Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Notations and Conventions

Logic

1. **Halmos' iff.** **iff** is a short for "if and only if".

2. **Definitions (of values) with $\triangleq$.** Given variables a and b, $\mathrm{a} \triangleq \mathrm{b}$ means that a is defined as equal to b.

3. **$\equiv$.** $\mathrm{a} \equiv \mathrm{b}$ means that there exists a "natural" bijection $\to$ that maps a to b; which let us identify a with b. In a metric space context, $\mathrm{a} \equiv \mathrm{b}$ means that $\to$ is isometric.

4. **Definitions (formulæ).** Definitions use the **iff** format. In other words, every definition has a "only if".

5. **Iverson notation.** Given a boolean expression Φ, $[\,\Phi\,]$ returns the truth value of Φ, encoded as follows,

$$[\,\Phi\,] \triangleq \begin{cases} 0 & \text{if } \Phi \text{ is false;} \\ 1 & \text{if } \Phi \text{ is true.} \end{cases}$$

   For example, $[\,1 > 0\,] = 1$ but $\left[\,\sqrt{2} \in \mathbf{Q}\,\right] = 0$.

Topological vector spaces

1. **Product space**

2. **Scalar field.** The usual (complete) scalar field is $\mathbf{C}$. A property, *e.g.* linearity, that is true on $\mathbf{C}$ is also true on $\mathbf{R}$. The complex case is then a *special case* of the real one. Sometimes, this specialization is not purely formal. For example, theorem 12.7 of [3] asserts that, in a Hilbert space H equipped with the inner product $\langle\,\cdot\,|\,\cdot\,\rangle$, every nonzero linear continuous operator T "breaks orthogonality", in the sense that there always exists $\mathrm{x} = \mathrm{x}(\mathrm{T})$ in H that satisfies $\langle \mathrm{Tx}|\mathrm{x}\rangle \neq 0$. The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exists. To see that, consider the 90° rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitely mentioned*, the exension to the real case will always be obvious. So, taking $\mathbf{C}$ as the scalar field shall mean

   > *Instead of letting the scalar field undefined, we choose* $\mathbf{C}$ *for the sake of expessivity. But considering* $\mathbf{R}$ *instead of* $\mathbf{C}$ *would actually make no difference here* .

3. **Finite dimensional spaces**. It may be customary to identify, without loss of generality, any vector space of finite dimension n with $\mathbf{C}^n$. Such identification is relevant in the sense that all vector spaces that share common dimension n are actually topological vector spaces that are homeomorphic each others.

   To see that, let Y run through all n-dimensional subspaces of a given complex topological vector space X. From now on, $\mathbf{C}^n$ is equipped with the Euclian norm topology. It is easy to get an isomorphism f of $\mathbf{C}^n$ onto Y. To do so, we skip the trivial case $n = 0$ then pick a base F_Y of Y. There so exists a one-to-one mapping of $F_{\mathbf{C}^n}$ onto F_Y that extends to an isomorphism $f : \mathbf{C}^n \to Y$. We now use the Section 1.21 of [3] to conclude that f is more specifically an homeomorphism. Given two possibles copies (Y_i, f_i), $(i = 1, 2)$ of such pairs (Y, f), we then obtain the following commutative diagram,

$$\begin{tikzcd} & \mathbf{C}^n \arrow[dr, "f_2"] & \\ Y_1 \arrow[ur, "f_1^{-1}"] \arrow[rr, "\phi"] & & Y_2 \end{tikzcd} \tag{1}$$

   It is now clear that all Y are homeomorphic each other: Since $\mathbf{C}^n$ is locally convex balanced, so is every Y. In other words, each τ_Y is induced by norms $\| \, \|_Y$; see Section 1.39 of [3]. The special case $Y_1 = Y_2$, $\phi : y \mapsto y$ expresses the equivalence of all such $\| \, \|_Y$, in the sense that

$$A\|y\|_{Y_1} \le \|y\|_{Y_2} \le B\|y\|_{Y_1} \quad (y \in Y_1) \tag{2}$$

   for some positive constants A, B. For instance, choose $B = \sup\{\|y\|_2 : \|y\|_1 < 1\}$.

Chapter 1

Topological Vector Spaces

1.1 Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X*. Prove the following statements from the axioms as given as in section 1.4.*

(a) *If* $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ *there is a unique* $\mathrm{z} \in \mathrm{X}$ *such that* $\mathrm{x} + \mathrm{z} = \mathrm{y}$.

(b) $0 \cdot \mathrm{x} = 0 = \alpha \cdot 0 \quad (\alpha \in \mathbf{C}, \mathrm{x} \in \mathrm{X})$.

(c) $2\mathrm{A} \subset \mathrm{A} + \mathrm{A}$.

(d) A *is convex if and only if* $(\mathrm{s}+\mathrm{t})\mathrm{A} = \mathrm{sA} + \mathrm{tA}$ *for all positive scalars* s *and* t.

(e) *Every union (and intersection) of balanced sets is balanced.*

(f) *Every intersection of convex sets is convex.*

(g) *If* Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

(h) *If* A *and* B *are convex, so is* $\mathrm{A} + \mathrm{B}$.

(i) *If* A *and* B *are balanced, so is* $\mathrm{A} + \mathrm{B}$.

(j) *Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.*

Proof. (a) Such property only depends on the group structure of X: Each x in X has an opposite -x. Let x' be any opposite of x, so that $\mathrm{x} - \mathrm{x} = 0 = \mathrm{x} + \mathrm{x}'$. Thus, $\text{-x} + \mathrm{x} - \mathrm{x} = \text{-x} + \mathrm{x} + \mathrm{x}'$, which is equivalent to $\text{-x} = \mathrm{x}'$. So is established the uniqueness of -x. It is now clear that $\mathrm{x} + \mathrm{z} = \mathrm{y}$ **iff** $\mathrm{z} = \text{-x} + \mathrm{y}$, which asserts both the existence and the uniqueness of z.

(b) Remark that

$$0 \cdot \mathrm{x} = (0+0) \cdot \mathrm{x} = 0 \cdot \mathrm{x} + 0 \cdot \mathrm{x} \tag{1.1}$$
$$= (0+0) \cdot \mathrm{x} = 0 + 0 \cdot \mathrm{x} \tag{1.2}$$

then conclude from (a) that $0 \cdot \mathrm{x} = 0$. So,

$$0 = 0 \cdot \mathrm{x} = (1-1) \cdot \mathrm{x} = \mathrm{x} + (\text{-}1) \cdot \mathrm{x} \Rightarrow \text{-}1 \cdot \mathrm{x} = \text{-x}. \tag{1.3}$$

Finally,

$$\alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0, \tag{1.4}$$

which proves (b).

(c) Remark that

$$2x = (1 + 1)x = x + x \tag{1.5}$$

for every x in X, and so conclude that

$$2A = \{2x : x \in A\} = \{x + x : x \in A\} \subset \{x + y : (x, y) \in A^2\} = A + A \tag{1.6}$$

for all subsets A of X; which proves (c).

(d) If A is convex, then

$$A \subset \frac{s}{s+t}A + \frac{t}{s+t}A \subset A; \tag{1.7}$$

which is

$$sA + tA = (s + t)A. \tag{1.8}$$

Conversely, the special case $s + t = 1$ is

$$sA + (1 - s)A = A. \tag{1.9}$$

The latter extends to $s = 0$, since

$$0A + A \overset{(b)}{=} \{0\} + A = A. \tag{1.10}$$

The extension to $s = 1$ is analogously established (or simply use the fact that + is commutative!). So ends the proof.

(e) Let A range over B a collection of balanced subsets, so that

$$\alpha \bigcap B \subset \alpha A \subset A \subset \bigcup B \tag{1.11}$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap B \subset A$ establishes the first part. Now remark that

$$\alpha A \subset \bigcup B \tag{1.12}$$

implies

$$\alpha \bigcup B \subset \bigcup B; \tag{1.13}$$

which achieves the proof.

(f) Let A range over C a collection of convex subsets, so that

$$(s + t) \bigcap C \subset s \bigcap C + t \bigcap C \subset sA + tA \overset{(d)}{\subset} (s + t)A \tag{1.14}$$

for all positives scalars s, t. Inclusions at both extremities force

$$s \bigcap C + t \bigcap C = (s + t) \bigcap C. \tag{1.15}$$

We now conclude from (d) that the intersection of C is convex. So ends the proof.

(g) Skip all trivial cases $\Gamma = \emptyset$, $\{\emptyset\}$, $\{\{x\}\}$, $\{\emptyset, \{x\}\}$ then pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i $(i = 1, 2)$ lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_i's is then in $C_2 \subset \bigcup \Gamma$. Hence (g).

(h) Simply remark that

$$s(A + B) + t(A + B) = sA + tA + sB + tB = (s + t)(A + B) \tag{1.16}$$

for all positive scalars s and t, then conclude from (d) that $A + B$ is convex.

(i) Given any α from the closed unit disc,

$$\alpha(A + B) = \alpha A + \alpha B \subset A + B. \tag{1.17}$$

There is no more to prove: $A + B$ is balanced.

(j) Our proof will be based on the following lemma,

If S *is nonempty, then each of the following three properties*

(i) S *is a vector subspace of* X*;*

(ii) S *is convex balanced such that* $S + S = S$*;*

(iii) S *is convex balanced such that* $\lambda S = S \quad (\lambda > 0)$

implies the other two.

To prove the lemma, let S run through all nonempty subsets of X. First, assume that (i) holds: Clearly, every S is convex balanced. Moreover, $S+S \subset S$. Conversely, $S = S + \{0\} \subset S + S$; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$nS = (n - 1)S + S = S + S = S \quad (n = 1, 2, 3, \dots) \tag{1.18}$$

with the help of (b) and (d). Pick $\lambda > 0$ then choose n so large that $1 < n\lambda < n^2$. Thus,

$$nS \overset{(1.18)}{\subset} S \subset n\lambda S \subset n^2 S, \tag{1.19}$$

since S is balanced. For instance, set $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$. Dividing the latter inclusions by n shows that

$$S \subset \lambda S \subset nS \overset{(1.18)}{\subset} S, \tag{1.20}$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$\alpha S + \beta S \overset{(a)}{=} |\alpha|S + |\beta|S \overset{(d)}{=} (|\alpha| + |\beta|)S \overset{(iii)}{=} S \quad (|\alpha| + |\beta| > 0); \tag{1.21}$$

where the equality at the left holds as S is balanced. Moreover (under the sole assumption that S is balanced), this extends to $|\alpha| + |\beta| = 0$, as follows,

$$\alpha S + \beta S = 0S + 0S \overset{(b)}{=} \{0\} \overset{(b)}{=} 0S \subset S. \tag{1.22}$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let V be a collection of vector spaces of X, of intersection I and union U. First, remark that every member of V is convex balanced: So is I (combine (e) with (f)). Next, let Y range over V, so that

$$\mathrm{I} + \mathrm{I} \subset \mathrm{Y} + \mathrm{Y} \subset \mathrm{Y}; \tag{1.23}$$

which yields

$$\mathrm{I} + \mathrm{I} = \mathrm{I} \tag{1.24}$$

(the fact that $\mathrm{I} = \mathrm{I} + \{0\} \subset \mathrm{I} + \mathrm{I}$ was tacitely used). It now follows from the lemma's (ii) $\Rightarrow$ (i) that I is a vector subspace of X. Now temporarily assume that S is totally ordered by set inclusion: Combining (e) with (g) establishes that U is convex balanced. To show that U is more specifically a vector subspace, we first remark that such total order implies that either $\mathrm{Z} \subset \mathrm{Y}$ or $\mathrm{Y} \subset \mathrm{Z}$, as Z ranges over V. A straightforward consequence is that

$$\mathrm{Y} \subset \mathrm{Y} + \mathrm{Z} \subset \mathrm{Y} \cup \mathrm{Z}. \tag{1.25}$$

Another one is that $\mathrm{Y} \cup \mathrm{Z}$ ranges over V as well. Combined with the latter inclusions, this leads to

$$\mathrm{U} \subset \mathrm{U} + \mathrm{U} \subset \mathrm{U}. \tag{1.26}$$

It then follows from the lemma's (ii) $\Rightarrow$ (i) that U is a vector subspace of X. Finally, let A, B run through all vector subspaces of X: Combining (h) with (i) proves that $\mathrm{A} + \mathrm{B}$ is convex balanced as well. Furthermore,

$$\mathrm{A} + \mathrm{B} \overset{(i) \Rightarrow (ii)}{=} (\mathrm{A} + \mathrm{A}) + (\mathrm{B} + \mathrm{B}) = (\mathrm{A} + \mathrm{B}) + (\mathrm{A} + \mathrm{B}), \tag{1.27}$$

where the equality at the right holds as X is an abelian group. We now conclude from (ii) that any $\mathrm{A} + \mathrm{B}$ is a vector subspace of X. So ends the proof.

$\square$

1.2 Exercise 2. Convex hull

The convex hull of a set A *in a vector space* X *is the set of all convex combinations of members of* A*, that is the set of all sums* $\mathrm{t_1x_1}+\cdots+\mathrm{t_nx_n}$ *in which* $\mathrm{x_i}\in\mathrm{A}$, $\mathrm{t_i}\geq 0$, $\sum\mathrm{t_i}=1$*;* n *is arbitrary. Prove that the convex hull of a set* A *is convex and that is the intersection of all convex sets that contain* A.

Proof. The convex hull of a set S will be denoted by $\mathrm{co(S)}$. Remark that $\mathrm{S}\subset\mathrm{co(S)}$ (to see that, take $\mathrm{t_1}=1$ for each $\mathrm{x_1}$ in S) and that $\mathrm{co(A)}\subset\mathrm{co(B)}$ where $\mathrm{A}\subset\mathrm{B}$ (obvious).

Our proof will directly derive from (i) $\Rightarrow$ (iv) in the following lemma,

> *Let* S *be a subset of a vector space* X*: Its convex hull* $\mathrm{co(S)}$ *is convex and the following statements*
>
> *(i)* S *is convex;*
>
> *(ii)* $\mathrm{s_1S}+\cdots+\mathrm{s_nS}=(\mathrm{s_1}+\cdots+\mathrm{s_n})\mathrm{S}$ *for all positive scalar variables* $s_1,\ldots,s_n$*;*
>
> *(iii)* $t_1\mathrm{S}+\cdots+t_n\mathrm{S}=\mathrm{S}$ *for all positive scalar variables* $s_1,\ldots,s_n$ *such that* $\mathrm{s_1}+\cdots+\mathrm{s_n}=1$*;*
>
> *(iv)* $\mathrm{co(S)}=\mathrm{S}$
>
> *are equivalent.*

From now on, we skip the trivial case $\mathrm{S}=\emptyset$ then only consider nonempty sets. To prove the first part, let a, b range over $\mathrm{co(S)}$, so that $\mathrm{a}=\mathrm{t_1x_1}+\cdots+\mathrm{t_nx_n}$ and $\mathrm{b}=\mathrm{t_{n+1}x_{n+1}}+\cdots+\mathrm{t_{n+p}x_{n+p}}$ for some (t_i,x_i). Every sum $\mathrm{sa}+(1-\mathrm{s})\mathrm{b}$ $(0\leq\mathrm{s}\leq 1)$ is then in the convex hull of $\{x_1,\ldots,x_{n+p}\}$, since

$$\mathrm{sa}+(1-\mathrm{s})\mathrm{b}=\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{st_ix_i}+\sum_{\mathrm{i=n+1}}^{\mathrm{n+p}}(1-\mathrm{s})\mathrm{t_ix_i} \tag{1.28}$$

and

$$\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{st_i}+\sum_{\mathrm{i=n+1}}^{\mathrm{n+p}}(1-\mathrm{s})\mathrm{t_i}=\mathrm{s}\sum_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{t_i}+(1-\mathrm{s})\sum_{\mathrm{i=n+1}}^{\mathrm{n+p}}\mathrm{t_i}=1. \tag{1.29}$$

In terms of sets S, this reads

$$\mathrm{s\,co(S)}+(1-\mathrm{s})\,\mathrm{co(S)}\subset\mathrm{co(S)}; \tag{1.30}$$

which was our fist goal. We now aim at the equivalence (i) $\Rightarrow\cdots\Rightarrow$ (iv) $\Rightarrow$ (i): An easy proof by induction makes the implication (i) $\Rightarrow$ (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii) $\Rightarrow$ (iv) derives from the definition of the convex hull. We now close the chain with (iv) $\Rightarrow$ (i), by remarking that S is convex whether $\mathrm{S}=\mathrm{co(S)}$. The lemma being proved, let us establish the second part.

To do so, we start from the convexity of $\mathrm{co(A)}$ then set $\mathrm{F}=\{\mathrm{co(A)}\}$. We may enrich F as follows,

$$\mathrm{B}\in\mathrm{F}\Rightarrow\mathrm{B}\text{ is convex and contains }\mathrm{A}. \tag{1.31}$$

Note that our initial predicate "[F only encompasses] *all convex sets that contain A*", is now the special case

$$\text{B} \in \text{F} \Leftrightarrow \text{B is convex and contains A.} \tag{1.32}$$

In any case, the key ingredient is that co(A) $\in$ F implies

$$\mathrm{co(A)} \supset \bigcap_{\mathrm{B} \in \mathrm{F}} \mathrm{B}. \tag{1.33}$$

Conversely, the next formula

$$\mathrm{co(A)} \subset \mathrm{co(B)} \overset{\text{(i)}\Rightarrow\text{(iv)}}{=} \mathrm{B} \quad (\mathrm{B} \in \mathrm{F}) \tag{1.34}$$

is valid and implies

$$\mathrm{co(A)} \subset \bigcap_{\mathrm{B} \in \mathrm{F}} \mathrm{B}. \tag{1.35}$$

So ends the proof □

1.3 Exercise 3. Other basic results

Let be X as topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements:

(a) The convex hull of every open set is open.

(b) If X is locally convex then the convex hull of every bounded set is bounded.

(c) If A and B are bounded, so is A+B.

(d) If A and B are compact, so is A+B.

(e) If A is compact and B is closed, then A+B is closed.

(f) The sum of two closed sets may fail to be closed.

Proof. (a) Pick a nonempty open set A then let all variables x_i ($\mathrm{i} = 1, 2, \dots$) range over A, so that, at each i,

$$\mathrm{x_i \in V_i \subset A} \tag{1.36}$$

for some neighborhood $\mathrm{V_i}$ of $\mathrm{x_i}$. Hence

$$\mathrm{\sum t_i x_i \in \sum t_i V_i \subset co(A)} \tag{1.37}$$

at arbitrary convex combination $\mathrm{\sum_i t_i x_i}$. Now remark that $\mathrm{\sum t_i V_i}$ is open; see Section 1.7 of [3]; which achieves the proof (the case $\mathrm{A} = \emptyset$ is trivial).

(b) Provided a bounded set E, pick V a neighbourhood of 0: By (b) of Section 1.14 in [3], V contains a convex neighbourhood of 0, say W. There so exists a positive scalar s such that

$$\mathrm{E \subset tW \subset tV \quad (t > s);} \tag{1.38}$$

which yields

$$\mathrm{co(E) \subset co(tW) = t\,co(W) = tW \subset tV.} \tag{1.39}$$

So ends the proof.

(c) At fixed V, neighbourhood of the origin, we combine the continuousness of + with Section 1.14 of [3] to conclude that there exists U a balanced neighborhood of the origin such that

$$\mathrm{U + U \subset V.} \tag{1.40}$$

Moreover, by the very definition of boundedness, $\mathrm{A \subset rU}$ for some positive scalar r. Similarly, $\mathrm{B \subset sU}$ for some positive s. Finally,

$$\mathrm{A + B \subset rU + sU \subset tU + tU \subset tV \quad (t > r, s),} \tag{1.41}$$

since U is balanced. So ends the proof.

(d) First, A and B are compact: So is $\mathrm{A \times B}$. Next, + maps continuously $\mathrm{A \times B}$ onto $\mathrm{A + B}$. In conclusion, $\mathrm{A + B}$ is compact.

(e) From now on, we assume that neither A nor B is empty, since otherwise the result is trivial. Now pick $c \in X$ outside $A + B$: The result will be established by showing that c is not in the closure of $A + B$.

To do so, we let the variable a range over A: Every set $a + B$ is closed as well; see Section 1.7 of [3]. Trivially, $a + B \neq c$: By Section 1.10 of [3], there so exists $V = V(a)$ a neighborhood of the origin such that

$$(a + B + V) \cap (c + V) = \emptyset. \tag{1.42}$$

Moreover, there are finitely many $a + V$, say $a_1 + V_1, a_2 + V_2, \ldots$, whose union U contains the compact set A. Therefore,

$$A + B \subset U + B. \tag{1.43}$$

Now define

$$W \triangleq V_1 \cap V_2 \cap \cdots, \tag{1.44}$$

so that

$$(a_i + B + V_i) \cap (c + W) \overset{(1.42)}{=} \emptyset \quad (i = 1, 2, \ldots). \tag{1.45}$$

As a conclusion, c is not in the closure of $U + B$. Finally, (1.43) asserts that c is not in $\overline{A + B}$ either; which achieves the proof.

Corollary: If B is the closure of a set S, then

$$A + B \subset \overline{A + S} \subset \overline{A + B} = A + B \tag{1.46}$$

by (b) of Section 1.13 of [3] (since A is closed; see Section 1.12, from the same source). The special case $A = \{x\}$, $B = X$ will occur in the proof of Exercise 15 in chapter 2.

(f) The last proof will consist in exibhiting a counterexample. To do so, let f be any continuous mapping of the real line such that

(i) $f(x) + f(\text{-}x) \neq 0 \quad (x \in \mathbf{R})$;

(ii) f vanishes at infinity.

For instance, we may combine (ii) with f even and $f > 0$ by setting $f(x) = 2^{-|x|}$, $f(x) = e^{-x^2}$, $f(x) = 1/(1 + |x|)$, ..., and so on.

As a continous function, f has closed graph G; see [2.14] of [3]. Moreover, (i) implies that the origin $(0, 0) \neq (x - x, f(x) + f(\text{-}x))$ is not in $G + G$. On the other hand,

$$\{(0, f(n) + f(\text{-}n)) : n = 1, 2, \ldots\} \subset G + G. \tag{1.47}$$

Now the key ingredient is that

$$(0, f(n) + f(\text{-}n)) \xrightarrow[n \to \infty]{(ii)} (0, 0). \tag{1.48}$$

We have so constructed a sequence in $G + G$ that converges outside $G + G$. So ends the proof.

$\square$

1.4 Exercise 4. A nonempty set whose interior is not

Let be $\mathrm{B} = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$. *Show that* B *is balanced but that its interior is not.*

Proof. It is obvious that the nonempty set B contains the origin $(0, 0)$. Additionally, its interior B° is nonempty as well. Indeed, the following set

$$\{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2] < 1/2\} \subset \mathrm{B} \tag{1.49}$$

is a neighborhood of $(1, 2) \in \mathrm{B}$. Moreover, B is balanced, since

$$|\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \quad (|\alpha| \leq 1) \tag{1.50}$$

for all (z_1, z_2) in B. Nevertheless, the nonempty set B° is not balanced, what we now establish by showing that $(0, 0) \notin \mathrm{B}^\circ$. To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$\mathrm{U} \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2] < r\} \subset \mathrm{B} \tag{1.51}$$

for some positive r. Clearly, U contains $(r/2, 0)$, and that special case $(r/2, 0) \in \mathrm{B}$ now contradicts the definition of B. So ends the proof. $\square$

1.5 Exercise 5. A first restatement of boundedness

Consider the definition of "bounded set" given in Section 1.6. Would the content of this definition be altered if it were required merely required that to every neighbourhood V *of 0 corresponds* some $\mathrm{t} > 0$ *such that* $\mathrm{E} \subset \mathrm{tV}$*?*

Proof. The answer is: No. To prove it, start from (a) of Section 1.14: V contains W, a balanced neighbourhood of 0. Assume that E is bounded in this weaker sense, *i.e.* there exists a positive t that satisfies

$$\mathrm{E} \subset \mathrm{tW}. \tag{1.52}$$

Thus,

$$\mathrm{E} \subset \mathrm{tW} \subset \mathrm{sW} \subset \mathrm{sV} \quad (\mathrm{s} > \mathrm{t}), \tag{1.53}$$

since W is balanced. We so reach the definition given in Section 1.6: The two ones are equivalent. $\square$

1.6 Exercise 6. A second restatement of boundedness

Prove that a set E *in a topological vector space is bounded if and only if every countable subset of E is bounded.*

Proof. It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighbourhood of the origin: No counting number $\mathrm{n} = 1, 2, \ldots,$ verifies $\mathrm{E} \subset \mathrm{nV}$ (see Exercise 1 in Chapter 1). In other words, there exists a sequence $\{\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}, \ldots\} \subset \mathrm{E}$ such that

$$\mathrm{x}_\mathrm{n} \notin \mathrm{nV}. \tag{1.54}$$

As a consequence, $\mathrm{x}_\mathrm{n}/\mathrm{n}$ fails to converge to 0 as n tends to ∞. In contrast, $1/\mathrm{n}$ succeeds. It then follows from Section 1.30 that $\{\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}, \ldots\}$ is not bounded. So ends the proof. $\square$

1.7 Exercise 7. Metrizability & number theory

Let be X *the vector space of all complex functions on the unit interval* $[0,1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x) \quad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ *in X such that (a)* $\{f_n\}$ *converges to* 0 *as* $n \to \infty$*, but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

Proof. The family of the seminorms p_x is separating: The collection $\mathscr{B}$ of all finite intersections of the sets

$$V(x,k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0,1], k = 1,2,3,\dots) \tag{1.55}$$

is therefore a local base for a topology τ on X; see Section 1.37 of [3]. So,

$$\sum_{n=1}^{\infty}[f_n \notin \cap_{i=1}^{m} U_i] \le \sum_{n=1}^{\infty}\sum_{i=1}^{m}[f_n \notin U_i] = \sum_{i=1}^{m}\sum_{n=1}^{\infty}[f_n \notin U_i] \quad (f_n \in X, U_i \in \tau). \tag{1.56}$$

Now assume that $\{f_n\}$ τ-converges to some f, *i.e.*

$$\sum_{n=1}^{\infty}[f_n \notin f + W] < \infty \quad (W \in \mathscr{B}). \tag{1.57}$$

The special case $W = V(x,k)$ means that, given k, $|f_n(x) - f(x)| < 2^{-k}$ for almost all n. I other words, $\{f_n(x)\}$ converges to f(x). Conversely, assume that $\{f_n\}$ does not τ-converges in X, *i.e.*

$$\forall f \in X, \exists W \in \mathscr{B} : \sum_{n=1}^{\infty}[f_n \notin f + W] = \infty. \tag{1.58}$$

W is now the (nonempty) intersection of finitely many V(x, k), say $V(x_1,k_1),\dots,V(x_m,k_m)$. Thus,

$$\sum_{i=1}^{m}\sum_{n=1}^{\infty}[f_n \notin f + V(x_i,k_i)] \overset{(1.56)}{\ge} \sum_{n=1}^{\infty}[f_n \notin f + W] \overset{(1.58)}{=} \infty. \tag{1.59}$$

We can now conclude that, for some index i,

$$\sum_{n=1}^{\infty}[f_n \notin f + V(x_i,k_i)] = \infty. \tag{1.60}$$

In other word, $\{f_n(x_i)\}$ fails to converge to $f(x_i)$. We have so proved that τ-convergence is a rewording of pointwise convergence. We now establish the second part by constructing a specific sequence $\{f_n\}$ that satisfies both (a) and (b).

The proof will be based on the following well-known result: Each irrational number α has a *unique* binary expansion. More precisely, there exists a bijection

$$b : [0,1] \setminus \mathbf{Q} \to \{\beta \in \{0,1\}^{\mathbf{N}_+} : \beta \text{ is not eventually periodic}\} \tag{1.61}$$

where $b(\alpha) = (\beta_1, \beta_2, \dots)$ is the only bit stream such that

$$\alpha = \sum_{k=1}^{\infty} \beta_k 2^{-k}. \tag{1.62}$$

First, remark that $b(\alpha)_1 + \cdots + b(\alpha)_n \underset{n\to\infty}{\longrightarrow} \infty$, since $b(\alpha)$ has infinite support. Next, fix

$$f_n(\alpha) \triangleq \frac{1}{b(\alpha)_1 + \cdots + b(\alpha)_n} \underset{n\to\infty}{\longrightarrow} 0 \tag{1.63}$$

wherever $b(\alpha)_1 + \cdots + b(\alpha)_n > 0$. All other values $f_n(x)$ are of no interest. For instance, put $f_n(x) = 0$. Now take an arbitrary $\gamma_n \longrightarrow \infty$: Given any counting number p, γ_n is greater than p for all but finitely many n. Next, we choose n_p among those *almost all* n that are large enough to additionally satisfy

$$n_p - n_{p-1} > p \longrightarrow \infty, \tag{1.64}$$

as $n_0 = 0$. This way, the distribution of $n_1, n_2, \dots,$ *displays no periodic pattern.* In other words, the *characteristic function* $\chi : k \mapsto [k \in \{n_1, n_2, \dots\}]$ is not eventually periodic. Combined with (1.62), this establishes that

$$\alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \tag{1.65}$$

is irrational. Conversely, still with (1.62),

$$b(\alpha_\gamma)_k = \chi_k. \tag{1.66}$$

Moreover, it follows from the very definition of χ that

$$\chi_1 + \cdots + \chi_{n_1} + \cdots + \chi_{n_p} = p. \tag{1.67}$$

Hence

$$\gamma_{n_p} f_{n_p}(\alpha_\gamma) = \frac{\gamma_{n_p}}{p} > 1. \tag{1.68}$$

There so exists a subsequence $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. Since $\{\gamma_n\}$ was arbitrary, this proves (b). □

1.9 Exercise 9. Quotient map

Suppose

(a) X *and* Y *are topological vector spaces,*

(b) $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear.*

(c) N *is a closed subspace of* X*,*

(d) $\pi : \mathrm{X} \to \mathrm{X}/\mathrm{N}$ *is the quotient map, and*

(e) $\Lambda x = 0$ *for every* $x \in \mathrm{N}$*.*

Prove that there is a unique $\mathrm{f} : \mathrm{X}/\mathrm{N} \to \mathrm{Y}$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = \mathrm{f} \circ \pi$ has necessarily a unique solution, which is the binary relation

$$\mathrm{f} \triangleq \{(\pi \mathrm{x}, \Lambda \mathrm{x}) : \mathrm{x} \in \mathrm{X}\} \subset \mathrm{X}/\mathrm{N} \times \mathrm{Y}. \tag{1.69}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda \mathrm{x} \neq \Lambda \mathrm{x}' \Rightarrow \pi \mathrm{x}' \neq \pi \mathrm{x}'. \tag{1.70}$$

It straightforwardly derives from (1.69) that f inherits linearity from π and Λ.

Remark. The special case $\mathrm{N} = \{\Lambda = 0\}$, *i.e.* $\Lambda \mathrm{x} = 0$ **iff** $\mathrm{x} \in \mathrm{N}$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$\mathrm{f}(\pi \mathrm{x}) = 0 \overset{(1.69)}{\Rightarrow} \Lambda \mathrm{x} = 0 \Rightarrow \mathrm{x} \in \mathrm{N} \Rightarrow \pi \mathrm{x} = \mathrm{N} \tag{1.71}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = \mathrm{f} \circ \pi$, by 1.41 (a) of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_{Y} there exists a neighborhood U of 0_{X} such that

$$\Lambda(\mathrm{U}) = \mathrm{f}(\pi(\mathrm{U})) \subset \mathrm{V}. \tag{1.72}$$

Since π is open (1.41 (a) of [3]), $\pi(\mathrm{U})$ is a neighborhood of $\mathrm{N} = 0_{\mathrm{X}/\mathrm{N}}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = \mathrm{f} \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{\mathrm{X}/\mathrm{N}}$ satisfies

$$\mathrm{W} = \pi(\mathrm{V}) \tag{1.73}$$

for some neighborhood V of 0_{X}. So,

$$\mathrm{f}(\mathrm{W}) = \mathrm{f}(\pi(\mathrm{V})) = \Lambda(\mathrm{V}). \tag{1.74}$$

As a consequence, if Λ is open, then $\mathrm{f}(\mathrm{W})$ is a neighborhood of 0_{Y}. So ends the proof. $\square$

1.10 Exercise 10. An open mapping theorem

1.12 Exercise 12. Topology stays, completeness leaves

Suppose $d_1(x, y) = |x - y|, d_2(x, y) = |\varphi(x) - \varphi(y)|$, *where* $\varphi(x) = x/(1 + |x|)$. *Prove that* d_1 *and* d_2 *are metrics on* $\mathbf{R}$ *which induce the same topology, although* d_1 *is complete and* d_2 *is not.*

Proof. First, each d_i (i = 1, 2) induces a topology τ_i spanned by set of open balls

$$B_i(a, r) \triangleq \{x \in \mathbf{R} : \ d_i(a, x) < r\} \quad (a \in \mathbf{R}, \ r \in \mathbf{R}_+) \quad . \tag{1.75}$$

Next, remark that the mapping $\varphi : \mathbf{R} \to (-1\,;\,1)$ is odd and that

$$1 > \varphi(x) = 1 - \frac{1}{x+1} \underset{x\infty}{\uparrow} 1 \qquad (x\,0) \quad . \tag{1.76}$$

φ is then an τ_1-homeomorphism of $\mathbf{R}$ onto (-1 ; 1). Pick a in $\mathbf{R}$: given any positive scalar ε the τ_1-continuity of φ supplies a positive scalar $\eta = \eta(\varepsilon)$ so that

$$\forall x \in \mathbf{R} : \ (|a - x| < \eta \ \Rightarrow \ |\varphi(a) - \varphi(x)| < \varepsilon) \quad , \tag{1.77}$$

i.e.

$$B_1(a, \eta) \subset B_2(a, \varepsilon) \quad . \tag{1.78}$$

Keep a and deduce from the τ_1-continuity of $\varphi^{-1} : (-1;\,1) \to \mathbf{R}$ that there exists a positive scalar ε' such that

$$B_2(a, \varepsilon') \subset B_1(a, \ \eta') \quad , \tag{1.79}$$

provided a positive scalar η'. The special case $\eta' \triangleq \eta(\varepsilon)$ leads us to

$$B_2(a, \varepsilon') \overset{(1.79)}{\subset} B_1(a, \eta) \overset{(1.78)}{\subset} B_2(a, \varepsilon) \quad . \tag{1.80}$$

This yields

$$\tau_1 = \tau_2 \quad . \tag{1.81}$$

Finally, let m and n range $\mathbf{N}$, so that

$$d_2(m, \ n) = |\varphi(m) - \varphi(n)| \underset{m>n\infty}{\longrightarrow} 0 \quad . \tag{1.82}$$

The natural numbers sequence is then a τ_2-Cauchy one that τ_2-diverges, since

$$d_2(0, \ n) = \varphi(n) \underset{n\infty}{\longrightarrow} 1 \ \notin \ \varphi(\mathbf{R}) \quad . \tag{1.83}$$

Hence d_2 fails to be complete. □

1.14 Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms

Put $\mathrm{K} = [0,1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0,1,2,\dots$*) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

(a) $\|\mathrm{D^n f}\|_\infty = \sup\{|\mathrm{D^n f(x)}| : \infty < \mathrm{x} < \infty\}$

(b) $\|\mathrm{D^n f}\|_1 = \int_0^1 |\mathrm{D^n f}(x)|\,\mathrm{d}x$

(c) $\|\mathrm{D^n f}\|_2 = \left\{\int_0^1 |\mathrm{D^n f}(x)|^2 \mathrm{d}x\right\}^{1/2}$.

Proof. First, remark that

$$\|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty < \infty \tag{1.84}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of $\mathrm{D^n f}$ lies in K; which yields

$$|\,\mathrm{D^n f(x)}\,| = \left| \int_0^{\mathrm{x}} \mathrm{D^{n+1} f} \right| \le \int_0^{\mathrm{x}} \left|\,\mathrm{D^{n+1} f}\,\right| \le \|\mathrm{D^{n+1} f}\|_1. \tag{1.85}$$

So,

$$\|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1. \tag{1.86}$$

We now combine (1.84) with (1.86) and so obtain

$$\|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1 \le \cdots \quad (\mathrm{n} = 0,1,2,\dots). \tag{1.87}$$

Put

$$\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \triangleq \{\mathrm{f} \in \mathscr{D}_{\mathrm{K}} : \|\,\mathrm{f}\,\|_{\mathrm{i}} < 2^{-\mathrm{n}}\} \quad (\mathrm{i} = 1,2,\infty) \tag{1.88}$$

$$\mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} : \mathrm{n} = 0,1,2,\dots\}, \tag{1.89}$$

so that (1.87) is mirrored in terms of neighborhood inclusions, as follows,

$$\mathrm{V}_{\mathrm{n}}^{(1)} \supset \mathrm{V}_{\mathrm{n}}^{(2)} \supset \mathrm{V}_{\mathrm{n}}^{(\infty)} \supset \mathrm{V}_{\mathrm{n+1}}^{(1)} \supset \cdots. \tag{1.90}$$

Since $\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \supset \mathrm{V}_{\mathrm{n+1}}^{(\mathrm{i})}$, $\mathscr{B}^{(\mathrm{i})}$ is a local base of a topology τ_{i}. But the chain (1.90) forces

$$\tau_1 = \tau_2 = \tau_\infty. \tag{1.91}$$

To see that, choose a set S that is τ_1-open at f, *i.e.* $\mathrm{V}_{\mathrm{n}}^{(1)} \subset \mathrm{S} - \mathrm{f}$ for some n. Next, concatenate this with $\mathrm{V}_{\mathrm{n}}^{(2)} \subset \mathrm{V}_{\mathrm{n}}^{(1)}$ (see (1.90)) and so obtain $\mathrm{V}_{\mathrm{n}}^{(2)} \subset \mathrm{S} - \mathrm{f}$; which implies that S is τ_2-open at f. Similarly, we deduce, still from (1.90), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.92}$$

So ends the proof. □

1.16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $\mathrm{p_n}$, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1 *Let* X *be a topological space with a countable local base* $\{V_n : n = 1, 2, 3, \dots\}$*. If* $\tilde{V}_n = V_1 \cap \cdots \cap V_n$*, then every subsequence* $\{\tilde{V}_{\varrho(n)}\}$ *is a decreasing (i.e.* $\tilde{V}_{\varrho(n)} \supset \tilde{V}_{\varrho(n+1)}$*) local base of X.*

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_n} \supset \tilde{\mathrm{V}}_\mathrm{n}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{n}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$, since $\tilde{\mathrm{V}}_\mathrm{n} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n})}$. □

The following special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$ is one of the key ingredients:

Corollary 1 (special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$**)** *Under the same notations of Lemma 1, if* $\{V_n\}$ *is a decreasing local base, then so is* $\{V_{\varrho(n)}\}$*.*

Corollary 2 *If* $\{Q_n\}$ *is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence* $\{Q_{\varrho(n)}\}$ *also satisfies theses conditions. Furthermore, if* τ_Q *is the* $C(\Omega)$*'s (respectively* $C^\infty(\Omega)$*'s) topology of the seminorms* p_n*, as defined in section 1.44 (respectively 1.46), then the seminorms* $p_{\varrho(n)}$ *define the same topology* τ_Q*.*

Proof. Let X be $\mathrm{C}(\Omega)$ topologized by the seminorms $\mathrm{p_n}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_n} = \{\mathrm{p_n} < 1/\mathrm{n}\}$, then $\{\mathrm{V_n}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n}+1)}. \tag{1.93}$$

Thus,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n}+1)}. \tag{1.94}$$

In other words, $\mathrm{Q}_{\rho(\mathrm{n})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\rho(\mathrm{n})}\}$ then defines a topology τ_{Q_ρ} for which $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\rho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base of τ_Q, which yields $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\rho}$. □

Lemma 2 *If a sequence of compact sets* $\{Q_n\}$ *satisfies the conditions specified in section 1.44, then every compact set* K *lies in allmost all* Q_n°*, i.e. there exists* m *such that*

$$K \subset \mathring{Q}_m \subset \mathring{Q}_{m+1} \subset \mathring{Q}_{m+2} \subset \cdots. \tag{1.95}$$

Proof. The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \tag{1.96}$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset Q_m^\circ$, for some m. Finally,

$$K \subset \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots . \tag{1.97}$$

□

We are now in a fair position to establish the following:

Theorem *The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.*

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subset L_{m_i}^\circ \subset L_{m_i+1}^\circ \subset \cdots$ for some m_i. In particular, the special case $\varkappa_i = m_i + i$ is

$$K_i \subset \overset{\circ}{L}_{\varkappa_i}. \tag{1.98}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \tag{1.99}$$

Combine (1.98) with (1.99) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\varkappa_1} \subset L_{\varkappa_1} \subset \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subset K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots , \tag{1.100}$$

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L. \tag{1.101}$$

So ends the proof □

[1] See (b) of 2.5 of [2].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [2].

1.17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$*, for every multi-index* α.

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{\varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \le N, x \in K_N\} < 1/N\right\}. \tag{1.102}$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^\gamma g(x)| : |\gamma| \le N + |\alpha|, x \in K_N\} < \frac{1}{N+|\alpha|}. \tag{1.103}$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^\beta D^\alpha g(x)| : |\beta| \le N, x \in K_N\} < \frac{1}{N}. \tag{1.104}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad \textit{i.e.} \quad D^\alpha\left(V_{N+|\alpha|}\right) \subset V_N, \tag{1.105}$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^\alpha|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^\alpha|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) = D^\alpha\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) \tag{1.106}$$

$$\subset D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha(\mathscr{D}_K) \tag{1.107}$$

$$\subset V_N \cap \mathscr{D}_K \quad \text{(see (1.105))} \tag{1.108}$$

So ends the proof. □

Chapter 2

Completeness

2.3 Exercise 3. An equicontinous sequence of measures

Put $\mathrm{K} = [-1, 1]$*; define* $\mathscr{D}_{\mathrm{K}}$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^{\mathrm{n}}$*). Supose* $\{\mathrm{f_n}\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\varphi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ *exists for every* $\varphi \in \mathscr{D}_{\mathrm{K}}$*. Show that* Λ *is a continuous linear functional on* $\mathscr{D}_{\mathrm{K}}$*. Show that there is a positive integer* p *and a number* $\mathrm{M} < \infty$ *such that*

$$\left| \int_{-1}^{1} f_n(t)\varphi(t)dt \right| \leq M \| D^p \|_\infty$$

for all n*. For example, if* $f_n(t) = n^3 t$ *on* $[-1/n, 1/n]$ *and* 0 *elsewhere, show that this can be done with* $\mathrm{p} = 1$*. Construct an example where it can be done with* $\mathrm{p} = 2$ *but not with* $\mathrm{p} = 1$*.*

We will also consider the case $\mathrm{p} = 0$. Since all supports of $\varphi, \varphi', \varphi'', \ldots,$ are in K, we make a specialization of the mean value theorem:

Lemma If $\varphi \in \mathscr{D}_{[a,b]}$, then

$$\| D^\alpha \varphi \|_\infty \leq \| D^p \varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (\alpha = 0, 1, \ldots, p) \tag{2.1}$$

at every order $p = 0, 1, 2, \ldots$; where λ is the length $|\,b - a\,|$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 $(a < x_1 < x_0)$, such that

$$\varphi(x_0) = \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a). \tag{2.2}$$

Since every $D^p\varphi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$\begin{aligned}
\varphi(x_0) &= D^0\varphi(x_0) && (2.3)\\
&= D^1\varphi(x_1)(x_0 - a) && (2.4)\\
&= \cdots \\
&= D^p\varphi(x_p)(x_0 - a)\cdots(x_{p-1} - a), && (2.5)
\end{aligned}$$

for all p. More compactly,

$$D^{\alpha}\varphi(x_0) = D^{p}\varphi(x_p)\prod_{k=\alpha}^{p-1}(x_k - a); \tag{2.6}$$

which yields,

$$|D^{\alpha}\varphi(x)| \leq \| D^{p}\varphi \|_{\infty} \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c]) \tag{2.7}$$

The case $x_0 \geq c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^{\alpha}\varphi(x)| \leq \| D^{p}\varphi \|_{\infty} \left(\frac{\lambda}{2}\right)^{p-\alpha} \tag{2.8}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\| D^{\alpha}\varphi \|_{\infty} \leq \| D^{p}\varphi \|_{\infty} \left(\frac{\lambda}{2}\right)^{p-\alpha}. \tag{2.9}$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\varphi\rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})). \tag{2.10}$$

The following inequalities

$$|\langle u|\varphi\rangle| \leq \int_{\mathbf{R}} |u\varphi| \leq \| u \|_{L^1} \quad (\| \varphi \|_{\infty} \leq 1) \tag{2.11}$$

imply that every linear functional

$$\begin{aligned} \langle u| : C_0(\mathbf{R}) &\to \mathbf{C} \\ \varphi &\mapsto \langle u|\varphi\rangle \end{aligned} \tag{2.12}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u|$, since u is determined (a.e) by the integrals $\langle u|\varphi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\sup\{|\langle u|\varphi\rangle| : \| \varphi \|_{\infty} = 1\} = \| u \|_{L^1}. \tag{2.13}$$

Note that, in the latter equality, $\leq \| u \|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$\begin{aligned} g_n : \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & \left(x \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right) \\ 0 & \left(x \notin \left[-\frac{1}{n}, \frac{1}{n}\right]\right). \end{cases} \end{aligned} \tag{2.14}$$

[1] see 1.32, 4.1 of [3]

[2] See 7.26 of [1].

First, remark that $g_n(x) \longrightarrow 0$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$\langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi, \tag{2.15}$$

where φ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\varphi_1^+, \ldots, \varphi_j^+, \ldots$, from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\varphi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

(ii) $\varphi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

(iii) $0 \leq \varphi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those φ_j^+. Let $\varphi_1^-, \ldots, \varphi_j^-, \ldots$, mirror the φ_j^+, in the sense that $\varphi_j^-(x) = \varphi_j^+(-x)$, so that

(iv) $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$ is odd, as g_n is;

(v) every φ_j is in $C_K^\infty(\mathbf{R})$;

(vi) The sequence $\{\varphi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\| \varphi_j \|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n|\varphi_j\rangle = 2\int_0^1 g_n(t)\varphi_j^+(t)dt \underset{j\to\infty}{\longrightarrow} 2\int_0^1 g_n(t)dt = \| g_n \|_{L^1} = n. \tag{2.16}$$

Finally,

$$\| g_n \|_{L^1} \overset{(2.16)}{\leq} \sup\{| \langle g_n|\varphi\rangle | : \| \varphi \|_\infty = 1\} \overset{(2.13)}{\leq} \| g_n \|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$|\langle g_n|\varphi\rangle| \leq C_n \quad (\| \varphi \|_\infty = 1); \tag{2.18}$$

see (2.11). Furthermore, $\| g_n \|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\varphi_{\rho(n)}\rangle\}$ such that $\langle g_n|\varphi_{\rho(n)}\rangle$ is greater than, say, $n - 0.01$, as $\| \varphi_{\rho(n)} \|_\infty = 1$. Consequently, there is no bound M such that

$$|\langle g_n|\varphi\rangle| \leq M \quad (\| \varphi \|_\infty = 1; n = 1, 2, 3, \ldots). \tag{2.19}$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi \quad (\varphi \in C_0(\mathbf{R})) \tag{2.20}$$

[3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that φ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$\Lambda\varphi = \lim_{n\to\infty} \int_K f_n\varphi \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.21}$$

for some linear form Λ. Since φ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$\Lambda\varphi = \lim_{n\to\infty} \langle f_n|\varphi\rangle \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.22}$$

Let $K_1, K_2, \ldots,$ be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha\varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\varphi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\varphi \in \mathscr{D}_K : \|\varphi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\varphi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|f_n\|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [3].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi$ for all φ, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$|\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i, \tag{2.23}$$

$$|\Lambda\varphi - \langle f_n|\varphi\rangle| < 1/2 \quad \text{if } \varphi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\varphi\rangle| \leq |\,\Lambda\varphi - \langle f_n|\varphi\rangle\,| + |\,\Lambda\varphi\,| < 1 \qquad \text{if } \varphi \in V_p. \tag{2.25}$$

Now remark that every $\psi = \psi[\mu, \varphi]$, where

$$\psi[\mu,\varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\varphi\|_p)\varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases} \tag{2.26}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\psi\rangle| < 1 \tag{2.27}$$

$$|\langle f_n|\varphi\rangle| < 2^p \|\varphi\|_p \cdot \mu \tag{2.28}$$

$$|\langle f_n|\varphi\rangle| \leq 2^p \|\varphi\|_p \tag{2.29}$$

$$|\langle f_n|\varphi\rangle| \leq 2^p \{\| D^0\varphi\|_\infty + \cdots + \| D^p\varphi\|_\infty\}. \tag{2.30}$$

Finally, with the help of (2.1),

$$|\langle f_n|\varphi\rangle| \leq 2^p(p+1)\| D^p\varphi\|_\infty. \tag{2.31}$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\varphi = \lim\limits_{n\to\infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\varphi\rangle| \leq M\| D^p\varphi\|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let φ keep running on $\mathscr{D}_K$ and so define

$$B_n(\varphi) \triangleq \max\{|\varphi(x)| : x \in [-1/n, 1/n]\}, \tag{2.32}$$

$$\Delta_n(\varphi) \triangleq \max\{|\varphi(x) - \varphi(0)| : x \in [-1/n, 1/n]\}. \tag{2.33}$$

The mean value asserts that

$$|\varphi(1/n) - \varphi(-1/n)| \leq B_n(\varphi')\,|1/n - (-1/n)| = \frac{2}{n}B_n(\varphi'). \tag{2.34}$$

Independently, an integration by parts shows that

$$\langle f_n|\varphi\rangle = \left[\frac{n^3t^2}{2}\varphi(t)\right]_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\varphi'(t)dt \tag{2.35}$$

$$= \frac{n}{2}\left(\varphi(1/n) - \varphi(-1/n)\right) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\varphi'(t)dt. \tag{2.36}$$

Combine (2.34) with (2.36) and so obtain

$$|\langle f_n|\varphi\rangle| \leq \frac{n}{2}|\varphi(1/n) - \varphi(-1/n)| + \frac{n^3}{2}\int_{-1/n}^{1/n} t^2|\varphi'(t)|dt \tag{2.37}$$

$$\leq B_n(\varphi') + \frac{n^3}{2}B_n(\varphi')\int_{-1/n}^{1/n} t^2 dt \tag{2.38}$$

$$\leq \frac{4}{3}B_n(\varphi') \tag{2.39}$$

$$\leq \frac{4}{3}\|\varphi'\|_\infty. \tag{2.40}$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\varphi')$ tends to $|\,\varphi'(0)\,|$, we may expect that f_n tends to $\frac{4}{3}\varphi'(0)$. This is actually true: A straightforward computation shows that

$$(2.41)\qquad \langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0) \overset{(2.36)}{=} \frac{\varphi(1/n)-\varphi(\text{-}1/n)}{1/n-(\text{-}1/n)} - \varphi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n}(\varphi'-\varphi'(0))t^2dt.$$

So,

$$(2.42)\qquad \left|\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0)\right| \le \left|\frac{\varphi(1/n)-\varphi(\text{-}1/n)}{1/n-(\text{-}1/n)} - \varphi'(0)\right| + \frac{1}{3}\Delta_n(\varphi') \underset{n\to\infty}{\longrightarrow} 0.$$

We have just proved that

$$(2.43)\qquad \langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\varphi'(0) \quad (\varphi\in\mathscr{D}_K).$$

In other words,

$$(2.44)\qquad \langle f_n| \underset{n\to\infty}{\longrightarrow} \text{-}\frac{4}{3}\delta',$$

where δ is the *Dirac measure* and $\delta', \delta'', \dots$, its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $\text{-}\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$(2.45)\qquad |\langle f_n|\varphi\rangle| \le \frac{4}{3}\|\,\varphi'\,\|_\infty \quad (n=1,2,3,\dots)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$(2.46)\qquad \{\langle f_n|\varphi_{\rho(n)}\rangle : \|\,\varphi_{\rho(n)}\,\|_\infty = 1; n=1,2,3,\dots\}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$(2.47)\qquad |\langle f_n|\varphi\rangle| \le M\|\,\varphi\,\|_\infty \quad (\varphi\in C_K^\infty(\mathbf{R})).$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* $p=1$ is the smallest possible p and, given $p=1$, $M=\frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*[4] ${f_n}'$, as follows

$$(2.48)\qquad \begin{aligned} {f_n}' : \mathscr{D}_K &\to \mathbf{C} \\ \varphi &\mapsto \text{-}\left\langle f_n|\varphi'\right\rangle. \end{aligned}$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$(2.49)\qquad \begin{aligned} D : \mathscr{D}_K &\to \mathscr{D}_K \\ \varphi &\mapsto \varphi'; \end{aligned}$$

[4] See 6.1 of [3] for a further discussion.

see Exercise 1.17. ${f_n}'$ is therefore continuous. Now apply (2.43) with φ' and so obtain

$$-\langle f_n|\varphi'\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\varphi''(0) \quad (\varphi \in \mathscr{D}_K),$$

i.e.

$$(2.50) \qquad {f_n}' \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\delta''.$$

It follows from (2.40) that,

$$(2.51) \qquad |\langle f_n|\varphi'\rangle| \leq \frac{4}{3}\| \varphi'' \|_\infty \quad (n = 1, 2, 3, \dots).$$

It is therefore possible to uniformly bound ${f_n}'$ with respect to a norm $\| D^p \cdot \|_\infty$, namely $\| D^2 \cdot \|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$(2.52) \qquad |\langle f_n|\varphi'\rangle| \leq M\| \varphi' \|_\infty \quad (n = 1, 2, 3, \dots).$$

Define

$$(2.53) \qquad \Phi_j(x) = \int_{-1}^{x} \varphi_j.$$

The oddness of φ_j forces Φ_j to vanish outside [-1, 1]: φ_j is therefore in $\mathscr{D}_K$. So, under our assumption,

$$(2.54) \qquad |\langle f_n|\Phi_j'\rangle| \leq M\| \Phi_j' \|_\infty \quad (n = 1, 2, 3, \dots);$$

which is

$$(2.55) \qquad |\langle f_n|\varphi_j\rangle| \leq M \quad (n = 1, 2, 3, \dots).$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n|\varphi_{\rho(n)}\rangle\}$) and so conclude that there is no constant M such that

$$(2.56) \qquad |\langle|f_n\varphi'\rangle| \leq M\| \varphi' \|_\infty \quad (n = 1, 2, 3, \dots).$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$(2.57) \qquad |\langle f_n|\varphi'\rangle| \leq M\| \varphi \|_\infty.$$

The mean value theorem (see (2.1)) asserts that

$$(2.58) \qquad |\langle f_n|\varphi'\rangle| \leq M\| \varphi \|_\infty \leq M\| \varphi' \|_\infty;$$

which is, again, a desired contradiction. So ends the proof.

$\square$

2.6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ *of a function* $f \in L^2(T)$ *(T is the unit circle) by*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

for all $n \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f \text{ exists}\}$ *is a dense subspace of* $L^2(T)$ *of the first category.*

Proof. Let $f(\theta)$ stand for $f(e^{i\theta})$, so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta. \tag{2.59}$$

We believe it is customary to write

$$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n). \tag{2.60}$$

Moreover, a well known (and easy to prove) result is

$$(e_n, e_{n'}) = [n = n'], \textit{ i.e. } \{e_n : n \in \mathbf{Z}\} \text{ is an orthormal subset of } L^2(T). \tag{2.61}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

$$\| \Lambda_n \|^2 \overset{(2.60)}{=} \| e_{-n} + \cdots + e_n \|^2 \overset{(2.61)}{=} \| e_{-n} \|^2 + \cdots + \| e_n \|^2 \overset{(2.61)}{=} 2n + 1. \tag{2.62}$$

We now assume, to reach a contradiction, that

$$B \triangleq \{f \in L^2(T) : \sup\{| \Lambda_n f | : n = 1, 2, 3, \ldots\} < \infty\} \tag{2.63}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$\| \Lambda_n \| \overset{(2.62)}{=} \sqrt{2n+1} \underset{n\to\infty}{\longrightarrow} \infty. \tag{2.64}$$

We have just established that B is actually of the first category; and so is its subset $L = \{f \in L^2(T) : \lim_{n\longrightarrow\infty} \Lambda_n f \text{ exists}\}$. We now prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be span$\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$P \subset L \subset L^2(T). \tag{2.65}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.65), that

$$L^2(T) = \overline{P} = \overline{L}. \tag{2.66}$$

So ends the proof □

2.9 Exercise 9. Boundedness without closedness

Suppose $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ *are Banach spaces and*

$$\mathrm{B} : \mathrm{X} \times \mathrm{Y} \to \mathrm{Z}$$

is bilinear and continuous. Prove that there exists $\mathrm{M} < \infty$ *such that*

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| \leq \mathrm{M} \| \mathrm{x} \| \| \mathrm{x} \| \quad (\mathrm{x} \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Since B is continous at the origin, there exists a positive r such that

$$\| \mathrm{x} \| + \| \mathrm{y} \| < \mathrm{r} \Rightarrow \| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| < 1. \tag{2.67}$$

Given nonzero x, y, let s range over $]0, \mathrm{r}[$, so that the folllowing bound

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| = \frac{4 \| \mathrm{x} \| \| \mathrm{y} \|}{\mathrm{s}^2} \left\| \mathrm{B}\left(\frac{\mathrm{s}}{2 \| \mathrm{x} \|} \mathrm{x}, \frac{\mathrm{s}}{2 \| \mathrm{y} \|} \mathrm{y} \right) \right\| \overset{(2.67)}{<} \frac{4 \| \mathrm{x} \| \| \mathrm{y} \|}{\mathrm{s}^2} \tag{2.68}$$

is effective. It is now obvious that

$$\mathrm{B}(\mathrm{x}, \mathrm{y}) \leq \frac{4}{\mathrm{s}^2} \| \mathrm{x} \| \| \mathrm{y} \| \underset{\mathrm{s} \to \mathrm{r}}{\longrightarrow} \frac{4}{\mathrm{r}^2} \| \mathrm{x} \| \| \mathrm{y} \| \quad ((\mathrm{x}, \mathrm{y}) \in \mathrm{X} \times \mathrm{Y}); \tag{2.69}$$

which achieves the proof.

As a concrete example, choose $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = \mathrm{C_c}(\mathbf{R})$, topologized by the supremum norm. $\mathrm{C_c}(\mathbf{R})$ is not complete (see 5.4.4 of [4]), nevertheless the bilinear product

$$\begin{array}{llll} \mathrm{B} : & \mathrm{C_c}(\mathbf{R})^2 & \to & \mathrm{C_c}(\mathbf{R}) \\ & (\mathrm{f}, \mathrm{g}) & \mapsto & \mathrm{f} \cdot \mathrm{g} \end{array}$$

is bounded (since $\| \mathrm{f} \cdot \mathrm{g} \|_\infty \leq \| \mathrm{f} \|_\infty \cdot \| \mathrm{g} \|_\infty$), and continuous. To show this, pick a positive scalar ε smaller than 1, provided any (f, g). Next, define

$$\mathrm{r} \triangleq \frac{\varepsilon}{1 + \| \mathrm{f} \|_\infty + \| \mathrm{g} \|_\infty} < 1. \tag{2.70}$$

We now restrict (u, v) to a particular neighborhood of (f, g). More specifically,

$$\| \mathrm{f} - \mathrm{u} \|_\infty + \| \mathrm{g} - \mathrm{v} \|_\infty < \mathrm{r}. \tag{2.71}$$

Next, remark that $\| \mathrm{u} \|_\infty \leq \mathrm{r} + \| \mathrm{f} \|_\infty$ and so obtain (bear in mind that $\mathrm{r} < 1$)

$$\begin{align} \| \mathrm{fg} - \mathrm{uv} \|_\infty &= \| (\mathrm{f} - \mathrm{u}) \cdot \mathrm{g} + \mathrm{u} \cdot (\mathrm{g} - \mathrm{v}) \|_\infty \tag{2.72} \\ &\leq \| \mathrm{f} - \mathrm{u} \|_\infty \cdot \| \mathrm{g} \|_\infty + \| \mathrm{u} \|_\infty \cdot \| \mathrm{g} - \mathrm{v} \|_\infty \tag{2.73} \\ &< \mathrm{r} \cdot \| \mathrm{g} \|_\infty + (\mathrm{r} + \| \mathrm{f} \|_\infty) \cdot \mathrm{r} \tag{2.74} \\ &< \mathrm{r} \cdot (\mathrm{r} + \| \mathrm{f} \|_\infty + \| \mathrm{g} \|_\infty) \tag{2.75} \\ &< \varepsilon. \tag{2.76} \end{align}$$

Since ε was arbitrary, it is now established that B continuous at every (f, g). □

2.10 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let (X_1, X_2, Z) be topological spaces and B a bilinear mapping

$$B : X_1 \times X_2 \to Z. \tag{2.77}$$

From now on, $x = (x_1, x_2)$ denotes an arbitrary element of $X_1 \times X_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $X_1 \times X_2$, *i.e.* given an arbitrary **balanced** open subset W of Z, there exists in X_i $(i = 1, 2)$ a **balanced** open subset U_i such that

$$B(U_1 \times U_2) \subset W. \tag{2.78}$$

In such context, $\lambda_i(x)$ is chosen greater than $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$; see [1.33] of [3] for further reading about the *Minkowski functionals* μ. In other words, x_i lies in $\lambda_i(x)U_i$, since U_i is balanced. Thus,

$$B(x_1, x_2) = \lambda_1(x)\lambda_2(x) \cdot B(x_1/\lambda_1(x), x_2/\lambda_2(x)) \tag{2.79}$$
$$\in \lambda_1(x)\lambda_2(x) \cdot B(U_1 \times U_2) \tag{2.80}$$
$$\subset \lambda_1(x)\lambda_2(x) \cdot W. \tag{2.81}$$

Pick $p = (p_1, p_2)$ in $X_1 \times X_2$, and let $q = (q_1, q_2)$ range over $X \times Y$, as a first step: It directly follows from (2.81) that

$$B(p) - B(q) = B(p_1, p_2 - q_2) + B(p_1, q_2) - B(q_1, q_2) \tag{2.82}$$
$$= B(p_1, p_2 - q_2) + B(p_1 - q_1, q_2) \tag{2.83}$$
$$= B(p_1, p_2 - q_2) + B(p_1 - q_1, q_2 - p_2) + B(p_1 - q_1, p_2) \tag{2.84}$$
$$\in \lambda_1(p)\lambda_2(p - q)W + \lambda_1(p - q)\lambda_2(q - p)W + \lambda_1(p - q)\lambda_2(p)W. \tag{2.85}$$

We now restrict q to a particular neighborhood of p. More specifically,

$$p_i - q_i \in \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 2} U_i; \tag{2.86}$$

which implies

$$\mu_i(q_i - p_i) = \mu_i(p_i - q_i) \le \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 2} \tag{2.87}$$

(the equality at the left is valid, since $U_i = -U_i$). The special case

$$\lambda_i(p) \triangleq \mu_1(p_1) + \mu_2(p_2) + 1, \tag{2.88}$$
$$\lambda_i(p - q) \triangleq \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 1} \triangleq \lambda_i(q - p) \tag{2.89}$$

implies that

$$B(p) - B(q) \in W + W + W, \tag{2.90}$$

since W is balanced. W being arbitrary, we have so established the continuousness of B at arbitrary p; which achieves the proof. □

2.12 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\| f \| = \int_0^1 |f(t)| \, dt.$$

Put $B(f, g) = \int_0^1 f(t)g(t)dt$*, and show that* B *is a bilinear continuous functional on* $X \times X$ *which is separately but not continuous.*

Proof. Let f denote the first variable, g the second one. Remark that

$$| B(f, g) | < \| f \| \cdot \max_{[0,1]} | g | ; \tag{2.91}$$

which is sufficient (1.18 of [3]) to assert that any $f \mapsto B(f, g)$ is continuous. The continuity of all $g \mapsto B(f, g)$ follows (Put $C(g, f) = B(f, g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$| B(f, g) | < M\| f \| \| g \|. \tag{2.92}$$

Put

$$f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \qquad (n = 1, 2, 3, \dots), \tag{2.93}$$

so that

$$\| f_n \| = \frac{2\sqrt{n}}{n + 1} \underset{n\to\infty}{\longrightarrow} 0. \tag{2.94}$$

On the other hand,

$$B(f_n, f_n) = \frac{4n}{2n + 1} > 1. \tag{2.95}$$

Finally, we combine (2.95) and (2.92) with (2.94) and so obtain

$$1 < B(f_n, f_n) < M\| f_n \|^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{2.96}$$

Our continuousness assumption is then contradicted. So ends the proof. □

2.15 Exercise 15. Baire cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* $\mathrm{Y}=\mathrm{X}$. Hint: Y *must intersect* $\mathrm{x}+\mathrm{Y}$ *for every* $\mathrm{x}\in\mathrm{X}$.

Proof. Assume Y is a subgroup of X. Under our assumptions, there exists a sequence $\{\mathrm{E_n} : \mathrm{n}=1,2,3,\dots\}$ of X such that

(i) $(\overline{\mathrm{E}}_\mathrm{n})^\circ=\emptyset$;

(ii) $\mathrm{X}\setminus\mathrm{Y}=\bigcup_{\mathrm{n}=1}^{\infty}\mathrm{E_n}$.

By (i), the complement $\mathrm{V_n}$ of $\overline{\mathrm{E}}_\mathrm{n}$ is a dense open set. Since X is an F-space, it follows from the Baire's theorem that the intersection S of the $\mathrm{V_n}$'s is dense in X: So is $\mathrm{x}+\mathrm{S}$ $(\mathrm{x}\in\mathrm{X})$. To see that, remark that

$$\mathrm{X}=\mathrm{x}+\overline{\mathrm{S}}\subset\overline{\mathrm{x}+\mathrm{S}} \tag{2.97}$$

follows from 1.3 (b) of [3]. Since S and $\mathrm{x}+\mathrm{S}$ are both dense open subsets of X, the Baire's theorem asserts that

$$\overline{(\mathrm{x}+\mathrm{S})\cap\mathrm{S}}=\mathrm{X}. \tag{2.98}$$

Thus,

$$(\mathrm{x}+\mathrm{S})\cap\mathrm{S}\neq\emptyset. \tag{2.99}$$

Moreover, it follows from (ii) that $\mathrm{X}\setminus\mathrm{Y}\subset\bigcup_\mathrm{n}\overline{\mathrm{E}}_\mathrm{n}$, *i.e.* $\mathrm{Y}\supset\mathrm{S}$. Combined with (2.99), this shows that $\mathrm{x}+\mathrm{Y}$ cuts Y. Therefore, our arbitrary x is an element of the subgroup Y. We have thus established that $\mathrm{X}\subset\mathrm{Y}$, which achieves the proof. □

2.16 Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $\mathrm{f} : \mathrm{X} \to \mathrm{K}$ *is a closed subset of* $\mathrm{X} \times \mathrm{K}$*. Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

Proof. Choose a sequence $\{\mathrm{x_n} : \mathrm{n} = 1, 2, 3, \dots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(\mathrm{x}_{\rho(\mathrm{n})}, \mathrm{f}(\mathrm{x}_{\rho(\mathrm{n})}))\}$ of $\{(\mathrm{x_n}, \mathrm{f}(\mathrm{x_n}))\}$ that converges to some (a, b) of $\mathrm{X} \times \mathrm{K}$. G is closed; therefore, $\{(\mathrm{x}_{\rho(\mathrm{n})}, \mathrm{f}(\mathrm{x}_{\rho(\mathrm{n})}))\}$ converges in G. So, $\mathrm{b} = \mathrm{f}(\mathrm{a})$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$\begin{aligned} \mathrm{f} : [0, \infty) &\to [0, \infty) \\ \mathrm{x} &\mapsto \begin{cases} 1/\mathrm{x} & (\mathrm{x} > 0) \\ 0 & (\mathrm{x} = 0). \end{cases} \end{aligned} \tag{2.100}$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$\mathrm{G} = \{(\mathrm{x}, 1/\mathrm{x}) : \mathrm{x} > 0\} \cup \{(0, 0)\}. \tag{2.101}$$

Next, let $\{(\mathrm{x_n}, 1/\mathrm{x_n})\}$ be a sequence in $\mathrm{G}_+ = \{(\mathrm{x}, 1/\mathrm{x}) : \mathrm{x} > 0\}$ that converges to (a, b). To be more specific: $\mathrm{a} = 0$ contradicts the boundedness of $\{(\mathrm{x_n}, 1/\mathrm{x_n})\}$: a is necessarily positive and $\mathrm{b} = 1/\mathrm{a}$, since $\mathrm{x} \mapsto 1/\mathrm{x}$ is continuous on R_+. This establishes that $(\mathrm{a}, \mathrm{b}) \in \mathrm{G}_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. ∎

Chapter 3

Convexity

3.3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subset X$ *an internal point of* A *if* $A - x_0$ *is an absorbing set.*

(a) Suppose A *and* B *are disjoint convex sets in* X*, and* A *has an internal point. Prove that there is a nonconstant linear functional* Λ *such that* $\Lambda(A) \cap \Lambda(B)$ *contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) Show (with $X = \mathbf{R}^2$*, for example) that it may not possible to have* $\Lambda(A)$ *and* $\Lambda(B)$ *disjoint, under the hypotheses of (a).*

Proof. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [3] to establish the existence of a linear functional $\Lambda : X \to \mathbf{R}$ such that

$$\Lambda \leq p \tag{3.1}$$

and

$$\Lambda(b_0 - x_0) = 1. \tag{3.2}$$

Then

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B). \tag{3.3}$$

Hence

$$\Lambda a \leq \Lambda b. \tag{3.4}$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There so exists (a_i, b_i) in $A \times B$ $(i = 1, 2)$ such that

$$\Lambda a_i = \Lambda b_i = y_i. \tag{3.5}$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad . \tag{3.6}$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

$$\Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1 \quad ; \tag{3.7}$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\}, \tag{3.8}$$

$$S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}, \tag{3.9}$$

$$A \triangleq S_1 \cup S_2, \tag{3.10}$$

$$B \triangleq X \setminus A. \tag{3.11}$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subset A. \tag{3.12}$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.* A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a linear functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$\Lambda(L) = \{0\} \subset \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad . \tag{3.13}$$

So ends the proof. □

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [2].

3.11 Exercise 11. Meagerness of the polar

Let X be an infinite-dimensional Fréchet space. Prove that X^*, with its weak-topology, is of the first category in itself.*

This is actually a consequence of the below lemma, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual X^* separates points on X, then the polar*

$$K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\} \tag{3.14}$$

of any absorbing subset A is a weak-closed set that has empty interior.*

Proof. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous; see 3.14 of [3]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see 3.11 of [3]). Put

$$W_{F,x} \triangleq \bigcap_{x\in F}\{\Lambda \in X^* : |\Lambda x| < r_x\} \qquad (r_x > 0) \tag{3.15}$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$M \triangleq \text{span}(F) \tag{3.16}$$

is finite dimensional. Assume, to reach a contradiction, that $A \subset M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subset M$: Now pick a in $A \setminus M$ and so conclude that

$$b \triangleq \frac{a}{t_a} \in A \tag{3.17}$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see 1.21 (b) of [3] for a proof): By the Hahn-Banach theorem 3.5 of [3], there exists Λ_a in X^* such that

$$\Lambda_a b > 2 \tag{3.18}$$

and

$$\Lambda_a(M) = \{0\}. \tag{3.19}$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$|\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1. \tag{3.20}$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

$$\Lambda + W \not\subset K_A. \tag{3.21}$$

Since W and Λ are both arbitrary, this achieves the proof. □

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$\mathrm{B_n} \triangleq \{\mathrm{x} \in \mathrm{X} : \mathrm{d}(0, \mathrm{x}) < 1/\mathrm{n}\} \qquad (\mathrm{n} = 1, 2, 3, \dots) \tag{3.22}$$

form a local base of X. If Λ is in X^*, then

$$|\,\Lambda\,| \leq \mathrm{m} \text{ on } \mathrm{B_n} \tag{3.23}$$

for some $(\mathrm{n}, \mathrm{m}) \in \{1, 2, 3, \dots\}^2$; see 1.18 of [3]. Hence, X^* is the countable union of all

$$\mathrm{m} \cdot \mathrm{K_n} \qquad (\mathrm{m}, \mathrm{n} = 1, 2, 3, \dots), \tag{3.24}$$

where $\mathrm{K_n}$ is the polar of $\mathrm{B_n}$. Clearly, showing that every $\mathrm{m} \cdot \mathrm{K_n}$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [3]. As a consequence, the above lemma implies

$$\left(\overline{\mathrm{K_n}}\right)^\circ = (\mathrm{K_n})^\circ = \emptyset. \tag{3.25}$$

Since the multiplication by m is an homeomorphism (see 1.7 of [3]), this is equivalent to

$$\left(\overline{\mathrm{m} \cdot \mathrm{K_n}}\,\right)^\circ = \mathrm{m} \cdot (\mathrm{K_n})^\circ = \emptyset. \tag{3.26}$$

So ends the proof. □

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

4.1 Exercise 1. Basic results

Let φ be the embedding of X *into* X^{**} *decribed in Section 4.5. Let τ be the weak topology of* X*, and let σ be the weak*- topology of* X^{**}*- the one induced by* X^**.*

(a) Prove that φ is an homeomorphism of (X, τ) *onto a dense subspace of* (X^{**}, σ)*.*

(b) If B *is the closed unit ball of* X*, prove that $\varphi(B)$ is σ-dense in the closed unit ball of* X^{**}*. (Use the Hahn-Banach separation theorem.)*

(c) Use (a), (b), and the Banach-Alaoglu theorem to prove that X *is reflexive if and only if* B *is weakly compact.*

(d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.

(e) If X *is reflexive and* Y *is a closed subspace of* X*, prove that* X/Y *is reflexive.*

(f) Prove that X *is reflexive if and only* X^* *if reflexive.*
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\varphi(X)$ *of* X^{**}*.*

Proof. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$\varphi^{-1} : \varphi(X) \to X \tag{4.1}$$

$$\varphi(x) \mapsto x \tag{4.2}$$

is an homeomorphism (with respect to τ and σ). We first consider

$$V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}|\psi x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0); \tag{4.3}$$

$$U \triangleq \{x \in X : |\langle x|x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0). \tag{4.4}$$

and remark that the so defined V 's (respectively U 's) shape a local subbase $\mathscr{S}_\sigma$ (respectively $\mathscr{S}_\tau$) of σ (respectively τ). We now observe that

$$U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathscr{S}_\sigma,\ U \in \mathscr{S}_\tau) \quad , \tag{4.5}$$

since φ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathscr{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that φ^{-1} and its inverse φ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$\overline{\varphi(X)}_\sigma \subset X^{**} \quad . \tag{4.6}$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\varphi(X)$. By [3.5], there so exists y^* in X^* such that

$$\langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ; \tag{4.7}$$

$$\langle z^{**}, \psi y^* \rangle = 1 \tag{4.8}$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$\varphi(B) \subset B^{**} \quad , \tag{4.9}$$

since φ is isometric. Hence

$$\overline{\varphi(B)}_\sigma \subset \overline{(B^{**})}_\sigma = B^{**} \quad . \tag{4.10}$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\varphi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$|\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_\sigma \quad ; \tag{4.11}$$

$$\langle z^{**}, \psi y^* \rangle > 1 \quad . \tag{4.12}$$

It follows from (4.11) that

$$|\psi y^*| \leq 1 \text{ on } \varphi(B), \ \textit{i.e.} |y^*| \leq 1 \text{ on } B \quad . \tag{4.13}$$

We have so proved that

$$y^* \in B^* \quad . \tag{4.14}$$

Since z^{**} lies in B^{**}, it is now clear that

$$|\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ; \tag{4.15}$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

(4.16) B is weakly compact if and only if $\varphi(B)$ is weak*-compact.

If B is weakly compact, then $\varphi(B)$ is weak*-closed. So,

$$\varphi(B) = \overline{\varphi(B)}_\sigma \overset{(b)}{=} B^{**} \quad . \tag{4.17}$$

φ is therefore onto, *i.e.* X is reflexive.
Conversely, keep φ as onto: one easily checks that $\varphi(\mathrm{B}) = \mathrm{B}^{**}$. The image $\varphi(\mathrm{B})$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $\mathrm{B} \cap \mathrm{Y}$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(\mathrm{X}/\mathrm{Y})^* \equiv \mathrm{Y}^{\perp} \quad , \tag{4.18}$$

next,

$$(\mathrm{X}/\mathrm{Y})^{**} \equiv (\mathrm{Y}^{\perp})^* \equiv \mathrm{X}^{**}/(\mathrm{Y}^{\perp})^{\perp} \equiv \mathrm{X}/\mathrm{Y} \quad . \tag{4.19}$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z*, the weak*-topology of* Z^* *is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\varphi(\mathrm{X})$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\varphi(\mathrm{X})$ is reflexive, by (d). It now follows from (c) that $\mathrm{B}^{**} \cap \varphi(\mathrm{X})$ is weakly compact, *i.e.* weak*-compact (to see this, apply (L) with $\mathrm{Z} = \mathrm{X}^*$).

By (a), B is therefore weakly compact, *i.e.* X is reflexive; see (c). So ends the proof. $\square$

4.13 Exercise 13. Operator compactness in a Hilbert space

4.15 Exercise 15. Hilbert-Schmidt operators

Chapter 6

Distributions

6.1 Exercise 1. Test functions are almost polynomial

6.6 Exercise 6. Around the supports of some distributions

6.9 Exercise 9. Convergence in $\mathscr{D}(\Omega)$ vs. convergence in $\mathscr{D}'(\Omega)$

6.17 Exercise 17.

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