

Solutions to some exercises from Walter Rudin's  
*Functional Analysis*

gitcordier

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## Chapter 1

# Topological Vector Spaces

## 1.1 Exercise 7. Metrizable & number theory

Let be  $X$  the vector space of all complex functions on the unit interval  $[0, 1]$ , topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence  $\{f_n\}$  in  $X$  such that (a)  $\{f_n\}$  converges to 0 as  $n \rightarrow \infty$ , but (b) if  $\{\gamma_n\}$  is any sequence of scalars such that  $\gamma_n \rightarrow \infty$  then  $\{\gamma_n f_n\}$  does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as  $[0, 1]$ .) This shows that metrizable cannot be omitted in (b) of Theorem 1.28.

*Proof.* Our justification consists in proving that  $\tau$ -convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms  $p_x$  is separating. By [1.37], the collection  $\mathcal{B}$  of all finite intersections of the sets

$$V_{x,k} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k \in \mathbf{N}) \quad (1.1)$$

is therefore a local base for a topology  $\tau$  on  $X$ . Given  $\{f_n : n = 1, 2, 3, \dots\}$ , we put

$$\text{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \quad (1.2)$$

with the convention “ $\Sigma = \infty$ ” whether the sum has no finite support. So,

$$\sum_{i=1}^m \text{off}(U_i) = \sum_{n=1}^{\infty} \sum_{i=1}^m [f_n \notin U_i] \geq \text{off}(U_1 \cap \dots \cap U_m). \quad (1.3)$$

We first assume that  $\{f_n\}$   $\tau$ -converges to some  $f$  in  $X$ , *i.e.*

$$\text{off}(f + V) < \infty \quad (V \in \mathcal{B}). \quad (1.4)$$

The special cases  $V_{x,1}, V_{x,2}, \dots$ , mean the pointwise convergence  $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ . Conversely, assume that  $\{f_n\}$  does not  $\tau$ -converges to any  $g$  in  $X$ , *i.e.*

$$\forall g \in X, \exists W \in \mathcal{B} : \text{off}(g + W) = \infty. \quad (1.5)$$

Given  $g$ , such  $W$  is, by definition, a finite intersection  $V_{x_1,k_1} \cap \dots \cap V_{x_m,k_m}$ . Thus,

$$\sum_{i=1}^m \text{off}(g + V_{x_i,k_i}) \stackrel{(1.3)}{\geq} \text{off}(g + W) \stackrel{(1.5)}{=} \infty. \quad (1.6)$$

One of the sum  $\text{off}(g + V_{x_i,k_i})$  must then be  $\infty$ . In other words, there exists a point  $x_i$  for which  $\{f_n(x_i)\}$  does not converge to  $g(x_i)$ .  $g$  being arbitrary, we so conclude that  $f_n$  does not converge pointwise. We have just proved that  $\tau$ -convergence is a rewording of pointwise convergence. We now prove the second part. From now on, we let  $k, n$  and  $p$  run on  $\mathbf{N}_+$ , as  $\text{dyadic}(x)$  denotes the usual dyadic expansion of  $x$ , so that  $\text{dyadic}(x)$  is an aperiodic binary sequence **iff**  $x$  is irrational. Define

$$f_n(x) \triangleq \begin{cases} \exp_2(-\sum_{k=1}^n \text{dyadic}(x)_{-k}) & (x \in [0, 1] \setminus \mathbf{Q}) \\ 0 & (x \in [0, 1] \cap \mathbf{Q}), \end{cases} \quad (1.7)$$

so that  $f_n(x) \xrightarrow{n \rightarrow \infty} 0$ , and take  $\gamma_n \xrightarrow{n \rightarrow \infty} \infty$ , *i.e.* at fixed  $p$ ,  $\gamma_n$  is greater than  $2^p$  for almost all  $n$ . Next, choose  $n_p$  among those *almost all*  $n$  that are large enough to satisfy

$$n_{p-1} - n_{p-2} < n_p - n_{p-1} \quad (1.8)$$

(start with  $n_{-1} = n_0 = 0$ ) and so obtain

$$2^p < \gamma_{n_p} : 0 < n_p - n_{p-1} \xrightarrow{p \rightarrow \infty} \infty. \quad (1.9)$$

The indicator  $\chi$  of  $\{n_1, n_2, \dots\}$  in  $\mathbf{Z}$  is then aperiodic, *i.e.*

$$\alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \in [0, 1] \setminus \mathbf{Q}. \quad (1.10)$$

Hence,  $\chi$  is not a the infinite-support expansion of a rational number; which forces

$$\text{dyadic}(\alpha_\gamma)_{-k} = \chi_k. \quad (1.11)$$

The key ingredient is that

$$\chi_1 + \dots + \chi_{n_p} = p. \quad (1.12)$$

Combined with (1.7), it yields

$$f_{n_p}(\alpha_\gamma) = 2^{-p}. \quad (1.13)$$

Finally,

$$\gamma_{n_p} f_{n_p}(\alpha_\gamma) > 1. \quad (1.14)$$

There so exists  $\{\gamma_{n_p}\}$  such that  $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$  fails to converge pointwise to 0. In other words, (b) holds, which is in violent contrast with 1.28 of [3]:  $X$  is therefore not metrizable. So ends the proof.  $\square$

## 1.2 Exercise 9. Quotient map

Suppose

- (a)  $X$  and  $Y$  are topological vector spaces,
- (b)  $\Lambda : X \rightarrow Y$  is linear.
- (c)  $N$  is a closed subspace of  $X$ ,
- (d)  $\pi : X \rightarrow X/N$  is the quotient map, and
- (e)  $\Lambda x = 0$  for every  $x \in N$ .

Prove that there is a unique  $f : X/N \rightarrow Y$  which satisfies  $\Lambda = f \circ \pi$ , that is,  $\Lambda x = f(\pi(x))$  for all  $x \in X$ . Prove that  $f$  is linear and that  $\Lambda$  is continuous if and only if  $f$  is continuous. Also,  $\Lambda$  is open if and only if  $f$  is open.

*Proof.* Bear in mind that  $\pi$  continuously maps  $X$  onto the topological (Hausdorff) space  $X/N$ , since  $N$  is closed (see 1.41 of [3]). Moreover, the equation  $\Lambda = f \circ \pi$  has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \quad (1.15)$$

To ensure that  $f$  is actually a mapping, simply remark that the linearity of  $\Lambda$  implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x. \quad (1.16)$$

It straightforwardly derives from (1.15) that  $f$  inherits linearity from  $\pi$  and  $\Lambda$ .

**Remark.** The special case  $N = \{\Lambda = 0\}$ , i.e.  $\Lambda x = 0$  iff  $x \in N$  (cf. (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$f(\pi x) = 0 \stackrel{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \quad (1.17)$$

and so conclude that  $f$  is also one-to-one.

Now assume  $f$  to be continuous. Then so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [3]. Conversely, if  $\Lambda$  is continuous, then for each neighborhood  $V$  of  $0_Y$  there exists a neighborhood  $U$  of  $0_X$  such that

$$\Lambda(U) = f(\pi(U)) \subset V. \quad (1.18)$$

Since  $\pi$  is open (1.41 (a) of [3]),  $\pi(U)$  is a neighborhood of  $N = 0_{X/N}$ : This is sufficient to establish that the linear mapping  $f$  is continuous. If  $f$  is open, so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood  $W$  of  $0_{X/N}$  satisfies

$$W = \pi(V) \quad (1.19)$$

for some neighborhood  $V$  of  $0_X$ . So,

$$f(W) = f(\pi(V)) = \Lambda(V). \quad (1.20)$$

As a consequence, if  $\Lambda$  is open, then  $f(W)$  is a neighborhood of  $0_Y$ . So ends the proof.  $\square$



### 1.3 Exercise 10. An open mapping theorem

Suppose that  $X$  and  $Y$  are topological vector spaces,  $\dim Y < \infty$ ,  $\Lambda : X \rightarrow Y$  is linear, and  $\Lambda(X) = Y$ .

(a) Prove that  $\Lambda$  is an open mapping.

(b) Assume, in addition, that the null space of  $\Lambda$  is closed, and prove that  $\Lambda$  is continuous.

*Proof.* We discard the trivial case  $\dim Y = 0$  then henceforth assume that  $\dim Y$  has positive dimension  $n$ .

Let  $e$  range over a base of  $Y$ : For each  $e$ , there exists  $x_e$  in  $X$  such that  $\Lambda(x_e) = e$ , since  $\Lambda$  is onto. So,

$$y = \sum_e y_e \Lambda x_e \quad (y \in Y). \quad (1.21)$$

The sequence  $\{x_e\}$  is finite; therefore it is bounded: Given  $V$  a balanced neighborhood of the origin, there exists a positive scalar  $s$  such that

$$x_e \in sV \text{ for all } x_e. \quad (1.22)$$

Combining this with (1.21) shows that

$$y \in \sum_e \Lambda(V) \quad (y \in Y : |y_e| < s^{-1}), \quad (1.23)$$

which proves (a).

To prove (b), assume that the null space  $\{\Lambda = 0\}$  is closed and let  $f, \pi$  be as in Exercise 1.9, with  $\{\Lambda = 0\}$  playing the role of  $N$ . Since  $\Lambda$  is onto, the first isomorphism theorem (see Exercise 1.9) asserts that  $f$  is an isomorphism of  $X/N$  onto  $Y$ . Consequently,

$$\dim X/N = n. \quad (1.24)$$

$f$  is then an homeomorphism of  $X/N \equiv \mathbf{C}^n$  onto  $Y$ ; see 1.21 of [3]. We have thus established that  $f$  is continuous: So is  $\Lambda = f \circ \pi$ .  $\square$

### 1.4 Exercise 14. $\mathcal{D}_K$ equipped with other seminorms

Put  $K = [0, 1]$  and define  $\mathcal{D}_K$  as in Section 1.46. Show that the following three families of seminorms (where  $n = 0, 1, 2, \dots$ ) define the same topology on  $\mathcal{D}_K$ . If  $D = d/dx$ :

$$(a) \|D^n f\|_\infty = \sup\{|D^n f(x)| : 0 < x < 1\}$$

$$(b) \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$(c) \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

*Proof.* First, remark that

$$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty \quad (1.25)$$

holds, since  $K$  has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of  $D^n f$  lies in  $K$ ; which yields

$$|D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1. \quad (1.26)$$

So,

$$\|D^n f\|_\infty \leq \|D^{n+1} f\|_1. \quad (1.27)$$

We now combine (1.25) with (1.27) and so obtain

$$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots \quad (n = 0, 1, 2, \dots). \quad (1.28)$$

Put

$$V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\} \quad (i = 1, 2, \infty) \quad (1.29)$$

$$\mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\}, \quad (1.30)$$

so that (1.28) is mirrored in terms of neighborhood inclusions, as follows,

$$V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \dots. \quad (1.31)$$

Since  $V_n^{(i)} \supset V_{n+1}^{(i)}$ ,  $\mathcal{B}^{(i)}$  is a local base of a topology  $\tau_i$ . But the chain (1.31) forces

$$\tau_1 = \tau_2 = \tau_\infty. \quad (1.32)$$

To see that, choose a set  $S$  that is  $\tau_1$ -open at  $p$ , i.e.  $V_n^{(1)} \subset S - p$  for some  $n$ . Next, concatenate this with  $V_n^{(2)} \subset V_n^{(1)}$  (see (1.31)) and so obtain  $V_n^{(2)} \subset S - p$ , which implies that  $S$  is  $\tau_2$ -open at  $p$ . Similarly, we deduce, still from (1.31), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \quad (1.33)$$

So ends the proof.  $\square$

## 1.5 Exercise 16. Uniqueness of topology for test functions

*Prove that the topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Do the same for  $C^\infty(\Omega)$  (Section 1.46).*

**Comment** This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms  $p_n$ , then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (e.g. [2.6] in [2]) about intersection of nonempty compact sets.

**Lemma 1** Let  $X$  be a topological space with a countable local base  $\{V_n : n = 1, 2, 3, \dots\}$ . If  $\tilde{V}_n = V_1 \cap \dots \cap V_n$ , then every subsequence  $\{\tilde{V}_{\rho(n)}\}$  is a decreasing (i.e.  $\tilde{V}_{\rho(n)} \supset \tilde{V}_{\rho(n+1)}$ ) local base of  $X$ .

*Proof.* The decreasing property is trivial. Now remark that  $V_n \supset \tilde{V}_n$ : This shows that  $\{\tilde{V}_n\}$  is a local base of  $X$ . Then so is  $\{\tilde{V}_{\rho(n)}\}$ , since  $\tilde{V}_n \supset \tilde{V}_{\rho(n)}$ .  $\square$

The following special case  $V_n = \tilde{V}_n$  is one of the key ingredients:

**Corollary 1 (special case  $V_n = \tilde{V}_n$ )** Under the same notations of Lemma 1, if  $\{V_n\}$  is a decreasing local base, then so is  $\{V_{\rho(n)}\}$ .

**Corollary 2** If  $\{Q_n\}$  is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence  $\{Q_{\rho(n)}\}$  also satisfies theses conditions. Furthermore, if  $\tau_Q$  is the  $C(\Omega)$ 's (respectively  $C^\infty(\Omega)$ 's) topology of the seminorms  $p_n$ , as defined in section 1.44 (respectively 1.46), then the seminorms  $p_{\rho(n)}$  define the same topology  $\tau_Q$ .

*Proof.* Let  $X$  be  $C(\Omega)$  topologized by the seminorms  $p_n$  (the case  $X = C^\infty(\Omega)$  is proved the same way). If  $V_n = \{p_n < 1/n\}$ , then  $\{V_n\}$  is a decreasing local base of  $X$ . Moreover,

$$Q_{\rho(n)} \subset \overset{\circ}{Q}_{\rho(n)+1} \subset Q_{\rho(n)+1} \subset Q_{\rho(n+1)}. \quad (1.34)$$

Thus,

$$Q_{\rho(n)} \subset \overset{\circ}{Q}_{\rho(n+1)}. \quad (1.35)$$

In other words,  $Q_{\rho(n)}$  satisfies the conditions specified in section 1.44.  $\{p_{\rho(n)}\}$  then defines a topology  $\tau_{Q_\rho}$  for which  $\{V_{\rho(n)}\}$  is a local base. So,  $\tau_{Q_\rho} \subset \tau_Q$ . Conversely, the above corollary asserts that  $\{V_{\rho(n)}\}$  is a local base of  $\tau_Q$ , which yields  $\tau_Q \subset \tau_{Q_\rho}$ .  $\square$

**Lemma 2** If a sequence of compact sets  $\{Q_n\}$  satisfies the conditions specified in section 1.44, then every compact set  $K$  lies in almost all  $Q_n^\circ$ , i.e. there exists  $m$  such that

$$K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \dots \quad (1.36)$$

*Proof.* The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \quad (n = 1, 2, 3, \dots) \quad (1.37)$$

shapes  $\{C_n\}$  as a decreasing sequence of compact<sup>1</sup> sets. We now suppose (to reach a contradiction) that no  $C_n$  is empty and so conclude<sup>2</sup> that the  $C_n$ 's intersection contains a point that is not in any  $\overset{\circ}{Q}_n$ . On the other hand, the conditions specified in [1.44] force the  $\overset{\circ}{Q}_n$ 's collection to be an open cover. This contradiction reveals that  $C_m = \emptyset$ , *i.e.*  $K \subset \overset{\circ}{Q}_m$ , for some  $m$ . Finally,

$$K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \dots \quad (1.38)$$

□

We are now in a fair position to establish the following:

**Theorem** The topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of  $C^\infty(\Omega)$ , as long as this sequence satisfies the conditions specified in section 1.44.

*Proof.* With the second corollary's notations,  $\tau_K = \tau_{K_\lambda}$ , for every subsequence  $\{K_{\lambda(n)}\}$ . Similarly, let  $\{L_n\}$  be another sequence of compact subsets of  $\Omega$  that satisfies the condition specified in [1.44], so that  $\tau_L = \tau_{L_\kappa}$  for every subsequence  $\{L_{\kappa(n)}\}$ . Now apply the above Lemma 2 with  $K_i$  ( $i = 1, 2, 3, \dots$ ) and so conclude that  $K_i \subset \overset{\circ}{L}_{m_i} \subset \overset{\circ}{L}_{m_i+1} \subset \dots$  for some  $m_i$ . In particular, the special case  $\kappa_i = m_i + i$  is

$$K_i \subset \overset{\circ}{L}_{\kappa_i}. \quad (1.39)$$

Let us reiterate the above proof with  $K_n$  and  $L_n$  in exchanged roles then similarly find a subsequence  $\{\lambda_j : j = 1, 2, 3, \dots\}$  such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \quad (1.40)$$

Combine (1.39) with (1.40) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{L}_{\kappa_{\lambda_{\kappa_1}}} \subset \dots, \quad (1.41)$$

which means that the sequence  $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$  satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L. \quad (1.42)$$

So ends the proof

□

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<sup>1</sup> See (b) of 2.5 of [2].

<sup>2</sup> In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of 2.6 of [2].

## 1.6 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that  $f \mapsto D^\alpha f$  is a continuous mapping of  $C^\infty(\Omega)$  into  $C^\infty(\Omega)$  and also of  $\mathcal{D}_K$  into  $\mathcal{D}_K$ , for every multi-index  $\alpha$ .

*Proof.* In both cases,  $D^\alpha$  is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the  $C^\infty(\Omega)$  case.

Let  $U$  be an arbitrary neighborhood of the origin. There so exists  $N$  such that  $U$  contains

$$V_N = \left\{ \phi \in C^\infty(\Omega) : \max\{|D^\beta \phi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}. \quad (1.43)$$

Now pick  $g$  in  $V_{N+|\alpha|}$ , so that

$$\max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}. \quad (1.44)$$

(the fact that  $K_N \subset K_{N+|\alpha|}$  was tacitely used). The special case  $\gamma = \beta + \alpha$  yields

$$\max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}. \quad (1.45)$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad i.e. \quad D^\alpha(V_{N+|\alpha|}) \subset V_N, \quad (1.46)$$

which establishes the continuity of  $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$ .

To prove the continuousness of the restriction  $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$ , we first remark that the collection of the  $V_N \cap \mathcal{D}_K$  is a local base of the subspace topology of  $\mathcal{D}_K$ .  $V_{N+|\alpha|} \cap \mathcal{D}_K$  is then a neighborhood of 0 in this topology. Furthermore,

$$D^\alpha|_{\mathcal{D}_K}(V_{N+|\alpha|} \cap \mathcal{D}_K) = D^\alpha(V_{N+|\alpha|} \cap \mathcal{D}_K) \quad (1.47)$$

$$\subset D^\alpha(V_{N+|\alpha|}) \cap D^\alpha(\mathcal{D}_K) \quad (1.48)$$

$$\subset V_N \cap \mathcal{D}_K \quad (\text{see (1.46)}) \quad (1.49)$$

So ends the proof.  $\square$

## Chapter 2

# Completeness

### 2.1 Exercise 3. An equicontinuous sequence of measures

Put  $K = [-1, 1]$ ; define  $\mathcal{D}_K$  as in section 1.46 (with  $\mathbf{R}$  in place of  $\mathbf{R}^n$ ). Suppose  $\{f_n\}$  is a sequence of Lebesgue integrable functions such that  $\Lambda\phi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\phi(t)dt$  exists for every  $\phi \in \mathcal{D}_K$ . Show that  $\Lambda$  is a continuous linear functional on  $\mathcal{D}_K$ . Show that there is a positive integer  $p$  and a number  $M < \infty$  such that

$$\left| \int_{-1}^1 f_n(t)\phi(t)dt \right| \leq M \|D^p \phi\|_\infty$$

for all  $n$ . For example, if  $f_n(t) = n^3 t$  on  $[-1/n, 1/n]$  and 0 elsewhere, show that this can be done with  $p = 1$ . Construct an example where it can be done with  $p = 2$  but not with  $p = 1$ .

We will also consider the case  $p = 0$ . Since all supports of  $\phi, \phi', \phi'', \dots$ , are in  $K$ , we make a specialization of the mean value theorem:

**Lemma** If  $\phi \in \mathcal{D}_{[a,b]}$ , then

$$\|D^\alpha \phi\|_\infty \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (\alpha = 0, 1, \dots, p) \quad (2.1)$$

at every order  $p = 0, 1, 2, \dots$ ; where  $\lambda$  is the length  $|b - a|$ .

*Proof.* Let  $x_0$  be in  $(a, b)$ . We first consider the case  $x_0 \leq c = (a + b)/2$ : The mean value theorem asserts that there exists  $x_1$  ( $a < x_1 < x_0$ ), such that

$$\phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a). \quad (2.2)$$

Since every  $D^p \phi$  lies in  $\mathcal{D}_{[a,b]}$ , a straightforward proof by induction shows that there exists a partition  $a < \dots < x_p < \dots < x_0$  such that

$$\phi(x_0) = D^0 \phi(x_0) \quad (2.3)$$

$$= D^1 \phi(x_1)(x_0 - a) \quad (2.4)$$

$$= \dots$$

$$= D^p \phi(x_p)(x_0 - a) \cdots (x_{p-1} - a), \quad (2.5)$$

for all  $p$ . More compactly,

$$D^\alpha \phi(x_0) = D^p \phi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a); \quad (2.6)$$

which yields,

$$|D^\alpha \phi(x)| \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c]) \quad (2.7)$$

The case  $x_0 \geq c$  outputs a “reversed” result, with  $b > \dots > x_p > \dots > x_0$  and  $x_k - b$  playing the role of  $x_k - a$ : So,

$$|D^\alpha \phi(x)| \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (2.8)$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\|D^\alpha \phi\|_\infty \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}. \quad (2.9)$$

□

*Proof.* We first consider  $C_0(\mathbf{R})$  topologized by the supremum norm. Given a Lebesgue integrable function  $u$ , we put

$$\langle u | \phi \rangle \triangleq \int_{\mathbf{R}} u \phi \quad (\phi \in C_0(\mathbf{R})). \quad (2.10)$$

The following inequalities

$$|\langle u | \phi \rangle| \leq \int_{\mathbf{R}} |u \phi| \leq \|u\|_{L^1} \quad (\|\phi\|_\infty \leq 1) \quad (2.11)$$

imply that every linear functional

$$\begin{aligned} \langle u | : C_0(\mathbf{R}) &\rightarrow \mathbf{C} \\ \phi &\mapsto \langle u | \phi \rangle \end{aligned} \quad (2.12)$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely,  $u$  can be identified with  $\langle u |$ , since  $u$  is determined (a.e) by the integrals  $\langle u | \phi \rangle$ . In the Banach spaces terminology,  $u$  is then (identified with) a linear *bounded*<sup>1</sup> operator  $\langle u |$ , of norm

$$\sup\{|\langle u | \phi \rangle| : \|\phi\|_\infty = 1\} = \|u\|_{L^1}. \quad (2.13)$$

Note that, in the latter equality,  $\leq \|u\|_{L^1}$  comes from (2.11), as the converse comes from the Stone-Weierstrass theorem<sup>2</sup>. We now consider the special cases  $u = g_n$ , where  $g_n$  is

$$\begin{aligned} g_n : \mathbf{R} &\rightarrow \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) \end{cases} \end{aligned} \quad (2.14)$$

<sup>1</sup> see 1.32, 4.1 of [3]

<sup>2</sup> See 7.26 of [1].

First, remark that  $g_n(x) \xrightarrow{n \rightarrow \infty} 0$  ( $x \in \mathbf{R}$ ), as the sequence  $\{g_n\}$  fails to converge in  $C_0(\mathbf{R})$  (since  $g_n(1/n) = n^2 \geq 1$ ), and also in  $L^1$  (since  $\int_{\mathbf{R}} |g_n| = n^2 \rightarrow \infty$ ). Nevertheless, we will show that the  $\langle g_n |$  converge pointwise<sup>3</sup> on  $\mathcal{D}_K$  *i.e.* there exists a  $\tau_K$ -continuous linear form  $\Lambda$  such that

$$\langle g_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \phi, \quad (2.15)$$

where  $\phi$  ranges over  $\mathcal{D}_K$ . We now prove (2.13) in the special cases  $u = g_n$ . To do so, we fetch  $\phi_1^+, \dots, \phi_j^+, \dots$ , from  $C_K^\infty(\mathbf{R})$ . More specifically,

$$(i) \quad \phi_j^+ = 1 \text{ on } [e^{-j}, 1 - e^{-j}];$$

$$(ii) \quad \phi_j^+ = 0 \text{ on } \mathbf{R} \setminus [-1, 1];$$

$$(iii) \quad 0 \leq \phi_j^+ \leq 1 \text{ on } \mathbf{R};$$

see [1.46] of [3] for a possible construction of those  $\phi_j^+$ . Let  $\phi_1^-, \dots, \phi_j^-, \dots$ , mirror the  $\phi_j^+$ , in the sense that  $\phi_j^-(x) = \phi_j^+(-x)$ , so that

$$(iv) \quad \phi_j \triangleq \phi_j^+ - \phi_j^- \text{ is odd, as } g_n \text{ is};$$

$$(v) \quad \text{every } \phi_j \text{ is in } C_K^\infty(\mathbf{R});$$

$$(vi) \quad \text{The sequence } \{\phi_j\} \text{ converges (pointwise) to } 1_{[0,1]} - 1_{[-1,0]}, \text{ and } \|\phi_j\|_\infty = 1.$$

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n | \phi_j \rangle = 2 \int_0^1 g_n(t) \phi_j^+(t) dt \xrightarrow{j \rightarrow \infty} 2 \int_0^1 g_n(t) dt = \|g_n\|_{L^1} = n. \quad (2.16)$$

Finally,

$$\|g_n\|_{L^1} \stackrel{(2.16)}{\leq} \sup\{|\langle g_n | \phi \rangle| : \|\phi\|_\infty = 1\} \stackrel{(2.13)}{\leq} \|g_n\|_{L^1}; \quad (2.17)$$

which is the desired result. So, in terms of boundedness constants: Given  $n$ , there exists  $C_n < \infty$  such that

$$|\langle g_n | \phi \rangle| \leq C_n \quad (\|\phi\|_\infty = 1); \quad (2.18)$$

see (2.11). Furthermore,  $\|g_n\|_{L^1}$  is actually the best, *i.e.* lowest, possible  $C_n$ ; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence  $\{\langle g_n | \phi_{\rho(n)} \rangle\}$  such that  $\langle g_n | \phi_{\rho(n)} \rangle$  is greater than, say,  $n - 0.01$ , as  $\|\phi_{\rho(n)}\|_\infty = 1$ . Consequently, there is no bound  $M$  such that

$$|\langle g_n | \phi \rangle| \leq M \quad (\|\phi\|_\infty = 1; n = 1, 2, 3, \dots). \quad (2.19)$$

In other words, the  $g_n$  have no *uniform bound* in  $L^1$ , *i.e.* the collection of all continuous linear mappings  $\langle g_n |$  is not equicontinuous (see discussion in 2.6 of [3]). As a consequence, the  $\langle g_n |$  do not converge pointwise (or “vaguely”, in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \phi \quad (\phi \in C_0(\mathbf{R})) \quad (2.20)$$

<sup>3</sup> See 3.14 of [3] for a definition of the related topology.



for some  $\Lambda \in C_0(\mathbf{R})^*$ , which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the  $\langle g_n |$  do not converge pointwise, in a relevant space, to some  $\Lambda$  (see (2.15)).

From now on, unless the contrary is explicitly stated, we assume that  $\phi$  only denotes an element of  $C_K^\infty(\mathbf{R})$ . Let  $f_n$  be a Lebesgue integrable function such that

$$\Lambda\phi = \lim_{n \rightarrow \infty} \int_K f_n \phi \quad (\phi \in C_K^\infty(\mathbf{R})). \quad (2.21)$$

for some linear form  $\Lambda$ . Since  $\phi$  vanishes outside  $K$ , we can suppose without loss of generality that the support of  $f_n$  lies in  $K$ . So, (2.21) can be restated as follows,

$$\Lambda\phi = \lim_{n \rightarrow \infty} \langle f_n | \phi \rangle \quad (\phi \in C_K^\infty(\mathbf{R})). \quad (2.22)$$

Let  $K_1, K_2, \dots$ , be compact sets that satisfy the conditions specified in 1.44 of [3].  $\mathcal{D}_K$  is  $C_K^\infty(\mathbf{R})$  topologized by the related seminorms  $p_1, p_2, \dots$ ; see 1.46, 6.2 of [3] and Exercise 1.16. We know that  $K \subset K_m$  for some index  $m$  (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices  $N \geq m$ , so that

- (a)  $p_N(\phi) = \|\phi\|_N \triangleq \max\{|D^\alpha \phi(x)| : \alpha \leq N, x \in \mathbf{R}\}$ , for  $\phi \in \mathcal{D}_K$ ;
- (b) The collection of the sets  $V_N = \{\phi \in \mathcal{D}_K : \|\phi\|_N < 2^{-N}\}$  is a (decreasing) local base of  $\tau_K$ , the subspace topology of  $\mathcal{D}_K$ ; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with  $u = f_n$  and  $\phi \in V_m$  then conclude that  $\langle f_n |$  is bounded by  $\|f_n\|_{L^1}$  on  $V_m$ : Every linear functional  $\langle f_n |$  is therefore  $\tau_K$ -continuous; see 1.18 of [3].

To sum it up:

- (i)  $\mathcal{D}_K$ , equipped the topology  $\tau_K$ , is a Fréchet space (see section 1.46 of [3]);
- (ii) Every linear functional  $\langle f_n |$  is continuous with respect to this topology;
- (iii)  $\langle f_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \Lambda\phi$  for all  $\phi$ , i.e.  $\Lambda - \langle f_n | \xrightarrow{n \rightarrow \infty} 0$ .

With the help of [2.6] and [2.8] of [3], we conclude that  $\Lambda$  is continuous and that the sequence  $\{\langle f_n | \}$  is equicontinuous. So is the sequence  $\{\Lambda - \langle f_n | \}$ , since addition is continuous. There so exists  $i, j$  such that, for all  $n$ ,

$$|\Lambda\phi| < 1/2 \quad \text{if } \phi \in V_i, \quad (2.23)$$

$$|\Lambda\phi - \langle f_n | \phi \rangle| < 1/2 \quad \text{if } \phi \in V_j. \quad (2.24)$$

Choose  $p = \max\{i, j\}$ , so that  $V_p = V_i \cap V_j$ : The latter inequalities imply that

$$|\langle f_n | \phi \rangle| \leq |\Lambda\phi - \langle f_n | \phi \rangle| + |\Lambda\phi| < 1 \quad \text{if } \phi \in V_p. \quad (2.25)$$

Now remark that every  $\psi = \psi[\mu, \phi]$ , where

$$\psi[\mu, \phi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\phi\|_p) \phi & (\phi \neq 0, \mu > 1) \\ 0 & (\phi = 0, \mu > 1), \end{cases} \quad (2.26)$$

keeps in  $V_p$ . Finally, it is clear that each below statement implies the following one.

$$|\langle f_n | \phi \rangle| < 1 \quad (2.27)$$

$$|\langle f_n | \phi \rangle| < 2^p \| \phi \|_p \cdot \mu \quad (2.28)$$

$$|\langle f_n | \phi \rangle| \leq 2^p \| \phi \|_p \quad (2.29)$$

$$|\langle f_n | \phi \rangle| \leq 2^p \{ \| D^0 \phi \|_\infty + \cdots + \| D^p \phi \|_\infty \}. \quad (2.30)$$

Finally, with the help of (2.1),

$$|\langle f_n | \phi \rangle| \leq 2^p(p+1) \| D^p \phi \|_\infty. \quad (2.31)$$

The first part is so proved, with *some*  $p$  and  $M = 2^p(p+1)$ .

We now come back to the special case  $f_n = g_n$  (see the first part). From now on,  $f_n(x) = n^3 x$  on  $[-1/n, 1/n]$ , 0 elsewhere. Actually, we will prove that

(a)  $\Lambda \phi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t) \phi(t) dt$  exists for every  $\phi \in \mathcal{D}_K$ ;

(b) A *uniform* bound  $|\langle f_n | \phi \rangle| \leq M \| D^p \phi \|_\infty$  ( $n = 1, 2, 3, \dots$ ) exists for all those  $f_n$ , with  $p = 1$  as the smallest possible  $p$ .

Bear in mind that  $K \subset K_m$  and shift the  $K_N$ 's indices, so that  $K_{m+1}$  becomes  $K_1$ ,  $K_{m+2}$  becomes  $K_2$ , and so on. The resulting topology  $\tau_K$  remains unchanged (see Exercise 1.16). We let  $\phi$  keep running on  $\mathcal{D}_K$  and so define

$$B_n(\phi) \triangleq \max\{|\phi(x)| : x \in [-1/n, 1/n]\}, \quad (2.32)$$

$$\Delta_n(\phi) \triangleq \max\{|\phi(x) - \phi(0)| : x \in [-1/n, 1/n]\}. \quad (2.33)$$

The mean value asserts that

$$|\phi(1/n) - \phi(-1/n)| \leq B_n(\phi') |1/n - (-1/n)| = \frac{2}{n} B_n(\phi'). \quad (2.34)$$

Independently, an integration by parts shows that

$$\langle f_n | \phi \rangle = \left[ \frac{n^3 t^2}{2} \phi(t) \right]_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt \quad (2.35)$$

$$= \frac{n}{2} (\phi(1/n) - \phi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt. \quad (2.36)$$

Combine (2.34) with (2.36) and so obtain

$$|\langle f_n | \phi \rangle| \leq \frac{n}{2} |\phi(1/n) - \phi(-1/n)| + \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\phi'(t)| dt \quad (2.37)$$

$$\leq B_n(\phi') + \frac{n^3}{2} B_n(\phi') \int_{-1/n}^{1/n} t^2 dt \quad (2.38)$$

$$\leq \frac{4}{3} B_n(\phi') \quad (2.39)$$

$$\leq \frac{4}{3} \| \phi' \|_\infty. \quad (2.40)$$

Futhermore, (2.39) gives a hint about the convergence of  $f_n$ : Since  $B_n(\phi')$  tends to  $|\phi'(0)|$ , we may expect that  $f_n$  tends to  $\frac{4}{3}\phi'(0)$ . This is actually true: A straightforward computation shows that

$$\langle f_n | \phi \rangle - \frac{4}{3}\phi'(0) \stackrel{(2.36)}{=} \frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} (\phi' - \phi'(0))t^2 dt. \quad (2.41)$$

So,

$$\left| \langle f_n | \phi \rangle - \frac{4}{3}\phi'(0) \right| \leq \left| \frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0) \right| + \frac{1}{3}\Delta_n(\phi') \xrightarrow{n \rightarrow \infty} 0. \quad (2.42)$$

We have just proved that

$$\langle f_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3}\phi'(0) \quad (\phi \in \mathcal{D}_K). \quad (2.43)$$

In other words,

$$\langle f_n | \xrightarrow{n \rightarrow \infty} -\frac{4}{3}\delta', \quad (2.44)$$

where  $\delta$  is the *Dirac measure* and  $\delta', \delta'', \dots$ , its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that  $-\frac{4}{3}\delta'$  is  $\tau_K$ -continuous, and from (2.40) that

$$|\langle f_n | \phi \rangle| \leq \frac{4}{3} \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots) \quad (2.45)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n | \phi_{\rho(n)} \rangle : \|\phi_{\rho(n)}\|_\infty = 1; n = 1, 2, 3, \dots\} \quad (2.46)$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant  $M$  such that

$$|\langle f_n | \phi \rangle| \leq M \|\phi\|_\infty \quad (\phi \in C_K^\infty(\mathbf{R})). \quad (2.47)$$

The previous bound of  $\langle f_n |$  - see (2.40), is therefore the best possible one, *i.e.*  $p = 1$  is the smallest possible  $p$  and, given  $p = 1$ ,  $M = \frac{4}{3}$  is the smallest possible  $M$  (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give  $f_n$  a *derivative*<sup>4</sup>  $f_n'$ , as follows

$$\begin{aligned} f_n' : \mathcal{D}_K &\rightarrow \mathbf{C} \\ \phi &\mapsto -\langle f_n | \phi' \rangle. \end{aligned} \quad (2.48)$$

It has been proved that every  $\langle f_n |$  is continuous. So is

$$\begin{aligned} D : \mathcal{D}_K &\rightarrow \mathcal{D}_K \\ \phi &\mapsto \phi'; \end{aligned} \quad (2.49)$$

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<sup>4</sup> See 6.1 of [3] for a further discussion.

see Exercise 1.17.  $f_n'$  is therefore continuous. Now apply (2.43) with  $\phi'$  and so obtain

$$-\langle f_n | \phi' \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3} \phi''(0) \quad (\phi \in \mathcal{D}_K),$$

*i.e.*

$$f_n' \xrightarrow{n \rightarrow \infty} \frac{4}{3} \delta''. \quad (2.50)$$

It follows from (2.40) that,

$$|\langle f_n | \phi' \rangle| \leq \frac{4}{3} \|\phi''\|_\infty \quad (n = 1, 2, 3, \dots). \quad (2.51)$$

It is therefore possible to uniformly bound  $f_n'$  with respect to a norm  $\|D^p \cdot\|_\infty$ , namely  $\|D^2 \cdot\|_\infty$ . Then arises a question: Is 2 the smallest  $p$ ? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant  $M$  such that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots). \quad (2.52)$$

Define

$$\Phi_j(x) = \int_{-1}^x \phi_j. \quad (2.53)$$

The oddness of  $\phi_j$  forces  $\Phi_j$  to vanish outside  $[-1, 1]$ :  $\phi_j$  is therefore in  $\mathcal{D}_K$ . So, under our assumption,

$$|\langle f_n | \Phi_j' \rangle| \leq M \|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots); \quad (2.54)$$

which is

$$|\langle f_n | \phi_j \rangle| \leq M \quad (n = 1, 2, 3, \dots). \quad (2.55)$$

We have thus reached a contradiction (again with the sequence  $\{\langle f_n | \phi_{e(n)} \rangle\}$ ) and so conclude that there is no constant  $M$  such that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots). \quad (2.56)$$

Finally, assume, to reach a contradiction, that there exists a constant  $M$  such that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi\|_\infty. \quad (2.57)$$

The mean value theorem (see (2.1)) asserts that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi\|_\infty \leq M \|\phi'\|_\infty; \quad (2.58)$$

which is, again, a desired contradiction. So ends the proof.  $\square$

## 2.2 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient  $\hat{f}(n)$  of a function  $f \in L^2(T)$  ( $T$  is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all  $n \in \mathbf{Z}$  (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^n \hat{f}(k).$$

Prove that  $\{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$  is a dense subspace of  $L^2(T)$  of the first category.

*Proof.* Let  $f(\theta)$  stand for  $f(e^{i\theta})$ , so that  $L^2(T)$  is identified with a closed subset of  $L^2([-\pi, \pi])$ , hence the inner product

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta. \quad (2.59)$$

We believe it is customary to write

$$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n). \quad (2.60)$$

Moreover, a well known (and easy to prove) result is

$$(e_n, e_{n'}) = [n = n'], \text{ i.e. } \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T). \quad (2.61)$$

For the sake of brevity, we assume the isometric ( $\equiv$ ) identification  $L^2 \equiv (L^2)^*$ . So,

$$\|\Lambda_n\|^2 \stackrel{(2.60)}{=} \|e_{-n} + \cdots + e_n\|^2 \stackrel{(2.61)}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \stackrel{(2.61)}{=} 2n + 1. \quad (2.62)$$

We now assume, to reach a contradiction, that

$$B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \dots\} < \infty\} \quad (2.63)$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence  $\{\Lambda_n\}$  is norm-bounded; which is a desired contradiction, since

$$\|\Lambda_n\| \stackrel{(2.62)}{=} \sqrt{2n+1} \xrightarrow{n \rightarrow \infty} \infty. \quad (2.64)$$

We have just established that  $B$  is actually of the first category; and so is its subset  $L = \{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$ . We now prove that  $L$  is nevertheless dense in  $L^2(T)$ . To do so, we let  $P$  be  $\text{span}\{e_k : k \in \mathbf{Z}\}$ , the collection of the trigonometric polynomials  $p(\theta) = \sum \lambda_k e^{ik\theta}$ . Combining (2.60) with (2.61) shows that  $\Lambda_n(p) = \sum \lambda_k$  for almost all  $n$ . Thus,

$$P \subset L \subset L^2(T). \quad (2.65)$$

We know from the Fejér theorem (the Lebesgue variant) that  $P$  is dense in  $L^2(T)$ . We then conclude, with the help of (2.65), that

$$L^2(T) = \overline{P} = \overline{L}. \quad (2.66)$$

So ends the proof □

## 2.3 Exercise 9. Boundedness without closedness

Suppose  $X, Y, Z$  are Banach spaces and

$$B : X \times Y \rightarrow Z$$

is bilinear and continuous. Prove that there exists  $M < \infty$  such that

$$\|B(x, y)\| \leq M \|x\| \|y\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

*Proof.* The answer is: No. To prove this, we only assume that  $X, Y, Z$  are normed spaces. Let  $(x, y)$  range over  $X \times Y$ .  $B$  is continuous at the origin; thus, there exists a positive  $r$  such that

$$\|B(x, y)\| < 1 \quad (\max\{\|x\|, \|y\|\} < r). \quad (2.67)$$

Given  $(x, y)$ , we choose two scalars  $\alpha, \beta$  such that  $r > \alpha^{-1}\|x\|$  and  $r > \beta^{-1}\|y\|$ . Thus,

$$\|B(x, y)\| = \alpha\beta \|B(\alpha^{-1}x, \beta^{-1}y)\| \quad (2.68)$$

$$< \alpha\beta. \quad (2.69)$$

Since the latter inequality holds for all  $\alpha > \|x\|/r$  and all  $\beta > \|y\|/r$  we conclude that

$$\|B(x, y)\| \leq r^2 \|x\| \|y\|. \quad (2.70)$$

So ends the proof.  $\square$

As a concrete example, choose  $X = Y = Z = C_c(\mathbf{R})$ , topologized by the supremum norm.  $C_c(\mathbf{R})$  is not complete<sup>5</sup>, nevertheless the bilinear product

$$B(f, g) = f \times g \quad ((f, g) \in C_c(\mathbf{R})^2) \quad (2.71)$$

is bounded, (since  $\|B(f, g)\|_\infty = \|f\|_\infty \|g\|_\infty$ ) and continuous. To see that, pick  $(u, v)$  in  $C_c(\mathbf{R})^2$ : Given any positive scalar  $\varepsilon$ , there exists another positive scalar  $r$  such that  $r(r + \|u\| + \|v\|) < \varepsilon$ . So, under the following assumption

$$\max\{\|f - u\|_\infty, \|g - v\|_\infty\} < r, \quad (2.72)$$

we reach

$$\|fg - uv\|_\infty \leq \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty \quad (2.73)$$

$$< r(r + \|v\|) + \|u\|r \quad (2.74)$$

$$< r(r + \|u\| + \|v\|) \quad (2.75)$$

$$< \varepsilon. \quad (2.76)$$

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<sup>5</sup> See 5.4.4 [4]

## 2.4 Exercise 10. Continuousness of bilinear mappings

*Prove that a bilinear mapping is continuous if it is continuous at the origin  $(0, 0)$ .*

*Proof.* Let  $(X_1, X_2, Z)$  be topological spaces and  $B$  a bilinear mapping

$$B : X_1 \times X_2 \rightarrow Z \quad (2.77)$$

From now on,  $x = (x_1, x_2)$  denotes an arbitrary element of  $X_1 \times X_2$ . We henceforth assume that  $B$  is continuous at the origin  $(0, 0)$  of  $X_1 \times X_2$ , *i.e.* given an arbitrary balanced open subset  $W$  of  $Z$ , there exists in  $X_i$  ( $i = 1, 2$ ) a balanced open subset  $U_i$  such that

$$B(U_1 \times U_2) \subset W. \quad (2.78)$$

Let  $\nu_i(x)$  denote any scalar that is greater than  $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$ . So,

$$B(x_1, x_2) = \nu_1(x)\nu_2(x) \cdot B(\nu_1(x)^{-1}x_1, \nu_2(x)^{-1}x_2) \quad (2.79)$$

$$\in \nu_1(x)\nu_2(x) \cdot B(U_1 \times U_2) \quad (2.80)$$

$$\subset \nu_1(x)\nu_2(x) \cdot W. \quad (2.81)$$

Now pick  $p = (p_1, p_2)$  in  $X_1 \times X_2$ : It directly follows from (2.81) that

$$B(p_1, p_2) - B(x_1, x_2) = B(p_1, p_2 - x_2) + B(p_1 - x_1, x_2 - p_2) + B(p_1 - x_1, p_2) \quad (2.82)$$

$$\in \nu_1(p)\nu_2(p - x) \cdot W + \nu_1(p - x)\nu_2(x - p) \cdot W + \nu_1(p - x)\nu_2(p) \cdot W. \quad (2.83)$$

Let us henceforth assume that

$$p_i - x_i \in [\mu_1(p) + \mu_2(p) + 2]^{-1} \cdot U_i; \quad (2.84)$$

which yields

$$\mu_i(p_i - x_i) \leq [\mu_1(p) + \mu_2(p) + 2]^{-1}. \quad (2.85)$$

Finally, combine the special case

$$\nu_i(p - x) = [\mu_1(p) + \mu_2(p) + 1]^{-1}, \quad (2.86)$$

$$\nu_i(p) = \mu_1(p) + \mu_2(p) + 1 \quad (2.87)$$

with (2.83) and so obtain

$$B(p_1, p_2) - B(x_1, x_2) \in W + W + W. \quad (2.88)$$

$W$  being arbitrary, we have so established the continuousness of  $B$  at  $(p_1, p_2)$ . Since  $(p_1, p_2)$  is also arbitrary, the proof is complete.  $\square$

# Bibliography

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