrm{B} \subseteq \mathrm{rU} + \mathrm{sU} \subseteq \mathrm{tU} + \mathrm{tU} \subseteq \mathrm{tV} \qquad (\mathrm{t} > \mathrm{r}, \mathrm{s}), \tag{1.35}$$

since U is balanced. So ends the proof.

(d) First, A and B are compact: So is $\mathrm{A} \times \mathrm{B}$. Next, + maps continuously $\mathrm{A} \times \mathrm{B}$ onto $\mathrm{A} + \mathrm{B}$. In conclusion, $\mathrm{A} + \mathrm{B}$ is compact.

(e) From now on, we assume that neither A nor B is empty, since otherwise the result is trivial. Now pick $\mathrm{c} \in \mathrm{X}$ outside $\mathrm{A} + \mathrm{B}$: The result will be established by showing that c is not in the closure of $\mathrm{A} + \mathrm{B}$.

To do so, we let the variable a range over A: Every set $\mathrm{a} + \mathrm{B}$ is closed as well, see [1.7] of [4]. Trivially, $\mathrm{a} + \mathrm{B} \neq \mathrm{c}$: By Section [1.10] of [4], there exists $\mathrm{V} = \mathrm{V}(\mathrm{a})$ a neighborhood of the origin such that

$$(\mathrm{a} + \mathrm{B} + \mathrm{V}) \cap (\mathrm{c} + \mathrm{V}) = \emptyset. \tag{1.36}$$

Moreover, there are finitely many $\mathrm{a} + \mathrm{V}$, say $\mathrm{a}_1 + \mathrm{V}_1, \mathrm{a}_2 + \mathrm{V}_2, \dots$, whose union U contains the compact set A. Therefore,

$$\mathrm{A} + \mathrm{B} \subseteq \mathrm{U} + \mathrm{B}. \tag{1.37}$$

Now define

$$\mathrm{W} \triangleq \mathrm{V}_1 \cap \mathrm{V}_2 \cap \cdots, \tag{1.38}$$

so that

$$(\mathrm{a}_\mathrm{i} + \mathrm{B} + \mathrm{V}_\mathrm{i}) \cap (\mathrm{c} + \mathrm{W}) \overset{(1.36)}{=} \emptyset \qquad (\mathrm{i} = 1, 2, \dots). \tag{1.39}$$

In conclusion, c is not in the closure of $\mathrm{U} + \mathrm{B}$. Finally, (1.37) asserts that c is not in $\overline{\mathrm{A} + \mathrm{B}}$ either; which achieves the proof.

Corollary: If B is the closure of a set S, then

$$\mathrm{A} + \mathrm{B} \subseteq \overline{\mathrm{A} + \mathrm{S}} \subseteq \overline{\mathrm{A} + \mathrm{B}} = \mathrm{A} + \mathrm{B} \tag{1.40}$$

by [(b) of 1.13] of [4] (since A is closed; see Section 1.12, from the same source). The special case $\mathrm{A} = \{\mathrm{x}\}$, $\mathrm{B} = \mathrm{X}$ will occur in the proof of Exercise 15 in chapter 2.

(f) The final proof consists of exhibiting a counterexample. To do so, let f be any continuous mapping of the real line such that

(i) $\mathrm{f}(\mathrm{x}) + \mathrm{f}(\text{-}\mathrm{x}) \neq 0 \qquad (\mathrm{x} \in \mathbf{R})$;
(ii) f vanishes at infinity.

For instance, we may combine (ii) with f even and $f > 0$ by setting $f(x) = 2^{-|x|}$, $f(x) = e^{-x^2}$, $f(x) = 1/(1+|x|)$, ..., and so on.

As a continuous function, f has closed graph G, see [2.14] of [4]. Moreover, (i) implies that the origin $(0,0) \neq (x-x, f(x)+f(-x))$ is not in $G+G$. On the other hand,

$$\{(0, f(n) + f(-n)) : n = 1, 2, \dots\} \subseteq G + G. \tag{1.41}$$

Now the key ingredient is that

$$(0, f(n) + f(-n)) \xrightarrow[n\to\infty]{(ii)} (0,0). \tag{1.42}$$

We have so constructed a sequence in $G+G$ that converges outside $G+G$. So ends the proof.

□

4 Exercise 4. A balanced set whose interior is not balanced

Let be $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$. *Show that* B *is balanced but that its interior is not.*

PROOF. It is obvious that the nonempty set B contains the origin $(0,0)$. Additionally, its interior B° is nonempty as well. To see that, observe that the following set

$$\{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2] < 1/2\} \subseteq B \tag{1.43}$$

is a neighborhood of $(1,2) \in B$. Moreover, B is balanced, since

$$|\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \qquad (|\alpha| \leq 1) \tag{1.44}$$

for all (z_1, z_2) in B. However, the nonempty set B° is not balanced, which we establish by showing that $(0,0) \notin B^\circ$. To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$U \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2] < r\} \subseteq B \tag{1.45}$$

for some positive r. Clearly, U contains $(r/2, 0)$, and that special case $(r/2, 0) \in B$ now contradicts the definition of B. So ends the proof. □

5 Exercise 5. A first restatement of boundedness

Consider the definition of "bounded set" given in Section 1.6. Would the content of this definition be altered if it merely required that to every neighborhood V *of 0 corresponds* some $t > 0$ *such that* $E \subseteq tV$*?*

PROOF. The answer is: No. To prove this, start from (a) of Section 1.14: V contains W, a balanced neighborhood of 0. Assume that E is bounded in this weaker sense, *i.e.,* there exists a positive t that satisfies

$$E \subseteq tW. \tag{1.46}$$

Thus,

$$E \subseteq tW \subseteq sW \subseteq sV \qquad (s > t), \tag{1.47}$$

since W is balanced. Thus, we recover the definition given in Section 1.6: The two definitions are equivalent. □

6 Exercise 6. A second restatement of boundedness

Prove that a set E *in a topological vector space is bounded if and only if every countable subset of E is bounded.*

PROOF. It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighborhood of the origin: No integer $n = 1, 2, \dots$ satisfies $E \subseteq nV$ (see Exercise 1 in Chapter 1). In other words, there exists a sequence $\{x_1, \dots, x_n, \dots\} \subseteq E$ such that

$$x_n \notin nV. \tag{1.48}$$

As a consequence, x_n/n fails to converge to 0 as n tends to ∞. In contrast, $1/n$ succeeds. It then follows from Section 1.30 that $\{x_1, \dots, x_n, \dots\}$ is not bounded. So ends the proof. □

7 Exercise 7. Metrizability and number theory

Let X *be the vector space of all complex functions on the unit interval* $[0,1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \qquad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology.

Show that there is a sequence $\{f_n\}$ *in* X *such that (a)* $\{f_n\}$ *converges to* 0 *as* $n \to \infty$*, but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omitted in (b) of Theorem 1.28.*

PROOF. **Justifying the terminology.** Since the family of seminorms p_x is separating, the collection $\mathcal{B}$ of all finite intersections of the following

$$V(x,k) \triangleq \{p_x < 2^{-k}\} \qquad (x \in [0,1], k = 1,2,3,\dots) \tag{1.49}$$

forms a local base for a topology τ on X, see Theorem [1.37] of [4]. Hence

$$\sum_{n=1}^{\infty}\Big[g_n \notin \cap_{i=1}^{m} U_i\Big] \le \sum_{n=1}^{\infty}\sum_{i=1}^{m}\Big[g_n \notin U_i\Big] = \sum_{i=1}^{m}\sum_{n=1}^{\infty}\Big[g_n \notin U_i\Big], \tag{1.50}$$

see [iii] for Iverson bracket notation. Now assume that $\{f_n\}$ τ-converges to some f. By definition,

$$\sum_{n=1}^{\infty}\Big[f_n - f \notin W\Big] < \infty \qquad (W \in \mathcal{B}). \tag{1.51}$$

The special case $W = V(x,k)$ implies that, for a fixed k, $|f_n(x) - f(x)| < 2^{-k}$ for all but finitely many n. In other words, $\{f_n(x)\}$ converges to $f(x)$. Conversely, assume that $\{f_n\}$ does not converge in τ. This implies that, for each f there exists finitely many $V(x_1,k_1),\dots,V(x_m,k_m)$ such that

$$\sum_{n=1}^{\infty}\Big[f_n - f \notin \cap_{i=1}^{m} V(x_i,k_i)\Big] = \infty. \tag{1.52}$$

Substituting $f_n - f$ for f_n in (1.50), with $U_i = V(x_i,k_i)$, yields

$$\sum_{n=1}^{\infty}\Big[f_n - f \notin \cap_{i=1}^{m} V(x_i,k_i)\Big] \le \sum_{i=1}^{m}\sum_{n=1}^{\infty}\Big[f_n - f \notin V(x_i,k_i)\Big] = \infty. \tag{1.53}$$

It is now obvious that

$$\sum_{n=1}^{\infty}[f_n - f \notin V(x_i,k_i)] = \infty \tag{1.54}$$

for some i, which shows that $\{f_n(x_i)\}$ does not converge to $f(x_i)$. Thus, τ-convergence coincides with pointwise convergence on X.

Proof with the given hint. We prove the second part by constructing a specific sequence $\{f_n\}$ that satisfies both (a) and (b). The hint suggests that there exists a bijection

$$\begin{aligned} \phi : \Big\{\theta_n : \theta_n \xrightarrow{n\infty} 0\Big\} &\to [0,1] \\ (\theta_1,\dots,\theta_n,\dots) &\mapsto x. \end{aligned} \tag{1.55}$$

We set

$$f_n(x) \triangleq \theta_n \qquad (x = \phi(\theta_1,\dots,\theta_n,\dots)) \tag{1.56}$$

so that $\{f_n\}$ tends pointwise to 0. Note that, with this construction, the following

$$x_\gamma \triangleq \phi\Big(1/\sqrt{1+|\gamma_1|},\dots,1/\sqrt{1+|\gamma_n|},\dots\Big) \tag{1.57}$$

outputs

$$\gamma_n f_n(x_\gamma) = \gamma_n/\sqrt{1+|\gamma_n|} \xrightarrow[n\to\infty]{} \infty \tag{1.58}$$

when $\gamma_n \to \infty$. This proves (b), since $\{\gamma_n f_n(x_\gamma)\}$ diverges. We now give an alternative construction of $\{f_n\}$ that requires no cardinality argument.

Proof with binary expansions (no hint) We rely on the following assertion: Every irrational number has a binary expansion that is not eventually periodic. More precisely, there exists a bijective sum

$$\begin{aligned} \sigma : \left\{\beta \in \{0,1\}^{N_+} : \beta \text{ is not eventually periodic}\right\} &\to [0,1] \setminus \mathbf{Q} \\ (\beta_1, \dots, \beta_n, \dots) &\mapsto \sum_{k=1}^{\infty} \beta_k 2^{-k}. \end{aligned} \tag{1.59}$$

A suitable $\{f_n\}$ is defined as follows:

$$f_n(x) \triangleq \begin{cases} 2^{-(\beta_1+\cdots+\beta_n)} & (x = \sigma(\beta_1, \dots, \beta_n, \dots) \notin \mathbf{Q}) \\ 0 & (x \in \mathbf{Q}). \end{cases} \tag{1.60}$$

Indeed, every bit stream $\sigma^{-1}(x)$ has infinitely many 1's, which implies that $f_n(x) \xrightarrow{n\infty} 0$. Next, pick an arbitrary $\gamma_n \to \infty$. Thus, for any positive integer k, $\gamma_n > 4^k$ for all sufficiently large n, say $n > N_k$. We select $n_k > N_k$ so large that

$$n_{k+1} - n_k > k. \tag{1.61}$$

The crucial point is that the sequence $1_{\{n_1,n_2,\dots\}}$ is not eventually periodic. Moreover, the particular choice

$$\beta^\gamma \triangleq 1_{\{n_1,n_2,\dots\}} \tag{1.62}$$

implies

$$\beta_1^\gamma + \cdots + \beta_{n_1}^\gamma + \cdots + \beta_{n_k}^\gamma = k. \tag{1.63}$$

Finally, (1.60) and (1.63) together yield

$$\gamma_{n_k} f_{n_k}(\sigma(\beta^\gamma)) = \gamma_{n_k}/2^k > 2^k \xrightarrow[k\to\infty]{} \infty. \tag{1.64}$$

In conclusion, every sequence of scalars γ_n such that $\gamma_n \to \infty$ contains a subsequence $\{\gamma_{n_k}\}$ that causes $\{\gamma_{n_k} f_{n_k}\}$ to diverge. This is (b). $\square$

9 Exercise 9. Quotient map

Suppose

(a) X *and* Y *are topological vector spaces,*

(b) $\Lambda : X \to Y$ *is linear.*

(c) N *is a closed subspace of* X,

(d) $: X \to X/N$ *is the quotient map, and*

(e) $\Lambda x = 0$ *for every* $x \in N$.

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ$ *, that is,* $\Lambda x = f((x))$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

PROOF. Bear in mind that π continuously maps X onto the topological (Hausdorff) space X/N, since N is closed (see [1.41] of [4]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y. \tag{1.65}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'. \tag{1.66}$$

It straightforwardly derives from (1.65) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.,* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$f(\pi x) = 0 \overset{1.65}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \tag{1.67}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by [1.41 (a)] of [4]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subseteq V. \tag{1.68}$$

Since π is open ([1.41 (a)] of [4]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by [1.41 (a)] of [4]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \tag{1.69}$$

for some neighborhood V of 0_X. So,

$$f(W) = f(\pi(V)) = \Lambda(V). \tag{1.70}$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y. So ends the proof. □

10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \to Y$ *is linear, and* $\Lambda(X) = Y$.

(a) Prove that Λ *is an open mapping.*

(b) Assume, in addition, that the null space of Λ *is closed, and prove that* Λ *is continuous.*

PROOF. Let $B = \{e, e', ...\}$ be a basis for Y, and let $W \subseteq X$ be an arbitrary neighborhood of the origin. Since addition is continuous in X, there exists a balanced open V such that

$$\sum_e V \subseteq W. \tag{1.71}$$

Note that $\Lambda(V)$ is balanced as well. Moreover, the surjective Λ provides a vector x_e such that $\Lambda x_e = e$. Therefore, given the coordinate representation $\sum_e y_e e$ of $y \in Y$, we have

$$y = \sum_e y_e e = \sum_e y_e \Lambda x_e. \tag{1.72}$$

Note that $\{x_e : e \in B\}$ is bounded (as a finite set). Hence

$$\{x_e : e \in B\} \subseteq sV \tag{1.73}$$

for some $s > 0$. Combining this with (1.72) yields

$$y \in \sum_e s y_e \Lambda(V). \tag{1.74}$$

Using (1.71) and the balancedness of $\Lambda(V)$, we conclude that $|y_e| < 1/s$ implies

$$y \in \sum_e \Lambda(V) \subseteq \Lambda(W). \tag{1.75}$$

This establishes (a) for $Y = \mathbf{C}^n$ equipped with $\|\cdot\|_\infty$, when B is the standard basis. The general case $\dim Y = n$ now follows from [iii]. The case $Y = \{0\}$ is trivial.
To prove (b), assume that the null space $N = \{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. By [iii], f is also a homeomorphism. We have thus established that f is continuous; so is $\Lambda = f \circ \pi$. □

12 Exercise 12. Topology stays, completeness leaves

Suppose $d_1(x,y) = |x-y|, d_2(x,y) = |\phi(x) - \phi(y)|$, *where* $\phi(x) = x/(1+|x|)$. *Prove that* d_1 *and* d_2 *are metrics on* $\mathbf{R}$ *which induce the same topology, although* d_1 *is complete and* d_2 *is not.*

PROOF. First, each d_i $(i = 1,2)$ induces a topology τ_i whose open balls are all

$$B_i(a,r) \triangleq \{x \in \mathbf{R} : d_i(a,x) < r\} \qquad (a \in \mathbf{R}, r > 0). \tag{1.76}$$

Next, remark that the monotonically increasing mapping $\phi : \mathbf{R} \to]-1,1[$ is odd and that

$$\phi(x) \xrightarrow[x\to\infty]{} 1. \tag{1.77}$$

ϕ is therefore a τ_1-homeomorphsim of $\mathbf{R}$ onto $]-1,1[$. A first consequence is that, at fixed $a \in \mathbf{R}$, given any positive scalar ε, the τ_1-continuity of ϕ supplies an open ball $B_1(a,\eta)$ on which $|\phi(a) - \phi| < \varepsilon$. In terms of balls B_i, this reads as follows,

$$B_1(a,\eta) \subseteq B_2(a,\varepsilon). \tag{1.78}$$

The second consequence is that the τ_1-continuity of ϕ^{-1} yields similar inclusions

$$B_2(a,\varepsilon') \subseteq B_1(a,\eta') \tag{1.79}$$

provided $\eta' > 0$. At arbitrary ε, the special case $\eta' = \eta$ is the concatenation

$$B_2(a,\varepsilon') \subseteq B_1(a,\eta) \subseteq B_2(a,\varepsilon); \tag{1.80}$$

which proves that $\tau_1 = \tau_2$. Finally, all inequalities $n < i < j$ over $\mathbf{N}$ together yield

$$d_2(i,j) = |\phi(i) - \phi(j)| \xrightarrow[n\to\infty]{} 0. \tag{1.81}$$

The sequence $n = 0,1,2,\ldots$ is therefore τ_2-Cauchy. We will nevertheless establish that it τ_2-diverges. To do so, we start by assuming the τ_2-convergence to some λ: The triangle inequality immediately dismisses that assumption, as follows,

$$d_2(0,\lambda) \geq d_2(0,n) - d_2(\lambda,n) = \phi(n) - d_2(\lambda,n) \xrightarrow[n\to\infty]{} 1. \tag{1.82}$$

We then conclude that d_2 fails to be complete. □

14 Exercise 14. $\mathscr{D}_K$ equipped with other seminorms

Put $K = [0,1]$ *and define* $\mathscr{D}_K$ *as in Section 1.46. Show that the following three families of seminorms (where* $n = 0,1,2,\ldots$*) define the same topology on* $\mathscr{D}_K$. *If* $D = d/dx$*:*

(a) $\|D^n f\|_\infty = \sup\{|D^n f(x)| : \infty < x < \infty\}$

(b) $\|D^n f\|_1 = \int_0^1 |D^n f(x)|\, dx$

(c) $\|D^n f\|_2 = \left\{\int_0^1 |D^n f(x)|^2 dx\right\}^{1/2}$.

PROOF. Let us equip $\mathcal{D}_K$ with the inner product $\langle f|g\rangle = \int_0^1 f\,\bar{g}$, so that $\langle f|f\rangle = \|f\|_2^2$. The following

$$\int_0^1 1\,|D^n f| \leq \|1\|_2 \|D^n f\|_2 \tag{1.83}$$

is then a Cauchy-Schwarz inequality, see Theorem [12.2] of [4]FA. We so obtain

$$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty \tag{1.84}$$

since K has length 1. Obviously, the support of $D^n f$ lies in K, hence the below equality

$$|D^n f(x)| = \left|\int_0^x D^{n+1} f\right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1. \tag{1.85}$$

Take the supremum over all $|D^n f(x)|$: Combining (1.84) with (1.85) now reads as follows,

$$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \cdots < \infty. \tag{1.86}$$

Finally, put

$$V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\}, \tag{1.87}$$

$$\mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\}, \tag{1.88}$$

so that (1.86) is mirrored by neighborhood inclusions, provided $i = 1, 2, \infty$:

$$V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \cdots. \tag{1.89}$$

Their subchains $V_n^{(i)} \supseteq V_{n+1}^{(i)}$ turn $\mathcal{B}^{(i)}$ into a local base of a topology τ_i. The whole chain (1.89) then forces

$$\tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1; \tag{1.90}$$

which achieves the proof. □

16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Lemma 1 *Let X be a topological space with a countable local base $\{V_n : n = 1, 2, 3, \dots\}$. If $\tilde{V}_n = V_1 \cap \cdots \cap V_n$, then every subsequence $\{\tilde{V}_{(n)}\}$ is a decreasing (, i.e., $\tilde{V}_{(n)} \supseteq \tilde{V}_{(n+1)}$) local base of X.*

PROOF. The proof consists in combining trivial consequences of the local base definition with a well-known result (for instance, see [2.6] of [3]) about intersection of nonempty compact sets.

The decreasing property is trivial. Now remark that $V_n \supseteq \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X. Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supseteq \tilde{V}_{\rho(n)}$. □

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$) *Under the same notations of Lemma 1, if $\{V_n\}$ is a decreasing local base, then so is $\{V_{(n)}\}$.*

Corollary 2 *If $\{Q_n\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{(n)}\}$ also satisfies theses conditions. Furthermore, if $_Q$ is the $C()$'s (respectively $C^\infty()$'s) topology of the seminorms p_n, as defined in section 1.44 (respectively 1.46), then the seminorms $p_{(n)}$ define the same topology $_Q$.*

PROOF. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X. Moreover,

$$Q_{\rho(n)} \subseteq Q^\circ_{\rho(n)+1} \subseteq Q_{\rho(n)+1} \subseteq Q_{\rho(n+1)}. \tag{1.91}$$

Thus,

$$Q_{\rho(n)} \subseteq Q^\circ_{\rho(n+1)}. \tag{1.92}$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(n)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(n)}\}$ is a local base. So, $\tau_{Q_\rho} \subseteq \tau_Q$. Conversely, the above corollary asserts that $\{V_{\rho(n)}\}$ is a local base of τ_Q, which yields $\tau_Q \subseteq \tau_{Q_\rho}$. □

Lemma 2 *If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in almost all Q°_n, i.e., there exists m such that*

$$K \subseteq Q^\circ_m \subseteq Q^\circ_{m+1} \subseteq Q^\circ_{m+2} \subseteq \cdots. \tag{1.93}$$

PROOF. The following definition

$$C_n \triangleq K \setminus Q^\circ_n \tag{1.94}$$

yields a decreasing sequence of compact[1] sets $\{C_n\}$. Suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q°_n. On the other hand, the conditions specified in [1.44] force the Q°_n's collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, , *i.e.*, $K \subseteq Q^\circ_m$, for some m. Finally,

$$K \subseteq Q^\circ_m \subseteq Q_m \subseteq Q^\circ_{m+1} \subseteq Q_{m+1} \subseteq Q^\circ_{m+2} \subseteq \cdots. \tag{1.95}$$

□

We are now in a fair position to establish the following:

Theorem *The topology of $C()$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty()$, as long as this sequence satisfies the conditions specified in section 1.44.*

PROOF. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\kappa}$ for every subsequence $\{L_{\kappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subseteq L^\circ_{m_i} \subseteq L^\circ_{m_i+1} \subseteq \cdots$ for some m_i. In particular, the special case $\kappa_i = m_i + i$ is

$$K_i \subseteq L^\circ_{\kappa_i}. \tag{1.96}$$

We now reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subseteq K^\circ_{\lambda_j} \tag{1.97}$$

Combine (1.96) with (1.97) and so obtain

$$K_1 \subseteq L^\circ_{\kappa_1} \subseteq L_{\kappa_1} \subseteq K^\circ_{\lambda_{\kappa_1}} \subseteq K_{\lambda_{\kappa_1}} \subseteq L^\circ_{\kappa_{\lambda_{\kappa_1}}} \subseteq \cdots, \tag{1.98}$$

which means that the sequence $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L. \tag{1.99}$$

So ends the proof □

[1] See [(b) of 2.5] of [3].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of [2.6] of [3].

17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathcal{D}_K$ *into* $\mathcal{D}_K$*, for every multi-index* α.

PROOF. In both cases, D^α is a linear mapping. It is then sufficient to establish continuity at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an arbitrary neighborhood of the origin. It exists N such that U contains

$$V_N = \left\{\phi \in C^\infty(\Omega) : \max\left\{|D^\beta \phi(x)| : |\beta| \le N, x \in K_N\right\} < 1/N\right\}. \tag{1.100}$$

Now pick g in $V_{N+|\alpha|}$ so that

$$\max\left\{|D^\gamma g(x)| : |\gamma| \le N + |\alpha|, x \in K_N\right\} < \frac{1}{N+|\alpha|}. \tag{1.101}$$

(the fact that $K_N \subseteq K_{N+|\alpha|}$ was tacitly used). The special case $\gamma = \beta + \alpha$ yields

$$\max\left\{|D^\beta D^\alpha g(x)| : |\beta| \le N, x \in K_N\right\} < \frac{1}{N}. \tag{1.102}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \qquad , \textit{i.e.,} \qquad D^\alpha\left(V_{N+|\alpha|}\right) \subseteq V_N, \tag{1.103}$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuity of the restriction $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \to \mathcal{D}_K$, we first remark that the collection of the $V_N \cap \mathcal{D}_K$ is a local base of the subspace topology of $\mathcal{D}_K$. $V_{N+|\alpha|} \cap \mathcal{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^\alpha|_{\mathcal{D}_K}\left(V_{N+|\alpha|} \cap \mathcal{D}_K\right) = D^\alpha\left(V_{N+|\alpha|} \cap \mathcal{D}_K\right) \tag{1.104}$$

$$\subseteq D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha(\mathcal{D}_K) \tag{1.105}$$

$$\subseteq V_N \cap \mathcal{D}_K \qquad \text{(see (1.103))} \tag{1.106}$$

So ends the proof. □

Chapter 2

Completeness

3 Exercise 3. An equicontinuous sequence of measures that does not converge vaguely

Put $K = [-1, 1]$*; define* $\mathscr{D}_K$ *as in Section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^n$*). Suppose* $\{f_n\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\phi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ *exists for every* $\phi \in \mathscr{D}_K$*. Show that* Λ *is a continuous functional on* $\mathscr{D}_K$*. Show that there is a positive integer* p *and a number* $M < \infty$ *such that*

$$\left| \int_{-1}^{1} f_n(t)\phi(t)\, dt \right| \leq M \| D^p \phi \|_\infty$$

for all n*. For example, if* $f_n(t) = n^3 t$ *on* $[-1/n, 1/n]$ *and* 0 *elsewhere, show that this can be done with* $p = 1$*. Construct an example where it can be done with* $p = 2$ *but not with* $p = 1$*.*

PROOF. Equipped with the supremum norm, $\mathscr{C}_K = C(\mathbf{R}) \cap \{\phi : \operatorname{supp}\phi \subseteq K\}$ is the copy of $C_c(K)$ in $C(\mathbf{R})$. Each density f_n is then identified with the following

$$\begin{aligned} \Lambda_n : \mathscr{C}_K &\to \mathbf{C} \\ \phi &\mapsto \int_{-1}^{1} f_n(t)\phi(t)\, dt, \end{aligned} \tag{2.1}$$

seen as a Radon measure. Every Λ_n is continuous since $\| \Lambda_n \| = \| f_n \|_1$ is finite; *cf.* [6.19] of [4]. Note that the dual space $\mathscr{C}_K^*$ is a Banach space as well, by [4.1] of [4]. Here, we consider only pointwise convergence, which is weaker than norm convergence. We prove that the assumed pointwise convergence

$$\Lambda_n\phi \xrightarrow[n\to\infty]{} \Lambda\phi \qquad (\phi \in \mathscr{D}_K) \tag{2.2}$$

implies the continuity of Λ in the topology of $\mathscr{D}_K$. We also construct a specific sequence $\{\Lambda_n\}$ whose pointwise limit Λ is not bounded with respect to the supremum norm. By contraposition, Theorem [2.8] of [4] implies that pointwise convergence on $\mathscr{D}_K$ does not extend to $\mathscr{C}_K$. This conclusion also derives from the following bounds:

$$| \Lambda_n\phi | \leq M \| \phi' \|_\infty, \tag{2.3}$$

$$| \Lambda_n\phi' | \leq M \| \phi'' \|_\infty. \tag{2.4}$$

Combined with the impossibility of boundedness at order $p = 0$, *cf.* (2.3), the contraposition of Theorem [2.6] of [4] implies that $\{\Lambda_n\}$ does not converge pointwise on $\mathscr{C}_K$. In Radon measure theory, pointwise convergence is known as *vague convergence*. In the next paragraph, ϕ is restricted to $\mathscr{D}_K$.

Continuity of Λ. We equip $\mathscr{D}_K$ with derivative norms $\| \phi \|_N \triangleq \| \phi \|_\infty + \| D^1\phi \|_\infty + \cdots + \| D^N\phi \|_\infty$. The induced topology τ_K of $\mathscr{D}_K$ is the weakest topology that makes all norms $\| \cdot \|_N$ continuous, *cf.* [1.46, 6.2] of [4] and Exercise [1.16]. Equivalently, the collection of all convex balanced sets

$$V_N \triangleq \left\{ \| \ \|_N < 1/N \right\} \tag{2.5}$$

forms a local base of τ_K, *cf.* [6.2] of [4]. Note that $\|\phi\|_N < 1$ implies $\|\phi\|_\infty < 1$: Every Λ_n is then τ_K-continuous by [1.18] of [4], since $|\Lambda_n|$ is bounded by $\|\Lambda_n\|$ on V_1. In summary:

(a) $\mathcal{D}_K$, equipped with the topology τ_K, is a Fréchet space; see [1.46] of [4].

(b) Every functional Λ_n is τ_K-continuous.

(c) $\Lambda_n\phi \to \Lambda\phi$ pointwise on $\mathcal{D}_K$ (our premise).

By [2.6, 2.8] of [4], the equicontinuous sequence $\{\Lambda_n\}$ converges pointwise to a continuous Λ. Furthermore, the equicontinuity of $\{\Lambda_n\}$ ensures that all $|\Lambda_n|$ remain below 1 on a common *balanced* neighborhood V_p. So,

$$\frac{1}{p}\cdot\frac{\phi}{\|\phi\|_p+\varepsilon}\in V_p \tag{2.6}$$

for all $\varepsilon > 0$. This yields $|\Lambda_n\phi| < p(\|\phi\|_p + \varepsilon)$, which reduces to $|\Lambda_n\phi| \le p\|\phi\|_p$. Applying Lemma [A.2] with $\phi, D\phi, \dots, D^p\phi$ outputs

$$|\Lambda_n\phi| \le p(p+1)\|D^p\phi\|_\infty. \tag{2.7}$$

This completes the first part of the proof, with some p and a positive constant $M = M(p)$.

The counterexample: A sequence of $\{\Lambda_n\}$ that does not converge pointwise on $\mathcal{C}_K$. Let u be a smooth, even mapping that equals 1 on $[-1/2, 1/2]$, vanishes outside $[-1, 1]$, and satisfies $0 \le u \le 1$ on $\mathbf{R}$. The function u belongs to the general construction in [1.46] of [4]. Alternatively, u can be the derivative of ϕ from Lemma [A.3], with $\tau = 1/2$, $\omega = 2$, and $A = 1$. We set

$$f_n(t) \triangleq n^3 t\Big[\,\text{-}1/n \le t \le 1/n\Big]. \tag{2.8}$$

Under the identification $C_c(K) \equiv \mathcal{C}_K$, Λ_n reads as the difference of two (positive) Radon measures Λ_n^+ and Λ_n^-, since

$$\Lambda_n\phi = \underbrace{n^3\int_0^{1/n} t\phi(t)\,dt}_{\Lambda_n^+\phi} - \underbrace{n^3\int_{-1/n}^{0} \text{-}t\phi(t)\,dt}_{\Lambda_n^-\phi} \qquad (\phi\in\mathcal{C}_K). \tag{2.9}$$

Thus, Λ_n is a signed Radon measure, whose compact support $[-1/n, 1/n]$ shrinks to $\{0\}$ as $n \to \infty$. We see that $\|\Lambda_n\| \le \|\Lambda_n^+\| + \|\Lambda_n^-\|$. However, the collection of all Λ_n is not uniformly bounded, since

$$\|\Lambda_n\| = \|f_n\|_1 = n = \|\Lambda_n^+\| + \|\Lambda_n^-\|. \tag{2.10}$$

The *logistic function* $\sigma_\lambda : t \mapsto 1/(1+\exp(\text{-}\lambda t))$ provides a direct proof of this. It is a standard approximation of the Heaviside step function[1]. Thus, Lebesgue's dominated convergence theorem implies

$$\Lambda_n(u\cdot(\underbrace{\sigma_\lambda - 1/2}_{\text{odd}})) = 2n^3\int_0^{1/n}\underbrace{t\,(\sigma_\lambda(t)-1/2)}_{\text{even}}\,dt \xrightarrow[\lambda\to\infty]{} n \qquad (n\ge 2). \tag{2.11}$$

We refer to [6.19] of [4] for a more general scope. A first point is that there is no vague convergence for the current sequence $\{\Lambda_n\}$: this would, by Theorem [2.6] of [4], imply that $\sup_n\|\Lambda_n\| < \infty$, which would contradict (2.10). However, we investigate further to establish weaker convergence and boundedness in τ_K. From now on, we bound ϕ to $\mathcal{D}_K$: The mean value theorem implies that

$$\phi(1/n) - \phi(\text{-}1/n) = \frac{2}{n}\phi'(t_n) \tag{2.12}$$

for some $-1/n < t_n < 1/n$. Moreover, integration by parts yields

$$\Lambda_n\phi = \frac{n^3}{2}t^2\phi(t)\Big|_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt \tag{2.13}$$

$$= \frac{n}{2}(\phi(1/n)-\phi(\text{-}1/n)) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt \tag{2.14}$$

$$= \phi'(t_n) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt. \tag{2.15}$$

[1] σ_λ connects machine learning with statistical mechanics; see [1]

Note that when $\phi' = 1$ in a neighborhood of 0, *e.g.*, for $\phi(t) = tu(t)$, the latter equality reduces to

$$\Lambda_n\phi = 1 - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\,dt = \frac{2}{3}. \tag{2.16}$$

This suggests that continuity of ϕ' dictates $\Lambda_n\phi \to \frac{2}{3}\phi'(0)$. We establish this convergence in two steps. First,

$$\Lambda_n\phi - \frac{2}{3}\phi'(0) = \phi'(t_n) - \phi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt + \phi'(0)\underbrace{\frac{n^3}{2}\int_{-1/n}^{1/n} t^2\,dt}_{1/3} \tag{2.17}$$

$$= \phi'(t_n) - \phi'(0) - \frac{n^3}{2}\left(\int_{-1/n}^{1/n} t^2\Big((\phi'(t) - \phi'(0))\Big)\,dt\right). \tag{2.18}$$

Next, taking absolute values gives

$$\left|\Lambda_n\phi - \frac{2}{3}\phi'(0)\right| \le |\,\phi'(t_n) - \phi'(0)\,| + \frac{1}{3}\max_{[-1/n,1/n]}|\,\phi' - \phi(0)\,| \xrightarrow[n\to\infty]{} 0. \tag{2.19}$$

As a result,

$$\Lambda_n\phi \xrightarrow[n\to\infty]{} -\frac{2}{3}\delta'\phi \qquad (\phi \in \mathscr{D}_K), \tag{2.20}$$

where $\delta' : \phi \mapsto -\phi'(0)$ is the *derivative* of the *Dirac measure* $\delta : \phi \mapsto \phi(0)$; see [6.1, 6.9] of [4] and Section [A.3]. The reasoning from the previous part shows that the limit $\Lambda = -\frac{2}{3}\delta'$ is τ_K-continuous. As a complement, absolute values from (2.15) provide

$$|\,\Lambda_n\phi\,| \le |\,\phi'(t_n)\,| + \frac{1}{3}\max_{[-1/n,1/n]}|\,\phi'\,|. \tag{2.21}$$

A simpler bound is

$$|\,\Lambda_n\phi\,| \le \frac{4}{3}\|\,\phi'\,\|_\infty. \tag{2.22}$$

This is a concrete instance of (2.7), with $p = 1$ and $M = 4/3$. To establish it as (2.3), we need to prove that no reduction to order $p = 0$ is possible. To do so, we first assume, to reach a contradiction, that there exists M such that

$$|\,\Lambda_n\phi\,| \le M\|\,\phi\,\|_\infty \qquad (\phi \in \mathscr{D}_K,\ n = 1, 2, 3, \dots). \tag{2.23}$$

Next, we choose

$$\phi_n \triangleq \tilde{\phi}_n u, \tag{2.24}$$

where $\tilde{\phi}_n$ is ϕ from Lemma [A.3] with $\tau = 1/n = 1/\omega = 1/A$. So, $\|\,\phi_n\,\|_\infty < 2$. In contrast, for $n \ge 4$, $\Lambda_n\phi_n$ is now

$$\frac{n}{2}(\,\underbrace{\tilde{\phi}_n(1/n)}_{1} - \underbrace{\tilde{\phi}_n(-1/n)}_{-1}\,) - \underbrace{\frac{n^3}{2}\int_{-1/n}^{1/n} t^2\tilde{\phi}'(t)\,dt}_{\frac{1}{3}n} = \frac{2}{3}n. \tag{2.25}$$

Hence

$$2M \overset{(2.23)}{\ge} |\,\Lambda_n\phi_n\,| \xrightarrow[n\to\infty]{} \infty, \tag{2.26}$$

which brings the desired contradiction. We now combine (2.20) with (2.26) to establish that

$$|\,\Lambda\phi_n\,| \ge |\,\Lambda_n\phi_n\,| - |\,\Lambda_n\phi_n - \Lambda\phi_n\,| \xrightarrow[n\to\infty]{} \infty. \tag{2.27}$$

Therefore, Λ is not bounded either. A direct way to see this is to pick ϕ_ω from Lemma [A.3] so that

$$\Lambda(u\phi_\omega) = \frac{2}{3}\omega \xrightarrow[\omega\to\infty]{} \infty \tag{2.28}$$

contrasts with $\|\,u\phi_\omega\,\|_\infty = 1$. Thus, we have exhibited a sequence of Radon measures Λ_n that

(a) does not converge vaguely to any Radon measure,

(b) but converges pointwise on $\mathcal{D}_K$, in the specific $\mathcal{D}_K$'s topology; see (2.5), (2.20), and (2.22).

As a second example, we present the *derivative*

$$\begin{aligned} \Lambda_n' : \mathcal{D}_K &\to \mathbf{C} \\ \phi &\mapsto -\Lambda_n \phi'; \end{aligned} \tag{2.29}$$

see [6.1] of [4]. We have proved that every Λ_n is continuous. So is the derivative operator in $\mathcal{D}_K$ - see Exercise [1.17]. Therefore, Λ_n' is continuous. Now apply (2.20) with ϕ' and so obtain

$$\Lambda_n' \phi \xrightarrow[n\to\infty]{} -\frac{2}{3}\phi''(0).$$

Furthermore, Theorem [2.8] of [4] implies that the limit $-\frac{2}{3}\phi''(0)$ is τ_K continuous. Additionally, it follows from (2.22) that the bound (2.4) is

$$|\Lambda_n' \phi| \leq \frac{4}{3}\|\phi''\|_\infty. \tag{2.30}$$

To prove this, it now suffices to show that 2 is the smallest suitable p. First, we assume, to reach a contradiction, that

$$|\Lambda_n \phi'| \leq M\|\phi'\|_\infty \qquad (\phi \in \mathcal{D}_K,\ n = 1, 2, 3, \dots). \tag{2.31}$$

Next, let Φ_n be the primitive of ϕ_n that vanishes at -1; see (2.24). The oddness of Φ_n (u is even) implies that $\operatorname{supp}\Phi_n \subseteq [-1, 1]$. So, under our assumption,

$$|\Lambda_n' \Phi_n| = |\Lambda_n \Phi_n'| \leq M\|\Phi_n'\|_\infty. \tag{2.32}$$

Equivalently,

$$|\Lambda_n \phi_n| \leq M\|\phi_n\|_\infty, \tag{2.33}$$

which has already been disproved. Finally, to reach a last contradiction, assume that there exists M attached to order $p = 0$ so that

$$|\Lambda_n' \phi| \leq M\|\phi\|_\infty \qquad (\phi \in \mathcal{D}_K,\ n = 1, 2, 3, \dots). \tag{2.34}$$

Lemma [A.3] implies that

$$|\Lambda_n' \phi| \leq M\|\phi'\|_\infty. \tag{2.35}$$

This contradiction concludes the proof. □

6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ *of a function* $f \in L^2(T)$ *(*T *is the unit circle) by*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

for all $n \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f \textit{ exists}\}$ *is a dense subspace of* $L^2(T)$ *of the first category.*

PROOF. Let $f(\theta)$ stand for $f(e^{i\theta})$so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi,\pi])$, hence the inner product

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta. \tag{2.36}$$

We express $\Lambda_n(f)$ as follows:

$$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n). \tag{2.37}$$

Moreover, a well-known (and easy to prove) result is

$$(e_n, e_{n'}) = [n = n'],, \textit{ i.e.}, \{e_n : n \in \mathbf{Z}\} \text{ is an orthormal subset of } L^2(T). \tag{2.38}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

$$\|\Lambda_n\|^2 \overset{2.37}{=} \|e_{-n} + \cdots + e_n\|^2 \overset{2.38}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \overset{2.38}{=} 2n + 1. \tag{2.39}$$

Now suppose, to reach a contradiction, that

$$B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, ...\} < \infty\} \tag{2.40}$$

is of the second category. So, the Banach-Steinhaus theorem [2.5] of [4] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$\|\Lambda_n\| \overset{2.39}{=} \sqrt{2n+1} \xrightarrow[n\to\infty]{} \infty. \tag{2.41}$$

This establishes that B is actually of the first category; and so is its subset L=$\{f \in L^2(T) : \lim_{n\longrightarrow\infty} \Lambda_n f \text{ exists}\}$. It remains to prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be span$\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.37) with (2.38) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$P \subseteq L \subseteq L^2(T). \tag{2.42}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.42), that

$$L^2(T) = \overline{P} = \overline{L}. \tag{2.43}$$

So ends the proof □

9 Exercise 9. Boundedness without closedness

Suppose X, Y, Z *are Banach spaces and*

$$B : X \times Y \to Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ *such that*

$$\|B(x, y)\| \le M\|x\|\|y\| \qquad (x \in X, y \in Y).$$

Is completeness needed here?

PROOF. Completeness is not required. To show this, we only assume that X, Y, and Z are normed spaces. Since B is continuous, there exists $r > 0$ such that

$$\|B(x, y)\| < 1 \tag{2.44}$$

whenever $\|x\| + \|y\| < r$. Therefore, any $0 < s < r$ yields

$$\|B(x, y)\| = \frac{4\|x\|\|y\|}{s^2} \cdot \left\| B\left(\frac{s}{2} \cdot \frac{x}{\|x\|}, \frac{s}{2} \cdot \frac{y}{\|y\|}\right) \right\| < \frac{4\|x\|\|y\|}{s^2} \qquad (x \neq 0, y \neq 0) \tag{2.45}$$

when $\left\| \frac{s}{2} \cdot \frac{x}{\|x\|} \right\| + \left\| \frac{s}{2} \cdot \frac{y}{\|y\|} \right\| = s < r$. This establishes the existence of $M = 4/r^2$, since

$$\|B(x,y)\| \leq \frac{4}{s^2}\|x\|\|y\| \xrightarrow[s\to r]{} \frac{4}{r^2}\|x\|\|y\|. \tag{2.46}$$

Consider the example where $X = Y = Z = C_c(\mathbf{R})$ equipped with the supremum norm. The existence of $M = 1$ follows from

$$\|fg\|_\infty \leq \|F\|_\infty \cdot \|g\|_\infty. \tag{2.47}$$

However, $C_c(\mathbf{R})$ is not complete; see [5.4.4] of [5]. We prove that the bilinear product

$$\begin{aligned} B : C_c(\mathbf{R})^2 &\to C_c(\mathbf{R}) \\ (f,g) &\mapsto f \cdot g \end{aligned} \tag{2.48}$$

is nevertheless continuous. To do so, we start with any

$$0 < r < \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty} < \varepsilon < 1. \tag{2.49}$$

Next, we choose $(u,v) \in C_c(\mathbf{R})^2$ with

$$\|f-u\|_\infty + \|g-v\|_\infty < r, \tag{2.50}$$

so that $\|fg-uv\|_\infty < \varepsilon$. Explicitly, we have

$$\begin{aligned}
\|fg-uv\|_\infty &= \|(f-u)\cdot g + u\cdot(g-v)\|_\infty && (2.51)\\
&\leq \|f-u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g-v\|_\infty && (2.52)\\
&< r\cdot\|g\|_\infty + \left(r + \|f\|_\infty\right)\cdot r \qquad \left(\text{using } \|u\|_\infty \leq r + \|f\|_\infty\right) && (2.53)\\
&< r\cdot\left(r + \|f\|_\infty + \|g\|_\infty\right) && (2.54)\\
&< \varepsilon\frac{r + \|f\|_\infty + \|g\|_\infty}{1 + \|f\|_\infty + \|g\|_\infty} && (2.55)\\
&< \varepsilon \qquad (\text{because } r < 1). && (2.56)
\end{aligned}$$

Since ε is arbitrary, this establishes that B is continuous. □

10 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

PROOF. Let $B : X_1 \times X_2 \to Z$ be a bilinear mapping that is continuous at $(0,0)$, where X_i $(i \in \{1,2\})$ and Z are topological vector spaces. This implies that, for any balanced open W, X_i contains a balanced open U_i such that

$$B(U_1 \times U_2) \subseteq W. \tag{2.57}$$

Let a_i be in X_i. Therefore, $a_i \in r_iU_i$ for some positive r_i. We now choose b_i in $a_i + (1 + r_1 + r_2)^{-1}U_i$. Hence

$$\begin{aligned}
B(b_1,b_2) - B(a_1,a_2) &= B(b_1-a_1,b_2) + B(a_1,b_2) - B(a_1,a_2) && (2.58)\\
&= B(b_1-a_1,b_2) + B(a_1,b_2-a_2) && (2.59)\\
&= \underbrace{B(b_1-a_1,b_2-a_2)}_{\in\frac{1}{(1+r_1+r_2)^2}W} + \underbrace{B(b_1-a_1,a_2)}_{\in\frac{r_2}{1+r_1+r_2}W} + \underbrace{B(a_1,b_2-a_2)}_{\in\frac{r_1}{1+r_1+r_2}W} && (2.60)\\
&\in W + W + W. && (2.61)
\end{aligned}$$

We conclude that B is continuous at every (a_1,a_2), since W was arbitrary. □

12 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)|\, dt.$$

Put $B(f,g) = \int_0^1 f(t)g(t)dt$, *and show that* B *is a bilinear continuous functional on* $X \times X$ *which is separately continuous but is not continuous.*

PROOF. Let f denote the first variable, g the second one. Remark that

$$(2.62) \qquad |B(f,g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient ([1.18] of [4]) to assert that any $f \mapsto B(f,g)$ is continuous. The continuity of all $g \mapsto B(f,g)$ follows (Define $C(g,f) = B(f,g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. Thus, there exists a positive M such that,

$$(2.63) \qquad |B(f,g)| < M\|f\|\,\|g\|.$$

Put

$$(2.64) \qquad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \qquad (n = 1,2,3,\dots),$$

so that

$$(2.65) \qquad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow[n\to\infty]{} 0.$$

On the other hand,

$$(2.66) \qquad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.66) and (2.63) with (2.65) and so obtain

$$(2.67) \qquad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow[n\to\infty]{} 0.$$

Our continuousness assumption is then contradicted. So ends the proof. □

15 Exercise 15. Baire's cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* $Y = X$. Hint: Y *must intersect* $x + Y$ *for every* $x \in X$.

PROOF. Let $\{E_n : n = 1,2,3,\dots\}$ be a collection whose union is $X \setminus Y$. Additionally, assume that every E_n is nowhere dense, *i.e.,* $V_n = X \setminus \overline{E_n}$ is dense in X. Now pick an arbitrary $x \in X$. Since the translation by x is a self-homeomorphism, $x + V_n$ is open and dense[2].We now apply Baire's theorem twice to establish that

(a) every intersection $W_n = V_n \cap [x + V_n]$ is dense in X,

(b) so is the nonempty intersection $\bigcap_{n=1}^{\infty} W_n$.

Moreover, the intersection $\bigcap_{n=1}^{\infty} V_n$ is disjoint from every E_n. In summary,

$$(2.68) \qquad w \in \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq Y$$

[2]Alternatively, observe that $X = x + X \subseteq \overline{x + V_n}$, as a special case of [1.3 (b)] of [4].

for some $w = w(x)$. Furthermore, w also lies in every $x + V_n$, by (a). This implies

$$w - x \in \bigcap_{n=1}^{\infty} V_n \subseteq Y. \tag{2.69}$$

Finally, (2.68) and (2.69) together yield

$$x = w - (w - x) \in Y - Y = Y, \tag{2.70}$$

as Y is a subgroup of X. This establishes that

$$X \subseteq Y, \tag{2.71}$$

since x was arbitrary. □

16 Exercise 16. An elementary closed graph theorem

Suppose that X and K are metric spaces, that K is compact, and that the graph of $f : X \to K$ is a closed subset of $X \times K$. Prove that f is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of K cannot be omitted from the hypothese, even when X is compact.

PROOF. Choose a sequence $\{x_n : n = 1, 2, 3, ...\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of $X \times K$. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous, see [[A6]] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we present the following counterexample,

$$\begin{aligned} f : [0, \infty) &\to [0, \infty) \\ x &\mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \end{aligned} \tag{2.72}$$

Clearly, f has a discontinuity at 0. In contrast, the graph G of f is closed. To see that, first remark that

$$G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}. \tag{2.73}$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b). To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on R_+. This establishes that $(a, b) \in G_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. □

Chapter 3

Convexity

3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $\mathrm{x}_0 \in \mathrm{A} \subseteq \mathrm{X}$ *an internal point of* A *if* $\mathrm{A} - \mathrm{x}_0$ *is an absorbing set.*

(a) Suppose A *and* B *are disjoint convex sets in* X*, and* A *has an internal point. Prove that there is a nonconstant functional* Λ *such that* $\Lambda(\mathrm{A}) \cap \Lambda(\mathrm{B})$ *contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) Show (with $\mathrm{X} = \mathbf{R}^2$*, for example) that it may not possible to have* $\Lambda(\mathrm{A})$ *and* $\Lambda(\mathrm{B})$ *disjoint, under the hypotheses of (a).*

PROOF. Take A and B as in (a); the trivial case $\mathrm{B} = \emptyset$ is discarded. Since $\mathrm{A} - \mathrm{x}_0$ is absorbing, so is its convex superset $\mathrm{C} = \mathrm{A} - \mathrm{B} - \mathrm{x}_0 + \mathrm{b}_0$ ($\mathrm{b}_0 \in \mathrm{B}$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $\mathrm{b}_0 - \mathrm{x}_0$ is not in C, hence $\mathrm{p}(\mathrm{b}_0 - \mathrm{x}_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem [3.4] of [4] to establish the existence of a functional $\Lambda : \mathrm{X} \to \mathbf{R}$ such that

$$\Lambda \leq \mathrm{p} \tag{3.1}$$

and

$$\Lambda(\mathrm{b}_0 - \mathrm{x}_0) = 1. \tag{3.2}$$

Then

$$\Lambda\mathrm{a} - \Lambda\mathrm{b} + 1 = \Lambda(\mathrm{a} - \mathrm{b} + \mathrm{b}_0 - \mathrm{x}_0) \leq \mathrm{p}(\mathrm{a} - \mathrm{b} + \mathrm{b}_0 - \mathrm{x}_0) \leq 1 \qquad (\mathrm{a} \in \mathrm{A}, \mathrm{b} \in \mathrm{B}). \tag{3.3}$$

Hence

$$\Lambda\mathrm{a} \leq \Lambda\mathrm{b}. \tag{3.4}$$

We now prove that $\Lambda(\mathrm{A}) \cap \Lambda(\mathrm{B})$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There exists $(\mathrm{a}_\mathrm{i}, \mathrm{b}_\mathrm{i})$ in $\mathrm{A} \times \mathrm{B}$ $(\mathrm{i} = 1, 2)$ such that

$$\Lambda\mathrm{a}_\mathrm{i} = \Lambda\mathrm{b}_\mathrm{i} = \mathrm{y}_\mathrm{i}. \tag{3.5}$$

Assume without loss of generality that $\mathrm{y}_1 < \mathrm{y}_2$. Then,

$$2 \cdot \mathrm{y}_1 = \Lambda\mathrm{b}_1 + \Lambda\mathrm{b}_1 < \Lambda(\mathrm{a}_1 + \mathrm{a}_2) = (\mathrm{y}_1 + \mathrm{y}_2) \qquad . \tag{3.6}$$

Remark that $\mathrm{a}_3 = \frac{1}{2}(\mathrm{a}_1 + \mathrm{a}_2)$ lies in the convex set A. This implies

$$\Lambda\mathrm{b}_1 \overset{(3.6)}{<} \Lambda\mathrm{a}_3 \overset{(3.4)}{\leq} \Lambda\mathrm{b}_1 \qquad ; \tag{3.7}$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$S_1 \triangleq \{(x,y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\}, \tag{3.8}$$

$$S_2 \triangleq \{(x,y) \in \mathbf{R}^2 : x > 0, y > 0\}, \tag{3.9}$$

$$A \triangleq S_1 \cup S_2, \tag{3.10}$$

$$B \triangleq X \setminus A. \tag{3.11}$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A. \tag{3.12}$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle*, i.e.,* A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$\Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad . \tag{3.13}$$

So ends the proof. □

11 Exercise 11. Meagerness of the polar

Let X *be an infinite-dimensional Fréchet space. Prove that* X^**, with its weak*-topology, is of the first category in itself.*

This is actually a consequence of the lemma below, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual X^* separates points on* X*, then the polar*

$$K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \ on\ A\} \tag{3.14}$$

of any absorbing subset A is a weak-closed set that has empty interior.*

PROOF. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous, see [3.14] of [4]. Therefore, $P_x = \Big\{\Lambda \in X^* : |\Lambda x| \leq 1\Big\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see [3.11] of [4]). Put

$$W_{F,x} \triangleq \bigcap_{x \in F} \Big\{\Lambda \in X^* : |\Lambda x| < r_x\Big\} \qquad (r_x > 0) \tag{3.15}$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$M \triangleq \operatorname{span}(F) \tag{3.16}$$

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] of [4] or [2.5] of [3].

is finite dimensional. Assume, to reach a contradiction, that $A \subseteq M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subseteq M$: Now pick a in $A \setminus M$ and so conclude that

$$b \triangleq \frac{a}{t_a} \in A \tag{3.17}$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see [1.21 (b)] of [4] for a proof): By the Hahn-Banach theorem [3.5] of [4], there exists Λ_a in X^* such that

$$\Lambda_a b > 2 \tag{3.18}$$

and

$$\Lambda_a(M) = \{0\}. \tag{3.19}$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$|\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1. \tag{3.20}$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

$$\Lambda + W \not\subseteq K_A. \tag{3.21}$$

Since W and Λ are both arbitrary, this achieves the proof. □

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

PROOF. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$B_n \triangleq \{x \in X : d(0, x) < 1/n\} \qquad (n = 1, 2, 3, \dots) \tag{3.22}$$

form a local base of X. If Λ is in X^*, then

$$|\Lambda| \leq m \text{ on } B_n \tag{3.23}$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$, see [1.18] of [4]. Hence, X^* is the countable union of all

$$m \cdot K_n \qquad (m, n = 1, 2, 3, \dots), \tag{3.24}$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see [3.4] of [4]. As a consequence, the above lemma implies

$$(\overline{K_n})^\circ = (K_n)^\circ = \emptyset. \tag{3.25}$$

Since the multiplication by m is a homeomorphism (see [1.7] of [4]), this is equivalent to

$$(\overline{m \cdot K_n})^\circ = m \cdot (K_n)^\circ = \emptyset. \tag{3.26}$$

So ends the proof. □

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

1 Exercise 1. Basic results

Let ϕ be the embedding of X *into* X^{**} *decribed in Section 4.5. Let τ be the weak topology of* X*, and let σ be the weak***- topology of* X^{**}*- the one induced by* X^**.*

(a) Prove that ϕ is a homeomorphism of (X, τ) *onto a dense subspace of* (X^{**}, σ)*.*

(b) If B *is the closed unit ball of* X*, prove that* $\phi(B)$ *is σ-dense in the closed unit ball of* X^{**}*. (Use the Hahn-Banach separation theorem.)*

(c) Use (a), (b), and the Banach-Alaoglu theorem to prove that X *is reflexive if and only if* B *is weakly compact.*

(d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.

(e) If X *is reflexive and* Y *is a closed subspace of* X*, prove that* X/Y *is reflexive.*

(f) Prove that X *is reflexive if and only* X^* *if reflexive.*
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\phi(X)$ *of* X^{**}*.*

PROOF. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$\phi^{-1} : \phi(X) \to X \tag{4.1}$$
$$\phi(x) \mapsto x \tag{4.2}$$

is a homeomorphism (with respect to τ and σ). We first consider

$$V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}|\psi x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0); \tag{4.3}$$
$$U \triangleq \{x \in X : |\langle x|x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0). \tag{4.4}$$

and remark that the so defined V 's (respectively U 's) shape a local subbase $\mathcal{S}_\sigma$ (respectively $\mathcal{S}_\tau$) of σ (respectively τ). We now observe that

$$U = \phi^{-1}(V \cap \phi(X)) = \phi^{-1}(V) \cap X \qquad (V \in \mathcal{S}_\sigma,\ U \in \mathcal{S}_\tau) \qquad , \tag{4.5}$$

since ϕ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathcal{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that ϕ^{-1} and its inverse ϕ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$\overline{\phi(X)}_\sigma \subseteq X^{**} \qquad . \tag{4.6}$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\phi(X)$. By [3.5], there exists y^* in X^* such that

$$(4.7) \qquad \langle \phi x, \psi y^* \rangle = \langle y^*, \phi x \rangle = \langle x, y^* \rangle = 0 \qquad (x \in X) \qquad ;$$

$$(4.8) \qquad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \qquad \phi(B) \subseteq B^{**} \qquad ,$$

since ϕ is isometric. Hence

$$(4.10) \qquad \overline{\phi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} \qquad .$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\phi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$(4.11) \qquad |\psi y^*| \leq 1 \qquad \text{on } \overline{\phi(B)}_\sigma \qquad ;$$

$$(4.12) \qquad \langle z^{**}, \psi y^* \rangle > 1 \qquad .$$

It follows from (4.11) that

$$(4.13) \qquad |\psi y^*| \leq 1 \text{ on } \phi(B), \ , \textit{ i.e., } |y^*| \leq 1 \text{ on } B \qquad .$$

We have so proved that

$$(4.14) \qquad y^* \in B^* \qquad .$$

Since z^{**} lies in B^{**}, it is now clear that

$$(4.15) \qquad |\langle z^{**}, \psi y^* \rangle| \leq 1 \qquad ;$$

what it contradicts (4.12), and thus proves (b). We now prove (c).

It follows from (a) that

(4.16) B is weakly compact if and only if $\phi(B)$ is weak*-compact.

If B is weakly compact, then $\phi(B)$ is weak*-closed. So,

$$(4.17) \qquad \phi(B) = \overline{\phi(B)}_\sigma \overset{(b)}{=} B^{**} \qquad .$$

ϕ is therefore onto, *i.e.,* X is reflexive.
Conversely, keep ϕ as onto: one easily checks that $\phi(B) = B^{**}$. The image $\phi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(4.18) \qquad (X/Y)^* \equiv Y^\perp \qquad ,$$

next,

$$(4.19) \qquad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \qquad .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z, *the weak***-topology of* Z^* *is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\phi(X)$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\phi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \phi(X)$ is weakly compact, *i.e.,* weak*-compact (to see this, apply (L) with $Z = X^*$).

By (a), B is therefore weakly compact, *i.e.,* X is reflexive, see (c). So ends the proof. □

13 Exercise 13. Operator compactness in a Hilbert space

(a) Suppose $T \in \mathcal{B}(X, Y), T_n \in \mathcal{B}(X, Y)$ *for* $n = 1, 2, 3, \dots$, *each* T_n *has finite-dimensional range, and* $\lim \|T - T_n\| = 0$. *Prove that* T *is compact.*

(b) Assume Y *is a Hilbert space, and prove the converse of (a): Every compact* $T \in \mathcal{B}(X, Y)$ *can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)*

PROOF. Since each T_n is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every T_n as a compact operator. Let $\{x_0^i : i \in \mathbf{N}\}$ be in U the open unit ball of X. Since T_1 is compact, $\{x_0^i\}$ contains a subsequence $\{x_1^i : i \in \mathbf{N}\}$ such that $\{T_1 x_1^i\}$ converges to a point y_1 of Y. The same reasoning can be recursively applied to T_n and $\{x_{n-1}^i\} \subseteq U$ so that $\{T_n x_n^i\}$ tends to some y_n of Y, as $\{x_n^i\}$ is a subsequence of $\{x_{n-1}^i\}$. Then

(4.20) $$T_n x_p^i \underset{i \to \infty}{\longrightarrow} y_n \qquad (p\, n = 1, 2, 3, \cdots) \quad .$$

Applied with $\{x_n^i : (n, i) \in \mathbf{N}^2\}$, a Cantor's diagonal process therefore provides a subsequence $\{\tilde{x}_j : j \in \mathbf{N}\}$ such that

(4.21) $$T_j \tilde{x}_k \underset{k \to \infty}{\longrightarrow} y_j \quad ;$$

(4.22) $$T_j \tilde{x}_j \underset{j \to \infty}{\longrightarrow} y_j \quad .$$

We now easily obtain

(4.23) $$\|T_j \tilde{x}_j - T_k \tilde{x}_k\| \leq \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \underset{k > j \to \infty}{\longrightarrow} 0 \quad .$$

$\{T_j \tilde{x}_j\}$ is then a Cauchy sequence. So is $\{T \tilde{x}_j\}$, since $\|T - T_j\| \to 0$. On the other hand, Y is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix ε as a positive scalar. Since T is compact, Y contains a finite set C such that

(4.24) $$T(U) \subseteq \bigcup_{c \in C} B(c, \varepsilon) \quad .$$

As a Hilbert space, Y contains a maximal orthonormal set (or Hilbert basis) M. This implies that $\operatorname{span}(M)$ is dense in Y; *cf.* 4.18 & [4.22] of [3]. The finiteness of C forces M to enclose a finite set S so that

(4.25) $$\forall c \in C, \exists s(c) \in \operatorname{span}(S) : \|c - s(c)\| < \varepsilon \quad .$$

Let x be in U. It follows from (4.24) that

(4.26) $$\|Tx - c_x\| < \varepsilon$$

for some c_x of C. We now combine (4.25) and (4.26) to obtain

(4.27) $$\|Tx - s(c_x)\| \leq \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon$$

As a finite-dimensional subspace, $\mathrm{span}(S)$ is closed (see footnote 4, Exercise 1.10). We so obtain

(4.28)
$$Y = \mathrm{span}(S) \oplus \mathrm{span}(S)^{\perp} \qquad ,$$

by [12.4]. There exists a unique projection $\pi = \pi(\varepsilon)$ of Y onto itself (see [5.6] for the definition) such that

(4.29)
$$\pi(Y) = \mathrm{span}(S)\,,\ (I - \pi)(Y) = \mathrm{span}(S)^{\perp} \qquad .$$

It is easily checked that π has norm 1. Moreover,

(4.30)
$$\pi s = s \qquad (s \in \mathrm{span}(S)) \qquad .$$

Thus,

(4.31)
$$(I - \pi)(Tx) = (I - \pi)(Tx - s(c_x)) \qquad (x \in U) \qquad .$$

Then,

(4.32)
$$\|(I - \pi)(Tx)\| \le \|I - \pi\|\, \|Tx - s(c_x)\| < 4\varepsilon \qquad (x \in U)$$

(the fact that π has norm 1 is hidden in the right side inequality). We have just so proved that

(4.33)
$$\|T - \pi \circ T\| \in \underset{\varepsilon \sim 0}{O}(\varepsilon) \qquad .$$

That is particularly true when $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$. Let so T_n be $\pi(\varepsilon_n) \circ T$ and conclude that these (compact) operators approximate T in the desired fashion, *i.e.*,

(4.34)
$$\|T - T_n\| \underset{n\to\infty}{\longrightarrow} 0 \qquad .$$

□

15 Exercise 15. Hilbert-Schmidt operators

Suppose μ *is a finite (or* σ*-finite) positive measure on a measure space* Ω*,* $\mu \times \mu$ *is the corresponding product measure on* $\Omega \times \Omega$*, and* $K \in L^2(\mu \times \mu)$*. Define*

$$(Tf)(s) = \int_{\Omega} K(s,t) f(t)\, d\mu(t) \qquad [f \in L^2(\mu)].$$

(a) Prove that $T \in \mathcal{B}(L^2(\mu))$ *and that*

$$\|T\|^2 \le \int_{\Omega}\int_{\Omega} |K(s,t)|^2 d\mu(s)\, d\mu(t).$$

(b) Suppose a_i*,* b_i *are members of* $L^2(\mu)$*, for* $1 \le i \le n$*, put* $K_1 = \sum a_i(s) b_i(t)$ *and define* T_1 *in terms of* K_1 *a* T *was defined in terms of* K*. Prove that* $\dim \mathcal{R}(T_1) \le n$.

(c) Deduce that T *is a compact operator in* $L^2(\mu)$*. Hint: Use exercise 13.*

(d) Suppose $\lambda \in \mathbf{C}$*,* $\lambda \ne 0$*. Prove: Either the equation*

$$Tf - \lambda f = g$$

has a unique solution $f \in L^2(\mu)$ *for every* $g \in L^2(\mu)$ *or there are infinitely many solutions for some* g *and none for others. (This is known as the* Fredholm alternative.*).*

(e) Describe the adjoint of T.

PROOF. Let X (respectively P) be the Banach space $L^2(\mu)$ (respectively $L^2(\mu \times \mu)$). A consequence of the Radon-Nikodym theorem (*cf.* [6.16] of [3]) is that there exists a group isomorphism $\rho : X \to X^*$, $f \mapsto f^*$ such that

$$\langle u, f^* \rangle = \int_\Omega u \cdot f \, d\mu \qquad (u \in X, f \in X) \quad . \tag{4.35}$$

Define a.e $K_s, K_t : \Omega \to \mathbf{C}$ by setting

$$K_s(t) \triangleq K_t(s) \triangleq K(s, t) \text{ a.e} \qquad ((s,t) \in \Omega) \quad . \tag{4.36}$$

T is clearly linear. Moreover,

$$|(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \qquad (\|f\|_X < 1) \tag{4.37}$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with $|K|^2$ to obtain

$$\|Tf\|_X^2 \leq \int_\Omega \|K_s\|_X^2 \; \mu(s) = \|K\|_P^2 < \infty \qquad (\|f\|_X < 1) \quad . \tag{4.38}$$

(a) is then proved.

To show (b), remark that

$$\int_\Omega a_i(s) \cdot b_i \cdot f \; d\mu \in \mathbf{C} \cdot a_i(s) \text{ a.e} \qquad (f \in X, s \in \Omega) \quad . \tag{4.39}$$

It is now clear that T maps any f of X into $\mathbf{C} \cdot a_1 + \cdots + \mathbf{C} \cdot a_n$. We so conclude that $\dim R(T_1) \leq n$.

We now prove (c). The current part refers to Exercise 4.13. X is also a Hilbert space and so contains a Hilbert basis M. Define a.e

$$\begin{aligned} a_b : \Omega &\to \mathbf{C} \\ s &\mapsto (K_s, b) \end{aligned} \tag{4.40}$$

whenever b ranges M. Hence,

$$K_s = \sum_{b \in M} a_b(s) \cdot b \text{ a.e} \qquad (s \in \Omega) \quad . \tag{4.41}$$

Provided any positive scalar ε, there exists a finite subset $S = S(\varepsilon)$ of M such that

$$\|K_s - \sum_{b \in S} a_b(s) \cdot b\|_X < \varepsilon \qquad (s \in \Omega) \quad . \tag{4.42}$$

Remark that $\sum_{b \in S} a_b \cdot b$ matches the definition of K_1; *cf.* (b): from now on,

$$K_1 \triangleq \sum_{b \in S} a_b \cdot b \quad . \tag{4.43}$$

It follows from (b) that

$$\dim R(K_1) < \infty \quad . \tag{4.44}$$

Now turn back to (a), with $K - K_1$ playing the role of K, and so obtain

$$\|T - T_1\| < \varepsilon \mu(\Omega) \leq \infty \quad . \tag{4.45}$$

For if μ is finite, use (a) of Exercise 4.13 to conclude that T is compact. Assume henceforth that μ is not (necessarily) finite and pick δ in $\mathbf{R}_+$. The simple functions (with finite measure support) form a dense family of an L^p space $(1 \leq p < \infty)$; *cf.* [3.13] of [3]. It then exists a simple function K_δ of $L^2(\mu \times \mu)$ such that

$$(\mu \times \mu)(\{K_\delta \neq 0\}) < \infty \,, \;\; \|K - K_\delta\|_P < \delta \quad . \tag{4.46}$$

Define an operator T_δ in terms of K_δ as T was defined in terms of K, and proceed as in (a) with $T - T_\delta$ instead of T. Then

$$\|T - T_\delta\| < \delta \quad . \tag{4.47}$$

The key ingredient is that K_δ can be identified with an element of the finite measure space $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$. What we have attempted to approximate T by T_1 can therefore be reiterated (with K_δ playing the role of K) to achieve an approximation $T_{\delta,1}$ of T_δ so that

$$\|T_\delta - T_{\delta,1}\| < \varepsilon \quad . \tag{4.48}$$

It now follows from (4.47) and (4.48) that

$$\|T - T_{\delta,1}\| \leq \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta \quad . \tag{4.49}$$

Since ε and δ were arbitrary, the σ-finite case is proved. We now establish (d).

Provided g of X, let E_g be the following equation on X

$$Tf - \lambda f = g \quad , \tag{4.50}$$

whose solution set is denoted by S_g. Note that S_0 is $\ker(T - \lambda)$ and discard the trivial case $S_0 = X$[1]: each f of X lies in $S_{Tf-\lambda f}$, as some $Tf - \lambda f$'s are nonzero. Some S_g's are then nonempty. Remark that

$$S_g = f + S_0 \qquad (f \in S_g) \tag{4.51}$$

in such case. Furthermore, the equality $\beta = \alpha$ of [4.25] yields

$$(T - \lambda I)(X) \neq X, , \textit{ i.e.}, S_0 \neq \{0\} \quad . \tag{4.52}$$

So if $T - \lambda I$ is not onto, not only some S_g's are empty, but also $S_0 \neq \{0\}$. Every nonempty S_g (such sets always exist, see above) is then infinite, by (4.51).
Otherwise, $T - \lambda I$ is bijective and every equation E_g has then a unique solution f. The Fredholm alternative is so proved.

Our last step is the description of T^*. Let $S : X \to X$ be such that

$$(Sf)(t) \triangleq \int_\Omega K_t \cdot f \text{ a.e} \qquad (f \in X,\ t \in \Omega) \tag{4.53}$$

Proceed as in (a), with S instead of T: S lies in $\mathcal{B}(X)$. Next, we claim that

$$\langle u, T^*f^* \rangle = \langle Tu, f^* \rangle \tag{4.54}$$

$$= \int_\Omega (Tu) \cdot f \ \ d\mu \tag{4.55}$$

$$= \int_{\Omega^2} K \cdot f \cdot u \ \ d(\mu \times \mu) \tag{4.56}$$

$$= \int_\Omega (Sf) \cdot u \ \ d\mu \tag{4.57}$$

$$= \langle u, (Sf)^* \rangle \quad , \tag{4.58}$$

whenever u and f run through the closed unit ball of X. Since $\|T\|$, $\|T^*\|$ are equal and finite, only exactness of (4.56) is possibly in doubt; the justification below dissipates it. In conclusion,

$$T^* = \rho S \rho^{-1} \quad . \tag{4.59}$$

Informally,

$$T^* = S \quad . \tag{4.60}$$

[1], *e.g.*, $X = L^2(\{0\}, \delta)$ so that $I = \lambda^{-1}T$ is compact.

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep u and f as above. Let us introduce

$$A(s) \triangleq \int_{\Omega} |K_s(t) \cdot u(t)| \ d\mu(t) \ \text{a.e} \qquad (s \in \Omega) \qquad , \tag{4.61}$$

to make hold the following Cauchy-Schwarz inequality

$$A(s) \leq \|K_s\|_X \qquad (s \in \Omega) \qquad . \tag{4.62}$$

Thus,

$$\int_{\Omega^2} |K(s,\ t)\, u(t)\, f(s)| \ d\mu(s)\, d\mu(t) = \int_{\Omega} |f(s)|\, A(s) \ d\mu(s) \tag{4.63}$$

$$\leq \int_{\Omega} |f(s)|\, \|K_s\|_X \ d\mu(s) \tag{4.64}$$

$$\leq \left[\int_{\Omega} \|K_s\|_X^2 \ d\mu(s) \right]^{\frac{1}{2}} = \|K\|_P < \infty \qquad . \tag{4.65}$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof. $\square$

Chapter 6

Distributions

1 Exercise 1. Test functions are almost polynomial

Suppose f *is a complex continuous function in* $\mathbf{R}^n$*, with compact support. Prove that* $\psi P_j \to f$ *uniformly on* $\mathbf{R}^n$*, for some* $\psi \in \mathcal{D}$ *and for some sequence* $\{P_j\}$ *of polynomials.*

PROOF. According to 1.16, Ω is union of a compact sets sequence $\{K_i\}$ and supp(f) lies in some $K = K_i$ so that f is embedded in $\mathcal{D}(\Omega)$. We can apply [1.10] to ensure that Ω encloses a compact set $S = \overline{K + B(\varepsilon)}$ for sufficiently small $\varepsilon > 0$.

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra $\{g \in C(S) : g \text{ is polynomial}\}$ of $C(S)$. There exists a sequence $\{P_j : j \in \mathbf{N}\}$ of $\mathbf{R}[X_1, \dots, X_n]$ such that

$$\sup_S |f - P_j| \underset{j\infty}{\longrightarrow} 0 \qquad . \tag{6.1}$$

By [6.20], the open set $K + B(\varepsilon)$ has a local partition of unity $\{\psi_i\} \subseteq \mathcal{D}(\Omega)$. Moreover, there exists an integer l such that $\psi = \psi_1 + \dots + \psi_l$ equals 1 on K. Hence

$$\| f - \psi P_j \|_\infty = \| \psi f - \psi P_j \|_\infty = \sup_S |\psi f - \psi P_j| \tag{6.2}$$

$$= \sup_S | f - P_j | \underset{j\infty}{\overset{(6.1)}{\longrightarrow}} 0 \qquad . \tag{6.3}$$

□

We will actually prove more by showing that $\mathcal{D}(\Omega)$ is separable for each nonempty open subset Ω of $\mathbf{R}^n$.

PROOF. The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the separability of $\mathcal{D}(\Omega)$. To do so, we keep (α, j) in $\mathbf{N}^n \times \mathbf{N}$. Remark that S encloses $\text{supp}(D^\alpha f)$: according to the first part, there exists a sequence $\{P_{\alpha,j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \dots, X_n]$ such that

$$\|D^\alpha f - \psi P_{\alpha,j}\|_\infty \underset{j\infty}{\longrightarrow} 0 \qquad . \tag{6.4}$$

Now let m range over $\{1, 2, 3, \dots\}$ and set $W_{m,j}$ in $\mathcal{D}(\Omega)$ as follows

$$D^{-\alpha}\phi \in \mathcal{D}(\Omega) : D^\alpha D^{-\alpha}\phi = \phi \qquad . \tag{6.5}$$

$$W_{m,j}(x) \triangleq D^{-(m,\dots,m)}(\psi P_{(m,\dots,m),j}) \tag{6.6}$$

By (6.4), there exists a natural number k(m) such that

$$\|D^{(m,\dots,m)}(f - W_{m,j})\|_\infty < 1/m \qquad (j \geq k(m)) \qquad . \tag{6.7}$$

Assume without loss of generality that S has diameter 1so that (6.7) yields

$$\|D^\lambda(f - W_{m,k(m)})\|_\infty < 1/m \qquad (|\lambda| \leq m) \qquad , \tag{6.8}$$

by the mean value theorem. In other words (remark that $\mathrm{supp}(f - W_{m,k(m)}) \subseteq S$),

$$f - W_{m,k(m)} \in U_m \triangleq \{\phi \in \mathcal{D}_S : \|\phi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \cdots \qquad (m = 1, 2, 3, \dots) \quad . \tag{6.9}$$

Pick W in β (see (b) of [6.3]): $W \cap \mathcal{D}_S$ contains a neighborhood of 0. Hence W contains some U_m, for m sufficiently large. Thus

$$W_{m,k(m)} \xrightarrow[m\infty]{} f \qquad (\text{in } \mathcal{D}(\Omega)) \quad . \tag{6.10}$$

We have so established that the $W_{m,k(m)}$'s family is dense in $\mathcal{D}(\Omega)$. We now aim to disclose a countable set $\tilde{W}$ that has the same property.

Choose δ in $\mathbf{R}_+$ and fetch any $W_{m,k(m)}$. Let X be $(X_1, \dots, X_n)$ and express $P_{(m,\dots,m),k(m)}$ as

$$P(X) = \sum_{|\gamma|\le d} p_\gamma \cdot X^\gamma \quad . \tag{6.11}$$

Since $\bar{\mathbf{Q}} = \mathbf{R}$, $\mathbf{Q}[X]$ hosts some $\mathbf{Q}(X) = \sum_{|\gamma|\le d} q_\gamma \cdot X^\gamma$ such that $|p_\gamma - q_\gamma| < \delta$ for all γ. Thus,

$$|P(x) - Q(x)| \le \sum_{|\gamma|\le d} |p_\gamma - q_\gamma|\, |x|^{|\gamma|} \le \delta \sum_{l\le d} \binom{l+n-1}{n-1} \|x\|_\infty^l \qquad (x \in \mathbf{R}^n) \quad . \tag{6.12}$$

Since S is bounded, we so obtain

$$\|\psi(P - Q)\|_\infty \in O(\delta) \quad . \tag{6.13}$$

Now define $\tilde{W}_m$ in terms of Q as $W_{m,k(m)}$ was defined in terms of P, and consider the integrations made in (6.6): each $D^\lambda \tilde{W}_m$ ($|\lambda| \le m$) can be obtained from some of them. So (6.13) yields

$$\|D^\lambda(W_{m,k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \qquad (|\lambda| \le m) \quad . \tag{6.14}$$

To be more specific, these λ's only exist in finite amount, so the big O can be assumed to be the same for all them. Since δ was arbitrary, combining (6.10) with (6.14) establishes the density of the all $\tilde{W}_m$'s family $\tilde{W}$.

Furthermore, each member of $\tilde{W}$ is only made of two ingredients: ψ and a polynomial of $\mathbf{Q}[X]$. The mapping ψ is attached to some K_i and $\mathbf{Q}[X]$ inherits countableness from $\mathbf{Q}$. Note that the "integrations packs" of (6.6) only exist in countable amount. Our $\tilde{W}$ is then countable. □

6 Exercise 6. Around the supports of some distributions

(a) Suppose that $c_m = \exp\{-(m!)!\}$, $m = 0, 1, 2, \dots$ *. Does the series*

$$\sum_{m=0}^{\infty} c_m (D^m\phi)(0)$$

converges for every $\phi \in C^\infty(R)$*?*

(b) Let Ω *be open in* $\mathbf{R}^n$*, suppose* $\Lambda_i \in \mathcal{D}'(\Omega)$*, and suppose that all* Λ_i *have their supports in some fixed compact* $K \subseteq \Omega$*. Prove that the sequence* $\{\Lambda_i\}$ *cannot converge in* $\mathcal{D}'(\Omega)$ *unless the orders of the* Λ_j *are bounded. Hint: Use the Banach-Steinhaus theorem.*

(c) Can the assumption about the supports be dropped in (b)?

PROOF. The answer is: no. To establish this assertion, we first assume, to reach a contradiction, that the above series converges for every smooth $\phi : \mathbf{R} \to \mathbf{C}$.

The sequence $\{c_m (D^m\phi)(0)\}$ so converges to 0. Nevertheless, it is proved in [1.46] that $C^\infty(\Omega)$ is not locally bounded. In other words, it is always possible to excavate a ϕ for which the magnitude of the m-th derivative at 0 is as large as we please[1], *e.g.,* greater than $1/c_m$. A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume $\{\Lambda_j\}$ to converge to some Λ of $\mathcal{D}'(\Omega)$ and we let Q run through the compact sets of Ω. Next, we define

$$S(T,\,Q) \triangleq \{N \in \mathbf{N},\, \exists C \in \mathbf{R}_+ : \; |T\phi| \le C\,\|\phi\|_N \text{ for all } \phi \text{ of } \mathcal{D}_Q\} \qquad (T \in \mathcal{D}(\Omega)) \quad . \tag{6.15}$$

Such subset of $\mathbf{N}$ has a minimum $\omega(T,\,Q)$. The following value

$$\omega(T) \triangleq \max\{\omega(T,\,Q) : \; Q \subseteq \Omega,\; Q \text{ compact }\} \le \infty \tag{6.16}$$

is then the order of T. Assume, to reach a contradiction, that, after passage to a subsequence,

$$\omega(\Lambda_j\,,\,Q_j) = j \qquad (j = 1,\,2,\,3,\,\dots) \tag{6.17}$$

for some compact $Q = Q_j$. By (a) of [6.24], Q_j cuts $\mathrm{supp}\Lambda_j$, say in p_j. Since K encloses $\mathrm{supp}\Lambda_j$, $\{p_j\}$ tends, after passage to a subsequence, to some p of K. Choose a positive scalar r so that

$$\overline{B}(p,\,r) \triangleq \{x \in \mathbf{R}^n : \; |x - p| \le r\} \subseteq \Omega \quad . \tag{6.18}$$

Such closed ball $\overline{B}(p,\,r)$ is a compact subset of Ω. By (b) of [6.5] (which refers to [1.46]) $\mathcal{D}_{\overline{B}(p,\,r)}$ is then a Fréchet space. It now follows from [2.6] that $\{\Lambda_j\}$ is equicontinuous on $\mathcal{D}_{\overline{B}(p,\,r)}$. There exists[2] a nonnegative integer N such that

$$|\Lambda\phi| \le C\,\|\phi\|_N \qquad (\phi \in \mathcal{D}_{\overline{B}(p,\,r)}) \tag{6.19}$$

for some positive constant C. On the other hand, $\overline{B}(p,\,r)$ contains almost all the p_j's. Hence

$$|\Lambda_N\,\phi| > C\,\|\phi\|_N \tag{6.20}$$

for some ϕ of $\mathcal{D}_{\overline{B}(p,r)}$. (b) is then established.

To prove (c), we introduce a sequence $\{x_m : m \in \mathbf{Z}\}$ of Ω that has no limit point. Let $\{\alpha_m : m \in \mathbf{Z}\}$ be in $\mathbf{N}$ and so define[3]

$$\begin{aligned} \Lambda :\; &\mathcal{D}(\Omega) \to \mathbf{C} \\ &\phi \mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m}\phi)(x_m) \end{aligned} \quad . \tag{6.21}$$

Λ belongs to $\mathcal{D}'(\Omega)$, since $\{x_m\}$ has no limit point. Next, we easily check that

$$\begin{aligned} \Lambda_j :\; &\mathcal{D}(\Omega) \to \mathbf{C} \\ &\phi \mapsto \sum_{|m|\le j} (D^{\alpha_m}\phi)(x_m) \end{aligned} \qquad (j \in \mathbf{N}) \tag{6.22}$$

is also a distribution and that $\{\Lambda_j\}$ tends to Λ in $\mathcal{D}'(\Omega)$. Nevertheless, no Λ_j's can have common support because $\{x_m\}$ has no limit point. Our assumption can therefore be dropped. □

9 Exercise 9. Convergence in $\mathcal{D}$ vs. convergence in $\mathcal{D}'$

(a) Prove that a set $E \subseteq \mathcal{D}(\Omega)$ *is bounded if and only if*

$$\sup\{|\Lambda\phi| : \; \phi \in E\} < \infty$$

for every $\Lambda \in \mathcal{D}(\Omega)$.

[1] indeed [1.46] provides sufficient tools for constructive proof of this, see the $\phi_j - \check{\phi}_j$ involved in (??).

[2] For more details, see Exercise 2.3.

[3] As $\Omega = \mathbf{R}$, the case $\alpha_m = m$ is the "counterpart" of the series of (a) and the case $(x_m,\,\alpha_m) = (m,\,0)$ is the Dirac comb.

(b) *Suppose* $\{\phi_j\}$ *is a sequence in* $\mathcal{D}(\Omega)$ *such that* $\{\Lambda\phi_j\}$ *is a bounded sequence of numbers, for every* $\Lambda \in \mathcal{D}'(\Omega)$*. Prove that some subsequence of* $\{\phi_j\}$ *converges, in the topology of* $\mathcal{D}(\Omega)$*.*

(c) *Suppose* $\{\Lambda_j\}$ *is a sequence in* $\mathcal{D}'(\Omega)$ *such that* $\{\Lambda_j\,\phi\}$ *is bounded, for every* $\phi \in \mathcal{D}(\Omega)$*. Prove that some subsequence of* $\{\Lambda_j\}$ *converges in* $\mathcal{D}'(\Omega)$ *and that the convergence is uniform on every bounded subset of* $\mathcal{D}(\Omega)$*. Hint: By the Banach-Steinhaus theorem, the restrictions of the* Λ_j *to* $\mathcal{D}_K$ *are equicontinuous. Apply Ascoli's theorem.*

PROOF. Since $\mathcal{D}(\Omega)$ is a locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $E = \{\phi_j : j \in \mathbf{N}\}$ is bounded: so is $\overline{E}$. By (c) of [6.5], there exists some $\mathcal{D}_K$ that contains $\overline{E}$. Since $\mathcal{D}_K$ has the Heine-Borel property (see [1.46]), $\overline{E}$ is τ_K-compact. Apply [A4] with the metrizable space $\mathcal{D}_K$ (see [1.46]) to conclude that $\overline{E}$ has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds. □

Annex

A.1 Vector spaces

Lemma A.1 Vector subspaces as convex and balanced sets.

Given a vector space X, the following are equivalent for any nonempty $S \subseteq X$.

(a) S is a vector subspace of X,

(b) S is convex and balanced, and $S + S \subseteq S$,

(c) S is convex and balanced, and $\lambda S = S$ for all $\lambda > 0$.

PROOF. It suffices to show that (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a). Assume (a), which implies $S + S \subseteq S$. Furthermore, S is convex and balanced. Hence (a) $\Rightarrow$ (b). Next, assume (b): By convexity of S, we have[4]:

$$2S = S + S \tag{A.1}$$
$$nS = (n-1)S + S = S + S. \qquad \text{(by induction on } n = 2, 3, 4, \dots) \tag{A.2}$$

The assumption $S + S \subseteq S$ then yields $nS \subseteq S$ for $n = 1, 2, 3, \dots$. Now choose $\lambda > 0$ and observe that

$$1 \leq \gamma = \max\{\lambda, 1/\lambda\} \leq \lceil \gamma \rceil. \tag{A.3}$$

Since S is balanced, this implies

$$S \subseteq \gamma S \subseteq \lceil \gamma \rceil S \subseteq S. \tag{A.4}$$

Thus, $\gamma S = S$. Furthermore, multiplying both sides by $1/\gamma$ gives

$$S = (1/\gamma)S. \tag{A.5}$$

This proves (c), because $\lambda \in \{\gamma, 1/\gamma\}$. Finally, assume (c). For any $(\alpha_1, \alpha_2) \in \mathbf{C}^2$, we have:

$$\alpha_1 \cdot S + \alpha_2 \cdot S \subseteq \{1 + |\alpha_1|\} \cdot S + \{1 + |\alpha_2|\} \cdot S \qquad \text{(by balancedness)} \tag{A.6}$$
$$\subseteq S + S \qquad \text{(by the assumption } \lambda S = S) \tag{A.7}$$
$$= 2S \qquad \text{(by convexity)} \tag{A.8}$$
$$= S. \qquad \text{(by the assumption } \lambda S = S) \tag{A.9}$$

In conclusion, S is a vector subspace of X. □

A.2 Mean value and bounded derivatives

Lemma A.2 A mean value inequality for higher-order derivatives.

If $\phi \in \mathcal{D}_{[a,b]}$, then

$$\| D^k \phi \|_\infty \leq \| D^p \phi \|_\infty \left(\frac{b-a}{2} \right)^{p-k} \tag{A.10}$$

for all $k \leq p$ in $\mathbf{N}$.

[4]See Exercise 1(d), equation (1.8).

PROOF. First, consider $a < x_0 \le (a+b)/2$. By the mean value theorem, there exists $a < x_1 < x_0$ such that

$$\phi(x_0) - \underbrace{\phi(a)}_{0} = D\phi(x_1)(x_0 - a). \tag{A.11}$$

If $p > 1$, repeating the same reasoning, first for $D\phi$, then $D^2\phi$, and so on, yields

$$\phi(x_0) = D^1\phi(x_1)(x_0 - a) \tag{A.12}$$
$$= D^2\phi(x_2)(x_1 - a)(x_0 - a) \tag{A.13}$$
$$\vdots$$
$$= \underbrace{D^p\phi(x_p)(x_{p-1} - a)}_{D^{p-1}\phi(x_{p-1})}(x_{p-2} - a)\cdots(x_0 - a) \tag{A.14}$$

for some points $a < x_p < \cdots < x_1 < x_0$. Hence

$$|\phi(x_0)| \le \|D^p\phi\|_\infty \left(\frac{b-a}{2}\right)^p \qquad (p = 0, 1, 2, \dots). \tag{A.15}$$

Similarly, if $(a+b)/2 < x_0 < b$ (with b playing the role of a), the same inequality holds. Thus,

$$|\phi(x_0)| \le \|D^p\phi\|_\infty \left(\frac{b-a}{2}\right)^p \qquad (a < x_0 < b), \tag{A.16}$$

which establishes the result when $k = 0$. Finally, applying the latter inequality to $D^k\phi$ in place of ϕ shows that

$$\|D^k\phi\|_\infty \le \|D^p\phi\|_\infty \left(\frac{b-a}{2}\right)^{p-k} \tag{A.17}$$

for all $0 \le k \le p$. □

Lemma A.3 Higher derivatives cannot be bounded by lower derivatives.

There is no general formula to estimate higher-order derivatives from lower-order derivatives. This immediately implies that no reversed mean value theorem exists.

PROOF. For *angular frequency* $\omega > 1$, we consider

$$\begin{aligned} \phi_\omega : \mathbf{R} &\to [-1, 1] \\ t &\mapsto \sin(\omega t), \end{aligned} \tag{A.18}$$

so that

$$\frac{\|D^p\phi_\omega\|_\infty}{\|D^k\phi_\omega\|_\infty} = \omega^{p-k} \qquad (0 \le k < p) \tag{A.19}$$

is unbounded as p or ω tends to ∞. Note that no pointwise estimation holds either. Indeed, when it exists, the quotient $Q_\omega(t) = \left|\frac{D^p\phi_\omega}{D^k\phi_\omega}(t)\right|$ is ω^{p-k} if p and k have the same parity. Otherwise, $Q_\omega(t)$ is either $\omega^{p-k}|\tan(\omega t)|$ or $\omega^{p-k}|\cot(\omega t)|$. In all cases, $Q_\omega(t) \to \infty$ as $\omega \to \infty$ at fixed t. This example rules out any general inequality in the reversed direction. However, the following smooth example ϕ maintains a constant derivative around 0, which is the simplest possible behavior.

Let $\rho \in C^\infty(\mathbf{R})$ be 1 on $]-\infty, 0]$, 0 on $[1, \infty[$, and strictly decaying on $]0, 1[$. A standard choice is $\rho = 1 - h$ with

$$h(t) \triangleq \frac{e^{-1/t}}{e^{-1/t} + e^{-1/(1-t)}} \tag{A.20}$$

for all $0 < t < 1$. Inspired by signal processing, in addition to the angular frequency ω we introduce two positive parameters: maximum amplitude (A) and delay (τ). We now define ϕ as the time-dependent solution of

$$\begin{cases} \phi(0) &= 0 \\ d\phi &= A\rho(\omega(|t| - \tau))\,dt. \end{cases} \tag{A.21}$$

Equivalently, ϕ is odd and, for $s > 0$,

$$\phi(s) = A\int_0^s \rho(\omega(t-\tau))\,dt = A\min(s,\tau) + A\int_\tau^{\max(s,\tau)} \rho(\omega(t-\tau))\,dt. \tag{A.22}$$

Hence

$$\|\phi\|_\infty = \phi(\tau + 1/\omega) \tag{A.23}$$

$$= \tau A + \frac{A}{\omega}\int_0^1 \rho(u)\,du \tag{A.24}$$

$$< \tau A + \frac{A}{\omega}. \tag{A.25}$$

The special case $\tau = 1/\omega = 1/A$ is of great interest. Indeed, $D\phi|_{[-\tau,\tau]} = A \to \infty$ as $\tau \to 0$. In contrast, $\|\phi\|_\infty < 2$. $\square$

A.3 Dirac's impulse, a physicist's detour

Consider a physical example: a particle colliding with a surface, which absorbs a unit of energy at impact time $t = 0$. We start with H the *Heaviside step function* $t \mapsto [t \geq 0]$, so that $H(t)$ indicates whether the particle has contributed its energy by time t. This formalism expresses that

(a) The energy is transferred by an instantaneous jump at time $t = 0$.

(b) The energy is conserved over time.

Heuristically, we write,

$$\int_{\mathbf{R}} \mathcal{H}' = 1, \qquad \mathcal{H}'(t) = \begin{cases} \infty & (t = 0) \\ 0 & (t \neq 0) \end{cases} \tag{A.26}$$

These properties cannot coexist in standard calculus. Nevertheless, the informal density $\mathcal{H}'$ describes the *Dirac* δ *function*. When identified with a positive Borel measure, δ has total mass 1 because

$$\int_{\mathbf{R}} d\delta = \int_{\mathbf{R}} \mathcal{H}' = \Big[H\Big]_{-\infty}^{\infty} = 1. \tag{A.27}$$

Physically, integrating over time recaptures all the energy. Let W be the observation window, which is adjusted so that either $0 \in W^\circ$ or $0 \notin \overline{W}$. Next, consider any smooth real function ϕ with (nonempty) compact support in W° as a test signal. In this formalism, the integral $\int_{\mathbf{R}} \phi\, d\delta$ represents the detector's response to the collision. If $\max|\phi| = 1$, then Lebesgue's dominated convergence theorem ensures

$$\sup_\phi \int_{W^\circ} |\phi|\,d\delta = \int_{W^\circ} d\delta = \Big[0 \in W^\circ\Big]. \tag{A.28}$$

We now make the model rigorous by eliminating the heuristic $\mathcal{H}'$, as follows:

$$\int_{\mathbf{R}} \phi\,d\delta = \int_{\mathbf{R}} \phi\,\mathcal{H}' \qquad \text{(generalization of (A.27))} \tag{A.29}$$

$$= \Big[H\phi\Big]_{-\infty}^{\infty} - \int_{\mathbf{R}} H\phi' \qquad \text{(integration by parts)} \tag{A.30}$$

$$= -\int_{\mathbf{R}} H\phi' \qquad (\operatorname{supp}\phi \text{ is compact}) \tag{A.31}$$

$$= \phi(0). \tag{A.32}$$

The key point is that the right-hand side in (A.31) is valid in standard calculus. Moreover, we obtain all filtered responses as the evaluation functional $\phi \mapsto \phi(0)$. This motivates the following definitions:

$$\Lambda_H(\phi) \triangleq \int_{\mathbf{R}} H\phi, \qquad \text{(expresses H)} \tag{A.33}$$

$$\Lambda'_H(\phi) \triangleq -\int_{\mathbf{R}} H\phi', \qquad \text{(the } \textit{weak derivative} \text{ of H)} \tag{A.34}$$

$$\delta(\phi) \triangleq \Lambda'_H(\phi) = \phi(0). \qquad \text{(impulse at 0: now } \delta \text{ has a rigorous definition)} \tag{A.35}$$

The functional $\delta : \phi \mapsto \phi(0)$ represents an instantaneous energy injection at $t = 0$. Its extension to all $\phi \in C_c(\mathbf{R})$ turns δ into a positive Radon measure of norm/total variation $\| \delta \| = 1$ and support $\{0\}$. In the sense of distribution theory, δ is a (tempered) distribution of order 0; see [Chapters 6 and 7] of [4]. Notably, its Borel-measure counterpart is the *Dirac measure*

$$\delta : E \mapsto \Big[0 \in E \Big] \tag{A.36}$$

restricted to Borel sets in $\mathbf{R}$. Hence the special case of (15)

$$\int_{\mathbf{R}} \phi \, d\delta = \delta(\phi) = \phi(0). \tag{A.37}$$

Convolution of δ with translated signal $\phi_t : s \mapsto \phi(t - s)$ extends (A.37), as follows:

$$[\delta * \phi](t) \triangleq \int_{\mathbf{R}} \phi_t \, d\delta = \delta(\phi_t) = \phi(t). \tag{A.38}$$

The Radon measure δ now serves as the convolution identity.

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