Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$*, topologized by the family of seminorms*

$$\mathrm{p_x(f) = |f(x)|} \quad (0 \leq \mathrm{x} \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{\mathrm{f_n}\}$ *in X such that (a)* $\{\mathrm{f_n}\}$ *converges to* 0 *as* $\mathrm{n} \to \infty$*, but (b) if* $\{\gamma_\mathrm{n}\}$ *is any sequence of scalars such that* $\gamma_\mathrm{n} \to \infty$ *then* $\{\gamma_\mathrm{n} \mathrm{f_n}\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms $\mathrm{p_x}$ is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$\mathrm{V^{((x,k)}} \triangleq \{\mathrm{p_x} < 2^{\text{-k}}\} \quad (\mathrm{x} \in [0,1], \mathrm{k} \in \mathbf{N}) \tag{1.1}$$

is then a local base for a topology τ on X. Given $\{\mathrm{f_n} : \mathrm{n} = 1,2,3,\dots\}$, we set

$$\mathtt{off}(\mathrm{U}) \triangleq \sum_{\mathrm{n}=1}^{\infty} [\mathrm{f_n} \notin \mathrm{U}] \quad (\mathrm{U} \in \tau), \tag{1.2}$$

with the convention $\mathtt{off}(\mathrm{U}) = \infty$ whether the sum has no finite support. So,

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{U^{(i)}}) = \sum_{\mathrm{n}=1}^{\infty} \sum_{\mathrm{i}=1}^{\mathrm{m}} [\mathrm{f_n} \notin \mathrm{U^{(i)}}] \geq \mathtt{off}(\mathrm{U^{(1)}} \cap \dots \cap \mathrm{U^{(m)}}) \tag{1.3}$$

We first assume that $\{\mathrm{f_n}\}$ τ-converges to some f in X, *i.e.*

$$\mathtt{off}(\mathrm{f} + \mathrm{V}) < \infty \quad (\mathrm{V} \in \mathscr{B}). \tag{1.4}$$

The special cases $\mathrm{V} = \mathrm{V^{(x,k)}}$ mean the pointwise convergence of $\{\mathrm{f_n}\}$. Conversely, assume that $\{\mathrm{f_n}\}$ does not τ-converges to any g in X, *i.e.*

$$\forall \mathrm{g} \in \mathrm{X}, \exists \mathrm{V^{(g)}} \in \mathscr{B} : \mathtt{off}(\mathrm{g} + \mathrm{V^{(g)}}) = \infty. \tag{1.5}$$

Given g, $\mathrm{V^{(g)}}$ is then an intersection $\mathrm{V^{(x^{(1)},k^{(1)})}} \cap \dots \cap \mathrm{V^{(x^{(m)},k^{(m)})}}$. Thus

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{g} + \mathrm{V^{(x^{(i)},k^{(i)})}}) \overset{(1.3)}{\geq} \mathtt{off}(\mathrm{g} + \mathrm{V^{(g)}}) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\mathtt{off}(\mathrm{g} + \mathrm{V^{(x^{(i)},k^{(i)})}})$ must then be ∞. This implies that convergence of $\mathrm{f_n}$ to g fails at point $\mathrm{x_i}$. g being arbitrary, we so conclude that $\mathrm{f_n}$ does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on, k, n and p run on $\mathbf{N}_+$. Let $\mathtt{dyadic}(\mathrm{x})$ be the usual dyadic expansion of a real number x, so that $\mathtt{dyadic}(\mathrm{x})$ is an aperiodic binary sequence **iff** x is irrational. Define

$$\mathrm{f_n(x)} \triangleq \begin{cases} 2^{-\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathtt{dyadic}(\mathrm{x})_{-\mathrm{k}}} & (\mathrm{x} \in [0,1] \setminus \mathbf{Q}) \\ 0 & (\mathrm{x} \in [0,1] \cap \mathbf{Q}) \end{cases} \tag{1.7}$$

so that $\mathrm{f_n(x)} \underset{\mathrm{n}\to\infty}{\longrightarrow} 0$ and take scalars γ_n such that $\underset{\mathrm{n}\to\infty}{\longrightarrow} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose $\mathrm{n^{(p)}}$ among those *almost all* n that are large enough to satisfy

$$\mathrm{n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)}} \tag{1.8}$$

(start with $\mathrm{n^{(\text{-}1)} = n^{(0)} = 0}$) and so obtain

$$2^\mathrm{p} < \gamma_{\mathrm{n^{(p)}}} : 0 < \mathrm{n^{(p)} - n^{(p-1)}} \underset{\mathrm{p}\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator χ of $\{\mathrm{n^{(1)}, n^{(2)}}, \dots\}$ is then aperiodic, *i.e.*

$$\mathrm{x}^{(\gamma)} \triangleq \sum_{\mathrm{k}=1}^{\infty} \chi_\mathrm{k} 2^{\text{-}\mathrm{k}} \tag{1.10}$$

is irrational. Consequently,

$$\mathtt{dyadic}(\mathrm{x}^{(\gamma)})_{\text{-}\mathrm{k}} = \chi_\mathrm{k}. \tag{1.11}$$

We now easily see that

$$\chi_1 + \dots + \chi_{\mathrm{n^{(p)}}} = \mathrm{p}, \tag{1.12}$$

which, combined with (1.7), yields

$$\mathrm{f}_{\mathrm{n^{(p)}}}(\mathrm{x}^{(\gamma)}) = 2^{\text{-}\mathrm{p}}. \tag{1.13}$$

Finally,

$$\gamma_{\mathrm{n^{(p)}}} \mathrm{f}_{\mathrm{n^{(p)}}}(\mathrm{x}^{(\gamma)}) > 1. \tag{1.14}$$

We have so established that the subsequence $\{\gamma_{\mathrm{n^{(p)}}} \mathrm{f}_{\mathrm{n^{(p)}}}\}$ does not tend pointwise to 0, hence neither does the whole sequence $\{\gamma_\mathrm{n} \mathrm{f_n}\}$. In other words, (b) holds, which is in violent contrast with [1.28]: X is then not metrizable. So ends the proof. □

1.2 Exercise 9. Quotient map

Suppose

(*a*) X *and* Y *are topological vector spaces,*

(*b*) $\Lambda : X \to Y$ *is linear.*

(*c*) N *is a closed subspace of* X*,*

(*d*) $\pi : X \to X/N$ *is the quotient map, and*

(*e*) $\Lambda x = 0$ *for every* $x \in N$*.*

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. The equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \tag{1.15}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'. \tag{1.16}$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ. Now remark that

$$\pi x = N \overset{(\text{f linear})}{\Rightarrow} f(\pi x) = 0 \overset{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow \pi x = N \tag{1.17}$$

and so conclude that f is also one-to-one. Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by (a) of [1.41]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subset V. \tag{1.18}$$

Since π is open (see (a) of [1.41]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by (a) of [1.41]. Conversely, let

$$W \triangleq \pi(V) \subset X/N \quad (\text{V neighborhood of } 0_X) \tag{1.19}$$

range over all neighborhoods of N, as Λ is kept open: So is

$$\Lambda(V) = f(\pi(V)) = f(W). \tag{1.20}$$

The linear mapping f is then open. $\square$

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$*,* $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$*.*

(*a*) *Prove that* Λ *is an open mapping.*

(*b*) *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. (*a*) Let e range over a base of Y: For each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, since Λ is onto.So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \Lambda \mathrm{x_e} \quad (\mathrm{y} \in \mathrm{Y}). \tag{1.21}$$

The sequence $\{\mathrm{x_e}\}$ is finite hence bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$\mathrm{x_e} \in \mathrm{sV} \tag{1.22}$$

for all $\mathrm{x_e}$. Combining this with (1.21) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \quad (\mathrm{y} \in \mathrm{Y} : |\mathrm{y_e}| < \mathrm{s}^{-1}). \tag{1.23}$$

(*b*) Since N is closed, π continously maps X onto X/N, another topological (Hausdorf) vector space, see [1.41]. Now take f as in Exercise 9: Since Λ is onto, the first isomorphism theorem asserts that f is an isomorphism of X/N onto Y. Consequently, X/N has dimension $\mathrm{n} = \dim \mathrm{Y}$. f is then an homeomorphism of $\mathrm{X/N} \equiv \mathbf{C}^{\mathrm{n}}$ onto Y; see [1.21]. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$.

□

1.4 Exercise 14.

Put $\mathrm{K} = [0, 1]$ *and define* $\mathscr{D}_\mathrm{K}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0, 1, 2, \ldots$*) define the same topology on* $\mathscr{D}_\mathrm{K}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

(*a*) $\|\mathrm{D^n f}\|_\infty = \sup\{|\mathrm{D^n f(x)}| : \infty < \mathrm{x} < \infty\}$

(*b*) $\|\mathrm{D^n f}\|_1 = \int_0^1 |\mathrm{D^n f(x)}|$

(*c*) $\|\mathrm{D^n f}\|_2 = \left\{\int_0^1 |\mathrm{D^n f(x)}|^2 \,\mathrm{d}\,\delta\mathrm{x}\right\}^{1/2}$.

Proof. First, remark that

$$\|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty < \infty \tag{1.24}$$

(the inequality on the left is a Cauchy-Schwarz one), since K has length 1. Next, start from

$$\mathrm{D^n f(x)} = \int_{-\infty}^{\mathrm{x}} \mathrm{D^{n+1} f} \tag{1.25}$$

(which is true, since f has a bounded support) to obtain

$$|\mathrm{D^n f(x)}| \le \int_{-\infty}^{\mathrm{x}} \left|\mathrm{D^{n+1} f}\right| \le \|\mathrm{D^{n+1} f}\|_1 \tag{1.26}$$

hence

$$\|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1. \tag{1.27}$$

Combining (1.24) with (1.27) yields

$$\|\mathrm{Df}\|_1 \le \cdots \le \|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1 \le \cdots \tag{1.28}$$

We now define

$$\mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V_n^{(i)}} \triangleq \{\mathrm{f} \in \mathscr{D}_\mathrm{K} : \|\mathrm{f}\|_\mathrm{i} < 1/\mathrm{n}\} : \mathrm{n} = 1, 2, 3, \ldots\} \quad (\mathrm{i} = 1, 2, \infty), \tag{1.29}$$

so that (1.28) is mirrored in terms of neighborhood inclusions, as follows,

$$\mathrm{V_1^{(1)}} \supset \cdots \supset \mathrm{V_n^{(1)}} \supset \mathrm{V_n^{(2)}} \supset \mathrm{V_n^{(\infty)}} \supset \mathrm{V_{n+1}^{(1)}} \supset \cdots \tag{1.30}$$

Since $\mathrm{V_n^{(i)}} \supset \mathrm{V_{n+1}^{(i)}}$, $\mathscr{B}_\mathrm{i}$ is the local base of a topology τ_i. But the chain (1.30) forces the τ_i's to be equals. To see that, choose a set S that is τ_1-open at, say a: So, $\mathrm{V_n^{(1)}} \subset \mathrm{S} - \mathrm{a}$ for some n. Now $\mathrm{V_n^{(1)}} \supset \mathrm{V_n^{(2)}}$ (see (1.30)) forces $\mathrm{V_n^{(2)}} \subset \mathrm{S} - \mathrm{a}$, which implies that S is τ_2-open at a. Similarly, we deduce, still from (1.30), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.31}$$

So ends the proof. □

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $|\mathrm{f}|_\mathrm{N} = \max|\mathrm{f}|$, then, eventually, only on the ambient space itself. This should then be regarded as a very part of the textbook [2] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [1]) about intersection of nonempty compact sets.

Lemma Let X be a topological space with a countable local base $\{\mathrm{V_N} : \mathrm{N} = 1, 2, 3, \dots\}$. If $\tilde{\mathrm{V}}_\mathrm{N} = \mathrm{V}_1 \cap \dots \cap \mathrm{V_N}$, then every subsequence $\{\tilde{\mathrm{V}}_{\rho(\mathrm{N})}\}$ is a also a decreasing (*i.e.* $\tilde{\mathrm{V}}_{\rho(\mathrm{N})} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{N}+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_N} \supset \tilde{\mathrm{V}}_\mathrm{N}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{N}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\rho(\mathrm{N})}\}$, since $\tilde{\mathrm{V}}_\mathrm{N} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{N})}$. □

The following special case $\mathrm{V_N} = \tilde{\mathrm{V}}_\mathrm{N}$ is one of the key ingredients:

Corollary 1 (special case) With the same notations, if $\{\mathrm{V_N}\}$ is a decreasing local base, then so is $\{\mathrm{V}_{\rho(\mathrm{N})}\}$.

Corollary 2 If $\{\mathrm{Q_N}\}$ is a sequence of compacts that satisfies the conditions specified in section 1.44, then every subsequence $\{\mathrm{Q}_{\rho(\mathrm{N})}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $\mathrm{C}(\Omega)$'s (respectively $\mathrm{C}^\infty(\Omega)$'s) topology of the seminorms $\mathrm{p_N}$, as defined in section 1.44 (respectively 1.46), then the seminorms $\mathrm{p}_{\rho(\mathrm{N})}$ define the same topology τ_Q.

Proof. Let X be $\mathrm{C}(\Omega)$ topologized with the seminorms $\mathrm{p_N}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_N} = \{\mathrm{p_N} < 1/\mathrm{N}\}$, then $\{\mathrm{V_N}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\rho(\mathrm{N})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{N})+1} \subset \mathrm{Q}_{\rho(\mathrm{N})+1} \subset \mathrm{Q}_{\rho(\mathrm{N}+1)}. \tag{1.32}$$

Thus,

$$\mathrm{Q}_{\rho(\mathrm{N})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{N}+1)}. \tag{1.33}$$

In other words, $\mathrm{Q}_{\rho(\mathrm{N})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\rho(\mathrm{N})}\}$ then defines a topology τ_{Q_ρ} for which $\{\mathrm{V}_{\rho(\mathrm{N})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\rho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V}_{\rho(\mathrm{N})}\}$ is also a local base of τ_Q, which yields $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\rho}$. □

We are now in a fair position to establish the following:

Theorem The topology of $\mathrm{C}(\Omega)$ does not depend on the particular choice of $\{\mathrm{K_n}\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $\mathrm{C}^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.46.

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\varkappa}$, for every subsequence $\{K_{\varkappa(n)}\}$. Similarly, let $\{L_n\}$ be a sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\lambda}$ for every subsequence $\{L_{\lambda(n)}\}$. The following definition

$$C_{i,j} \triangleq K_i \setminus \overset{\circ}{L_j} \quad (i, j = 1, 2, 3, \dots) \tag{1.34}$$

shapes $\{C_{i,j} : j = 1, 2, 3, \dots\}$ as a decreasing sequence of compacts. We now suppose (to reach a contradiction) that no $C_{i,j}$ is empty and so conclude[1] that $C_{i,1} \cap C_{i,2} \cap \cdots$ contains a point that is not in any $\mathrm{int}(L_j)$. On the other hand, the conditions specified in [1.44] force the collection $\{\mathrm{int}(L_j) : j = 1, 2, 3, \dots\}$ to be an open cover. This contradiction reveals that $C_{i,j} = C_{i,j+1} = C_{i,j+2} = \cdots = \emptyset$ for some $j = j^{(i)}$. We now define $\lambda_i = i + j^{(i)}$, so that

$$K_i \setminus \overset{\circ}{L}_{\lambda_i} = \emptyset, \quad \textit{i.e.} \quad K_i \subset \overset{\circ}{L}_{\lambda_i}. \tag{1.35}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\varkappa_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\varkappa_j} \tag{1.36}$$

Combine (1.35) with (1.36) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\lambda_1} \subset L_{\lambda_1} \subset \overset{\circ}{K}_{\varkappa\circ\lambda_1} \subset K_{\varkappa\circ\lambda_1} \subset \overset{\circ}{L}_{\lambda_{\varkappa\circ\lambda_1}} \subset \cdots \tag{1.37}$$

Thus the sequence $Q = (K_1, L_{\lambda_1}, K_{\varkappa\circ\lambda_1}, L_{\lambda_{\varkappa\circ\lambda_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\varkappa} = \tau_Q = \tau_{L_\lambda} = \tau_L. \tag{1.38}$$

So ends the proof □

[1] The intersection of a decreasing sequence of nomempty Hausdorf compacts is nonempty. This is a corollary of 2.6 of [1].

Bibliography

[1] Walter Rudin. *Real and Complex Analysis.* McGraw-Hill, 1986.

[2] Walter Rudin. *Functional Analysis.* McGraw-Hill, 1991.