

Solutions to some exercises from Walter Rudin's  
*Functional Analysis*

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# Notations and Conventions

## I Logic

### i Propositional logic

Given propositional variables  $p$ ,  $q$ , the boolean operators  $\neg$ ,  $\vee$ ,  $\wedge$ ,  $\Leftrightarrow$ ,  $\Rightarrow$ ,  $\Leftarrow$ , assign boolean *truth values* as follows,

$\neg$   $\neg p$  has not the truth value of  $p$ .

$\vee$  The *conjunction*  $p \vee q$  is true, unless:  $p$  false,  $q$  false.

$\wedge$  The *disjunction* is false, unless:  $p$  true,  $q$  true.

$\Leftrightarrow$  The *logical equivalence* expresses *tautologies*:  $p \Leftrightarrow q$  is true, unless:  $p$  has not the truth value of  $q$ . It is easily checked that  $(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (p \Leftarrow q))$ ; see the below definitions.

$\Rightarrow$  The logical implication is denoted by  $\Rightarrow$ :  $p \Rightarrow q$  means *if (criterion/premise)  $p$  then (conclusion)  $q$* , or, alternatively,  $p$  *implies*  $q$ .  $p \Rightarrow q$  is formally defined as  $\neg p \vee q$ . Remark that the “reasoning”  $p \Rightarrow q$  is always valid, unless:  $p$  true,  $q$  false. Moreover,  $p \wedge (p \Rightarrow q) \Rightarrow q$  is always true.

$\Leftarrow$   $q \Leftarrow p$  is  $p \Rightarrow q$  read backward. A common pronunciation is  $q$  *since*  $p$ .

For a subtle introduction to proposition logic, see Section 1.3 and Subsection 16.1.3 of [1].

### ii Iverson notation

Given a boolean expression  $\phi$ ,  $[\phi]$  returns the truth value of  $\phi$ , encoded as follows,

$$[\phi] \triangleq \begin{cases} 0 & \text{if } \phi \text{ is false;} \\ 1 & \text{if } \phi \text{ is true.} \end{cases}$$

For example,  $[1 > 0] = 1$  but  $[\sqrt{2} \in \mathbf{Q}] = 0$ .

## II Special terms

### i Halmos’ iff and definitions

**iff** is a short for “if and only if”. Splitting **iff** into *if-then* clauses shows that it is just a rewording of the logical equivalence  $\Leftrightarrow$ . All definitions will use the **iff** format; which is consistent with the fact that every definition expresses a tautology.

## ii Assigning values

Given variables  $a$  and  $b$ ,  $\triangleq$  is a specialization of  $=$ . We say that  $a \triangleq b$  **iff**  $a$  and  $b$  are assumed to be equal. Usually,  $a \triangleq b$  means that  $a$  is assigned the previously known value  $b$  (some authors write  $a := b$ ) but this is not a limitation. Definitions can be redundant and may overlap. The only restriction is that  $a \triangleq b$  is inconsistent whether  $a \neq b$ .

## iii Equinumerosity

$a \equiv b$  means that there exists a bijection  $\rightarrow$  that maps  $a$  to  $b$ ; which let us identify  $a$  with  $b$ . In a metric space context,  $a \equiv b$  means that  $\rightarrow$  is isometric.

# III Sets

## i Subsets and supersets

Provided a pair  $(A, X)$ ,  $\subseteq$  and  $\supseteq$  are the regular symbols for set ordering, as follows:

- (1)  $A \subseteq X$  **iff**  $a \in A \Rightarrow a \in X$
- (2)  $X \supseteq A$  **iff**  $A \subseteq X$ .

Note that there is no specific symbol for a strict version: If needed, we will explicitly state that  $A \neq X$ .

# IV Topological vector spaces

## i Scalar field

The usual (complete) scalar field is  $\mathbf{C}$ . A property, *e.g.* linearity, that is true on  $\mathbf{C}$  is also true on  $\mathbf{R}$ . The complex case is then a *special case* of the real one. Sometimes, this specialization is not harmless. For example, theorem 12.7 of [4] asserts that, in a Hilbert space  $H$  equipped with the inner product  $\langle \cdot | \cdot \rangle$ , every nonzero linear continuous operator  $T$  “breaks orthogonality”, in the sense that there always exists  $x = x(T)$  in  $H$  that satisfies  $\langle Tx | x \rangle \neq 0$ . The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exist. To see that, consider the 90° rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitly mentioned*, the extension to the real case will always be obvious. So, taking  $\mathbf{C}$  as the scalar field shall mean

*Instead of letting the scalar field undefined, we choose  $\mathbf{C}$  for the sake of expressivity. But considering  $\mathbf{R}$  instead of  $\mathbf{C}$  would actually make no difference here.*

## ii Vector space bases

Given a vector space  $X$  over  $\mathbf{C}$  (or, more generally, over a field), a subset  $B$  of  $X$  is a basis of  $X$  **iff** the *finite sum*

$$(3) \quad \left\{ (z_u)_{u \in B} : z_u \in \mathbf{C}, \{u : z_u \neq 0\} \text{ is finite} \right\} \rightarrow X$$

$$(z_u) \mapsto \sum_{z_u \neq 0} z_u u$$

bijectively maps all *almost null*  $(z_u)$  onto  $X$ . The axiom of choice (AC) forces

- (a) the existence of such  $B$  (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [4] with  $B$  playing the role of  $\Lambda$ );
- (b) all bases to have the same cardinal, which is called the *dimension* of  $X$  and is denoted as  $\dim(X)$ .

We now come to the *finite-dimensional* case, *i.e.*  $\dim(X)$  is a nonnegative integer  $n$ . Remark that  $n = 0$ , *i.e.*  $B = \emptyset$ , means that  $X$  is a singleton. Our first step consists in studying  $\mathbf{C}^n$ , which is the standard  $n$ -dimensional vector space.

## iii Finite-dimensional spaces

*From now on, the zero-dimensional case, which is trivial, shall be skipped.*

### The product topology of $\mathbf{C}^n$

As the  $n$ -th power of  $\mathbf{C}$ ,  $\mathbf{C}^n$  has a standard base  $\{e_k : k = 1, \dots, n\}$  (where  $e_k = 1_{\{k\}}$ ). Furthermore, it is topologized by the *polydiscs*

$$(4) \quad \prod_{i=1}^n D_{r_i} \quad (D_{r_i} \triangleq \{z_i \in \mathbf{C} : |z_i| < r_i\}),$$

Equivalently, we may equip  $\mathbf{C}^n$  with the euclidian norm

$$(5) \quad \|z\|_2 \triangleq \sqrt{|z_1|^2 + \dots + |z_n|^2} \quad (z = (z_1, \dots, z_n) \in \mathbf{C}^n),$$

whose open balls centered at the origin are all nonempty

$$(6) \quad B_r \triangleq \{z \in \mathbf{C}^n : \|z\|_2 < r\}.$$

To show the equivalence, we pick a positive  $r$  then set  $r_i = r/\sqrt{n}$ , so that

$$(7) \quad \prod_{i=1}^n D_{r_i} \subseteq B_r.$$

Conversely, let us choose  $r = \min\{r_1, \dots, r_n\}$ . Hence

$$(8) \quad B_r \subseteq \prod_{i=1}^n D_{r_i}.$$

### Topology of a finite-dimensional vector space

It is customary to identify any  $n$ -dimensional vector space with  $\mathbf{C}^n$  topologized by the euclidian norm; see [iii]. To see this, pick a  $n$ -dimensional vector space  $Y$ , of basis  $\{u_1, \dots, u_n\}$ ; see [ii]. Setting  $u_k = f(e_k)$  means a special case of [1.20] of [4], where  $f$  is an isomorphism of  $\mathbf{C}^n$  onto  $Y$ . Actually,  $Y$  is endowed with the topology  $\{f(U) : U \text{ open}\}$ , and [1.21] of [4] states that  $f$  is an homeomorphism, which implies that

$\{f(U) : U \text{ open}\}$  is the only vector space topology for  $Y$ .

As a consequence,  $Y$  is necessarily locally convex and bounded; *i.e.* normable; see [1.39] of [4]. Moreover, provided a norm  $\|\cdot\|$  on  $Y$ , there exists a positive modulus of continuity  $C = C_f$  such that

$$(9) \quad \|y\| \leq C\|z\|_2 \quad ((z, y) \in f),$$

since  $f$  is continuous. Now pick a  $n$ -dimensional topological vector space  $W$  then repeat the same reasoning, first with  $g : \mathbf{C}^n \rightarrow W$ , next with  $h = g \circ f^{-1}$ , in the role of  $f$ , and so conclude that the homeomorphism  $h$  maps  $Y$ 's topology onto  $Y$ 's topology and that  $W$  is normable. To sum it up,

$\dim(Y) = \dim(W)$ , *i.e.*  $Y$  and  $W$  are isomorphic each other, means that  $Y$  and  $W$  are two normable spaces that are homeomorphic each other.

We then equip  $W$  with a norm  $|||\cdot|||$ , so that

$$(10) \quad |||w||| \leq C_h\|y\| \quad ((y, w) \in h)$$

for some positive  $C_h$ . The special case  $g = f$  means that  $Y$ 's norms are equivalent, in the sense that there exists a positive  $C_{id}$  such that

$$(11) \quad |||y||| \leq C_{id}\|y\|.$$

### The standard norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$

In all special cases  $Y = \mathbf{C}^n$  topologized by the standard norms 1, 2,  $\infty$ , the optimal modulus, *i.e.* the smallest  $C = C_{i,j}$  such that

$$(12) \quad \|z\|_j \leq C_{i,j}\|z\|_i \quad (z \in \mathbf{C}^n),$$

is easily derived from definitions - see [1.19] of [4] - with the noticeable exception of  $C_{2,1} = \sqrt{n}$ , which is usually seen as a special *Cauchy-Schwarz inequality*; see (1) in [12.2] of [4]. The very steps of this classical hack are left to the reader.



# Chapter 1

## Topological Vector Spaces

### 1 Exercise 1. Basic results

*Suppose  $X$  is a vector space. All sets mentioned below are understood to be subsets of  $X$ . Prove the following statements from the axioms as given as in section 1.4.*

- (a) *If  $x, y \in X$  there is a unique  $z \in X$  such that  $x + z = y$ .*
- (b)  $0 \cdot x = 0 = \alpha \cdot 0 \quad (\alpha \in \mathbf{C}, x \in X)$ .
- (c)  $2A \subseteq A + A$ .
- (d)  *$A$  is convex if and only if  $(s + t)A = sA + tA$  for all positive scalars  $s$  and  $t$ .*
- (e) *Every union (and intersection) of balanced sets is balanced.*
- (f) *Every intersection of convex sets is convex.*
- (g) *If  $\Gamma$  is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of  $\Gamma$  is convex.*
- (h) *If  $A$  and  $B$  are convex, so is  $A + B$ .*
- (i) *If  $A$  and  $B$  are balanced, so is  $A + B$ .*
- (j) *Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.*

*Proof.* (a) Such property only depends on the group structure of  $X$ : Each  $x$  in  $X$  has an opposite  $-x$ . Let  $x'$  be any opposite of  $x$ , so that  $x - x = 0 = x + x'$ . Thus,  $-x + x - x = -x + x + x'$ , which is equivalent to  $-x = x'$ . So is established the uniqueness of  $-x$ . It is now clear that  $x + z = y$  iff  $z = -x + y$ , which asserts both the existence and the uniqueness of  $z$ .

- (b) Remark that

$$(1.1) \quad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$

$$(1.2) \quad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that  $0 \cdot x = 0$ . So,

$$(1.3) \quad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x.$$

Finally,

$$(1.4) \quad \alpha \cdot 0 \stackrel{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

(c) Remark that

$$(1.5) \quad 2x = (1 + 1)x = x + x$$

for every  $x$  in  $X$ , and so conclude that

$$(1.6) \quad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets  $A$  of  $X$ ; which proves (c).

(d) If  $A$  is convex, then

$$(1.7) \quad A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A;$$

which is

$$(1.8) \quad sA + tA = (s+t)A.$$

Conversely, the special case  $s + t = 1$  is

$$(1.9) \quad sA + (1-s)A = A.$$

The latter extends to  $s = 0$ , since

$$(1.10) \quad 0A + A \stackrel{(b)}{=} \{0\} + A = A.$$

The extension to  $s = 1$  is analogously established (or simply use the fact that  $+$  is commutative!). So ends the proof.

(e) Let  $A$  range over  $B$  a collection of balanced subsets, so that

$$(1.11) \quad \alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B$$

for all scalars  $\alpha$  of magnitude  $\leq 1$ . The inclusion  $\alpha \bigcap B \subseteq A$  establishes the first part. Now remark that

$$(1.12) \quad \alpha A \subseteq \bigcup B$$

implies

$$(1.13) \quad \alpha \bigcup B \subseteq \bigcup B;$$

which achieves the proof.

(f) Let  $A$  range over  $C$  a collection of convex subsets, so that

$$(1.14) \quad (s+t) \bigcap C \subseteq s \bigcap C + t \bigcap C \subseteq sA + tA \stackrel{(d)}{\subseteq} (s+t)A$$

for all positive scalars  $s, t$ . Inclusions at both extremities force

$$(1.15) \quad s \bigcap C + t \bigcap C = (s+t) \bigcap C.$$

We now conclude from (d) that the intersection of  $C$  is convex. So ends the proof.

(g) Skip all trivial cases  $\Gamma = \emptyset, \{\emptyset\}, \{\{x\}\}, \{\emptyset, \{x\}\}$  then pick  $x_1, x_2$  in  $\bigcup \Gamma$ , so that each  $x_i$  ( $i = 1, 2$ ) lies in some  $C_i \in \Gamma$ . Since  $\Gamma$  is totally ordered by set inclusion, we henceforth assume without loss of generality that  $C_1$  is a subset of  $C_2$ . So,  $x_1, x_2$  are now elements of the convex set  $C_2$ . Every convex combination of our  $x_i$ 's is then in  $C_2 \subseteq \bigcup \Gamma$ . Hence (g).

(h) Simply remark that

$$(1.16) \quad s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B)$$

for all positive scalars  $s$  and  $t$ , then conclude from (d) that  $A+B$  is convex.

(i) Given any  $\alpha$  from the closed unit disc,

$$(1.17) \quad \alpha(A+B) = \alpha A + \alpha B \subseteq A+B.$$

There is no more to prove:  $A+B$  is balanced.

(j) Our proof will be based on the following lemma,

*If  $S$  is nonempty, then each of the following three properties*

*(i)  $S$  is a vector subspace of  $X$ ;*

*(ii)  $S$  is convex balanced such that  $S+S=S$ ;*

*(iii)  $S$  is convex balanced such that  $\lambda S = S$  ( $\lambda > 0$ )*

*implies the other two.*

To prove the lemma, let  $S$  run through all nonempty subsets of  $X$ . First, assume that (i) holds: Clearly, every  $S$  is convex balanced. Moreover,  $S+S \subseteq S$ . Conversely,  $S = S + \{0\} \subseteq S+S$ ; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \quad nS = (n-1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). Pick  $\lambda > 0$  then choose  $n$  so large that  $1 < n\lambda < n^2$ . Thus,

$$(1.19) \quad nS \stackrel{(1.18)}{\subseteq} S \subseteq n\lambda S \subseteq n^2 S,$$

since  $S$  is balanced. For instance, set  $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$ . Dividing the latter inclusions by  $n$  shows that

$$(1.20) \quad S \subseteq \lambda S \subseteq nS \stackrel{(1.18)}{\subseteq} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$(1.21) \quad \alpha S + \beta S \stackrel{(a)}{=} |\alpha|S + |\beta|S \stackrel{(d)}{=} (|\alpha| + |\beta|)S \stackrel{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as  $S$  is balanced. Moreover (under the sole assumption that  $S$  is balanced), this extends to  $|\alpha| + |\beta| = 0$ , as follows,

$$(1.22) \quad \alpha S + \beta S = 0S + 0S \stackrel{(b)}{=} \{0\} \stackrel{(b)}{=} 0S \subseteq S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let  $V$  be a collection of vector spaces of  $X$ , of intersection  $I$  and union  $U$ . First, remark that every member of  $V$  is convex balanced: So is  $I$  (combine (e) with (f)). Next, let  $Y$  range over  $V$ , so that

$$(1.23) \quad I + I \subseteq Y + Y \subseteq Y;$$

which yields

$$(1.24) \quad I + I = I$$

(the fact that  $I = I + \{0\} \subseteq I + I$  was tacitly used). It now follows from the lemma's (ii)  $\Rightarrow$  (i) that  $I$  is a vector subspace of  $X$ . Now temporarily assume that  $S$  is totally ordered by set inclusion: Combining (e) with (g) establishes that  $U$  is convex balanced. To show that  $U$  is more specifically a vector subspace, we first remark that such total order implies that either  $Z \subseteq Y$  or  $Y \subseteq Z$ , as  $Z$  ranges over  $V$ . A straightforward consequence is that

$$(1.25) \quad Y \subseteq Y + Z \subseteq Y \cup Z.$$

Another one is that  $Y \cup Z$  ranges over  $V$  as well. Combined with the latter inclusions, this leads to

$$(1.26) \quad U \subseteq U + U \subseteq U.$$

It then follows from the lemma's (ii)  $\Rightarrow$  (i) that  $U$  is a vector subspace of  $X$ . Finally, let  $A, B$  run through all vector subspaces of  $X$ : Combining (h) with (i) proves that  $A + B$  is convex balanced as well. Furthermore,

$$(1.27) \quad A + B \stackrel{(i) \Rightarrow (ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as  $X$  is an abelian group. We now conclude from (ii) that any  $A + B$  is a vector subspace of  $X$ . So ends the proof. □

## 2 Exercise 2. Convex hull

The convex hull of a set  $A$  in a vector space  $X$  is the set of all convex combinations of members of  $A$ , that is the set of all sums  $t_1x_1 + \cdots + t_nx_n$  in which  $x_i \in A$ ,  $t_i \geq 0$ ,  $\sum t_i = 1$ ;  $n$  is arbitrary. Prove that the convex hull of a set  $A$  is convex and that is the intersection of all convex sets that contain  $A$ .

*Proof.* The convex hull of a set  $S$  will be denoted by  $\text{co}(S)$ . Remark that  $S \supseteq \text{eq co}(S)$  (to see that, take  $t_1 = 1$  for each  $x_1$  in  $S$ ) and that  $\text{co}(A) \supseteq \text{eq co}(B)$  where  $A \supseteq \text{eq} B$  (obvious).

Our proof will directly derive from (i)  $\Rightarrow$  (iv) in the following lemma,

*Let  $S$  be a subset of a vector space  $X$ : Its convex hull  $\text{co}(S)$  is convex and the following statements*

- (i)  $S$  is convex;*
  - (ii)  $s_1S + \cdots + s_nS = (s_1 + \cdots + s_n)S$  for all positive scalar variables  $s_1, \dots, s_n$ ;*
  - (iii)  $t_1S + \cdots + t_nS = S$  for all positive scalar variables  $s_1, \dots, s_n$  such that  $s_1 + \cdots + s_n = 1$ ;*
  - (iv)  $\text{co}(S) = S$*
- are equivalent.*

From now on, we skip the trivial case  $S = \emptyset$  then only consider nonempty sets. To prove the first part, let  $a, b$  range over  $\text{co}(S)$ , so that  $a = t_1x_1 + \cdots + t_nx_n$  and  $b = t_{n+1}x_{n+1} + \cdots + t_{n+p}x_{n+p}$  for some  $(t_i, x_i)$ . Every sum  $sa + (1-s)b$  ( $0 \leq s \leq 1$ ) is then in the convex hull of  $\{x_1, \dots, x_{n+p}\}$ , since

$$(1.28) \quad sa + (1-s)b = \sum_{i=1}^n st_ix_i + \sum_{i=n+1}^{n+p} (1-s)t_ix_i$$

and

$$(1.29) \quad \sum_{i=1}^n st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s \sum_{i=1}^n t_i + (1-s) \sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets  $S$ , this reads as follows,

$$(1.30) \quad s \text{co}(S) + (1-s) \text{co}(S) \subseteq \text{co}(S);$$

which was our first goal. We now aim at the equivalence (i)  $\Rightarrow \cdots \Rightarrow$  (iv)  $\Rightarrow$  (i): An easy proof by induction makes the implication (i)  $\Rightarrow$  (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii)  $\Rightarrow$  (iv) derives from the definition of the convex hull. We now close the chain with (iv)  $\Rightarrow$  (i), by remarking that  $S$  is convex whether  $S = \text{co}(S)$ . The lemma being proved, let us establish the second part.

To do so, we start from the convexity of  $\text{co}(A)$  then set  $F = \{\text{co}(A)\}$ . We may enrich  $F$  as follows,

$$(1.31) \quad B \in F \Rightarrow B \text{ is convex and contains } A.$$

Note that our initial predicate “[ $F$  only encompasses] *all convex sets that contain  $A$* ”, is now the special case

$$(1.32) \quad B \in F \Leftrightarrow B \text{ is convex and contains } A.$$

In any case, the key ingredient is that  $\text{co}(A) \in F$  implies

$$(1.33) \quad \text{co}(A) \supseteq \bigcap_{B \in F} B.$$

Conversely, the next formula

$$(1.34) \quad \text{co}(A) \subseteq \text{co}(B) \stackrel{(i) \Rightarrow (iv)}{=} B \quad (B \in F)$$

is valid and implies

$$(1.35) \quad \text{co}(A) \subseteq \bigcap_{B \in F} B.$$

So ends the proof

□

### 3 Exercise 3. Other basic results

Let  $X$  be a topological vector space. All sets mentioned below are understood to be subsets of  $X$ . Prove the following statements:

- (a) The convex hull of every open set is open.
- (b) If  $X$  is locally convex then the convex hull of every bounded set is bounded.
- (c) If  $A$  and  $B$  are bounded, so is  $A+B$ .
- (d) If  $A$  and  $B$  are compact, so is  $A+B$ .
- (e) If  $A$  is compact and  $B$  is closed, then  $A+B$  is closed.
- (f) The sum of two closed sets may fail to be closed.

*Proof.* (a) Pick an open set  $A$  then let the variables  $V_i$  ( $i = 1, 2, \dots$ ) run through all open subsets of  $A$ , so that

$$(1.36) \quad \text{co}(A) \subseteq \bigcup_{t_i} (t_1 V_1 + \dots + t_i V_i + \dots) \subseteq \text{co}(A)$$

given all convex combinations  $t_1 V_1 + \dots + t_i V_i + \dots$ . We know from Section 1.7 of [4] that those sums are open; which achieves the proof.

- (b) Provided a bounded set  $E$ , pick  $V$  a neighbourhood of 0: By (b) of Section 1.14 in [4],  $V$  contains a convex neighbourhood of 0, say  $W$ . There so exists a positive scalar  $s$  such that

$$(1.37) \quad E \subseteq tW \subseteq tV \quad (t > s);$$

which yields

$$(1.38) \quad \text{co}(E) \subseteq \text{co}(tW) = t \text{co}(W) = tW \subseteq tV.$$

So ends the proof.

- (c) At fixed  $V$ , neighbourhood of the origin, we combine the continuousness of  $+$  with Section 1.14 of [4] to conclude that there exists  $U$  a balanced neighborhood of the origin such that

$$(1.39) \quad U + U \subseteq V.$$

Moreover, by the very definition of boundedness,  $A \subseteq rU$  for some positive scalar  $r$ . Similarly,  $B \subseteq sU$  for some positive  $s$ . Finally,

$$(1.40) \quad A + B \subseteq rU + sU \subseteq tU + tU \subseteq tV \quad (t > r, s),$$

since  $U$  is balanced. So ends the proof.

- (d) First,  $A$  and  $B$  are compact: So is  $A \times B$ . Next,  $+$  maps continuously  $A \times B$  onto  $A + B$ . In conclusion,  $A + B$  is compact.

- (e) From now on, we assume that neither  $A$  nor  $B$  is empty, since otherwise the result is trivial. Now pick  $c \in X$  outside  $A + B$ : The result will be established by showing that  $c$  is not in the closure of  $A + B$ .

To do so, we let the variable  $a$  range over  $A$ : Every set  $a + B$  is closed as well; see Section 1.7 of [4]. Trivially,  $a + B \neq c$ : By Section 1.10 of [4], there so exists  $V = V(a)$  a neighborhood of the origin such that

$$(1.41) \quad (a + B + V) \cap (c + V) = \emptyset.$$

Moreover, there are finitely many  $a + V$ , say  $a_1 + V_1, a_2 + V_2, \dots$ , whose union  $U$  contains the compact set  $A$ . Therefore,

$$(1.42) \quad A + B \subseteq U + B.$$

Now define

$$(1.43) \quad W \triangleq V_1 \cap V_2 \cap \dots,$$

so that

$$(1.44) \quad (a_i + B + V_i) \cap (c + W) \stackrel{(1.41)}{=} \emptyset \quad (i = 1, 2, \dots).$$

As a conclusion,  $c$  is not in the closure of  $U + B$ . Finally, (1.42) asserts that  $c$  is not in  $\overline{A + B}$  either; which achieves the proof.

**Corollary:** If  $B$  is the closure of a set  $S$ , then

$$(1.45) \quad A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B$$

by (b) of Section 1.13 of [4] (since  $A$  is closed; see Section 1.12, from the same source). The special case  $A = \{x\}$ ,  $B = X$  will occur in the proof of Exercise 15 in chapter 2.

- (f) The last proof will consist in exhibiting a counterexample. To do so, let  $f$  be any continuous mapping of the real line such that

- (i)  $f(x) + f(-x) \neq 0 \quad (x \in \mathbf{R})$ ;
- (ii)  $f$  vanishes at infinity.

For instance, we may combine (ii) with  $f$  even and  $f > 0$  by setting  $f(x) = 2^{-|x|}$ ,  $f(x) = e^{-x^2}$ ,  $f(x) = 1/(1 + |x|)$ , ..., and so on.

As a continuous function,  $f$  has closed graph  $G$ ; see [2.14] of [4]. Moreover, (i) implies that the origin  $(0, 0) \neq (x - x, f(x) + f(-x))$  is not in  $G + G$ . On the other hand,

$$(1.46) \quad \{(0, f(n) + f(-n)) : n = 1, 2, \dots\} \subseteq G + G.$$

Now the key ingredient is that

$$(1.47) \quad (0, f(n) + f(-n)) \xrightarrow[n \rightarrow \infty]{(ii)} (0, 0).$$

We have so constructed a sequence in  $G + G$  that converges outside  $G + G$ . So ends the proof. □



## 4 Exercise 4. A nonempty set whose interior is not

Let be  $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$ . Show that  $B$  is balanced but that its interior is not.

*Proof.* It is obvious that the nonempty set  $B$  contains the origin  $(0, 0)$ . Additionally, its interior  $B^\circ$  is nonempty as well. Indeed, the following set

$$(1.48) \quad \{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2| < 1/2\} \subseteq B$$

is a neighborhood of  $(1, 2) \in B$ . Moreover,  $B$  is balanced, since

$$(1.49) \quad |\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \quad (|\alpha| \leq 1)$$

for all  $(z_1, z_2)$  in  $B$ . Nevertheless, the nonempty set  $B^\circ$  is not balanced, what we now establish by showing that  $(0, 0) \notin B^\circ$ . To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$(1.50) \quad U \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2| < r\} \subseteq B$$

for some positive  $r$ . Clearly,  $U$  contains  $(r/2, 0)$ , and that special case  $(r/2, 0) \in B$  now contradicts the definition of  $B$ . So ends the proof.  $\square$

## 5 Exercise 5. A first restatement of boundedness

*Consider the definition of “bounded set” given in Section 1.6. Would the content of this definition be altered if it were required merely required that to every neighbourhood  $V$  of  $0$  corresponds some  $t > 0$  such that  $E \subseteq tV$ ?*

*Proof.* The answer is: No. To prove it, start from (a) of Section 1.14:  $V$  contains  $W$ , a balanced neighbourhood of  $0$ . Assume that  $E$  is bounded in this weaker sense, *i.e.* there exists a positive  $t$  that satisfies

$$(1.51) \quad E \subseteq tW.$$

Thus,

$$(1.52) \quad E \subseteq tW \subseteq sW \subseteq sV \quad (s > t),$$

since  $W$  is balanced. We so reach the definition given in Section 1.6: The two ones are equivalent.  $\square$

## 6 Exercise 6. A second restatement of boundedness

*Prove that a set  $E$  in a topological vector space is bounded if and only if every countable subset of  $E$  is bounded.*

*Proof.* It is clear that every subset of a bounded set is bounded. Conversely, assume that  $E$  is not bounded then pick  $V$  a neighbourhood of the origin: No counting number  $n = 1, 2, \dots$ , verifies  $E \subseteq nV$  (see Exercise 1 in Chapter 1). In other words, there exists a sequence  $\{x_1, \dots, x_n, \dots\} \subseteq E$  such that

$$(1.53) \quad x_n \notin nV.$$

As a consequence,  $x_n/n$  fails to converge to 0 as  $n$  tends to  $\infty$ . In contrast,  $1/n$  succeeds. It then follows from Section 1.30 that  $\{x_1, \dots, x_n, \dots\}$  is not bounded. So ends the proof.  $\square$

## 7 Exercise 7. Metrizability & number theory

Let  $X$  be the vector space of all complex functions on the unit interval  $[0, 1]$ , topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence  $\{f_n\}$  in  $X$  such that (a)  $\{f_n\}$  converges to 0 as  $n \rightarrow \infty$ , but (b) if  $\{\gamma_n\}$  is any sequence of scalars such that  $\gamma_n \rightarrow \infty$  then  $\{\gamma_n f_n\}$  does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as  $[0, 1]$ .) This shows that metrizability cannot be omitted in (b) of Theorem 1.28.

### 1 Justifying the terminology

*Proof.* The family of seminorms  $p_x$  is separating: The collection  $\mathcal{B}$  of all finite intersections of the sets

$$(1.54) \quad V(x, k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k = 1, 2, 3, \dots)$$

is therefore a local base for a topology  $\tau$  on  $X$ ; see Section 1.37 of [4]. So,

$$(1.55) \quad \sum_{n=1}^{\infty} [f_n \notin \cap_{i=1}^m U_i] \leq \sum_{n=1}^{\infty} \sum_{i=1}^m [f_n \notin U_i] = \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n \notin U_i] \quad (f_n \in X, U_i \in \tau).$$

Now assume that  $\{f_n\}$   $\tau$ -converges to some  $f$ , *i.e.*

$$(1.56) \quad \sum_{n=1}^{\infty} [f_n \notin f + W] < \infty \quad (W \in \mathcal{B}).$$

The special case  $W = V(x, k)$  means that, for any  $k$ ,  $|f_n(x) - f(x)| < 2^{-k}$  for almost all  $n$ . In other words,  $\{f_n(x)\}$  converges to  $f(x)$ . Conversely, assume that  $\{f_n\}$  does not  $\tau$ -converge in  $X$ , *i.e.*

$$(1.57) \quad \forall f \in X, \exists W \in \mathcal{B} : \sum_{n=1}^{\infty} [f_n \notin f + W] = \infty.$$

$W$  is then the intersection of finitely many  $V(x_1, k_1), \dots, V(x_m, k_m)$ . Hence

$$(1.58) \quad \infty \stackrel{(1.57)}{=} \sum_{n=1}^{\infty} [f_n \notin f + W] \stackrel{(1.55)}{\leq} \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)].$$

It is now clear that

$$(1.59) \quad \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)] = \infty$$

for some  $i$ . In other words,  $\{f_n(x_i)\}$  fails to converge to  $f(x_i)$ . The overall conclusion is that  $\tau$ -convergence is  $X$ 's version of pointwise convergence.  $\square$

## 2 Proof (with the given hint)

We now prove the second part by constructing a specific sequence  $\{f_n\}$  that simultaneously satisfies (a) and (b). Indeed, the hint says that there exists a one-to-one and *onto* mapping

$$(1.60) \quad \begin{aligned} \phi : [0, 1] &\rightarrow \{\theta : \theta \in \mathbf{R}^{N_+}, \lim_{n \rightarrow \infty} \theta = 0\}. \\ x &\mapsto (\theta_1, \dots, \theta_n, \dots) \end{aligned}$$

*Proof.* We set (under the same notation)

$$(1.61) \quad f_n(x) \triangleq \theta_n \xrightarrow{n \rightarrow \infty} 0 \quad (x = \phi^{-1}(\theta_1, \dots, \theta_n, \dots))$$

so that  $x_\gamma = \phi^{-1}\left(1/\sqrt{1+|\gamma_1|}, \dots, 1/\sqrt{1+|\gamma_n|}, \dots\right)$  implies

$$(1.62) \quad \gamma_n f_n(x_\gamma) = \gamma_n / \sqrt{1+|\gamma_n|} \underset{\infty}{\sim} \sqrt{\gamma_n} \xrightarrow{n \rightarrow \infty} \infty,$$

provided  $\gamma_n \rightarrow \infty$ . This proves (b), since  $\{\gamma_n f_n(x_\gamma) : n = 1, 2, 3, \dots\}$  diverges.  $\square$

## 3 Proving it the hard way (no hint)

We will use the following simple proposition about binary expansions: Every irrational has an eventually aperiodic binary expansion.

*Proof.* Formally, there exists a one-to-one and *onto* mapping

$$(1.63) \quad \begin{aligned} \text{bin} : [0, 1] \setminus \mathbf{Q} &\rightarrow \{\beta \in \{0, 1\}^{N_+} : \beta \text{ is eventually aperiodic}\} \\ v &\mapsto (\beta_1, \dots, \beta_n, \dots) \end{aligned}$$

where  $(\beta_1, \dots, \beta_n, \dots)$  satisfies

$$(1.64) \quad v = \beta_1/2 + \dots + \beta_n/2^n + \dots$$

First, note that  $\beta_1 + \beta_2 + \beta_3 + \dots = \infty$ ; then define (under the same notation)

$$(1.65) \quad f_n(v) \triangleq 1/2^{\beta_1 + \dots + \beta_n} \xrightarrow{n \rightarrow \infty} 0.$$

Now pick an arbitrary  $\gamma_n \rightarrow \infty$ : Given a positive integer  $k$ ,  $\gamma_n > 4^k$  for all sufficiently large  $n$ , say  $n > \tilde{n}$ . Next, choose  $n_k > \tilde{n}$  from  $\{\tilde{n}\}$  so large that

$$(1.66) \quad n_{k+1} - n_k > k.$$

This way,  $\chi = 1_{\{n_1, n_2, \dots\}}$  is *not eventually periodic on  $\mathbf{N}_+$* ! Moreover, one easily checks that the specialization  $\beta = \chi = \text{bin}(v_\gamma)$  yields

$$(1.67) \quad \beta_1 + \dots + \beta_{n_1} + \dots + \beta_{n_k} = k.$$

Finally, combining (1.65) with (1.67) shows that

$$(1.68) \quad \gamma_{n_k} f_{n_k}(v_\gamma) = \gamma_{n_k} / 2^k > 4^k / 2^k > 2^k \xrightarrow{k \rightarrow \infty} \infty.$$

As a conclusion, every  $\gamma_n \rightarrow \infty$  contains a subsequence  $\{\gamma_{n_k}\}$  that keeps  $\gamma_{n_k} f_{n_k}(v)$  away from 0 (for some  $v = v_\gamma$ ) when  $k \rightarrow \infty$ . This is (b).  $\square$

## 9 Exercise 9. Quotient map

*Suppose*

- (a)  $X$  and  $Y$  are topological vector spaces,
- (b)  $\Lambda : X \rightarrow Y$  is linear.
- (c)  $N$  is a closed subspace of  $X$ ,
- (d)  $\pi : X \rightarrow X/N$  is the quotient map, and
- (e)  $\Lambda x = 0$  for every  $x \in N$ .

*Prove that there is a unique  $f : X/N \rightarrow Y$  which satisfies  $\Lambda = f \circ \pi$ , that is,  $\Lambda x = f(\pi(x))$  for all  $x \in X$ . Prove that  $f$  is linear and that  $\Lambda$  is continuous if and only if  $f$  is continuous. Also,  $\Lambda$  is open if and only if  $f$  is open.*

*Proof.* Bear in mind that  $\pi$  continuously maps  $X$  onto the topological (Hausdorff) space  $X/N$ , since  $N$  is closed (see 1.41 of [4]). Moreover, the equation  $\Lambda = f \circ \pi$  has necessarily a unique solution, which is the binary relation

$$(1.69) \quad f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y.$$

To ensure that  $f$  is actually a mapping, simply remark that the linearity of  $\Lambda$  implies

$$(1.70) \quad \Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x.$$

It straightforwardly derives from (1.69) that  $f$  inherits linearity from  $\pi$  and  $\Lambda$ .

**Remark.** The special case  $N = \{\Lambda = 0\}$ , *i.e.*  $\Lambda x = 0$  **iff**  $x \in N$  (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$(1.71) \quad f(\pi x) = 0 \stackrel{(1.69)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that  $f$  is also one-to-one.

Now assume  $f$  to be continuous. Then so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [4]. Conversely, if  $\Lambda$  is continuous, then for each neighborhood  $V$  of  $0_Y$  there exists a neighborhood  $U$  of  $0_X$  such that

$$(1.72) \quad \Lambda(U) = f(\pi(U)) \subseteq V.$$

Since  $\pi$  is open (1.41 (a) of [4]),  $\pi(U)$  is a neighborhood of  $N = 0_{X/N}$ . This is sufficient to establish that the linear mapping  $f$  is continuous. If  $f$  is open, so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [4]. To prove the converse, remark that every neighborhood  $W$  of  $0_{X/N}$  satisfies

$$(1.73) \quad W = \pi(V)$$

for some neighborhood  $V$  of  $0_X$ . So,

$$(1.74) \quad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if  $\Lambda$  is open, then  $f(W)$  is a neighborhood of  $0_Y$ . So ends the proof.  $\square$

## 10 Exercise 10. An open mapping theorem

Suppose that  $X$  and  $Y$  are topological vector spaces,  $\dim Y < \infty$ ,  $\Lambda : X \rightarrow Y$  is linear, and  $\Lambda(X) = Y$ .

- (a) Prove that  $\Lambda$  is an open mapping.
- (b) Assume, in addition, that the null space of  $\Lambda$  is closed, and prove that  $\Lambda$  is continuous.

*Proof.* Discard the trivial case  $\Lambda = 0$  and assume that  $\dim Y = n$  for some positive  $n$ . Let  $e$  range over a basis of  $B$  of  $Y$  then pick in  $X$  an arbitrary neighborhood  $W$  of the origin: There so exists  $V$  a balanced neighborhood of the origin of  $X$  such that

$$(1.75) \quad \sum_e V \subseteq W,$$

since addition is continuous. Moreover, for each  $e$ , there exists  $x_e$  in  $X$  such that  $\Lambda(x_e) = e$ , simply because  $\Lambda$  is onto: Given  $y$  in  $Y$ , of  $e$ -component(s)  $y_e$ , we now obtain

$$(1.76) \quad y = \sum_e y_e \Lambda(x_e).$$

As a finite set,  $\{x_e : e \in B\}$  is bounded: There so exists a positive scalar  $s$  such that

$$(1.77) \quad \forall e \in B, x_e \in sV.$$

Combining this with (1.76) shows that

$$(1.78) \quad y \in \sum_e y_e s\Lambda(V).$$

We now come back to (1.75) and so conclude that

$$(1.79) \quad y \in \sum_e \Lambda(V) \subseteq \Lambda(W)$$

for if  $|y_e| < 1/s$ ; which proves (a) whether  $B$  is the standard basis of  $Y = \mathbf{C}^n$  equipped with  $\|\cdot\|_\infty$ . The general case is now provided for free by [iii].

To prove (b), assume that the null space  $\{\Lambda = 0\}$  is closed and let  $f, \pi$  be as in Exercise 1.9,  $\{\Lambda = 0\}$  playing the role of  $N$ . Since  $\Lambda$  is onto, the first isomorphism theorem (see Exercise 1.9) asserts that  $f$  is an isomorphism of  $X/N$  onto  $Y$ . We now conclude with the help of [iii] that  $f$  is a homeomorphism of  $X/N$  onto  $Y$ . We have thus established that  $f$  is continuous: So is  $\Lambda = f \circ \pi$ . □

## 12 Exercise 12. Topology stays, completeness leaves

Suppose  $d_1(x, y) = |x - y|$ ,  $d_2(x, y) = |\phi(x) - \phi(y)|$ , where  $\phi(x) = x/(1 + |x|)$ . Prove that  $d_1$  and  $d_2$  are metrics on  $\mathbf{R}$  which induce the same topology, although  $d_1$  is complete and  $d_2$  is not.

*Proof.* First, each  $d_i$  ( $i = 1, 2$ ) induces a topology  $\tau_i$  whose open balls are all

$$(1.80) \quad B_i(a, r) \triangleq \{x \in \mathbf{R} : d_i(a, x) < r\} \quad (a \in \mathbf{R}, r > 0).$$

Next, remark that the monotonically increasing mapping  $\phi : \mathbf{R} \rightarrow ]-1, 1[$  is odd and that

$$(1.81) \quad \phi(x) \xrightarrow{x \rightarrow \infty} 1.$$

$\phi$  is therefore a  $\tau_1$ -homeomorphism of  $\mathbf{R}$  onto  $]-1, 1[$ . A first consequence is that, at fixed  $a \in \mathbf{R}$ , given any positive scalar  $\varepsilon$ , the  $\tau_1$ -continuousness of  $\phi$  supplies an open ball  $B_1(a, \eta)$  on which  $|\phi(a) - \phi| < \varepsilon$ . In terms of balls  $B_i$ , this reads as follows,

$$(1.82) \quad B_1(a, \eta) \subseteq B_2(a, \varepsilon).$$

The second consequence is that the  $\tau_1$ -continuousness of  $\phi^{-1}$  yields similar inclusions

$$(1.83) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta')$$

provided  $\eta' > 0$ . At arbitrary  $\varepsilon$ , the special case  $\eta' = \eta$  is the concatenation

$$(1.84) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta) \subseteq B_2(a, \varepsilon);$$

which proves that  $\tau_1 = \tau_2$ . Finally, all inequalities  $n < i < j$  over  $\mathbf{N}$  together yield

$$(1.85) \quad d_2(i, j) = |\phi(i) - \phi(j)| \xrightarrow{n \rightarrow \infty} 0.$$

The sequence  $n = 0, 1, 2, \dots$  is therefore  $\tau_2$ -Cauchy. We will nevertheless establish that it  $\tau_2$ -diverges. To do so, we start by offering the  $\tau_2$ -convergence to some  $\lambda$ : The triangle inequality immediately dismiss that assumption, as follows,

$$(1.86) \quad d_2(0, \lambda) \geq d_2(0, n) - d_2(\lambda, n) = \phi(n) - d_2(\lambda, n) \xrightarrow{n \rightarrow \infty} 1.$$

We then conclude that  $d_2$  fails to be complete. □



## 14 Exercise 14. $\mathcal{D}_K$ equipped with other seminorms

Put  $K = [0, 1]$  and define  $\mathcal{D}_K$  as in Section 1.46. Show that the following three families of seminorms (where  $n = 0, 1, 2, \dots$ ) define the same topology on  $\mathcal{D}_K$ . If  $D = d/dx$ :

$$(a) \|D^n f\|_\infty = \sup\{|D^n f(x)| : 0 < x < 1\}$$

$$(b) \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$(c) \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

*Proof.* Let us equip  $\mathcal{D}_K$  with the inner product  $\langle f|g \rangle = \int_0^1 f \bar{g}$ , so that  $\langle f|f \rangle = \|f\|_2^2$ . The following

$$(1.87) \quad \int_0^1 |D^n f|^2 \leq \|1\|_2 \|D^n f\|_2$$

is then a Cauchy-Schwarz inequality; see Theorem 12.2 of [4]. We so obtain

$$(1.88) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

since  $K$  has length 1. Obviously, the support of  $D^n f$  lies in  $K$ , hence the below equality

$$(1.89) \quad |D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1.$$

Take the supremum over all  $|D^n f(x)|$ : Combining (1.88) with (1.89) now reads as follows,

$$(1.90) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots < \infty.$$

Finally, put

$$(1.91) \quad V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\},$$

$$(1.92) \quad \mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\},$$

so that (1.90) is mirrored by neighborhood inclusions, provided  $i = 1, 2, \infty$ :

$$(1.93) \quad V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \dots$$

Their subchains  $V_n^{(i)} \supseteq V_{n+1}^{(i)}$  turn  $\mathcal{B}^{(i)}$  into a local base of a topology  $\tau_i$ . The whole chain (1.93) then forces

$$(1.94) \quad \tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1;$$

which achieves the proof.  $\square$

## 16 Exercise 16. Uniqueness of topology for test functions

*Prove that the topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Do the same for  $C^\infty(\Omega)$  (Section 1.46).*

**Comment** This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms  $p_n$ , then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [4] The proof consists in combining trivial consequences of the local base definition with a well-known result (e.g. [2.6] in [3]) about intersection of nonempty compact sets.

**Lemma 1** *Let  $X$  be a topological space with a countable local base  $\{V_n : n = 1, 2, 3, \dots\}$ . If  $\tilde{V}_n = V_1 \cap \dots \cap V_n$ , then every subsequence  $\{\tilde{V}_{\varrho(n)}\}$  is a decreasing (i.e.  $\tilde{V}_{\varrho(n)} \supseteq \tilde{V}_{\varrho(n+1)}$ ) local base of  $X$ .*

*Proof.* The decreasing property is trivial. Now remark that  $V_n \supseteq \tilde{V}_n$ : This shows that  $\{\tilde{V}_n\}$  is a local base of  $X$ . Then so is  $\{\tilde{V}_{\varrho(n)}\}$ , since  $\tilde{V}_n \supseteq \tilde{V}_{\varrho(n)}$ .  $\square$

The following special case  $V_n = \tilde{V}_n$  is one of the key ingredients:

**Corollary 1 (special case  $V_n = \tilde{V}_n$ )** *Under the same notations of Lemma 1, if  $\{V_n\}$  is a decreasing local base, then so is  $\{V_{\varrho(n)}\}$ .*

**Corollary 2** *If  $\{Q_n\}$  is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence  $\{Q_{\varrho(n)}\}$  also satisfies theses conditions. Furthermore, if  $\tau_Q$  is the  $C(\Omega)$ 's (respectively  $C^\infty(\Omega)$ 's) topology of the seminorms  $p_n$ , as defined in section 1.44 (respectively 1.46), then the seminorms  $p_{\varrho(n)}$  define the same topology  $\tau_Q$ .*

*Proof.* Let  $X$  be  $C(\Omega)$  topologized by the seminorms  $p_n$  (the case  $X = C^\infty(\Omega)$  is proved the same way). If  $V_n = \{p_n < 1/n\}$ , then  $\{V_n\}$  is a decreasing local base of  $X$ . Moreover,

$$(1.95) \quad Q_{\varrho(n)} \subseteq \overset{\circ}{Q}_{\varrho(n)+1} \subseteq Q_{\varrho(n)+1} \subseteq Q_{\varrho(n+1)}.$$

Thus,

$$(1.96) \quad Q_{\varrho(n)} \subseteq \overset{\circ}{Q}_{\varrho(n+1)}.$$

In other words,  $Q_{\varrho(n)}$  satisfies the conditions specified in section 1.44.  $\{p_{\varrho(n)}\}$  then defines a topology  $\tau_{Q_\varrho}$  for which  $\{V_{\varrho(n)}\}$  is a local base. So,  $\tau_{Q_\varrho} \subseteq \tau_Q$ . Conversely, the above corollary asserts that  $\{V_{\varrho(n)}\}$  is a local base of  $\tau_Q$ , which yields  $\tau_Q \subseteq \tau_{Q_\varrho}$ .  $\square$

**Lemma 2** *If a sequence of compact sets  $\{Q_n\}$  satisfies the conditions specified in section 1.44, then every compact set  $K$  lies in almost all  $Q_n^\circ$ , i.e. there exists  $m$  such that*

$$(1.97) \quad K \subseteq \overset{\circ}{Q}_m \subseteq \overset{\circ}{Q}_{m+1} \subseteq \overset{\circ}{Q}_{m+2} \subseteq \cdots .$$

*Proof.* The following definition

$$(1.98) \quad C_n \triangleq K \setminus \overset{\circ}{Q}_n$$

shapes  $\{C_n\}$  as a decreasing sequence of compact<sup>1</sup> sets. We now suppose (to reach a contradiction) that no  $C_n$  is empty and so conclude<sup>2</sup> that the  $C_n$ 's intersection contains a point that is not in any  $Q_n^\circ$ . On the other hand, the conditions specified in [1.44] force the  $Q_n^\circ$ 's collection to be an open cover. This contradiction reveals that  $C_m = \emptyset$ , i.e.  $K \subseteq Q_m^\circ$ , for some  $m$ . Finally,

$$(1.99) \quad K \subseteq \overset{\circ}{Q}_m \subseteq Q_m \subseteq \overset{\circ}{Q}_{m+1} \subseteq Q_{m+1} \subseteq \overset{\circ}{Q}_{m+2} \subseteq \cdots .$$

□

We are now in a fair position to establish the following:

**Theorem** *The topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of  $C^\infty(\Omega)$ , as long as this sequence satisfies the conditions specified in section 1.44.*

*Proof.* With the second corollary's notations,  $\tau_K = \tau_{K_\lambda}$ , for every subsequence  $\{K_{\lambda(n)}\}$ . Similarly, let  $\{L_n\}$  be another sequence of compact subsets of  $\Omega$  that satisfies the condition specified in [1.44], so that  $\tau_L = \tau_{L_\kappa}$  for every subsequence  $\{L_{\kappa(n)}\}$ . Now apply the above Lemma 2 with  $K_i$  ( $i = 1, 2, 3, \dots$ ) and so conclude that  $K_i \subseteq L_{m_i}^\circ \subseteq L_{m_i+1}^\circ \subseteq \cdots$  for some  $m_i$ . In particular, the special case  $\kappa_i = m_i + i$  is

$$(1.100) \quad K_i \subseteq \overset{\circ}{L}_{\kappa_i}.$$

Let us reiterate the above proof with  $K_n$  and  $L_n$  in exchanged roles then similarly find a subsequence  $\{\lambda_j : j = 1, 2, 3, \dots\}$  such that

$$(1.101) \quad L_j \subseteq \overset{\circ}{K}_{\lambda_j}$$

Combine (1.100) with (1.101) and so obtain

$$(1.102) \quad K_1 \subseteq \overset{\circ}{L}_{\kappa_1} \subseteq L_{\kappa_1} \subseteq \overset{\circ}{K}_{\lambda_{\kappa_1}} \subseteq K_{\lambda_{\kappa_1}} \subseteq \overset{\circ}{L}_{\kappa_{\lambda_{\kappa_1}}} \subseteq \cdots ,$$

which means that the sequence  $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$  satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$(1.103) \quad \tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L.$$

So ends the proof

□

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<sup>1</sup>See (b) of 2.5 of [3].

<sup>2</sup>In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of 2.6 of [3].

## 17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that  $f \mapsto D^\alpha f$  is a continuous mapping of  $C^\infty(\Omega)$  into  $C^\infty(\Omega)$  and also of  $\mathcal{D}_K$  into  $\mathcal{D}_K$ , for every multi-index  $\alpha$ .

*Proof.* In both cases,  $D^\alpha$  is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the  $C^\infty(\Omega)$  case.

Let  $U$  be an arbitrary neighborhood of the origin. There so exists  $N$  such that  $U$  contains

$$(1.104) \quad V_N = \{ \varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \}.$$

Now pick  $g$  in  $V_{N+|\alpha|}$ , so that

$$(1.105) \quad \max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}.$$

(the fact that  $K_N \subseteq K_{N+|\alpha|}$  was tacitely used). The special case  $\gamma = \beta + \alpha$  yields

$$(1.106) \quad \max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

$$(1.107) \quad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad \text{i.e.} \quad D^\alpha(V_{N+|\alpha|}) \subseteq V_N,$$

which establishes the continuity of  $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$ .

To prove the continuousness of the restriction  $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$ , we first remark that the collection of the  $V_N \cap \mathcal{D}_K$  is a local base of the subspace topology of  $\mathcal{D}_K$ .  $V_{N+|\alpha|} \cap \mathcal{D}_K$  is then a neighborhood of 0 in this topology. Furthermore,

$$(1.108) \quad D^\alpha|_{\mathcal{D}_K}(V_{N+|\alpha|} \cap \mathcal{D}_K) = D^\alpha(V_{N+|\alpha|} \cap \mathcal{D}_K)$$

$$(1.109) \quad \subseteq D^\alpha(V_{N+|\alpha|}) \cap D^\alpha(\mathcal{D}_K)$$

$$(1.110) \quad \subseteq V_N \cap \mathcal{D}_K \quad (\text{see (1.107)})$$

So ends the proof. □

# Chapter 2

## Completeness

### 3 Exercise 3. An equicontinuous sequence of measures

Put  $K = [-1, 1]$ ; define  $\mathcal{D}_K$  as in section 1.46 (with  $\mathbf{R}$  in place of  $\mathbf{R}^n$ ). Suppose  $\{f_n\}$  is a sequence of Lebesgue integrable functions such that  $\Lambda\varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\varphi(t)dt$  exists for every  $\varphi \in \mathcal{D}_K$ . Show that  $\Lambda$  is a continuous linear functional on  $\mathcal{D}_K$ . Show that there is a positive integer  $p$  and a number  $M < \infty$  such that

$$\left| \int_{-1}^1 f_n(t)\varphi(t)dt \right| \leq M\|D^p\|_\infty$$

for all  $n$ . For example, if  $f_n(t) = n^3t$  on  $[-1/n, 1/n]$  and 0 elsewhere, show that this can be done with  $p = 1$ . Construct an example where it can be done with  $p = 2$  but not with  $p = 1$ .

We will also consider the case  $p = 0$ . Since all supports of  $\varphi, \varphi', \varphi'', \dots$ , are in  $K$ , we make a specialization of the mean value theorem:

**Lemma** If  $\varphi \in \mathcal{D}_{[a,b]}$ , then

$$(2.1) \quad \|D^\alpha\varphi\|_\infty \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (\alpha = 0, 1, \dots, p)$$

at every order  $p = 0, 1, 2, \dots$ ; where  $\lambda$  is the length  $|b - a|$ .

*Proof.* Let  $x_0$  be in  $(a, b)$ . We first consider the case  $x_0 \leq c = (a + b)/2$ : The mean value theorem asserts that there exists  $x_1$  ( $a < x_1 < x_0$ ), such that

$$(2.2) \quad \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a).$$

Since every  $D^p\varphi$  lies in  $\mathcal{D}_{[a,b]}$ , a straightforward proof by induction shows that there exists a partition  $a < \cdots < x_p < \cdots < x_0$  such that

$$(2.3) \quad \varphi(x_0) = D^0\varphi(x_0)$$

$$(2.4) \quad \begin{aligned} &= D^1\varphi(x_1)(x_0 - a) \\ &= \cdots \end{aligned}$$

$$(2.5) \quad = D^p\varphi(x_p)(x_0 - a) \cdots (x_{p-1} - a),$$

for all  $p$ . More compactly,

$$(2.6) \quad D^\alpha\varphi(x_0) = D^p\varphi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a);$$

which yields,

$$(2.7) \quad |D^\alpha\varphi(x)| \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c])$$

The case  $x_0 \geq c$  outputs a “reversed” result, with  $b > \cdots > x_p > \cdots > x_0$  and  $x_k - b$  playing the role of  $x_k - a$ : So,

$$(2.8) \quad |D^\alpha\varphi(x)| \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$(2.9) \quad \|D^\alpha\varphi\|_\infty \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}.$$

□

*Proof.* We first consider  $C_0(\mathbf{R})$  topologized by the supremum norm. Given a Lebesgue integrable function  $u$ , we put

$$(2.10) \quad \langle u|\varphi \rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})).$$

The following inequalities

$$(2.11) \quad |\langle u|\varphi \rangle| \leq \int_{\mathbf{R}} |u\varphi| \leq \|u\|_{L^1} \quad (\|\varphi\|_\infty \leq 1)$$

imply that every linear functional

$$(2.12) \quad \begin{aligned} \langle u| : C_0(\mathbf{R}) &\rightarrow \mathbf{C} \\ \varphi &\mapsto \langle u|\varphi \rangle \end{aligned}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [4]. Conversely,  $u$  can be identified with  $\langle u|$ , since  $u$  is determined (a.e) by the

integrals  $\langle u|\varphi \rangle$ . In the Banach spaces terminology,  $u$  is then (identified with) a linear *bounded*<sup>1</sup> operator  $\langle u|$ , of norm

$$(2.13) \quad \sup\{|\langle u|\varphi \rangle| : \|\varphi\|_\infty = 1\} = \|u\|_{L^1}.$$

Note that, in the latter equality,  $\leq \|u\|_{L^1}$  comes from (2.11), as the converse comes from the Stone-Weierstrass theorem<sup>2</sup>. We now consider the special cases  $u = g_n$ , where  $g_n$  is

$$(2.14) \quad g_n : \mathbf{R} \rightarrow \mathbf{R} \\ x \mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) \end{cases}.$$

First, remark that  $g_n(x) \rightarrow 0$ , as the sequence  $\{g_n\}$  fails to converge in  $C_0(\mathbf{R})$  (since  $g_n(1/n) = n^2 \geq 1$ ), and also in  $L^1$  (since  $\int_{\mathbf{R}} |g_n| = n^2 \rightarrow \infty$ ). Nevertheless, we will show that the  $\langle g_n|$  converge pointwise<sup>3</sup> on  $\mathcal{D}_K$  *i.e.* there exists a  $\tau_K$ -continuous linear form  $\Lambda$  such that

$$(2.15) \quad \langle g_n|\varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda\varphi,$$

where  $\varphi$  ranges over  $\mathcal{D}_K$ . We now prove (2.13) in the special cases  $u = g_n$ . To do so, we fetch  $\varphi_1^+, \dots, \varphi_j^+, \dots$ , from  $C_K^\infty(\mathbf{R})$ . More specifically,

- (i)  $\varphi_j^+ = 1$  on  $[e^{-j}, 1 - e^{-j}]$ ;
- (ii)  $\varphi_j^+ = 0$  on  $\mathbf{R} \setminus [-1, 1]$ ;
- (iii)  $0 \leq \varphi_j^+ \leq 1$  on  $\mathbf{R}$ ;

see [1.46] of [4] for a possible construction of those  $\varphi_j^+$ . Let  $\varphi_1^-, \dots, \varphi_j^-, \dots$ , mirror the  $\varphi_j^+$ , in the sense that  $\varphi_j^-(x) = \varphi_j^+(-x)$ , so that

- (iv)  $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$  is odd, as  $g_n$  is;
- (v) every  $\varphi_j$  is in  $C_K^\infty(\mathbf{R})$ ;
- (vi) The sequence  $\{\varphi_j\}$  converges (pointwise) to  $1_{[0,1]} - 1_{[-1,0]}$ , and  $\|\varphi_j\|_\infty = 1$ .

Thus, with the help of the Lebesgue's convergence theorem,

$$(2.16) \quad \langle g_n|\varphi_j \rangle = 2 \int_0^1 g_n(t) \varphi_j^+(t) dt \xrightarrow{j \rightarrow \infty} 2 \int_0^1 g_n(t) dt = \|g_n\|_{L^1} = n.$$

Finally,

$$(2.17) \quad \|g_n\|_{L^1} \stackrel{(2.16)}{\leq} \sup\{|\langle g_n|\varphi \rangle| : \|\varphi\|_\infty = 1\} \stackrel{(2.13)}{\leq} \|g_n\|_{L^1};$$

which is the desired result. So, in terms of boundedness constants: Given  $n$ , there exists  $C_n < \infty$  such that

$$(2.18) \quad |\langle g_n|\varphi \rangle| \leq C_n \quad (\|\varphi\|_\infty = 1);$$

<sup>1</sup>see 1.32, 4.1 of [4]

<sup>2</sup>See 7.26 of [2].

<sup>3</sup>See 3.14 of [4] for a definition of the related topology.

see (2.11). Furthermore,  $\|g_n\|_{L^1}$  is actually the best, *i.e.* lowest, possible  $C_n$ ; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence  $\{\langle g_n | \varphi_{\rho(n)} \rangle\}$  such that  $\langle g_n | \varphi_{\rho(n)} \rangle$  is greater than, say,  $n - 0.01$ , as  $\|\varphi_{\rho(n)}\|_\infty = 1$ . Consequently, there is no bound  $M$  such that

$$(2.19) \quad |\langle g_n | \varphi \rangle| \leq M \quad (\|\varphi\|_\infty = 1; n = 1, 2, 3, \dots).$$

In other words, the  $g_n$  have no *uniform bound* in  $L^1$ , *i.e.* the collection of all continuous linear mappings  $\langle g_n |$  is not equicontinuous (see discussion in 2.6 of [4]). As a consequence, the  $\langle g_n |$  do not converge pointwise (or “vaguely”, in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$(2.20) \quad \langle g_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \varphi \quad (\varphi \in C_0(\mathbf{R}))$$

for some  $\Lambda \in C_0(\mathbf{R})^*$ , which would make (2.19) hold; see 2.6, 2.8 of [4]. This by no means says that the  $\langle g_n |$  do not converge pointwise, in a relevant space, to some  $\Lambda$  (see (2.15)).

From now on, unless the contrary is explicitly stated, we assume that  $\varphi$  only denotes an element of  $C_K^\infty(\mathbf{R})$ . Let  $f_n$  be a Lebesgue integrable function such that

$$(2.21) \quad \Lambda \varphi = \lim_{n \rightarrow \infty} \int_K f_n \varphi \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

for some linear form  $\Lambda$ . Since  $\varphi$  vanishes outside  $K$ , we can suppose without loss of generality that the support of  $f_n$  lies in  $K$ . So, (2.21) can be restated as follows,

$$(2.22) \quad \Lambda \varphi = \lim_{n \rightarrow \infty} \langle f_n | \varphi \rangle \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

Let  $K_1, K_2, \dots$ , be compact sets that satisfy the conditions specified in 1.44 of [4].  $\mathcal{D}_K$  is  $C_K^\infty(\mathbf{R})$  topologized by the related seminorms  $p_1, p_2, \dots$ ; see 1.46, 6.2 of [4] and Exercise 1.16. We know that  $K \subseteq K_m$  for some index  $m$  (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices  $N \geq m$ , so that

- (a)  $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha \varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$ , for  $\varphi \in \mathcal{D}_K$ ;
- (b) The collection of the sets  $V_N = \{\varphi \in \mathcal{D}_K : \|\varphi\|_N < 2^{-N}\}$  is a (decreasing) local base of  $\tau_K$ , the subspace topology of  $\mathcal{D}_K$ ; see 6.2 of [4] for a more complete discussion.

Let us specialize (2.11) with  $u = f_n$  and  $\varphi \in V_m$  then conclude that  $\langle f_n |$  is bounded by  $\|f_n\|_{L^1}$  on  $V_m$ : Every linear functional  $\langle f_n |$  is therefore  $\tau_K$ -continuous; see 1.18 of [4].

To sum it up:

- (i)  $\mathcal{D}_K$ , equipped the topology  $\tau_K$ , is a Fréchet space (see section 1.46 of [4]);



- (ii) Every linear functional  $\langle f_n |$  is continuous with respect to this topology;
- (iii)  $\langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \varphi$  for all  $\varphi$ , i.e.  $\Lambda - \langle f_n | \xrightarrow{n \rightarrow \infty} 0$ .

With the help of [2.6] and [2.8] of [4], we conclude that  $\Lambda$  is continuous and that the sequence  $\{\langle f_n | \}$  is equicontinuous. So is the sequence  $\{\Lambda - \langle f_n | \}$ , since addition is continuous. There so exists  $i, j$  such that, for all  $n$ ,

$$(2.23) \quad |\Lambda \varphi| < 1/2 \quad \text{if } \varphi \in V_i,$$

$$(2.24) \quad |\Lambda \varphi - \langle f_n | \varphi \rangle| < 1/2 \quad \text{if } \varphi \in V_j.$$

Choose  $p = \max\{i, j\}$ , so that  $V_p = V_i \cap V_j$ : The latter inequalities imply that

$$(2.25) \quad |\langle f_n | \varphi \rangle| \leq |\Lambda \varphi - \langle f_n | \varphi \rangle| + |\Lambda \varphi| < 1 \quad \text{if } \varphi \in V_p.$$

Now remark that every  $\psi = \psi[\mu, \varphi]$ , where

$$(2.26) \quad \psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\varphi\|_p) \varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases}$$

keeps in  $V_p$ . Finally, it is clear that each below statement implies the following one.

$$(2.27) \quad |\langle f_n | \psi \rangle| < 1$$

$$(2.28) \quad |\langle f_n | \varphi \rangle| < 2^p \|\varphi\|_p \cdot \mu$$

$$(2.29) \quad |\langle f_n | \varphi \rangle| \leq 2^p \|\varphi\|_p$$

$$(2.30) \quad |\langle f_n | \varphi \rangle| \leq 2^p \{ \|D^0 \varphi\|_\infty + \cdots + \|D^p \varphi\|_\infty \}.$$

Finally, with the help of (2.1),

$$(2.31) \quad |\langle f_n | \varphi \rangle| \leq 2^p (p+1) \|D^p \varphi\|_\infty.$$

The first part is so proved, with *some*  $p$  and  $M = 2^p(p+1)$ .

We now come back to the special case  $f_n = g_n$  (see the first part). From now on,  $f_n(x) = n^3 x$  on  $[-1/n, 1/n]$ , 0 elsewhere. Actually, we will prove that

- (a)  $\Lambda \varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t) \varphi(t) dt$  exists for every  $\varphi \in \mathcal{D}_K$ ;
- (b) A *uniform* bound  $|\langle f_n | \varphi \rangle| \leq M \|D^p \varphi\|_\infty$  ( $n = 1, 2, 3, \dots$ ) exists for all those  $f_n$ , with  $p = 1$  as the smallest possible  $p$ .

Bear in mind that  $K \subseteq K_m$  and shift the  $K_N$ 's indices, so that  $K_{m+1}$  becomes  $K_1$ ,  $K_{m+2}$  becomes  $K_2$ , and so on. The resulting topology  $\tau_K$  remains unchanged (see Exercise 1.16). We let  $\varphi$  keep running on  $\mathcal{D}_K$  and so define

$$(2.32) \quad B_n(\varphi) \triangleq \max\{|\varphi(x)| : x \in [-1/n, 1/n]\},$$

$$(2.33) \quad \Delta_n(\varphi) \triangleq \max\{|\varphi(x) - \varphi(0)| : x \in [-1/n, 1/n]\}.$$

The mean value asserts that

$$(2.34) \quad |\varphi(1/n) - \varphi(-1/n)| \leq B_n(\varphi') |1/n - (-1/n)| = \frac{2}{n} B_n(\varphi').$$

Independently, an integration by parts shows that

$$(2.35) \quad \langle f_n | \varphi \rangle = \left[ \frac{n^3 t^2}{2} \varphi(t) \right]_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt$$

$$(2.36) \quad = \frac{n}{2} (\varphi(1/n) - \varphi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt.$$

Combine (2.34) with (2.36) and so obtain

$$(2.37) \quad |\langle f_n | \varphi \rangle| \leq \frac{n}{2} |\varphi(1/n) - \varphi(-1/n)| + \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\varphi'(t)| dt$$

$$(2.38) \quad \leq B_n(\varphi') + \frac{n^3}{2} B_n(\varphi') \int_{-1/n}^{1/n} t^2 dt$$

$$(2.39) \quad \leq \frac{4}{3} B_n(\varphi')$$

$$(2.40) \quad \leq \frac{4}{3} \|\varphi'\|_\infty.$$

Futhermore, (2.39) gives a hint about the convergence of  $f_n$ : Since  $B_n(\varphi')$  tends to  $|\varphi'(0)|$ , we may expect that  $f_n$  tends to  $\frac{4}{3}\varphi'(0)$ . This is actually true: A straightforward computation shows that

$$(2.41) \quad \langle f_n | \varphi \rangle - \frac{4}{3} \varphi'(0) \stackrel{(2.36)}{=} \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} (\varphi' - \varphi'(0)) t^2 dt.$$

So,

$$(2.42) \quad \left| \langle f_n | \varphi \rangle - \frac{4}{3} \varphi'(0) \right| \leq \left| \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0) \right| + \frac{1}{3} \Delta_n(\varphi') \xrightarrow{n \rightarrow \infty} 0.$$

We have just proved that

$$(2.43) \quad \langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3} \varphi'(0) \quad (\varphi \in \mathcal{D}_K).$$

In other words,

$$(2.44) \quad \langle f_n | \xrightarrow{n \rightarrow \infty} -\frac{4}{3} \delta',$$

where  $\delta$  is the *Dirac measure* and  $\delta', \delta'', \dots$ , its *derivatives*; see 6.1 and 6.9 of [4].

It follows from the previous part that  $-\frac{4}{3}\delta'$  is  $\tau_K$ -continuous, and from (2.40) that

$$(2.45) \quad |\langle f_n | \varphi \rangle| \leq \frac{4}{3} \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$(2.46) \quad \{ \langle f_n | \varphi_{\rho(n)} \rangle : \|\varphi_{\rho(n)}\|_\infty = 1; n = 1, 2, 3, \dots \}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant  $M$  such that

$$(2.47) \quad |\langle f_n | \varphi \rangle| \leq M \|\varphi\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

The previous bound of  $\langle f_n |$  - see (2.40), is therefore the best possible one, *i.e.*  $p = 1$  is the smallest possible  $p$  and, given  $p = 1$ ,  $M = \frac{4}{3}$  is the smallest possible  $M$  (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give  $f_n$  a *derivative*<sup>4</sup>  $f_n'$ , as follows

$$(2.48) \quad \begin{aligned} f_n' : \mathcal{D}_K &\rightarrow \mathbf{C} \\ \varphi &\mapsto -\langle f_n | \varphi' \rangle. \end{aligned}$$

It has been proved that every  $\langle f_n |$  is continuous. So is

$$(2.49) \quad \begin{aligned} D : \mathcal{D}_K &\rightarrow \mathcal{D}_K \\ \varphi &\mapsto \varphi'; \end{aligned}$$

see Exercise 1.17.  $f_n'$  is therefore continuous. Now apply (2.43) with  $\varphi'$  and so obtain

$$-\langle f_n | \varphi' \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3} \varphi''(0) \quad (\varphi \in \mathcal{D}_K),$$

*i.e.*

$$(2.50) \quad f_n' \xrightarrow{n \rightarrow \infty} \frac{4}{3} \delta''.$$

It follows from (2.40) that,

$$(2.51) \quad |\langle f_n | \varphi' \rangle| \leq \frac{4}{3} \|\varphi''\|_\infty \quad (n = 1, 2, 3, \dots).$$

It is therefore possible to uniformly bound  $f_n'$  with respect to a norm  $\|D^p \cdot\|_\infty$ , namely  $\|D^2 \cdot\|_\infty$ . Then arises a question: Is 2 the smallest  $p$ ? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant  $M$  such that

$$(2.52) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Define

$$(2.53) \quad \Phi_j(x) = \int_{-1}^x \varphi_j.$$

---

<sup>4</sup>See 6.1 of [4] for a further discussion.

The oddness of  $\varphi_j$  forces  $\Phi_j$  to vanish outside  $[-1, 1]$ :  $\varphi_j$  is therefore in  $\mathcal{D}_K$ . So, under our assumption,

$$(2.54) \quad |\langle f_n | \Phi_j' \rangle| \leq M \|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots);$$

which is

$$(2.55) \quad |\langle f_n | \varphi_j \rangle| \leq M \quad (n = 1, 2, 3, \dots).$$

We have thus reached a contradiction (again with the sequence  $\{\langle f_n | \varphi_{\rho(n)} \rangle\}$ ) and so conclude that there is no constant  $M$  such that

$$(2.56) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Finally, assume, to reach a contradiction, that there exists a constant  $M$  such that

$$(2.57) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi\|_\infty.$$

The mean value theorem (see (2.1)) asserts that

$$(2.58) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi\|_\infty \leq M \|\varphi'\|_\infty;$$

which is, again, a desired contradiction. So ends the proof.

□

## 6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient  $\hat{f}(n)$  of a function  $f \in L^2(T)$  ( $T$  is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all  $n \in \mathbf{Z}$  (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^n \hat{f}(k).$$

Prove that  $\{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$  is a dense subspace of  $L^2(T)$  of the first category.

*Proof.* Let  $f(\theta)$  stand for  $f(e^{i\theta})$ , so that  $L^2(T)$  is identified with a closed subset of  $L^2([-\pi, \pi])$ , hence the inner product

$$(2.59) \quad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

$$(2.60) \quad \Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well known (and easy to prove) result is

$$(2.61) \quad (e_n, e_{n'}) = [n = n'], \text{ i.e. } \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric ( $\equiv$ ) identification  $L^2 \equiv (L^2)^*$ . So,

$$(2.62) \quad \|\Lambda_n\|^2 \stackrel{(2.60)}{=} \|e_{-n} + \cdots + e_n\|^2 \stackrel{(2.61)}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \stackrel{(2.61)}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

$$(2.63) \quad B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \dots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [4] asserts that the sequence  $\{\Lambda_n\}$  is norm-bounded; which is a desired contradiction, since

$$(2.64) \quad \|\Lambda_n\| \stackrel{(2.62)}{=} \sqrt{2n+1} \xrightarrow{n \rightarrow \infty} \infty.$$

We have just established that  $B$  is actually of the first category; and so is its subset  $L = \{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$ . We now prove that  $L$  is nevertheless dense in  $L^2(T)$ . To do so, we let  $P$  be  $\text{span}\{e_k : k \in \mathbf{Z}\}$ , the collection of the trigonometric polynomials  $p(\theta) = \sum \lambda_k e^{ik\theta}$ . Combining (2.60) with (2.61) shows that  $\Lambda_n(p) = \sum \lambda_k$  for almost all  $n$ . Thus,

$$(2.65) \quad P \subseteq L \subseteq L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that  $P$  is dense in  $L^2(T)$ . We then conclude, with the help of (2.65), that

$$(2.66) \quad L^2(T) = \overline{P} = \overline{L}.$$

So ends the proof □

## 9 Exercise 9. Boundedness without closedness

Suppose  $X, Y, Z$  are Banach spaces and

$$B : X \times Y \rightarrow Z$$

is bilinear and continuous. Prove that there exists  $M < \infty$  such that

$$\|B(x, y)\| \leq M\|x\|\|y\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

*Proof.* The answer is: No. To prove this, we only assume that  $X, Y, Z$  are normed spaces. Since  $B$  is continuous at the origin, there exists a positive  $r$  such that

$$(2.67) \quad \|x\| + \|y\| < r \Rightarrow \|B(x, y)\| < 1.$$

Given nonzero  $x, y$ , let  $s$  range over  $]0, r[$ , so that the following bound

$$(2.68) \quad \|B(x, y)\| = \frac{4\|x\|\|y\|}{s^2} \left\| B\left(\frac{s}{2\|x\|}x, \frac{s}{2\|y\|}y\right) \right\| \stackrel{(2.67)}{<} \frac{4\|x\|\|y\|}{s^2}$$

is effective. It is now obvious that

$$(2.69) \quad B(x, y) \leq \frac{4}{s^2}\|x\|\|y\| \xrightarrow{s \rightarrow r} \frac{4}{r^2}\|x\|\|y\| \quad ((x, y) \in X \times Y);$$

which achieves the proof.

As a concrete example, choose  $X = Y = Z = C_c(\mathbf{R})$ , topologized by the supremum norm.  $C_c(\mathbf{R})$  is not complete (see 5.4.4 of [5]), nevertheless the bilinear product

$$\begin{aligned} B : C_c(\mathbf{R})^2 &\rightarrow C_c(\mathbf{R}) \\ (f, g) &\mapsto f \cdot g \end{aligned}$$

is bounded (since  $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$ ), and continuous. To show this, pick a positive scalar  $\varepsilon$  smaller than 1, provided any  $(f, g)$ . Next, define

$$(2.70) \quad r \triangleq \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty} < 1.$$

We now restrict  $(u, v)$  to a particular neighborhood of  $(f, g)$ . More specifically,

$$(2.71) \quad \|f - u\|_\infty + \|g - v\|_\infty < r.$$

Next, remark that  $\|u\|_\infty \leq r + \|f\|_\infty$  and so obtain (bear in mind that  $r < 1$ )

$$(2.72) \quad \|fg - uv\|_\infty = \|(f - u) \cdot g + u \cdot (g - v)\|_\infty$$

$$(2.73) \quad \leq \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty$$

$$(2.74) \quad < r \cdot \|g\|_\infty + (r + \|f\|_\infty) \cdot r$$

$$(2.75) \quad < r \cdot (r + \|f\|_\infty + \|g\|_\infty)$$

$$(2.76) \quad < \varepsilon.$$

Since  $\varepsilon$  was arbitrary, it is now established that  $B$  continuous at every  $(f, g)$ .  $\square$

## 10 Exercise 10. Continuousness of bilinear mappings

*Prove that a bilinear mapping is continuous if it is continuous at the origin  $(0, 0)$ .*

*Proof.* Let  $(X_1, X_2, Z)$  be topological spaces and  $B$  a bilinear mapping

$$(2.77) \quad B : X_1 \times X_2 \rightarrow Z.$$

From now on,  $x = (x_1, x_2)$  denotes an arbitrary element of  $X_1 \times X_2$ . We henceforth assume that  $B$  is continuous at the origin  $(0, 0)$  of  $X_1 \times X_2$ , *i.e.* given an arbitrary **balanced** open subset  $W$  of  $Z$ , there exists in  $X_i$  ( $i = 1, 2$ ) a **balanced** open subset  $U_i$  such that

$$(2.78) \quad B(U_1 \times U_2) \subseteq W.$$

In such context,  $\lambda_i(x)$  is chosen greater than  $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$ ; see [1.33] of [4] for further reading about the *Minkowski functionals*  $\mu$ . In other words,  $x_i$  lies in  $\lambda_i(x)U_i$ , since  $U_i$  is balanced. Thus,

$$(2.79) \quad B(x_1, x_2) = \lambda_1(x)\lambda_2(x) \cdot B(x_1/\lambda_1(x), x_2/\lambda_2(x))$$

$$(2.80) \quad \in \lambda_1(x)\lambda_2(x) \cdot B(U_1 \times U_2)$$

$$(2.81) \quad \subseteq \lambda_1(x)\lambda_2(x) \cdot W.$$

Pick  $p = (p_1, p_2)$  in  $X_1 \times X_2$ , and let  $q = (q_1, q_2)$  range over  $X \times Y$ , as a first step: It directly follows from (2.81) that

$$(2.82)$$

$$B(p) - B(q) = B(p_1, p_2 - q_2) + B(p_1, q_2) - B(q_1, q_2)$$

$$(2.83) \quad = B(p_1, p_2 - q_2) + B(p_1 - q_1, q_2)$$

$$(2.84) \quad = B(p_1, p_2 - q_2) + B(p_1 - q_1, q_2 - p_2) + B(p_1 - q_1, p_2)$$

$$(2.85) \quad \in \lambda_1(p)\lambda_2(p - q)W + \lambda_1(p - q)\lambda_2(q - p)W + \lambda_1(p - q)\lambda_2(p)W.$$

We now restrict  $q$  to a particular neighborhood of  $p$ . More specifically,

$$(2.86) \quad p_i - q_i \in \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 2} U_i;$$

which implies

$$(2.87) \quad \mu_i(q_i - p_i) = \mu_i(p_i - q_i) \leq \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 2}$$

(the equality at the left is valid, since  $U_i = -U_i$ ). The special case

$$(2.88) \quad \lambda_i(p) \triangleq \mu_1(p_1) + \mu_2(p_2) + 1,$$

$$(2.89) \quad \lambda_i(p - q) \triangleq \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 1} \triangleq \lambda_i(q - p)$$

implies that

$$(2.90) \quad B(p) - B(q) \in W + W + W,$$

since  $W$  is balanced.  $W$  being arbitrary, we have so established the continuousness of  $B$  at arbitrary  $p$ ; which achieves the proof.  $\square$

## 12 Exercise 12. A bilinear mapping that is not continuous

Let  $X$  be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| \, dt.$$

Put  $B(f, g) = \int_0^1 f(t)g(t)dt$ , and show that  $B$  is a bilinear continuous functional on  $X \times X$  which is separately but not continuous.

*Proof.* Let  $f$  denote the first variable,  $g$  the second one. Remark that

$$(2.91) \quad |B(f, g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient (1.18 of [4]) to assert that any  $f \mapsto B(f, g)$  is continuous. The continuity of all  $g \mapsto B(f, g)$  follows (Put  $C(g, f) = B(f, g)$  and proceed as above). Suppose, to reach a contradiction, that  $B$  is continuous. There so exists a positive  $M$  such that,

$$(2.92) \quad |B(f, g)| < M\|f\|\|g\|.$$

Put

$$(2.93) \quad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \quad (n = 1, 2, 3, \dots),$$

so that

$$(2.94) \quad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand,

$$(2.95) \quad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.95) and (2.92) with (2.94) and so obtain

$$(2.96) \quad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Our continuousness assumption is then contradicted. So ends the proof.  $\square$



## 15 Exercise 15. Baire's cut

Suppose  $X$  is an  $F$ -space and  $Y$  is a subspace of  $X$  whose complement is of the first category. Prove that  $Y = X$ . Hint:  $Y$  must intersect  $x + Y$  for every  $x \in X$ .

*Proof.* Assume that  $X = V_n \sqcup \overline{E}_n$  ( $n = 1, 2, 3, \dots$ ) such that

- (i)  $X = \overline{V}_n$ ;
- (ii)  $X \setminus Y = \bigcup_{n=1}^{\infty} E_n$ .

First, let  $x$  range over  $X$ , so that  $x + V_n$  is open and dense as well (because<sup>5</sup> the translation by  $x$  is a homeomorphism of  $X$  onto  $X$ ). Next, apply the Baire's theorem twice to establish that

- (a) every intersection  $W_n = V_n \cap [x + V_n]$  is dense in  $X$ ;
- (b) so is the (then nonempty) intersection  $\bigcap_{n=1}^{\infty} W_n$ .

Moreover, the intersection  $\bigcap_{n=1}^{\infty} V_n$  cuts no  $E_n$ . To sum up,

$$(2.97) \quad w \in \bigcap_{n=1}^{\infty} W_n \stackrel{(a)}{\subseteq} \bigcap_{n=1}^{\infty} V_n \stackrel{(ii)}{\subseteq} Y$$

for some  $w = w(x) \in Y$ . Note that  $w$  also lies in every  $x + V_n$ , by (a). Hence

$$(2.98) \quad w - x \in \bigcap_{n=1}^{\infty} V_n \stackrel{(2.97)}{\subseteq} Y.$$

Finally, combining (2.97) with (2.98) yields

$$(2.99) \quad x = w - (w - x) \in Y - Y = Y,$$

since  $Y$  is a subspace (subgroup) of  $X$ . We have so established that

$$(2.100) \quad X \subseteq Y,$$

which achieves the proof. □

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<sup>5</sup>This is also a special case of 1.3 (b) of [4], where:  $X = x + X \subseteq \overline{x + V_n}$ .

## 16 Exercise 16. An elementary closed graph theorem

Suppose that  $X$  and  $K$  are metric spaces, that  $K$  is compact, and that the graph of  $f : X \rightarrow K$  is a closed subset of  $X \times K$ . Prove that  $f$  is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of  $K$  cannot be omitted from the hypothesis, even when  $X$  is compact.

*Proof.* Choose a sequence  $\{x_n : n = 1, 2, 3, \dots\}$  whose limit is an arbitrary  $a$ . By compactness of  $K$ , the graph  $G$  of  $f$  contains a subsequence  $\{(x_{p(n)}, f(x_{p(n)}))\}$  of  $\{(x_n, f(x_n))\}$  that converges to some  $(a, b)$  of  $X \times K$ .  $G$  is closed; therefore,  $\{(x_{p(n)}, f(x_{p(n)}))\}$  converges in  $G$ . So,  $b = f(a)$ ; which establishes that  $f$  is sequentially continuous. Since  $X$  is metrizable,  $f$  is also continuous; see [A6] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.101) \quad f : [0, \infty) \rightarrow [0, \infty)$$

$$x \mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases}$$

Clearly,  $f$  has a discontinuity at 0. Nevertheless the graph  $G$  of  $f$  is closed. To see that, first remark that

$$(2.102) \quad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let  $\{(x_n, 1/x_n)\}$  be a sequence in  $G_+ = \{(x, 1/x) : x > 0\}$  that converges to  $(a, b)$ . To be more specific:  $a = 0$  contradicts the boundedness of  $\{(x_n, 1/x_n)\}$ :  $a$  is necessarily positive and  $b = 1/a$ , since  $x \mapsto 1/x$  is continuous on  $\mathbb{R}_+$ . This establishes that  $(a, b) \in G_+$ , hence the closedness  $G_+$ . Finally, we conclude that  $G$  is closed, as a finite union of closed sets.  $\square$

# Chapter 3

## Convexity

### 3 Exercise 3.

Suppose  $X$  is a real vector space (without topology). Call a point  $x_0 \in A \subseteq X$  an internal point of  $A$  if  $A - x_0$  is an absorbing set.

- (a) Suppose  $A$  and  $B$  are disjoint convex sets in  $X$ , and  $A$  has an internal point. Prove that there is a nonconstant linear functional  $\Lambda$  such that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. (The proof is similar to that of Theorem 3.4)
- (b) Show (with  $X = \mathbf{R}^2$ , for example) that it may not be possible to have  $\Lambda(A)$  and  $\Lambda(B)$  disjoint, under the hypotheses of (a).

*Proof.* Take  $A$  and  $B$  as in (a); the trivial case  $B = \emptyset$  is discarded. Since  $A - x_0$  is absorbing, so is its convex superset  $C = A - B - x_0 + b_0$  ( $b_0 \in B$ ). Note that  $C$  contains the origin. Let  $p$  be the Minkowski functional of  $C$ . Since  $A$  and  $B$  are disjoint,  $b_0 - x_0$  is not in  $C$ , hence  $p(b_0 - x_0) \geq 1$ . We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [4] to establish the existence of a linear functional  $\Lambda : X \rightarrow \mathbf{R}$  such that

$$(3.1) \quad \Lambda \leq p$$

and

$$(3.2) \quad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \quad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

$$(3.4) \quad \Lambda a \leq \Lambda b.$$

We now prove that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. Suppose, to reach a contradiction, that this intersection contains  $y_1$  and  $y_2$ . There so exists  $(a_i, b_i)$  in  $A \times B$  ( $i = 1, 2$ ) such that

$$(3.5) \quad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that  $y_1 < y_2$ . Then,

$$(3.6) \quad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad .$$

Remark that  $a_3 = \frac{1}{2}(a_1 + a_2)$  lies in the convex set  $A$ . This implies

$$(3.7) \quad \Lambda b_1 \stackrel{(3.6)}{<} \Lambda a_3 \stackrel{(3.4)}{\leq} \Lambda b_1 \quad ;$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space  $X$  is  $\mathbf{R}^2$ . Fetch

$$(3.8) \quad S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$

$$(3.9) \quad S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$$

$$(3.10) \quad A \triangleq S_1 \cup S_2,$$

$$(3.11) \quad B \triangleq X \setminus A.$$

Pick  $(x_i, y_i)$  in  $S_i$ . Let  $t$  range over the unit interval, and so obtain

$$(3.12) \quad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A.$$

Thus, every segment that has an extremity in  $S_1$  and the other one in  $S_2$  lies in  $A$ . Moreover, each  $S_i$  is convex. We can now conclude that  $A$  is so. The convexity of  $B$  is proved in the same manner. Furthermore,  $A$  hosts a non degenerate triangle, *i.e.*  $A^\circ$  is nonempty<sup>1</sup>:  $A$  contains an internal point.

Let  $L$  be a vector line of  $\mathbf{R}^2$ . In other words,  $L$  is the null space of a linear functional  $\Lambda : \mathbf{R}^2 \rightarrow \mathbf{R}$  (to see this, take some nonzero  $u$  in  $L^\perp$  and set  $\Lambda x = (x, u)$  for all  $x$  in  $\mathbf{R}^2$ ). One easily checks that both  $A$  and  $B$  cut  $L$ . Hence

$$(3.13) \quad \Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad .$$

So ends the proof. □

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<sup>1</sup>For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [3].

## 11 Exercise 11. Meagerness of the polar

Let  $X$  be an infinite-dimensional Fréchet space. Prove that  $X^*$ , with its weak\*-topology, is of the first category in itself.

This is actually a consequence of the below lemma, which we prove first. The proof that  $X^*$  is of the first category in itself comes right after, as a corollary.

**Lemma.** *If  $X$  is an infinite dimensional topological vector space whose dual  $X^*$  separates points on  $X$ , then the polar*

$$(3.14) \quad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\}$$

*of any absorbing subset  $A$  is a weak\*-closed set that has empty interior.*

*Proof.* Let  $x$  range over  $X$ . The linear form  $\Lambda \mapsto \Lambda x$  is weak\*-continuous; see 3.14 of [4]. Therefore,  $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$  is weak\*-closed. As the intersection of  $\{P_a : a \in A\}$ ,  $K_A$  is also a weak\*-closed set. We now prove the second half of the statement.

From now on,  $X$  is assumed to be endowed with its weak topology:  $X$  is then locally convex, but its dual space is still  $X^*$  (see 3.11 of [4]). Put

$$(3.15) \quad W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \quad (r_x > 0)$$

where  $F$  runs through the nonempty finite subsets of  $X$ . Clearly, the collection of all such  $W$  is a local base of  $X^*$ . Pick one of those  $W$  and remark that the following subspace

$$(3.16) \quad M \triangleq \text{span}(F)$$

is finite dimensional. Assume, to reach a contradiction, that  $A \subseteq M$ . So, every  $x$  lies in  $t_x M = M$  for some  $t_x > 0$ , since  $A$  is absorbing. As a consequence,  $X$  is the finite dimensional space  $M$ , which is a desired contradiction. We have just established that  $A \not\subseteq M$ : Now pick  $a$  in  $A \setminus M$  and so conclude that

$$(3.17) \quad b \triangleq \frac{a}{t_a} \in A$$

Remark that  $b \notin M$  (otherwise,  $a = t_a b \in t_a M = M$  would hold) and that  $M$ , as a finite dimensional space, is closed (see 1.21 (b) of [4] for a proof): By the Hahn-Banach theorem 3.5 of [4], there exists  $\Lambda_a$  in  $X^*$  such that

$$(3.18) \quad \Lambda_a b > 2$$

and

$$(3.19) \quad \Lambda_a(M) = \{0\}.$$

The latter equality implies that  $\Lambda_a$  vanishes on  $F$ ; hence  $\Lambda_a$  is an element of  $W$ . On the other hand, given an arbitrary  $\Lambda \in K_A$ , the following inequalities

$$(3.20) \quad |\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that  $\Lambda + \Lambda_a$  is not in  $K_A$ . We have thus proved that

$$(3.21) \quad \Lambda + W \not\subseteq K_A.$$

Since  $W$  and  $\Lambda$  are both arbitrary, this achieves the proof.  $\square$

We now give a proof of the original statement.

**Corollary.** *If  $X$  is an infinite-dimensional Fréchet space, then  $X^*$  is meager in itself.*

*Proof.* From now on,  $X^*$  is only endowed with its weak\*-topology. Let  $d$  be an invariant distance that is compatible with the topology of  $X$ , so that the following sets

$$(3.22) \quad B_n \triangleq \{x \in X : d(0, x) < 1/n\} \quad (n = 1, 2, 3, \dots)$$

form a local base of  $X$ . If  $\Lambda$  is in  $X^*$ , then

$$(3.23) \quad |\Lambda| \leq m \text{ on } B_n$$

for some  $(n, m) \in \{1, 2, 3, \dots\}^2$ ; see 1.18 of [4]. Hence,  $X^*$  is the countable union of all

$$(3.24) \quad m \cdot K_n \quad (m, n = 1, 2, 3, \dots),$$

where  $K_n$  is the polar of  $B_n$ . Clearly, showing that every  $m \cdot K_n$  is nowhere dense is now sufficient. To do so, we use the fact that  $X^*$  separates points; see 3.4 of [4]. As a consequence, the above lemma implies

$$(3.25) \quad (\overline{K_n})^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by  $m$  is an homeomorphism (see 1.7 of [4]), this is equivalent to

$$(3.26) \quad (\overline{m \cdot K_n})^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof.  $\square$

# Chapter 4

## Banach Spaces

Throughout this set of exercises,  $X$  and  $Y$  denote Banach spaces, unless the contrary is explicitly stated.

### 1 Exercise 1. Basic results

Let  $\varphi$  be the embedding of  $X$  into  $X^{**}$  described in Section 4.5. Let  $\tau$  be the weak topology of  $X$ , and let  $\sigma$  be the weak\*-topology of  $X^{**}$  - the one induced by  $X^*$ .

- (a) Prove that  $\varphi$  is an homeomorphism of  $(X, \tau)$  onto a dense subspace of  $(X^{**}, \sigma)$ .
- (b) If  $B$  is the closed unit ball of  $X$ , prove that  $\varphi(B)$  is  $\sigma$ -dense in the closed unit ball of  $X^{**}$ . (Use the Hahn-Banach separation theorem.)
- (c) Use (a), (b), and the Banach-Alaoglu theorem to prove that  $X$  is reflexive if and only if  $B$  is weakly compact.
- (d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.
- (e) If  $X$  is reflexive and  $Y$  is a closed subspace of  $X$ , prove that  $X/Y$  is reflexive.
- (f) Prove that  $X$  is reflexive if and only if  $X^*$  is reflexive.  
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace  $\varphi(X)$  of  $X^{**}$ .

*Proof.* Let  $\psi$  be the isometric embedding of  $X^*$  into  $X^{***}$ . The dual space of  $(X^{**}, \sigma)$  is then  $\psi(X^*)$ .

It is sufficient to prove that

$$(4.1) \quad \varphi^{-1} : \varphi(X) \rightarrow X$$

$$(4.2) \quad \varphi(x) \mapsto x$$

is an homeomorphism (with respect to  $\tau$  and  $\sigma$ ). We first consider

$$(4.3) \quad V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}, \psi x^* \rangle| < r\} \quad (x^* \in X^*, r > 0);$$

$$(4.4) \quad U \triangleq \{x \in X : |\langle x, x^* \rangle| < r\} \quad (x^* \in X^*, r > 0).$$

and remark that the so defined  $V$ 's (respectively  $U$ 's) shape a local subbase  $\mathcal{S}_\sigma$  (respectively  $\mathcal{S}_\tau$ ) of  $\sigma$  (respectively  $\tau$ ). We now observe that

$$(4.5) \quad U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathcal{S}_\sigma, U \in \mathcal{S}_\tau) \quad ,$$

since  $\varphi^{-1}$  is one-to-one. This remains true whether we enrich each subbase  $\mathcal{S}$  with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak\*-topology that  $\varphi^{-1}$  and its inverse  $\varphi$  are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$(4.6) \quad \overline{\varphi(X)}_\sigma \subseteq X^{**} \quad .$$

and we now assume- to reach a contradiction- that  $(X^{**}, \sigma)$  contains a point  $z^{**}$  outside the  $\sigma$ -closure of  $\varphi(X)$ . By [3.5], there so exists  $y^*$  in  $X^*$  such that

$$(4.7) \quad \langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ;$$

$$(4.8) \quad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces  $y^*$  to be a the zero of  $X^*$ . The functional  $\psi y^*$  is then the zero of  $X^{**}$ : (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball  $B^{**}$  of  $X^{**}$  is weak\*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \quad \varphi(B) \subseteq B^{**} \quad ,$$

since  $\varphi$  is isometric. Hence

$$(4.10) \quad \overline{\varphi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} \quad .$$

Now suppose, to reach a contradiction, that  $B^{**} \setminus \overline{\varphi(B)}_\sigma$  contains a vector  $z^{**}$ . By [3.7], there exists  $y^*$  in  $X^*$  such that

$$(4.11) \quad |\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_\sigma \quad ;$$

$$(4.12) \quad \langle z^{**}, \psi y^* \rangle > 1 \quad .$$

It follows from (4.11) that

$$(4.13) \quad |\psi y^*| \leq 1 \quad \text{on } \varphi(B), \text{ i.e. } |y^*| \leq 1 \quad \text{on } B \quad .$$

We have so proved that

$$(4.14) \quad y^* \in B^* \quad .$$



Since  $z^{**}$  lies in  $B^{**}$ , it is now clear that

$$(4.15) \quad |\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ;$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

$$(4.16) \quad B \text{ is weakly compact if and only if } \varphi(B) \text{ is weak}^* \text{-compact.}$$

If  $B$  is weakly compact, then  $\varphi(B)$  is weak\*-closed. So,

$$(4.17) \quad \varphi(B) = \overline{\varphi(B)}_\sigma \stackrel{(b)}{=} B^{**} \quad .$$

$\varphi$  is therefore onto, *i.e.*  $X$  is reflexive.

Conversely, keep  $\varphi$  as onto: one easily checks that  $\varphi(B) = B^{**}$ . The image  $\varphi(B)$  is then weak\*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let  $X$  be a reflexive space  $X$ , whose closed unit ball is  $B$ . Let  $Y$  be a norm-closed subspace of  $X$ :  $Y$  is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that  $B$  is weakly compact. We now conclude that the closed unit ball  $B \cap Y$  of  $Y$  is weakly compact. We again use (c) to conclude that  $Y$  is reflexive. (d) is therefore established. Now proceed to (e).

Let  $\equiv$  stand for “isometrically isomorphic” and apply twice [4.9] to obtain, first

$$(4.18) \quad (X/Y)^* \equiv Y^\perp \quad ,$$

next,

$$(4.19) \quad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \quad .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

*Given a reflexive Banach space  $Z$ , the weak\*-topology of  $Z^*$  is its weak one.*

Assume first that  $X$  is reflexive. Since  $B^*$  is weak\* compact, by (c) of [4.3], (L) implies that  $B^*$  is also weakly compact. Then (c) turns  $X^*$  into a reflexive space.

Conversely, let  $X^*$  be reflexive. What we have just proved that makes  $X^{**}$  reflexive. On the other hand,  $\varphi(X)$  is a norm-closed subspace of  $X^{**}$ ; *cf.* [4.5]. Hence  $\varphi(X)$  is reflexive, by (d). It now follows from (c) that  $B^{**} \cap \varphi(X)$  is weakly compact, *i.e.* weak\*-compact (to see this, apply (L) with  $Z = X^*$ ).

By (a),  $B$  is therefore weakly compact, *i.e.*  $X$  is reflexive; see (c). So ends the proof.  $\square$

### 13 Exercise 13. Operator compactness in a Hilbert space

- (a) Suppose  $T \in \mathcal{B}(X, Y)$ ,  $T_n \in \mathcal{B}(X, Y)$  for  $n = 1, 2, 3, \dots$ , each  $T_n$  has finite-dimensional range, and  $\lim \|T - T_n\| = 0$ . Prove that  $T$  is compact.
- (b) Assume  $Y$  is a Hilbert space, and prove the converse of (a): Every compact  $T \in \mathcal{B}(X, Y)$  can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)

*Proof.* Since each  $T_n$  is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every  $T_n$  as a compact operator. Let  $\{x_0^i : i \in \mathbf{N}\}$  be in  $U$  the open unit ball of  $X$ . Since  $T_1$  is compact,  $\{x_0^i\}$  contains a subsequence  $\{x_1^i : i \in \mathbf{N}\}$  such that  $\{T_1 x_1^i\}$  converges to a point  $y_1$  of  $Y$ . The same reasoning can be recursively applied to  $T_n$  and  $\{x_{n-1}^i\} \subseteq U$  so that  $\{T_n x_n^i\}$  tends to some  $y_n$  of  $Y$ , as  $\{x_n^i\}$  is a subsequence of  $\{x_{n-1}^i\}$ . Then

$$(4.20) \quad T_n x_p^i \xrightarrow{i \rightarrow \infty} y_n \quad (p = 1, 2, 3, \dots) \quad .$$

Applied with  $\{x_n^i : (n, i) \in \mathbf{N}^2\}$ , a Cantor's diagonal process therefore provides a subsequence  $\{\tilde{x}_j : j \in \mathbf{N}\}$  such that

$$(4.21) \quad T_j \tilde{x}_k \xrightarrow{k \rightarrow \infty} y_j \quad ;$$

$$(4.22) \quad T_j \tilde{x}_j \xrightarrow{j \rightarrow \infty} y_j \quad .$$

We now easily obtain

$$(4.23) \quad \|T_j \tilde{x}_j - T_k \tilde{x}_k\| \leq \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \xrightarrow{k > j \rightarrow \infty} 0 \quad .$$

$\{T_j \tilde{x}_j\}$  is then a Cauchy sequence. So is  $\{T \tilde{x}_j\}$ , since  $\|T - T_j\| \rightarrow 0$ . On the other hand,  $Y$  is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix  $\varepsilon$  as a positive scalar. Since  $T$  is compact,  $Y$  contains a finite set  $C$  such that

$$(4.24) \quad T(U) \subseteq \bigcup_{c \in C} B(c, \varepsilon) \quad .$$

As a Hilbert space,  $Y$  contains a *maximal orthonormal set* (or *Hilbert basis*)  $M$ . This implies that  $\text{span}(M)$  is dense in  $Y$ ; cf. 4.18 & 4.22 of [3]. The finiteness of  $C$  forces  $M$  to enclose a finite set  $S$  so that

$$(4.25) \quad \forall c \in C, \exists s(c) \in \text{span}(S) : \|c - s(c)\| < \varepsilon \quad .$$

Let  $x$  be in  $U$ . It follows from (4.24) that

$$(4.26) \quad \|Tx - c_x\| < \varepsilon$$

for some  $c_x$  of  $C$ . We now combine (4.25) and (4.26) to obtain

$$(4.27) \quad \|Tx - s(c_x)\| \leq \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon$$

As a finite-dimensional subspace,  $\text{span}(S)$  is closed (see footnote 4, Exercise 1.10). We so obtain

$$(4.28) \quad Y = \text{span}(S) \oplus \text{span}(S)^\perp,$$

by [12.4]. There so exists a unique projection  $\pi = \pi(\varepsilon)$  of  $Y$  onto itself (see [5.6] for the definition) such that

$$(4.29) \quad \pi(Y) = \text{span}(S), \quad (I - \pi)(Y) = \text{span}(S)^\perp.$$

It is easily checked that  $\pi$  has norm 1. Moreover,

$$(4.30) \quad \pi s = s \quad (s \in \text{span}(S)).$$

Thus,

$$(4.31) \quad (I - \pi)(Tx) = (I - \pi)(Tx - s(c_x)) \quad (x \in U).$$

Then,

$$(4.32) \quad \|(I - \pi)(Tx)\| \leq \|I - \pi\| \|Tx - s(c_x)\| < 4\varepsilon \quad (x \in U)$$

(the fact that  $\pi$  has norm 1 is hidden in the right side inequality). We have just so proved that

$$(4.33) \quad \|T - \pi \circ T\| \in O_{\varepsilon \sim 0}(\varepsilon).$$

That is particularly true whether  $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \xrightarrow{n \rightarrow \infty} 0$ . Let so  $T_n$  be  $\pi(\varepsilon_n) \circ T$  and conclude that these (compact) operators approximate  $T$  in the desired fashion, *i.e.*

$$(4.34) \quad \|T - T_n\| \xrightarrow{n \rightarrow \infty} 0.$$

□

## 15 Exercise 15. Hilbert-Schmidt operators

Suppose  $\mu$  is a finite (or  $\sigma$ -finite) positive measure on a measure space  $\Omega$ ,  $\mu \times \mu$  is the corresponding product measure on  $\Omega \times \Omega$ , and  $K \in L^2(\mu \times \mu)$ . Define

$$(Tf)(s) = \int_{\Omega} K(s, t) f(t) d\mu(t) \quad [f \in L^2(\mu)].$$

(a) Prove that  $T \in \mathcal{B}(L^2(\mu))$  and that

$$\|T\|^2 \leq \int_{\Omega} \int_{\Omega} |K(s, t)|^2 d\mu(s) d\mu(t).$$

(b) Suppose  $a_i, b_i$  are members of  $L^2(\mu)$ , for  $1 \leq i \leq n$ , put  $K_1 = \sum a_i(s) b_i(t)$  and define  $T_1$  in terms of  $K_1$  as  $T$  was defined in terms of  $K$ . Prove that  $\dim \mathcal{R}(T_1) \leq n$ .

(c) Deduce that  $T$  is a compact operator in  $L^2(\mu)$ . Hint: Use exercise 13.

(d) Suppose  $\lambda \in \mathbf{C}, \lambda \neq 0$ . Prove: Either the equation

$$Tf - \lambda f = g$$

has a unique solution  $f \in L^2(\mu)$  for every  $g \in L^2(\mu)$  or there are infinitely many solutions for some  $g$  and none for others. (This is known as the Fredholm alternative.).

(e) Describe the adjoint of  $T$ .

*Proof.* Let  $X$  (respectively  $P$ ) be the Banach space  $L^2(\mu)$  (respectively  $L^2(\mu \times \mu)$ ). A consequence of the Radon-Nikodym theorem (cf. 6.16 of [3]) is that there exists a group isomorphism  $\rho : X \rightarrow X^*, f \mapsto f^*$  such that

$$(4.35) \quad \langle u, f^* \rangle = \int_{\Omega} u \cdot f d\mu \quad (u \in X, f \in X) \quad .$$

Define a.e  $K_s, K_t : \Omega \rightarrow \mathbf{C}$  by setting

$$(4.36) \quad K_s(t) \triangleq K_t(s) \triangleq K(s, t) \quad \text{a.e} \quad ((s, t) \in \Omega) \quad .$$

$T$  is clearly linear. Moreover,

$$(4.37) \quad |(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \quad (\|f\|_X < 1)$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with  $|K|^2$  to obtain

$$(4.38) \quad \|Tf\|_X^2 \leq \int_{\Omega} \|K_s\|_X^2 d\mu(s) = \|K\|_P^2 < \infty \quad (\|f\|_X < 1) \quad .$$

(a) is then proved.

To show (b), remark that

$$(4.39) \quad \int_{\Omega} a_i(s) \cdot b_i \cdot f d\mu \in \mathbf{C} \cdot a_i(s) \quad \text{a.e} \quad (f \in X, s \in \Omega) \quad .$$

It is now clear that  $T$  maps any  $f$  of  $X$  into  $\mathbf{C} \cdot a_1 + \cdots + \mathbf{C} \cdot a_n$ . We so conclude that  $\dim R(T_1) \leq n$ .

We now aim at (c). The current part refers to Exercise 4.13.  $X$  is also a Hilbert space and so contains a Hilbert basis  $M$ . Define a.e

$$(4.40) \quad \begin{aligned} a_b : \Omega &\rightarrow \mathbf{C} \\ s &\mapsto (K_s, b) \end{aligned}$$

whenever  $b$  ranges  $M$ . Hence,

$$(4.41) \quad K_s = \sum_{b \in M} a_b(s) \cdot b \quad \text{a.e.} \quad (s \in \Omega) \quad .$$

Provided any positive scalar  $\varepsilon$ , there so exists a finite subset  $S = S(\varepsilon)$  of  $M$  such that

$$(4.42) \quad \|K_s - \sum_{b \in S} a_b(s) \cdot b\|_X < \varepsilon \quad (s \in \Omega) \quad .$$

Remark that  $\sum_{b \in S} a_b \cdot b$  matches the definition of  $K_1$ ; *cf.* (b): from now on,

$$(4.43) \quad K_1 \triangleq \sum_{b \in S} a_b \cdot b \quad .$$

It follows from (b) that

$$(4.44) \quad \dim R(K_1) < \infty \quad .$$

Now turn back to (a), with  $K - K_1$  playing the role of  $K$ , and so obtain

$$(4.45) \quad \|T - T_1\| < \varepsilon \mu(\Omega) \leq \infty \quad .$$

For if  $\mu$  is finite, use (a) of Exercise 4.13 to conclude that  $T$  is compact. Assume henceforth that  $\mu$  is not (necessarily) finite and pick  $\delta$  in  $\mathbf{R}_+$ . The simple functions (with finite measure support) form a dense family of an  $L^p$  space ( $1 \leq p < \infty$ ); *cf.* 3.13 of [3]. It then exists a simple function  $K_\delta$  of  $L^2(\mu \times \mu)$  such that

$$(4.46) \quad (\mu \times \mu)(\{K_\delta \neq 0\}) < \infty, \quad \|K - K_\delta\|_P < \delta \quad .$$

Define an operator  $T_\delta$  in terms of  $K_\delta$  as  $T$  was defined in terms of  $K$ , and proceed as in (a) with  $T - T_\delta$  instead of  $T$ . Then

$$(4.47) \quad \|T - T_\delta\| < \delta \quad .$$

The key ingredient is that  $K_\delta$  can be identified with an element of the finite measure space  $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$ . What we have attempted to approximate  $T$  by  $T_1$  can therefore be reiterated (with  $K_\delta$  playing the role of  $K$ ) to achieve an approximation  $T_{\delta,1}$  of  $T_\delta$  so that

$$(4.48) \quad \|T_\delta - T_{\delta,1}\| < \varepsilon \quad .$$

It now follows from (4.47) and (4.48) that

$$(4.49) \quad \|T - T_{\delta,1}\| \leq \|T - T_{\delta}\| + \|T_{\delta} - T_{\delta,1}\| < \varepsilon + \delta \quad .$$

Since  $\varepsilon$  and  $\delta$  were arbitrary, the  $\sigma$ -finite case is proved. We now establish (d).

Provided  $g$  of  $X$ , let  $E_g$  be the following equation on  $X$

$$(4.50) \quad Tf - \lambda f = g \quad ,$$

whose solution set is denoted by  $S_g$ . Note that  $S_0$  is  $\ker(T - \lambda)$  and discard the trivial case  $S_0 = X$ <sup>1</sup>: each  $f$  of  $X$  lies in  $S_{Tf - \lambda f}$ , as some  $Tf - \lambda f$ 's are nonzero. Some  $S_g$ 's are then nonempty. Remark that

$$(4.51) \quad S_g = f + S_0 \quad (f \in S_g)$$

in such case. Furthermore, the equality  $\beta = \alpha$  of [4.25] yields

$$(4.52) \quad (T - \lambda I)(X) \neq X, \text{ i.e. } S_0 \neq \{0\} \quad .$$

So if  $T - \lambda I$  is not onto, not only some  $S_g$ 's are empty, but also  $S_0 \neq \{0\}$ . Every nonempty  $S_g$  (such sets always exist, see above) is then infinite, by (4.51).

Otherwise,  $T - \lambda I$  is bijective and every equation  $E_g$  has then a unique solution  $f$ . The Fredholm alternative is so proved.

Our last step is the description of  $T^*$ . Let  $S : X \rightarrow X$  be such that

$$(4.53) \quad (Sf)(t) \triangleq \int_{\Omega} K_t \cdot f \text{ a.e.} \quad (f \in X, t \in \Omega)$$

Proceed as in (a), with  $S$  instead of  $T$ :  $S$  lies in  $\mathcal{B}(X)$ . Next, we claim that

$$(4.54) \quad \langle u, T^* f^* \rangle = \langle Tu, f^* \rangle$$

$$(4.55) \quad = \int_{\Omega} (Tu) \cdot f \, \mu$$

$$(4.56) \quad = \int_{\Omega^2} K \cdot f \cdot u \, (\mu \times \mu)$$

$$(4.57) \quad = \int_{\Omega} (Sf) \cdot u \, \mu$$

$$(4.58) \quad = \langle u, (Sf)^* \rangle \quad ,$$

whenever  $u$  and  $f$  run through the closed unit ball of  $X$ . Since  $\|T\|$ ,  $\|T^*\|$  are equal and finite, only exactness of (4.56) is possibly in doubt; the below justification dissipates it. In conclusion,

$$(4.59) \quad T^* = \rho S \rho^{-1} \quad .$$

---

<sup>1</sup>e.g.  $X = L^2(\{0\}, \delta)$ , so that  $I = \lambda^{-1}T$  is compact.

Informally,

$$(4.60) \quad T^* = S \quad .$$

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep  $u$  and  $f$  as above. Let us introduce

$$(4.61) \quad A(s) \triangleq \int_{\Omega} |K_s(t) \cdot u(t)| \, \mu(t) \quad \text{a.e.} \quad (s \in \Omega) \quad ,$$

to make hold the following Cauchy-Schwarz inequality

$$(4.62) \quad A(s) \leq \|K_s\|_X \quad (s \in \Omega) \quad .$$

Thus,

$$(4.63) \quad \int_{\Omega^2} |K(s, t) u(t) f(s)| \, \mu(s) \mu(t) = \int_{\Omega} |f(s)| A(s) \, \mu(s)$$

$$(4.64) \quad \leq \int_{\Omega} |f(s)| \|K_s\|_X \, \mu(s)$$

$$(4.65) \quad \leq \left[ \int_{\Omega} \|K_s\|_X^2 \, \mu(s) \right]^{\frac{1}{2}} = \|K\|_P < \infty \quad .$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof.  $\square$

# Chapter 6

## Distributions

### 1 Exercise 1. Test functions are almost polynomial

*Suppose  $f$  is a complex continuous function in  $\mathbf{R}^n$ , with compact support. Prove that  $\psi P_j \rightarrow f$  uniformly on  $\mathbf{R}^n$ , for some  $\psi \in \mathcal{D}$  and for some sequence  $\{P_j\}$  of polynomials.*

*Proof.* According to 1.16,  $\Omega$  is union of a compact sets sequence  $\{K_i\}$  and  $\text{supp}(f)$  lies in some  $K = K_i$ , so that  $f$  is embedded in  $\mathcal{D}(\Omega)$ . We can apply [1.10] to ensure that  $\Omega$  encloses a compact set  $S = K + B(\varepsilon)$  for sufficiently small  $\varepsilon > 0$ .

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra  $\{g \in C(S) : g \text{ is polynomial}\}$  of  $C(S)$ . There so exists a sequence  $\{P_j : j \in \mathbf{N}\}$  of  $\mathbf{R}[X_1, \dots, X_n]$  such that

$$(6.1) \quad \sup_S |f - P_j| \xrightarrow{j \rightarrow \infty} 0 \quad .$$

By [6.20], the open set  $K + B(\varepsilon)$  has a local partition of unity  $\{\psi_i\} \subseteq \mathcal{D}(\Omega)$ . Moreover, there exists an integer  $l$  such that  $\psi = \psi_1 + \dots + \psi_l$  equals 1 on  $K$ . Hence

$$(6.2) \quad \|f - \psi P_j\|_\infty = \|\psi f - \psi P_j\|_\infty = \sup_S |\psi f - \psi P_j|$$

$$(6.3) \quad = \sup_S |f - P_j| \xrightarrow{j \rightarrow \infty} 0 \quad . \quad \stackrel{(6.1)}{}$$

□

We will actually prove more by showing that  $\mathcal{D}(\Omega)$  is separable for each nonempty open subset  $\Omega$  of  $\mathbf{R}^n$ .

*Proof.* The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the



separability of  $\mathcal{D}(\Omega)$ . To do so, we keep  $(\alpha, j)$  in  $\mathbf{N}^n \times \mathbf{N}$ . Remark that  $S$  encloses  $\text{supp}(D^\alpha f)$ : according to the first part, there exists a sequence  $\{P_{\alpha, j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \dots, X_n]$  such that

$$(6.4) \quad \|D^\alpha f - \psi P_{\alpha, j}\|_\infty \xrightarrow{j \rightarrow \infty} 0 \quad .$$

Now let  $m$  range over  $\{1, 2, 3, \dots\}$  and set  $W_{m, j}$  in  $\mathcal{D}(\Omega)$  as follows

$$(6.5) \quad D^{-\alpha} \varphi \in \mathcal{D}(\Omega) : D^\alpha D^{-\alpha} \varphi = \varphi \quad .$$

$$(6.6) \quad W_{m, j}(x) \triangleq D^{-(m, \dots, m)}(\psi P_{(m, \dots, m), j})$$

By (6.4), there exists a natural number  $k(m)$  such that

$$(6.7) \quad \|D^{(m, \dots, m)}(f - W_{m, j})\|_\infty < 1/m \quad (j \geq k(m)) \quad .$$

Assume without loss of generality that  $S$  has diameter 1, so that (6.7) yields

$$(6.8) \quad \|D^\lambda(f - W_{m, k(m)})\|_\infty < 1/m \quad (|\lambda| \leq m) \quad ,$$

by the mean value theorem. In other words (remark that  $\text{supp}(f - W_{m, k(m)}) \subseteq S$ ),

$$(6.9) \quad f - W_{m, k(m)} \in U_m \triangleq \{\varphi \in \mathcal{D}_S : \|\varphi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \dots \quad (m = 1, 2, 3, \dots) \quad .$$

Pick  $W$  in  $\beta$  (see (b) of [6.3]):  $W \cap \mathcal{D}_S$  contains a neighbourhood of 0. Hence  $W$  contains some  $U_m$ , for  $m$  sufficiently large. Thus

$$(6.10) \quad W_{m, k(m)} \xrightarrow{m \rightarrow \infty} f \quad (\text{in } \mathcal{D}(\Omega)) \quad .$$

We have so established that the  $W_{m, k(m)}$ 's family is dense in  $\mathcal{D}(\Omega)$ . We now aim to disclose a countable set  $\tilde{W}$  that has the same property.

Choose  $\delta$  in  $\mathbf{R}_+$  and fetch any  $W_{m, k(m)}$ . Let  $X$  be  $(X_1, \dots, X_n)$  and express  $P_{(m, \dots, m), k(m)}$  as

$$(6.11) \quad P(X) = \sum_{|\gamma| \leq d} p_\gamma \cdot X^\gamma \quad .$$

Since  $\bar{\mathbf{Q}} = \mathbf{R}$ ,  $\mathbf{Q}[X]$  hosts some  $\mathbf{Q}(X) = \sum_{|\gamma| \leq d} q_\gamma \cdot X^\gamma$  such that  $|p_\gamma - q_\gamma| < \delta$  for all  $\gamma$ . Thus,

$$(6.12) \quad |P(x) - Q(x)| \leq \sum_{|\gamma| \leq d} |p_\gamma - q_\gamma| |x|^{|\gamma|} \leq \delta \sum_{|\gamma| \leq d} \binom{1+n-1}{n-1} \|x\|_\infty^1 \quad (x \in \mathbf{R}^n) \quad .$$

Since  $S$  is bounded, we so obtain

$$(6.13) \quad \|\psi(P - Q)\|_\infty \in O(\delta) \quad .$$

Now define  $\tilde{W}_m$  in terms of  $\mathbf{Q}$  as  $W_{m, k(m)}$  was defined in terms of  $\mathbf{P}$ , and consider the integrations made in (6.6): each  $D^\lambda \tilde{W}_m$  ( $|\lambda| \leq m$ ) can be obtained from some of them. So (6.13) yields

$$(6.14) \quad \|D^\lambda(W_{m, k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \quad (|\lambda| \leq m) \quad .$$

To be more specific, these  $\lambda$ 's only exist in finite amount, so the big  $O$  can be assumed to be the same for all them. Since  $\delta$  was arbitrary, combining (6.10) with (6.14) establishes the density of the all  $\tilde{W}_m$ 's family  $\tilde{W}$ .

Furthermore, each member of  $\tilde{W}$  is only made of two ingredients:  $\psi$  and a polynomial of  $\mathbf{Q}[X]$ . The mapping  $\psi$  is attached to some  $K_i$  and  $\mathbf{Q}[X]$  inherits countableness from  $\mathbf{Q}$ . Note that the “integrations packs” of (6.6) only exist in countable amount. Our  $\tilde{W}$  is then countable.  $\square$

## 6 Exercise 6. Around the supports of some distributions

(a) Suppose that  $c_m = \exp\{-(m!)!\}$ ,  $m = 0, 1, 2, \dots$ . Does the series

$$\sum_{m=0}^{\infty} c_m (D^m \varphi)(0)$$

converges for every  $\varphi \in C^\infty(\mathbf{R})$ ?

(b) Let  $\Omega$  be open in  $\mathbf{R}^n$ , suppose  $\Lambda_i \in \mathcal{D}'(\Omega)$ , and suppose that all  $\Lambda_i$  have their supports in some fixed compact  $K \subseteq \Omega$ . Prove that the sequence  $\{\Lambda_i\}$  cannot converge in  $\mathcal{D}'(\Omega)$  unless the orders of the  $\Lambda_j$  are bounded. Hint: Use the Banach-Steinhaus theorem.

(c) Can the assumption about the supports be dropped in (b)?

*Proof.* The answer is: no. Let us establish this assertion. Assume, to reach a contradiction, that the above series converges for every smooth  $\varphi : \mathbf{R} \rightarrow \mathbf{C}$ .

The sequence  $\{c_m (D^m \varphi)(0)\}$  so converges to 0. Nevertheless, it is proved in [1.46] that  $C^\infty(\Omega)$  is not locally bounded. In other words, it is always possible to excavate a  $\varphi$  for which the magnitude of the  $m$ -th derivative at 0 is as large as we please<sup>1</sup>, e.g. greater than  $1/c_m$ . A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume  $\{\Lambda_j\}$  to converge to some  $\Lambda$  of  $\mathcal{D}'(\Omega)$  and we let  $\mathbf{Q}$  run through the compact sets of  $\Omega$ . Next, we define

$$(6.15) \quad S(\mathbf{T}, \mathbf{Q}) \triangleq \{N \in \mathbf{N}, \exists C \in \mathbf{R}_+ : |T\varphi| \leq C \|\varphi\|_N \text{ for all } \varphi \text{ of } \mathcal{D}_Q\} \quad (\mathbf{T} \in \mathcal{D}(\Omega)) \quad .$$

<sup>1</sup>indeed [1.46] provides sufficient tools for constructive proof of this; see the  $\varphi_j - \tilde{\varphi}_j$  involved in (??).

Such subset of  $\mathbf{N}$  has a minimum  $\omega(T, Q)$ . The following value

$$(6.16) \quad \omega(T) \triangleq \max\{\omega(T, Q) : Q \subseteq \Omega, Q \text{ compact}\} \leq \infty$$

is then the order of  $T$ . Assume, to reach a contradiction, that, after passage to a subsequence,

$$(6.17) \quad \omega(\Lambda_j, Q_j) = j \quad (j = 1, 2, 3, \dots)$$

for some compact  $Q = Q_j$ . By (a) of [6.24],  $Q_j$  cuts  $\text{supp } \Lambda_j$ , say in  $p_j$ . Since  $K$  encloses  $\text{supp } \Lambda_j$ ,  $\{p_j\}$  tends, after passage to a subsequence, to some  $p$  of  $K$ . Choose a positive scalar  $r$  so that

$$(6.18) \quad \overline{B}(p, r) \triangleq \{x \in \mathbf{R}^n : |x - p| \leq r\} \subseteq \Omega.$$

Such closed ball  $\overline{B}(p, r)$  is a compact subset of  $\Omega$ . By (b) of [6.5] (which refers to [1.46])  $\mathcal{D}_{\overline{B}(p, r)}$  is then a Fréchet space. It now follows from [2.6] that  $\{\Lambda_j\}$  is equicontinuous on  $\mathcal{D}_{\overline{B}(p, r)}$ . There so exists<sup>2</sup> a nonnegative integer  $N$  such that

$$(6.19) \quad |\Lambda \varphi| \leq C \|\varphi\|_N \quad (\varphi \in \mathcal{D}_{\overline{B}(p, r)})$$

for some positive constant  $C$ . On the other hand,  $\overline{B}(p, r)$  contains almost all the  $p_j$ 's. Hence

$$(6.20) \quad |\Lambda_N \varphi| > C \|\varphi\|_N$$

for some  $\varphi$  of  $\mathcal{D}_{\overline{B}(p, r)}$ . (b) is then established.

To prove (c), we introduce a sequence  $\{x_m : m \in \mathbf{Z}\}$  of  $\Omega$  that has no limit point. Let  $\{\alpha_m : m \in \mathbf{Z}\}$  be in  $\mathbf{N}$  and so define<sup>3</sup>

$$(6.21) \quad \Lambda : \mathcal{D}(\Omega) \rightarrow \mathbf{C} \quad .$$

$$\varphi \mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m} \varphi)(x_m)$$

$\Lambda$  belongs to  $\mathcal{D}'(\Omega)$ , since  $\{x_m\}$  has no limit point. Next, we easily check that

$$(6.22) \quad \Lambda_j : \mathcal{D}(\Omega) \rightarrow \mathbf{C} \quad (j \in \mathbf{N})$$

$$\varphi \mapsto \sum_{|m| \leq j} (D^{\alpha_m} \varphi)(x_m)$$

is also a distribution and that  $\{\Lambda_j\}$  tends to  $\Lambda$  in  $\mathcal{D}'(\Omega)$ . Nevertheless, no  $\Lambda_j$ 's can have common support because  $\{x_m\}$  has no limit point. Our assumption can therefore be dropped.  $\square$

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<sup>2</sup>For more details, see Exercise 2.3.

<sup>3</sup>As  $\Omega = \mathbf{R}$ , the case  $\alpha_m = m$  is the "counterpart" of the series of (a) and the case  $(x_m, \alpha_m) = (m, 0)$  is the *Dirac comb*.

## 9 Exercise 9. Convergence in $\mathcal{D}(\Omega)$ vs. convergence in $\mathcal{D}'(\Omega)$

(a) Prove that a set  $E \subseteq \mathcal{D}(\Omega)$  is bounded if and only if

$$\sup\{|\Lambda\varphi| : \varphi \in E\} < \infty$$

for every  $\Lambda \in \mathcal{D}'(\Omega)$ .

(b) Suppose  $\{\varphi_j\}$  is a sequence in  $\mathcal{D}(\Omega)$  such that  $\{\Lambda\varphi_j\}$  is a bounded sequence of numbers, for every  $\Lambda \in \mathcal{D}'(\Omega)$ . Prove that some subsequence of  $\{\varphi_j\}$  converges, in the topology of  $\mathcal{D}(\Omega)$ .

(c) Suppose  $\{\Lambda_j\}$  is a sequence in  $\mathcal{D}'(\Omega)$  such that  $\{\Lambda_j\varphi\}$  is bounded, for every  $\varphi \in \mathcal{D}(\Omega)$ . Prove that some subsequence of  $\{\Lambda_j\}$  converges in  $\mathcal{D}'(\Omega)$  and that the convergence is uniform on every bounded subset of  $\mathcal{D}(\Omega)$ . *Hint: By the Banach-Steinhaus theorem, the restrictions of the  $\Lambda_j$  to  $\mathcal{D}_K$  are equicontinuous. Apply Ascoli's theorem.*

*Proof.* Since  $\mathcal{D}(\Omega)$  locally convex space (see (b) of [6.4]), [3.18] states that  $E$  is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that  $E = \{\varphi_j : j \in \mathbf{N}\}$  is bounded: so is  $\bar{E}$ . By (c) of [6.5], there exists some  $\mathcal{D}_K$  that contains  $\bar{E}$ . Since  $\mathcal{D}_K$  has the Heine-Borel property (see [1.46]),  $\bar{E}$  is  $\tau_K$ -compact. Apply [A4] with the metrizable space  $\mathcal{D}_K$  (see [1.46]) to conclude that  $\bar{E}$  has a  $\tau_K$  limit point. It then follows from (b) of [6.5] that (b) holds.  $\square$

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