

Solutions to some exercises from Walter Rudin's
Functional Analysis

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Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizable & number theory

Let be X the vector space of all complex functions on the unit interval $[0, 1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \rightarrow \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \rightarrow \infty$ then $\{\gamma_n f_n\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0, 1]$.) This shows that metrizable cannot be omitted in (b) of Theorem 1.28.

Proof. Our justification consists in proving that τ -convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms p_x is separating. By [1.37], the collection \mathcal{B} of all finite intersections of the sets

$$V_{x,k} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k \in \mathbf{N}) \quad (1.1)$$

is therefore a local base for a topology τ on X . Given $\{f_n : n = 1, 2, 3, \dots\}$, we put

$$\text{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \quad (1.2)$$

with the convention “ $\Sigma = \infty$ ” whether the sum has no finite support. So,

$$\sum_{i=1}^m \text{off}(U_i) = \sum_{n=1}^{\infty} \sum_{i=1}^m [f_n \notin U_i] \geq \text{off}(U_1 \cap \dots \cap U_m). \quad (1.3)$$

We first assume that $\{f_n\}$ τ -converges to some f in X , *i.e.*

$$\text{off}(f + V) < \infty \quad (V \in \mathcal{B}). \quad (1.4)$$

The special cases $V_{x,1}, V_{x,2}, \dots$, mean the pointwise convergence $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$. Conversely, assume that $\{f_n\}$ does not τ -converges to any g in X , *i.e.*

$$\forall g \in X, \exists W \in \mathcal{B} : \text{off}(g + W) = \infty. \quad (1.5)$$

Given g , such W is, by definition, a finite intersection $V_{x_1,k_1} \cap \dots \cap V_{x_m,k_m}$. Thus,

$$\sum_{i=1}^m \text{off}(g + V_{x_i,k_i}) \stackrel{(1.3)}{\geq} \text{off}(g + W) \stackrel{(1.5)}{=} \infty. \quad (1.6)$$

One of the sum $\text{off}(g + V_{x_i,k_i})$ must then be ∞ . In other words, there exists a point x_i for which $\{f_n(x_i)\}$ does not converge to $g(x_i)$. g being arbitrary, we so conclude that f_n does not converge pointwise. We have just proved that τ -convergence is a rewording of pointwise convergence. We now prove the second part. From now on, we let k, n and p run on \mathbf{N}_+ , as $\text{dyadic}(x)$ denotes the usual dyadic expansion of x , so that $\text{dyadic}(x)$ is an aperiodic binary sequence **iff** x is irrational. Define

$$f_n(x) \triangleq \begin{cases} \exp_2(-\sum_{k=1}^n \text{dyadic}(x)_{-k}) & (x \in [0, 1] \setminus \mathbf{Q}) \\ 0 & (x \in [0, 1] \cap \mathbf{Q}), \end{cases} \quad (1.7)$$

so that $f_n(x) \xrightarrow{n \rightarrow \infty} 0$, and take $\gamma_n \xrightarrow{n \rightarrow \infty} \infty$, *i.e.* at fixed p , γ_n is greater than 2^p for almost all n . Next, choose n_p among those *almost all* n that are large enough to satisfy

$$n_{p-1} - n_{p-2} < n_p - n_{p-1} \quad (1.8)$$

(start with $n_{-1} = n_0 = 0$) and so obtain

$$2^p < \gamma_{n_p} : 0 < n_p - n_{p-1} \xrightarrow{p \rightarrow \infty} \infty. \quad (1.9)$$

The indicator χ of $\{n_1, n_2, \dots\}$ in \mathbf{Z} is then aperiodic, *i.e.*

$$\alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \in [0, 1] \setminus \mathbf{Q}. \quad (1.10)$$

Hence, χ is not a the infinite-support expansion of a rational number; which forces

$$\text{dyadic}(\alpha_\gamma)_{-k} = \chi_k. \quad (1.11)$$

The key ingredient is that

$$\chi_1 + \dots + \chi_{n_p} = p. \quad (1.12)$$

Combined with (1.7), it yields

$$f_{n_p}(\alpha_\gamma) = 2^{-p}. \quad (1.13)$$

Finally,

$$\gamma_{n_p} f_{n_p}(\alpha_\gamma) > 1. \quad (1.14)$$

There so exists $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. In other words, (b) holds, which is in violent contrast with 1.28 of [3]: X is therefore not metrizable. So ends the proof. \square

1.2 Exercise 9. Quotient map

Suppose

- (a) X and Y are topological vector spaces,
- (b) $\Lambda : X \rightarrow Y$ is linear.
- (c) N is a closed subspace of X ,
- (d) $\pi : X \rightarrow X/N$ is the quotient map, and
- (e) $\Lambda x = 0$ for every $x \in N$.

Prove that there is a unique $f : X/N \rightarrow Y$ which satisfies $\Lambda = f \circ \pi$, that is, $\Lambda x = f(\pi(x))$ for all $x \in X$. Prove that f is linear and that Λ is continuous if and only if f is continuous. Also, Λ is open if and only if f is open.

Proof. Bear in mind that π continuously maps X onto the topological (Hausdorff) space X/N , since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \quad (1.15)$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x. \quad (1.16)$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ .

Remark. The special case $N = \{\Lambda = 0\}$, i.e. $\Lambda x = 0$ iff $x \in N$ (cf. (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$f(\pi x) = 0 \stackrel{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \quad (1.17)$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by (a) of [1.41]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subset V. \quad (1.18)$$

Since π is open (see (a) of [1.41]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$. This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by (a) of [1.41]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \quad (1.19)$$

for some neighborhood V of 0_X . So,

$$f(W) = f(\pi(V)) = \Lambda(V). \quad (1.20)$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y . So ends the proof. \square

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \rightarrow Y$ is linear, and $\Lambda(X) = Y$.

(a) Prove that Λ is an open mapping.

(b) Assume, in addition, that the null space of Λ is closed, and prove that Λ is continuous.

Proof. We discard the trivial case $\dim Y = 0$ then henceforth assume that $\dim Y$ has positive dimension n .

Let e range over a base of Y : For each e , there exists x_e in X such that $\Lambda(x_e) = e$, since Λ is onto. So,

$$y = \sum_e y_e \Lambda x_e \quad (y \in Y). \quad (1.21)$$

The sequence $\{x_e\}$ is finite; therefore it is bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$x_e \in sV \text{ for all } x_e. \quad (1.22)$$

Combining this with (1.21) shows that

$$y \in \sum_e \Lambda(V) \quad (y \in Y : |y_e| < s^{-1}), \quad (1.23)$$

which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, with $\{\Lambda = 0\}$ playing the role of N . Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y . Consequently,

$$\dim X/N = n. \quad (1.24)$$

f is then an homeomorphism of $X/N \equiv \mathbf{C}^n$ onto Y ; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = f \circ \pi$. \square

1.4 Exercise 14. \mathcal{D}_K equipped with other seminorms

Put $K = [0, 1]$ and define \mathcal{D}_K as in Section 1.46. Show that the following three families of seminorms (where $n = 0, 1, 2, \dots$) define the same topology on \mathcal{D}_K . If $D = d/dx$:

$$(a) \|D^n f\|_\infty = \sup\{|D^n f(x)| : 0 < x < 1\}$$

$$(b) \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$(c) \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

Proof. First, remark that

$$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty \quad (1.25)$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, start from

$$D^n f(x) = \int_{-1}^x D^{n+1} f \quad (1.26)$$

(which is true, since f has support K) to obtain

$$|D^n f(x)| \leq \int_{-1}^x |D^{n+1} f| \leq \|D^{n+1} f\|_1, \quad (1.27)$$

hence

$$\|D^n f\|_\infty \leq \|D^{n+1} f\|_1. \quad (1.28)$$

Combining (1.25) with (1.28) yields

$$\|D^0 f\|_1 \leq \dots \leq \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots \quad (1.29)$$

We now put

$$V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 1/n\} \quad (i = 1, 2, \infty) \quad (1.30)$$

$$\mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 1, 2, 3, \dots\}, \quad (1.31)$$

so that (1.29) is mirrored in terms of neighborhood inclusions, as follows,

$$V_1^{(1)} \supset \dots \supset V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \dots \quad (1.32)$$

Since $V_n^{(i)} \supset V_{n+1}^{(i)}$, $\mathcal{B}^{(i)}$ is the local base of a topology τ_i . But the chain (1.32) forces the τ_i to be equals. To see that, choose a set S that is τ_1 -open at, say a , *i.e.* $V_n^{(1)} \subset S - a$ for some n . Next, concatenate this with $V_n^{(2)} \subset V_n^{(1)}$ (see (1.32)) and so obtain $V_n^{(2)} \subset S - a$, which implies that S is τ_2 -open at a . Similarly, we deduce, still from (1.32), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \quad (1.33)$$

So ends the proof. \square

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms p_n , then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (e.g. [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1 Let X be a topological space with a countable local base $\{V_n : n = 1, 2, 3, \dots\}$. If $\tilde{V}_n = V_1 \cap \dots \cap V_n$, then every subsequence $\{\tilde{V}_{\rho(n)}\}$ is a decreasing (i.e. $\tilde{V}_{\rho(n)} \supset \tilde{V}_{\rho(n+1)}$) local base of X .

Proof. The decreasing property is trivial. Now remark that $V_n \supset \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X . Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supset \tilde{V}_{\rho(n)}$. \square

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$) Under the same notations of Lemma 1, if $\{V_n\}$ is a decreasing local base, then so is $\{V_{\rho(n)}\}$.

Corollary 2 If $\{Q_n\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{\rho(n)}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $C(\Omega)$'s (respectively $C^\infty(\Omega)$'s) topology of the seminorms p_n , as defined in section 1.44 (respectively 1.46), then the seminorms $p_{\rho(n)}$ define the same topology τ_Q .

Proof. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X . Moreover,

$$Q_{\rho(n)} \subset \overset{\circ}{Q}_{\rho(n)+1} \subset Q_{\rho(n)+1} \subset Q_{\rho(n+1)}. \quad (1.34)$$

Thus,

$$Q_{\rho(n)} \subset \overset{\circ}{Q}_{\rho(n+1)}. \quad (1.35)$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(n)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(n)}\}$ is a local base. So, $\tau_{Q_\rho} \subset \tau_Q$. Conversely, the above corollary asserts that $\{V_{\rho(n)}\}$ is a local base of τ_Q , which yields $\tau_Q \subset \tau_{Q_\rho}$. \square

Lemma 2 If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in almost all Q_n° , i.e. there exists m such that

$$K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \dots \quad (1.36)$$

Proof. The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \quad (n = 1, 2, 3, \dots) \quad (1.37)$$

shapes $\{C_n\}$ as a decreasing sequence of compact¹ sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude² that the C_n 's intersection contains a point that is not in any $\overset{\circ}{Q}_n$. On the other hand, the conditions specified in [1.44] force the $\overset{\circ}{Q}_n$'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset \overset{\circ}{Q}_m$, for some m . Finally,

$$K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \dots \quad (1.38)$$

□

We are now in a fair position to establish the following:

Theorem The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\kappa}$ for every subsequence $\{L_{\kappa(n)}\}$. Now apply the above Lemma 2 with K_i ($i = 1, 2, 3, \dots$) and so conclude that $K_i \subset \overset{\circ}{L}_{m_i} \subset \overset{\circ}{L}_{m_i+1} \subset \dots$ for some m_i . In particular, the special case $\kappa_i = m_i + i$ is

$$K_i \subset \overset{\circ}{L}_{\kappa_i}. \quad (1.39)$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \quad (1.40)$$

Combine (1.39) with (1.40) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{L}_{\kappa_1} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{K}_{\lambda_{\kappa_1}} \subset \overset{\circ}{L}_{\kappa_{\lambda_{\kappa_1}}} \subset \dots, \quad (1.41)$$

which means that the sequence $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L. \quad (1.42)$$

So ends the proof

□

¹ See (b) of 2.5 of [2].

² The intersection of a decreasing sequence of nonempty Hausdorff compact sets is nonempty. This is a corollary of 2.6 of [2].

1.6 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ is a continuous mapping of $C^\infty(\Omega)$ into $C^\infty(\Omega)$ and also of \mathcal{D}_K into \mathcal{D}_K , for every multi-index α .

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an arbitrary neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{ \phi \in C^\infty(\Omega) : \max\{|D^\beta \phi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}. \quad (1.43)$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}. \quad (1.44)$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}. \quad (1.45)$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad i.e. \quad D^\alpha(V_{N+|\alpha|}) \subset V_N, \quad (1.46)$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$, we first remark that the collection of the $V_N \cap \mathcal{D}_K$ is a local base of the subspace topology of \mathcal{D}_K . $V_{N+|\alpha|} \cap \mathcal{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^\alpha|_{\mathcal{D}_K}(V_{N+|\alpha|} \cap \mathcal{D}_K) = D^\alpha(V_{N+|\alpha|} \cap \mathcal{D}_K) \quad (1.47)$$

$$\subset D^\alpha(V_{N+|\alpha|}) \cap D^\alpha(\mathcal{D}_K) \quad (1.48)$$

$$\subset V_N \cap \mathcal{D}_K \quad (\text{see (1.46)}) \quad (1.49)$$

So ends the proof. \square

Chapter 2

Completeness

2.1 Exercise 3. An equicontinuous sequence of measures

Put $K = [-1, 1]$; define \mathcal{D}_K as in section 1.46 (with \mathbf{R} in place of \mathbf{R}^n). Suppose $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\Lambda\phi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\phi(t)dt$ exists for every $\phi \in \mathcal{D}_K$. Show that Λ is a continuous linear functional on \mathcal{D}_K . Show that there is a positive integer p and a number $M < \infty$ such that

$$\left| \int_{-1}^1 f_n(t)\phi(t)dt \right| \leq M \|D^p \phi\|_\infty$$

for all n . For example, if $f_n(t) = n^3 t$ on $[-1/n, 1/n]$ and 0 elsewhere, show that this can be done with $p = 1$. Construct an example where it can be done with $p = 2$ but not with $p = 1$.

We will also consider the case $p = 0$. The following version of the mean value theorem will be of a great deal of help.

Lemma If $\phi \in \mathcal{D}_{[a,b]}$, then

$$\|D^\alpha \phi\|_\infty \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (\alpha = 0, 1, \dots, p) \quad (2.1)$$

at every order $p = 0, 1, 2, \dots$; where λ is the length $|b - a|$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 ($a < x_1 < x_0$), such that

$$\phi(x_0) = \phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a). \quad (2.2)$$

Since every $D^p \phi$ lies in $\mathcal{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \dots < x_p < \dots < x_0$ such that

$$\phi(x_0) = D^0 \phi(x_0) \quad (2.3)$$

$$= D^1 \phi(x_1)(x_0 - a) \quad (2.4)$$

$$= \dots$$

$$= D^p \phi(x_p)(x_0 - a) \cdots (x_{p-1} - a), \quad (2.5)$$

for all p . More compactly,

$$D^\alpha \phi(x_0) = D^p \phi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a); \quad (2.6)$$

which yields,

$$|D^\alpha \phi(x)| \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c]) \quad (2.7)$$

The case $x_0 \geq c$ outputs a “reversed” result, with $b > \dots > x_p > \dots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^\alpha \phi(x)| \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [c, b]). \quad (2.8)$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\|D^\alpha \phi\|_\infty \leq \|D^p \phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}. \quad (2.9)$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u , we put

$$\langle u | \phi \rangle \triangleq \int_{\mathbf{R}} u \phi \quad (\phi \in C_0(\mathbf{R})). \quad (2.10)$$

The following inequalities

$$|\langle u | \phi \rangle| \leq \int_{\mathbf{R}} |u \phi| \leq \|u\|_{L^1} \quad (\|\phi\|_\infty \leq 1) \quad (2.11)$$

imply that every linear functional

$$\begin{aligned} \langle u | : C_0(\mathbf{R}) &\rightarrow \mathbf{C} \\ \phi &\mapsto \langle u | \phi \rangle \end{aligned} \quad (2.12)$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u |$, since u is determined (a.e) by the integrals $\langle u | \phi \rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*¹ operator $\langle u |$, of norm

$$\sup\{|\langle u | \phi \rangle| : \|\phi\|_\infty \leq 1\} = \|u\|_{L^1}. \quad (2.13)$$

Note that, in the latter equality, $\leq \|u\|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem². We now consider the special cases $u = g_n$, where g_n is

$$\begin{aligned} g_n : \mathbf{R} &\rightarrow \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) \end{cases} \end{aligned} \quad (2.14)$$

¹ see 1.32, 4.1 of [3]

² See 7.26 of [1].

First, remark that $g_n(x) \xrightarrow{n \rightarrow \infty} 0$ ($x \in \mathbf{R}$), as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \rightarrow \infty$). Nevertheless, we will show that the $\langle g_n |$ converge pointwise³ on \mathcal{D}_K *i.e.* there exists a τ_K -continuous linear form Λ such that

$$\langle g_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \phi, \quad (2.15)$$

where ϕ ranges over \mathcal{D}_K . We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\phi_1^+, \dots, \phi_j^+, \dots$, from $C_K^\infty(\mathbf{R})$. More specifically,

$$(i) \quad \phi_j^+ = 1 \text{ on } [e^{-j}, 1 - e^{-j}];$$

$$(ii) \quad \phi_j^+ = 0 \text{ on } \mathbf{R} \setminus [-1, 1];$$

$$(iii) \quad 0 \leq \phi_j^+ \leq 1 \text{ on } \mathbf{R};$$

see [1.46] of [3] for a possible construction of those ϕ_j^+ . Let $\phi_1^-, \dots, \phi_j^-, \dots$, mirror the ϕ_j^+ , in the sense that $\phi_j^-(x) = \phi_j^+(-x)$, so that

$$(iv) \quad \phi_j \triangleq \phi_j^+ - \phi_j^- \text{ is odd, as } g_n \text{ is};$$

$$(v) \quad \text{every } \phi_j \text{ is in } C_K^\infty(\mathbf{R});$$

$$(vi) \quad \text{The sequence } \{\phi_j\} \text{ converges (pointwise) to } 1_{[0,1]} - 1_{[-1,0]}, \text{ and } |\phi_j| \leq 1.$$

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n | \phi_j \rangle = 2 \int_0^1 g_n(t) \phi_j^+(t) dt \xrightarrow{j \rightarrow \infty} 2 \int_0^1 g_n(t) dt = \|g_n\|_{L^1} = n. \quad (2.16)$$

Finally,

$$\|g_n\|_{L^1} \stackrel{(2.13)}{\geq} \sup\{|\langle g_n | \phi \rangle| : \|\phi\| \leq 1\} \stackrel{(2.16)}{\geq} \|g_n\|_{L^1}; \quad (2.17)$$

which is the desired result. So, in terms of boundedness constants: Given n , there exists $C_n < \infty$ such that

$$|\langle g_n | \phi \rangle| \leq C_n \quad (\|\phi\|_\infty \leq 1); \quad (2.18)$$

see (2.11). Furthermore, $\|g_n\|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n ; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n | \phi_{\rho(n)} \rangle\}$ such that $\langle g_n | \phi_{\rho(n)} \rangle$ is greater than, say, \sqrt{n} ; as $\|\phi_{\rho(n)}\|_\infty = 1$. Consequently, there is no bound M such that

$$|\langle g_n | \phi \rangle| \leq M \quad (\|\phi\|_\infty \leq 1; n = 1, 2, 3, \dots). \quad (2.19)$$

In other words, the g_n have no *uniform bound* in L^1 , *i.e.* the collection of all continuous linear mappings $\langle g_n |$ is not equicontinuous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n |$ do not converge pointwise (or “vaguely”, in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \phi \quad (\phi \in C_0(\mathbf{R})) \quad (2.20)$$

³ See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n |$ do not converge pointwise, in a relevant space, to some Λ (see (2.15)).

From now on, unless the contrary is explicitly stated, we assume that ϕ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$\Lambda\phi = \lim_{n \rightarrow \infty} \int_K f_n \phi \quad (\phi \in C_K^\infty(\mathbf{R})). \quad (2.21)$$

for some linear form Λ . Since ϕ vanishes outside K , we can suppose without loss of generality that the support of f_n lies in K . So, (2.21) can be restated as follows,

$$\Lambda\phi = \lim_{n \rightarrow \infty} \langle f_n | \phi \rangle \quad (\phi \in C_K^\infty(\mathbf{R})). \quad (2.22)$$

Let K_1, K_2, \dots , be compact sets that satisfy the conditions specified in 1.44 of [3]. \mathcal{D}_K is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms p_1, p_2, \dots ; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

- (a) $p_N(\phi) = \|\phi\|_N \triangleq \max\{|D^\alpha \phi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\phi \in \mathcal{D}_K$;
- (b) The collection of the sets $V_N = \{\phi \in \mathcal{D}_K : \|\phi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K , the subspace topology of \mathcal{D}_K ; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\phi \in V_m$ then conclude that $\langle f_n |$ is bounded by $\|f_n\|_{L^1}$ on V_m : Every linear functional $\langle f_n |$ is therefore τ_K -continuous; see 1.18 of [3].

To sum it up:

- (i) \mathcal{D}_K , equipped the topology τ_K , is a Fréchet space (see section 1.46 of [3]);
- (ii) Every linear functional $\langle f_n |$ is continuous with respect to this topology;
- (iii) $\langle f_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \Lambda\phi$ for all ϕ , i.e. $\Lambda - \langle f_n | \xrightarrow{n \rightarrow \infty} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n | \}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n | \}$, since addition is continuous. There so exists i, j such that, for all n ,

$$|\Lambda\phi| < 1/2 \quad \text{if } \phi \in V_i, \quad (2.23)$$

$$|\Lambda\phi - \langle f_n | \phi \rangle| < 1/2 \quad \text{if } \phi \in V_j. \quad (2.24)$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n | \phi \rangle| \leq |\Lambda\phi - \langle f_n | \phi \rangle| + |\Lambda\phi| < 1 \quad \text{if } \phi \in V_p. \quad (2.25)$$

Now remark that every $\psi = \psi[\mu, \phi]$, where

$$\psi[\mu, \phi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\phi\|_p) \phi & (\phi \neq 0, \mu > 1) \\ 0 & (\phi = 0, \mu > 1), \end{cases} \quad (2.26)$$

keeps in V_p . Finally, it is clear that each below statement implies the following one.

$$|\langle f_n | \phi \rangle| < 1 \quad (2.27)$$

$$|\langle f_n | \phi \rangle| < 2^p \|\phi\|_p \cdot \mu \quad (2.28)$$

$$|\langle f_n | \phi \rangle| \leq 2^p \|\phi\|_p \quad (2.29)$$

$$|\langle f_n | \phi \rangle| \leq 2^p \{ \|D^0 \phi\|_\infty + \cdots + \|D^p \phi\|_\infty \}. \quad (2.30)$$

Finally, with the help of (2.1),

$$|\langle f_n | \phi \rangle| \leq 2^p (p+1) \|D^p \phi\|_\infty. \quad (2.31)$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3 x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda \phi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t) \phi(t) dt$ exists for every $\phi \in \mathcal{D}_K$;

(b) A *uniform* bound $|\langle f_n | \phi \rangle| \leq M \|D^p \phi\|$ ($n = 1, 2, 3, \dots$) exists for all those f_n , with $p = 1$ as the smallest possible p .

Bear in mind that $K \subset K_m$ and shift the K_N 's indices, so that K_{m+1} becomes K_1 , K_{m+2} becomes K_2 , and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let ϕ keep running on \mathcal{D}_K and so define

$$B_n(\phi) \triangleq \max\{|\phi(x)| : x \in [-1/n, 1/n]\}, \quad (2.32)$$

$$\Delta_n(\phi) \triangleq \max\{|\phi(x) - \phi(0)| : x \in [-1/n, 1/n]\}. \quad (2.33)$$

The mean value asserts that

$$|\phi(1/n) - \phi(-1/n)| \leq B_n(\phi') |1/n - (-1/n)| = \frac{2}{n} B_n(\phi'). \quad (2.34)$$

Independently, an integration by parts shows that

$$\langle f_n | \phi \rangle = \left[\frac{n^3 t^2}{2} \phi(t) \right]_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt \quad (2.35)$$

$$= \frac{n}{2} (\phi(1/n) - \phi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt. \quad (2.36)$$

Combine (2.34) with (2.36) and so obtain

$$|\langle f_n | \phi \rangle| \leq \frac{n}{2} |\phi(1/n) - \phi(-1/n)| + \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\phi'(t)| dt \quad (2.37)$$

$$\leq B_n(\phi') + \frac{n^3}{2} B_n(\phi') \int_{-1/n}^{1/n} t^2 dt \quad (2.38)$$

$$\leq \frac{4}{3} B_n(\phi') \quad (2.39)$$

$$\leq \frac{4}{3} \|\phi'\|_\infty. \quad (2.40)$$

Futhermore, (2.39) gives a hint about the convergence of f_n : Since $B_n(\phi')$ tends to $|\phi'(0)|$, we may expect that f_n tends to $\frac{4}{3}\phi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n | \phi \rangle - \frac{4}{3}\phi'(0) \stackrel{(2.36)}{=} \frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} (\phi' - \phi'(0))t^2 dt. \quad (2.41)$$

So,

$$\left| \langle f_n | \phi \rangle - \frac{4}{3}\phi'(0) \right| \leq \left| \frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0) \right| + \frac{1}{3}\Delta_n(\phi') \xrightarrow{n \rightarrow \infty} 0. \quad (2.42)$$

We have just proved that

$$\langle f_n | \phi \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3}\phi'(0) \quad (\phi \in \mathcal{D}_K). \quad (2.43)$$

In other words,

$$\langle f_n | \xrightarrow{n \rightarrow \infty} -\frac{4}{3}\delta', \quad (2.44)$$

where δ is the *Dirac measure* and δ', δ'', \dots , its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $-\frac{4}{3}\delta'$ is τ_K -continuous. Moreover, we have a bound

$$|\langle f_n | \phi \rangle| \leq M \|D^p \phi\|_\infty \quad (n = 1, 2, 3, \dots), \quad (2.45)$$

with $p = 1$ and $M = \frac{4}{3}$ (which is a constructive version of (2.40)). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n | \phi_{\rho(n)} \rangle : \|\phi_{\rho(n)}\|_\infty = 1; n = 1, 2, 3, \dots\} \quad (2.46)$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle f_n | \phi \rangle| \leq M \|\phi\|_\infty \quad (\phi \in C_K^\infty(\mathbf{R})). \quad (2.47)$$

The previous bound of $\langle f_n |$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*⁴ f_n' , as follows

$$\begin{aligned} f_n' : \mathcal{D}_K &\rightarrow \mathbf{C} \\ \phi &\mapsto -\langle f_n | \phi' \rangle. \end{aligned} \quad (2.48)$$

It has been proved that every $\langle f_n |$ is continuous. So is

$$\begin{aligned} D : \mathcal{D}_K &\rightarrow \mathcal{D}_K \\ \phi &\mapsto \phi'; \end{aligned} \quad (2.49)$$

⁴ See 6.1 of [3] for a further discussion.

see Exercise 1.17. f_n' is therefore continuous. Now apply (2.43) with ϕ' and so obtain

$$|\langle f_n | \phi' \rangle| \xrightarrow{n \rightarrow \infty} \frac{4}{3} \phi''(0) \quad (\phi \in \mathcal{D}_K),$$

i.e.

$$f_n' \xrightarrow{n \rightarrow \infty} \frac{4}{3} \delta''. \quad (2.50)$$

It follows from (2.40) that,

$$|\langle f_n | \phi' \rangle| \leq \frac{4}{3} \|\phi''\|_\infty \quad (n = 1, 2, 3, \dots). \quad (2.51)$$

It is therefore possible to uniformly bound f_n' with respect to a norm $\|D^p \cdot\|_\infty$, namely $\|D^2 \cdot\|$. Then arises a question: Is 2 the smallest p ? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots). \quad (2.52)$$

Define

$$\Phi_j(x) = \int_{-1}^x \phi_j. \quad (2.53)$$

The oddness of ϕ_j forces Φ_j to vanish outside $[-1, 1]$: ϕ_j is therefore in \mathcal{D}_K . So, under our assumption,

$$|\langle f_n | \Phi_j' \rangle| \leq M \|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots); \quad (2.54)$$

which is

$$|\langle f_n | \phi_j \rangle| \leq M \quad (n = 1, 2, 3, \dots). \quad (2.55)$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n | \phi_{\rho(n)} \rangle\}$) and so conclude that there is no constant M such that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots). \quad (2.56)$$

Finally, assume, to reach a contradiction, that there exists a constant M such that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi\|_\infty. \quad (2.57)$$

The mean value theorem (see (2.1)) asserts that

$$|\langle f_n | \phi' \rangle| \leq M \|\phi\|_\infty \leq M \|\phi'\|_\infty; \quad (2.58)$$

which is, again, a desired contradiction. So ends the proof. \square

Bibliography

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