

Solutions to some exercises from Walter Rudin's  
*Functional Analysis*

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# Notations and Conventions

## 0.1 Logic

1. **Halmos' iff.** **iff** is a short for “if and only if”.
2. **Definitions (of values) with  $\triangleq$ .** Given variables  $a$  and  $b$ ,  $a \triangleq b$  means that  $a$  is defined as equal to  $b$ .
3.  **$\equiv$ .**  $a \equiv b$  means that there exists a “natural” bijection  $\rightarrow$  that maps  $a$  to  $b$ ; which let us identify  $a$  with  $b$ . In a metric space context,  $a \equiv b$  means that  $\rightarrow$  is isometric.
4. **Definitions (formulæ).** Definitions use the **iff** format. In other words, every definition has a “only if”.
5. **Iverson notation.** Given a boolean expression  $\Phi$ ,  $[\Phi]$  returns the truth value of  $\Phi$ , encoded as follows,

$$[\Phi] \triangleq \begin{cases} 0 & \text{if } \Phi \text{ is false;} \\ 1 & \text{if } \Phi \text{ is true.} \end{cases}$$

For example,  $[1 > 0] = 1$  but  $[\sqrt{2} \in \mathbf{Q}] = 0$ .

## 0.2 Topological vector spaces

1. **Product space**
2. **Scalar field.** The usual (complete) scalar field is  $\mathbf{C}$ . A property, *e.g.* linearity, that is true on  $\mathbf{C}$  is also true on  $\mathbf{R}$ . The complex case is then a *special case* of the real one. Sometimes, this specialization is not purely formal. For example, theorem 12.7 of [3] asserts that, in a Hilbert space  $H$  equipped with the inner product  $\langle \cdot | \cdot \rangle$ , every nonzero linear continuous operator  $T$  “breaks orthogonality”, in the sense that there always exists  $x = x(T)$  in  $H$  that satisfies  $\langle Tx | x \rangle \neq 0$ . The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exist. To see that, consider the  $90^\circ$  rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitly mentioned*, the extension to the real case will always be obvious. So, taking  $\mathbf{C}$  as the scalar field shall mean “*Instead of letting the scalar field undefined, we choose  $\mathbf{C}$  for the sake of expressivity. But considering  $\mathbf{R}$  instead of  $\mathbf{C}$  would actually make no difference here*”.
3. **Finite dimensional spaces.** Let  $Y$  be a finite dimensional space. If  $\dim Y = 0$ , *i.e.*  $Y$  is a group of order 1, then  $\{\emptyset, Y\}$  is the only possible topology for  $Y$ . For instance, in a quotient space  $X/N$ , the zero is  $N$  and  $\{N\}$  is zero-dimensional in  $X/N$ .

Assume henceforth that  $\dim Y > 0$ , *i.e.*  $Y$  has a base  $F_n = \{f_i : i = 1, \dots, n\}$  for some positive  $n$ . The cartesian power  $\mathbf{C}^n = \prod_{j=1}^n \mathbf{C}$  is the vector space of all lists  $(z_1, \dots, z_n)$ , where  $z_j \in \mathbf{C}$  (identify  $\mathbf{C}^1$  with  $\mathbf{C}$ ). The subset  $E_n = \{e_j : j = 1, \dots, n\}$  is the *standard base* of  $\mathbf{C}^n$ , *i.e.*  $e_j = 1_{\{j\}}$ . So,  $\dim \mathbf{C}^n = n$ . Let  $u : \mathbf{C}^n \rightarrow Y$  be the only linear mapping that verifies all  $u(e_j) = f_j$ . Since  $u$  is encoded as the identity matrix, both  $u$  and  $v = u^{-1}$  exist as isomorphisms. Additionally,  $\mathbf{C}^n$  can be equipped with various norms, *e.g.* the *p-norms*  $\|\cdot\|_p$  (where  $\|(z_1, \dots, z_n)\|_p^p = |z_1|^p + \dots + |z_n|^p$ ;  $p \geq 1$ ) or  $\|\cdot\|_\infty$  (where  $\|(z_1, \dots, z_n)\|_\infty = \max |z_j|$ ). Note that  $Y$  inherits any norm  $\|\cdot\|$  of  $\mathbf{C}^n$ , with  $\|u(z_1, \dots, z_n)\| = \|(z_1, \dots, z_n)\|$ ; which turns  $u$  into a isometry of  $\mathbf{C}^n$  onto  $Y$ . Let  $\tau_{\|\cdot\|}$  denote the topology of a norm  $\|\cdot\|$ . We now go back to the proof of 1.21 of [3] and so equip  $Y$  with a its own norm  $\|\cdot\|_2$ ; which turns  $u$  into a isometric isomorphism of  $\mathbf{C}^n$  onto  $Y$ .  $Y$  can now be seen as a topological vector space, in at least one fashion; namely, the space  $(Y, \tau_{\|\cdot\|_2})$ . Let  $\tau = \tau_Y$  stand for any arbitrary topology of  $Y$ . Hence the following commutative diagram

$$\begin{array}{ccc}
 & (\mathbf{C}^n, \tau_{\|\cdot\|_2}) & \\
 u \swarrow & & \nwarrow v \\
 (Y, \tau_{\|\cdot\|_2}) & & (Y, \tau) \\
 v \searrow & & \nearrow u \\
 & (\mathbf{C}^n, \tau_{\|\cdot\|_2}) &
 \end{array}$$

(1)

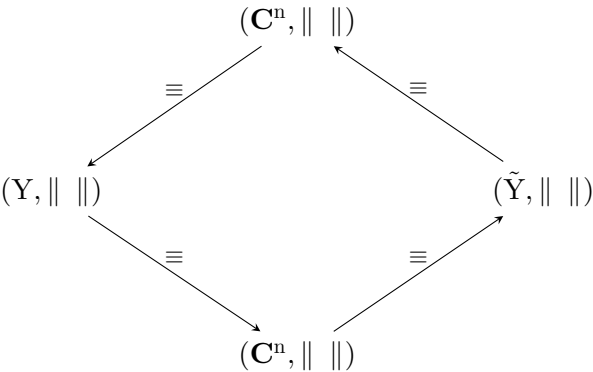
It is now clear that the *identity mapping*  $u \circ v$  is an homeomorphism of  $Y$  onto  $Y$ , which implies that  $\tau = \tau_{\|\cdot\|_2}$ . In other words, there is only one topology  $\tau$  for  $Y$ , as a topological vector space. This topology is normable, since  $\tau = \tau_{\|\cdot\|_2}$ . Let  $\|\cdot\|_Y$  stand for any norm of  $Y$ . The special case  $Y = \mathbf{C}^n, F_n = E_n, u = i$  is of considerable interest. TOTO. Now take  $\tilde{Y}$  of dimension  $n$  then similarly define (obvious notations)  $\tilde{u}, \tilde{v}$  and  $\tilde{\tau}$ .

$$\begin{array}{ccc}
 & (\mathbf{C}^n, \tau_{\mathbf{C}^n}) & \\
 u \swarrow & & \nwarrow \tilde{v} \\
 (Y, \tau) & & (\tilde{Y}, \tilde{\tau}) \\
 v \searrow & & \nearrow \tilde{u} \\
 & (\mathbf{C}^n, \tau_{\mathbf{C}^n}) &
 \end{array}$$

(2)

The homeomorphism between  $Y$  and  $\tilde{Y}$  leads to *the equivalence of norms at fixed dimension*  $n$ , as follows  $A\|y\|_Y \leq \|\tilde{u} \circ v(y)\|_{\tilde{Y}} \leq B\|y\|_Y$  ( $y \in Y$ ) for some positive

A,B. Equip  $Y$  and  $\tilde{Y}$  with the inherited norm  $\| \cdot \|$ .  $Y$  and  $\tilde{Y}$  are homeomorphically isomorphic ( $\equiv$ ) to  $\mathbf{C}^n, \| \cdot \|$ .



(3)

From now the default norm will be  $\| \cdot \|_\infty$ .





# Chapter 1

## Topological Vector Spaces

### 1.1 Exercise 1. Basic results

Suppose  $X$  is a vector space. All sets mentioned below are understood to be subsets of  $X$ . Prove the following statements from the axioms as given as in section 1.4.

- (a) If  $x, y \in X$  there is a unique  $z \in X$  such that  $x + z = y$ .
- (b)  $0 \cdot x = 0 = \alpha \cdot 0$  ( $\alpha \in \mathbf{C}, x \in X$ ).
- (c)  $2A \subset A + A$ .
- (d)  $A$  is convex if and only if  $(s + t)A = sA + tA$  for all positive scalars  $s$  and  $t$ .
- (e) Every union (and intersection) of balanced sets is balanced.
- (f) Every intersection of convex sets is convex.
- (g) If  $\Gamma$  is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of  $\Gamma$  is convex.
- (h) If  $A$  and  $B$  are convex, so is  $A + B$ .
- (i) If  $A$  and  $B$  are balanced, so is  $A + B$ .
- (j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

*Proof.* 1. Such property only depends on the group structure of  $X$ : Each  $x$  in  $X$  has an opposite  $-x$ . Let  $x'$  be any opposite of  $x$ , so that  $x - x = 0 = x + x'$ . Thus,  $-x + x - x = -x + x + x'$ , which is equivalent to  $-x = x'$ . So is established the uniqueness of  $-x$ . It is now clear that  $x + z = y$  **iff**  $z = -x + y$ , which asserts both the existence and the uniqueness of  $z$ .

2. Remark that

$$(1.1) \quad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$

$$(1.2) \quad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that  $0 \cdot x = 0$ . So,

$$(1.3) \quad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x.$$

Finally,

$$(1.4) \quad \alpha \cdot 0 \stackrel{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

3. Remark that

$$(1.5) \quad 2x = (1 + 1)x = x + x$$

for every  $x$  in  $X$ , and so conclude that

$$(1.6) \quad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subset \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets  $A$  of  $X$ ; which proves (c).

4. If  $A$  is convex, then

$$(1.7) \quad A \subset \frac{s}{s+t}A + \frac{t}{s+t}A \subset A;$$

which is

$$(1.8) \quad sA + tA = (s+t)A.$$

Conversely, the special case  $s + t = 1$  is

$$(1.9) \quad sA + (1-s)A = A.$$

The latter extends to  $s = 0$ , since

$$(1.10) \quad 0A + A \stackrel{(b)}{=} \{0\} + A = A.$$

The extension to  $s = 1$  is analogously established (or simply use the fact that  $+$  is commutative!). So ends the proof.

5. Let  $A$  range over  $B$  a collection of balanced subsets, so that

$$(1.11) \quad \alpha \bigcap B \subset \alpha A \subset A \subset \bigcup B$$

for all scalars  $\alpha$  of magnitude  $\leq 1$ . The inclusion  $\alpha \bigcap B \subset A$  establishes the first part. Now remark that

$$(1.12) \quad \alpha A \subset \bigcup B$$

implies

$$(1.13) \quad \alpha \bigcup B \subset \bigcup B;$$

which achieves the proof.

6. Let  $A$  range over  $C$  a collection of convex subsets, so that

$$(1.14) \quad (s+t) \bigcap C \subset s \bigcap C + t \bigcap C \subset sA + tA \stackrel{(d)}{=} (s+t)A$$

for all positive scalars  $s, t$ . Thus,

$$(1.15) \quad (s+t) \bigcap C \subset s \bigcap C + t \bigcap C \subset (s+t) \bigcap C.$$

We now conclude from (d) that the intersection of  $C$  is convex. So ends the proof.

7. Pick  $x_1, x_2$  in  $\bigcup \Gamma$ , so that each  $x_i$  ( $i = 1, 2$ ) lies in some  $C_i \in \Gamma$ . Since  $\Gamma$  is totally ordered by set inclusion, we henceforth assume without loss of generality that  $C_1$  is a subset of  $C_2$ . So,  $x_1, x_2$  are now elements of the convex set  $C_2$ . Every convex combination of our  $x_1$ 's is then in  $C_2 \subset \bigcup \Gamma$ , hence (g).

8. Simply remark that

$$(1.16) \quad s(A + B) + t(A + B) = sA + tA + sB + tB = (s + t)(A + B)$$

for all positive scalars  $s$  and  $t$ , then conclude from (d) that  $A + B$  is convex.

9. Given any  $\alpha$  from the closed unit disc,

$$(1.17) \quad \alpha(A + B) = \alpha A + \alpha B \subset A + B.$$

There is no more to prove.

10. Our proof will be based on the following lemma,

*If  $S \subset X$ , then any assertion*

- (i)  $S$  is a vector subspace of  $X$ ;*
  - (ii)  $S$  is convex balanced such that  $S + S = S$ ;*
  - (iii)  $S$  is convex balanced such that  $\lambda S = S \quad (\lambda > 0)$*
- implies the other ones.*

To prove the lemma, let  $S$  range over the subsets of  $X$ . First, assume that (i) holds: Clearly, every  $S$  is convex balanced. Moreover,  $S + S \subset S$ . Conversely,  $S = S + \{0\} \subset S + S$ ; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \quad nS = (n - 1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). The special case  $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$  ( $\lambda > 0$ ) yields

$$(1.19) \quad nS \stackrel{(1.18)}{\subset} S \subset n\lambda S \subset n^2 S,$$

since  $S$  is balanced and that  $1 < n\lambda < n^2$ . Dividing the latter inclusions by  $n$  shows that

$$(1.20) \quad S \subset \lambda S \subset nS \stackrel{(1.18)}{\subset} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$(1.21) \quad \alpha S + \beta S = |\alpha|S + |\beta|S \stackrel{(d)}{=} (|\alpha| + |\beta|)S \stackrel{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as  $S$  is balanced. Moreover (under the sole assumption that  $S$  is balanced), this extends to  $|\alpha| + |\beta| = 0$ , as follows,

$$(1.22) \quad \alpha S + \beta S = 0S + 0S \stackrel{(b)}{=} \{0\} \stackrel{(b)}{=} 0S \subset S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let  $V$  be a collection of vector spaces of  $X$ , of intersection  $I$  and union  $U$ . First, remark that every member of  $V$  is convex balanced: So is  $I$  (combine (e) with (f)). Next, let  $Y$  range over  $V$ , so that

$$(1.23) \quad I + I \subset Y + Y \subset Y;$$

which yields

$$(1.24) \quad I + I \subset I.$$

Conversely,

$$(1.25) \quad I = I + \{0\} \subset I + I.$$

It now follows from the lemma's (ii)  $\Rightarrow$  (i) that  $I$  is a vector subspace of  $X$ . Now temporarily assume that  $S$  is totally ordered by set inclusion: Combining (e) with (g) establishes that  $U$  is convex balanced. To show that  $U$  is more specifically a vector subspace, we first remark that such total order implies that either  $Z \subset Y$  or  $Y \subset Z$ , as  $Z$  ranges over  $V$ . A straightforward consequence is that

$$(1.26) \quad Y \subset Y + Z \subset Y \cup Z.$$

Another one is that  $Y \cup Z$  ranges over  $V$  as well. Combined with the latter inclusions, this leads to

$$(1.27) \quad U \subset U + U \subset U.$$

It then follows from the lemma's (ii)  $\Rightarrow$  (i) that  $U$  is a vector subspace of  $X$ . Finally, let  $A, B$  run through the vector subspaces of  $X$ : Combining (h) with (i) proves that  $A + B$  is convex balanced as well. Furthermore,

$$(1.28) \quad A + B \stackrel{(i) \Rightarrow (ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as  $X$  is an abelian group. We now conclude from (ii) that any  $A + B$  is a vector subspace of  $X$ . So ends the proof.  $\square$

## 1.2 Exercise 2. Convex hull

The convex hull of a set  $A$  in a vector space  $X$  is the set of all convex combinations of members of  $A$ , that is the set of all sums  $t_1x_1 + \cdots + t_nx_n$  in which  $x_i \in A$ ,  $t_i \geq 0$ ,  $\sum t_i = 1$ ;  $n$  is arbitrary. Prove that the convex hull of a set  $A$  is convex and that is the intersection of all convex sets that contain  $A$ .

*Proof.* The convex hull of a set  $S$  will be denoted by  $\text{co}(S)$ . Remark that  $S \subset \text{co}(S)$  (to see that, take  $t_1 = 1$  for each  $x_1$  in  $S$ ) and that  $\text{co}(A) \subset \text{co}(B)$  where  $A \subset B$  (obvious). Our proof will directly derive from the following lemma,

*Let  $S$  be a subset of a vector space  $X$ : Its convex hull  $\text{co}(S)$  is convex and the following statements*

- (i)  $S$  is convex;
  - (ii)  $s_1S + \cdots + s_nS = (s_1 + \cdots + s_n)S$  for all positive scalar variables  $s_1, \dots, s_n$ ;
  - (iii)  $t_1S + \cdots + t_nS = S$  for all positive scalar variables  $s_1, \dots, s_n$  such that  $s_1 + \cdots + s_n = 1$ ;
  - (iv)  $\text{co}(S) = S$
- are equivalent.*

More specifically, our proof of the second part will only depend on (i)  $\Rightarrow$  (iv).

From now on, we skip the trivial case  $S = \emptyset$  then only consider nonempty sets. To prove the first part, let  $a, b$  run through the convex combination(s) of  $S$ , so that  $a = t_1x_1 + \cdots + t_nx_n$  and  $b = t_{n+1}x_{n+1} + \cdots + t_{n+p}x_{n+p}$  for some  $(t_i, x_i)$ . Every sum  $sa + (1-s)b$  ( $0 \leq s \leq 1$ ) is then a convex combination of  $x_1, \dots, x_{n+p}$ , since

$$(1.29) \quad sa + (1-s)b = \sum_{i=1}^n st_ix_i + \sum_{i=n+1}^{n+p} (1-s)t_ix_i$$

and

$$(1.30) \quad \sum_{i=1}^n st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s \sum_{i=1}^n t_i + (1-s) \sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets  $S$ , this reads

$$(1.31) \quad s\text{co}(S) + (1-s)\text{co}(S) \subset \text{co}(S);$$

which was our first goal. We now aim at the equivalence (i)  $\Rightarrow \cdots \Rightarrow$  (iv)  $\Rightarrow$  (i): An easy proof by induction makes the implication (i)  $\Rightarrow$  (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii)  $\Rightarrow$  (iv) derives from the definition of the convex hull. We now close the chain with (iv)  $\Rightarrow$  (i), by remarking that  $S$  is convex whether  $S = \text{co}(S)$ . The lemma being proved, let us establish the second part. To do so, start from  $F \ni \text{co}(A)$  then possibly enrich  $F$  the following way:

$$(1.32) \quad B \in F \Rightarrow B \text{ is convex and contains } A.$$

Note that our definition of  $F$  is weaker than the primary assumption “[ $F$  only encompasses] all convex sets that contain  $A$ ”, which is the special case

$$(1.33) \quad B \in F \Leftrightarrow B \text{ is convex and contains } A.$$

In any case, the key ingredient is that  $\text{co}(A) \in \mathcal{F}$  implies

$$(1.34) \quad \text{co}(A) \supset \bigcap_{B \in \mathcal{F}} B.$$

Conversely, the next formula

$$(1.35) \quad \text{co}(A) \subset \text{co}(B) \stackrel{(i) \Rightarrow (iv)}{=} B \quad (B \in \mathcal{F})$$

is valid and implies

$$(1.36) \quad \text{co}(A) \subset \bigcap_{B \in \mathcal{F}} B.$$

So ends the proof

□

### **1.3 Exercise 3. Other basic results**

**1.4 Exercise 4. A nonempty set whose interior is not**



## 1.5 Exercise 5. A first restatement of boundedness

**1.6 Exercise 6. A second restatement of boundedness**

## 1.7 Exercise 7. Metrizability & number theory

Let be  $X$  the vector space of all complex functions on the unit interval  $[0, 1]$ , topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence  $\{f_n\}$  in  $X$  such that (a)  $\{f_n\}$  converges to 0 as  $n \rightarrow \infty$ , but (b) if  $\{\gamma_n\}$  is any sequence of scalars such that  $\gamma_n \rightarrow \infty$  then  $\{\gamma_n f_n\}$  does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as  $[0, 1]$ .) This shows that metrizability cannot be omitted in (b) of Theorem 1.28.

*Proof.* The family of the seminorms  $p_x$  is separating: By 1.37 of [3], the collection  $\mathcal{B}$  of all finite intersections of the sets

$$(1.37) \quad V(x, k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k = 1, 2, 3, \dots)$$

is therefore a local base for a topology  $\tau$  on  $X$ . So,

$$(1.38) \quad \sum_{n=1}^{\infty} [f_n \notin \cap_{i=1}^m U_i] \leq \sum_{n=1}^{\infty} \sum_{i=1}^m [f_n \notin U_i] = \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n \notin U_i] \quad (f_n \in X, U_i \in \tau).$$

Now assume that  $\{f_n\}$   $\tau$ -converges to some  $f$ , *i.e.*

$$(1.39) \quad \sum_{n=1}^{\infty} [f_n \notin f + W] < \infty \quad (W \in \mathcal{B}).$$

The special case  $W = V(x, k)$  means that, given  $k$ ,  $|f_n(x) - f(x)| < 2^{-k}$  for almost all  $n$ , *i.e.*  $\{f_n(x)\}$  converges to  $f(x)$ . Conversely, assume that  $\{f_n\}$  does not  $\tau$ -converges in  $X$ , *i.e.*

$$(1.40) \quad \forall f \in X, \exists W \in \mathcal{B} : \sum_{n=1}^{\infty} [f_n \notin f + W] = \infty.$$

$W$  is now the (nonempty) intersection of finitely many  $V(x, k)$ , say  $V(x_1, k_1), \dots, V(x_m, k_m)$ . Thus,

$$(1.41) \quad \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)] \stackrel{(1.38)}{\geq} \sum_{n=1}^{\infty} [f_n \notin f + W] \stackrel{(1.40)}{=} \infty.$$

We can now conclude that, for some index  $i$ ,

$$(1.42) \quad \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)] = \infty.$$

In other word,  $\{f_n(x_i)\}$  fails to converge to  $f(x_i)$ . We have so proved that  $\tau$ -convergence is a rewording of pointwise convergence. We now establish the second part.

To do so, we split  $x$  into two variables:  $r$  if  $x$  is rational,  $\alpha$  otherwise. The proof is based on the following well-known result: Each  $\alpha$  has a *unique* binary expansion. More precisely,

there exists a bijection  $b : [0, 1] \setminus \mathbf{Q} \rightarrow \{\beta \in \{0, 1\}^{\mathbf{N}^+} : \beta \text{ is not eventually periodic}\}$  where  $b(\alpha) = (\beta_1, \beta_2, \dots)$  is the only bit stream such that

$$(1.43) \quad \alpha = \sum_{k=1}^{\infty} \beta_k \cdot 2^{-k}.$$

Remark that  $b(\alpha)_1 + \dots + b(\alpha)_n \xrightarrow{n \rightarrow \infty} \infty$ , since  $b(\alpha)$  has infinite support, then fix

$$(1.44) \quad f_n(\alpha) \triangleq \frac{1}{b(\alpha)_1 + \dots + b(\alpha)_n} \xrightarrow{n \rightarrow \infty} 0.$$

The actual values  $f_n(r)$  are of no interest, as long as every sequence  $\{f_n(r) : n = 1, 2, 3, \dots\}$  converges to 0. For example, put  $f_n(r) = r/n$ , or just  $f_n(r) = 0$ . We also take  $\gamma_n \rightarrow \infty$ , *i.e.* given any counting number  $p$ ,  $\gamma_n$  is greater than  $p$  for almost all  $n$ . Next, we choose  $n_p$  among those *almost all*  $n$  that are large enough to satisfy

$$(1.45) \quad n_p - n_{p-1} > p$$

(start with  $n_0 = 0$ ). So, every list  $n_p, n_{p'}, n_{p''}, \dots$  that satisfies  $n_{p'} - n_p = n_{p''} - n_{p'} = \dots$  is finite (otherwise,  $n_{p'} - n_p \geq n_{p+1} - n_p > p \rightarrow \infty$  would hold; see (1.45)). In other words, *the distribution of  $n_1, n_2, \dots$  displays no periodic pattern*. As a consequence, the *characteristic function*  $\chi : k \mapsto [k \in \{n_1, n_2, \dots\}]$  is not eventually periodic. Combined with (1.43), this establishes that

$$(1.46) \quad \alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k}$$

is irrational. Conversely, still with (1.43),

$$(1.47) \quad b(\alpha_\gamma)_k = \chi_k.$$

Now remark that

$$(1.48) \quad \chi_1 + \dots + \chi_{n_1} = 1$$

$$(1.49) \quad \chi_1 + \dots + \chi_{n_1} + \dots + \chi_{n_2} = 2$$

$$\vdots$$

$$(1.50) \quad \chi_1 + \dots + \chi_{n_1} + \dots + \chi_{n_2} + \dots + \chi_{n_p} = p.$$

Combined with (1.44), this yields

$$(1.51) \quad \gamma_{n_p} f_{n_p}(\alpha_\gamma) = \frac{\gamma_{n_p}}{p} > 1.$$

There so exists a subsequence  $\{\gamma_{n_p}\}$  such that  $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$  fails to converge pointwise to 0. Since  $\{\gamma_n\}$  was arbitrary, this proves (b).  $\square$

## 1.9 Exercise 9. Quotient map

Suppose

1.  $X$  and  $Y$  are topological vector spaces,
2.  $\Lambda : X \rightarrow Y$  is linear.
3.  $N$  is a closed subspace of  $X$ ,
4.  $\pi : X \rightarrow X/N$  is the quotient map, and
5.  $\Lambda x = 0$  for every  $x \in N$ .

Prove that there is a unique  $f : X/N \rightarrow Y$  which satisfies  $\Lambda = f \circ \pi$ , that is,  $\Lambda x = f(\pi(x))$  for all  $x \in X$ . Prove that  $f$  is linear and that  $\Lambda$  is continuous if and only if  $f$  is continuous. Also,  $\Lambda$  is open if and only if  $f$  is open.

*Proof.* Bear in mind that  $\pi$  continuously maps  $X$  onto the topological (Hausdorff) space  $X/N$ , since  $N$  is closed (see 1.41 of [3]). Moreover, the equation  $\Lambda = f \circ \pi$  has necessarily a unique solution, which is the binary relation

$$(1.52) \quad f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y.$$

To ensure that  $f$  is actually a mapping, simply remark that the linearity of  $\Lambda$  implies

$$(1.53) \quad \Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x.$$

It straightforwardly derives from (1.52) that  $f$  inherits linearity from  $\pi$  and  $\Lambda$ .

**Remark.** The special case  $N = \{\Lambda = 0\}$ , i.e.  $\Lambda x = 0$  iff  $x \in N$  (cf. (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$(1.54) \quad f(\pi x) = 0 \stackrel{(1.52)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that  $f$  is also one-to-one.

Now assume  $f$  to be continuous. Then so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [3]. Conversely, if  $\Lambda$  is continuous, then for each neighborhood  $V$  of  $0_Y$  there exists a neighborhood  $U$  of  $0_X$  such that

$$(1.55) \quad \Lambda(U) = f(\pi(U)) \subset V.$$

Since  $\pi$  is open (1.41 (a) of [3]),  $\pi(U)$  is a neighborhood of  $N = 0_{X/N}$ : This is sufficient to establish that the linear mapping  $f$  is continuous. If  $f$  is open, so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood  $W$  of  $0_{X/N}$  satisfies

$$(1.56) \quad W = \pi(V)$$

for some neighborhood  $V$  of  $0_X$ . So,

$$(1.57) \quad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if  $\Lambda$  is open, then  $f(W)$  is a neighborhood of  $0_Y$ . So ends the proof.  $\square$

### 1.10 Exercise 10. An open mapping theorem

Suppose that  $X$  and  $Y$  are topological vector spaces,  $\dim Y < \infty$ ,  $\Lambda : X \rightarrow Y$  is linear, and  $\Lambda(X) = Y$ .

1. Prove that  $\Lambda$  is an open mapping.

2. Assume, in addition, that the null space of  $\Lambda$  is closed, and prove that  $\Lambda$  is continuous.

*Proof.* Discard the trivial case  $\Lambda = 0$  and so assume that  $\dim Y = n$  for some positive  $n$ . Let  $e$  range over a base of  $B$  of  $Y$  then pick  $W$  an arbitrary neighborhood of the origin: There so exists  $V$  a balanced neighborhood of the origin such that

$$(1.58) \quad \sum_e V \subset W,$$

since addition is continuous. Moreover, for each  $e$ , there exists  $x_e$  in  $X$  such that  $\Lambda(x_e) = e$ , simply because  $\Lambda$  is onto. So,

$$(1.59) \quad y = \sum_e y_e \cdot \Lambda x_e \quad (y \in Y).$$

As a finite set,  $\{x_e : e \in B\}$  is bounded: There so exists a positive scalar  $s$  such that

$$(1.60) \quad \forall e \in B, x_e \in s \cdot V.$$

Combining this with (1.59) shows that

$$(1.61) \quad y \in \sum_e y_e \cdot s \cdot \Lambda(V).$$

We now come back to (1.58) and so conclude that

$$(1.62) \quad y \in \sum_e \Lambda(V) \subset \Lambda(W)$$

whether  $|y_e| < 1/s$ ; which proves (a).

To prove (b), assume that the null space  $\{\Lambda = 0\}$  is closed and let  $f, \pi$  be as in Exercise 1.9,  $\{\Lambda = 0\}$  playing the role of  $N$ . Since  $\Lambda$  is onto, the first isomorphism theorem (see Exercise 1.9) asserts that  $f$  is an isomorphism of  $X/N$  onto  $Y$ . Consequently,

$$(1.63) \quad \dim X/N = n.$$

$f$  is then an homeomorphism of  $X/N \equiv \mathbf{C}^n$  onto  $Y$ ; see 1.21 of [3]. We have thus established that  $f$  is continuous: So is  $\Lambda = f \circ \pi$ .  $\square$

**1.12 Exercise 12. Topology stays, completeness leaves**

### 1.14 Exercise 14. $\mathcal{D}_K$ equipped with other seminorms

Put  $K = [0, 1]$  and define  $\mathcal{D}_K$  as in Section 1.46. Show that the following three families of seminorms (where  $n = 0, 1, 2, \dots$ ) define the same topology on  $\mathcal{D}_K$ . If  $D = d/dx$ :

$$1. \|D^n f\|_\infty = \sup\{|D^n f(x)| : 0 < x < 1\}$$

$$2. \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$3. \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

*Proof.* First, remark that

$$(1.64) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

holds, since  $K$  has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of  $D^n f$  lies in  $K$ ; which yields

$$(1.65) \quad |D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1.$$

So,

$$(1.66) \quad \|D^n f\|_\infty \leq \|D^{n+1} f\|_1.$$

We now combine (1.64) with (1.66) and so obtain

$$(1.67) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots \quad (n = 0, 1, 2, \dots).$$

Put

$$(1.68) \quad V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\} \quad (i = 1, 2, \infty)$$

$$(1.69) \quad \mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\},$$

so that (1.67) is mirrored in terms of neighborhood inclusions, as follows,

$$(1.70) \quad V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \dots.$$

Since  $V_n^{(i)} \supset V_{n+1}^{(i)}$ ,  $\mathcal{B}^{(i)}$  is a local base of a topology  $\tau_i$ . But the chain (1.70) forces

$$(1.71) \quad \tau_1 = \tau_2 = \tau_\infty.$$

To see that, choose a set  $S$  that is  $\tau_1$ -open at  $f$ , i.e.  $V_n^{(1)} \subset S - f$  for some  $n$ . Next, concatenate this with  $V_n^{(2)} \subset V_n^{(1)}$  (see (1.70)) and so obtain  $V_n^{(2)} \subset S - f$ ; which implies that  $S$  is  $\tau_2$ -open at  $f$ . Similarly, we deduce, still from (1.70), that

$$(1.72) \quad \tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}.$$

So ends the proof. □



## 1.16 Exercise 16. Uniqueness of topology for test functions

*Prove that the topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Do the same for  $C^\infty(\Omega)$  (Section 1.46).*

**Comment** This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms  $p_n$ , then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (e.g. [2.6] in [2]) about intersection of nonempty compact sets.

**Lemma 1** *Let  $X$  be a topological space with a countable local base  $\{V_n : n = 1, 2, 3, \dots\}$ . If  $\tilde{V}_n = V_1 \cap \dots \cap V_n$ , then every subsequence  $\{\tilde{V}_{\varrho(n)}\}$  is a decreasing (i.e.  $\tilde{V}_{\varrho(n)} \supset \tilde{V}_{\varrho(n+1)}$ ) local base of  $X$ .*

*Proof.* The decreasing property is trivial. Now remark that  $V_n \supset \tilde{V}_n$ : This shows that  $\{\tilde{V}_n\}$  is a local base of  $X$ . Then so is  $\{\tilde{V}_{\varrho(n)}\}$ , since  $\tilde{V}_n \supset \tilde{V}_{\varrho(n)}$ .  $\square$

The following special case  $V_n = \tilde{V}_n$  is one of the key ingredients:

**Corollary 1 (special case  $V_n = \tilde{V}_n$ )** *Under the same notations of Lemma 1, if  $\{V_n\}$  is a decreasing local base, then so is  $\{V_{\varrho(n)}\}$ .*

**Corollary 2** *If  $\{Q_n\}$  is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence  $\{Q_{\varrho(n)}\}$  also satisfies these conditions. Furthermore, if  $\tau_Q$  is the  $C(\Omega)$ 's (respectively  $C^\infty(\Omega)$ 's) topology of the seminorms  $p_n$ , as defined in section 1.44 (respectively 1.46), then the seminorms  $p_{\varrho(n)}$  define the same topology  $\tau_Q$ .*

*Proof.* Let  $X$  be  $C(\Omega)$  topologized by the seminorms  $p_n$  (the case  $X = C^\infty(\Omega)$  is proved the same way). If  $V_n = \{p_n < 1/n\}$ , then  $\{V_n\}$  is a decreasing local base of  $X$ . Moreover,

$$(1.73) \quad Q_{\varrho(n)} \subset \overset{\circ}{Q}_{\varrho(n)+1} \subset Q_{\varrho(n)+1} \subset Q_{\varrho(n+1)}.$$

Thus,

$$(1.74) \quad Q_{\varrho(n)} \subset \overset{\circ}{Q}_{\varrho(n+1)}.$$

In other words,  $Q_{\varrho(n)}$  satisfies the conditions specified in section 1.44.  $\{p_{\varrho(n)}\}$  then defines a topology  $\tau_{Q_\varrho}$  for which  $\{V_{\varrho(n)}\}$  is a local base. So,  $\tau_{Q_\varrho} \subset \tau_Q$ . Conversely, the above corollary asserts that  $\{V_{\varrho(n)}\}$  is a local base of  $\tau_Q$ , which yields  $\tau_Q \subset \tau_{Q_\varrho}$ .  $\square$

**Lemma 2** *If a sequence of compact sets  $\{Q_n\}$  satisfies the conditions specified in section 1.44, then every compact set  $K$  lies in almost all  $Q_n^\circ$ , i.e. there exists  $m$  such that*

$$(1.75) \quad K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \dots$$

*Proof.* The following definition

$$(1.76) \quad C_n \triangleq K \setminus \overset{\circ}{Q}_n$$

shapes  $\{C_n\}$  as a decreasing sequence of compact<sup>1</sup> sets. We now suppose (to reach a contradiction) that no  $C_n$  is empty and so conclude<sup>2</sup> that the  $C_n$ 's intersection contains a point that is not in any  $\overset{\circ}{Q}_n$ . On the other hand, the conditions specified in [1.44] force the  $\overset{\circ}{Q}_n$ 's collection to be an open cover. This contradiction reveals that  $C_m = \emptyset$ , *i.e.*  $K \subset \overset{\circ}{Q}_m$ , for some  $m$ . Finally,

$$(1.77) \quad K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots .$$

□

We are now in a fair position to establish the following:

**Theorem** *The topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of  $C^\infty(\Omega)$ , as long as this sequence satisfies the conditions specified in section 1.44.*

*Proof.* With the second corollary's notations,  $\tau_K = \tau_{K_\lambda}$ , for every subsequence  $\{K_{\lambda(n)}\}$ . Similarly, let  $\{L_n\}$  be another sequence of compact subsets of  $\Omega$  that satisfies the condition specified in [1.44], so that  $\tau_L = \tau_{L_\kappa}$  for every subsequence  $\{L_{\kappa(n)}\}$ . Now apply the above Lemma 2 with  $K_i$  ( $i = 1, 2, 3, \dots$ ) and so conclude that  $K_i \subset \overset{\circ}{L}_{m_i} \subset \overset{\circ}{L}_{m_i+1} \subset \cdots$  for some  $m_i$ . In particular, the special case  $\kappa_i = m_i + i$  is

$$(1.78) \quad K_i \subset \overset{\circ}{L}_{\kappa_i}.$$

Let us reiterate the above proof with  $K_n$  and  $L_n$  in exchanged roles then similarly find a subsequence  $\{\lambda_j : j = 1, 2, 3, \dots\}$  such that

$$(1.79) \quad L_j \subset \overset{\circ}{K}_{\lambda_j}$$

Combine (1.78) with (1.79) and so obtain

$$(1.80) \quad K_1 \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{L}_{\kappa_{\lambda_{\kappa_1}}} \subset \cdots ,$$

which means that the sequence  $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$  satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$(1.81) \quad \tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L.$$

So ends the proof

□

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<sup>1</sup> See (b) of 2.5 of [2].

<sup>2</sup> In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of 2.6 of [2].

### 1.17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that  $f \mapsto D^\alpha f$  is a continuous mapping of  $C^\infty(\Omega)$  into  $C^\infty(\Omega)$  and also of  $\mathcal{D}_K$  into  $\mathcal{D}_K$ , for every multi-index  $\alpha$ .

*Proof.* In both cases,  $D^\alpha$  is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the  $C^\infty(\Omega)$  case.

Let  $U$  be an arbitrary neighborhood of the origin. There so exists  $N$  such that  $U$  contains

$$(1.82) \quad V_N = \left\{ \varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}.$$

Now pick  $g$  in  $V_{N+|\alpha|}$ , so that

$$(1.83) \quad \max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}.$$

(the fact that  $K_N \subset K_{N+|\alpha|}$  was tacitly used). The special case  $\gamma = \beta + \alpha$  yields

$$(1.84) \quad \max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

$$(1.85) \quad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad i.e. \quad D^\alpha(V_{N+|\alpha|}) \subset V_N,$$

which establishes the continuity of  $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$ .

To prove the continuousness of the restriction  $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$ , we first remark that the collection of the  $V_N \cap \mathcal{D}_K$  is a local base of the subspace topology of  $\mathcal{D}_K$ .  $V_{N+|\alpha|} \cap \mathcal{D}_K$  is then a neighborhood of 0 in this topology. Furthermore,

$$(1.86) \quad D^\alpha|_{\mathcal{D}_K}(V_{N+|\alpha|} \cap \mathcal{D}_K) = D^\alpha(V_{N+|\alpha|} \cap \mathcal{D}_K)$$

$$(1.87) \quad \subset D^\alpha(V_{N+|\alpha|}) \cap D^\alpha(\mathcal{D}_K)$$

$$(1.88) \quad \subset V_N \cap \mathcal{D}_K \quad (\text{see (1.85)})$$

So ends the proof. □

## Chapter 2

# Completeness

### 2.3 Exercise 3. An equicontinuous sequence of measures

Put  $K = [-1, 1]$ ; define  $\mathcal{D}_K$  as in section 1.46 (with  $\mathbf{R}$  in place of  $\mathbf{R}^n$ ). Suppose  $\{f_n\}$  is a sequence of Lebesgue integrable functions such that  $\Lambda\varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\varphi(t)dt$  exists for every  $\varphi \in \mathcal{D}_K$ . Show that  $\Lambda$  is a continuous linear functional on  $\mathcal{D}_K$ . Show that there is a positive integer  $p$  and a number  $M < \infty$  such that

$$\left| \int_{-1}^1 f_n(t)\varphi(t)dt \right| \leq M \|D^p \varphi\|_{\infty}$$

for all  $n$ . For example, if  $f_n(t) = n^3 t$  on  $[-1/n, 1/n]$  and 0 elsewhere, show that this can be done with  $p = 1$ . Construct an example where it can be done with  $p = 2$  but not with  $p = 1$ .

We will also consider the case  $p = 0$ . Since all supports of  $\varphi, \varphi', \varphi'', \dots$ , are in  $K$ , we make a specialization of the mean value theorem:

**Lemma** If  $\varphi \in \mathcal{D}_{[a,b]}$ , then

$$(2.1) \quad \|D^{\alpha}\varphi\|_{\infty} \leq \|D^p\varphi\|_{\infty} \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (\alpha = 0, 1, \dots, p)$$

at every order  $p = 0, 1, 2, \dots$ ; where  $\lambda$  is the length  $|b - a|$ .

*Proof.* Let  $x_0$  be in  $(a, b)$ . We first consider the case  $x_0 \leq c = (a + b)/2$ : The mean value theorem asserts that there exists  $x_1$  ( $a < x_1 < x_0$ ), such that

$$(2.2) \quad \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a).$$

Since every  $D^p\varphi$  lies in  $\mathcal{D}_{[a,b]}$ , a straightforward proof by induction shows that there exists a partition  $a < \dots < x_p < \dots < x_0$  such that

$$(2.3) \quad \varphi(x_0) = D^0\varphi(x_0)$$

$$(2.4) \quad = D^1\varphi(x_1)(x_0 - a)$$

$$= \dots$$

$$(2.5) \quad = D^p\varphi(x_p)(x_0 - a) \cdots (x_{p-1} - a),$$

for all  $p$ . More compactly,

$$(2.6) \quad D^\alpha \varphi(x_0) = D^p \varphi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a);$$

which yields,

$$(2.7) \quad |D^\alpha \varphi(x)| \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c])$$

The case  $x_0 \geq c$  outputs a “reversed” result, with  $b > \dots > x_p > \dots > x_0$  and  $x_k - b$  playing the role of  $x_k - a$ : So,

$$(2.8) \quad |D^\alpha \varphi(x)| \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$(2.9) \quad \|D^\alpha \varphi\|_\infty \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}.$$

□

*Proof.* We first consider  $C_0(\mathbf{R})$  topologized by the supremum norm. Given a Lebesgue integrable function  $u$ , we put

$$(2.10) \quad \langle u | \varphi \rangle \triangleq \int_{\mathbf{R}} u \varphi \quad (\varphi \in C_0(\mathbf{R})).$$

The following inequalities

$$(2.11) \quad |\langle u | \varphi \rangle| \leq \int_{\mathbf{R}} |u \varphi| \leq \|u\|_{L^1} \quad (\|\varphi\|_\infty \leq 1)$$

imply that every linear functional

$$(2.12) \quad \begin{aligned} \langle u | : C_0(\mathbf{R}) &\rightarrow \mathbf{C} \\ \varphi &\mapsto \langle u | \varphi \rangle \end{aligned}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely,  $u$  can be identified with  $\langle u |$ , since  $u$  is determined (a.e) by the integrals  $\langle u | \varphi \rangle$ . In the Banach spaces terminology,  $u$  is then (identified with) a linear *bounded*<sup>1</sup> operator  $\langle u |$ , of norm

$$(2.13) \quad \sup\{|\langle u | \varphi \rangle| : \|\varphi\|_\infty = 1\} = \|u\|_{L^1}.$$

Note that, in the latter equality,  $\leq \|u\|_{L^1}$  comes from (2.11), as the converse comes from the Stone-Weierstrass theorem<sup>2</sup>. We now consider the special cases  $u = g_n$ , where  $g_n$  is

$$(2.14) \quad \begin{aligned} g_n : \mathbf{R} &\rightarrow \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) \end{cases} \end{aligned}$$

<sup>1</sup> see 1.32, 4.1 of [3]

<sup>2</sup> See 7.26 of [1].

First, remark that  $g_n(x) \rightarrow 0$ , as the sequence  $\{g_n\}$  fails to converge in  $C_0(\mathbf{R})$  (since  $g_n(1/n) = n^2 \geq 1$ ), and also in  $L^1$  (since  $\int_{\mathbf{R}} |g_n| = n^2 \rightarrow \infty$ ). Nevertheless, we will show that the  $\langle g_n |$  converge pointwise<sup>3</sup> on  $\mathcal{D}_K$  *i.e.* there exists a  $\tau_K$ -continuous linear form  $\Lambda$  such that

$$(2.15) \quad \langle g_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \varphi,$$

where  $\varphi$  ranges over  $\mathcal{D}_K$ . We now prove (2.13) in the special cases  $u = g_n$ . To do so, we fetch  $\varphi_1^+, \dots, \varphi_j^+, \dots$ , from  $C_K^\infty(\mathbf{R})$ . More specifically,

$$(i) \quad \varphi_j^+ = 1 \text{ on } [e^{-j}, 1 - e^{-j}];$$

$$(ii) \quad \varphi_j^+ = 0 \text{ on } \mathbf{R} \setminus [-1, 1];$$

$$(iii) \quad 0 \leq \varphi_j^+ \leq 1 \text{ on } \mathbf{R};$$

see [1.46] of [3] for a possible construction of those  $\varphi_j^+$ . Let  $\varphi_1^-, \dots, \varphi_j^-, \dots$ , mirror the  $\varphi_j^+$ , in the sense that  $\varphi_j^-(x) = \varphi_j^+(-x)$ , so that

$$(iv) \quad \varphi_j \triangleq \varphi_j^+ - \varphi_j^- \text{ is odd, as } g_n \text{ is};$$

$$(v) \quad \text{every } \varphi_j \text{ is in } C_K^\infty(\mathbf{R});$$

$$(vi) \quad \text{The sequence } \{\varphi_j\} \text{ converges (pointwise) to } 1_{[0,1]} - 1_{[-1,0]}, \text{ and } \|\varphi_j\|_\infty = 1.$$

Thus, with the help of the Lebesgue's convergence theorem,

$$(2.16) \quad \langle g_n | \varphi_j \rangle = 2 \int_0^1 g_n(t) \varphi_j^+(t) dt \xrightarrow{j \rightarrow \infty} 2 \int_0^1 g_n(t) dt = \|g_n\|_{L^1} = n.$$

Finally,

$$(2.17) \quad \|g_n\|_{L^1} \stackrel{(2.16)}{\leq} \sup\{|\langle g_n | \varphi \rangle| : \|\varphi\|_\infty = 1\} \stackrel{(2.13)}{\leq} \|g_n\|_{L^1};$$

which is the desired result. So, in terms of boundedness constants: Given  $n$ , there exists  $C_n < \infty$  such that

$$(2.18) \quad |\langle g_n | \varphi \rangle| \leq C_n \quad (\|\varphi\|_\infty = 1);$$

see (2.11). Furthermore,  $\|g_n\|_{L^1}$  is actually the best, *i.e.* lowest, possible  $C_n$ ; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence  $\{\langle g_n | \varphi_{\rho(n)} \rangle\}$  such that  $\langle g_n | \varphi_{\rho(n)} \rangle$  is greater than, say,  $n - 0.01$ , as  $\|\varphi_{\rho(n)}\|_\infty = 1$ . Consequently, there is no bound  $M$  such that

$$(2.19) \quad |\langle g_n | \varphi \rangle| \leq M \quad (\|\varphi\|_\infty = 1; n = 1, 2, 3, \dots).$$

In other words, the  $g_n$  have no *uniform bound* in  $L^1$ , *i.e.* the collection of all continuous linear mappings  $\langle g_n |$  is not equicontinuous (see discussion in 2.6 of [3]). As a consequence, the  $\langle g_n |$  do not converge pointwise (or “vaguely”, in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$(2.20) \quad \langle g_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \varphi \quad (\varphi \in C_0(\mathbf{R}))$$

<sup>3</sup> See 3.14 of [3] for a definition of the related topology.

for some  $\Lambda \in C_0(\mathbf{R})^*$ , which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the  $\langle g_n |$  do not converge pointwise, in a relevant space, to some  $\Lambda$  (see (2.15)).

From now on, unless the contrary is explicitly stated, we assume that  $\varphi$  only denotes an element of  $C_K^\infty(\mathbf{R})$ . Let  $f_n$  be a Lebesgue integrable function such that

$$(2.21) \quad \Lambda\varphi = \lim_{n \rightarrow \infty} \int_K f_n \varphi \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

for some linear form  $\Lambda$ . Since  $\varphi$  vanishes outside  $K$ , we can suppose without loss of generality that the support of  $f_n$  lies in  $K$ . So, (2.21) can be restated as follows,

$$(2.22) \quad \Lambda\varphi = \lim_{n \rightarrow \infty} \langle f_n | \varphi \rangle \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

Let  $K_1, K_2, \dots$ , be compact sets that satisfy the conditions specified in 1.44 of [3].  $\mathcal{D}_K$  is  $C_K^\infty(\mathbf{R})$  topologized by the related seminorms  $p_1, p_2, \dots$ ; see 1.46, 6.2 of [3] and Exercise 1.16. We know that  $K \subset K_m$  for some index  $m$  (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices  $N \geq m$ , so that

- (a)  $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha \varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$ , for  $\varphi \in \mathcal{D}_K$ ;
- (b) The collection of the sets  $V_N = \{\varphi \in \mathcal{D}_K : \|\varphi\|_N < 2^{-N}\}$  is a (decreasing) local base of  $\tau_K$ , the subspace topology of  $\mathcal{D}_K$ ; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with  $u = f_n$  and  $\varphi \in V_m$  then conclude that  $\langle f_n |$  is bounded by  $\|f_n\|_{L^1}$  on  $V_m$ : Every linear functional  $\langle f_n |$  is therefore  $\tau_K$ -continuous; see 1.18 of [3].

To sum it up:

- (i)  $\mathcal{D}_K$ , equipped the topology  $\tau_K$ , is a Fréchet space (see section 1.46 of [3]);
- (ii) Every linear functional  $\langle f_n |$  is continuous with respect to this topology;
- (iii)  $\langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda\varphi$  for all  $\varphi$ , i.e.  $\Lambda - \langle f_n | \xrightarrow{n \rightarrow \infty} 0$ .

With the help of [2.6] and [2.8] of [3], we conclude that  $\Lambda$  is continuous and that the sequence  $\{\langle f_n | \}$  is equicontinuous. So is the sequence  $\{\Lambda - \langle f_n | \}$ , since addition is continuous. There so exists  $i, j$  such that, for all  $n$ ,

$$(2.23) \quad |\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i,$$

$$(2.24) \quad |\Lambda\varphi - \langle f_n | \varphi \rangle| < 1/2 \quad \text{if } \varphi \in V_j.$$

Choose  $p = \max\{i, j\}$ , so that  $V_p = V_i \cap V_j$ : The latter inequalities imply that

$$(2.25) \quad |\langle f_n | \varphi \rangle| \leq |\Lambda\varphi - \langle f_n | \varphi \rangle| + |\Lambda\varphi| < 1 \quad \text{if } \varphi \in V_p.$$

Now remark that every  $\psi = \psi[\mu, \varphi]$ , where

$$(2.26) \quad \psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\varphi\|_p) \varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases}$$

keeps in  $V_p$ . Finally, it is clear that each below statement implies the following one.

$$(2.27) \quad |\langle f_n | \psi \rangle| < 1$$

$$(2.28) \quad |\langle f_n | \varphi \rangle| < 2^p \|\varphi\|_p \cdot \mu$$

$$(2.29) \quad |\langle f_n | \varphi \rangle| \leq 2^p \|\varphi\|_p$$

$$(2.30) \quad |\langle f_n | \varphi \rangle| \leq 2^p \{\|D^0 \varphi\|_\infty + \cdots + \|D^p \varphi\|_\infty\}.$$

Finally, with the help of (2.1),

$$(2.31) \quad |\langle f_n | \varphi \rangle| \leq 2^p(p+1)\|D^p \varphi\|_\infty.$$

The first part is so proved, with *some*  $p$  and  $M = 2^p(p+1)$ .

We now come back to the special case  $f_n = g_n$  (see the first part). From now on,  $f_n(x) = n^3 x$  on  $[-1/n, 1/n]$ , 0 elsewhere. Actually, we will prove that

$$(a) \quad \Lambda \varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t) \varphi(t) dt \text{ exists for every } \varphi \in \mathcal{D}_K;$$

$$(b) \quad \text{A uniform bound } |\langle f_n | \varphi \rangle| \leq M \|D^p \varphi\|_\infty \text{ (} n = 1, 2, 3, \dots \text{) exists for all those } f_n, \text{ with } p = 1 \text{ as the smallest possible } p.$$

Bear in mind that  $K \subset K_m$  and shift the  $K_N$ 's indices, so that  $K_{m+1}$  becomes  $K_1$ ,  $K_{m+2}$  becomes  $K_2$ , and so on. The resulting topology  $\tau_K$  remains unchanged (see Exercise 1.16). We let  $\varphi$  keep running on  $\mathcal{D}_K$  and so define

$$(2.32) \quad B_n(\varphi) \triangleq \max\{|\varphi(x)| : x \in [-1/n, 1/n]\},$$

$$(2.33) \quad \Delta_n(\varphi) \triangleq \max\{|\varphi(x) - \varphi(0)| : x \in [-1/n, 1/n]\}.$$

The mean value asserts that

$$(2.34) \quad |\varphi(1/n) - \varphi(-1/n)| \leq B_n(\varphi') |1/n - (-1/n)| = \frac{2}{n} B_n(\varphi').$$

Independently, an integration by parts shows that

$$(2.35) \quad \langle f_n | \varphi \rangle = \left[ \frac{n^3 t^2}{2} \varphi(t) \right]_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt$$

$$(2.36) \quad = \frac{n}{2} (\varphi(1/n) - \varphi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt.$$

Combine (2.34) with (2.36) and so obtain

$$(2.37) \quad |\langle f_n | \varphi \rangle| \leq \frac{n}{2} |\varphi(1/n) - \varphi(-1/n)| + \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\varphi'(t)| dt$$

$$(2.38) \quad \leq B_n(\varphi') + \frac{n^3}{2} B_n(\varphi') \int_{-1/n}^{1/n} t^2 dt$$

$$(2.39) \quad \leq \frac{4}{3} B_n(\varphi')$$

$$(2.40) \quad \leq \frac{4}{3} \|\varphi'\|_\infty.$$



Futhermore, (2.39) gives a hint about the convergence of  $f_n$ : Since  $B_n(\varphi')$  tends to  $|\varphi'(0)|$ , we may expect that  $f_n$  tends to  $\frac{4}{3}\varphi'(0)$ . This is actually true: A straightforward computation shows that

$$(2.41) \quad \langle f_n | \varphi \rangle - \frac{4}{3}\varphi'(0) \stackrel{(2.36)}{=} \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} (\varphi' - \varphi'(0))t^2 dt.$$

So,

$$(2.42) \quad \left| \langle f_n | \varphi \rangle - \frac{4}{3}\varphi'(0) \right| \leq \left| \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0) \right| + \frac{1}{3}\Delta_n(\varphi') \xrightarrow{n \rightarrow \infty} 0.$$

We have just proved that

$$(2.43) \quad \langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3}\varphi'(0) \quad (\varphi \in \mathcal{D}_K).$$

In other words,

$$(2.44) \quad \langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} -\frac{4}{3}\delta',$$

where  $\delta$  is the *Dirac measure* and  $\delta', \delta'', \dots$ , its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that  $-\frac{4}{3}\delta'$  is  $\tau_K$ -continuous, and from (2.40) that

$$(2.45) \quad |\langle f_n | \varphi \rangle| \leq \frac{4}{3} \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$(2.46) \quad \{\langle f_n | \varphi_{p(n)} \rangle : \|\varphi_{p(n)}\|_\infty = 1; n = 1, 2, 3, \dots\}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant  $M$  such that

$$(2.47) \quad |\langle f_n | \varphi \rangle| \leq M \|\varphi\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

The previous bound of  $\langle f_n |$  - see (2.40), is therefore the best possible one, *i.e.*  $p = 1$  is the smallest possible  $p$  and, given  $p = 1$ ,  $M = \frac{4}{3}$  is the smallest possible  $M$  (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give  $f_n$  a *derivative*<sup>4</sup>  $f_n'$ , as follows

$$(2.48) \quad \begin{aligned} f_n' : \mathcal{D}_K &\rightarrow \mathbf{C} \\ \varphi &\mapsto -\langle f_n | \varphi' \rangle. \end{aligned}$$

It has been proved that every  $\langle f_n |$  is continuous. So is

$$(2.49) \quad \begin{aligned} D : \mathcal{D}_K &\rightarrow \mathcal{D}_K \\ \varphi &\mapsto \varphi'; \end{aligned}$$

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<sup>4</sup> See 6.1 of [3] for a further discussion.

see Exercise 1.17.  $f_n'$  is therefore continuous. Now apply (2.43) with  $\varphi'$  and so obtain

$$-\langle f_n | \varphi' \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3} \varphi''(0) \quad (\varphi \in \mathcal{D}_K),$$

*i.e.*

$$(2.50) \quad f_n' \xrightarrow{n \rightarrow \infty} \frac{4}{3} \delta''.$$

It follows from (2.40) that,

$$(2.51) \quad |\langle f_n | \varphi' \rangle| \leq \frac{4}{3} \|\varphi''\|_\infty \quad (n = 1, 2, 3, \dots).$$

It is therefore possible to uniformly bound  $f_n'$  with respect to a norm  $\|D^p \cdot\|_\infty$ , namely  $\|D^2 \cdot\|_\infty$ . Then arises a question: Is 2 the smallest  $p$ ? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant  $M$  such that

$$(2.52) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Define

$$(2.53) \quad \Phi_j(x) = \int_{-1}^x \varphi_j.$$

The oddness of  $\varphi_j$  forces  $\Phi_j$  to vanish outside  $[-1, 1]$ :  $\varphi_j$  is therefore in  $\mathcal{D}_K$ . So, under our assumption,

$$(2.54) \quad |\langle f_n | \Phi_j' \rangle| \leq M \|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots);$$

which is

$$(2.55) \quad |\langle f_n | \varphi_j \rangle| \leq M \quad (n = 1, 2, 3, \dots).$$

We have thus reached a contradiction (again with the sequence  $\{\langle f_n | \varphi_{\rho(n)} \rangle\}$ ) and so conclude that there is no constant  $M$  such that

$$(2.56) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Finally, assume, to reach a contradiction, that there exists a constant  $M$  such that

$$(2.57) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi\|_\infty.$$

The mean value theorem (see (2.1)) asserts that

$$(2.58) \quad |\langle f_n | \varphi' \rangle| \leq M \|\varphi\|_\infty \leq M \|\varphi'\|_\infty;$$

which is, again, a desired contradiction. So ends the proof. □

## 2.6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient  $\hat{f}(n)$  of a function  $f \in L^2(T)$  ( $T$  is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all  $n \in \mathbf{Z}$  (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^n \hat{f}(k).$$

Prove that  $\{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$  is a dense subspace of  $L^2(T)$  of the first category.

*Proof.* Let  $f(\theta)$  stand for  $f(e^{i\theta})$ , so that  $L^2(T)$  is identified with a closed subset of  $L^2([-\pi, \pi])$ , hence the inner product

$$(2.59) \quad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

$$(2.60) \quad \Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well known (and easy to prove) result is

$$(2.61) \quad (e_n, e_{n'}) = [n = n'], \text{ i.e. } \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric ( $\equiv$ ) identification  $L^2 \equiv (L^2)^*$ . So,

$$(2.62) \quad \|\Lambda_n\|^2 \stackrel{(2.60)}{=} \|e_{-n} + \cdots + e_n\|^2 \stackrel{(2.61)}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \stackrel{(2.61)}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

$$(2.63) \quad B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \dots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence  $\{\Lambda_n\}$  is norm-bounded; which is a desired contradiction, since

$$(2.64) \quad \|\Lambda_n\| \stackrel{(2.62)}{=} \sqrt{2n+1} \xrightarrow{n \rightarrow \infty} \infty.$$

We have just established that  $B$  is actually of the first category; and so is its subset  $L = \{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$ . We now prove that  $L$  is nevertheless dense in  $L^2(T)$ . To do so, we let  $P$  be  $\text{span}\{e_k : k \in \mathbf{Z}\}$ , the collection of the trigonometric polynomials  $p(\theta) = \sum \lambda_k e^{ik\theta}$ . Combining (2.60) with (2.61) shows that  $\Lambda_n(p) = \sum \lambda_k$  for almost all  $n$ . Thus,

$$(2.65) \quad P \subset L \subset L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that  $P$  is dense in  $L^2(T)$ . We then conclude, with the help of (2.65), that

$$(2.66) \quad L^2(T) = \bar{P} = \bar{L}.$$

So ends the proof □

## 2.9 Exercise 9. Boundedness without closedness

Suppose  $X, Y, Z$  are Banach spaces and

$$B : X \times Y \rightarrow Z$$

is bilinear and continuous. Prove that there exists  $M < \infty$  such that

$$\|B(x, y)\| \leq M \|x\| \|y\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

*Proof.* The answer is: No. To prove this, we only assume that  $X, Y, Z$  are normed spaces. Let  $(x, y)$  range over  $X \times Y$ : Since  $B$  is continuous at the origin, there exists a positive  $r$  such that

$$(2.67) \quad \|x\| + \|y\| < r \Rightarrow \|B(x, y)\| < 1.$$

Now consider all scalars  $s, t$  such that  $2\|x\| < rs$  and  $2\|y\| < rt$ : The following bound

$$(2.68) \quad \|B(x, y)\| = st \|B(x/s, y/t)\| \stackrel{(2.67)}{<} st$$

is effective, since  $r > \|x\|/s + \|y\|/t$ . Finally, remark that  $s, t$  have infimum bound  $2\|x\|/r, 2\|y\|/r$  then so obtain

$$(2.69) \quad \|B(x, y)\| \leq \frac{4}{r^2} \|x\| \|y\|;$$

which achieves the proof.

As a concrete example, choose  $X = Y = Z = C_c(\mathbf{R})$ , topologized by the supremum norm.  $C_c(\mathbf{R})$  is not complete (see 5.4.4 of [4]), nevertheless the bilinear product

$$\begin{aligned} B : C_c(\mathbf{R})^2 &\rightarrow C_c(\mathbf{R}) \\ (f, g) &\mapsto f \cdot g \end{aligned}$$

is bounded (since  $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$ ), and continuous. To show this, pick a positive scalar  $\varepsilon$  smaller than 1 then put

$$(2.70) \quad r \triangleq \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty}$$

(bear in mind that  $r < 1$ ). The Stone-Weierstrass theorem (see 7.26 of [1]) asserts the existence of  $(u, v)$  in  $C_c(\mathbf{R})^2$  such that

$$(2.71) \quad \|f - u\|_\infty + \|g - v\|_\infty < r.$$

Next, remark that  $\|u\|_\infty \leq r + \|f\|_\infty$  and so obtain

$$(2.72) \quad \|fg - uv\|_\infty = \|(f - u) \cdot g + u \cdot (g - v)\|_\infty$$

$$(2.73) \quad \leq \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty$$

$$(2.74) \quad < r \cdot \|g\|_\infty + (r + \|f\|_\infty) \cdot r$$

$$(2.75) \quad < r \cdot (r + \|f\|_\infty + \|g\|_\infty)$$

$$(2.76) \quad < \varepsilon.$$

Since  $\varepsilon$  was arbitrary, it is now established that  $B$  continuous. □

## 2.10 Exercise 10. Continuousness of bilinear mappings

*Prove that a bilinear mapping is continuous if it is continuous at the origin  $(0, 0)$ .*

*Proof.* Let  $(X_1, X_2, Z)$  be topological spaces and  $B$  a bilinear mapping

$$(2.77) \quad B : X_1 \times X_2 \rightarrow Z$$

From now on,  $x = (x_1, x_2)$  denotes an arbitrary element of  $X_1 \times X_2$ . We henceforth assume that  $B$  is continuous at the origin  $(0, 0)$  of  $X_1 \times X_2$ , *i.e.* given an arbitrary balanced open subset  $W$  of  $Z$ , there exists in  $X_i$  ( $i = 1, 2$ ) a balanced open subset  $U_i$  such that

$$(2.78) \quad B(U_1 \times U_2) \subset W.$$

Let  $\nu_i(x)$  denote any scalar that is greater than  $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$ . So,

$$(2.79) \quad B(x_1, x_2) = \nu_1(x)\nu_2(x) \cdot B(\nu_1(x)^{-1}x_1, \nu_2(x)^{-1}x_2)$$

$$(2.80) \quad \in \nu_1(x)\nu_2(x) \cdot B(U_1 \times U_2)$$

$$(2.81) \quad \subset \nu_1(x)\nu_2(x) \cdot W.$$

Now pick  $p = (p_1, p_2)$  in  $X_1 \times X_2$ : It directly follows from (2.81) that

$$(2.82) \quad B(p_1, p_2) - B(x_1, x_2) = B(p_1, p_2 - x_2) + B(p_1 - x_1, x_2 - p_2) + B(p_1 - x_1, p_2)$$

$$(2.83) \quad \in \nu_1(p)\nu_2(p - x) \cdot W + \nu_1(p - x)\nu_2(x - p) \cdot W + \nu_1(p - x)\nu_2(p) \cdot W.$$

Let us henceforth assume that

$$(2.84) \quad p_i - x_i \in [\mu_1(p) + \mu_2(p) + 2]^{-1} \cdot U_i;$$

which yields

$$(2.85) \quad \mu_i(p_i - x_i) \leq [\mu_1(p) + \mu_2(p) + 2]^{-1}.$$

Finally, combine the special case

$$(2.86) \quad \nu_i(p - x) = [\mu_1(p) + \mu_2(p) + 1]^{-1},$$

$$(2.87) \quad \nu_i(p) = \mu_1(p) + \mu_2(p) + 1$$

with (2.83) and so obtain

$$(2.88) \quad B(p_1, p_2) - B(x_1, x_2) \in W + W + W.$$

$W$  being arbitrary, we have so established the continuousness of  $B$  at  $(p_1, p_2)$ . Since  $(p_1, p_2)$  is also arbitrary, the proof is complete.  $\square$

## 2.12 Exercise 12. A bilinear mapping that is not continuous

Let  $X$  be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| \, dt.$$

Put  $B(f, g) = \int_0^1 f(t)g(t)dt$ , and show that  $B$  is a bilinear continuous functional on  $X \times X$  which is separately but not continuous.

*Proof.* Let  $f$  denote the first variable,  $g$  the second one. Remark that

$$(2.89) \quad |B(f, g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient (1.18 of [3]) to assert that any  $f \mapsto B(f, g)$  is continuous. The continuity of all  $g \mapsto B(f, g)$  follows (Put  $C(g, f) = B(f, g)$  and proceed as above). Suppose, to reach a contradiction, that  $B$  is continuous. There so exists a positive  $M$  such that,

$$(2.90) \quad |B(f, g)| < M\|f\|\|g\|.$$

Put

$$(2.91) \quad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \quad (n = 1, 2, 3, \dots),$$

so that

$$(2.92) \quad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand,

$$(2.93) \quad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.93) and (2.90) with (2.92) and so obtain

$$(2.94) \quad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Our continuousness assumption is then contradicted. So ends the proof.  $\square$

## 2.15 Exercise 15. Baire cut

Suppose  $X$  is an  $F$ -space and  $Y$  is a subspace of  $X$  whose complement is of the first category. Prove that  $Y = X$ . Hint:  $Y$  must intersect  $x + Y$  for every  $x \in X$ .

*Proof.* Assume  $Y$  is a subgroup of  $X$ . Under our assumptions, there exists a sequence  $\{E_n : n = 1, 2, 3, \dots\}$  of  $X$  such that

$$(i) \quad (\overline{E_n})^\circ = \emptyset;$$

$$(ii) \quad X \setminus Y = \bigcup_{n=1}^{\infty} E_n.$$

By (i), the complement  $V_n$  of  $\overline{E_n}$  is a dense open set. Since  $X$  is an  $F$ -space, it follows from the Baire's theorem that the intersection  $S$  of the  $V_n$ 's is dense in  $X$ : So is  $x + S$  ( $x \in X$ ). To see that, remark that

$$(2.95) \quad X = x + \overline{S} \subset \overline{x + S}$$

follows from 1.3 (b) of [3]. Since  $S$  and  $x + S$  are both dense open subsets of  $X$ , the Baire's theorem asserts that

$$(2.96) \quad \overline{(x + S) \cap S} = X.$$

Thus,

$$(2.97) \quad (x + S) \cap S \neq \emptyset.$$

Moreover, it follows from (ii) that  $X \setminus Y \subset \bigcup_n \overline{E_n}$ , i.e.  $Y \supset S$ . Combined with (2.97), this shows that  $x + Y$  cuts  $Y$ . Therefore, our arbitrary  $x$  is an element of the subgroup  $Y$ . We have thus established that  $X \subset Y$ , which achieves the proof.  $\square$

## 2.16 Exercise 16. An elementary closed graph theorem

Suppose that  $X$  and  $K$  are metric spaces, that  $K$  is compact, and that the graph of  $f : X \rightarrow K$  is a closed subset of  $X \times K$ . Prove that  $f$  is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of  $K$  cannot be omitted from the hypothesis, even when  $X$  is compact.

*Proof.* Choose a sequence  $\{x_n : n = 1, 2, 3, \dots\}$  whose limit is an arbitrary  $a$ . By compactness of  $K$ , the graph  $G$  of  $f$  contains a subsequence  $\{(x_{p(n)}, f(x_{p(n)}))\}$  of  $\{(x_n, f(x_n))\}$  that converges to some  $(a, b)$  of  $X \times K$ .  $G$  is closed; therefore,  $\{(x_{p(n)}, f(x_{p(n)}))\}$  converges in  $G$ . So,  $b = f(a)$ ; which establishes that  $f$  is sequentially continuous. Since  $X$  is metrizable,  $f$  is also continuous; see [A6] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.98) \quad \begin{aligned} f : [0, \infty) &\rightarrow [0, \infty) \\ x &\mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \end{aligned}$$

Clearly,  $f$  has a discontinuity at 0. Nevertheless the graph  $G$  of  $f$  is closed. To see that, first remark that

$$(2.99) \quad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let  $\{(x_n, 1/x_n)\}$  be a sequence in  $G_+ = \{(x, 1/x) : x > 0\}$  that converges to  $(a, b)$ . To be more specific:  $a = 0$  contradicts the boundedness of  $\{(x_n, 1/x_n)\}$ :  $a$  is necessarily positive and  $b = 1/a$ , since  $x \mapsto 1/x$  is continuous on  $\mathbb{R}_+$ . This establishes that  $(a, b) \in G_+$ , hence the closedness  $G_+$ . Finally, we conclude that  $G$  is closed, as a finite union of closed sets.  $\square$



# Chapter 3

## Convexity

### 3.3 Exercise 3.

Suppose  $X$  is a real vector space (without topology). Call a point  $x_0 \in A \subset X$  an *internal point* of  $A$  if  $A - x_0$  is an absorbing set.

- (a) Suppose  $A$  and  $B$  are disjoint convex sets in  $X$ , and  $A$  has an internal point. Prove that there is a nonconstant linear functional  $\Lambda$  such that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. (The proof is similar to that of Theorem 3.4)
- (b) Show (with  $X = \mathbf{R}^2$ , for example) that it may not be possible to have  $\Lambda(A)$  and  $\Lambda(B)$  disjoint, under the hypotheses of (a).

*Proof.* Take  $A$  and  $B$  as in (a); the trivial case  $B = \emptyset$  is discarded. Since  $A - x_0$  is absorbing, so is its convex superset  $C = A - B - x_0 + b_0$  ( $b_0 \in B$ ). Note that  $C$  contains the origin. Let  $p$  be the Minkowski functional of  $C$ . Since  $A$  and  $B$  are disjoint,  $b_0 - x_0$  is not in  $C$ , hence  $p(b_0 - x_0) \geq 1$ . We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [3] to establish the existence of a linear functional  $\Lambda : X \rightarrow \mathbf{R}$  such that

$$(3.1) \quad \Lambda \leq p$$

and

$$(3.2) \quad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \quad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

$$(3.4) \quad \Lambda a \leq \Lambda b.$$

We now prove that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. Suppose, to reach a contradiction, that this intersection contains  $y_1$  and  $y_2$ . There so exists  $(a_i, b_i)$  in  $A \times B$  ( $i = 1, 2$ ) such that

$$(3.5) \quad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that  $y_1 < y_2$ . Then,

$$(3.6) \quad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad .$$

Remark that  $a_3 = \frac{1}{2}(a_1 + a_2)$  lies in the convex set  $A$ . This implies

$$(3.7) \quad \Lambda b_1 \stackrel{(3.6)}{<} \Lambda a_3 \stackrel{(3.4)}{\leq} \Lambda b_1 \quad ;$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space  $X$  is  $\mathbf{R}^2$ . Fetch

$$(3.8) \quad S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$

$$(3.9) \quad S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$$

$$(3.10) \quad A \triangleq S_1 \cup S_2,$$

$$(3.11) \quad B \triangleq X \setminus A.$$

Pick  $(x_i, y_i)$  in  $S_i$ . Let  $t$  range over the unit interval, and so obtain

$$(3.12) \quad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subset A.$$

Thus, every segment that has an extremity in  $S_1$  and the other one in  $S_2$  lies in  $A$ . Moreover, each  $S_i$  is convex. We can now conclude that  $A$  is so. The convexity of  $B$  is proved in the same manner. Furthermore,  $A$  hosts a non degenerate triangle, *i.e.*  $A^\circ$  is nonempty<sup>1</sup>:  $A$  contains an internal point.

Let  $L$  be a vector line of  $\mathbf{R}^2$ . In other words,  $L$  is the null space of a linear functional  $\Lambda : \mathbf{R}^2 \rightarrow \mathbf{R}$  (to see this, take some nonzero  $u$  in  $L^\perp$  and set  $\Lambda x = (x, u)$  for all  $x$  in  $\mathbf{R}^2$ ). One easily checks that both  $A$  and  $B$  cut  $L$ . Hence

$$(3.13) \quad \Lambda(L) = \{0\} \subset \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad .$$

So ends the proof. □

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<sup>1</sup>For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [2].

### 3.11 Exercise 11. Meagerness of the polar

Let  $X$  be an infinite-dimensional Fréchet space. Prove that  $X^*$ , with its weak\*-topology, is of the first category in itself.

This is actually a consequence of the below lemma, which we prove first. The proof that  $X^*$  is of the first category in itself comes right after, as a corollary.

**Lemma.** *If  $X$  is an infinite dimensional topological vector space whose dual  $X^*$  separates points on  $X$ , then the polar*

$$(3.14) \quad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\}$$

*of any absorbing subset  $A$  is a weak\*-closed set that has empty interior.*

*Proof.* Let  $x$  range over  $X$ . The linear form  $\Lambda \mapsto \Lambda x$  is weak\*-continuous; see 3.14 of [3]. Therefore,  $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$  is weak\*-closed. As the intersection of  $\{P_a : a \in A\}$ ,  $K_A$  is also a weak\*-closed set. We now prove the second half of the statement.

From now on,  $X$  is assumed to be endowed with its weak topology:  $X$  is then locally convex, but its dual space is still  $X^*$  (see 3.11 of [3]). Put

$$(3.15) \quad W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \quad (r_x > 0)$$

where  $F$  runs through the nonempty finite subsets of  $X$ . Clearly, the collection of all such  $W$  is a local base of  $X^*$ . Pick one of those  $W$  and remark that the following subspace

$$(3.16) \quad M \triangleq \text{span}(F)$$

is finite dimensional. Assume, to reach a contradiction, that  $A \subset M$ . So, every  $x$  lies in  $t_x M = M$  for some  $t_x > 0$ , since  $A$  is absorbing. As a consequence,  $X$  is the finite dimensional space  $M$ , which is a desired contradiction. We have just established that  $A \not\subset M$ : Now pick  $a$  in  $A \setminus M$  and so conclude that

$$(3.17) \quad b \triangleq \frac{a}{t_a} \in A$$

Remark that  $b \notin M$  (otherwise,  $a = t_a b \in t_a M = M$  would hold) and that  $M$ , as a finite dimensional space, is closed (see 1.21 (b) of [3] for a proof): By the Hahn-Banach theorem 3.5 of [3], there exists  $\Lambda_a$  in  $X^*$  such that

$$(3.18) \quad \Lambda_a b > 2$$

and

$$(3.19) \quad \Lambda_a(M) = \{0\}.$$

The latter equality implies that  $\Lambda_a$  vanishes on  $F$ ; hence  $\Lambda_a$  is an element of  $W$ . On the other hand, given an arbitrary  $\Lambda \in K_A$ , the following inequalities

$$(3.20) \quad |\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that  $\Lambda + \Lambda_a$  is not in  $K_A$ . We have thus proved that

$$(3.21) \quad \Lambda + W \not\subset K_A.$$

Since  $W$  and  $\Lambda$  are both arbitrary, this achieves the proof.  $\square$

We now give a proof of the original statement.

**Corollary.** *If  $X$  is an infinite-dimensional Fréchet space, then  $X^*$  is meager in itself.*

*Proof.* From now on,  $X^*$  is only endowed with its weak\*-topology. Let  $d$  be an invariant distance that is compatible with the topology of  $X$ , so that the following sets

$$(3.22) \quad B_n \triangleq \{x \in X : d(0, x) < 1/n\} \quad (n = 1, 2, 3, \dots)$$

form a local base of  $X$ . If  $\Lambda$  is in  $X^*$ , then

$$(3.23) \quad |\Lambda| \leq m \text{ on } B_n$$

for some  $(n, m) \in \{1, 2, 3, \dots\}^2$ ; see 1.18 of [3]. Hence,  $X^*$  is the countable union of all

$$(3.24) \quad m \cdot K_n \quad (m, n = 1, 2, 3, \dots),$$

where  $K_n$  is the polar of  $B_n$ . Clearly, showing that every  $m \cdot K_n$  is nowhere dense is now sufficient. To do so, we use the fact that  $X^*$  separates points; see 3.4 of [3]. As a consequence, the above lemma implies

$$(3.25) \quad (\overline{K_n})^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by  $m$  is an homeomorphism (see 1.7 of [3]), this is equivalent to

$$(3.26) \quad (\overline{m \cdot K_n})^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof. □

## Chapter 4

# Banach Spaces

Throughout this set of exercises,  $X$  and  $Y$  denote Banach spaces, unless the contrary is explicitly stated.

### 4.1 Exercise 1. Basic results

Let  $\varphi$  be the embedding of  $X$  into  $X^{**}$  described in Section 4.5. Let  $\tau$  be the weak topology of  $X$ , and let  $\sigma$  be the weak\*-topology of  $X^{**}$  - the one induced by  $X^*$ .

- (a) Prove that  $\varphi$  is an homeomorphism of  $(X, \tau)$  onto a dense subspace of  $(X^{**}, \sigma)$ .
- (b) If  $B$  is the closed unit ball of  $X$ , prove that  $\varphi(B)$  is  $\sigma$ -dense in the closed unit ball of  $X^{**}$ . (Use the Hahn-Banach separation theorem.)
- (c) Use (a), (b), and the Banach-Alaoglu theorem to prove that  $X$  is reflexive if and only if  $B$  is weakly compact.
- (d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.
- (e) If  $X$  is reflexive and  $Y$  is a closed subspace of  $X$ , prove that  $X/Y$  is reflexive.
- (f) Prove that  $X$  is reflexive if and only if  $X^*$  is reflexive.  
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace  $\varphi(X)$  of  $X^{**}$ .

*Proof.* Let  $\psi$  be the isometric embedding of  $X^*$  into  $X^{***}$ . The dual space of  $(X^{**}, \sigma)$  is then  $\psi(X^*)$ .

It is sufficient to prove that

$$(4.1) \quad \varphi^{-1} : \varphi(X) \rightarrow X$$

$$(4.2) \quad \varphi(x) \mapsto x$$

is an homeomorphism (with respect to  $\tau$  and  $\sigma$ ). We first consider

$$(4.3) \quad V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}, \psi x^* \rangle| < r\} \quad (x^* \in X^*, r > 0);$$

$$(4.4) \quad U \triangleq \{x \in X : |\langle x, x^* \rangle| < r\} \quad (x^* \in X^*, r > 0).$$

and remark that the so defined  $V$ 's (respectively  $U$ 's) shape a local subbase  $\mathcal{S}_\sigma$  (respectively  $\mathcal{S}_\tau$ ) of  $\sigma$  (respectively  $\tau$ ). We now observe that

$$(4.5) \quad U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathcal{S}_\sigma, U \in \mathcal{S}_\tau) \quad ,$$

since  $\varphi^{-1}$  is one-to-one. This remains true whether we enrich each subbase  $\mathcal{S}$  with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak\*-topology that  $\varphi^{-1}$  and its inverse  $\varphi$  are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$(4.6) \quad \overline{\varphi(X)}_{\sigma} \subset X^{**} \quad .$$

and we now assume- to reach a contradiction- that  $(X^{**}, \sigma)$  contains a point  $z^{**}$  outside the  $\sigma$ -closure of  $\varphi(X)$ . By [3.5], there so exists  $y^*$  in  $X^*$  such that

$$(4.7) \quad \langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ;$$

$$(4.8) \quad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces  $y^*$  to be a the zero of  $X^*$ . The functional  $\psi y^*$  is then the zero of  $X^{***}$ : (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball  $B^{**}$  of  $X^{**}$  is weak\*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \quad \varphi(B) \subset B^{**} \quad ,$$

since  $\varphi$  is isometric. Hence

$$(4.10) \quad \overline{\varphi(B)}_{\sigma} \subset \overline{(B^{**})}_{\sigma} = B^{**} \quad .$$

Now suppose, to reach a contradiction, that  $B^{**} \setminus \overline{\varphi(B)}_{\sigma}$  contains a vector  $z^{**}$ . By [3.7], there exists  $y^*$  in  $X^*$  such that

$$(4.11) \quad |\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_{\sigma} \quad ;$$

$$(4.12) \quad \langle z^{**}, \psi y^* \rangle > 1 \quad .$$

It follows from (4.11) that

$$(4.13) \quad |\psi y^*| \leq 1 \quad \text{on } \varphi(B) \quad , \quad \text{i.e.} \quad |y^*| \leq 1 \quad \text{on } B \quad .$$

We have so proved that

$$(4.14) \quad y^* \in B^* \quad .$$

Since  $z^{**}$  lies in  $B^{**}$ , it is now clear that

$$(4.15) \quad |\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ;$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

$$(4.16) \quad B \text{ is weakly compact if and only if } \varphi(B) \text{ is weak*-compact.}$$

If  $B$  is weakly compact, then  $\varphi(B)$  is weak\*-closed. So,

$$(4.17) \quad \varphi(B) = \overline{\varphi(B)}_{\sigma} \stackrel{(b)}{=} B^{**} \quad .$$

$\varphi$  is therefore onto, *i.e.*  $X$  is reflexive.

Conversely, keep  $\varphi$  as onto: one easily checks that  $\varphi(B) = B^{**}$ . The image  $\varphi(B)$  is then weak\*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let  $X$  be a reflexive space  $X$ , whose closed unit ball is  $B$ . Let  $Y$  be a norm-closed subspace of  $X$ :  $Y$  is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that  $B$  is weakly compact. We now conclude that the closed unit ball  $B \cap Y$  of  $Y$  is weakly compact. We again use (c) to conclude that  $Y$  is reflexive. (d) is therefore established. Now proceed to (e).

Let  $\equiv$  stand for “isometrically isomorphic” and apply twice [4.9] to obtain, first

$$(4.18) \quad (X/Y)^* \equiv Y^\perp \quad ,$$

next,

$$(4.19) \quad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \quad .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

*Given a reflexive Banach space  $Z$ , the weak\*-topology of  $Z^*$  is its weak one.*

Assume first that  $X$  is reflexive. Since  $B^*$  is weak\* compact, by (c) of [4.3], (L) implies that  $B^*$  is also weakly compact. Then (c) turns  $X^*$  into a reflexive space.

Conversely, let  $X^*$  be reflexive. What we have just proved that makes  $X^{**}$  reflexive. On the other hand,  $\varphi(X)$  is a norm-closed subspace of  $X^{**}$ ; *cf.* [4.5]. Hence  $\varphi(X)$  is reflexive, by (d). It now follows from (c) that  $B^{**} \cap \varphi(X)$  is weakly compact, *i.e.* weak\*-compact (to see this, apply (L) with  $Z = X^*$ ).

By (a),  $B$  is therefore weakly compact, *i.e.*  $X$  is reflexive; see (c). So ends the proof.  $\square$

**4.13 Exercise 13. Operator compactness in a Hilbert space**



## 4.15 Exercise 15. Hilbert-Schmidt operators

## Chapter 6

# Distributions

### 6.1 Exercise 1. Test functions are almost polynomial

### 6.6 Exercise 6. Around the supports of some distributions

### 6.9 Exercise 9. Convergence in $\mathcal{D}(\Omega)$ vs. convergence in $\mathcal{D}'(\Omega)$

(a) Prove that a set  $E \subset \mathcal{D}(\Omega)$  is bounded if and only if

$$\sup\{|\Lambda\varphi| : \varphi \in E\} < \infty$$

for every  $\Lambda \in \mathcal{D}'(\Omega)$ .

(b) Suppose  $\{\varphi_j\}$  is a sequence in  $\mathcal{D}(\Omega)$  such that  $\{\Lambda\varphi_j\}$  is a bounded sequence of numbers, for every  $\Lambda \in \mathcal{D}'(\Omega)$ . Prove that some subsequence of  $\{\varphi_j\}$  converges, in the topology of  $\mathcal{D}(\Omega)$ .

(c) Suppose  $\{\Lambda_j\}$  is a sequence in  $\mathcal{D}'(\Omega)$  such that  $\{\Lambda_j\varphi\}$  is bounded, for every  $\varphi \in \mathcal{D}(\Omega)$ . Prove that some subsequence of  $\{\Lambda_j\}$  converges in  $\mathcal{D}'(\Omega)$  and that the convergence is uniform on every bounded subset of  $\mathcal{D}(\Omega)$ . *Hint: By the Banach-Steinhaus theorem, the restrictions of the  $\Lambda_j$  to  $\mathcal{D}_K$  are equicontinuous. Apply Ascoli's theorem.*

**PROOF.** Since  $\mathcal{D}(\Omega)$  locally convex space (see (b) of [6.4]), [3.18] states that  $E$  is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that  $E = \{\varphi_j : j \in \mathbf{N}\}$  is bounded: so is  $\overline{E}$ . By (c) of [6.5], there exists some  $\mathcal{D}_K$  that contains  $\overline{E}$ . Since  $\mathcal{D}_K$  has the Heine-Borel property (see [1.46]),  $\overline{E}$  is  $\tau_K$ -compact. Apply [A4] with the metrizable space  $\mathcal{D}_K$  (see [1.46]) to conclude that  $\overline{E}$  has a  $\tau_K$  limit point. It then follows from (b) of [6.5] that (b) holds.

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