Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \quad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ *in X such that (a)* $\{f_n\}$ *converges to* 0 *as* $n \to \infty$*, but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms p_x is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$V_{x,k} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0,1], k \in \mathbf{N}) \tag{1.1}$$

is therefore a local base for a topology τ on X. Given $\{f_n : n = 1, 2, 3, \dots\}$, we put

$$\texttt{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \tag{1.2}$$

with the convention "$\Sigma = \infty$" whether the sum has no finite support. So,

$$\sum_{i=1}^{m} \texttt{off}(U_i) = \sum_{n=1}^{\infty} \sum_{i=1}^{m} [f_n \notin U_i] \ge \texttt{off}(U_1 \cap \dots \cap U_m). \tag{1.3}$$

We first assume that $\{f_n\}$ τ-converges to some f in X, *i.e.*

$$\texttt{off}(f + V) < \infty \quad (V \in \mathscr{B}). \tag{1.4}$$

The special cases $V_{x,1}, V_{x,2}, \dots,$ mean the pointwise convergence $f_n(x) \overset{n\infty}{\longrightarrow} f(x)$. Conversely, assume that $\{f_n\}$ does not τ-converges to any g in X, *i.e.*

$$\forall g \in X, \exists W \in \mathscr{B} : \texttt{off}(g + W) = \infty. \tag{1.5}$$

Given g, such W is, by definition, a finite intersection $V_{x_1,k_1} \cap \dots \cap V_{x_m,k_m}$. Thus,

$$\sum_{i=1}^{m} \texttt{off}(g + V_{x_i,k_i}) \overset{(1.3)}{\ge} \texttt{off}(g + W) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\texttt{off}(g + V_{x_i,k_i})$ must then be ∞. In other words, there exists a point x_i for which $\{f_n(x_i)\}$ does not converge to $g(x_i)$. g being arbitrary, we so conclude that f_n does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now prove the second part. From now on, we let *k*, *n* and *p* run on $\mathbf{N}_+$, as $\texttt{dyadic}(x)$ denotes the usual dyadic expansion of x, so that $\texttt{dyadic}(x)$ is an aperiodic binary sequence **iff** x is irrational. Define

$$f_n(x) \triangleq \begin{cases} \exp_2\left(-\sum_{k=1}^{n} \texttt{dyadic}(x)_{-k}\right) & (x \in [0,1] \setminus \mathbf{Q}) \\ 0 & (x \in [0,1] \cap \mathbf{Q}), \end{cases} \tag{1.7}$$

so that $f_n(x) \overset{n\infty}{\longrightarrow} 0$, and take $\gamma_n \overset{n\infty}{\longrightarrow} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose n_p among those *almost all* n that are large enough to satisfy

$$n_{p-1} - n_{p-2} < n_p - n_{p-1} \tag{1.8}$$

(start with $n_{-1} = n_0 = 0$) and so obtain

$$2^p < \gamma_{n_p} : 0 < n_p - n_{p-1} \underset{p\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator χ of $\{n_1, n_2, \dots\}$ in $\mathbf{Z}$ is then aperiodic, *i.e.*

$$\alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \in [0,1] \setminus \mathbf{Q}. \tag{1.10}$$

Hence, χ is not a the infinite-support expansion of a rational number; which forces

$$\texttt{dyadic}(\alpha_\gamma)_{-k} = \chi_k. \tag{1.11}$$

The key ingredient is that

$$\chi_1 + \cdots + \chi_{n_p} = p\,. \tag{1.12}$$

Combined with (1.7), it yields

$$f_{n_p}(\alpha_\gamma) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n_p} f_{n_p}(\alpha_\gamma) > 1. \tag{1.14}$$

There so exists $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. In other words, (b) holds, which is in violent contrast with 1.28 of [3]: X is therefore not metrizable. So ends the proof. $\square$

1.2 Exercise 9. Quotient map

Suppose

(*a*) X *and* Y *are topological vector spaces,*

(*b*) $\Lambda : X \to Y$ *is linear.*

(*c*) N *is a closed subspace of* X*,*

(*d*) $\pi : X \to X/N$ *is the quotient map, and*

(*e*) $\Lambda x = 0$ *for every* $x \in N$.

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \tag{1.15}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'. \tag{1.16}$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$f(\pi x) = 0 \overset{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \tag{1.17}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subset V. \tag{1.18}$$

Since π is open (1.41 (a) of [3]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \tag{1.19}$$

for some neighborhood V of 0_X. So,

$$f(W) = f(\pi(V)) = \Lambda(V). \tag{1.20}$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y. So ends the proof. □

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$, $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$.

(*a*) *Prove that* Λ *is an open mapping.*

(*b*) *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. We discard the trivial case $\dim \mathrm{Y} = 0$ then henceforth assume that $\dim \mathrm{Y}$ has positive dimension n.

Let e range over a base of Y: For each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, since Λ is onto. So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \Lambda \mathrm{x_e} \quad (\mathrm{y} \in \mathrm{Y}). \tag{1.21}$$

The sequence $\{\mathrm{x_e}\}$ is finite; therefore it is bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$\mathrm{x_e} \in \mathrm{sV} \text{ for all } \mathrm{x_e}. \tag{1.22}$$

Combining this with (1.21) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \quad (\mathrm{y} \in \mathrm{Y} : |\mathrm{y_e}| < \mathrm{s^{-1}}), \tag{1.23}$$

which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, with $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$\dim \mathrm{X/N} = \mathrm{n}. \tag{1.24}$$

f is then an homeomorphism of $\mathrm{X/N} \equiv \mathbf{C}^{\mathrm{n}}$ onto Y; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$. ☐

1.4 Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms

Put $\mathrm{K} = [0,1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0, 1, 2, \ldots$*) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

(*a*) $\|\mathrm{D^n f}\|_\infty = \sup\{|\mathrm{D^n f(x)}| : \infty < \mathrm{x} < \infty\}$

(*b*) $\|\mathrm{D^n f}\|_1 = \int_0^1 |\mathrm{D^n f}(x)|\,\mathrm{d}x$

(*c*) $\|\mathrm{D^n f}\|_2 = \left\{\int_0^1 |\mathrm{D^n f}(x)|^2 \mathrm{d}x\right\}^{1/2}$.

Proof. First, remark that

$$\|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty < \infty \tag{1.25}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, start from

$$\mathrm{D^n f(x)} = \int_{-1}^{\mathrm{x}} \mathrm{D^{n+1} f} \tag{1.26}$$

(which is true, since f has support K) to obtain

$$|\mathrm{D^n f(x)}| \le \int_{-1}^{\mathrm{x}} \left|\mathrm{D^{n+1} f}\right| \le \|\mathrm{D^{n+1} f}\|_1, \tag{1.27}$$

hence

$$\|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1. \tag{1.28}$$

Combining (1.25) with (1.28) yields

$$\|\mathrm{D^0 f}\|_1 \le \cdots \le \|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1 \le \cdots. \tag{1.29}$$

We now put

$$\mathrm{V_n^{(i)}} \triangleq \{\mathrm{f} \in \mathscr{D}_{\mathrm{K}} : \|\mathrm{f}\|_{\mathrm{i}} < 1/\mathrm{n}\} \quad (\mathrm{i} = 1, 2, \infty) \tag{1.30}$$

$$\mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V_n^{(i)}} : \mathrm{n} = 1, 2, 3, \ldots\}, \tag{1.31}$$

so that (1.29) is mirrored in terms of neighborhood inclusions, as follows,

$$\mathrm{V_1^{(1)}} \supset \cdots \supset \mathrm{V_n^{(1)}} \supset \mathrm{V_n^{(2)}} \supset \mathrm{V_n^{(\infty)}} \supset \mathrm{V_{n+1}^{(1)}} \supset \cdots. \tag{1.32}$$

Since $\mathrm{V_n^{(i)}} \supset \mathrm{V_{n+1}^{(i)}}$, $\mathscr{B}^{(\mathrm{i})}$ is the local base of a topology τ_{i}. But the chain (1.32) forces the τ_{i} to be equals. To see that, choose a set S that is τ_1-open at, say a, *i.e.* $\mathrm{V_n^{(1)}} \subset \mathrm{S} - \mathrm{a}$ for some n. Next, concatenate this with $\mathrm{V_n^{(2)}} \subset \mathrm{V_n^{(1)}}$ (see (1.32)) and so obtain $\mathrm{V_n^{(2)}} \subset \mathrm{S} - \mathrm{a}$, which implies that S is τ_2-open at a. Similarly, we deduce, still from (1.32), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.33}$$

So ends the proof. □

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $\mathrm{p_n}$, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1 Let X be a topological space with a countable local base $\{\mathrm{V_n} : \mathrm{n} = 1, 2, 3, \dots\}$. If $\tilde{\mathrm{V}}_\mathrm{n} = \mathrm{V_1} \cap \cdots \cap \mathrm{V_n}$, then every subsequence $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$ is a decreasing (*i.e.* $\tilde{\mathrm{V}}_{\rho(\mathrm{n})} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n}+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_n} \supset \tilde{\mathrm{V}}_\mathrm{n}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{n}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$, since $\tilde{\mathrm{V}}_\mathrm{n} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n})}$. □

The following special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$ is one of the key ingredients:

Corollary 1 (special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$) Under the same notations of Lemma 1, if $\{\mathrm{V_n}\}$ is a decreasing local base, then so is $\{\mathrm{V}_{\rho(\mathrm{n})}\}$.

Corollary 2 If $\{\mathrm{Q_n}\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{\mathrm{Q}_{\rho(\mathrm{n})}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $\mathrm{C}(\Omega)$'s (respectively $\mathrm{C}^\infty(\Omega)$'s) topology of the seminorms $\mathrm{p_n}$, as defined in section 1.44 (respectively 1.46), then the seminorms $\mathrm{p}_{\rho(\mathrm{n})}$ define the same topology τ_Q.

Proof. Let X be $\mathrm{C}(\Omega)$ topologized by the seminorms $\mathrm{p_n}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_n} = \{\mathrm{p_n} < 1/\mathrm{n}\}$, then $\{\mathrm{V_n}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n}+1)}. \tag{1.34}$$

Thus,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n}+1)}. \tag{1.35}$$

In other words, $\mathrm{Q}_{\rho(\mathrm{n})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\rho(\mathrm{n})}\}$ then defines a topology τ_{Q_ρ} for which $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\rho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base of τ_Q, which yields $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\rho}$. □

Lemma 2 If a sequence of compact sets $\{\mathrm{Q_n}\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in allmost all $\mathrm{Q}_\mathrm{n}^\circ$, *i.e.* there exists m such that

$$\mathrm{K} \subset \mathring{\mathrm{Q}}_\mathrm{m} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+1} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+2} \subset \cdots. \tag{1.36}$$

Proof. The following definition

$$\mathrm{C_n} \triangleq \mathrm{K} \setminus \overset{\circ}{\mathrm{Q}}_\mathrm{n} \quad (\mathrm{n} = 1, 2, 3, \dots) \tag{1.37}$$

shapes $\{\mathrm{C_n}\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no $\mathrm{C_n}$ is empty and so conclude[2] that the $\mathrm{C_n}$'s intersection contains a point that is not in any $\mathrm{Q}^\circ_\mathrm{n}$. On the other hand, the conditions specified in [1.44] force the $\mathrm{Q}^\circ_\mathrm{n}$'s collection to be an open cover. This contradiction reveals that $\mathrm{C_m} = \emptyset$, *i.e.* $\mathrm{K} \subset \mathrm{Q}^\circ_\mathrm{m}$, for some m. Finally,

$$\mathrm{K} \subset \overset{\circ}{\mathrm{Q}}_\mathrm{m} \subset \mathrm{Q_m} \subset \overset{\circ}{\mathrm{Q}}_\mathrm{m+1} \subset \mathrm{Q_{m+1}} \subset \overset{\circ}{\mathrm{Q}}_\mathrm{m+2} \subset \cdots. \tag{1.38}$$

□

We are now in a fair position to establish the following:

Theorem The topology of $\mathrm{C}(\Omega)$ does not depend on the particular choice of $\{\mathrm{K_n}\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $\mathrm{C}^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

Proof. With the second corollary's notations, $\tau_\mathrm{K} = \tau_{\mathrm{K}_\lambda}$, for every subsequence $\{\mathrm{K}_{\lambda(\mathrm{n})}\}$. Similarly, let $\{\mathrm{L_n}\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_\mathrm{L} = \tau_{\mathrm{L}_\varkappa}$ for every subsequence $\{\mathrm{L}_{\varkappa(\mathrm{n})}\}$. Now apply the above Lemma 2 with $\mathrm{K_i}$ $(\mathrm{i} = 1, 2, 3, \dots)$ and so conclude that $\mathrm{K_i} \subset \mathrm{L}^\circ_{\mathrm{m_i}} \subset \mathrm{L}^\circ_{\mathrm{m_i}+1} \subset \cdots$ for some $\mathrm{m_i}$. In particular, the special case $\varkappa_\mathrm{i} = \mathrm{m_i} + \mathrm{i}$ is

$$\mathrm{K_i} \subset \overset{\circ}{\mathrm{L}}_{\varkappa_\mathrm{i}}. \tag{1.39}$$

Let us reiterate the above proof with $\mathrm{K_n}$ and $\mathrm{L_n}$ in exchanged roles then similarly find a subsequence $\{\lambda_\mathrm{j} : \mathrm{j} = 1, 2, 3, \dots\}$ such that

$$\mathrm{L_j} \subset \overset{\circ}{\mathrm{K}}_{\lambda_\mathrm{j}} \tag{1.40}$$

Combine (1.39) with (1.40) and so obtain

$$\mathrm{K_1} \subset \overset{\circ}{\mathrm{L}}_{\varkappa_1} \subset \mathrm{L}_{\varkappa_1} \subset \overset{\circ}{\mathrm{K}}_{\lambda_{\varkappa_1}} \subset \mathrm{K}_{\lambda_{\varkappa_1}} \subset \overset{\circ}{\mathrm{L}}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots, \tag{1.41}$$

which means that the sequence $\mathrm{Q} = (\mathrm{K_1}, \mathrm{L}_{\varkappa_1}, \mathrm{K}_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_\mathrm{K} = \tau_{\mathrm{K}_\lambda} = \tau_\mathrm{Q} = \tau_{\mathrm{L}_\varkappa} = \tau_\mathrm{L}. \tag{1.42}$$

So ends the proof □

[1] See (b) of 2.5 of [2].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [2].

1.6 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^{\alpha} f$ *is a continuous mapping of* $C^{\infty}(\Omega)$ *into* $C^{\infty}(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$*, for every multi-index* α.

Proof. In both cases, D^{α} is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^{\infty}(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{ \phi \in C^{\infty}(\Omega) : \max\{|D^{\beta}\phi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}. \tag{1.43}$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^{\gamma} g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}. \tag{1.44}$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^{\beta} D^{\alpha} g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}. \tag{1.45}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^{\alpha} g \in V_N, \quad \textit{i.e.} \quad D^{\alpha}\left(V_{N+|\alpha|}\right) \subset V_N, \tag{1.46}$$

which establishes the continuity of $D^{\alpha} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$.

To prove the continuousness of the restriction $D^{\alpha}|_{\mathscr{D}_K}$: $\mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^{\alpha}|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) = D^{\alpha}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) \tag{1.47}$$

$$\subset D^{\alpha}\left(V_{N+|\alpha|}\right) \cap D^{\alpha}(\mathscr{D}_K) \tag{1.48}$$

$$\subset V_N \cap \mathscr{D}_K \quad \text{(see (1.46))} \tag{1.49}$$

So ends the proof. □

Chapter 2

Completeness

2.1 Exercise 3. An equicontinous sequence of measures

Put $K = [-1, 1]$*; define* $\mathscr{D}_K$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^n$*). Supose* $\{f_n\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\phi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ *exists for every* $\phi \in \mathscr{D}_K$*. Show that* Λ *is a continuous linear functional on* $\mathscr{D}_K$*. Show that there is a positive integer* p *and a number* $M < \infty$ *such that*

$$\left| \int_{-1}^{1} f_n(t)\phi(t)dt \right| \leq M \| D^p \|_\infty$$

for all n*. For example, if* $f_n(t) = n^3 t$ *on* $[-1/n, 1/n]$ *and* 0 *elsewhere, show that this can be done with* $p = 1$*. Construct an example where it can be done with* $p = 2$ *but not with* $p = 1$*.*

We will also consider the case $p = 0$. Since all supports of $\phi, \phi', \phi'', \ldots,$ are in K, we make a specialization of the mean value theorem:

Lemma If $\phi \in \mathscr{D}_{[a,b]}$, then

$$\| D^\alpha \phi \|_\infty \leq \| D^p \phi \|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (\alpha = 0, 1, \ldots, p) \tag{2.1}$$

at every order $p = 0, 1, 2, \ldots$; where λ is the length $|b - a|$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 $(a < x_1 < x_0)$, such that

$$\phi(x_0) = \phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a). \tag{2.2}$$

Since every $D^p\phi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$\begin{aligned} \phi(x_0) &= D^0\phi(x_0) && (2.3)\\ &= D^1\phi(x_1)(x_0 - a) && (2.4)\\ &= \cdots \\ &= D^p\phi(x_p)(x_0 - a)\cdots(x_{p-1} - a), && (2.5) \end{aligned}$$

for all p. More compactly,

$$D^{\alpha}\phi(x_0) = D^{p}\phi(x_p)\prod_{k=\alpha}^{p-1}(x_k - a); \tag{2.6}$$

which yields,

$$|D^{\alpha}\phi(x)| \leq \| D^{p}\phi \|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (x \in [a, c]) \tag{2.7}$$

The case $x_0 \geq c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^{\alpha}\phi(x)| \leq \| D^{p}\phi \|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha} \tag{2.8}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\| D^{\alpha}\phi \|_{\infty} \leq \| D^{p}\phi \|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha}. \tag{2.9}$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\phi\rangle \triangleq \int_{\mathbf{R}} u\phi \quad (\phi \in C_0(\mathbf{R})). \tag{2.10}$$

The following inequalities

$$|\langle u|\phi\rangle| \leq \int_{\mathbf{R}} |u\phi| \leq \| u \|_{L^1} \quad (\| \phi \|_{\infty} \leq 1) \tag{2.11}$$

imply that every linear functional

$$\begin{aligned} \langle u| : C_0(\mathbf{R}) &\to \mathbf{C} \\ \phi &\mapsto \langle u|\phi\rangle \end{aligned} \tag{2.12}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u|$, since u is determined (a.e) by the integrals $\langle u|\phi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\sup\{|\langle u|\phi\rangle| : \| \phi \|_{\infty} = 1\} = \| u \|_{L^1}. \tag{2.13}$$

Note that, in the latter equality, $\leq \| u \|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$\begin{aligned} g_n : \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & \left(x \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right) \\ 0 & \left(x \notin \left[-\frac{1}{n}, \frac{1}{n}\right]\right). \end{cases} \end{aligned} \tag{2.14}$$

[1] see 1.32, 4.1 of [3]

[2] See 7.26 of [1].

First, remark that $g_n(x) \underset{n\to\infty}{\longrightarrow} 0$ $(x \in \mathbf{R})$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$\langle g_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi, \tag{2.15}$$

where ϕ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\phi_1^+, \dots, \phi_j^+, \dots,$ from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\phi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

(ii) $\phi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

(iii) $0 \leq \phi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those ϕ_j^+. Let $\phi_1^-, \dots, \phi_j^-, \dots,$ mirror the ϕ_j^+, in the sense that $\phi_j^-(x) = \phi_j^+(-x)$, so that

(iv) $\phi_j \triangleq \phi_j^+ - \phi_j^-$ is odd, as g_n is;

(v) every ϕ_j is in $C_K^\infty(\mathbf{R})$;

(vi) The sequence $\{\phi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\| \phi_j \|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n|\phi_j\rangle = 2\int_0^1 g_n(t)\phi_j^+(t)dt \underset{j\to\infty}{\longrightarrow} 2\int_0^1 g_n(t)dt = \| g_n \|_{L^1} = n. \tag{2.16}$$

Finally,

$$\| g_n \|_{L^1} \overset{(2.16)}{\leq} \sup\{|\langle g_n|\phi\rangle| : \| \phi \|_\infty = 1\} \overset{(2.13)}{\leq} \| g_n \|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$|\langle g_n|\phi\rangle| \leq C_n \quad (\| \phi \|_\infty = 1); \tag{2.18}$$

see (2.11). Furthermore, $\| g_n \|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\phi_{\rho(n)}\rangle\}$ such that $\langle g_n|\phi_{\rho(n)}\rangle$ is greater than, say, $n - 0.01$, as $\| \phi_{\rho(n)} \|_\infty = 1$. Consequently, there is no bound M such that

$$|\langle g_n|\phi\rangle| \leq M \quad (\| \phi \|_\infty = 1; n = 1, 2, 3, \dots). \tag{2.19}$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi \quad (\phi \in C_0(\mathbf{R})) \tag{2.20}$$

[3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that ϕ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$\Lambda\phi = \lim_{n\to\infty} \int_K f_n\phi \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.21}$$

for some linear form Λ. Since ϕ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$\Lambda\phi = \lim_{n\to\infty} \langle f_n|\phi\rangle \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.22}$$

Let $K_1, K_2, \ldots,$ be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\phi) = \|\phi\|_N \triangleq \max\{|D^\alpha\phi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\phi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\phi \in \mathscr{D}_K : \|\phi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\phi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|f_n\|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [3].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi$ for all ϕ, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$|\Lambda\phi| < 1/2 \quad \text{if } \phi \in V_i, \tag{2.23}$$

$$|\Lambda\phi - \langle f_n|\phi\rangle| < 1/2 \quad \text{if } \phi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\phi\rangle| \leq |\Lambda\phi - \langle f_n|\phi\rangle| + |\Lambda\phi| < 1 \quad \text{if } \phi \in V_p. \tag{2.25}$$

Now remark that every $\psi = \psi[\mu, \phi]$, where

$$\psi[\mu, \phi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\phi\|_p)\phi & (\phi \neq 0, \mu > 1) \\ 0 & (\phi = 0, \mu > 1), \end{cases} \tag{2.26}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\psi\rangle| < 1 \tag{2.27}$$
$$|\langle f_n|\phi\rangle| < 2^p\|\,\phi\,\|_p\cdot\mu \tag{2.28}$$
$$|\langle f_n|\phi\rangle\,| \le 2^p\|\,\phi\,\|_p \tag{2.29}$$
$$|\langle f_n|\phi\rangle\,| \le 2^p\{\|\,D^0\phi\,\|_\infty + \cdots + \|\,D^p\phi\,\|_\infty\}. \tag{2.30}$$

Finally, with the help of (2.1),

$$|\langle f_n|\phi\rangle\,| \le 2^p(p+1)\|\,D^p\phi\,\|_\infty. \tag{2.31}$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\phi = \lim\limits_{n\to\infty}\int_{-1}^{1} f_n(t)\phi(t)dt$ exists for every $\phi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\phi\rangle| \le M\|\,D^p\phi\,\|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let ϕ keep running on $\mathscr{D}_K$ and so define

$$B_n(\phi) \triangleq \max\{|\phi(x)\,| : x \in [-1/n, 1/n]\}, \tag{2.32}$$
$$\Delta_n(\phi) \triangleq \max\{|\phi(x) - \phi(0)\,| : x \in [-1/n, 1/n]\}. \tag{2.33}$$

The mean value asserts that

$$|\phi(1/n) - \phi(-1/n)\,| \le B_n(\phi')\,|1/n - (-1/n)\,| = \frac{2}{n}B_n(\phi'). \tag{2.34}$$

Independently, an integration by parts shows that

$$\langle f_n|\phi\rangle = \left[\frac{n^3t^2}{2}\phi(t)\right]_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)dt \tag{2.35}$$
$$= \frac{n}{2}\left(\phi(1/n) - \phi(-1/n)\right) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)dt. \tag{2.36}$$

Combine (2.34) with (2.36) and so obtain

$$|\langle f_n|\phi\rangle\,| \le \frac{n}{2}\,|\phi(1/n) - \phi(-1/n)\,| + \frac{n^3}{2}\int_{-1/n}^{1/n} t^2|\phi'(t)|dt \tag{2.37}$$
$$\le B_n(\phi') + \frac{n^3}{2}B_n(\phi')\int_{-1/n}^{1/n} t^2dt \tag{2.38}$$
$$\le \frac{4}{3}B_n(\phi') \tag{2.39}$$
$$\le \frac{4}{3}\|\,\phi'\,\|_\infty. \tag{2.40}$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\phi')$ tends to $|\phi'(0)|$, we may expect that f_n tends to $\frac{4}{3}\phi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0) \overset{(2.36)}{=} \frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n}(\phi' - \phi'(0))t^2 dt. \tag{2.41}$$

So,

$$\left|\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0)\right| \leq \left|\frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0)\right| + \frac{1}{3}\Delta_n(\phi') \underset{n\to\infty}{\longrightarrow} 0. \tag{2.42}$$

We have just proved that

$$\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\phi'(0) \quad (\phi \in \mathscr{D}_K). \tag{2.43}$$

In other words,

$$\langle f_n| \underset{n\to\infty}{\longrightarrow} -\frac{4}{3}\delta', \tag{2.44}$$

where δ is the *Dirac measure* and $\delta', \delta'', \ldots$, its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $-\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$|\langle f_n|\phi\rangle| \leq \frac{4}{3}\| \phi' \|_\infty \quad (n = 1, 2, 3, \ldots) \tag{2.45}$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n|\phi_{\rho(n)}\rangle : \| \phi_{\rho(n)} \|_\infty = 1; n = 1, 2, 3, \ldots\} \tag{2.46}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle f_n|\phi\rangle| \leq M\| \phi \|_\infty \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.47}$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*[4] ${f_n}'$, as follows

$$\begin{aligned} {f_n}' : \mathscr{D}_K &\to \mathbf{C} \\ \phi &\mapsto -\langle f_n|\phi'\rangle . \end{aligned} \tag{2.48}$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$\begin{aligned} D : \mathscr{D}_K &\to \mathscr{D}_K \\ \phi &\mapsto \phi'; \end{aligned} \tag{2.49}$$

[4] See 6.1 of [3] for a further discussion.

see Exercise 1.17. ${f_n}'$ is therefore continuous. Now apply (2.43) with ϕ' and so obtain

$$-\langle f_n|\phi'\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\phi''(0) \quad (\phi \in \mathscr{D}_K),$$

i.e.

$$ {f_n}' \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\delta''. \tag{2.50}$$

It follows from (2.40) that,

$$|\langle f_n|\phi'\rangle| \leq \frac{4}{3}\| \phi'' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.51}$$

It is therefore possible to uniformly bound ${f_n}'$ with respect to a norm $\| D^p \cdot \|_\infty$, namely $\| D^2 \cdot \|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle f_n|\phi'\rangle| \leq M\| \phi' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.52}$$

Define

$$\Phi_j(x) = \int_{-1}^{x} \phi_j. \tag{2.53}$$

The oddness of ϕ_j forces Φ_j to vanish outside [-1, 1]: ϕ_j is therefore in $\mathscr{D}_K$. So, under our assumption,

$$|\langle f_n|\Phi_j'\rangle| \leq M\| \Phi_j' \|_\infty \quad (n = 1, 2, 3, \dots); \tag{2.54}$$

which is

$$|\langle f_n|\phi_j\rangle| \leq M \quad (n = 1, 2, 3, \dots). \tag{2.55}$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n|\phi_{\rho(n)}\rangle\}$) and so conclude that there is no constant M such that

$$|\langle|f_n\phi'\rangle| \leq M\| \phi' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.56}$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$|\langle f_n|\phi'\rangle| \leq M\| \phi \|_\infty. \tag{2.57}$$

The mean value theorem (see (2.1)) asserts that

$$|\langle f_n|\phi'\rangle| \leq M\| \phi \|_\infty \leq M\| \phi' \|_\infty; \tag{2.58}$$

which is, again, a desired contradiction. So ends the proof.

$\square$

2.2 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ *of a function* $f \in L^2(T)$ *(*T *is the unit circle) by*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

for all $n \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f$ *exists*$\}$ *is a dense subspace of* $L^2(T)$ *of the first category.*

Proof. Let $f(\theta)$ stand for $f(e^{i\theta})$, so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta. \tag{2.59}$$

We believe it is customary to write

$$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n). \tag{2.60}$$

Moreover, a well known (and easy to prove) result is

$$(e_n, e_{n'}) = [n = n'], \textit{i.e.}\ \{e_n : n \in \mathbf{Z}\} \text{ is an orthormal subset of } L^2(T). \tag{2.61}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

$$\| \Lambda_n \|^2 \overset{(2.60)}{=} \| e_{-n} + \cdots + e_n \|^2 \overset{(2.61)}{=} \| e_{-n} \|^2 + \cdots + \| e_n \|^2 \overset{(2.61)}{=} 2n + 1. \tag{2.62}$$

We now assume, to reach a contradiction, that

$$B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \ldots\} < \infty\} \tag{2.63}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$\| \Lambda_n \| \overset{(2.62)}{=} \sqrt{2n+1} \underset{n\to\infty}{\longrightarrow} \infty. \tag{2.64}$$

We have just established that B is actually of the first category; and so is its subset L=$\{f \in L^2(T) : \lim_{n\longrightarrow\infty} \Lambda_n f \text{ exists}\}$. We now prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be span$\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$P \subset L \subset L^2(T). \tag{2.65}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.65), that

$$L^2(T) = \overline{P} = \overline{L}. \tag{2.66}$$

So ends the proof $\square$

2.3 Exercise 9. Boundedness without closedness

Suppose X, Y, Z *are Banach spaces and*

$$B : X \times Y \to Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ *such that*

$$\| B(x, y \| \leq M \| x \| \| y \| \quad (x \in X, y \in Y).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Let (x, y) range over $X \times Y$. B is continous at the origin; thus, there exists a positive r such that

$$\| B(x, y) \| < 1 \quad (\max\{\| x \|, \| y \|\} < r). \tag{2.67}$$

Given (x, y), we choose two scalars α, β such that $r > \alpha^{-1} \| x \|$ and $r > \beta^{-1} \| y \|$. Thus,

$$\begin{aligned} \| B(x, y) \| &= \alpha\beta \| B\left(\alpha^{-1}x, \beta^{-1}y\right) \| & (2.68)\\ &< \alpha\beta. & (2.69) \end{aligned}$$

Since the latter inequality holds for all $\alpha > \| x \|/r$ and all $\beta > \| y \|/r$ we conclude that

$$B(x, y) \leq r^{-2} \| x \| \| y \|. \tag{2.70}$$

So ends the proof. □

As a concrete example, choose $X = Y = Z = C_c(\mathbf{R})$, topologized by the supremum norm. $C_c(\mathbf{R})$ is not complete[5], nevertheless the bilinear product

$$B(f, g) = f \times g \quad ((f, g) \in C_c(\mathbf{R})^2) \tag{2.71}$$

is bounded, (since $\| B(f, g) \|_\infty = \| f \|_\infty \| g \|_\infty$) and continuous. To see that, pick (u, v) in $C_c(\mathbf{R})^2$: Given any positive scalar ε, there exists another positive scalar r such that $r(r + \| u \| + \| v \|) < \varepsilon$. So, under the following assumption

$$\max\{\| f - u \|_\infty, \| g - v \|_\infty\} < r, \tag{2.72}$$

we reach

$$\begin{aligned} \| fg - uv \|_\infty &\leq \| f - u \|_\infty \cdot \| g \|_\infty + \| u \|_\infty \cdot \| g - v \|_\infty & (2.73)\\ &< r(r + \| v \|) + \| u \| r & (2.74)\\ &< r(r + \| u \| + \| v \|) & (2.75)\\ &< \varepsilon. & (2.76) \end{aligned}$$

[5] See 5.4.4 [4]

2.4 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let $(\mathrm{X}_1, \mathrm{X}_2, \mathrm{Z})$ be topological spaces and B a bilinear mapping

$$\mathrm{B} : \mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{Z} \tag{2.77}$$

From now on, $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)$ denotes an arbitrary element of $\mathrm{X}_1 \times \mathrm{X}_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $\mathrm{X}_1 \times \mathrm{X}_2$, *i.e.* given an arbitrary balanced open subset W of Z, there exists in X_i $(\mathrm{i} = 1,2)$ a balanced open subset U_i such that

$$\mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) \subset \mathrm{W}. \tag{2.78}$$

Let $\nu_\mathrm{i}(\mathrm{x})$ denote any scalar that is greater than $\mu_\mathrm{i}(\mathrm{x}_\mathrm{i}) = \inf\{\mathrm{r} > 0 : \mathrm{x}_\mathrm{i} \in \mathrm{r} \cdot \mathrm{U}_\mathrm{i}\}$. So,

$$\begin{aligned} \mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) &= \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{B}\left(\nu_1(\mathrm{x})^{-1}\mathrm{x}_1, \nu_2(\mathrm{x})^{-1}\mathrm{x}_2\right) && (2.79)\\ &\in \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) && (2.80)\\ &\subset \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{W}. && (2.81) \end{aligned}$$

Now pick $\mathrm{p} = (\mathrm{p}_1, \mathrm{p}_2)$ in $\mathrm{X}_1 \times \mathrm{X}_2$: It directly follows from (2.81) that

$$\begin{aligned} \mathrm{B}(\mathrm{p}_1, \mathrm{p}_2) - \mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) &= \mathrm{B}(\mathrm{p}_1, \mathrm{p}_2 - \mathrm{x}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{x}_1, \mathrm{x}_2 - \mathrm{p}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{x}_1, \mathrm{p}_2) && (2.82)\\ &\in \nu_1(\mathrm{p})\nu_2(\mathrm{p} - \mathrm{x}) \cdot \mathrm{W} + \nu_1(\mathrm{p} - \mathrm{x})\nu_2(\mathrm{x} - \mathrm{p}) \cdot \mathrm{W} + \nu_1(\mathrm{p} - \mathrm{x})\nu_2(\mathrm{p}) \cdot \mathrm{W}. && (2.83) \end{aligned}$$

Let us henceforth assume that

$$\mathrm{p}_\mathrm{i} - \mathrm{x}_\mathrm{i} \in [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 2]^{-1} \cdot \mathrm{U}_\mathrm{i}; \tag{2.84}$$

which yields

$$\mu_\mathrm{i}(\mathrm{p}_\mathrm{i} - \mathrm{x}_\mathrm{i}) \leq [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 2]^{-1}. \tag{2.85}$$

Finally, combine the special case

$$\begin{aligned} \nu_\mathrm{i}(\mathrm{p} - \mathrm{x}) &= [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 1]^{-1}, && (2.86)\\ \nu_\mathrm{i}(\mathrm{p}) &= \ \mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 1 && (2.87) \end{aligned}$$

with (2.83) and so obtain

$$\mathrm{B}(\mathrm{p}_1, \mathrm{p}_2) - \mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{W} + \mathrm{W} + \mathrm{W}. \tag{2.88}$$

W being arbitrary, we have so established the continuousness of B at $(\mathrm{p}_1, \mathrm{p}_2)$. Since $(\mathrm{p}_1, \mathrm{p}_2)$ is also arbitrary, the proof is complete. □

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