Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Notations and Assumptions

I   Special terms

i   The iff convention

iff is a shorthand for "if and only if". Splitting **iff** into *if-then* clauses shows that it is a natural language version of the logical equivalence $\Leftrightarrow$. All definitions are understood to be **iff** clauses, which is consistent with the fact that every definition expresses an equivalence.

ii   The assignment operator

Given variables a and b, $a \triangleq b$ is a special form of $a = b$. We say that $a \triangleq b$ **iff** a and b are *assumed* to be equal. Typically, $a \triangleq b$ is used to indicate that a is assigned the value b (some authors write $a := b$) but, in a different context, $a \triangleq b$ may also denote $a =: b$, *i.e.,* , $a := b$.

iii   Iverson brackets

Given a boolean expression ϕ, the boolean value $[\phi]$ encodes the truth value of ϕ as a `bit`:

$$[\phi] \triangleq \begin{cases} 0 & (\phi \text{ is false}) \\ 1 & (\phi \text{ is true}). \end{cases}$$

For example, $[1 > 0] = 1$ but $[\sqrt{2} \in \mathbf{Q}] = 0$. For interpretations of Boolean operators in logic, see [1].

II   Sets

i   Subsets and supersets

$\subseteq$ and $\supseteq$ are the standard symbols for set ordering. No specific symbol is reserved for strict ordering, every constraint $X \neq Y$ will be explicitly stated when a strict subset is intended. Given A and B, $A \cup B$ is the union of A and B. More generally, for any collection C, the union U of C is expressed as follows:

$$\text{(1)} \qquad\qquad U \triangleq \bigcup C \triangleq \bigcup_{S \in C} S.$$

Similarly, $\cap$ denotes intersection in the same manner.

ii   Special mappings

The identity I (or id) is the mapping $\{(x, x) : x \in X\}$. Similarly, the projection $\pi = \{((x, y), x) : x \in X, y \in Y\}$ always exists. Observe that I is the diagonal of $X \times X$.

iii   Equinumerosity

In a metric space context, $X \equiv Y$ means that ϕ is a surjective isometry.

III   Topological vector spaces

i   Scalar field

$\mathbf{C}$ extends $\mathbf{R}$, which implies that a property, *e.g.,* , linearity, that holds over $\mathbf{C}$ also holds over $\mathbf{R}$. The complex case is therefore *stronger* than the real case. This restriction may be significant in some contexts. However, the standard scalar field is $\mathbf{C}$. Unless the field is explicitly given, as in [3.1 and 12.7] of [3], we assume that results for $\mathbf{C}$ apply equally to the real case.

ii   Vector space bases

Given a vector space X, a subset B of X is a basis of X **iff** the sum

(2)
$$\left\{ (z_u)_{u \in B} : z_u \in \mathbf{C}, \{u : z_u \neq 0\} \text{ is finite} \right\} \to X$$
$$(z_u) \mapsto \sum_{u \in B} z_u u$$

is a bijection from all *finitely supported* families (z_u) onto X. The axiom of choice (AC) forces

(a) the existence of such B (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [3] with B playing the role of Λ);

(b) all bases to have the same cardinality, which is called the *dimension* of X and is denoted as $\dim X$.

We now turn to the finite-dimensional case over the field $\mathbf{C}$. The zero-dimensional case is $B = \emptyset$, *i.e.,* , $X = \{0\}$. Our first step is to study $\mathbf{C}^n$, the standard n-dimensional vector space, when $n = 1, 2, 3, \ldots$.

iii   Finite-dimensional spaces

The product topology of $\mathbf{C}^n$

$\mathbf{C}^n$ has the standard basis $1_{\{1\}}, \ldots, 1_{\{n\}} : \{1, \ldots, n\} \to \{0, 1\}$, so that z_k is the k-th component of

(3)
$$(z_1, \ldots, z_n) = z_1 \cdot (\underbrace{1, 0, \ldots}_{1_{\{1\}}}) + \cdots + z_k \cdot (\underbrace{0, \ldots, 1, 0, \ldots}_{1_{\{k\}}}) + \cdots + z_n \cdot (\underbrace{0, \ldots, 1}_{1_{\{n\}}}) \in \mathbf{C}^n.$$

A common notation is $e_k = 1_{\{k\}}$. Moreover, $\mathbf{C}^n$ is endowed with the product topology generated by all polydiscs

(4)
$$\prod_{k=1}^{n} \{z_k : |z_k| < r_k\} \qquad (\min(r_1, \ldots, r_n) > 0).$$

In the next subsection, we prove that every n-dimensional topological vector space is linearly homeomorphic to $\mathbf{C}^n$ equipped with this product topology, or, equivalently, with any norm.

Topology of a finite-dimensional vector space

To show this, choose an isomorphism $f : \mathbf{C}^n \to Y$. For instance, set $f(e_k) = u_k$, as in [1.20] of [3], when $\{u_k\}$ is a basis of the n-dimensional Y; see [ii]. It follows from [1.21] of [3] that f is a homeomorphism. A striking consequence is that $\{f(U) : U \text{ open in } \mathbf{C}^n\}$ *is the unique topological vector space topology on* Y. Thus, Y is necessarily locally convex and locally bounded , *i.e.,* , normable; see [1.39] of [3]. Note that the equality $\|y\|_Y = \|f^{-1}(y)\|_2$ for all $y \in Y$ defines a norm on Y. Additionally, Y is locally compact, as the closed unit ball of $\mathbf{C}^n$ is compact. Choose an n-dimensional topological vector space W, and repeat the same reasoning with $g : \mathbf{C}^n \to W$, then $h = g \circ f^{-1}$, in place of f. Thus, $h : Y \to W$ is a homeomorphism and W is normable as well. The following assertions are then equivalent in the finite-dimensional context:

(i) $\dim W = \dim Y$,

(ii) W and Y are isomorphic,

(iii) W and Y are linearly homeomorphic and normable.

Moreover, the norms on W and Y are *equivalent*. That is, for any given norm $\|\cdot\|_Y$ on Y and any given norm $\||\cdot\||_W$ on W, there exists a positive constant $C = C_h$ such that

$$\tag{5} \||\, h(y)\, \||_W \leq C \|\, y\, \|_Y \qquad (y \in Y)$$

because h is continuous. When $h = \mathrm{id}_Y$, this means that all norms on Y are equivalent, in the sense that

$$\tag{6} \||\, y\, \||_Y \leq C \|\, y\, \|_Y.$$

We now introduce the versions of (6) where $\|\cdot\|_Y$ and $\||\cdot\||_Y$ denote the standard norms $i, j = 1, 2, \infty$; see [1.19] of [3].

The standard norms 1, 2, and ∞

When $\mathbf{C}^n$ is equipped with $\|\cdot\|_i$ and $\|\cdot\|_j$, the smallest possible constants $C_{i,j}$ such that

$$\tag{7} \|\, z\, \|_j \leq C_{i,j} \|\, z\, \|_i$$

are easily derived from the definitions of these norms, except for the case $C_{2,1} = \sqrt{n}$, which requires the Cauchy-Schwarz inequality; see (1) in [12.2] of [3]. The table below records these sharp $C_{i,j}$:

i \ j	1	2	∞
1	1	1	1
2	$\sqrt{n}$	1	1
∞	n	$\sqrt{n}$	1

Table 1. Optimal $C_{i,j}$ for the norms $i, j = 1, 2, \infty$.

iv Continuity and boundedness in normed spaces

A linear mapping Λ is said to be *bounded* **iff** $\Lambda(E)$ *is a bounded set for every bounded set* E; see [1.31] of [3]. Linearity implies that $\Lambda : X \to Y$ is bounded **iff**

$$\tag{8} \|\Lambda\| \triangleq \sup\{\|\Lambda x\| : \|x\| < 1\} < \infty,$$

Given normed spaces X and Y, bounded linear maps Λ form a normed space $B(X, Y)$, where norm $\|\Lambda\|$ is given above; see [4.1] of [3]. It is now easy to see that, in the current context, boundedness and continuity coincide. This is a particular case of [2.8] of [3]. When Y is the scalar field, this equivalence also comes from [1.32] of [3]. Furthermore, we observe, that given a collection of bounded mappings $\Lambda \in B(X, Y)$, the property of equicontinuity now reads as

$$\tag{9} \sup_{\Lambda} \|\Lambda\| < \infty.$$

Thus, *uniform boundedness* and uniform continuity coincide as well; *cf.* [2.4] of [3].

IV Measure theory

i Radon measures

Given a locally compact Hausdorff space X, *e.g.,* , $\mathbf{R}^n$, a positive Radon measure Λ is a functional that is *positive* on $C_c(X)$, in the sense that

(i) $\phi \geq 0 \Rightarrow \Lambda\phi \geq 0 \qquad (\phi \in C_c(X)).$

Theorem [2.14] of [2] shows that positivity (i) implies the following *continuity* property (ii):

(ii) For each compact $K \subseteq X$ there exists a continuity bound M_K such that

$$|\Lambda\phi| \leq M_K \|\phi\|_\infty \qquad (\phi \in C_c(X) : \mathrm{supp}(\phi) \subseteq K).$$

Condition (ii) defines Radon measures in a more general sense; see [4]. Furthermore, Λ is bounded on $C_c(X)$ with respect to the supremum norm **iff** (ii) is strengthened by $\|\Lambda\|_\infty \leq \sup\{M_K : K \subseteq X \text{ compact}\} < \infty$. Such uniformly continuous functionals Λ constitute the space of *bounded* Radon measures. According to [6.19] of [2], each of them can be isometrically identified with a specific regular Borel measure μ. Conversely, when the ambient space X is compact, every Radon measure is $\|\cdot\|_\infty$-bounded. For instance, this case is addressed in Exercise 3 of Chapter 2. In measure theory, by *support* of μ, we mean:

$$(10) \qquad \operatorname{supp}(\mu) \triangleq X \setminus \bigcup V,$$

where V runs through all open sets of measure 0. A very important Radon measure of support $\{0\}$, the *Dirac delta function* is described in [A.2].

ii Lebesgue integration

Theorem [2.14] of [2] states that every positive Radon measure $\Lambda : C_c(X) \to \mathbf{C}$ is identified with a positive and regular Borel measure $\beta : X \to \mathbf{R}$. By "regular", we mean that

$$(11) \qquad \sup\{\beta(K) : K \subseteq E, K \text{ compact}\} = \beta(E) = \inf\{\beta(U) : E \subseteq U, U \text{ open}\}$$

when E is measurable. This justifies the following definition of the Lebesgue integral:

$$(12) \qquad \int_X \phi \, d\beta \triangleq \Lambda\phi \qquad (\phi \in C_c(X)),$$

The standard example is $X = \mathbf{R}$ with $\beta = \beta_{\mathbf{R}}$ the Lebesgue measure on Borel sets. The regularity property implies that β has total mass

$$(13) \qquad \int_X 1 \, d\beta \triangleq \sup\{\Lambda\phi : \phi \in C_c(X), 0 \leq \phi \leq 1\} \leq \infty.$$

When we substitute $f\beta$ for β, with f measurable, (12) takes the form

$$(14) \qquad \int_X \phi \, f \, d\beta = \Lambda\phi.$$

In particular, if f is positive, then $f\beta$ is a regular Borel measure such that

$$(15) \qquad \int_X f \, d\beta = \sup\{\Lambda\phi : \phi \in C_c(X), 0 \leq \phi \leq 1\}$$

Moreover, given $N = N(\beta)$ denoting $\{f : f = 0 \ \beta\text{-a.e.}\}$, we see that the *density* f and the measure Λ are identified modulo N in (14). Algebraically speaking:

$$(16) \qquad g\beta = f\beta \Leftrightarrow g - f \in N.$$

A central question in calculus is whether the integral of the modulus, namely

$$(17) \qquad |f|_1 \triangleq \int_X |f| \, d\beta \qquad (f \ X\text{-measurable}),$$

is finite. This motivates the following definitions:

$$(18) \qquad \mathcal{L}^1(X, \beta) \triangleq \{f : |f|_1 < \infty\} \subseteq \mathcal{L}^1_{\text{loc}}(X, \beta) \triangleq \bigcap\{\mathcal{L}^1(K, \beta) : K \subseteq X \text{ compact.}\}.$$

Standard notation $\mathcal{L}^1(\beta)$ is used to avoid redundancy, or simply $\mathcal{L}^1$ when the context is clear. Key points include:

  (a) $\mathcal{L}^1$, equipped with $|\cdot|_1$, is a seminormed space,

  (b) N is a closed subspace of the seminormed space $\mathcal{L}^1$.

Together, these two facts imply that $L^1 = \mathcal{L}^1/N$, equipped with the norm

$$\|f + N\|_1 \triangleq |f|_1 \qquad (f \in \mathcal{L}^1), \tag{19}$$

is a Banach space. These definitions and properties extend to L^p spaces ($p > 1$), as follows

$$\mathcal{L}^p \triangleq \{f : |f|^p \in \mathcal{L}^1\}, \tag{20}$$

$$L^p \triangleq \mathcal{L}^p/N, \tag{21}$$

$$\|f + N\|_p \triangleq \left(\int_X |f|^p\right)^{1/p} \qquad (f \in \mathcal{L}^p). \tag{22}$$

Similar *quasi-Banach* L^p spaces exist for $0 < p < 1$, with the notable difference that $\|f + N\|_p$ no longer defines a norm (the triangle inequality is lost). In contrast, $L^\infty = \{f + N : \|f + N\|_\infty < \infty\}$, equipped with the *quotient norm*

$$\|f + N\|_\infty \triangleq \inf\{M : |f| < M \ \beta\text{-a.e.}\} \tag{23}$$

is a Banach space.

Chapter 1

Topological Vector Spaces

1 Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X. *Prove the following statements from the axioms as given in section 1.4.*

(a) If x, y $\in$ X *there is a unique* z $\in$ X *such that* x + z = y.

(b) $0 \cdot x = 0 = \alpha \cdot 0$ $(\alpha \in \mathbf{C}, x \in \mathbf{X})$.

(c) $2A \subseteq A + A$.

(d) A *is convex if and only if* $(s + t)A = sA + tA$ *for all positive scalars* s *and* t.

(e) Every union (and intersection) of balanced sets is balanced.

(f) Every intersection of convex sets is convex.

(g) If Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

(h) If A *and* B *are convex, so is* $A + B$.

(i) If A *and* B *are balanced, so is* $A + B$.

(j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

PROOF.   (a) Such a property only depends on the group structure of X: Each x in X has an additive inverse -x. Let x' be any additive inverse of x, so that $x - x = 0 = x + x'$. Thus, $-x + x - x = -x + x + x'$, which is equivalent to $-x = x'$. Therefore, the inverse -x is unique. It is now clear that $x + z = y$ **iff** $z = -x + y$, which asserts both the existence and the uniqueness of z.

(b) Remark that

$$(1.1) \qquad\qquad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$
$$(1.2) \qquad\qquad\qquad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that $0 \cdot x = 0$. So,

$$(1.3) \qquad\qquad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (\text{-1}) \cdot x \Rightarrow \text{-1} \cdot x = \text{-x}.$$

Finally,

$$(1.4) \qquad\qquad \alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (x + (\text{-1} \cdot x)) = \alpha \cdot x + \alpha \cdot (\text{-1}) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

(c) Remark that

$$(1.5) \qquad\qquad\qquad\qquad 2x = (1+1)x = x + x$$

for every x in X, and so conclude that

$$(1.6) \qquad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets A of X; which proves (c).

(d) If A is convex, then

$$(1.7) \qquad\qquad\qquad\qquad A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A;$$

which is

$$(1.8) \qquad\qquad\qquad\qquad sA + tA = (s+t)A.$$

Conversely, the special case $s + t = 1$ is

$$(1.9) \qquad\qquad\qquad\qquad sA + (1 - s)A = A.$$

The latter extends to $s = 0$, since

$$(1.10) \qquad\qquad\qquad\qquad 0A + A \overset{(b)}{=} \{0\} + A = A.$$

The extension to $s = 1$ is analogously established (or simply use the fact that $+$ is commutative!). So ends the proof.

(e) Let A range over B a collection of balanced subsets, so that

$$(1.11) \qquad\qquad\qquad\qquad \alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap B \subseteq A$ establishes the first part. Now remark that

$$(1.12) \qquad\qquad\qquad\qquad \alpha A \subseteq \bigcup B$$

implies

$$(1.13) \qquad\qquad\qquad\qquad \alpha \bigcup B \subseteq \bigcup B;$$

which completes the proof.

(f) Let A range over C a collection of convex subsets, so that

$$(1.14) \qquad (s+t)\bigcap C \subseteq s\bigcap C + t\bigcap C \subseteq sA + tA \overset{(d)}{\subseteq} (s+t)A$$

for all positives scalars s, t. Inclusions at both extremities force

$$(1.15) \qquad\qquad\qquad\qquad s\bigcap C + t\bigcap C = (s+t)\bigcap C.$$

We then conclude from (d) that the intersection of C is convex. So ends the proof.

(g) We dismiss all trivial cases $\Gamma = \emptyset$, $\{\emptyset\}$, $\{\{x\}\}$, $\{\emptyset, \{x\}\}$ then pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i (i = 1, 2) lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_i's is then in $C_2 \subseteq \bigcup \Gamma$. Hence (g).

(h) Simply remark that

$$(1.16) \qquad s(A + B) + t(A + B) = sA + tA + sB + tB = (s+t)(A + B)$$

for all positive scalars s and t, then conclude from (d) that $A + B$ is convex.

(i) Given any α from the closed unit disc,

(1.17) $$\alpha(A + B) = \alpha A + \alpha B \subseteq A + B.$$

This completes the proof: $A + B$ is balanced.

(j) The proof is based on Lemma [A.1]. Let Γ be a collection of vector subspaces of X. Define $I = \bigcap \Gamma$ and $U = \bigcup \Gamma$. The intersection I is convex and balanced by (e) and (f), because every $Y \in \Gamma$ is convex and balanced. Next, observe that

(1.18) $$I + I \subseteq Y + Y \subseteq Y$$

for all $Y \in \Gamma$. Thus,

(1.19) $$I + I \subseteq I.$$

It now follows from the implication (b) $\Rightarrow$ (a) of Lemma [A.1] that I is a vector subspace of X. We now prove the counterpart of (g) when Γ is totally ordered by set inclusion. Combining (e) with (g) demonstrates that U is convex and balanced. To show that U is a vector subspace, we note that this total ordering of Γ implies

(1.20) $$Y_1 + Y_2 \subseteq \max(Y_1, Y_2),$$

where Y_1 and Y_2 run through Γ. Hence

(1.21) $$U + U \subseteq U.$$

It then follows from the implication (b) $\Rightarrow$ (a) of Lemma [A.1] that U is a vector subspace of X. To prove the counterpart of (h), let each Y_i be a vector subspace of X. Taken together, (h) and (i) imply that $Y_1 + Y_2$ is convex and balanced. Moreover,

(1.22) $$(Y_1 + Y_2) + (Y_1 + Y_2) = (Y_1 + Y_1) + (Y_2 + Y_2) \subseteq Y_1 + Y_2.$$

Finally, we conclude from (b) $\Rightarrow$ (a) of Lemma [A.1] that $Y_1 + Y_2$ is a vector subspace of X.

$\square$

2 Exercise 2. Convex hull

The convex hull of a set A *in a vector space* X *is the set of all convex combinations of members of* A, *that is the set of all sums* $t_1 x_1 + \cdots + t_n x_n$ *in which* $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$; n *is arbitrary. Prove that the convex hull of a set* A *is convex and that it is the intersection of all convex sets that contain* A.

PROOF. Our proof relies on the following equivalent properties:

(i) A is convex.

(ii) $s_1 A + \cdots + s_n A = (s_1 + \cdots + s_n) A$ when $s_i > 0$.

(iii) $t_1 A + \cdots + t_n A = A$ when $t_i > 0$ and $\sum t_i = 1$.

(iv) $\mathrm{co}(A) = A$.

Let y stand for $t_1 x_1 + \cdots + t_n x_n$. If $s \in [0, 1]$, then $sy + (1 - s)y' \in \mathrm{co}(A)$, because

(1.23) $$sy + (1 - s)y' = st_1 x_1 + \cdots + st_n x_n + (1 - s)t_1' x_1' + \cdots + (1 - s)t_{n'}' x_{n'}'$$

and

(1.24) $$\sum_i st_i + \sum_i (1 - s)t_i' = s \sum_i t_i + (1 - s) \sum_i t_i' = 1.$$

This shows that

(1.25) $$s\,\mathrm{co}(A) + (1 - s)\,\mathrm{co}(A) \subseteq \mathrm{co}(A) \qquad (0 \leq s \leq 1).$$

In other words, co(A) is convex (which implies (iv) → (i)). We now prove that (i) → (ii) → (iii) → (iv) to establish the intersection property as a corollary of (i) → (iv).

A proof by induction shows that (i) → (ii), since (d) of [1.1] gives the base case n = 2; see (1.8). Next, (iii) is a special case of (ii), and (iii) → (iv) derives from the definition of the convex hull. Finally, consider

(1.26) $$\Gamma \triangleq \{ B : A \subseteq B, \ B \text{ convex } \}.$$

We must prove that co(A) = $\bigcap \Gamma$. Observe that

(1.27) $$\bigcap \Gamma \subseteq co(A),$$

since co(A) ∈ Γ. Conversely, the implication (i) → (iv), with B ∈ Γ substituted for A, yields

(1.28) $$co(A) \subseteq co(B) = B$$

because A ⊆ B ⇒ co(A) ⊆ co(B). In conclusion,

(1.29) $$\bigcap \Gamma \subseteq co(A) \subseteq B,$$

where B runs through Γ. This completes the proof.   □

3   Exercise 3. Other basic results

Let be X as topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements:

   (a) The convex hull of every open set is open.

   (b) If X is locally convex then the convex hull of every bounded set is bounded.

   (c) If A and B are bounded, so is A+B.

   (d) If A and B are compact, so is A+B.

   (e) If A is compact and B is closed, then A+B is closed.

   (f) The sum of two closed sets may fail to be closed.

PROOF.   (a) Pick an open set A then let the variables V_i (i = 1, 2, ...) run through all open subsets of A so that

(1.30) $$co(A) \subseteq \bigcup_{t_i} (t_1 V_1 + \cdots + t_i V_i + \cdots) \subseteq co(A)$$

given all convex combinations $t_1 V_1 + \cdots + t_i V_i + \cdots$. We know from [1.7] of [3] that those sums are open; which achieves the proof.

   (b) Provided a bounded set E, pick V a neighborhood of 0: By (b) of Section 1.14 in [14] of [3], V contains a convex neighborhood of 0, say W. It follows that there exists a positive scalar s such that

(1.31) $$E \subseteq tW \subseteq tV \qquad (t > s).$$

   Hence

(1.32) $$co(E) \subseteq co(tW) = t \, co(W) = tW \subseteq tV,$$

   which completes the proof.

   (c) At fixed V, neighborhood of the origin, we combine the continuity of + with [1.14] of [3] to conclude that there exists U a balanced neighborhood of the origin such that

(1.33) $$U + U \subseteq V.$$

   Moreover, by the very definition of boundedness, A ⊆ rU for some positive scalar r. Similarly, B ⊆ sU for some positive s. Finally,

(1.34) $$A + B \subseteq rU + sU \subseteq tU + tU \subseteq tV \qquad (t > r, s),$$

   since U is balanced. So ends the proof.

(d) First, A and B are compact: So is $A \times B$. Next, $+$ maps continuously $A \times B$ onto $A + B$. In conclusion, $A + B$ is compact.

(e) From now on, we assume that neither A nor B is empty, since otherwise the result is trivial. Now pick $c \in X$ outside $A + B$: The result will be established by showing that c is not in the closure of $A + B$.

To do so, we let the variable a range over A: Every set $a + B$ is closed as well, see [1.7] of [3]. Trivially, $a + B \neq c$: By Section [1.10] of [3], there exists $V = V(a)$ a neighborhood of the origin such that

$$(1.35) \qquad (a + B + V) \cap (c + V) = \emptyset.$$

Moreover, there are finitely many $a + V$, say $a_1 + V_1, a_2 + V_2, \ldots$, whose union U contains the compact set A. Therefore,

$$(1.36) \qquad A + B \subseteq U + B.$$

Now define

$$(1.37) \qquad W \triangleq V_1 \cap V_2 \cap \cdots,$$

so that

$$(1.38) \qquad (a_i + B + V_i) \cap (c + W) \overset{(1.35)}{=} \emptyset \qquad (i = 1, 2, \ldots).$$

In conclusion, c is not in the closure of $U + B$. Finally, (1.36) asserts that c is not in $\overline{A + B}$ either; which achieves the proof.

Corollary: If B is the closure of a set S, then

$$(1.39) \qquad A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B$$

by [(b) of 1.13] of [3] (since A is closed; see Section 1.12, from the same source). The special case $A = \{x\}$, $B = X$ will occur in the proof of Exercise 15 in chapter 2.

(f) The final proof consists of exhibiting a counterexample. To do so, let f be any continuous mapping of the real line such that

   (i) $f(x) + f(-x) \neq 0 \qquad (x \in \mathbf{R})$;

  (ii) f vanishes at infinity.

For instance, we may combine (ii) with f even and $f > 0$ by setting $f(x) = 2^{-|x|}$, $f(x) = e^{-x^2}$, $f(x) = 1/(1 + |x|)$, ..., and so on.

As a continuous function, f has closed graph G, see [2.14] of [3]. Moreover, (i) implies that the origin $(0, 0) \neq (x - x, f(x) + f(-x))$ is not in $G + G$. On the other hand,

$$(1.40) \qquad \{(0, f(n) + f(-n)) : n = 1, 2, \ldots\} \subseteq G + G.$$

Now the key ingredient is that

$$(1.41) \qquad (0, f(n) + f(-n)) \xrightarrow[n \to \infty]{(ii)} (0, 0).$$

We have so constructed a sequence in $G + G$ that converges outside $G + G$. So ends the proof.

$\square$

4   Exercise 4. A balanced set whose interior is not balanced

Let be $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$. *Show that* B *is balanced but that its interior is not.*

PROOF. It is obvious that the nonempty set B contains the origin $(0,0)$. Additionally, its interior B° is nonempty as well. To see that, observe that the following set

$$(1.42) \qquad\qquad \{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2] < 1/2\} \subseteq B$$

is a neighborhood of $(1, 2) \in B$. Moreover, B is balanced, since

$$(1.43) \qquad\qquad |\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \qquad (|\alpha| \leq 1)$$

for all (z_1, z_2) in B. However, the nonempty set B° is not balanced, which we establish by showing that $(0,0) \notin$ B°. To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$(1.44) \qquad\qquad U \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2| < r\} \subseteq B$$

for some positive r. Clearly, U contains $(r/2, 0)$, and that special case $(r/2, 0) \in B$ now contradicts the definition of B. So ends the proof. $\qquad\square$

5   Exercise 5. A first restatement of boundedness

Consider the definition of "bounded set" given in Section 1.6. Would the content of this definition be altered if it merely required that to every neighborhood V *of 0 corresponds* some $t > 0$ *such that* $E \subseteq tV$?

PROOF. The answer is: No. To prove this, start from (a) of Section 1.14: V contains W, a balanced neighborhood of 0. Assume that E is bounded in this weaker sense, *i.e.,* , there exists a positive t that satisfies

$$(1.45) \qquad\qquad\qquad E \subseteq tW.$$

Thus,

$$(1.46) \qquad\qquad\qquad E \subseteq tW \subseteq sW \subseteq sV \qquad (s > t),$$

since W is balanced. Thus, we recover the definition given in Section 1.6: The two definitions are equivalent. $\qquad\square$

6   Exercise 6. A second restatement of boundedness

 Prove that a set E *in a topological vector space is bounded if and only if every countable subset of E is bounded.*

PROOF. It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighborhood of the origin: No integer $n = 1, 2, \dots$ satisfies $E \subseteq nV$ (see Exercise 1 in Chapter 1). In other words, there exists a sequence $\{x_1, \dots, x_n, \dots\} \subseteq E$ such that

$$(1.47) \qquad\qquad\qquad x_n \notin nV.$$

As a consequence, x_n/n fails to converge to 0 as n tends to ∞. In contrast, $1/n$ succeeds. It then follows from Section 1.30 that $\{x_1, \dots, x_n, \dots\}$ is not bounded. So ends the proof. $\qquad\square$

7   Exercise 7. Metrizability and number theory

Let X *be the vector space of all complex functions on the unit interval* $[0,1]$, *topologized by the family of seminorms*

$$p_x(f) = |f(x)| \qquad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology.

   Show that there is a sequence $\{f_n\}$ *in X such that (a)* $\{f_n\}$ *converges to 0 as* $n \to \infty$, *but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as* $[0,1]$.) *This shows that metrizability cannot be omitted in (b) of Theorem 1.28.*

PROOF. **Justifying the terminology.** The collection $\mathcal{B}$ of all finite intersections of $V(x, k)$ that are of the form

$$(1.48) \qquad V(x, k) \triangleq \{p_x < 1/k\} \qquad (x \in [0, 1], k = 1, 2, 3, \dots)$$

is a local base for a topology τ on X, since the family of seminorms p_x is separating; see Theorem [1.37] of [3]. Additionally, the number of occurrences of $f_n \in \cup_{i=1}^m S_i$ can be bounded by a sum of Iverson brackets, as follows:

$$(1.49) \qquad \sum_{n=1}^{\infty} \left[f_n \in \bigcup_{i=1}^m S_i \right] \leq \sum_{n=1}^{\infty} \sum_{i=1}^m \left[f_n \in S_i \right] = \sum_{i=1}^m \sum_{n=1}^{\infty} \left[f_n \in S_i \right];$$

see [iii] for the Iverson bracket notation. Now assume that $\{f_n\}$ τ-converges to some f. By definition,

$$(1.50) \qquad \sum_{n=1}^{\infty} \left[f_n - f \in X \setminus W \right] < \infty \qquad (W \in \mathcal{B}).$$

In the special case where $W = V(x, k)$, we have, for a fixed k, $|f_n(x) - f(x)| < 1/k$ for all but finitely many n. Equivalently, $f_n(x) \to f(x)$. Conversely, assume that $\{f_n\}$ diverges in τ. For each f, choose $\bigcap_{i=1}^m V(x_i, k_i)$ such that

$$(1.51) \qquad \sum_{n=1}^{\infty} \left[f_n - f \notin \bigcap_{i=1}^m V(x_i, k_i) \right] = \sum_{n=1}^{\infty} \left[f_n - f \in \bigcup_{i=1}^m X \setminus V(x_i, k_i) \right] = \infty.$$

Substituting $f_n - f$ for f_n in (1.49), with $S_i = X \setminus V(x_i, k_i)$, we see that

$$(1.52) \qquad \sum_{n=1}^{\infty} \left[f_n - f \in \bigcup_{i=1}^m X \setminus V(x_i, k_i) \right] \leq \sum_{i=1}^m \sum_{n=1}^{\infty} \left[f_n - f \in X \setminus V(x_i, k_i) \right] = \infty.$$

As a result,

$$(1.53) \qquad \sum_{n=1}^{\infty} \left[f_n - f \in X \setminus V(x_i, k_i) \right] = \infty$$

for some index i. This shows that $\{f_n(x_i)\}$ does not converge to $f(x_i)$. Thus, τ-convergence coincides with pointwise convergence on X. We prove the second part in two different ways. The first proof uses the suggested cardinality argument; the second one provides an alternative construction based on binary expansions.

Proof with the given hint. We prove the second part by constructing a specific sequence $\{f_n\}$ that satisfies both (a) and (b). The hint suggests that there exists a bijection

$$(1.54) \qquad \phi : \{\theta : \theta \in \mathbf{C}^{\mathbf{N}_+}, \lim_{\infty} \theta = 0\} \to [0, 1]$$
$$(\theta_1, \dots, \theta_n, \dots) \mapsto x.$$

We set

$$(1.55) \qquad f_n(x) \triangleq \theta_n \qquad (x = \phi(\theta_1, \dots, \theta_n, \dots)),$$

so that $\{f_n\}$ tends pointwise to 0. With this construction, the following choice

$$(1.56) \qquad x_\gamma \triangleq \phi(1/\sqrt{1 + |\gamma_1|}, \dots, 1/\sqrt{1 + |\gamma_n|}, \dots) \qquad (\gamma_n \xrightarrow{n \to \infty} \infty)$$

outputs

$$(1.57) \qquad \gamma_n f_n(x_\gamma) = \gamma_n / \sqrt{1 + |\gamma_n|} \xrightarrow{n \to \infty} \infty.$$

This proves (b), since $\{\gamma_n f_n(x_\gamma)\}$ diverges. We now construct a $\{f_n\}$ without the hint.

Proof with binary expansions (no hint).   The current proof is based on the following property: A real number is irrational **iff** its binary expansion is not eventually periodic. More precisely, there exists a bijective mapping

(1.58) $$\sigma : \{\beta : \beta \in \{0,1\}^{N_+}, \beta \text{ not eventually periodic}\} \to [0,1] \setminus \mathbf{Q}$$

$$(\beta_1, \dots, \beta_n, \dots) \mapsto \sum_{k=1}^{\infty} \beta_k 2^{-k}.$$

A suitable $\{f_n\}$ is defined as follows on $[0,1] \setminus \mathbf{Q}$:

(1.59) $$f_n(\alpha) \triangleq 2^{-(\beta_1 + \cdots + \beta_n)} \qquad (\alpha = \sigma(\beta_1, \dots, \beta_n, \dots) \in [0,1] \setminus \mathbf{Q}).$$

Indeed, every bit stream $(\beta_1, \dots, \beta_n, \dots) = \sigma^{-1}(\alpha)$ has infinitely many 1's, which implies $\beta_1 + \cdots + \beta_n \to \infty$. Hence $f_n(\alpha) \to 0$. The values of f_n on rationals are irrelevant. For simplicity, set $f_n = 0$ on $[0,1] \cap \mathbf{Q}$. Pick an arbitrary $\gamma_n \to \infty$. For each $k = 1, 2, 3, \dots$ choose m_k such that $\gamma_n > 4^k$ for all $n > m_k$. Next, select $n = n_k > m_k$ so large that

(1.60) $$n_k - n_{k-1} > k \qquad (\text{start with } n_0 = 0).$$

We have thus constructed a sequence $\beta^\gamma = 1_{\{n_1, n_2, \dots\}}$ that is not eventually periodic. Hence

(1.61) $$\sigma(\beta^\gamma) \in [0,1] \setminus \mathbf{Q}.$$

Moreover, a straightforward induction on k shows that

(1.62) $$\beta_1^\gamma + \cdots + \beta_{n_1}^\gamma + \cdots + \beta_{n_k}^\gamma = k.$$

Finally, (1.59) and (1.62) together yield

(1.63) $$\gamma_{n_k} f_{n_k}(\sigma(\beta^\gamma)) = \gamma_{n_k} \cdot 2^{-k} > 2^k \xrightarrow[k \to \infty]{} \infty.$$

In conclusion, every sequence of scalars $\gamma_n \to \infty$ contains a subsequence $\{\gamma_{n_k}\}$ such that $\{\gamma_{n_k} f_{n_k}\}$ does not converge pointwise to 0, which proves (b).   □

9   Exercise 9. Quotient map

Suppose

  (a) X *and* Y *are topological vector spaces,*

  (b) $\Lambda : X \to Y$ *is linear.*

  (c) N *is a closed subspace of* X,

  (d) $: X \to X/N$ *is the quotient map, and*

  (e) $\Lambda x = 0$ *for every* $x \in N$.

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ$, *that is,* $\Lambda x = f((x))$ *for all* $x \in X$. *Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

PROOF. Bear in mind that π continuously maps X onto the topological (Hausdorff) space X/N, since N is closed (see [1.41] of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

(1.64) $$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y.$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

(1.65) $$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'.$$

It straightforwardly derives from (1.64) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.,* , $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$(1.66) \qquad\qquad f(\pi x) = 0 \overset{1.64}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by [1.41 (a)] of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$(1.67) \qquad\qquad \Lambda(U) = f(\pi(U)) \subseteq V.$$

Since π is open ([1.41 (a)] of [3]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by [1.41 (a)] of [3]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$(1.68) \qquad\qquad W = \pi(V)$$

for some neighborhood V of 0_X. So,

$$(1.69) \qquad\qquad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if Λ is open, then f(W) is a neighborhood of 0_Y. So ends the proof. $\qquad\square$

10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \to Y$ is linear, and $\Lambda(X) = Y$.

(a) *Prove that Λ is an open mapping.*

(b) *Assume, in addition, that the null space of Λ is closed, and prove that Λ is continuous.*

PROOF. The case $Y = \{0\}$ is trivial. Let Y have dimension $n > 0$ and basis $B = \{e, e', ...\}$. Let $W \subseteq X$ be a neighborhood of the origin. Since addition is continuous in X, there exists a balanced open V such that

$$(1.70) \qquad\qquad \sum_e V \subseteq W.$$

Note that $\Lambda(V)$ is balanced as well. Moreover, by the surjectivity of Λ, there exists a vector $x_e \in X$ such that $\Lambda x_e = e$. Therefore, given the coordinate representation $\sum_e y_e e$ of $y \in Y$, we have

$$(1.71) \qquad\qquad y = \sum_e y_e e = \sum_e y_e \Lambda x_e.$$

Note that the finite set $\{x_e : e \in B\}$ is bounded. Hence

$$(1.72) \qquad\qquad \{x_e : e \in B\} \subseteq sV$$

for some $s > 0$. Equation (1.71) then yields

$$(1.73) \qquad\qquad y \in \sum_e s y_e \Lambda(V).$$

Using (1.70) and the balancedness of $\Lambda(V)$, we conclude that $|y_e| < 1/s$ implies

$$(1.74) \qquad\qquad y \in \sum_e \Lambda(V) \subseteq \Lambda(W).$$

This proves (a) for $Y = \mathbf{C}^n$ equipped with $\|\cdot\|_\infty$ and the standard basis. The general case now follows from [iii].

To prove (b), assume that the null space $N = \{x : \Lambda x = 0\}$ is closed and let f, π be as in Exercise 1.9. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. By [iii], f is also a homeomorphism. We have thus established that f is continuous; so is $\Lambda = f \circ \pi$. $\qquad\square$

12   Exercise 12. Topology stays, completeness leaves

Suppose $d_1(x, y) = |x - y|, d_2(x, y) = |\phi(x) - \phi(y)|$, *where* $\phi(x) = x/(1 + |x|)$. *Prove that* d_1 *and* d_2 *are metrics on* **R** *which induce the same topology, although* d_1 *is complete and* d_2 *is not.*

PROOF. First, each d_i $(i = 1, 2)$ induces a topology τ_i whose open balls are all

$$(1.75) \qquad\qquad B_i(a, r) \triangleq \{x \in \mathbf{R} : d_i(a, x) < r\} \qquad (a \in \mathbf{R}, r > 0).$$

Next, remark that the monotonically increasing mapping $\phi : \mathbf{R} \to]$ - $1, 1[$ is odd and that

$$(1.76) \qquad\qquad \phi(x) \xrightarrow[x \to \infty]{} 1.$$

ϕ is therefore a τ_1-homeomorphsim of **R** onto $]$ - $1, 1[$. A first consequence is that, at fixed $a \in \mathbf{R}$, given any positive scalar ε, the τ_1-continuity of ϕ supplies an open ball $B_1(a, \eta)$ on which $|\phi(a) - \phi| < \varepsilon$. In terms of balls B_i, this reads as follows,

$$(1.77) \qquad\qquad B_1(a, \eta) \subseteq B_2(a, \varepsilon).$$

The second consequence is that the τ_1-continuity of ϕ^{-1} yields similar inclusions

$$(1.78) \qquad\qquad B_2(a, \varepsilon') \subseteq B_1(a, \eta')$$

provided $\eta' > 0$. At arbitrary ε, the special case $\eta' = \eta$ is the concatenation

$$(1.79) \qquad\qquad B_2(a, \varepsilon') \subseteq B_1(a, \eta) \subseteq B_2(a, \varepsilon);$$

which proves that $\tau_1 = \tau_2$. Finally, all inequalities $n < i < j$ over **N** together yield

$$(1.80) \qquad\qquad d_2(i, j) = |\phi(i) - \phi(j)| \xrightarrow[n \to \infty]{} 0.$$

The sequence $n = 0, 1, 2, ...$ is therefore τ_2-Cauchy. We will nevertheless establish that it τ_2-diverges. To do so, we start by assuming the τ_2-convergence to some λ: The triangle inequality immediately dismisses that assumption, as follows,

$$(1.81) \qquad\qquad d_2(0, \lambda) \geq d_2(0, n) - d_2(\lambda, n) = \phi(n) - d_2(\lambda, n) \xrightarrow[n \to \infty]{} 1.$$

We then conclude that d_2 fails to be complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

14   Exercise 14. $\mathscr{D}_K$ equipped with other seminorms

Put $K = [0, 1]$ *and define* $\mathscr{D}_K$ *as in Section 1.46. Show that the following three families of seminorms (where* $n = 0, 1, 2, ...$ *) define the same topology on* $\mathscr{D}_K$. *If* $D = d/dx$:

(a) $\|D^n f\|_\infty = \sup\{|D^n f(x)| : \infty < x < \infty\}$

(b) $\|D^n f\|_1 = \int_0^1 |D^n f(x)| \, dx$

(c) $\|D^n f\|_2 = \left\{\int_0^1 |D^n f(x)|^2 dx\right\}^{1/2}$.

PROOF. Let us equip $\mathscr{D}_K$ with the inner product $\langle f | g \rangle = \int_0^1 f \bar{g}$, so that $\langle f | f \rangle = \|f\|_2^2$. The following

$$(1.82) \qquad\qquad \int_0^1 1 \, |D^n f| \leq \|1\|_2 \|D^n f\|_2$$

is then a Cauchy-Schwarz inequality, see Theorem [12.2] of [3]FA. We so obtain

$$(1.83) \qquad\qquad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

since K has length 1. Obviously, the support of $D^n f$ lies in K, hence the below equality

$$(1.84) \qquad |D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \le \int_0^x |D^{n+1} f| \le \|D^{n+1} f\|_1.$$

Take the supremum over all $|D^n f(x)|$: Combining (1.83) with (1.84) now reads as follows,

$$(1.85) \qquad \|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty \le \|D^{n+1} f\|_1 \le \cdots < \infty.$$

Finally, put

$$(1.86) \qquad V_n^{(i)} \triangleq \{ f \in \mathcal{D}_K : \|f\|_i < 2^{-n} \},$$

$$(1.87) \qquad \mathcal{B}^{(i)} \triangleq \{ V_n^{(i)} : n = 0, 1, 2, \dots \},$$

so that (1.85) is mirrored by neighborhood inclusions, provided $i = 1, 2, \infty$:

$$(1.88) \qquad V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \cdots.$$

Their subchains $V_n^{(i)} \supseteq V_{n+1}^{(i)}$ turn $\mathcal{B}^{(i)}$ into a local base of a topology τ_i. The whole chain (1.88) then forces

$$(1.89) \qquad \tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1;$$

which achieves the proof. $\qquad\qquad\square$

16   Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Lemma 1   *Let X be a topological space with a countable local base $\{ V_n : n = 1, 2, 3, \dots \}$. If $\tilde V_n = V_1 \cap \cdots \cap V_n$, then every subsequence $\{ \tilde V_{(n)} \}$ is a decreasing (equivalently, $\tilde V_{(n)} \supseteq \tilde V_{(n+1)}$) local base of X.*

PROOF. The proof consists in combining trivial consequences of the local base definition with a well-known result (for instance, see [2.6] of [2]) about intersection of nonempty compact sets.

The decreasing property is trivial. Now remark that $V_n \supseteq \tilde V_n$: This shows that $\{ \tilde V_n \}$ is a local base of X. Then so is $\{ \tilde V_{\rho(n)} \}$, since $\tilde V_n \supseteq \tilde V_{\rho(n)}$. $\qquad\qquad\square$

The following special case $V_n = \tilde V_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde V_n$)   *Under the same notations of Lemma 1, if $\{ V_n \}$ is a decreasing local base, then so is $\{ V_{(n)} \}$.*

Corollary 2   *If $\{ Q_n \}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{ Q_{(n)} \}$ also satisfies theses conditions. Furthermore, if $_Q$ is the $C()$'s (respectively $C^\infty()$'s) topology of the seminorms p_n, as defined in section 1.44 (respectively 1.46), then the seminorms $p_{(n)}$ define the same topology Q.*

PROOF. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{ p_n < 1/n \}$, then $\{ V_n \}$ is a decreasing local base of X. Moreover,

$$(1.90) \qquad Q_{\rho(n)} \subseteq Q^\circ_{\rho(n)+1} \subseteq Q_{\rho(n)+1} \subseteq Q_{\rho(n+1)}.$$

Thus,

$$(1.91) \qquad Q_{\rho(n)} \subseteq Q^\circ_{\rho(n+1)}.$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{ p_{\rho(n)} \}$ then defines a topology τ_{Q_ρ} for which $\{ V_{\rho(n)} \}$ is a local base. So, $\tau_{Q_\rho} \subseteq \tau_Q$. Conversely, the above corollary asserts that $\{ V_{\rho(n)} \}$ is a local base of τ_Q, which yields $\tau_Q \subseteq \tau_{Q_\rho}$. $\qquad\qquad\square$

Lemma 2   *If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set* K *lies in almost all* Q_n°, *i.e., , there exists m such that*

(1.92) $$K \subseteq Q_m^\circ \subseteq Q_{m+1}^\circ \subseteq Q_{m+2}^\circ \subseteq \cdots.$$

PROOF. The following definition

(1.93) $$C_n \triangleq K \setminus Q_n^\circ$$

yields a decreasing sequence of compact[1] sets $\{C_n\}$. Suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, , *i.e.,* , $K \subseteq Q_m^\circ$, for some m. Finally,

(1.94) $$K \subseteq Q_m^\circ \subseteq Q_m \subseteq Q_{m+1}^\circ \subseteq Q_{m+1} \subseteq Q_{m+2}^\circ \subseteq \cdots.$$

$\square$

We are now in a fair position to establish the following:

Theorem   *The topology of $C()$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty()$, as long as this sequence satisfies the conditions specified in section 1.44.*

PROOF. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\kappa}$ for every subsequence $\{L_{\kappa(n)}\}$. Now apply the above Lemma 2 with K_i (i = 1, 2, 3, ...) and so conclude that $K_i^\circ \subseteq L_{m_i}^\circ \subseteq L_{m_i+1}^\circ \subseteq \cdots$ for some m_i. In particular, the special case $\kappa_i = m_i + i$ is

(1.95) $$K_i \subseteq L_{\kappa_i}^\circ.$$

We now reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, ...\}$ such that

(1.96) $$L_j \subseteq K_{\lambda_j}^\circ$$

Combine (1.95) with (1.96) and so obtain

(1.97) $$K_1 \subseteq L_{\kappa_1}^\circ \subseteq L_{\kappa_1} \subseteq K_{\lambda_{\kappa_1}}^\circ \subseteq K_{\lambda_{\kappa_1}} \subseteq L_{\kappa_{\lambda_{\kappa_1}}}^\circ \subseteq \cdots,$$

which means that the sequence $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, ...)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

(1.98) $$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L.$$

So ends the proof $\square$

17   Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that f $\mapsto$ D^αf *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathcal{D}_K$ *into* $\mathcal{D}_K$, *for every multi-index* α.

PROOF. In both cases, D^α is a linear mapping. It is then sufficient to establish continuity at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an arbitrary neighborhood of the origin. It exists N such that U contains

(1.99) $$V_N = \left\{ \phi \in C^\infty(\Omega) : \max\{|D^\beta\phi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}.$$

---

[1]See [(b) of 2.5] of [2].

[2]In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of [2.6] of [2].

Now pick g in $V_{N+|\alpha|}$ so that

$$(1.100) \qquad \max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}.$$

(the fact that $K_N \subseteq K_{N+|\alpha|}$ was tacitly used). The special case $\gamma = \beta + \alpha$ yields

$$(1.101) \qquad \max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

$$(1.102) \qquad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \qquad , i.e., , \qquad D^\alpha\left(V_{N+|\alpha|}\right) \subseteq V_N,$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuity of the restriction $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \to \mathcal{D}_K$, we first remark that the collection of the $V_N \cap \mathcal{D}_K$ is a local base of the subspace topology of $\mathcal{D}_K$. $V_{N+|\alpha|} \cap \mathcal{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$(1.103) \qquad D^\alpha|_{\mathcal{D}_K}\left(V_{N+|\alpha|} \cap \mathcal{D}_K\right) = D^\alpha\left(V_{N+|\alpha|} \cap \mathcal{D}_K\right)$$

$$(1.104) \qquad \subseteq D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha(\mathcal{D}_K)$$

$$(1.105) \qquad \subseteq V_N \cap \mathcal{D}_K \qquad (\text{see } (1.102))$$

So ends the proof. $\qquad\qquad\qquad \square$

Chapter 2

Completeness

3   Exercise 3. Equicontinuity of Radon measures without vague convergence

Put K $= [\text{-}1, 1]$; *define* $\mathcal{D}_K$ *as in Section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^n$*). Suppose* $\{f_n\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda(\phi) = \lim_{n \to \infty} \int_{\text{-}1}^{1} f_n(t) \phi(t) \, dt$ *exists for every* $\phi \in \mathcal{D}_K$. *Show that* Λ *is a continuous functional on* $\mathcal{D}_K$. *Show that there is a positive integer* p *and a number* M $< \infty$ *such that*

$$\left| \int_{\text{-}1}^{1} f_n(t) \phi(t) \, dt \right| \leq M \| D^p \phi \|_\infty$$

for all n. *For example, if* $f_n(t) = n^3 t$ *on* $[\text{-}1, 1]$ *and* 0 *elsewhere, show that this can be done with* p $= 1$. *Construct an example where it can be done with* p $= 2$ *but not with* p $= 1$.

PROOF. Equipped with the supremum norm, $\mathcal{C}_K = C(\mathbf{R}) \cap \{u : \text{supp}(u) \subseteq K\}$ is the copy of $C_c(K)$ in $C(\mathbf{R})$. Each density f_n is then identified modulo $N(\beta_{\mathbf{R}})$ with the following Radon measure

(2.1)
$$\mu_n : \mathcal{C}_K \to \mathbf{C}$$
$$u \mapsto \int_{\text{-}1}^{1} f_n(t) u(t) \, dt;$$

see [i] and [ii] for definitions and notations. Every μ_n is continuous since $\| \mu_n \| = \| f_n \|_1$ is finite; *cf.* [6.19] of [3]. Note that the dual space $\mathcal{C}_K^*$ is a Banach space as well, by [4.1] of [3]. Here, we consider only pointwise convergence, which is weaker than norm convergence. Define

(2.2)
$$\Lambda_n \triangleq \mu_n|_{\mathcal{D}_K},$$

and assume the pointwise convergence of the sequence $\{\Lambda_n\}$, that is

(2.3)
$$\Lambda_n(\phi) \xrightarrow[n \to \infty]{} \Lambda(\phi) \qquad (\phi \in \mathcal{D}_K).$$

We show that the linear form Λ is continuous in the topology of $\mathcal{D}_K$. We also construct a particular sequence $\{\Lambda_n\}$ whose pointwise limit Λ is not norm-bounded. By contraposition, Theorem [2.8] of [3] implies that pointwise convergence on $\mathcal{D}_K$ does not extend to $\mathcal{C}_K$. This conclusion also follows from the following bounds:

(2.4)
$$\left| \Lambda_n(\phi) \right| \leq M \| \phi' \|_\infty,$$

(2.5)
$$\left| \Lambda_n(\phi') \right| \leq M \| \phi'' \|_\infty.$$

Combined with the lack of boundedness at order p $= 0$, *cf.* (2.4), the contraposition of Theorem [2.6] of [3] implies that $\{\Lambda_n\}$ does not converge pointwise on $\mathcal{C}_K$. In Radon measure theory, pointwise convergence is known as *vague convergence*. The rest of the current proof has been split into the following paragraphs,

Continuity of Λ: The sequence $\{\Lambda_n\}$ has pointwise limit $\Lambda \in \mathcal{D}_K^*$.

Existence of the uniform bound $|\Lambda_n(\phi)| \leq M\|D^p\phi\|$**:** $\{\Lambda_n\}$ is equicontinuous, hence $|\Lambda_n(\phi)| \leq M\|D^p\phi\|$.

Existence of $\{\Lambda_n\}$ **with optimal uniform bound** $|\Lambda_n(\phi)| \leq M\|D\phi\|$**:** In this example, no similar bound in $\|\phi\|$ exists. We show that Λ does not extend to a Radon measure.

Case of a uniform bound in $\|D^2\phi\|$**:** Where the smallest possible p is 2.

We first prove that $\lim\limits_{n\to\infty} \mu_n|_{\mathscr{D}_K} = \Lambda$ is continuous in $\mathscr{D}_K$'s topology.

Continuity of Λ**.** We equip $\mathscr{D}_K$ with derivative norms $\|\phi\|_N \triangleq \|\phi\|_\infty + \|D\phi\|_\infty + \cdots + \|D^N\phi\|_\infty$ for all $\phi \in \mathscr{D}_K$ and all $N = 1, 2, 3, \ldots$. The induced topology τ_K of $\mathscr{D}_K$ is the weakest topology that makes all norms $\|\cdot\|_N$ continuous, *cf.* [1.46, 6.2] of [3] and Exercise [1.16]. Equivalently, the collection of all convex balanced sets

$$(2.6) \qquad V_N \triangleq \left\{\phi : \phi \in \mathscr{D}_K, \|\phi\|_N < 1/N\right\}$$

forms a local base of τ_K, *cf.* [6.2] of [3]. Note that $\|\phi\|_N < 1$ implies $\|\phi\|_\infty < 1$: Every Λ_n is then τ_K-continuous by [1.18] of [3], since $|\Lambda_n|$ is bounded by $\|\Lambda_n\|$ on V_1. In summary:

(a) $\mathscr{D}_K$, equipped with the topology τ_K, is a Fréchet space; see [1.46] of [3].

(b) Every functional Λ_n is τ_K-continuous.

(c) $\Lambda_n(\phi) \to \Lambda(\phi)$ pointwise on $\mathscr{D}_K$ (our premise).

The hypotheses of the Banach-Steinhaus theorem are then satisfied, which supports the following existence proof.

Existence of the uniform bound $|\Lambda_n(\phi)| \leq M\|D^p\phi\|$**.** By [2.6, 2.8] of [3], the equicontinuous sequence $\{\Lambda_n\}$ converges pointwise to a continuous Λ. Furthermore, the equicontinuity of $\{\Lambda_n\}$ ensures that all $|\Lambda_n|$ remain below 1 on a common balanced neighborhood V_p. Hence

$$(2.7) \qquad \frac{1}{p} \cdot \frac{\phi}{\|\phi\|_p + \varepsilon} \in V_p \qquad (\varepsilon > 0).$$

This gives $|\Lambda_n(\phi)| < p \cdot \left\{\|\phi\|_p + \varepsilon\right\}$, which reduces to $|\Lambda_n(\phi)| \leq p\|\phi\|_p$. Use [A.2] with $I, D, \ldots, D^p$ to output

$$(2.8) \qquad |\Lambda_n(\phi)| \leq p(p+1)\|D^p\phi\|_\infty.$$

This completes the first part of the proof, with some p and a positive constant $M = M(p)$. We now construct an example of $\{\Lambda_n\}$.

Existence of $\{\Lambda_n\}$ **with optimal uniform bound** $|\Lambda_n(\phi)| \leq M\|D\phi\|$**.** Let ψ be a smooth even mapping that equals 1 on $[-1/2, 1/2]$, vanishes outside $[-1, 1]$, and satisfies $0 \leq \psi \leq 1$ on $\mathbf{R}$. ψ can be obtained from the general construction in [1.46] of [3]. Alternatively, take the derivative of ϕ from Lemma [A.3], with $\tau = 1/2$, $\omega = 2$, and $A = 1$. We set

$$(2.9) \qquad f_n(t) \triangleq n^3 t \left[\, |t| \leq 1/n \,\right] \qquad (t \in \mathbf{R}).$$

Under the identification $C_c(K) \equiv \mathcal{C}_K$, μ_n reads as the difference of two positive Radon measures μ_n^+ and μ_n^-, since

$$(2.10) \qquad \mu_n(u) = \underbrace{n^3 \int_0^{1/n} tu(t)\,dt}_{\mu_n^+(u)} - \underbrace{n^3 \int_{-1/n}^0 -tu(t)\,dt}_{\mu_n^-(u)} \qquad (u \in \mathcal{C}_K).$$

Thus, μ_n is a signed Radon measure, whose compact support $[-1, 1]$ shrinks to $\{0\}$ as $n \to \infty$. We see that $\|\mu_n\| \leq \|\mu_n^+\| + \|\mu_n^-\|$. However, the sequence $\{\mu_n\}$ is not uniformly bounded, since

$$(2.11) \qquad \|\mu_n\| = \|f_n\|_1 = n = \|\mu_n^+\| + \|\mu_n^-\|.$$

These equalities are justified by [6.19] of [3], but the logistic function $\sigma_\lambda : t \mapsto 1/(1 + \exp(-\lambda t))$ provides a direct proof of them. Indeed, this function is a standard smooth approximation of $H : t \mapsto [t \geq 0]$, the Heaviside step function; see [A.2]. Lebesgue's dominated convergence theorem implies

$$(2.12) \qquad \mu_n(u \cdot (\underbrace{2\sigma_\lambda - 1}_{odd})) = 2n^3 \int_0^{1/n} \underbrace{t\,(2\sigma_\lambda(t) - 1)}_{even}\,dt \xrightarrow[\lambda \to \infty]{} n \qquad (n \geq 2).$$

We first observe that this $\{\mu_n\}$ cannot converge vaguely: this would, by Theorem [2.6] of [3], imply $\sup_n \|\mu_n\| < \infty$, which would contradict (2.11). However, we investigate further to establish weaker convergence and boundedness in τ_K. The mean value theorem implies that, given $\phi \in \mathcal{D}_K$,

$$(2.13) \qquad \phi(1/n) - \phi(-1/n) = \frac{2}{n}\phi'(t_n)$$

for some $t_n = t_n(\phi) \in \,]-1/n, 1/n[$. Moreover, integration by parts yields

$$(2.14) \qquad \mu_n(\phi) = \Lambda_n(\phi) = \frac{n^3}{2}t^2\phi(t)\,\Big|_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt$$

$$(2.15) \qquad = \frac{n}{2}\big(\phi(1/n) - \phi(-1/n)\big) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt$$

$$(2.16) \qquad = \phi'(t_n) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt.$$

Note that when $\phi' = 1$ in a neighborhood of 0, *e.g.,* , for $\phi(t) = \psi(t) \cdot t$, the latter equality reduces to

$$(2.17) \qquad \Lambda_n(\phi) = 1 - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\,dt = \frac{2}{3}.$$

This suggests that continuity of ϕ' dictates $\Lambda_n(\phi) \to \frac{2}{3}\phi'(0)$. We establish this convergence in two steps. First,

$$(2.18) \qquad \Lambda_n(\phi) - \frac{2}{3}\phi'(0) = \phi'(t_n) - \phi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)\,dt + \phi'(0)\underbrace{\frac{n^3}{2}\int_{-1/n}^{1/n} t^2\,dt}_{1/3}$$

$$(2.19) \qquad = \phi'(t_n) - \phi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\big(\phi'(t) - \phi'(0)\big)\,dt.$$

Next, taking absolute values gives

$$(2.20) \qquad \left|\Lambda_n(\phi) - \frac{2}{3}\phi'(0)\right| \leq |\phi'(t_n) - \phi'(0)| + \frac{1}{3}\max_{[-1,1]}|\phi' - \phi'(0)| \xrightarrow[n \to \infty]{} 0.$$

As a result,

$$(2.21) \qquad \Lambda_n(\phi) \xrightarrow[n \to \infty]{} -\frac{2}{3}\delta'(\phi) \qquad (\phi \in \mathcal{D}_K),$$

where $\delta' : \phi \mapsto -\phi'(0)$ is the *derivative* of the *Dirac measure* $\delta : \phi \mapsto \phi(0)$; see [6.1, 6.9] of [3] and [A.2]. The previous reasoning shows that the limit $\Lambda = -\frac{2}{3}\delta'$ is τ_K-continuous. In addition, (2.16) justifies

$$(2.22) \qquad |\Lambda_n(\phi)| \leq |\phi'(t_n)| + \frac{1}{3}\max_{[-1,1]}|\phi'|.$$

A simpler bound is

$$(2.23) \qquad |\Lambda_n(\phi)| \leq \frac{4}{3}\|\phi'\|_\infty.$$

This is a concrete instance of (2.8), with $p = 1$ and $M = 4/3$. To establish (2.4), it suffices to prove that no reduction to order $p = 0$ is possible. To do so, we first assume, to reach a contradiction, that there exists M such that

$$(2.24) \qquad |\Lambda_n(\phi)| \leq M\|\phi\|_\infty \qquad (\phi \in \mathcal{D}_K,\ n = 1, 2, 3, \ldots).$$

Next, we choose

(2.25)
$$\phi_n \triangleq \psi \cdot \tilde{\phi}_n,$$

where $\tilde{\phi}_n$ is ϕ from Lemma [A.3] with $\tau = 1/n = 1/\omega = 1/A$. Hence $\|\phi_n\|_\infty < 2$. In contrast, for $n \geq 4$, $\Lambda_n(\phi_n)$ is

(2.26)
$$\frac{n}{2}\big(\underbrace{\tilde{\phi}_n(1/n)}_{1} - \underbrace{\tilde{\phi}_n(\text{-}1/n)}_{\text{-}1}\big) - \underbrace{\frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2\tilde{\phi}'(t)\,dt}_{\frac{1}{3}n} = \frac{2}{3}n.$$

Our assumption (2.24) then reads as

(2.27)
$$\frac{2}{3}n \leq 2M < \infty.$$

The behavior $\frac{2}{3}n \to \infty$ provides the desired contradiction. Combining (2.21) with (2.27) gives

(2.28)
$$|\Lambda(\phi_n)| \geq |\Lambda_n(\phi_n)| - |\Lambda_n(\phi_n) - \Lambda(\phi_n)| \xrightarrow[n\to\infty]{} \infty.$$

This shows that Λ has no continuity bound for K because $\|\phi_n\|_\infty < 2$ while $|\Lambda_n(\phi_n)| \to \infty$. Therefore, Λ cannot be extended as a Radon measure; see [i], equation (2). A direct way to see this is to pick ϕ_ω from Lemma [A.3], so that

(2.29)
$$\Lambda(\psi \cdot \phi_\omega) = \frac{2}{3}\omega \xrightarrow[\omega\to\infty]{} \infty$$

contrasts with $\|\psi \cdot \phi_\omega\|_\infty = 1$. Thus, we have exhibited a sequence of Radon measures μ_n that

   (a) does not converge vaguely to any Radon measure,

   (b) but converges pointwise on $\mathcal{D}_K$ to $\Lambda \in \mathcal{D}_K^*$, in the specific $\mathcal{D}_K$'s topology; see (2.6), (2.21), and (2.23).

We now propose a second example, with $p = 2$ as the smallest admissible order.

Case of a uniform bound in $\|D^2\phi\|$.   We define the *derivative*

(2.30)
$$\Lambda_n' : \mathcal{D}_K \to \mathbf{C}$$
$$\phi \mapsto \text{-}\Lambda_n(\phi');$$

see [6.1] of [3]. We have proved that every Λ_n is continuous. So is the derivative operator in $\mathcal{D}_K$; see Exercise [1.17]. Therefore, Λ_n' is continuous. Applying (2.21) with ϕ' yields

(2.31)
$$\Lambda_n'(\phi) \xrightarrow[n\to\infty]{} \text{-}\frac{2}{3}\phi''(0).$$

Furthermore, Theorem [2.8] of [3] implies that the limit $\text{-}\frac{2}{3}\phi''(0)$ is τ_K continuous. Additionally, it follows from (2.23) that the bound (2.5) is

(2.32)
$$|\Lambda_n'(\phi)| \leq \frac{4}{3}\|\phi''\|_\infty.$$

To prove this, it suffices to show that 2 is the smallest suitable p. First, we assume, to reach a contradiction, that

(2.33)
$$|\Lambda_n(\phi')| \leq M\|\phi'\|_\infty \qquad (\phi \in \mathcal{D}_K, \, n = 1,2,3,\dots).$$

Next, let Φ_n be the antiderivative of ϕ_n that vanishes at -1; see (2.25). The oddness of Φ_n (ψ is even) implies that $\mathrm{supp}(\Phi_n) \subseteq [\text{-}1, 1]$. So, under our assumption,

(2.34)
$$|\Lambda_n'(\Phi_n)| = |\Lambda_n(\Phi_n')| \leq M\|\Phi_n'\|_\infty.$$

Equivalently,

(2.35)
$$|\Lambda_n(\phi_n)| \leq M\|\phi_n\|_\infty,$$

which has already been disproved. To reach one final contradiction, assume that there exists $M = M(0)$ so that

(2.36)
$$|\Lambda_n'(\phi)| \leq M\|\phi\|_\infty \qquad (\phi \in \mathcal{D}_K, \, n = 1,2,3,\dots).$$

Lemma [A.3] implies that

(2.37)
$$|\Lambda_n'(\phi)| \leq M\|\phi'\|_\infty.$$

This contradiction concludes the proof. $\qquad\qquad\square$

6   Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ of a function $f \in L^2(T)$ (T is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all $n \in \mathbf{Z}$ (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f \text{ exists}\}$ is a dense subspace of $L^2(T)$ of the first category.

PROOF. Let $f(\theta)$ stand for $f(e^{i\theta})$ so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$(2.38) \qquad\qquad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We express $\Lambda_n(f)$ as follows:

$$(2.39) \qquad\qquad \Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well-known (and easy to prove) result is

$$(2.40) \qquad\qquad (e_n, e_{n'}) = [n = n'], , \textit{ i.e., }, \{e_n : n \in \mathbf{Z}\} \text{ is an orthormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

$$(2.41) \qquad\qquad \|\Lambda_n\|^2 \overset{2.39}{=} \|e_{-n} + \cdots + e_n\|^2 \overset{2.40}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \overset{2.40}{=} 2n + 1.$$

Now suppose, to reach a contradiction, that

$$(2.42) \qquad\qquad B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, ...\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem [2.5] of [3] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$(2.43) \qquad\qquad \|\Lambda_n\|^2 \overset{2.41}{=} \sqrt{2n + 1} \xrightarrow[n\to\infty]{} \infty.$$

This establishes that B is actually of the first category; and so is its subset $L = \{f \in L^2(T) : \lim_{n\to\infty} \Lambda_n f \text{ exists}\}$. It remains to prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be $\text{span}\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.39) with (2.40) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$(2.44) \qquad\qquad P \subseteq L \subseteq L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.44), that

$$(2.45) \qquad\qquad L^2(T) = \overline{P} = \overline{L}.$$

So ends the proof $\square$

9   Exercise 9. Boundedness without closedness

Suppose X, Y, Z are Banach spaces and

$$B : X \times Y \to Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ such that

$$\|B(x, y)\| \leq M \|x\| \|y\| \qquad (x \in X, y \in Y).$$

Is completeness needed here?

PROOF. Completeness is not required. To show this, we only assume that X, Y, and Z are normed spaces. Since B is continuous, there exists $r > 0$ such that

$$(2.46) \qquad \|B(x,y)\| < 1$$

whenever $\|x\| + \|y\| < r$. Now, for general $(x,y) \in X \times Y$ and for all $0 < r - s < \min(1, r)$, we substitute:

$$(2.47) \qquad x_s \triangleq \frac{s}{2} \cdot \frac{x}{\|x\| + r - s} \qquad \text{(for x)},$$

$$(2.48) \qquad y_s \triangleq \frac{s}{2} \cdot \frac{y}{\|y\| + r - s} \qquad \text{(for y)},$$

in (2.46). A straightforward computation shows that, at fixed (x, y),

$$(2.49) \qquad \|B(x,y)\| = \frac{4}{s^2}\Big(\|x\| + r - s\Big)\Big(\|y\| + r - s\Big) \underbrace{\|B(x_s, y_s)\|}_{<1} < \frac{4}{s^2}\Big(\|x\|\|y\| + O(r-s)\Big) \xrightarrow[s\to r]{} \frac{4}{r^2}\|x\|\|y\|;$$

which establishes the existence of a bound $M = 4/r^2$. To illustrate this result, we now turn to a concrete example.

Take $X = Y = Z = C_c(\mathbf{R})$ equipped with $\|\cdot\|_\infty$, which is noncomplete; see [5.4.4] of [4]. Nevertheless, the following bilinear mapping

$$(2.50) \qquad \begin{aligned} B : C_c(\mathbf{R})^2 &\to C_c(\mathbf{R}) \\ (f,g) &\mapsto f \cdot g \end{aligned}$$

is continuous. In addition, B has continuity bound $M = 1$, since

$$(2.51) \qquad \|f \cdot g\|_\infty \le \|f\|_\infty \cdot \|g\|_\infty.$$

To prove the continuity of B at every (f, g), fix $\varepsilon > 0$ and let r satisfy

$$(2.52) \qquad 0 < r < \min\Big(1, \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty}\Big).$$

Next, choose $(u, v) \in C_c(\mathbf{R})^2$ such that

$$(2.53) \qquad \underbrace{\|u\|_\infty - \|f\|_\infty \le \|f - u\|_\infty}_{\text{reversed triangle inequality}} + \|g - v\|_\infty < r.$$

Hence

$$(2.54) \quad \|\overbrace{(f-u)g + u(g-v)}^{fg-uv}\|_\infty \le \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty \quad \text{(by the triangle inequality and (2.51))}$$

$$(2.55) \qquad\qquad\qquad < r \cdot \|g\|_\infty + \Big(r + \|f\|_\infty\Big) \cdot r \qquad \text{(since } \|u\| \le r + \|f\|, \text{ see (2.53))}$$

$$(2.56) \qquad\qquad\qquad < \varepsilon \frac{r + \|f\|_\infty + \|g\|_\infty}{1 + \|f\|_\infty + \|g\|_\infty} \qquad \text{(by (2.52))}$$

$$(2.57) \qquad\qquad\qquad < \varepsilon. \qquad\qquad\qquad \text{(since } r < 1, \text{ see (2.52))}$$

In conclusion,

$$(2.58) \qquad \|B(f,g) - B(u,v)\|_\infty < \varepsilon.$$

Since ε was arbitrary, the continuity of B is established. We have thus constructed a continuous bilinear $B : X^2 \to X$ over a noncomplete normed space X. This concludes the proof. □

10 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

PROOF. Let $B : X_1 \times X_2 \to Z$ be a bilinear mapping that is continuous at $(0,0)$, where X_i ($i \in \{1,2\}$) and Z are topological vector spaces. This implies that, for any balanced open W, X_i contains a balanced open U_i such that

$$(2.59) \qquad\qquad\qquad\qquad B(U_1 \times U_2) \subseteq W.$$

Let a_i be in X_i. Therefore, $a_i \in r_i U_i$ for some positive r_i. We now choose b_i in $a_i + (1 + r_1 + r_2)^{-1} U_i$. Hence

$$(2.60) \qquad\qquad B(b_1, b_2) - B(a_1, a_2) = B(b_1 - a_1, b_2) + B(a_1, b_2) - B(a_1, a_2)$$

$$(2.61) \qquad\qquad\qquad\qquad = B(b_1 - a_1, b_2) + B(a_1, b_2 - a_2)$$

$$(2.62) \qquad\qquad\qquad\qquad = \underbrace{B(b_1 - a_1, b_2 - a_2)}_{\in \frac{1}{(1+r_1+r_2)^2} W} + \underbrace{B(b_1 - a_1, a_2)}_{\in \frac{r_2}{1+r_1+r_2} W} + \underbrace{B(a_1, b_2 - a_2)}_{\in \frac{r_1}{1+r_1+r_2} W}$$

$$(2.63) \qquad\qquad\qquad\qquad \in W + W + W.$$

We conclude that B is continuous at every (a_1, a_2), since W was arbitrary. □

12   Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| \, dt.$$

Put $B(f,g) = \int_0^1 f(t)g(t)dt$, *and show that* B *is a bilinear continuous functional on* $X \times X$ *which is separately continuous but is not continuous.*

PROOF. Let f denote the first variable, g the second one. Remark that

$$(2.64) \qquad\qquad\qquad\qquad |B(f,g)| < \|f\| \cdot \max_{[0,1]} |g| \, ;$$

which is sufficient ([1.18] of [3]) to assert that any $f \mapsto B(f,g)$ is continuous. The continuity of all $g \mapsto B(f,g)$ follows (Define $C(g,f) = B(f,g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. Thus, there exists a positive M such that,

$$(2.65) \qquad\qquad\qquad\qquad |B(f,g)| < M\|f\|\|g\|.$$

Put

$$(2.66) \qquad\qquad\qquad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \qquad\qquad (n = 1, 2, 3, \dots),$$

so that

$$(2.67) \qquad\qquad\qquad\qquad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow[n\to\infty]{} 0.$$

On the other hand,

$$(2.68) \qquad\qquad\qquad\qquad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.68) and (2.65) with (2.67) and so obtain

$$(2.69) \qquad\qquad\qquad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow[n\to\infty]{} 0.$$

Our continuousness assumption is then contradicted. So ends the proof. □

15 Exercise 15. Baire's cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* Y = X.
Hint: Y *must intersect* x + Y *for every* x ∈ X.

PROOF. Let $\{E_n : n = 1, 2, 3, ...\}$ be a collection whose union is $X \setminus Y$. Additionally, assume that every E_n is nowhere dense, *i.e.,* , $V_n = X \setminus \overline{E_n}$ is dense in X. Now pick an arbitrary $x \in X$. Since the translation by x is a self-homeomorphism, $x + V_n$ is open and dense[1].We now apply Baire's theorem twice to establish that

(a) every intersection $W_n = V_n \cap (x + V_n)$ is dense in X,

(b) so is the nonempty intersection $\bigcap_{n=1}^{\infty} W_n$.

Moreover, the intersection $\bigcap_{n=1}^{\infty} V_n$ is disjoint from every E_n. In summary,

$$(2.70) \qquad w \in \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq Y$$

for some w = w(x). Furthermore, w also lies in every $x + V_n$, by (a). This implies

$$(2.71) \qquad w - x \in \bigcap_{n=1}^{\infty} V_n \subseteq Y.$$

Finally, (2.70) and (2.71) together yield

$$(2.72) \qquad x = w - (w - x) \in Y - Y = Y,$$

because Y is a subgroup of X. This establishes that

$$(2.73) \qquad X \subseteq Y,$$

since x was arbitrary. □

16 Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $f : X \to K$ *is a closed subset of* X × K. *Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

PROOF. Choose a sequence $\{x_n : n = 1, 2, 3, ...\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of X × K. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, b = f(a); which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous, see [[A6]] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we present the following counterexample,

$$(2.74) \qquad f : [0, \infty) \to [0, \infty)$$

$$x \mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases}$$

Clearly, f has a discontinuity at 0. In contrast, the graph G of f is closed. To see that, first remark that

$$(2.75) \qquad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b). To be more specific: a = 0 contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and b = 1/a, since $x \mapsto 1/x$ is continuous on R_+. This establishes that $(a, b) \in G_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. □

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[1]Alternatively, observe that $X = x + X \subseteq \overline{x + V_n}$, as a special case of [1.3 (b)] of [3].

Chapter 3

Convexity

3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subseteq X$ an internal point of A if $A - x_0$ is an absorbing set.

(a) *Suppose A and B are disjoint convex sets in X, and A has an internal point. Prove that there is a nonconstant functional Λ such that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) *Show (with $X = \mathbf{R}^2$, for example) that it may not possible to have $\Lambda(A)$ and $\Lambda(B)$ disjoint, under the hypotheses of (a).*

PROOF. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem [3.4] of [3] to establish the existence of a functional $\Lambda : X \to \mathbf{R}$ such that

$$(3.1) \qquad \Lambda \leq p$$

and

$$(3.2) \qquad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \qquad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \qquad (a \in A, b \in B).$$

Hence

$$(3.4) \qquad \Lambda a \leq \Lambda b.$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

$$(3.5) \qquad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$(3.6) \qquad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \qquad .$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

$$(3.7) \qquad \Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1 \qquad ;$$

22

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$(3.8) \qquad S_1 \triangleq \{(x,y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$
$$(3.9) \qquad S_2 \triangleq \{(x,y) \in \mathbf{R}^2 : x > 0, y > 0\},$$
$$(3.10) \qquad A \triangleq S_1 \cup S_2,$$
$$(3.11) \qquad B \triangleq X \setminus A.$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$(3.12) \qquad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A.$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.,*, A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$(3.13) \qquad \Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset \qquad .$$

So ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

11 Exercise 11. Meagerness of the polar

Let X *be an infinite-dimensional Fréchet space. Prove that* X^*, *with its weak*-topology, is of the first category in itself.*

This is actually a consequence of the lemma below, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f* X *is an infinite dimensional topological vector space whose dual* X^* *separates points on* X, *then the polar*

$$(3.14) \qquad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \ on \ A\}$$

of any absorbing subset A is a weak*-closed set that has empty interior.

PROOF. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous, see [3.14] of [3]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see [3.11] of [3]). Put

$$(3.15) \qquad W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \qquad (r_x > 0)$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$(3.16) \qquad M \triangleq \text{span}(F)$$

---

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] of [3] or [2.5] of [2].

is finite dimensional. Assume, to reach a contradiction, that $A \subseteq M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subset M$: Now pick a in $A \setminus M$ and so conclude that

$$(3.17) \qquad\qquad\qquad\qquad\qquad b \triangleq \frac{a}{t_a} \in A$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see [1.21 (b)] of [3] for a proof): By the Hahn-Banach theorem [3.5] of [3], there exists Λ_a in X^* such that

$$(3.18) \qquad\qquad\qquad\qquad\qquad \Lambda_a b > 2$$

and

$$(3.19) \qquad\qquad\qquad\qquad\qquad \Lambda_a(M) = \{0\}.$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$(3.20) \qquad\qquad\qquad\qquad\qquad |\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

$$(3.21) \qquad\qquad\qquad\qquad\qquad \Lambda + W \not\subset K_A.$$

Since W and Λ are both arbitrary, this achieves the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now give a proof of the original statement.

Corollary.   *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

PROOF. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$(3.22) \qquad\qquad B_n \triangleq \{x \in X : d(0, x) < 1/n\} \qquad\qquad (n = 1, 2, 3, \dots)$$

form a local base of X. If Λ is in X^*, then

$$(3.23) \qquad\qquad\qquad\qquad\qquad |\Lambda| \leq m \text{ on } B_n$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$, see [1.18] of [3]. Hence, X^* is the countable union of all

$$(3.24) \qquad\qquad\qquad\qquad m \cdot K_n \qquad\qquad (m, n = 1, 2, 3, \dots),$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see [3.4] of [3]. As a consequence, the above lemma implies

$$(3.25) \qquad\qquad\qquad\qquad\qquad \left(\overline{K_n}\right)^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by m is a homeomorphism (see [1.7] of [3]), this is equivalent to

$$(3.26) \qquad\qquad\qquad\qquad\qquad \left(\overline{m \cdot K_n}\right)^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

1   Exercise 1. Basic results

Let ϕ be the embedding of X into X^{**} decribed in Section 4.5. Let τ be the weak topology of X, and let σ be the weak*- topology of X^{**}- the one induced by X^*.

(a) *Prove that ϕ is a homeomorphism of (X, τ) onto a dense subspace of (X^{**}, σ).*

(b) *If B is the closed unit ball of X, prove that $\phi(B)$ is σ-dense in the closed unit ball of X^{**}. (Use the Hahn-Banach separation theorem.)*

(c) *Use (a), (b), and the Banach-Alaoglu theorem to prove that X is reflexive if and only if B is weakly compact.*

(d) *Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.*

(e) *If X is reflexive and Y is a closed subspace of X, prove that X/Y is reflexive.*

(f) *Prove that X is reflexive if and only X^* if reflexive.*
   Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\phi(X)$ of X^{**}.

PROOF. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

(4.1) $$\phi^{-1} : \phi(X) \to X$$
(4.2) $$\phi(x) \mapsto x$$

is a homeomorphism (with respect to τ and σ). We first consider

(4.3) $$V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}|\psi x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0);$$
(4.4) $$U \triangleq \{x \in X : |\langle x|x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0).$$

and remark that the so defined V 's (respectively U 's) shape a local subbase $\mathcal{S}_\sigma$ (respectively $\mathcal{S}_\tau$) of σ (respectively τ). We now observe that

(4.5) $$U = \phi^{-1}(V \cap \phi(X)) = \phi^{-1}(V) \cap X \qquad (V \in \mathcal{S}_\sigma , \ U \in \mathcal{S}_\tau) \qquad ,$$

since ϕ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathcal{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that ϕ^{-1} and its inverse ϕ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

(4.6) $$\overline{\phi(X)}_\sigma \subseteq X^{**} \qquad .$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\phi(X)$. By [3.5], there exists y^* in X^* such that

$$(4.7) \qquad\qquad \langle \phi x, \psi y^* \rangle = \langle y^*, \phi x \rangle = \langle x, y^* \rangle = 0 \qquad (x \in X) \qquad ;$$

$$(4.8) \qquad\qquad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \qquad\qquad \phi(B) \subseteq B^{**} \qquad ,$$

since ϕ is isometric. Hence

$$(4.10) \qquad\qquad \overline{\phi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} \qquad .$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\phi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$(4.11) \qquad\qquad |\psi y^*| \le 1 \qquad \text{on } \overline{\phi(B)}_\sigma \qquad ;$$

$$(4.12) \qquad\qquad \langle z^{**}, \psi y^* \rangle > 1 \qquad .$$

It follows from (4.11) that

$$(4.13) \qquad\qquad |\psi y^*| \le 1 \text{ on } \phi(B), \text{ ie, } |y^*| \le 1 \text{ on } B \qquad .$$

We have so proved that

$$(4.14) \qquad\qquad y^* \in B^* \qquad .$$

Since z^{**} lies in B^{**}, it is now clear that

$$(4.15) \qquad\qquad |\langle z^{**}, \psi y^* \rangle| \le 1 \qquad ;$$

what it contradicts (4.12), and thus proves (b). We now prove (c).

It follows from (a) that

$$(4.16) \qquad\qquad B \text{ is weakly compact if and only if } \phi(B) \text{ is weak*-compact.}$$

If B is weakly compact, then $\phi(B)$ is weak*-closed. So,

$$(4.17) \qquad\qquad \phi(B) = \overline{\phi(B)}_\sigma \overset{(b)}{=} B^{**} \qquad .$$

ϕ is therefore onto, *i.e.,* , X is reflexive.
Conversely, keep ϕ as onto: one easily checks that $\phi(B) = B^{**}$. The image $\phi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(4.18) \qquad\qquad (X/Y)^* \equiv Y^\perp \qquad ,$$

next,

$$(4.19) \qquad\qquad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \qquad .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z, *the weak*-topology of* Z^* *is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\phi(X)$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\phi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \phi(X)$ is weakly compact, *i.e.,* , weak*-compact (to see this, apply (L) with $Z = X^*$).

By (a), B is therefore weakly compact, *i.e.,* , X is reflexive, see (c). So ends the proof. □

13   Exercise 13. Operator compactness in a Hilbert space

(a) *Suppose* $T \in \mathcal{B}(X, Y), T_n \in \mathcal{B}(X, Y)$ *for* n = 1, 2, 3, ... , *each* T_n *has finite-dimensional range, and* $\lim \|T - T_n\| = 0$. *Prove that* T *is compact.*

(b) *Assume* Y *is a Hilbert space, and prove the converse of (a): Every compact* $T \in \mathcal{B}(X, Y)$ *can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)*

PROOF. Since each T_n is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every T_n as a compact operator. Let $\{x_0^i : i \in \mathbf{N}\}$ be in U the open unit ball of X. Since T_1 is compact, $\{x_0^i\}$ contains a subsequence $\{x_1^i : i \in \mathbf{N}\}$ such that $\{T_1 x_1^i\}$ converges to a point y_1 of Y. The same reasoning can be recursively applied to T_n and $\{x_{n-1}^i\} \subseteq U$ so that $\{T_n x_n^i\}$ tends to some y_n of Y, as $\{x_n^i\}$ is a subsequence of $\{x_{n-1}^i\}$. Then

$$(4.20) \qquad\qquad T_n x_p^i \underset{i\to\infty}{\longrightarrow} y_n \qquad (\text{p } n = 1, 2, 3, \cdots) \qquad .$$

Applied with $\{x_n^i : (n, i) \in \mathbf{N}^2\}$, a Cantor's diagonal process therefore provides a subsequence $\{\tilde{x}_j : j \in \mathbf{N}\}$ such that

$$(4.21) \qquad\qquad T_j \tilde{x}_k \underset{k\to\infty}{\longrightarrow} y_j \qquad ;$$

$$(4.22) \qquad\qquad T_j \tilde{x}_j \underset{j\to\infty}{\longrightarrow} y_j \qquad .$$

We now easily obtain

$$(4.23) \qquad \|T_j \tilde{x}_j - T_k \tilde{x}_k\| \le \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \underset{k>j\to\infty}{\longrightarrow} 0 \qquad .$$

$\{T_j \tilde{x}_j\}$ is then a Cauchy sequence. So is $\{T \tilde{x}_j\}$, since $\|T - T_j\| \to 0$. On the other hand, Y is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix ε as a positive scalar. Since T is compact, Y contains a finite set C such that

$$(4.24) \qquad\qquad T(U) \subseteq \bigcup_{c \in C} B(c, \varepsilon) \qquad .$$

As a Hilbert space, Y contains a maximal orthonormal set (or Hilbert basis) M. This implies that span(M) is dense in Y; *cf.* 4.18 & [4.22] of [2]. The finiteness of C forces M to enclose a finite set S so that

$$(4.25) \qquad\qquad \forall c \in C, \exists s(c) \in \text{span}(S) : \|c - s(c)\| < \varepsilon \qquad .$$

Let x be in U. It follows from (4.24) that

$$(4.26) \qquad\qquad \|Tx - c_x\| < \varepsilon$$

for some c_x of C. We now combine (4.25) and (4.26) to obtain

$$(4.27) \qquad\qquad \|Tx - s(c_x)\| \le \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon$$

As a finite-dimensional subspace, span(S) is closed (see footnote 4, Exercise 1.10). We so obtain

$$(4.28) \qquad\qquad Y = \mathrm{span}(S) \oplus \mathrm{span}(S)^{\perp} \qquad,$$

by [12.4]. There exists a unique projection $\pi = \pi(\varepsilon)$ of Y onto itself (see [5.6] for the definition) such that

$$(4.29) \qquad\qquad \pi(Y) = \mathrm{span}(S), \ (I - \pi)(Y) = \mathrm{span}(S)^{\perp} \qquad.$$

It is easily checked that π has norm 1. Moreover,

$$(4.30) \qquad\qquad \pi s = s \qquad (s \in \mathrm{span}(S)) \qquad.$$

Thus,

$$(4.31) \qquad\qquad (I - \pi)(Tx) = (I - \pi)(Tx - s(c_x)) \qquad (x \in U) \qquad.$$

Then,

$$(4.32) \qquad\qquad \|(I - \pi)(Tx)\| \leq \|I - \pi\|\,\|Tx - s(c_x)\| < 4\varepsilon \qquad (x \in U)$$

(the fact that π has norm 1 is hidden in the right side inequality). We have just so proved that

$$(4.33) \qquad\qquad \|T - \pi \circ T\| \in \underset{\varepsilon \sim 0}{O}(\varepsilon) \qquad.$$

That is particularly true when $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, ..., \varepsilon_n \underset{n \to \infty}{\longrightarrow} 0$. Let so T_n be $\pi(\varepsilon_n) \circ T$ and conclude that these (compact) operators approximate T in the desired fashion, *i.e.,* ,

$$(4.34) \qquad\qquad \|T - T_n\| \underset{n \to \infty}{\longrightarrow} 0 \qquad.$$

$\square$

15   Exercise 15. Hilbert-Schmidt operators

Suppose μ is a finite (or σ-finite) positive measure on a measure space Ω, $\mu \times \mu$ is the corresponding product measure on $\Omega \times \Omega$, and $K \in L^2(\mu \times \mu)$. Define

$$(Tf)(s) = \int_{\Omega} K(s, t) f(t) d\mu(t) \qquad [f \in L^2(\mu)].$$

(a) Prove that $T \in \mathcal{B}(L^2(\mu))$ and that

$$\|T\|^2 \leq \int_{\Omega} \int_{\Omega} |K(s, t)|^2 d\mu(s) d\mu(t).$$

(b) Suppose a_i, b_i are members of $L^2(\mu)$, for $1 \leq i \leq n$, put $K_1 = \sum a_i(s) b_i(t)$ and define T_1 in terms of K_1 a T was defined in terms of K. Prove that $\dim \mathcal{R}(T_1) \leq n$.

(c) Deduce that T is a compact operator in $L^2(\mu)$. Hint: Use exercise 13.

(d) Suppose $\lambda \in \mathbf{C}$, $\lambda \neq 0$. Prove: Either the equation

$$Tf - \lambda f = g$$

has a unique solution $f \in L^2(\mu)$ for every $g \in L^2(\mu)$ or there are infinitely many solutions for some g and none for others. (This is known as the Fredholm alternative.).

(e) Describe the adjoint of T.

PROOF. Let X (respectively P) be the Banach space $L^2(\mu)$ (respectively $L^2(\mu \times \mu)$). A consequence of the Radon-Nikodym theorem (*cf.* [6.16] of [2]) is that there exists a group isomorphism $\rho : X \to X^*$, $f \mapsto f^*$ such that

$$(4.35) \qquad \langle u, f^* \rangle = \int_\Omega u \cdot f \, d\mu \qquad (u \in X, f \in X) \qquad .$$

Define a.e K_s, $K_t : \Omega \to \mathbf{C}$ by setting

$$(4.36) \qquad K_s(t) \triangleq K_t(s) \triangleq K(s, t) \ \ \text{a.e} \qquad ((s, t) \in \Omega) \qquad .$$

T is clearly linear. Moreover,

$$(4.37) \qquad |(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \qquad (\|f\|_X < 1)$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with $|K|^2$ to obtain

$$(4.38) \qquad \|Tf\|_X^2 \leq \int_\Omega \|K_s\|_X^2 \ \mu(s) = \|K\|_P^2 < \infty \qquad (\|f\|_X < 1) \qquad .$$

(a) is then proved.

To show (b), remark that

$$(4.39) \qquad \int_\Omega a_i(s) \cdot b_i \cdot f \, d\mu \ \in \mathbf{C} \cdot a_i(s) \ \ \text{a.e} \qquad (f \in X, s \in \Omega) \qquad .$$

It is now clear that T maps any f of X into $\mathbf{C} \cdot a_1 + \cdots + \mathbf{C} \cdot a_n$. We so conclude that $\dim R(T_1) \leq n$.

We now prove (c). The current part refers to Exercise 4.13. X is also a Hilbert space and so contains a Hilbert basis M. Define a.e

$$(4.40) \qquad \begin{aligned} a_b : \Omega &\to \mathbf{C} \\ s &\mapsto (K_s, b) \end{aligned}$$

whenever b ranges M. Hence,

$$(4.41) \qquad K_s = \sum_{b \in M} a_b(s) \cdot b \ \ \text{a.e} \qquad (s \in \Omega) \qquad .$$

Provided any positive scalar ε, there exists a finite subset $S = S(\varepsilon)$ of M such that

$$(4.42) \qquad \left\| K_s - \sum_{b \in S} a_b(s) \cdot b \right\|_X < \varepsilon \qquad (s \in \Omega) \qquad .$$

Remark that $\sum_{b \in S} a_b \cdot b$ matches the definition of K_1; *cf.* (b): from now on,

$$(4.43) \qquad K_1 \triangleq \sum_{b \in S} a_b \cdot b \qquad .$$

It follows from (b) that

$$(4.44) \qquad \dim R(K_1) < \infty \qquad .$$

Now turn back to (a), with $K - K_1$ playing the role of K, and so obtain

$$(4.45) \qquad \|T - T_1\| < \varepsilon \mu(\Omega) \leq \infty \qquad .$$

For if μ is finite, use (a) of Exercise 4.13 to conclude that T is compact. Assume henceforth that μ is not (necessarily) finite and pick δ in $\mathbf{R}_+$. The simple functions (with finite measure support) form a dense family of an L^p space $(1 \leq p < \infty)$; *cf.* [3.13] of [2]. It then exists a simple function K_δ of $L^2(\mu \times \mu)$ such that

$$(4.46) \qquad (\mu \times \mu)(\{K_\delta \neq 0\}) < \infty, \ \|K - K_\delta\|_P < \delta \qquad .$$

Define an operator T_δ in terms of K_δ as T was defined in terms of K, and proceed as in (a) with $T - T_\delta$ instead of T. Then

(4.47) $\|T - T_\delta\| < \delta$.

The key ingredient is that K_δ can be identified with an element of the finite measure space $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$. What we have attempted to approximate T by T_1 can therefore be reiterated (with K_δ playing the role of K) to achieve an approximation $T_{\delta,1}$ of T_δ so that

(4.48) $\|T_\delta - T_{\delta,1}\| < \varepsilon$.

It now follows from (4.47) and (4.48) that

(4.49) $\|T - T_{\delta,1}\| \leq \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta$.

Since ε and δ were arbitrary, the σ-finite case is proved. We now establish (d).

Provided g of X, let E_g be the following equation on X

(4.50) $Tf - \lambda f = g$,

whose solution set is denoted by S_g. Note that S_0 is $\ker(T - \lambda)$ and discard the trivial case $S_0 = X$ [1]: each f of X lies in $S_{Tf-\lambda f}$, as some $Tf - \lambda f$'s are nonzero. Some S_g's are then nonempty. Remark that

(4.51) $S_g = f + S_0$ $(f \in S_g)$

in such case. Furthermore, the equality $\beta = \alpha$ of [4.25] yields

(4.52) $(T - \lambda I)(X) \neq X,, \ i.e., , \ S_0 \neq \{0\}$.

So if $T - \lambda I$ is not onto, not only some S_g's are empty, but also $S_0 \neq \{0\}$. Every nonempty S_g (such sets always exist, see above) is then infinite, by (4.51).
Otherwise, $T - \lambda I$ is bijective and every equation E_g has then a unique solution f. The Fredholm alternative is so proved.

Our last step is the description of T^*. Let $S : X \to X$ be such that

(4.53) $(Sf)(t) \triangleq \int_\Omega K_t \cdot f$ a.e $(f \in X, t \in \Omega)$

Proceed as in (a), with S instead of T: S lies in $\mathcal{B}(X)$. Next, we claim that

(4.54) $\langle u, T^* f^* \rangle = \langle Tu, f^* \rangle$

(4.55) $= \int_\Omega (Tu) \cdot f \ d\mu$

(4.56) $= \int_{\Omega^2} K \cdot f \cdot u \ d(\mu \times \mu)$

(4.57) $= \int_\Omega (Sf) \cdot u \ d\mu$

(4.58) $= \langle u, (Sf)^* \rangle$,

whenever u and f run through the closed unit ball of X. Since $\|T\|$, $\|T^*\|$ are equal and finite, only exactness of (4.56) is possibly in doubt; the justification below dissipates it. In conclusion,

(4.59) $T^* = \rho S \rho^{-1}$.

Informally,

(4.60) $T^* = S$.

_____

[1] For instace, $X = L^2(\{0\}, \delta)$ so that $I = \lambda^{-1}T$ is compact.

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep u and f as above. Let us introduce

$$
(4.61) \qquad A(s) \triangleq \int_{\Omega} |K_s(t) \cdot u(t)| \ d\mu(t) \quad \text{a.e} \qquad (s \in \Omega) \qquad ,
$$

to make hold the following Cauchy-Schwarz inequality

$$
(4.62) \qquad A(s) \leq \|K_s\|_X \qquad (s \in \Omega) \qquad .
$$

Thus,

$$
(4.63) \qquad \int_{\Omega^2} |K(s, t) \, u(t) \, f(s)| \ d\mu(s) \, d\mu(t) = \int_{\Omega} |f(s)| \, A(s) \ d\mu(s)
$$

$$
(4.64) \qquad \qquad \leq \int_{\Omega} |f(s)| \, \|K_s\|_X \ d\mu(s)
$$

$$
(4.65) \qquad \qquad \leq \left[\int_{\Omega} \|K_s\|_X^2 \ d\mu(s) \right]^{\frac{1}{2}} = \|K\|_P < \infty \qquad .
$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof.   □

Chapter 6

Distributions

1 Exercise 1. Test functions are almost polynomial

Suppose f *is a complex continuous function in* $\mathbf{R}^n$*, with compact support. Prove that* $\psi P_j \to f$ *uniformly on* $\mathbf{R}^n$*, for some* $\psi \in \mathcal{D}$ *and for some sequence* $\{P_j\}$ *of polynomials.*

PROOF. According to 1.16, Ω is union of a compact sets sequence $\{K_i\}$ and supp(f) lies in some $K = K_i$ so that f is embedded in $\mathcal{D}(\Omega)$. We can apply [1.10] to ensure that Ω encloses a compact set $S = \overline{K + B(\varepsilon)}$ for sufficiently small $\varepsilon > 0$.

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra $\{g \in C(S) : g \text{ is polynomial}\}$ of $C(S)$. There exists a sequence $\{P_j : j \in \mathbf{N}\}$ of $\mathbf{R}[X_1, \dots, X_n]$ such that

$$(6.1) \qquad \sup_S |f - P_j| \xrightarrow[j\infty]{} 0 \qquad .$$

By [6.20], the open set $K + B(\varepsilon)$ has a local partition of unity $\{\psi_i\} \subseteq \mathcal{D}(\Omega)$. Moreover, there exists an integer l such that $\psi = \psi_1 + \cdots + \psi_l$ equals 1 on K. Hence

$$(6.2) \qquad \|f - \psi P_j\|_\infty = \|\psi f - \psi P_j\|_\infty = \sup_S |\psi f - \psi P_j|$$

$$(6.3) \qquad = \sup_S |f - P_j| \xrightarrow[j\infty]{(6.1)} 0 \qquad .$$

$\square$

We will actually prove more by showing that $\mathcal{D}(\Omega)$ is separable for each nonempty open subset Ω of $\mathbf{R}^n$.

PROOF. The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the separability of $\mathcal{D}(\Omega)$. To do so, we keep (α, j) in $\mathbf{N}^n \times \mathbf{N}$. Remark that S encloses supp($D^\alpha f$): according to the first part, there exists a sequence $\{P_{\alpha,j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \dots, X_n]$ such that

$$(6.4) \qquad \|D^\alpha f - \psi P_{\alpha,j}\|_\infty \xrightarrow[j\infty]{} 0 \qquad .$$

Now let m range over $\{1, 2, 3, \dots\}$ and set $W_{m,j}$ in $\mathcal{D}(\Omega)$ as follows

$$(6.5) \qquad D^{-\alpha}\phi \in \mathcal{D}(\Omega) : D^\alpha D^{-\alpha}\phi = \phi \qquad .$$

$$(6.6) \qquad W_{m,j}(x) \triangleq D^{-(m,\dots,m)}(\psi P_{(m,\dots,m),j})$$

By (6.4), there exists a natural number k(m) such that

$$(6.7) \qquad \|D^{(m,\dots,m)}(f - W_{m,j})\|_\infty < 1/m \qquad (j \geq k(m)) \qquad .$$

Assume without loss of generality that S has diameter 1 so that (6.7) yields

$$(6.8) \qquad \|D^\lambda(f - W_{m,k(m)})\|_\infty < 1/m \qquad (|\lambda| \leq m) \qquad ,$$

by the mean value theorem. In other words (remark that $\text{supp}(f - W_{m,k(m)}) \subseteq S$),

$$(6.9) \qquad f - W_{m,k(m)} \in U_m \triangleq \{\phi \in \mathcal{D}_S : \|\phi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \cdots \qquad (m = 1, 2, 3, \dots) \qquad .$$

Pick W in β (see (b) of [6.3]): $W \cap \mathcal{D}_S$ contains a neighborhood of 0. Hence W contains some U_m, for m sufficiently large. Thus

$$(6.10) \qquad W_{m,k(m)} \underset{m\infty}{\longrightarrow} f \qquad (\text{in } \mathcal{D}(\Omega)) \qquad .$$

We have so established that the $W_{m,k(m)}$'s family is dense in $\mathcal{D}(\Omega)$. We now aim to disclose a countable set $\tilde{W}$ that has the same property.

Choose δ in $\mathbf{R}_+$ and fetch any $W_{m,k(m)}$. Let X be $(X_1, \dots, X_n)$ and express $P_{(m,\dots,m),k(m)}$ as

$$(6.11) \qquad P(X) = \sum_{|\gamma| \leq d} p_\gamma \cdot X^\gamma \qquad .$$

Since $\bar{\mathbf{Q}} = \mathbf{R}$, $\mathbf{Q}[X]$ hosts some $Q(X) = \sum_{|\gamma| \leq d} q_\gamma \cdot X^\gamma$ such that $|p_\gamma - q_\gamma| < \delta$ for all γ. Thus,

$$(6.12) \qquad |P(x) - Q(x)| \leq \sum_{|\gamma| \leq d} |p_\gamma - q_\gamma| \, |x|^{|\gamma|} \leq \delta \sum_{l \leq d} \binom{l+n-1}{n-1} \|x\|_\infty^l \qquad (x \in \mathbf{R}^n) \qquad .$$

Since S is bounded, we so obtain

$$(6.13) \qquad \|\psi(P-Q)\|_\infty \in O(\delta) \qquad .$$

Now define $\tilde{W}_m$ in terms of Q as $W_{m,k(m)}$ was defined in terms of P, and consider the integrations made in (6.6): each $D^\lambda \tilde{W}_m$ $(|\lambda| \leq m)$ can be obtained from some of them. So (6.13) yields

$$(6.14) \qquad \|D^\lambda(W_{m,k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \qquad (|\lambda| \leq m) \qquad .$$

To be more specific, these λ's only exist in finite amount, so the big O can be assumed to be the same for all them. Since δ was arbitrary, combining (6.10) with (6.14) establishes the density of the all $\tilde{W}_m$'s family $\tilde{W}$.

Furthermore, each member of $\tilde{W}$ is only made of two ingredients: ψ and a polynomial of $\mathbf{Q}[X]$. The mapping ψ is attached to some K_i and $\mathbf{Q}[X]$ inherits countableness from $\mathbf{Q}$. Note that the "integrations packs" of (6.6) only exist in countable amount. Our $\tilde{W}$ is then countable. $\qquad \square$

6 Exercise 6. Around the supports of some distributions

(a) *Suppose that* $c_m = \exp\{-(m!)!\}$, $m = 0, 1, 2, \dots$. *Does the series*

$$\sum_{m=0}^{\infty} c_m (D^m \phi)(0)$$

converges for every $\phi \in C^\infty(\mathbf{R})$?

(b) *Let* Ω *be open in* $\mathbf{R}^n$, *suppose* $\Lambda_i \in \mathcal{D}'(\Omega)$, *and suppose that all* Λ_i *have their supports in some fixed compact* $K \subseteq \Omega$. *Prove that the sequence* $\{\Lambda_i\}$ *cannot converge in* $\mathcal{D}'(\Omega)$ *unless the orders of the* Λ_j *are bounded. Hint: Use the Banach-Steinhaus theorem.*

(c) *Can the assumption about the supports be dropped in (b)?*

PROOF. The answer is: no. To establish this assertion, we first assume, to reach a contradiction, that the above series converges for every smooth $\phi : \mathbf{R} \to \mathbf{C}$.

The sequence $\{c_m (D^m \phi)(0)\}$ so converges to 0. Nevertheless, it is proved in [1.46] that $C^\infty(\Omega)$ is not locally bounded. In other words, it is always possible to excavate a ϕ for which the magnitude of the m-th derivative at 0

is as large as we please[1], *e.g.,* , greater than $1/c_m$. A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume $\{\Lambda_j\}$ to converge to some Λ of $\mathcal{D}'(\Omega)$ and we let Q run through the compact sets of Ω. Next, we define

$$(6.15) \qquad S(T, Q) \triangleq \{N \in \mathbf{N}, \exists C \in \mathbf{R}_+ : |T\phi| \leq C \|\phi\|_N \text{ for all } \phi \text{ of } \mathcal{D}_Q\} \qquad (T \in \mathcal{D}(\Omega)) \qquad .$$

Such subset of $\mathbf{N}$ has a minimum $\omega(T, Q)$. The following value

$$(6.16) \qquad \omega(T) \triangleq \max\{\omega(T, Q) : Q \subseteq \Omega, \ Q \text{ compact }\} \leq \infty$$

is then the order of T. Assume, to reach a contradiction, that, after passage to a subsequence,

$$(6.17) \qquad \omega(\Lambda_j, Q_j) = j \qquad (j = 1, 2, 3, \dots)$$

for some compact $Q = Q_j$. By (a) of [6.24], Q_j cuts $\mathrm{supp}\Lambda_j$, say in p_j. Since K encloses $\mathrm{supp}\Lambda_j$, $\{p_j\}$ tends, after passage to a subsequence, to some p of K. Choose a positive scalar r so that

$$(6.18) \qquad \overline{B}(p, r) \triangleq \{x \in \mathbf{R}^n : |x - p| \leq r\} \subseteq \Omega \qquad .$$

Such closed ball $\overline{B}(p, r)$ is a compact subset of Ω. By (b) of [6.5] (which refers to [1.46]) $\mathcal{D}_{\overline{B}(p, r)}$ is then a Fréchet space. It now follows from [2.6] that $\{\Lambda_j\}$ is equicontinuous on $\mathcal{D}_{\overline{B}(p, r)}$. There exists[2] a nonnegative integer N such that

$$(6.19) \qquad |\Lambda\phi| \leq C \|\phi\|_N \qquad (\phi \in \mathcal{D}_{\overline{B}(p, r)})$$

for some positive constant C. On the other hand, $\overline{B}(p, r)$ contains almost all the p_j's. Hence

$$(6.20) \qquad |\Lambda_N \phi| > C \|\phi\|_N$$

for some ϕ of $\mathcal{D}_{\overline{B}(p,r)}$. (b) is then established.

To prove (c), we introduce a sequence $\{x_m : m \in \mathbf{Z}\}$ of Ω that has no limit point. Let $\{\alpha_m : m \in \mathbf{Z}\}$ be in $\mathbf{N}$ and so define[3]

$$(6.21) \qquad \Lambda : \mathcal{D}(\Omega) \to \mathbf{C} \qquad .$$
$$\phi \mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m}\phi)(x_m)$$

Λ belongs to $\mathcal{D}'(\Omega)$, since $\{x_m\}$ has no limit point. Next, we easily check that

$$(6.22) \qquad \Lambda_j : \mathcal{D}(\Omega) \to \mathbf{C} \qquad\qquad (j \in \mathbf{N})$$
$$\phi \mapsto \sum_{|m| \leq j} (D^{\alpha_m}\phi)(x_m)$$

is also a distribution and that $\{\Lambda_j\}$ tends to Λ in $\mathcal{D}'(\Omega)$. Nevertheless, no Λ_j's can have common support because $\{x_m\}$ has no limit point. Our assumption can therefore be dropped. $\qquad\qquad\square$

9   Exercise 9. Convergence in $\mathcal{D}$ vs. convergence in $\mathcal{D}'$

(a) *Prove that a set* $E \subseteq \mathcal{D}(\Omega)$ *is bounded if and only if*

$$\sup\{|\Lambda\phi| : \phi \in E\} < \infty$$

for every $\Lambda \in \mathcal{D}(\Omega)$.

---

[1] indeed [1.46] provides sufficient tools for constructive proof of this, see the $\phi_j - \check{\phi}_j$ involved in (??).

[2] For more details, see Exercise 2.3.

[3] As $\Omega = \mathbf{R}$, the case $\alpha_m = m$ is the "counterpart" of the series of (a) and the case $(x_m, \alpha_m) = (m, 0)$ is the Dirac comb.

(b) *Suppose $\{\phi_j\}$ is a sequence in $\mathcal{D}(\Omega)$ such that $\{\Lambda\phi_j\}$ is a bounded sequence of numbers, for every $\Lambda \in \mathcal{D}'(\Omega)$. Prove that some subsequence of $\{\phi_j\}$ converges, in the topology of $\mathcal{D}(\Omega)$.*

(c) *Suppose $\{\Lambda_j\}$ is a sequence in $\mathcal{D}'(\Omega)$ such that $\{\Lambda_j\phi\}$ is bounded, for every $\phi \in \mathcal{D}(\Omega)$. Prove that some subsequence of $\{\Lambda_j\}$ converges in $\mathcal{D}'(\Omega)$ and that the convergence is uniform on every bounded subset of $\mathcal{D}(\Omega)$. Hint: By the Banach-Steinhaus theorem, the restrictions of the Λ_j to $\mathcal{D}_K$ are equicontinuous. Apply Ascoli's theorem.*

PROOF. Since $\mathcal{D}(\Omega)$ is a locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $E = \{\phi_j : j \in \mathbf{N}\}$ is bounded: so is $\overline{E}$. By (c) of [6.5], there exists some $\mathcal{D}_K$ that contains $\overline{E}$. Since $\mathcal{D}_K$ has the Heine-Borel property (see [1.46]), $\overline{E}$ is τ_K-compact. Apply [A4] with the metrizable space $\mathcal{D}_K$ (see [1.46]) to conclude that $\overline{E}$ has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds. $\qquad\square$

Annex

A.1 Vector spaces

Lemma A.1 Vector subspaces as convex and balanced sets.

Given a vector space X, the following are equivalent for any nonempty $S \subseteq X$.

(a) S is a vector subspace of X,

(b) S is balanced, and $S + S \subseteq S$,

(c) S is convex and balanced, and $\lambda S \subseteq S$ for all $\lambda > 1$.

PROOF. It suffices to show that (a) $\to$ (b), (b) $\to$ (c), and (c) $\to$ (a).

Assume (a), which implies $S + S \subseteq S$ and that S is balanced. This establishes (b); we now work under (b) alone. Combining the assumptions of (b), we obtain the convexity of S:

$$(A.1) \qquad\qquad (1-t)\,S + t\,S \subseteq S + S \subseteq S \qquad (0 \le t \le 1).$$

Next, a straightforward induction on $n \in \{2, 3, 4, \dots\}$ shows that

$$(A.2) \qquad\qquad n\,S \subseteq (n-1)\,S + S \subseteq S + S \subseteq S.$$

Indeed, the base case $n = 2$ follows from $S + S \subseteq S$. Now choose $\lambda > 1$. Then

$$(A.3) \qquad\qquad 1 < \lambda \le \lceil \lambda \rceil.$$

The balancedness of S and $n\,S \subseteq S$ together yield

$$(A.4) \qquad\qquad S \subseteq \lambda S \subseteq \lceil \lambda \rceil\, S \subseteq S.$$

In summary,

$$(A.5) \qquad\qquad S = \lambda S,$$

which completes the proof of (b) $\to$ (c).

Finally, assume only (c): For any $(\alpha_1, \alpha_2) \in \mathbf{C}^2$, we have

$$(A.6) \qquad \alpha_1\,S + \alpha_2\,S \subseteq \{1 + |\alpha_1|\} \cdot S + \{1 + |\alpha_2|\} \cdot S \qquad \text{(by balancedness)}$$
$$(A.7) \qquad\qquad\qquad \subseteq S + S \qquad \text{(by the assumption that } \lambda S \subseteq S \text{ for all } \lambda > 1)$$
$$(A.8) \qquad\qquad\qquad \subseteq 2\,S \qquad \text{(by convexity; see (1.8))}$$
$$(A.9) \qquad\qquad\qquad \subseteq S. \qquad \text{(by the assumption that } \lambda S \subseteq S \text{ for all } \lambda > 1)$$

Therefore, S is a vector subspace of X. $\qquad\qquad\square$

A.2 Mean value and bounded derivatives

Lemma A.2 A mean value inequality for higher-order derivatives.

If $\phi \in \mathcal{D}_{[a,b]}$, then

$$(A.10) \qquad \left\| D^k \phi \right\|_\infty \leq \left\| D^p \phi \right\|_\infty \left(\frac{b-a}{2} \right)^{p-k}$$

for all $k \leq p$ in $\mathbf{N}$.

PROOF. First, consider $a < x_0 \leq (a+b)/2$. By the mean value theorem, there exists $a < x_1 < x_0$ such that

$$(A.11) \qquad \phi(x_0) - \underbrace{\phi(a)}_{0} = D\phi(x_1)(x_0 - a).$$

If $p > 1$, repeating the same reasoning, first for $D\phi$, then $D^2\phi$, and so on, yields

$$(A.12) \qquad \phi(x_0) = D^1\phi(x_1)(x_0 - a)$$
$$(A.13) \qquad \qquad = D^2\phi(x_2)(x_1 - a)(x_0 - a)$$
$$\vdots$$
$$(A.14) \qquad \qquad = \underbrace{D^p\phi(x_p)(x_{p-1} - a)}_{D^{p-1}\phi(x_{p-1})} (x_{p-2} - a) \cdots (x_0 - a)$$

for some points $a < x_p < \cdots < x_1 < x_0$. Hence

$$(A.15) \qquad \left| \phi(x_0) \right| \leq \left\| D^p\phi \right\|_\infty \left(\frac{b-a}{2} \right)^p \qquad (p = 0, 1, 2, \dots).$$

Similarly, if $(a+b)/2 < x_0 < b$ (with b playing the role of a), the same inequality holds. Thus,

$$(A.16) \qquad \left| \phi(x_0) \right| \leq \left\| D^p\phi \right\|_\infty \left(\frac{b-a}{2} \right)^p \qquad (a < x_0 < b),$$

which establishes the result when $k = 0$. Finally, applying the latter inequality to $D^k\phi$ in place of ϕ shows that

$$(A.17) \qquad \left\| D^k\phi \right\|_\infty \leq \left\| D^p\phi \right\|_\infty \left(\frac{b-a}{2} \right)^{p-k}$$

for all $0 \leq k \leq p$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Lemma A.3 Higher derivatives cannot be bounded by lower derivatives.

There is no general formula to estimate higher-order derivatives from lower-order derivatives. This immediately implies that no reversed mean value theorem exists.

PROOF. For *angular frequency* $\omega > 1$, we consider

$$(A.18) \qquad \begin{aligned} \phi_\omega : \mathbf{R} &\to [\text{-}1, 1] \\ t &\mapsto \sin(\omega t), \end{aligned}$$

so that

$$(A.19) \qquad \frac{\left\| D^p \phi_\omega \right\|_\infty}{\left\| D^k \phi_\omega \right\|_\infty} = \omega^{p-k} \qquad (0 \leq k < p)$$

is unbounded as p or ω tends to ∞. Note that no pointwise estimation holds either. Indeed, when it exists, the quotient $Q_\omega(t) = \left| \frac{D^p \phi_\omega}{D^k \phi_\omega}(t) \right|$ is ω^{p-k} if p and k have the same parity. Otherwise, $Q_\omega(t)$ is either $\omega^{p-k} \left| \tan(\omega t) \right|$ or $\omega^{p-k} \left| \cot(\omega t) \right|$. In all cases, $Q_\omega(t) \to \infty$ as $\omega \to \infty$ at fixed t. This example rules out any general inequality in the

reversed direction. However, the following smooth example ϕ maintains a constant derivative around 0, which is the simplest possible behavior.

Let $\rho \in C^\infty(\mathbf{R})$ be 1 on $]\text{-}\infty, 0]$, 0 on $[1, \infty[$, and strictly decaying on $]0, 1[$. A standard choice is $\rho = 1 - h$ with

$$(A.20) \qquad h(t) \triangleq \frac{e^{\text{-}1/t}}{e^{\text{-}1/t} + e^{\text{-}1/(1-t)}}$$

for all $0 < t < 1$. Inspired by signal processing, in addition to the angular frequency ω we introduce two positive parameters: maximum amplitude (A) and delay (τ). We now define ϕ as the time-dependent solution of

$$(A.21) \qquad \begin{cases} \phi(0) & = 0 \\ d\phi & = A\rho\big(\omega(|t| - \tau)\big)\, dt. \end{cases}$$

Equivalently, ϕ is odd and, for $s > 0$,

$$(A.22) \qquad \phi(s) = A \int_0^s \rho\big(\omega(t - \tau)\big)\, dt = A \min(s, \tau) + A \int_\tau^{\max(s,\tau)} \rho\big(\omega(t - \tau)\big)\, dt.$$

Hence

$$(A.23) \qquad \|\phi\|_\infty = \phi(\tau + 1/\omega)$$

$$(A.24) \qquad = \tau A + \frac{A}{\omega} \int_0^1 \rho(u)\, du$$

$$(A.25) \qquad < \tau A + \frac{A}{\omega}.$$

The special case $\tau = 1/\omega = 1/A$ is of great interest. Indeed, $D\phi|_{[\text{-}\tau,\tau]} = A \to \infty$ as $\tau \to 0$. In contrast, $\|\phi\|_\infty < 2$. $\quad\square$

A.3   Dirac's impulse, a physicist's detour

Consider a physical example: a particle colliding with a surface that absorbs a unit of energy at impact time $t = 0$. We start with H as the *Heaviside step function* $t \mapsto [t \geq 0]$, so that $H(t)$ indicates whether the particle has contributed its energy by time t. This formalism expresses that

  (a) The energy is transferred by an instantaneous jump at time $t = 0$.

  (b) The energy is conserved over time.

Heuristically, we write,

$$(A.26) \qquad H'(t) = \begin{cases} \infty & (t = 0) \\ 0 & (t \neq 0) \end{cases},$$

$$(A.27) \qquad \int_{\mathbf{R}} H' = 1,$$

These properties cannot coexist in standard calculus. Nevertheless, the informal density H′ describes the *Dirac δ function*. When identified with a positive Borel measure, δ has total mass 1 because

$$(A.28) \qquad \int_{\mathbf{R}} d\delta = \int_{\mathbf{R}} H' = \Big[H \Big]_{\text{-}\infty}^{\infty} = 1.$$

Physically, integrating over time recaptures all the energy. Let W be the observation window, which is adjusted so that either $0 \in W^\circ$ or $0 \notin \overline{W}$. Next, consider all smooth functions ϕ with (nonempty) compact support in W° as test signals. In this formalism, the integral $\int_{\mathbf{R}} \phi\, d\delta$ represents the detector's response to the collision. Lebesgue's dominated convergence theorem ensures

$$(A.29) \qquad \sup_{0 \leq \phi \leq 1} \int_{W^\circ} \phi\, d\delta = \int_{W^\circ} d\delta = [0 \in W^\circ].$$

We now make the model rigorous by eliminating the heuristic H', as follows:

$$(A.30) \qquad \int_{\mathbf{R}} \phi \, d\delta = \int_{\mathbf{R}} \phi \, H' \qquad \text{(generalization of (A.28))}$$

$$(A.31) \qquad = \Big[H\phi \Big]_{-\infty}^{\infty} - \int_{\mathbf{R}} H\phi' \qquad \text{(integration by parts)}$$

$$(A.32) \qquad = - \int_{\mathbf{R}} H\phi' \qquad \text{(supp}(\phi) \text{ is compact)}$$

$$(A.33) \qquad = \phi(0).$$

The key point is that the right-hand side in (A.32) is valid in standard calculus. Moreover, we obtain all filtered responses as the evaluation functional $\phi \mapsto \phi(0)$. This motivates the following definitions:

$$(A.34) \qquad \Lambda_H(\phi) \triangleq \int_{\mathbf{R}} H\phi, \qquad \text{(expresses H)}$$

$$(A.35) \qquad \Lambda'_H(\phi) \triangleq - \int_{\mathbf{R}} H\phi', \qquad \text{(the \emph{weak derivative} of H)}$$

and

$$(A.36) \qquad \delta(\phi) \triangleq \phi(0) = \Lambda'_H(\phi). \qquad \text{(impulse at 0: now } \delta \text{ has a rigorous definition)}$$

The functional $\delta : \phi \mapsto \phi(0)$ represents an instantaneous energy injection at $t = 0$. Extended to all $\phi \in C_c(\mathbf{R})$, δ is a positive Radon measure of norm/total variation $\|\delta\| = 1$ and support $\{0\}$. In the sense of distribution theory, δ is a (tempered) distribution of order 0; see [Chapters 6 and 7] of [3]. Notably, its Borel-measure counterpart is the *Dirac measure*

$$(A.37) \qquad \delta : E \mapsto \big[0 \in E \big]$$

restricted to Borel sets in $\mathbf{R}$. This yields the special case of (12):

$$(A.38) \qquad \int_{\mathbf{R}} \phi \, d\delta = \delta(\phi) = \phi(0).$$

Convolution of δ with the translated signal $\phi_t : s \mapsto \phi(t - s)$ extends (A.38), as follows:

$$(A.39) \qquad [\delta * \phi](t) \triangleq \int_{\mathbf{R}} \phi_t \, d\delta = \delta(\phi_t) = \phi(t).$$

The Radon measure δ now serves as the convolution identity.

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