

Solutions to some exercises from Walter Rudin's
Functional Analysis

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July 26, 2020

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Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizable & number theory

Let be X the vector space of all complex functions on the unit interval $[0, 1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \rightarrow \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \rightarrow \infty$ then $\{\gamma_n f_n\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0, 1]$.) This shows that metrizable cannot be omitted in (b) of Theorem 1.28.

Proof. Our justification consists in proving that τ -convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms p_x is separating. By [1.37], the collection \mathcal{B} of all finite intersections of the sets

$$V^{(x,k)} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k \in \mathbf{N}) \quad (1.1)$$

is then a local base for a topology τ on X . Given $\{f_n : n = 1, 2, 3, \dots\}$, we set

$$\text{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \quad (1.2)$$

with the convention $\text{off}(U) = \infty$ whether the sum has no finite support. So,

$$\sum_{i=1}^m \text{off}(U^{(i)}) = \sum_{n=1}^{\infty} \sum_{i=1}^m [f_n \notin U^{(i)}] \geq \text{off}(U^{(1)} \cap \dots \cap U^{(m)}) \quad (1.3)$$

We first assume that $\{f_n\}$ τ -converges to some f in X , *i.e.*

$$\text{off}(f + V) < \infty \quad (V \in \mathcal{B}). \quad (1.4)$$

The special cases $V = V^{(x,k)}$ mean the pointwise convergence of $\{f_n\}$. Conversely, assume that $\{f_n\}$ does not τ -converges to any g in X , *i.e.*

$$\forall g \in X, \exists V^{(g)} \in \mathcal{B} : \text{off}(g + V^{(g)}) = \infty. \quad (1.5)$$

Given g , $V^{(g)}$ is then an intersection $V^{(x^{(1)}, k^{(1)})} \cap \dots \cap V^{(x^{(m)}, k^{(m)})}$. Thus

$$\sum_{i=1}^m \text{off}(g + V^{(x^{(i)}, k^{(i)})}) \stackrel{(1.3)}{\geq} \text{off}(g + V^{(g)}) \stackrel{(1.5)}{=} \infty. \quad (1.6)$$

One of the sum $\text{off}(g + V^{(x^{(i)}, k^{(i)})})$ must then be ∞ . This implies that convergence of f_n to g fails at point x_i . g being arbitrary, we so conclude that f_n does not converge pointwise. We have just proved that τ -convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on, k , n and p run on \mathbf{N}_+ . Let $\text{dyadic}(x)$ be the usual dyadic expansion of a real number x , so that $\text{dyadic}(x)$ is an aperiodic binary sequence **iff** x is irrational. Define

$$f_n(x) \triangleq \begin{cases} 2^{-\sum_{k=1}^n \text{dyadic}(x)_k} & (x \in [0, 1] \setminus \mathbf{Q}) \\ 0 & (x \in [0, 1] \cap \mathbf{Q}) \end{cases} \quad (1.7)$$

so that $f_n(x) \xrightarrow{n \rightarrow \infty} 0$ and take scalars γ_n such that $\xrightarrow{n \rightarrow \infty} \infty$, *i.e.* at fixed p , γ_n is greater than 2^p for almost all n . Next, choose $n^{(p)}$ among those *almost all* n that are large enough to satisfy

$$n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)} \quad (1.8)$$

(start with $n^{(-1)} = n^{(0)} = 0$) and so obtain

$$2^p < \gamma_{n^{(p)}} : 0 < n^{(p)} - n^{(p-1)} \xrightarrow{p \rightarrow \infty} \infty. \quad (1.9)$$

The indicator χ of $\{n^{(1)}, n^{(2)}, \dots\}$ is then aperiodic, *i.e.*

$$x^{(\gamma)} \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \quad (1.10)$$

is irrational. Consequently,

$$\text{dyadic}(x^{(\gamma)})_{-k} = \chi_k. \quad (1.11)$$

We now easily see that

$$\chi_1 + \dots + \chi_{n^{(p)}} = p, \quad (1.12)$$

which, combined with (1.7), yields

$$f_{n^{(p)}}(x^{(\gamma)}) = 2^{-p}. \quad (1.13)$$

Finally,

$$\gamma_{n^{(p)}} f_{n^{(p)}}(x^{(\gamma)}) > 1. \quad (1.14)$$

We have so established that the subsequence $\{\gamma_{n^{(p)}} f_{n^{(p)}}\}$ does not tend pointwise to 0, hence neither does the whole sequence $\{\gamma_n f_n\}$. In other words, (b) holds, which is in violent contrast with [1.28]: X is then not metrizable. So ends the proof. \square

1.2 Exercise 9. Quotient map

Suppose

- (a) X and Y are topological vector spaces,
- (b) $\Lambda : X \rightarrow Y$ is linear.
- (c) N is a closed subspace of X ,
- (d) $\pi : X \rightarrow X/N$ is the quotient map, and
- (e) $\Lambda x = 0$ for every $x \in N$.

Prove that there is a unique $f : X/N \rightarrow Y$ which satisfies $\Lambda = f \circ \pi$, that is, $\Lambda x = f(\pi(x))$ for all $x \in X$. Prove that f is linear and that Λ is continuous if and only if f is continuous. Also, Λ is open if and only if f is open.

Proof. The equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \quad (1.15)$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x. \quad (1.16)$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ . Now remark that

$$\pi x = N \stackrel{(f \text{ linear})}{\Rightarrow} f(\pi x) = 0 \stackrel{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow \pi x = N \quad (1.17)$$

and so conclude that f is also one-to-one. Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by (a) of [1.41]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subset V. \quad (1.18)$$

Since π is open (see (a) of [1.41]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$. This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by (a) of [1.41]. Conversely, let

$$W \triangleq \pi(V) \subset X/N \quad (V \text{ neighborhood of } 0_X) \quad (1.19)$$

range over all neighborhoods of N , as Λ is kept open: So is

$$\Lambda(V) = f(\pi(V)) = f(W). \quad (1.20)$$

The linear mapping f is then open. □

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \rightarrow Y$ is linear, and $\Lambda(X) = Y$.

(a) Prove that Λ is an open mapping.

(b) Assume, in addition, that the null space of Λ is closed, and prove that Λ is continuous.

Proof. (a) Let e range over a base of Y : For each e , there exists x_e in X such that $\Lambda(x_e) = e$, since Λ is onto. So,

$$y = \sum_e y_e \Lambda x_e \quad (y \in Y). \quad (1.21)$$

The sequence $\{x_e\}$ is finite hence bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$x_e \in sV \quad (1.22)$$

for all x_e . Combining this with (1.21) shows that

$$y \in \sum_e \Lambda(V) \quad (y \in Y : |y_e| < s^{-1}). \quad (1.23)$$

(b) Since N is closed, π continuously maps X onto X/N , another topological (Hausdorff) vector space, see [1.41]. Now take f as in Exercise 9: Since Λ is onto, f is an isomorphism of X/N onto Y . X/N has then dimension $\dim Y$. The inverse isomorphism f^{-1} is then a homeomorphism of $Y \cong \mathbf{C}^{\dim Y}$ onto X/N ; see [1.21]. Consequently, both f and f^{-1} are continuous: So is $\Lambda = f \circ \pi$. □

1.4 Exercise 14.

Put $K = [0, 1]$ and define \mathcal{D}_K as in Section 1.4.6. Show that the following three families of seminorms (where $n = 0, 1, 2, \dots$) define the same topology on \mathcal{D}_K . If $D = d/dx$:

$$(a) \|D^n f\|_\infty = \sup\{|D^n f(x)| : 0 < x < 1\}$$

$$(b) \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$(c) \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

Proof. First, remark that

$$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty \quad (1.24)$$

(the inequality on the left is a Cauchy-Schwarz one), since K has length 1. Next, start from

$$D^n f(x) = \int_{-\infty}^x D^{n+1} f(t) dt \quad (1.25)$$

(which is true, since f has a bounded support) to obtain

$$|D^n f(x)| \leq \int_{-\infty}^x |D^{n+1} f(t)| dt \leq \|D^{n+1} f\|_1 \quad (1.26)$$

hence

$$\|D^n f\|_\infty \leq \|D^{n+1} f\|_1. \quad (1.27)$$

Combining (1.24) with (1.27) yields

$$\|Df\|_1 \leq \dots \leq \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots \quad (1.28)$$

We now define

$$\mathcal{B}^{(i)} \triangleq \{V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 1/n\} : n = 1, 2, 3, \dots\} \quad (i = 1, 2, \infty), \quad (1.29)$$

so that (1.28) is mirrored in terms of neighborhood inclusions, as follows,

$$V_1^{(1)} \supset \dots \supset V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \dots \quad (1.30)$$

Since $V_n^{(i)} \supset V_{n+1}^{(i)}$, \mathcal{B}_i is the local base of a topology τ_i . But the chain (1.30) forces the τ_i 's to be equals. To see that, choose a set S that is τ_1 -open at, say a : So, $V_n^{(1)} \subset S - a$ for some n . Now $V_n^{(1)} \supset V_n^{(2)}$ (see (1.30)) forces $V_n^{(2)} \subset S - a$, which implies that S is τ_2 -open at a . Similarly, we deduce, still from (1.30), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \quad (1.31)$$

So ends the proof. \square

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $|f|_N = \max |f|$, then, eventually, only on the ambient space itself. This should then be regarded as a very part of the textbook [2] The proof consists in combining trivial consequences of the local base definition with the well-known result (e.g. [2.6] in [1]) about intersection of nonempty compact sets.

Lemma Let X be a topological space with a countable local base $\{V_N : N = 1, 2, 3, \dots\}$. If $\tilde{V}_N = V_1 \cap \dots \cap V_N$, then every subsequence $\{\tilde{V}_{\rho(N)}\}$ is also a decreasing (i.e. $\tilde{V}_{\rho(N)} \supset \tilde{V}_{\rho(N+1)}$) local base of X .

Proof. The decreasing property is trivial. Now remark that $V_N \supset \tilde{V}_N \supset \tilde{V}_{N+1}$: The left inclusion shows that $\{\tilde{V}_N\}$ is a local base of X . Then so is $\{\tilde{V}_{\rho(N)}\}$, since $\tilde{V}_N \supset \tilde{V}_{\rho(N)}$. \square

The following special case $V_N = \tilde{V}_N$ is one of the key ingredients:

Corollary 1 (special case) With the same notations, if $\{V_N\}$ is a decreasing local base, then so is $\{V_{\rho(N)}\}$.

Corollary 2 If $\{Q_N\}$ is a sequence of compacts that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{\rho(N)}\}$ also satisfies these conditions. Furthermore, if τ_Q is the $C(\Omega)$'s (respectively $C^\infty(\Omega)$) topology of the seminorms p_N , as defined in section 1.44 (respectively 1.46), then the seminorms $p_{\rho(N)}$ define the same topology τ_Q .

Proof. Let X be $C(\Omega)$ topologized with the seminorms p_N (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_N = \{p_N < 1/N\}$, then $\{V_N\}$ is a decreasing local base of X . Moreover,

$$Q_{\rho(N)} \subset \overset{\circ}{Q}_{\rho(N)+1} \subset Q_{\rho(N)+1} \subset Q_{\rho(N+1)}, \quad (1.32)$$

and this yields

$$Q_{\rho(N)} \subset \overset{\circ}{Q}_{\rho(N+1)}. \quad (1.33)$$

In other words, $Q_{\rho(N)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(N)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(N)}\}$ is a local base. So, $\tau_{Q_\rho} \subset \tau_Q$. Conversely, the above corollary turns $\{V_{\rho(N)}\}$ into a local base of τ_Q . Hence $\tau_Q \subset \tau_{Q_\rho}$. \square

We are now in a fair position to establish the following:

Theorem The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.46.

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\kappa}$, for every subsequence $\{K_{\kappa(n)}\}$. Similarly, let $\{L_n\}$ be a sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\lambda}$ for every subsequence $\{L_{\lambda(n)}\}$. The following definition

$$C_{i,j} \triangleq K_i \setminus \overset{\circ}{L}_j \quad (i, j = 1, 2, 3, \dots) \quad (1.34)$$

shapes $\{C_{i,j} : j = 1, 2, 3, \dots\}$ as a decreasing sequence of compacts. We now suppose (to reach a contradiction) that no $C_{i,j}$ is empty and so conclude that $C_{i,1} \cap C_{i,2} \cap \dots$ contains a point that is not in any $\text{int}(L_j)$. But the conditions specified in [1.44] force $\{\text{int}(L_j) : j = 1, 2, 3, \dots\}$ to be an open cover. This contradiction reveals that $C_{i,j} = C_{i,j+1} = C_{i,j+2} = \dots = \emptyset$ for some $j = j^{(i)}$. We then define $\lambda_i = i + j^{(i)}$, so that

$$K_i \setminus \overset{\circ}{L}_{\lambda_i} = \emptyset, \quad i.e. \quad K_i \subset \overset{\circ}{L}_{\lambda_i}. \quad (1.35)$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\kappa_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\kappa_j} \quad (1.36)$$

Combine (1.35) with (1.36) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\lambda_1} \subset L_{\lambda_1} \subset \overset{\circ}{K}_{\kappa \circ \lambda_1} \subset K_{\kappa \circ \lambda_1} \subset \overset{\circ}{L}_{\lambda_{\kappa \circ \lambda_1}} \subset \dots \quad (1.37)$$

Thus the sequence $Q = (K_1, L_{\lambda_1}, K_{\kappa \circ \lambda_1}, L_{\lambda_{\kappa \circ \lambda_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\kappa} = \tau_Q = \tau_{L_\lambda} = \tau_L. \quad (1.38)$$

So ends the proof □

Bibliography

- [1] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, 1986.
- [2] Walter Rudin. *Functional Analysis*. McGraw-Hill, 1991.