Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$, topologized by the family of seminorms

$$\mathrm{p_x(f) = |f(x)|} \quad (0 \le \mathrm{x} \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{\mathrm{f_n}\}$ in X such that (a) $\{\mathrm{f_n}\}$ converges to 0 as $\mathrm{n} \to \infty$, but (b) if $\{\gamma_\mathrm{n}\}$ is any sequence of scalars such that $\gamma_\mathrm{n} \to \infty$ then $\{\gamma_\mathrm{n}\mathrm{f_n}\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0,1]$.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms $\mathrm{p_x}$ is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$\mathrm{V}^{((\mathrm{x,k})} \triangleq \{\mathrm{p_x} < 2^{-\mathrm{k}}\} \quad (\mathrm{x} \in [0,1], \mathrm{k} \in \mathbf{N}) \tag{1.1}$$

is then a local base for a topology τ on X. Given $\{\mathrm{f_n} : \mathrm{n} = 1,2,3,\dots\}$, we set

$$\mathtt{off}(\mathrm{U}) \triangleq \sum_{\mathrm{n}=1}^{\infty} [\mathrm{f_n} \notin \mathrm{U}] \quad (\mathrm{U} \in \tau), \tag{1.2}$$

with the convention $\mathtt{off}(\mathrm{U}) = \infty$ whether the sum has no finite support. So,

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{U}^{(\mathrm{i})}) = \sum_{\mathrm{n}=1}^{\infty}\sum_{\mathrm{i}=1}^{\mathrm{m}} [\mathrm{f_n} \notin \mathrm{U}^{(\mathrm{i})}] \ge \mathtt{off}(\mathrm{U}^{(1)} \cap \dots \cap \mathrm{U}^{(\mathrm{m})}) \tag{1.3}$$

We first assume that $\{\mathrm{f_n}\}$ τ-converges to some f in X, *i.e.*

$$\mathtt{off}(\mathrm{f} + \mathrm{V}) < \infty \quad (\mathrm{V} \in \mathscr{B}). \tag{1.4}$$

The special cases $\mathrm{V} = \mathrm{V}^{(\mathrm{x,k})}$ mean the pointwise convergence of $\{\mathrm{f_n}\}$. Conversely, assume that $\{\mathrm{f_n}\}$ does not τ-converges to any g in X, *i.e.*

$$\forall \mathrm{g} \in \mathrm{X}, \exists \mathrm{V}^{(\mathrm{g})} \in \mathscr{B} : \mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{g})}) = \infty. \tag{1.5}$$

Given g, $\mathrm{V}^{(\mathrm{g})}$ is then an intersection $\mathrm{V}^{(\mathrm{x}^{(1)},\mathrm{k}^{(1)})} \cap \dots \cap \mathrm{V}^{(\mathrm{x}^{(\mathrm{m})},\mathrm{k}^{(\mathrm{m})})}$. Thus

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{x}^{(\mathrm{i})},\mathrm{k}^{(\mathrm{i})})}) \overset{(1.3)}{\ge} \mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{g})}) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\mathtt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{x}^{(\mathrm{i})},\mathrm{k}^{(\mathrm{i})})})$ must then be ∞. This implies that convergence of $\mathrm{f_n}$ to g fails at point $\mathrm{x_i}$. g being arbitrary, we so conclude that $\mathrm{f_n}$ does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on, *k*, *n* and *p* run on $\mathbf{N}_+$. Let $\mathtt{dyadic}(\mathrm{x})$ be the usual dyadic expansion of a real number x, so that $\mathtt{dyadic}(\mathrm{x})$ is an aperiodic binary sequence **iff** x is irrational. Define

$$\mathrm{f_n(x)} \triangleq \begin{cases} 2^{-\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathtt{dyadic}(\mathrm{x})_{-\mathrm{k}}} & (\mathrm{x} \in [0,1] \setminus \mathbf{Q}) \\ 0 & (\mathrm{x} \in [0,1] \cap \mathbf{Q}) \end{cases} \tag{1.7}$$

so that $f_n(x) \underset{n\to\infty}{\longrightarrow} 0$ and take scalars γ_n such that $\underset{n\to\infty}{\longrightarrow} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose $n^{(p)}$ among those *almost all* n that are large enough to satisfy

$$n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)} \tag{1.8}$$

(start with $n^{(-1)} = n^{(0)} = 0$) and so obtain

$$2^p < \gamma_{n^{(p)}} : 0 < n^{(p)} - n^{(p-1)} \underset{p\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator χ of $\{n^{(1)}, n^{(2)}, \dots\}$ is then aperiodic, *i.e.*

$$x^{(\gamma)} \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \tag{1.10}$$

is irrational. Consequently,

$$\mathtt{dyadic}(x^{(\gamma)})_{-k} = \chi_k. \tag{1.11}$$

We now easily see that

$$\chi_1 + \cdots + \chi_{n^{(p)}} = p, \tag{1.12}$$

which, combined with (1.7), yields

$$f_{n^{(p)}}(x^{(\gamma)}) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n^{(p)}} f_{n^{(p)}}(x^{(\gamma)}) > 1. \tag{1.14}$$

We have so established that the subsequence $\{\gamma_{n^{(p)}} f_{n^{(p)}}\}$ does not tend pointwise to 0, hence neither does the whole sequence $\{\gamma_n f_n\}$. In other words, (b) holds, which is in violent contrast with [1.28]: X is then not metrizable. So ends the proof. □

1.2 Exercise 9. Quotient map

Suppose

(*a*) X *and* Y *are topological vector spaces,*

(*b*) $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear.*

(*c*) N *is a closed subspace of* X*,*

(*d*) $\pi : \mathrm{X} \to \mathrm{X/N}$ *is the quotient map, and*

(*e*) $\Lambda\mathrm{x} = 0$ *for every* $\mathrm{x} \in \mathrm{N}$*.*

Prove that there is a unique $\mathrm{f} : \mathrm{X/N} \to \mathrm{Y}$ *which satisfies* $\Lambda = \mathrm{f} \circ \pi$*, that is,* $\Lambda\mathrm{x} = \mathrm{f}(\pi(\mathrm{x}))$ *for all* $\mathrm{x} \in \mathrm{X}$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. The equation $\Lambda = \mathrm{f}\circ\pi$ has necessarily a unique solution, which is the binary relation

$$\mathrm{f} \triangleq \{(\pi\mathrm{x}, \Lambda\mathrm{x}) : \mathrm{x} \in \mathrm{X}\} \subset \mathrm{X/N} \times \mathrm{Y}. \tag{1.15}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda\mathrm{x} \neq \Lambda\mathrm{x}' \Rightarrow \pi\mathrm{x}' \neq \pi\mathrm{x}'. \tag{1.16}$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ. Now remark that

$$\pi\mathrm{x} = \mathrm{N} \overset{(\mathrm{f\ linear})}{\Rightarrow} \mathrm{f}(\pi\mathrm{x}) = 0 \overset{(1.15)}{\Rightarrow} \Lambda\mathrm{x} = 0 \Rightarrow \pi\mathrm{x} = \mathrm{N} \tag{1.17}$$

and so conclude that f is also one-to-one. Now assume f to be continuous. Then so is $\Lambda = \mathrm{f} \circ \pi$, by (a) of [1.41]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(\mathrm{U}) = \mathrm{f}\,(\pi(\mathrm{U})) \subset \mathrm{V}. \tag{1.18}$$

Since π is open (see (a) of [1.41]), $\pi(\mathrm{U})$ is a neighborhood of $\mathrm{N} = 0_{\mathrm{X/N}}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = \mathrm{f} \circ \pi$, by (a) of [1.41]. Conversely, let

$$\mathrm{W} \triangleq \pi(\mathrm{V}) \subset \mathrm{X/N} \quad (\mathrm{V\ neighborhood\ of\ } 0_\mathrm{X}) \tag{1.19}$$

range over all neighborhoods of N, as Λ is kept open: So is

$$\Lambda(\mathrm{V}) = \mathrm{f}\,(\pi(\mathrm{V})) = \mathrm{f}(\mathrm{W}). \tag{1.20}$$

The linear mapping f is then open. □

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$, $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$.

(*a*) *Prove that* Λ *is an open mapping.*

(*b*) *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. (*a*) Let e range over a base of Y: For each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, since Λ is onto.So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \Lambda \mathrm{x_e} \quad (\mathrm{y} \in \mathrm{Y}). \tag{1.21}$$

The sequence $\{\mathrm{x_e}\}$ is finite hence bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$\mathrm{x_e} \in \mathrm{sV} \tag{1.22}$$

for all $\mathrm{x_e}$. Combining this with (1.21) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \quad (\mathrm{y} \in \mathrm{Y} : |\mathrm{y_e}| < \mathrm{s}^{-1}). \tag{1.23}$$

(*b*) Since N is closed, π continously maps X onto X/N, another topological (Hausdorf) vector space, see [1.41]. Now take f as in Exercise 9: Since Λ is onto, f is an isomorphism of X/N onto Y. X/N has then dimension $\dim \mathrm{Y}$. The inverse isomorphism f^{-1} is then an homeomorphism of $\mathrm{Y} \equiv \mathbf{C}^{\dim \mathrm{Y}}$ onto X/N; see [1.21]. Consequently, both f and f^{-1} are continuous: So is $\Lambda = \mathrm{f} \circ \pi$.

□

1.4 Exercise 14.

Put $\mathrm{K} = [0,1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0, 1, 2, \ldots$ *) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

(*a*) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty = \sup\{|\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})| : \infty < \mathrm{x} < \infty\}$

(*b*) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_1 = \int_0^1 |\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})|$

(*c*) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_2 = \left\{\int_0^1 |\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})|^2 \,\mathrm{d}\,\delta\mathrm{x}\right\}^{1/2}$.

Proof. First, remark that

$$\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_1 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_2 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty < \infty \tag{1.24}$$

(the inequality on the left is a Cauchy-Schwarz one), since K has length 1. Next, start from

$$\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x}) = \int_{-\infty}^{\mathrm{x}} \mathrm{D}^{\mathrm{n}+1}\mathrm{f} \tag{1.25}$$

(which is true, since f has a bounded support) to obtain

$$|\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})| \le \int_{-\infty}^{\mathrm{x}} \left|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\right| \le \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_1 \tag{1.26}$$

hence

$$\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty \le \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_1. \tag{1.27}$$

Combining (1.24) with (1.27) yields

$$\|\mathrm{Df}\|_1 \le \cdots \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_1 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_2 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty \le \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_1 \le \cdots \tag{1.28}$$

We now define

$$\mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \triangleq \{\mathrm{f} \in \mathscr{D}_{\mathrm{K}} : \|\mathrm{f}\|_{\mathrm{i}} < 1/\mathrm{n}\} : \mathrm{n} = 1, 2, 3, \ldots\} \quad (\mathrm{i} = 1, 2, \infty), \tag{1.29}$$

so that (1.28) is mirrored in terms of neighborhood inclusions, as follows,

$$\mathrm{V}_1^{(1)} \supset \cdots \supset \mathrm{V}_{\mathrm{n}}^{(1)} \supset \mathrm{V}_{\mathrm{n}}^{(2)} \supset \mathrm{V}_{\mathrm{n}}^{(\infty)} \supset \mathrm{V}_{\mathrm{n}+1}^{(1)} \supset \cdots \tag{1.30}$$

Since $\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \supset \mathrm{V}_{\mathrm{n}+1}^{(\mathrm{i})}$, $\mathscr{B}_{\mathrm{i}}$ is the local base of a topology τ_{i}. But the chain (1.30) forces the τ_{i}'s to be equals. To see that, choose a set S that is τ_1-open at, say a: So, $\mathrm{V}_{\mathrm{n}}^{(1)} \subset \mathrm{S} - \mathrm{a}$ for some n. Now $\mathrm{V}_{\mathrm{n}}^{(1)} \supset \mathrm{V}_{\mathrm{n}}^{(2)}$ (see (1.30)) forces $\mathrm{V}_{\mathrm{n}}^{(2)} \subset \mathrm{S} - \mathrm{a}$, which implies that S is τ_2-open at a. Similarly, we deduce, still from (1.30), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.31}$$

So ends the proof. □

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $|\mathrm{f}|_\mathrm{N} = \max|\mathrm{f}|$, then, eventually, only on the ambient space itself. This should then be regarded as a very part of the textbook [2] The proof consists in combining trivial consequences of the local base definition with the well-known result (*e.g.* [2.6] in [1]) about intersection of nonempty compact sets.

Lemma Let X be a topological space with a countable local base $\{\mathrm{V_N} : \mathrm{N} = 1, 2, 3, \dots\}$. If $\tilde{\mathrm{V}}_\mathrm{N} = \mathrm{V}_1 \cap \cdots \cap \mathrm{V_N}$, then every subsequence $\{\tilde{\mathrm{V}}_{\rho(\mathrm{N})}\}$ is a also a decreasing (*i.e.* $\tilde{\mathrm{V}}_{\rho(\mathrm{N})} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{N}+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_N} \supset \tilde{\mathrm{V}}_\mathrm{N} \supset \tilde{\mathrm{V}}_{\mathrm{N}+1}$: The left inclusion shows that $\{\tilde{\mathrm{V}}_\mathrm{N}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\rho(\mathrm{N})}\}$, since $\tilde{\mathrm{V}}_\mathrm{N} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{N})}$. □

The following special case $\mathrm{V_N} = \tilde{\mathrm{V}}_\mathrm{N}$ is one of the key ingredients:

Corollary 1 (special case) With the same notations, if $\{\mathrm{V_N}\}$ is a decreasing local base, then so is $\{\mathrm{V}_{\rho(\mathrm{N})}\}$.

Corollary 2 If $\{\mathrm{Q_N}\}$ is a sequence of compacts that satisfies the conditions specified in section 1.44, then every subsequence $\{\mathrm{Q}_{\rho(\mathrm{N})}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $\mathrm{C}(\Omega)$'s (respectively $\mathrm{C}^\infty(\Omega)$'s) topology of the seminorms $\mathrm{p_N}$, as defined in section 1.44 (respectively 1.46), then the seminorms $\mathrm{p}_{\rho(\mathrm{N})}$ define the same topology τ_Q.

Proof. Let X be $\mathrm{C}(\Omega)$ topologized with the seminorms $\mathrm{p_N}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_N} = \{\mathrm{p_N} < 1/\mathrm{N}\}$, then $\{\mathrm{V_N}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\rho(\mathrm{N})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{N})+1} \subset \mathrm{Q}_{\rho(\mathrm{N})+1} \subset \mathrm{Q}_{\rho(\mathrm{N}+1)}, \tag{1.32}$$

and this yields

$$\mathrm{Q}_{\rho(\mathrm{N})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{N}+1)}. \tag{1.33}$$

In other words, $\mathrm{Q}_{\rho(\mathrm{N})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\rho(\mathrm{N})}\}$ then defines a topology τ_{Q_ρ} for which $\{\mathrm{V}_{\rho(\mathrm{N})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\rho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary turns $\{\mathrm{V}_{\rho(\mathrm{N})}\}$ into a local base of τ_Q. Hence $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\rho}$. □

We are now in a fair position to establish the following:

Theorem The topology of $\mathrm{C}(\Omega)$ does not depend on the particular choice of $\{\mathrm{K_n}\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $\mathrm{C}^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.46.

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\varkappa}$, for every subsequence $\{K_{\varkappa(n)}\}$. Similarly, let $\{L_n\}$ be a sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\lambda}$ for every subsequence $\{L_{\lambda(n)}\}$. The following definition

$$C_{i,j} \triangleq K_i \setminus \overset{\circ}{L}_j \quad (i, j = 1, 2, 3, \dots) \tag{1.34}$$

shapes $\{C_{i,j} : j = 1, 2, 3, \dots\}$ as a decreasing sequence of compacts. We now suppose (to reach a contradiction) that no $C_{i,j}$ is empty and so conclude that $C_{i,1} \cap C_{i,2} \cap \cdots$ contains a point that is not in any $\mathrm{int}(L_j)$. But the conditions specified in [1.44] force $\{\mathrm{int}(L_j) : j = 1, 2, 3, \dots\}$ to be an open cover. This contradiction reveals that $C_{i,\,j} = C_{i,\,j+1} = C_{i,\,j+2} = \cdots = \emptyset$ for some $j = j^{(i)}$. We then define $\lambda_i = i + j^{(i)}$, so that

$$K_i \setminus \overset{\circ}{L}_{\lambda_i} = \emptyset, \quad \textit{i.e.} \quad K_i \subset \overset{\circ}{L}_{\lambda_i}. \tag{1.35}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\varkappa_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\varkappa_j} \tag{1.36}$$

Combine (1.35) with (1.36) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\lambda_1} \subset L_{\lambda_1} \subset \overset{\circ}{K}_{\varkappa\circ\lambda_1} \subset K_{\varkappa\circ\lambda_1} \subset \overset{\circ}{L}_{\lambda_{\varkappa\circ\lambda_1}} \subset \cdots \tag{1.37}$$

Thus the sequence $Q = (K_1, L_{\lambda_1}, K_{\varkappa\circ\lambda_1}, L_{\lambda_{\varkappa\circ\lambda_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\varkappa} = \tau_Q = \tau_{L_\lambda} = \tau_L. \tag{1.38}$$

So ends the proof $\square$

Bibliography

[1] Walter Rudin. *Real and Complex Analysis.* McGraw-Hill, 1986.

[2] Walter Rudin. *Functional Analysis.* McGraw-Hill, 1991.