

Solutions to some exercises from Walter Rudin's *Functional
Analysis*

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Notations and Assumptions

I Special terms

i Halmos' iff and definitions

iff is a shorthand for “if and only if”. Splitting **iff** into *if-then* clauses shows that it is a rewording of the logical equivalence \Leftrightarrow . All definitions are understood to be **iff** clauses, which is consistent with the fact that every definition expresses an equivalence.

ii The assignment operator

Given a and b , \triangleq is a specific instance of $=$. We say that $a \triangleq b$ **iff** a and b are *assumed* to be equal. Typically, $a \triangleq b$ means that a is assigned the pre-existing value b (some authors write $a := b$) but this is not a strict requirement.

iii Iverson brackets

Given a boolean expression ϕ , $[\phi]$ encodes the truth value of ϕ as a **bit**:

$$[\phi] \triangleq \begin{cases} 0 & (\phi \text{ is false}) \\ 1 & (\phi \text{ is true}). \end{cases}$$

For example, $[1 > 0] = 1$ but $[\sqrt{2} \in \mathbf{Q}] = 0$. For interpretations of Boolean operators in logic, see [2].

II Sets

i Subsets and supersets

\subseteq and \supseteq are the standard symbols for set ordering. No specific symbol is reserved for strict ordering, every restriction $X \neq Y$ will be explicitly stated when a strict subset is intended. Given A and B , $A \cup B$ is the union of A and B . More generally, for any collection C , the union U of C is expressed as follows:

$$(1) \quad U \triangleq \bigcup C \triangleq \bigcup_{S \in C} S.$$

Similarly, \cap denotes intersection in the same manner.

ii Special mappings

The identity I (or id) is the mapping $\{(x, x) : x \in X\}$. Similarly, the projection $\pi = \{((x, y), x) : x \in X, y \in Y\}$ always exists. Observe that I is the diagonal of $X \times X$.

iii Equinumerosity

In a metric space context, $X \equiv Y$ means that ϕ is a surjective isometry.

III Topological vector spaces

i Scalar field

\mathbf{C} extends \mathbf{R} , which implies that a property, *e.g.*, linearity, that holds over \mathbf{C} also holds over \mathbf{R} . The complex case is therefore *stronger* than the real case. This restriction may be significant in some contexts. However, the standard scalar field is \mathbf{C} . Unless the field is explicitly given, as in [3.1 and 12.7] of [4], we assume that results for \mathbf{C} apply equally to the real case.

ii Vector space bases

Given a vector space X , a subset B of X is a basis of X **iff** the sum

$$(2) \quad \left\{ (z_u)_{u \in B} : z_u \in \mathbf{C}, \{u : z_u \neq 0\} \text{ is finite} \right\} \rightarrow X$$

$$(z_u) \mapsto \sum_{u \in B} z_u u$$

is a bijection of all *finitely supported* families (z_u) to X . The axiom of choice (AC) forces

- (a) the existence of such B (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [4] with B playing the role of Λ);
- (b) all bases to have the same cardinality, which is called the *dimension* of X and is denoted as $\dim X$.

We now turn to the finite-dimensional case over the field \mathbf{C} . The zero-dimensional case is $B = \emptyset$, *i.e.*, $X = \{0\}$. Our first step is to study \mathbf{C}^n the standard n -dimensional vector space when $n = 1, 2, 3, \dots$.

iii Finite-dimensional spaces

The product topology of \mathbf{C}^n

\mathbf{C}^n has a standard basis $1_{\{0\}}, \dots, 1_{\{n-1\}}$ so that scalar z_k is the k -th component of

$$(3) \quad (z_1, \dots, z_n) \triangleq z_1 \cdot (\underbrace{1, 0, \dots, 0}_{1_{\{0\}}}) + \dots + z_n \cdot (\underbrace{0, \dots, 0, 1}_{1_{\{n-1\}}}),$$

as $z = (z_1, \dots, z_n)$ ranges over \mathbf{C}^n . Moreover, \mathbf{C}^n is endowed with the topology generated by all polydiscs

$$(4) \quad \prod_{i=1}^n \underbrace{\{z_i \in \mathbf{C} : |z_i| < r_i\}}_{D_{r_i}} \quad (r_i > 0).$$

Equivalently, we may equip \mathbf{C}^n with the Euclidean norm

$$(5) \quad \|z\|_2 \triangleq \sqrt{|z_1|^2 + \dots + |z_n|^2},$$

whose open balls centered at the origin are all

$$(6) \quad B_r \triangleq \{z \in \mathbf{C}^n : \|z\|_2 < r\} \quad (r > 0).$$

To show the equivalence, first set $r_i = r/\sqrt{n}$. Hence

$$(7) \quad \prod_{i=1}^n D_{r_i} \subseteq B_r.$$

Conversely, put $r = \min(r_1, \dots, r_n)$ so that

$$(8) \quad B_r \subseteq \prod_{i=1}^n D_{r_i}.$$

Topology of a finite-dimensional vector space

It is customary to identify any n -dimensional vector space with \mathbf{C}^n equipped with the Euclidean norm, see [5]. To show this, choose an isomorphism $f : \mathbf{C}^n \rightarrow Y$. For instance, let $f(e_k)$ be u_k , as in [1.20] of [4], when $\{u_k\}$ is a basis of the n -dimensional Y ; see [ii]. It follows from [1.21] of [4] that f is a homeomorphism. A striking consequence is that $\{f(U) : U \text{ open in } \mathbf{C}^n\}$ is the unique topological vector space topology on Y . Thus, Y is necessarily locally convex and locally bounded, *i.e.*, normable; see [1.39] of [4]. Note that $\|y\|_Y = \|f^{-1}(y)\|_2$ for all $y \in Y$ defines a norm. Additionally, Y is locally compact, as the closed unit ball of \mathbf{C}^n is compact. Choose an n -dimensional topological vector space W , and repeat the same reasoning, first with $g : \mathbf{C}^n \rightarrow W$, next with $h = g \circ f^{-1}$, playing the role of f . Thus, $h : Y \rightarrow W$ is a homeomorphism and W is normable as well. The following assertions are then equivalent in the finite-dimensional context:

- (i) $\dim W = \dim Y$,
- (ii) W and Y are isomorphic,
- (iii) W and Y are homeomorphic and normable.

Furthermore, the norms on W and Y are *equivalent*. That is, for any given norm $\|\cdot\|_Y$ on Y and any given norm $\|\cdot\|_W$ on W , there exists a positive constant (termed the *modulus of continuity*) $C = C_h$ such that

$$(9) \quad \|w\|_W \leq C \|y\|_Y \quad ((y, w) \in h),$$

as h is continuous. When $W = Y$, this means that all norms on Y are equivalent, in the sense that

$$(10) \quad \|h(y)\|_Y \leq C \|y\|_Y.$$

The standard norms 1, 2, and ∞

When \mathbf{C}^n is equipped with standard norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$, the least $C_{i,j}$ such that

$$(11) \quad \|z\|_j \leq C_{i,j} \|z\|_i$$

is easily derived from definitions - see [1.19] of [4] - with the exception of $C_{2,1} = \sqrt{n}$. Indeed, $\|z\|_1 \leq \sqrt{n} \|z\|_2$ is a Cauchy-Schwarz inequality; see (1) in [12.2] of [4]. The table below records these sharp $C_{i,j}$,

i \ j	1	2	∞
1	1	1	1
2	\sqrt{n}	1	1
∞	n	\sqrt{n}	1

Table 1. Minimal $C_{i,j}$ for the standard norms $i, j = 1, 2, \infty$.

iv Continuity and boundedness in normed spaces

A linear mapping Λ is said to be *bounded* iff $\Lambda(E)$ is a bounded set for every bounded set E ; see [1.31] of [4]. Linearity implies that $\Lambda : X \rightarrow Y$ is bounded iff

$$(12) \quad \|\Lambda\| \triangleq \sup \left[\{\|\Lambda x\| : \|x\| < 1\} \right] < \infty,$$

Given normed spaces X and Y , bounded linear maps Λ form a normed space $B(X, Y)$, where norm $\|\Lambda\|$ is given above; see [4.1] of [4]. It is now easy to see that, in the current context, boundedness and continuity coincide. This is a particular case of [2.8] of [4]. When Y is the scalar field, this equivalence also comes from [1.32] of [4]. Furthermore, we observe, that given a collection of bounded mappings $\Lambda \in B(X, Y)$, the property of equicontinuity now reads as

$$(13) \quad \sup_{\Lambda} \|\Lambda\| < \infty.$$

Thus, *uniform boundedness* and uniform continuity coincide as well; cf. [2.4] of [4].

IV Measure theory

i Radon measures

Given a locally compact Hausdorff space X , *e.g.*, \mathbf{R}^n , a positive Radon measure Λ is a functional that is *positive* on $C_c(X)$, in the sense that

$$(i) \quad \phi \geq 0 \Rightarrow \Lambda\phi \geq 0 \quad (\phi \in C_c(X)).$$

Theorem [2.14] of [3] shows that positivity (i) implies the following *continuity* property (ii):

(ii) For each compact $K \subseteq X$ there exists a modulus of continuity M_K such that

$$|\Lambda\phi| \leq M_K \|\phi\|_\infty \quad (\phi \in C_c(X) : \text{supp } \phi \subseteq K).$$

Condition (ii) defines Radon measures in a weaker sense; see [5]. Moreover, Λ is bounded on $C_c(X)$ with respect to the supremum norm **iff** (ii) is satisfied with all $M_K \leq \|\Lambda\|_\infty < \infty$. These bounded functionals are then uniformly continuous Radon measures. According to [6.19] of [3], each of them can be isometrically identified with a specific regular Borel measure μ . Conversely, when the ambient space X is compact, every Radon measure is $\|\cdot\|_\infty$ -bounded. For instance, this case is addressed in Exercise 3 of Chapter 2. In measure theory context, by *support* of μ , we mean:

$$(14) \quad \text{supp } \mu \triangleq X \setminus \bigcup V,$$

where V runs through all open sets of measure 0. A very important Radon measure of support $\{0\}$, the *Dirac delta function* is described in [A.2].

ii Lebesgue integration

Theorem [2.14] of [3] states that every positive Radon measure $\Lambda : C_c(X) \rightarrow \mathbf{C}$ is identified with a specific positive and regular Borel measure $\beta : X \rightarrow \mathbf{R}$. Hence

$$(15) \quad \int_X \phi \, d\beta \triangleq \Lambda\phi \quad (\phi \in C_c(X)),$$

where the left side is a Lebesgue integral. The regularity property implies that β has total mass

$$(16) \quad \int_X 1 \, d\beta \triangleq \sup \left\{ \Lambda\phi : \phi \in C_c(X), \|\phi\|_\infty \leq 1 \right\}.$$

When we substitute $f d\beta$ for $d\beta$, with f X -measurable, (15) takes the form

$$(17) \quad \int_X f\phi \, d\beta = \Lambda(f\phi).$$

In particular, if f is positive,

$$(18) \quad \int_X f \, d\beta = \sup \left\{ \Lambda\phi : \phi \in C_c(X), \|\phi\|_\infty \leq 1 \right\}.$$

Moreover, given $N = N(\beta)$ denoting $\{f : f = 0 \text{ } \beta\text{-a.e.}\}$, we see that the density f and the measure Λ are identified modulo N in (17). Algebraically speaking:

$$(19) \quad g\beta = f\beta \Leftrightarrow g - f \in N.$$

A central question in calculus is whether the modulus integral $|f|_1$, namely

$$(20) \quad |f|_1 \triangleq \int_X |f| \, d\beta \quad (f \text{ } X\text{-measurable}),$$

is finite. This motivates the following definitions:

$$(21) \quad \mathcal{L}^1(X, \beta) \triangleq \left\{ f : |f|_1 < \infty \right\} \subseteq \mathcal{L}_{\text{loc}}^1(X, \beta) \triangleq \bigcap \left\{ \mathcal{L}^1(K, \beta) : K \subseteq X \text{ is compact.} \right\}$$

Standard notation $\mathcal{L}^1(\beta)$ is used to avoid redundancy, or simply \mathcal{L}^1 when the context is clear. Key points include:

(a) \mathcal{L}^1 , equipped with $|\cdot|_1$, is a seminormed space,

(b) N is a closed subspace of the seminormed space \mathcal{L}^1 .

Together, these two facts imply that $L^1 = \mathcal{L}^1/N$, equipped with the norm

$$(22) \quad \|f + N\|_1 \triangleq |f|_1 \quad (f \in \mathcal{L}^1),$$

is a Banach space. These definitions and properties extend to L^p spaces ($p > 1$), as follows

$$(23) \quad \mathcal{L}^p \triangleq \{f : |f|^p \in \mathcal{L}^1\},$$

$$(24) \quad L^p \triangleq \mathcal{L}^p/N,$$

$$(25) \quad \|f + N\|_p \triangleq \left(\int_X |f|^p \right)^{1/p} \quad (f \in \mathcal{L}^p).$$

Similar *quasi-Banach* L^p spaces exist for $0 < p < 1$, with the notable difference that $\|f + N\|_p$ no longer defines a norm (the triangle inequality is lost). In contrast, the counterpart $L^\infty = \{f + N : \|f + N\|_\infty < \infty\}$ is equipped with $\|\cdot\|_\infty$, where the *quotient-norm*

$$(26) \quad \|f + N\|_\infty \triangleq \inf \left\{ M : M > 0, |f| < M \text{ } \beta\text{-a.e.} \right\}$$

over bounded functions f turns it into a Banach space.

Chapter 1

Topological Vector Spaces

1 Exercise 1. Basic results

Suppose X is a vector space. All sets mentioned below are understood to be subsets of X . Prove the following statements from the axioms as given in section 1.4.

- (a) If $x, y \in X$ there is a unique $z \in X$ such that $x + z = y$.
- (b) $0 \cdot x = 0 = \alpha \cdot 0$ ($\alpha \in \mathbf{C}, x \in X$).
- (c) $2A \subseteq A + A$.
- (d) A is convex if and only if $(s + t)A = sA + tA$ for all positive scalars s and t .
- (e) Every union (and intersection) of balanced sets is balanced.
- (f) Every intersection of convex sets is convex.
- (g) If Γ is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of Γ is convex.
- (h) If A and B are convex, so is $A + B$.
- (i) If A and B are balanced, so is $A + B$.
- (j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

PROOF. (a) Such property only depends on the group structure of X : Each x in X has an opposite $-x$. Let x' be any opposite of x so that $x - x = 0 = x + x'$. Thus, $-x + x - x = -x + x + x'$, which is equivalent to $-x = x'$. So is established the uniqueness of $-x$. It is now clear that $x + z = y$ iff $z = -x + y$, which asserts both the existence and the uniqueness of z .

(b) Remark that

$$(1.1) \quad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$

$$(1.2) \quad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that $0 \cdot x = 0$. So,

$$(1.3) \quad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x.$$

Finally,

$$(1.4) \quad \alpha \cdot 0 \stackrel{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

(c) Remark that

$$(1.5) \quad 2x = (1 + 1)x = x + x$$

for every x in X , and so conclude that

$$(1.6) \quad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets A of X ; which proves (c).

(d) If A is convex, then

$$(1.7) \quad A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A;$$

which is

$$(1.8) \quad sA + tA = (s+t)A.$$

Conversely, the special case $s + t = 1$ is

$$(1.9) \quad sA + (1-s)A = A.$$

The latter extends to $s = 0$, since

$$(1.10) \quad 0A + A \stackrel{(b)}{=} \{0\} + A = A.$$

The extension to $s = 1$ is analogously established (or simply use the fact that $+$ is commutative!). So ends the proof.

(e) Let A range over B a collection of balanced subsets so that

$$(1.11) \quad \alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B$$

for all scalars α of magnitude ≤ 1 . The inclusion $\alpha \bigcap B \subseteq A$ establishes the first part. Now remark that

$$(1.12) \quad \alpha A \subseteq \bigcup B$$

implies

$$(1.13) \quad \alpha \bigcup B \subseteq \bigcup B;$$

which achieves the proof.

(f) Let A range over C a collection of convex subsets so that

$$(1.14) \quad (s+t) \bigcap C \subseteq s \bigcap C + t \bigcap C \subseteq sA + tA \stackrel{(d)}{\subseteq} (s+t)A$$

for all positive scalars s, t . Inclusions at both extremities force

$$(1.15) \quad s \bigcap C + t \bigcap C = (s+t) \bigcap C.$$

We now conclude from (d) that the intersection of C is convex. So ends the proof.

(g) We dismiss all trivial cases $\Gamma = \emptyset, \{\emptyset\}, \{\{x\}\}, \{\emptyset, \{x\}\}$ then pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i ($i = 1, 2$) lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2 . So, x_1, x_2 are now elements of the convex set C_2 . Every convex combination of our x_i 's is then in $C_2 \subseteq \bigcup \Gamma$. Hence (g).

(h) Simply remark that

$$(1.16) \quad s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B)$$

for all positive scalars s and t , then conclude from (d) that $A+B$ is convex.

- (i) Given any
- α
- from the closed unit disc,

$$(1.17) \quad \alpha(A + B) = \alpha A + \alpha B \subseteq A + B.$$

This completes the proof: $A + B$ is balanced.

- (j) Our proof will be based on the following lemma,

If S is nonempty, then each of the following three properties

- (i) S is a vector subspace of X ;*
- (ii) S is convex balanced such that $S + S = S$;*
- (iii) S is convex balanced such that $\lambda S = S$ ($\lambda > 0$)*

implies the other two.

To prove the lemma, let S run through all nonempty subsets of X . First, assume that (i) holds: Clearly, every S is convex balanced. Moreover, $S + S \subseteq S$. Conversely, $S = S + \{0\} \subseteq S + S$; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \quad nS = (n-1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). Pick $\lambda > 0$ then choose n so large that $1 < n\lambda < n^2$. Thus,

$$(1.19) \quad nS \stackrel{(1.18)}{\subseteq} S \subseteq n\lambda S \subseteq n^2S,$$

since S is balanced. For instance, set $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$. Dividing the latter inclusions by n shows that

$$(1.20) \quad S \subseteq \lambda S \subseteq nS \stackrel{(1.18)}{\subseteq} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$(1.21) \quad \alpha S + \beta S \stackrel{(a)}{=} |\alpha|S + |\beta|S \stackrel{(d)}{=} (|\alpha| + |\beta|)S \stackrel{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as S is balanced. Moreover (under the sole assumption that S is balanced), this extends to $|\alpha| + |\beta| = 0$, as follows,

$$(1.22) \quad \alpha S + \beta S = 0S + 0S \stackrel{(b)}{=} \{0\} \stackrel{(b)}{=} 0S \subseteq S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let V be a collection of vector spaces of X , of intersection I and union U . First, remark that every member of V is convex balanced: So is I (combine (e) with (f)). Next, let Y range over V so that

$$(1.23) \quad I + I \subseteq Y + Y \subseteq Y;$$

which yields

$$(1.24) \quad I + I = I$$

(the fact that $I = I + \{0\} \subseteq I + I$ was tacitly used). It now follows from the lemma's (ii) \Rightarrow (i) that I is a vector subspace of X . Now temporarily assume that S is totally ordered by set inclusion: Combining (e) with (g) establishes that U is convex balanced. To show that U is more specifically a vector subspace, we first remark that such total order implies that either $Z \subseteq Y$ or $Y \subseteq Z$, as Z ranges over V . A straightforward consequence is that

$$(1.25) \quad Y \subseteq Y + Z \subseteq Y \cup Z.$$

Another one is that $Y \cup Z$ ranges over V as well. Combined with the latter inclusions, this leads to

$$(1.26) \quad U \subseteq U + U \subseteq U.$$

It then follows from the lemma's (ii) \Rightarrow (i) that U is a vector subspace of X . Finally, let A, B run through all vector subspaces of X : Combining (h) with (i) proves that $A + B$ is convex balanced as well. Furthermore,

$$(1.27) \quad A + B \stackrel{(i) \Rightarrow (ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as X is an abelian group. We now conclude from (ii) that any $A + B$ is a vector subspace of X . So ends the proof. □

2 Exercise 2. Convex hull

The convex hull of a set A in a vector space X is the set of all convex combinations of members of A , that is the set of all sums $t_1x_1 + \dots + t_nx_n$ in which $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$; n is arbitrary. Prove that the convex hull of a set A is convex and that is the intersection of all convex sets that contain A .

PROOF. The convex hull of a set S will be denoted by $\text{co}(S)$. Remark that $S \supseteq \text{eq co}(S)$ (to see that, take $t_1 = 1$ for each x_1 in S) and that $\text{co}(A) \supseteq \text{eq co}(B)$ where $A \supseteq \text{eq} B$ (obvious).

Our proof will directly derive from (i) \Rightarrow (iv) in the following lemma,

Let S be a subset of a vector space X : Its convex hull $\text{co}(S)$ is convex and the following statements

- (i) S is convex;
- (ii) $s_1S + \dots + s_nS = (s_1 + \dots + s_n)S$ for all positive scalar variables s_1, \dots, s_n ;
- (iii) $t_1S + \dots + t_nS = S$ for all positive scalar variables s_1, \dots, s_n such that $s_1 + \dots + s_n = 1$;
- (iv) $\text{co}(S) = S$

are equivalent.

From now on, we skip the trivial case $S = \emptyset$ then only consider nonempty sets. To prove the first part, let a, b range over $\text{co}(S)$ so that $a = t_1x_1 + \dots + t_nx_n$ and $b = t_{n+1}x_{n+1} + \dots + t_{n+p}x_{n+p}$ for some (t_i, x_i) . Every sum $sa + (1-s)b$ ($0 \leq s \leq 1$) is then in the convex hull of $\{x_1, \dots, x_{n+p}\}$, since

$$(1.28) \quad sa + (1-s)b = \sum_{i=1}^n st_i x_i + \sum_{i=n+1}^{n+p} (1-s)t_i x_i$$

and

$$(1.29) \quad \sum_{i=1}^n st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s \sum_{i=1}^n t_i + (1-s) \sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets S , this reads as follows,

$$(1.30) \quad s \text{co}(S) + (1-s) \text{co}(S) \subseteq \text{co}(S);$$

which was our first goal. We now prove the equivalence (i) $\Rightarrow \dots \Rightarrow$ (iv) \Rightarrow (i): An easy proof by induction makes the implication (i) \Rightarrow (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii) \Rightarrow (iv) derives from the definition of the convex hull. We now close the chain with (iv) \Rightarrow (i), by remarking that S is convex whether $S = \text{co}(S)$. The lemma being proved, let us establish the second part.

To do so, we start from the convexity of $\text{co}(A)$ then set $F = \{\text{co}(A)\}$. We may enrich F as follows,

$$(1.31) \quad B \in F \Rightarrow B \text{ is convex and contains } A.$$

Note that our initial predicate “[F only encompasses] *all convex sets that contain A* ”, is now the special case

$$(1.32) \quad B \in F \Leftrightarrow B \text{ is convex and contains } A.$$

In any case, the key ingredient is that $\text{co}(A) \in F$ implies

$$(1.33) \quad \text{co}(A) \supseteq \bigcap_{B \in F} B.$$

Conversely, the next formula

$$(1.34) \quad \text{co}(A) \subseteq \text{co}(B) \stackrel{(i) \Rightarrow (iv)}{=} B \quad (B \in F)$$

is valid and implies

$$(1.35) \quad \text{co}(A) \subseteq \bigcap_{B \in F} B.$$

So ends the proof □

3 Exercise 3. Other basic results

Let X be a topological vector space. All sets mentioned below are understood to be subsets of X . Prove the following statements:

- (a) The convex hull of every open set is open.
- (b) If X is locally convex then the convex hull of every bounded set is bounded.
- (c) If A and B are bounded, so is $A+B$.
- (d) If A and B are compact, so is $A+B$.
- (e) If A is compact and B is closed, then $A+B$ is closed.
- (f) The sum of two closed sets may fail to be closed.

PROOF. (a) Pick an open set A then let the variables V_i ($i = 1, 2, \dots$) run through all open subsets of A so that

$$(1.36) \quad \text{co}(A) \subseteq \bigcup_{t_i} (t_1 V_1 + \dots + t_i V_i + \dots) \subseteq \text{co}(A)$$

given all convex combinations $t_1 V_1 + \dots + t_i V_i + \dots$. We know from [1.7] of [4] that those sums are open; which achieves the proof.

- (b) Provided a bounded set E , pick V a neighborhood of 0: By (b) of Section 1.14 in [14] of [4], V contains a convex neighborhood of 0, say W . There so exists a positive scalar s such that

$$(1.37) \quad E \subseteq tW \subseteq tV \quad (t > s);$$

which yields

$$(1.38) \quad \text{co}(E) \subseteq \text{co}(tW) = t \text{co}(W) = tW \subseteq tV.$$

So ends the proof.

- (c) At fixed V , neighborhood of the origin, we combine the continuity of $+$ with [1.14] of [4] to conclude that there exists U a balanced neighborhood of the origin such that

$$(1.39) \quad U + U \subseteq V.$$

Moreover, by the very definition of boundedness, $A \subseteq rU$ for some positive scalar r . Similarly, $B \subseteq sU$ for some positive s . Finally,

$$(1.40) \quad A + B \subseteq rU + sU \subseteq tU + tU \subseteq tV \quad (t > r, s),$$

since U is balanced. So ends the proof.

- (d) First, A and B are compact: So is $A \times B$. Next, $+$ maps continuously $A \times B$ onto $A + B$. In conclusion, $A + B$ is compact.
- (e) From now on, we assume that neither A nor B is empty, since otherwise the result is trivial. Now pick $c \in X$ outside $A + B$: The result will be established by showing that c is not in the closure of $A + B$.

To do so, we let the variable a range over A : Every set $a + B$ is closed as well, see [1.7] of [4]. Trivially, $a + B \neq c$: By Section [1.10] of [4], there so exists $V = V(a)$ a neighborhood of the origin such that

$$(1.41) \quad (a + B + V) \cap (c + V) = \emptyset.$$

Moreover, there are finitely many $a + V$, say $a_1 + V_1, a_2 + V_2, \dots$, whose union U contains the compact set A . Therefore,

$$(1.42) \quad A + B \subseteq U + B.$$

Now define

$$(1.43) \quad W \triangleq V_1 \cap V_2 \cap \dots,$$

so that

$$(1.44) \quad (a_i + B + V_i) \cap (c + W) \stackrel{(1.41)}{=} \emptyset \quad (i = 1, 2, \dots).$$

As a conclusion, c is not in the closure of $U + B$. Finally, (1.42) asserts that c is not in $\overline{A + B}$ either; which achieves the proof.

Corollary: If B is the closure of a set S , then

$$(1.45) \quad A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B$$

by [(b) of 1.13] of [4] (since A is closed; see Section 1.12, from the same source). The special case $A = \{x\}$, $B = X$ will occur in the proof of Exercise 15 in chapter 2.

- (f) The last proof will consist in exhibiting a counterexample. To do so, let f be any continuous mapping of the real line such that

$$(i) \quad f(x) + f(-x) \neq 0 \quad (x \in \mathbf{R});$$

$$(ii) \quad f \text{ vanishes at infinity.}$$

For instance, we may combine (ii) with f even and $f > 0$ by setting $f(x) = 2^{-|x|}$, $f(x) = e^{-x^2}$, $f(x) = 1/(1 + |x|)$, ..., and so on.

As a continuous function, f has closed graph G , see [2.14] of [4]. Moreover, (i) implies that the origin $(0, 0) \neq (x - x, f(x) + f(-x))$ is not in $G + G$. On the other hand,

$$(1.46) \quad \{(0, f(n) + f(-n)) : n = 1, 2, \dots\} \subseteq G + G.$$

Now the key ingredient is that

$$(1.47) \quad (0, f(n) + f(-n)) \xrightarrow[n \rightarrow \infty]{(ii)} (0, 0).$$

We have so constructed a sequence in $G + G$ that converges outside $G + G$. So ends the proof. □

4 Exercise 4. A nonempty set whose interior is not

Let be $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$. Show that B is balanced but that its interior is not.

PROOF. It is obvious that the nonempty set B contains the origin $(0, 0)$. Additionally, its interior B° is nonempty as well. Indeed, the following set

$$(1.48) \quad \{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2| < 1/2\} \subseteq B$$

is a neighborhood of $(1, 2) \in B$. Moreover, B is balanced, since

$$(1.49) \quad |\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \quad (|\alpha| \leq 1)$$

for all (z_1, z_2) in B . Nevertheless, the nonempty set B° is not balanced, what we now establish by showing that $(0, 0) \notin B^\circ$. To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$(1.50) \quad U \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2| < r\} \subseteq B$$

for some positive r . Clearly, U contains $(r/2, 0)$, and that special case $(r/2, 0) \in B$ now contradicts the definition of B . So ends the proof. □

5 Exercise 5. A first restatement of boundedness

Consider the definition of “bounded set” given in Section 1.6. Would the content of this definition be altered if it merely required that to every neighborhood V of 0 corresponds some $t > 0$ such that $E \subseteq tV$?

PROOF. The answer is: No. To prove it, start from (a) of Section 1.14: V contains W , a balanced neighborhood of 0 . Assume that E is bounded in this weaker sense, *i.e.*, there exists a positive t that satisfies

$$(1.51) \quad E \subseteq tW.$$

Thus,

$$(1.52) \quad E \subseteq tW \subseteq sW \subseteq sV \quad (s > t),$$

since W is balanced. We so reach the definition given in Section 1.6: The two ones are equivalent. \square

6 Exercise 6. A second restatement of boundedness

Prove that a set E in a topological vector space is bounded if and only if every countable subset of E is bounded.

PROOF. It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighborhood of the origin: No counting number $n = 1, 2, \dots$ satisfies $E \subseteq nV$ (see Exercise 1 in Chapter 1). In other words, there exists a sequence $\{x_1, \dots, x_n, \dots\} \subseteq E$ such that

$$(1.53) \quad x_n \notin nV.$$

As a consequence, x_n/n fails to converge to 0 as n tends to ∞ . In contrast, $1/n$ succeeds. It then follows from Section 1.30 that $\{x_1, \dots, x_n, \dots\}$ is not bounded. So ends the proof. \square

7 Exercise 7. Metrizability and number theory

Let X be the vector space of all complex functions on the unit interval $[0, 1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology.

Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \rightarrow \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \rightarrow \infty$ then $\{\gamma_n f_n\}$ does not converge to 0 . (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0, 1]$.) This shows that metrizability cannot be omitted in (b) of Theorem 1.28.

PROOF. **Justifying the terminology.** The family of seminorms p_x is separating: The collection \mathcal{B} of all finite intersections of the sets

$$(1.54) \quad V(x, k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k = 1, 2, 3, \dots)$$

is therefore a local base for a topology τ on X , see Theorem [1.37] of [4]. Hence

$$(1.55) \quad \sum_{n=1}^{\infty} [g_n \notin \cap_{i=1}^m U_i] \leq \sum_{n=1}^{\infty} \sum_{i=1}^m [g_n \notin U_i] = \sum_{i=1}^m \sum_{n=1}^{\infty} [g_n \notin U_i],$$

see [iii] for Iverson bracket notation. Now assume that $\{f_n\}$ τ -converges to some f . By definition,

$$(1.56) \quad \sum_{n=1}^{\infty} [f_n - f \notin W] < \infty \quad (W \in \mathcal{B}).$$

The special case $W = V(x, k)$ implies that, for a fixed k , $|f_n(x) - f(x)| < 2^{-k}$ for all but finitely many n . In other words, $\{f_n(x)\}$ converges to $f(x)$. Conversely, assume that $\{f_n\}$ does not τ -converge. This implies that, for every $f \in X$, there exist finitely many $V(x_1, k_1), \dots, V(x_m, k_m)$ such that

$$(1.57) \quad \sum_{n=1}^{\infty} [f_n - f \notin \cap_{i=1}^m V(x_i, k_i)] = \infty.$$

Substituting $f_n - f$ for f_n in (1.55), with $U_i = V(x_i, k_i)$, yields

$$(1.58) \quad \sum_{n=1}^{\infty} [f_n - f \notin \cap_{i=1}^m V(x_i, k_i)] \leq \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n - f \notin V(x_i, k_i)] = \infty.$$

It is now obvious that

$$(1.59) \quad \sum_{n=1}^{\infty} [f_n - f \notin V(x_i, k_i)] = \infty$$

for some i , which shows that $\{f_n(x_i)\}$ does not converge to $f(x_i)$. Thus, τ -convergence coincides with pointwise convergence on X .

Proof with the given hint. We now prove the second part by constructing a specific sequence $\{f_n\}$ that satisfies both (a) and (b). Indeed, the hint suggests that there exists a bijection

$$(1.60) \quad \begin{aligned} \phi : \left\{ \theta_n : \theta_n \xrightarrow{n\infty} 0 \right\} &\rightarrow [0, 1] \\ (\theta_1, \dots, \theta_n, \dots) &\mapsto x. \end{aligned}$$

We set

$$(1.61) \quad f_n(x) \triangleq \theta_n \quad (x = \phi(\theta_1, \dots, \theta_n, \dots))$$

so that $\{f_n\}$ tends pointwise to 0. Note that, with this construction, the special case

$$(1.62) \quad x_\gamma \triangleq \phi\left(1/\sqrt{1+|\gamma_1|}, \dots, 1/\sqrt{1+|\gamma_n|}, \dots\right)$$

outputs

$$(1.63) \quad \gamma_n f_n(x_\gamma) = \gamma_n / \sqrt{1+|\gamma_n|} \xrightarrow{n \rightarrow \infty} \infty,$$

given a sequence $\{\gamma_n\}$ diverging to ∞ . This proves (b), since $\{\gamma_n f_n(x_\gamma)\}$ diverges. We now give an alternative construction of $\{f_n\}$. This second proof does not need the suggested cardinality argument.

Proof with binary expansions (no hint) We rely on the following assertion: Every irrational number has a binary expansion that is not eventually periodic. More precisely, there exists a bijective sum

$$(1.64) \quad \begin{aligned} \sigma : \left\{ \beta \in \{0, 1\}^{\mathbb{N}^+} : \beta \text{ is not eventually periodic} \right\} &\rightarrow [0, 1] \setminus \mathbb{Q} \\ (\beta_1, \dots, \beta_n, \dots) &\mapsto \sum_{k=1}^{\infty} \beta_k 2^{-k}. \end{aligned}$$

A suitable $\{f_n\}$ can be defined as follows:

$$(1.65) \quad f_n(x) \triangleq \begin{cases} 2^{-(\beta_1 + \dots + \beta_n)} & (x = \sigma(\beta_1, \dots, \beta_n, \dots) \notin \mathbb{Q}) \\ 0 & (x \in \mathbb{Q}). \end{cases}$$

Indeed, we note that every bit stream $\sigma^{-1}(x)$ has infinitely many 1's, which implies that $f_n(x) \xrightarrow{n\infty} 0$. Next, pick an arbitrary $\gamma_n \rightarrow \infty$. Thus, for any positive integer k , $\gamma_n > 4^k$ for all sufficiently large n , say $n > N_k$. We select $n_k > N_k$ so large that

$$(1.66) \quad n_{k+1} - n_k > k.$$

The crucial point is that the sequence $1_{\{n_1, n_2, \dots\}}$ is not eventually periodic. Moreover, the particular choice

$$(1.67) \quad \beta^\gamma \triangleq 1_{\{n_1, n_2, \dots\}}$$

implies

$$(1.68) \quad \beta_1^\gamma + \dots + \beta_{n_1}^\gamma + \dots + \beta_{n_k}^\gamma = k.$$

Finally, (1.65) and (1.68) together yield

$$(1.69) \quad \gamma_{n_k} f_{n_k}(\sigma(\beta^\gamma)) = \gamma_{n_k} / 2^k > 2^k \xrightarrow{k \rightarrow \infty} \infty.$$

In conclusion, every sequence of scalars γ_n such that $\gamma_n \rightarrow \infty$ contains a subsequence $\{\gamma_{n_k}\}$ that causes $\{\gamma_{n_k} f_{n_k}\}$ to diverge. This is (b). \square

9 Exercise 9. Quotient map

Suppose

(a) X and Y are topological vector spaces,

(b) $\Lambda : X \rightarrow Y$ is linear.

(c) N is a closed subspace of X ,

(d) $\pi : X \rightarrow X/N$ is the quotient map, and

(e) $\Lambda x = 0$ for every $x \in N$.

Prove that there is a unique $f : X/N \rightarrow Y$ which satisfies $\Lambda = f \circ \pi$, that is, $\Lambda x = f(\pi(x))$ for all $x \in X$. Prove that f is linear and that Λ is continuous if and only if f is continuous. Also, Λ is open if and only if f is open.

PROOF. Bear in mind that π continuously maps X onto the topological (Hausdorff) space X/N , since N is closed (see [1.41] of [4]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$(1.70) \quad f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y.$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$(1.71) \quad \Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x.$$

It straightforwardly derives from (1.70) that f inherits linearity from π and Λ .

Remark. The special case $N = \{x : \Lambda x = 0\}$, i.e., $\Lambda x = 0$ iff $x \in N$ (cf. (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$(1.72) \quad f(\pi x) = 0 \stackrel{1.70}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by [1.41 (a)] of [4]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of $0_{X/N}$ such that

$$(1.73) \quad \Lambda(U) = f(\pi(U)) \subseteq V.$$

Since π is open ([1.41 (a)] of [4]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by [1.41 (a)] of [4]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$(1.74) \quad W = \pi(V)$$

for some neighborhood V of 0_X . So,

$$(1.75) \quad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y . So ends the proof. \square

10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \rightarrow Y$ is linear, and $\Lambda(X) = Y$.

(a) Prove that Λ is an open mapping.

(b) Assume, in addition, that the null space of Λ is closed, and prove that Λ is continuous.

PROOF. Let $B = \{e, e', \dots\}$ be a basis for Y , and let $W \subseteq X$ be an arbitrary neighborhood of the origin. Since addition is continuous in X , there exists a neighborhood V of the origin such that

$$(1.76) \quad \sum_e V \subseteq W.$$

We assume henceforth, without loss of generality, that V is balanced. Moreover, the surjectivity of Λ provides a vector x_e such that $\Lambda x_e = e$. Therefore, given the coordinate representation $\sum_e y_e e$ of $y \in Y$, the following holds:

$$(1.77) \quad y = \sum_e y_e e = \sum_e y_e \Lambda x_e.$$

Note that $\{x_e : e \in B\}$ is bounded (as a finite set). Hence

$$(1.78) \quad \{x_e : e \in B\} \subseteq sV$$

for some $s > 0$. Combining this with (1.77) yields

$$(1.79) \quad y \in \sum_e s y_e \Lambda(V).$$

Using (1.76) and the balancedness of $\Lambda(V)$, we conclude that $|y_e| < 1/s$ implies

$$(1.80) \quad y \in \sum_e \Lambda(V) \subseteq \Lambda(W).$$

This establishes (a) for $Y = \mathbf{C}^n$ and B the standard basis. The general case $\dim Y = n$ now follows from [iii]. The case $Y = \{0\}$ is trivial.

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, $\{\Lambda = 0\}$ playing the role of N . Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y . By [iii], f is also a homeomorphism. We have thus established that f is continuous; so is $\Lambda = f \circ \pi$. \square

12 Exercise 12. Topology stays, completeness leaves

Suppose $d_1(x, y) = |x - y|$, $d_2(x, y) = |\phi(x) - \phi(y)|$, where $\phi(x) = x/(1 + |x|)$. Prove that d_1 and d_2 are metrics on \mathbf{R} which induce the same topology, although d_1 is complete and d_2 is not.

PROOF. First, each d_i ($i = 1, 2$) induces a topology τ_i whose open balls are all

$$(1.81) \quad B_i(a, r) \triangleq \{x \in \mathbf{R} : d_i(a, x) < r\} \quad (a \in \mathbf{R}, r > 0).$$

Next, remark that the monotonically increasing mapping $\phi : \mathbf{R} \rightarrow]-1, 1[$ is odd and that

$$(1.82) \quad \phi(x) \xrightarrow{x \rightarrow \infty} 1.$$

ϕ is therefore a τ_1 -homeomorphism of \mathbf{R} onto $]-1, 1[$. A first consequence is that, at fixed $a \in \mathbf{R}$, given any positive scalar ε , the τ_1 -continuity of ϕ supplies an open ball $B_1(a, \eta)$ on which $|\phi(a) - \phi| < \varepsilon$. In terms of balls B_i , this reads as follows,

$$(1.83) \quad B_1(a, \eta) \subseteq B_2(a, \varepsilon).$$

The second consequence is that the τ_1 -continuity of ϕ^{-1} yields similar inclusions

$$(1.84) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta')$$

provided $\eta' > 0$. At arbitrary ε , the special case $\eta' = \eta$ is the concatenation

$$(1.85) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta) \subseteq B_2(a, \varepsilon);$$

which proves that $\tau_1 = \tau_2$. Finally, all inequalities $n < i < j$ over \mathbf{N} together yield

$$(1.86) \quad d_2(i, j) = |\phi(i) - \phi(j)| \xrightarrow{n \rightarrow \infty} 0.$$

The sequence $n = 0, 1, 2, \dots$ is therefore τ_2 -Cauchy. We will nevertheless establish that it τ_2 -diverges. To do so, we start by offering the τ_2 -convergence to some λ : The triangle inequality immediately dismisses that assumption, as follows,

$$(1.87) \quad d_2(0, \lambda) \geq d_2(0, n) - d_2(\lambda, n) = \phi(n) - d_2(\lambda, n) \xrightarrow{n \rightarrow \infty} 1.$$

We then conclude that d_2 fails to be complete. \square

14 Exercise 14. \mathcal{D}_K equipped with other seminorms

Put $K = [0, 1]$ and define \mathcal{D}_K as in Section 1.46. Show that the following three families of seminorms (where $n = 0, 1, 2, \dots$) define the same topology on \mathcal{D}_K . If $D = d/dx$:

$$(a) \|D^n f\|_\infty = \sup\{|D^n f(x)| : 0 < x < 1\}$$

$$(b) \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$(c) \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

PROOF. Let us equip \mathcal{D}_K with the inner product $\langle f|g \rangle = \int_0^1 f \bar{g}$ so that $\langle f|f \rangle = \|f\|_2^2$. The following

$$(1.88) \quad \int_0^1 |D^n f|^2 dx \leq \|1\|_2 \|D^n f\|_2$$

is then a Cauchy-Schwarz inequality, see Theorem [12.2] of [4]FA. We so obtain

$$(1.89) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

since K has length 1. Obviously, the support of $D^n f$ lies in K , hence the below equality

$$(1.90) \quad |D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1.$$

Take the supremum over all $|D^n f(x)|$: Combining (1.89) with (1.90) now reads as follows,

$$(1.91) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots < \infty.$$

Finally, put

$$(1.92) \quad V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\},$$

$$(1.93) \quad \mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\},$$

so that (1.91) is mirrored by neighborhood inclusions, provided $i = 1, 2, \infty$:

$$(1.94) \quad V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \dots.$$

Their subchains $V_n^{(i)} \supseteq V_{n+1}^{(i)}$ turn $\mathcal{B}^{(i)}$ into a local base of a topology τ_i . The whole chain (1.94) then forces

$$(1.95) \quad \tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1;$$

which achieves the proof. □

16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Lemma 1 Let X be a topological space with a countable local base $\{V_n : n = 1, 2, 3, \dots\}$. If $\tilde{V}_n = V_1 \cap \dots \cap V_n$, then every subsequence $\{\tilde{V}_{(n)}\}$ is a decreasing (, i.e., $\tilde{V}_{(n)} \supseteq \tilde{V}_{(n+1)}$) local base of X .

PROOF. The proof consists in combining trivial consequences of the local base definition with a well-known result (for instance, see [2.6] of [3]) about intersection of nonempty compact sets.

The decreasing property is trivial. Now remark that $V_n \supseteq \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X . Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supseteq \tilde{V}_{\rho(n)}$. □

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$) Under the same notations of Lemma 1, if $\{V_n\}$ is a decreasing local base, then so is $\{V_{(n)}\}$.

Corollary 2 If $\{Q_n\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{(n)}\}$ also satisfies these conditions. Furthermore, if Q is the $C()$'s (respectively $C^\infty()$'s) topology of the seminorms p_n , as defined in section 1.44 (respectively 1.46), then the seminorms $p_{(n)}$ define the same topology Q .

PROOF. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X . Moreover,

$$(1.96) \quad Q_{p(n)} \subseteq Q_{p(n)+1}^\circ \subseteq Q_{p(n)+1} \subseteq Q_{p(n+1)}.$$

Thus,

$$(1.97) \quad Q_{p(n)} \subseteq Q_{p(n+1)}^\circ.$$

In other words, $Q_{p(n)}$ satisfies the conditions specified in section 1.44. $\{p_{p(n)}\}$ then defines a topology τ_{Q_p} for which $\{V_{p(n)}\}$ is a local base. So, $\tau_{Q_p} \subseteq \tau_Q$. Conversely, the above corollary asserts that $\{V_{p(n)}\}$ is a local base of τ_Q , which yields $\tau_Q \subseteq \tau_{Q_p}$. \square

Lemma 2 If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in almost all Q_n° , i.e., there exists m such that

$$(1.98) \quad K \subseteq Q_m^\circ \subseteq Q_{m+1}^\circ \subseteq Q_{m+2}^\circ \subseteq \dots$$

PROOF. The following definition

$$(1.99) \quad C_n \triangleq K \setminus Q_n^\circ$$

shapes $\{C_n\}$ as a decreasing sequence of compact¹ sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude² that the C_n 's intersection contains a point that is not in any Q_n° . On the other hand, the conditions specified in [1.44] force the Q_n° 's collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, i.e., $K \subseteq Q_m^\circ$, for some m . Finally,

$$(1.100) \quad K \subseteq Q_m^\circ \subseteq Q_m \subseteq Q_{m+1}^\circ \subseteq Q_{m+1} \subseteq Q_{m+2}^\circ \subseteq \dots$$

\square

We are now in a fair position to establish the following:

Theorem The topology of $C()$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty()$, as long as this sequence satisfies the conditions specified in section 1.44.

PROOF. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\kappa}$ for every subsequence $\{L_{\kappa(n)}\}$. Now apply the above Lemma 2 with K_i ($i = 1, 2, 3, \dots$) and so conclude that $K_i \subseteq L_{m_i}^\circ \subseteq L_{m_i+1}^\circ \subseteq \dots$ for some m_i . In particular, the special case $\kappa_i = m_i + i$ is

$$(1.101) \quad K_i \subseteq L_{\kappa_i}^\circ.$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$(1.102) \quad L_j \subseteq K_{\lambda_j}^\circ$$

¹See [(b) of 2.5] of [3].

²In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of [2.6] of [3].

Combine (1.101) with (1.102) and so obtain

$$(1.103) \quad K_1 \subseteq L_{\kappa_1}^\circ \subseteq L_{\kappa_1} \subseteq K_{\lambda_{\kappa_1}}^\circ \subseteq K_{\lambda_{\kappa_1}} \subseteq L_{\kappa_{\lambda_{\kappa_1}}}^\circ \subseteq \dots,$$

which means that the sequence $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$(1.104) \quad \tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L.$$

So ends the proof □

17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ is a continuous mapping of $C^\infty(\Omega)$ into $C^\infty(\Omega)$ and also of \mathcal{D}_K into \mathcal{D}_K , for every multi-index α .

PROOF. In both cases, D^α is a linear mapping. It is then sufficient to establish continuity at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an arbitrary neighborhood of the origin. There so exists N such that U contains

$$(1.105) \quad V_N = \{\phi \in C^\infty(\Omega) : \max\{|D^\beta \phi(x)| : |\beta| \leq N, x \in K_N\} < 1/N\}.$$

Now pick g in $V_{N+|\alpha|}$ so that

$$(1.106) \quad \max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}.$$

(the fact that $K_N \subseteq K_{N+|\alpha|}$ was tacitly used). The special case $\gamma = \beta + \alpha$ yields

$$(1.107) \quad \max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

$$(1.108) \quad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad , \text{ i.e., } \quad D^\alpha \left(V_{N+|\alpha|} \right) \subseteq V_N,$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$.

To prove the continuity of the restriction $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$, we first remark that the collection of the $V_N \cap \mathcal{D}_K$ is a local base of the subspace topology of \mathcal{D}_K . $V_{N+|\alpha|} \cap \mathcal{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$(1.109) \quad D^\alpha|_{\mathcal{D}_K} \left(V_{N+|\alpha|} \cap \mathcal{D}_K \right) = D^\alpha \left(V_{N+|\alpha|} \cap \mathcal{D}_K \right)$$

$$(1.110) \quad \subseteq D^\alpha \left(V_{N+|\alpha|} \right) \cap D^\alpha(\mathcal{D}_K)$$

$$(1.111) \quad \subseteq V_N \cap \mathcal{D}_K \quad (\text{see (1.108)})$$

So ends the proof. □

Chapter 2

Completeness

3 Exercise 3. An equicontinuous sequence of measures that does not converge vaguely

Put $K = [-1, 1]$; define \mathcal{D}_K as in Section 1.46 (with \mathbf{R} in place of \mathbf{R}^n). Suppose $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\Lambda\phi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\phi(t)dt$ exists for every $\phi \in \mathcal{D}_K$. Show that Λ is a continuous functional on \mathcal{D}_K . Show that there is a positive integer p and a number $M < \infty$ such that

$$\left| \int_{-1}^1 f_n(t)\phi(t) dt \right| \leq M \|D^p \phi\|_{\infty}$$

for all n . For example, if $f_n(t) = n^3 t$ on $[-1/n, 1/n]$ and 0 elsewhere, show that this can be done with $p = 1$. Construct an example where it can be done with $p = 2$ but not with $p = 1$.

PROOF. Equipped with the supremum norm, $\mathcal{C}_K = C(\mathbf{R}) \cap \{\phi : \text{supp } \phi \subseteq K\}$ realizes the Banach space $C_c(K)$ in $C(\mathbf{R})$. Each density f_n is then identified with the following

$$(2.1) \quad \begin{aligned} \Lambda_n : \mathcal{C}_K &\rightarrow \mathbf{C} \\ \phi &\mapsto \int_{-1}^1 f_n(t)\phi(t) dt, \end{aligned}$$

seen as with a Radon measure. Every Λ_n is continuous since $\|\Lambda_n\| = \|f_n\|_1$ is finite; cf. [6.19] of [4]. Note that the dual space \mathcal{C}_K^* is a Banach space as well, by [4.1] of [4]. Here, we consider only pointwise convergence, which is weaker than norm convergence. We prove that the assumed pointwise convergence

$$(2.2) \quad \Lambda_n \phi \xrightarrow{n \rightarrow \infty} \Lambda \phi \quad (\phi \in \mathcal{D}_K)$$

implies the continuity of Λ in the topology of \mathcal{D}_K . We also construct a specific sequence $\{\Lambda_n\}$ whose pointwise limit Λ is not bounded with respect to the supremum norm. By contraposition, Theorem [2.8] of [4] implies that pointwise convergence on \mathcal{D}_K does not extend to \mathcal{C}_K . This conclusion also derives from the following bounds:

$$(2.3) \quad |\Lambda_n \phi| \leq M \|\phi'\|_{\infty},$$

$$(2.4) \quad |\Lambda_n \phi'| \leq M \|\phi''\|_{\infty}.$$

Combined with the impossibility of boundedness at order $p = 0$, cf. (2.3), the contraposition of Theorem [2.6] of [4] implies that $\{\Lambda_n\}$ does not converge pointwise on \mathcal{C}_K . In Radon measure theory, pointwise convergence is known as *vague convergence*. In the next paragraph, ϕ is restricted to \mathcal{D}_K .

Continuity of Λ . We equip \mathcal{D}_K with derivative norms $\|\phi\|_N \triangleq \|\phi\|_{\infty} + \|D^1 \phi\|_{\infty} + \dots + \|D^N \phi\|_{\infty}$. The induced topology τ_K of \mathcal{D}_K is the weakest topology that makes all norms $\|\cdot\|_N$ continuous, cf. [1.46, 6.2] of [4] and Exercise [1.16]. Equivalently, the collection of all convex balanced sets

$$(2.5) \quad V_N \triangleq \{\|\cdot\|_N < 1/N\}$$

forms a local base of τ_K , cf. [6.2] of [4]. Note that $\|\phi\|_N < 1$ implies $\|\phi\|_\infty < 1$: Every Λ_n is then τ_K -continuous by [1.18] of [4], since $|\Lambda_n|$ is bounded by $\|\Lambda_n\|$ on V_1 . In summary:

- (a) \mathcal{D}_K , equipped with the topology τ_K , is a Fréchet space; see [1.46] of [4].
- (b) Every functional Λ_n is τ_K -continuous.
- (c) $\Lambda_n\phi \rightarrow \Lambda\phi$ pointwise on \mathcal{D}_K (our premise).

By [2.6, 2.8] of [4], the equicontinuous sequence $\{\Lambda_n\}$ converges pointwise to a continuous Λ . Furthermore, the equicontinuity of $\{\Lambda_n\}$ ensures that all $|\Lambda_n|$ remain below 1 on a common *balanced* neighborhood V_p . So,

$$(2.6) \quad \frac{1}{p} \cdot \frac{\phi}{\|\phi\|_p + \varepsilon} \in V_p$$

for all $\varepsilon > 0$. This yields $|\Lambda_n\phi| < p(\|\phi\|_p + \varepsilon)$, which reduces to $|\Lambda_n\phi| \leq p\|\phi\|_p$. Applying Lemma [A.1] with $\phi, D\phi, \dots, D^p\phi$ outputs

$$(2.7) \quad |\Lambda_n\phi| \leq p(p+1)\|D^p\phi\|_\infty.$$

This completes the first part of the proof, with some p and a positive constant $M = M(p)$.

The counterexample: A sequence of $\{\Lambda_n\}$ that does not converge pointwise on \mathcal{C}_K . Let u be a smooth, even mapping that equals 1 on $[-1/2, 1/2]$, vanishes outside $[-1, 1]$, and satisfies $0 \leq u \leq 1$ on \mathbf{R} . The function u belongs to the general construction in [1.46] of [4]. Alternatively, u can be the derivative of ϕ from Lemma [A.2], with $\tau = 1/2$, $\omega = 2$, and $A = 1$. We set

$$(2.8) \quad f_n(t) \triangleq n^3 t [-1/n \leq t \leq 1/n].$$

Under the identification $C_c(K) \equiv \mathcal{C}_K$, Λ_n reads as the difference of two (positive) Radon measures Λ_n^+ and Λ_n^- , since

$$(2.9) \quad \Lambda_n\phi = \underbrace{n^3 \int_0^{1/n} t\phi(t) dt}_{\Lambda_n^+\phi} - \underbrace{n^3 \int_{-1/n}^0 -t\phi(t) dt}_{\Lambda_n^-\phi} \quad (\phi \in \mathcal{C}_K).$$

Thus, Λ_n is a signed Radon measure, whose compact support $[-1/n, 1/n]$ shrinks to $\{0\}$ as $n \rightarrow \infty$. We see that $\|\Lambda_n\| \leq \|\Lambda_n^+\| + \|\Lambda_n^-\|$. However, the collection of all Λ_n is not uniformly bounded, since

$$(2.10) \quad \|\Lambda_n\| = \|f_n\|_1 = n = \|\Lambda_n^+\| + \|\Lambda_n^-\|.$$

The *logistic function* $\sigma_\lambda : t \mapsto 1/(1 + \exp(-\lambda t))$ provides a direct proof of this. It is a standard approximation of the Heaviside step function¹. Thus, Lebesgue's dominated convergence theorem implies

$$(2.11) \quad \Lambda_n(u \cdot \underbrace{(\sigma_\lambda - 1/2)}_{\text{odd}}) = 2n^3 \int_0^{1/n} \underbrace{t(\sigma_\lambda(t) - 1/2)}_{\text{even}} dt \xrightarrow{\lambda \rightarrow \infty} n \quad (n \geq 2).$$

We refer to [6.19] of [4] for a more general scope. At this step, it is already clear that there is no vague convergence for the current sequence $\{\Lambda_n\}$: this would, by Theorem [2.6] of [4], imply that $\sup_n \|\Lambda_n\| < \infty$, which would contradict (2.10). However, we investigate further to establish weaker convergence and boundedness in τ_K . From now on, we bound ϕ to \mathcal{D}_K : The mean value theorem implies that

$$(2.12) \quad \phi(1/n) - \phi(-1/n) = \frac{2}{n} \phi'(t_n)$$

for some $-1/n < t_n < 1/n$. Moreover, integration by parts yields

$$(2.13) \quad \Lambda_n\phi = \frac{n^3}{2} t^2 \phi(t) \Big|_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt$$

$$(2.14) \quad = \frac{n}{2} (\phi(1/n) - \phi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt$$

$$(2.15) \quad = \phi'(t_n) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt.$$

¹ σ_λ connects machine learning with statistical mechanics; see [1]

Note that when $\phi' = 1$ in a neighborhood of 0, *e.g.*, for $\phi(t) = tu(t)$, the latter equality reduces to

$$(2.16) \quad \Lambda_n \phi = 1 - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 dt = \frac{2}{3}.$$

This suggests that continuity of ϕ' dictates $\Lambda_n \phi \rightarrow \frac{2}{3} \phi'(0)$. We establish this convergence in two steps. First,

$$(2.17) \quad \Lambda_n \phi - \frac{2}{3} \phi'(0) = \phi'(t_n) - \phi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt + \phi'(0) \underbrace{\frac{n^3}{2} \int_{-1/n}^{1/n} t^2 dt}_{1/3}$$

$$(2.18) \quad = \phi'(t_n) - \phi'(0) - \frac{n^3}{2} \left(\int_{-1/n}^{1/n} t^2 ((\phi'(t) - \phi'(0))) dt \right).$$

Next, taking absolute values gives

$$(2.19) \quad |\Lambda_n \phi - \frac{2}{3} \phi'(0)| \leq |\phi'(t_n) - \phi'(0)| + \frac{1}{3} \max_{[-1/n, 1/n]} |\phi' - \phi(0)| \xrightarrow{n \rightarrow \infty} 0.$$

As a result,

$$(2.20) \quad \Lambda_n \phi \xrightarrow{n \rightarrow \infty} -\frac{2}{3} \delta' \phi \quad (\phi \in \mathcal{D}_K),$$

where $\delta' : \phi \mapsto -\phi'(0)$ is the *derivative* of the *Dirac measure* $\delta : \phi \mapsto \phi(0)$; see [6.1, 6.9] of [4] and Section [A.2]. The reasoning from the previous part shows that the limit $\Lambda = -\frac{2}{3} \delta'$ is τ_K -continuous. As a complement, absolute values from (2.15) provide

$$(2.21) \quad |\Lambda_n \phi| \leq |\phi'(t_n)| + \frac{1}{3} \max_{[-1/n, 1/n]} |\phi'|.$$

A simpler bound is

$$(2.22) \quad |\Lambda_n \phi| \leq \frac{4}{3} \|\phi'\|_\infty.$$

This is a concrete instance of (2.7), with $p = 1$ and $M = 4/3$. To establish it as (2.3), we need to prove that no reduction to order $p = 0$ is possible. To do so, we first assume, to reach a contradiction, that there exists M such that

$$(2.23) \quad |\Lambda_n \phi| \leq M \|\phi\|_\infty \quad (\phi \in \mathcal{D}_K, n = 1, 2, 3, \dots).$$

Next, we choose

$$(2.24) \quad \phi_n \triangleq \tilde{\phi}_n u,$$

where $\tilde{\phi}_n$ is the function ϕ from Lemma [A.2], with $\tau = 1/n = 1/\omega = 1/A$. So, $\|\phi_n\|_\infty < 2$. Nevertheless, for all $n \geq 4$, $\Lambda_n \phi_n$ is now

$$(2.25) \quad \Lambda_n \phi_n = \frac{n}{2} (\underbrace{\phi_n(1/n)}_1 - \underbrace{\phi_n(-1/n)}_{-1}) - \underbrace{\frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'_n(t) dt}_{\frac{1}{3}n} = \frac{2}{3}n.$$

Hence

$$(2.26) \quad 2M \stackrel{(2.23)}{\geq} |\Lambda_n \phi_n| \xrightarrow{n \rightarrow \infty} \infty,$$

which brings the desired contradiction. We now combine (2.20) with (2.26) to establish that

$$(2.27) \quad |\Lambda \phi_n| \geq |\Lambda_n \phi_n| - |\Lambda_n \phi_n - \Lambda \phi_n| \xrightarrow{n \rightarrow \infty} \infty.$$

Therefore, Λ is not bounded either. A direct way to see this is to pick ϕ_ω from Lemma [A.2] so that

$$(2.28) \quad \Lambda(u\phi_\omega) = \frac{2}{3}\omega \xrightarrow{\omega \rightarrow \infty} \infty$$

contrasts with $\|u\phi_\omega\|_\infty = 1$. Thus, we have exhibited a sequence of Radon measures Λ_n that

(a) does not converge vaguely to any Radon measure,

(b) but converges pointwise on \mathcal{D}_K , in the specific \mathcal{D}_K 's topology; see (2.5), (2.20), and (2.22).

We now present a second example, the *derivative*; see [6.1] of [4].

$$(2.29) \quad \begin{aligned} \Lambda'_n : \mathcal{D}_K &\rightarrow \mathbf{C} \\ \phi &\mapsto -\Lambda_n \phi'. \end{aligned}$$

We have proved that every Λ_n is continuous. So is the derivative operator in \mathcal{D}_K - see Exercise [1.17]. Therefore, Λ'_n is continuous. Now apply (2.20) with ϕ' and so obtain

$$\Lambda'_n \phi \xrightarrow{n \rightarrow \infty} -\frac{2}{3} \phi''(0).$$

Furthermore, Theorem [2.8] of [4] implies that the limit $-\frac{2}{3} \phi''(0)$ is τ_K continuous. Additionally, it follows from (2.22) that the bound (2.4) is

$$(2.30) \quad |\Lambda'_n \phi| \leq \frac{4}{3} \|\phi''\|_\infty.$$

Indeed, we show that 2 is the smallest suitable p. First, we assume, to reach a contradiction, that

$$(2.31) \quad |\Lambda_n \phi'| \leq M \|\phi'\|_\infty \quad (\phi \in \mathcal{D}_K, n = 1, 2, 3, \dots).$$

Next, let Φ_n be the primitive of ϕ_n that vanishes at -1; see (2.24). The oddness of Φ_n (u is even) implies that $\text{supp } \Phi_n \subseteq [-1, 1]$. So, under our assumption,

$$(2.32) \quad |\Lambda'_n \Phi_n| = |\Lambda_n \Phi'_n| \leq M \|\Phi'_n\|_\infty.$$

Equivalently,

$$(2.33) \quad |\Lambda_n \phi_n| \leq M \|\phi_n\|_\infty,$$

which has already been disproved. Finally, to reach a last contradiction, assume that there exists M attached to order $p = 0$ so that

$$(2.34) \quad |\Lambda'_n \phi| \leq M \|\phi\|_\infty \quad (\phi \in \mathcal{D}_K, n = 1, 2, 3, \dots).$$

Lemma [A.2] implies that

$$(2.35) \quad |\Lambda'_n \phi| \leq M \|\phi'\|_\infty.$$

This contradiction concludes the proof. □

6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ of a function $f \in L^2(\mathbf{T})$ (\mathbf{T} is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all $n \in \mathbf{Z}$ (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^n \hat{f}(k).$$

Prove that $\{f \in L^2(\mathbf{T}) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$ is a dense subspace of $L^2(\mathbf{T})$ of the first category.

PROOF. Let $f(\theta)$ stand for $f(e^{i\theta})$ so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$(2.36) \quad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

$$(2.37) \quad \Lambda_n(f) = (f, e_n) + \cdots + (f, e_n).$$

Moreover, a well-known (and easy to prove) result is

$$(2.38) \quad (e_n, e_{n'}) = [n = n'], \text{ i.e., } \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric (\equiv) identification $L^2 \equiv (L^2)^*$. So,

$$(2.39) \quad \|\Lambda_n\|^2 \stackrel{2.37}{=} \|e_n + \cdots + e_n\|^2 \stackrel{2.38}{=} \|e_n\|^2 + \cdots + \|e_n\|^2 \stackrel{2.38}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

$$(2.40) \quad B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \dots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem [2.5] of [4] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$(2.41) \quad \|\Lambda_n\| \stackrel{2.39}{=} \sqrt{2n + 1} \xrightarrow{n \rightarrow \infty} \infty.$$

We have just established that B is actually of the first category; and so is its subset $L = \{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$. We now prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be $\text{span}\{e_k : k \in \mathbf{Z}\}$, the collection of the trigonometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$. Combining (2.37) with (2.38) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n . Thus,

$$(2.42) \quad P \subseteq L \subseteq L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.42), that

$$(2.43) \quad L^2(T) = \overline{P} = \overline{L}.$$

So ends the proof □

9 Exercise 9. Boundedness without closedness

Suppose X, Y, Z are Banach spaces and

$$B : X \times Y \rightarrow Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ such that

$$\|B(x, y)\| \leq M \|x\| \|y\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

PROOF. Completeness is not required. For the proof, we only assume that X, Y , and Z are normed spaces. Since B is continuous, there exists $r > 0$ such that

$$(2.44) \quad \|B(x, y)\| < 1$$

whenever $\|x\| + \|y\| < r$. Therefore, any $0 < s < r$ yields

$$(2.45) \quad \|B(x, y)\| = \frac{4\|x\|\|y\|}{s^2} \left\| B\left(\frac{s}{2} \cdot \frac{x}{\|x\|}, \frac{s}{2} \cdot \frac{y}{\|y\|}\right) \right\| < \frac{4\|x\|\|y\|}{s^2} \quad (x \neq 0, y \neq 0)$$

because $\|\frac{s}{2} \cdot \frac{x}{\|x\|}\| + \|\frac{s}{2} \cdot \frac{y}{\|y\|}\| = s < r$. This establishes the existence of $M = 4/r^2$, since

$$(2.46) \quad \|B(x, y)\| \leq \frac{4}{s^2} \|x\| \|y\| \xrightarrow{s \rightarrow r} \frac{4}{r^2} \|x\| \|y\| \quad ((x, y) \in X \times Y).$$

Consider the example where $X = Y = Z = C_c(\mathbf{R})$ equipped with the supremum norm. The existence of $M = 1$ follows from

$$(2.47) \quad \|fg\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty.$$

However, $C_c(\mathbf{R})$ is not complete; see [5.4.4] of [5]. We prove that the bilinear product

$$(2.48) \quad \begin{aligned} B : C_c(\mathbf{R})^2 &\rightarrow C_c(\mathbf{R}) \\ (f, g) &\mapsto f \cdot g \end{aligned}$$

is nevertheless continuous. To do so, we start with any

$$(2.49) \quad 0 < r < \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty} < \varepsilon < 1.$$

Next, we choose $(u, v) \in C_c(\mathbf{R})^2$ with

$$(2.50) \quad \|f - u\|_\infty + \|g - v\|_\infty < r,$$

so that $\|fg - uv\|_\infty < \varepsilon$. Explicitly, we have

$$(2.51) \quad \|fg - uv\|_\infty = \|(f - u) \cdot g + u \cdot (g - v)\|_\infty$$

$$(2.52) \quad \leq \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty$$

$$(2.53) \quad < r \cdot \|g\|_\infty + (r + \|f\|_\infty) \cdot r \quad \left(\text{using } \|u\|_\infty \leq r + \|f\|_\infty \right)$$

$$(2.54) \quad < r \cdot (r + \|f\|_\infty + \|g\|_\infty)$$

$$(2.55) \quad < \varepsilon \frac{r + \|f\|_\infty + \|g\|_\infty}{1 + \|f\|_\infty + \|g\|_\infty}$$

$$(2.56) \quad < \varepsilon \quad (\text{because } r < 1).$$

Since ε is arbitrary, this establishes that B is continuous. □

10 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0, 0)$.

PROOF. Let $B : X_1 \times X_2 \rightarrow Z$ be a bilinear mapping that is continuous at $(0, 0)$, where X_i ($i \in \{1, 2\}$) and Z are topological vector spaces. This implies that, for any balanced open W , X_i contains a balanced open U_i such that

$$(2.57) \quad B(U_1 \times U_2) \subseteq W.$$

Let a_i be in X_i . Therefore, $a_i \in r_i U_i$ for some positive r_i . We now choose b_i in $a_i + (1 + r_1 + r_2)^{-1} U_i$. Hence

$$(2.58) \quad B(b_1, b_2) - B(a_1, a_2) = B(b_1 - a_1, b_2) + B(a_1, b_2) - B(a_1, a_2)$$

$$(2.59) \quad = B(b_1 - a_1, b_2) + B(a_1, b_2 - a_2)$$

$$(2.60) \quad = \underbrace{B(b_1 - a_1, b_2 - a_2)}_{\in \frac{1}{(1+r_1+r_2)^2} W} + \underbrace{B(b_1 - a_1, a_2)}_{\in \frac{r_2}{1+r_1+r_2} W} + \underbrace{B(a_1, b_2 - a_2)}_{\in \frac{r_1}{1+r_1+r_2} W}$$

$$(2.61) \quad \in W + W + W.$$

We conclude that B is continuous at every (a_1, a_2) , since W was arbitrary. □

12 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| \, dt.$$

Put $B(f, g) = \int_0^1 f(t)g(t)dt$, and show that B is a bilinear continuous functional on $X \times X$ which is separately but not continuous.

PROOF. Let f denote the first variable, g the second one. Remark that

$$(2.62) \quad |B(f, g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient ([1.18] of [4]) to assert that any $f \mapsto B(f, g)$ is continuous. The continuity of all $g \mapsto B(f, g)$ follows (Put $C(g, f) = B(f, g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$(2.63) \quad |B(f, g)| < M\|f\|\|g\|.$$

Put

$$(2.64) \quad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \quad (n = 1, 2, 3, \dots),$$

so that

$$(2.65) \quad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand,

$$(2.66) \quad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.66) and (2.63) with (2.65) and so obtain

$$(2.67) \quad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Our continuousness assumption is then contradicted. So ends the proof. \square

15 Exercise 15. Baire's cut

Suppose X is an F -space and Y is a subspace of X whose complement is of the first category. Prove that $Y = X$.
Hint: Y must intersect $x + Y$ for every $x \in X$.

PROOF. Let $\{E_n : n = 1, 2, 3, \dots\}$ whose union is $X \setminus Y$. More specifically, assume that every E_n is nowhere dense, i.e., $V_n = X \setminus \overline{E_n}$ is dense in X . Now pick an arbitrary $x \in X$. Since the translation by x is a self-homeomorphism, $x + V_n$ is open and dense as well². We now apply Baire's theorem twice to establish that

- (a) every intersection $W_n = V_n \cap [x + V_n]$ is dense in X ,
- (b) so is the nonempty intersection $\bigcap_{n=1}^{\infty} W_n$.

Moreover, the intersection $\bigcap_{n=1}^{\infty} V_n$ is disjoint from every E_n . In summary,

$$(2.68) \quad w \in \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq Y$$

²Alternatively, observe that $X = x + X \subseteq \overline{x + V_n}$, as a special case of [1.3 (b)] of [4].

for some $w = w(x)$. Furthermore, w also lies in every $x + V_n$, by (a). This implies

$$(2.69) \quad w - x \in \bigcap_{n=1}^{\infty} V_n \subseteq Y.$$

Finally, (2.68) and (2.69) together yield

$$(2.70) \quad x = w - (w - x) \in Y - Y = Y,$$

as Y is a subgroup of X . This establishes that

$$(2.71) \quad X \subseteq Y,$$

since x was arbitrary. □

16 Exercise 16. An elementary closed graph theorem

Suppose that X and K are metric spaces, that K is compact, and that the graph of $f : X \rightarrow K$ is a closed subset of $X \times K$. Prove that f is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of K cannot be omitted from the hypothesis, even when X is compact.

PROOF. Choose a sequence $\{x_n : n = 1, 2, 3, \dots\}$ whose limit is an arbitrary a . By compactness of K , the graph G of f contains a subsequence $\{(x_{p(n)}, f(x_{p(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of $X \times K$. G is closed; therefore, $\{(x_{p(n)}, f(x_{p(n)}))\}$ converges in G . So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous, see [[A6]] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.72) \quad \begin{aligned} f : [0, \infty) &\rightarrow [0, \infty) \\ x &\mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \end{aligned}$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$(2.73) \quad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b) . To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on \mathbb{R}_+ . This establishes that $(a, b) \in G_+$, hence the closedness G_+ . Finally, we conclude that G is closed, as a finite union of closed sets. □

Chapter 3

Convexity

3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subseteq X$ an *internal point* of A if $A - x_0$ is an absorbing set.

- (a) Suppose A and B are disjoint convex sets in X , and A has an internal point. Prove that there is a nonconstant functional Λ such that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. (The proof is similar to that of Theorem 3.4)
- (b) Show (with $X = \mathbf{R}^2$, for example) that it may not be possible to have $\Lambda(A)$ and $\Lambda(B)$ disjoint, under the hypotheses of (a).

PROOF. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C . Since A and B are disjoint, $b_0 - x_0$ is not in C , hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem [3.4] of [4] to establish the existence of a functional $\Lambda : X \rightarrow \mathbf{R}$ such that

$$(3.1) \quad \Lambda \leq p$$

and

$$(3.2) \quad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \quad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

$$(3.4) \quad \Lambda a \leq \Lambda b.$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2 . There so exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

$$(3.5) \quad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$(3.6) \quad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad .$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A . This implies

$$(3.7) \quad \Lambda b_1 \stackrel{(3.6)}{<} \Lambda a_3 \stackrel{(3.4)}{\leq} \Lambda b_1 \quad ;$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is \mathbf{R}^2 . Fetch

$$(3.8) \quad S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$

$$(3.9) \quad S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$$

$$(3.10) \quad A \triangleq S_1 \cup S_2,$$

$$(3.11) \quad B \triangleq X \setminus A.$$

Pick (x_i, y_i) in S_i . Let t range over the unit interval, and so obtain

$$(3.12) \quad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A.$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A . Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.*, A° is nonempty¹: A contains an internal point.

Let L be a vector line of \mathbf{R}^2 . In other words, L is the null space of a functional $\Lambda : \mathbf{R}^2 \rightarrow \mathbf{R}$ (to see this, take some nonzero u in L^\perp and set $\Lambda x = (x, u)$ for all x in \mathbf{R}^2). One easily checks that both A and B cut L . Hence

$$(3.13) \quad \Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset.$$

So ends the proof. □

11 Exercise 11. Meagerness of the polar

Let X be an infinite-dimensional Fréchet space. Prove that X^ , with its weak*-topology, is of the first category in itself.*

This is actually a consequence of the lemma below, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual X^* separates points on X , then the polar*

$$(3.14) \quad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\}$$

of any absorbing subset A is a weak-closed set that has empty interior.*

PROOF. Let x range over X . The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous, see [3.14] of [4]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*-closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see [3.11] of [4]). Put

$$(3.15) \quad W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \quad (r_x > 0)$$

where F runs through the nonempty finite subsets of X . Clearly, the collection of all such W is a local base of X^* . Pick one of those W and remark that the following subspace

$$(3.16) \quad M \triangleq \text{span}(F)$$

¹For an immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] of [4] or [2.5] of [3].

is finite dimensional. Assume, to reach a contradiction, that $A \subseteq M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M , which is a desired contradiction. We have just established that $A \not\subseteq M$: Now pick a in $A \setminus M$ and so conclude that

$$(3.17) \quad b \triangleq \frac{a}{t_a} \in A$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M , as a finite dimensional space, is closed (see [1.21 (b)] of [4] for a proof): By the Hahn-Banach theorem [3.5] of [4], there exists Λ_a in X^* such that

$$(3.18) \quad \Lambda_a b > 2$$

and

$$(3.19) \quad \Lambda_a(M) = \{0\}.$$

The latter equality implies that Λ_a vanishes on F ; hence Λ_a is an element of W . On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$(3.20) \quad |\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that $\Lambda + \Lambda_a$ is not in K_A . We have thus proved that

$$(3.21) \quad \Lambda + W \not\subseteq K_A.$$

Since W and Λ are both arbitrary, this achieves the proof. \square

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

PROOF. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X , so that the following sets

$$(3.22) \quad B_n \triangleq \{x \in X : d(0, x) < 1/n\} \quad (n = 1, 2, 3, \dots)$$

form a local base of X . If Λ is in X^* , then

$$(3.23) \quad |\Lambda| \leq m \text{ on } B_n$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$, see [1.18] of [4]. Hence, X^* is the countable union of all

$$(3.24) \quad m \cdot K_n \quad (m, n = 1, 2, 3, \dots),$$

where K_n is the polar of B_n . Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see [3.4] of [4]. As a consequence, the above lemma implies

$$(3.25) \quad (\overline{K_n})^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by m is a homeomorphism (see [1.7] of [4]), this is equivalent to

$$(3.26) \quad (\overline{m \cdot K_n})^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof. \square

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

1 Exercise 1. Basic results

Let ϕ be the embedding of X into X^{**} described in Section 4.5. Let τ be the weak topology of X , and let σ be the weak*-topology of X^{**} - the one induced by X^* .

- (a) Prove that ϕ is a homeomorphism of (X, τ) onto a dense subspace of (X^{**}, σ) .
- (b) If B is the closed unit ball of X , prove that $\phi(B)$ is σ -dense in the closed unit ball of X^{**} . (Use the Hahn-Banach separation theorem.)
- (c) Use (a), (b), and the Banach-Alaoglu theorem to prove that X is reflexive if and only if B is weakly compact.
- (d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.
- (e) If X is reflexive and Y is a closed subspace of X , prove that X/Y is reflexive.
- (f) Prove that X is reflexive if and only if X^* is reflexive.
*Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\phi(X)$ of X^{**} .*

PROOF. Let ψ be the isometric embedding of X^* into X^{***} . The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$(4.1) \quad \phi^{-1} : \phi(X) \rightarrow X$$

$$(4.2) \quad \phi(x) \mapsto x$$

is a homeomorphism (with respect to τ and σ). We first consider

$$(4.3) \quad V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}, \psi x^* \rangle| < r\} \quad (x^* \in X^*, r > 0);$$

$$(4.4) \quad U \triangleq \{x \in X : |\langle x, x^* \rangle| < r\} \quad (x^* \in X^*, r > 0).$$

and remark that the so defined V 's (respectively U 's) shape a local subbase \mathcal{S}_σ (respectively \mathcal{S}_τ) of σ (respectively τ). We now observe that

$$(4.5) \quad U = \phi^{-1}(V \cap \phi(X)) = \phi^{-1}(V) \cap X \quad (V \in \mathcal{S}_\sigma, U \in \mathcal{S}_\tau) \quad ,$$

since ϕ^{-1} is one-to-one. This remains true whether we enrich each subbase \mathcal{S} with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that ϕ^{-1} and its inverse ϕ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$(4.6) \quad \overline{\phi(X)}_\sigma \subseteq X^{**} \quad .$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ -closure of $\phi(X)$. By [3.5], there so exists y^* in X^* such that

$$(4.7) \quad \langle \phi x, \psi y^* \rangle = \langle y^*, \phi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ;$$

$$(4.8) \quad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces y^* to be a the zero of X^* . The functional ψy^* is then the zero of X^{***} : (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \quad \phi(B) \subseteq B^{**} \quad ,$$

since ϕ is isometric. Hence

$$(4.10) \quad \overline{\phi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} \quad .$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\phi(B)}_\sigma$ contains a vector z^{**} . By [3.7], there exists y^* in X^* such that

$$(4.11) \quad |\psi y^*| \leq 1 \quad \text{on } \overline{\phi(B)}_\sigma \quad ;$$

$$(4.12) \quad \langle z^{**}, \psi y^* \rangle > 1 \quad .$$

It follows from (4.11) that

$$(4.13) \quad |\psi y^*| \leq 1 \text{ on } \phi(B) \quad , \quad \text{i.e., } |y^*| \leq 1 \text{ on } B \quad .$$

We have so proved that

$$(4.14) \quad y^* \in B^* \quad .$$

Since z^{**} lies in B^{**} , it is now clear that

$$(4.15) \quad |\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ;$$

what it contradicts (4.12), and thus proves (b). We now prove (c).

It follows from (a) that

$$(4.16) \quad B \text{ is weakly compact if and only if } \phi(B) \text{ is weak*-compact.}$$

If B is weakly compact, then $\phi(B)$ is weak*-closed. So,

$$(4.17) \quad \phi(B) = \overline{\phi(B)}_\sigma \stackrel{(b)}{=} B^{**} \quad .$$

ϕ is therefore onto, i.e., X is reflexive.

Conversely, keep ϕ as onto: one easily checks that $\phi(B) = B^{**}$. The image $\phi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X , whose closed unit ball is B . Let Y be a norm-closed subspace of X : Y is then weakly closed (cf. [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let \equiv stand for “isometrically isomorphic” and apply twice [4.9] to obtain, first

$$(4.18) \quad (X/Y)^* \equiv Y^\perp \quad ,$$

next,

$$(4.19) \quad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \quad .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z , the weak*-topology of Z^* is its weak one.

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\phi(X)$ is a norm-closed subspace of X^{**} ; cf. [4.5]. Hence $\phi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \phi(X)$ is weakly compact, i.e., weak*-compact (to see this, apply (L) with $Z = X^*$).

By (a), B is therefore weakly compact, i.e., X is reflexive, see (c). So ends the proof. \square

13 Exercise 13. Operator compactness in a Hilbert space

- (a) Suppose $T \in \mathcal{B}(X, Y)$, $T_n \in \mathcal{B}(X, Y)$ for $n = 1, 2, 3, \dots$, each T_n has finite-dimensional range, and $\lim \|T - T_n\| = 0$. Prove that T is compact.
- (b) Assume Y is a Hilbert space, and prove the converse of (a): Every compact $T \in \mathcal{B}(X, Y)$ can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)

PROOF. Since each T_n is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every T_n as a compact operator. Let $\{x_0^i : i \in \mathbf{N}\}$ be in U the open unit ball of X . Since T_1 is compact, $\{x_0^i\}$ contains a subsequence $\{x_1^i : i \in \mathbf{N}\}$ such that $\{T_1 x_1^i\}$ converges to a point y_1 of Y . The same reasoning can be recursively applied to T_n and $\{x_{n-1}^i\} \subseteq U$ so that $\{T_n x_n^i\}$ tends to some y_n of Y , as $\{x_n^i\}$ is a subsequence of $\{x_{n-1}^i\}$. Then

$$(4.20) \quad T_n x_p^i \xrightarrow{i \rightarrow \infty} y_n \quad (p, n = 1, 2, 3, \dots) \quad .$$

Applied with $\{x_n^i : (n, i) \in \mathbf{N}^2\}$, a Cantor's diagonal process therefore provides a subsequence $\{\tilde{x}_j : j \in \mathbf{N}\}$ such that

$$(4.21) \quad T_j \tilde{x}_k \xrightarrow{k \rightarrow \infty} y_j \quad ;$$

$$(4.22) \quad T_j \tilde{x}_j \xrightarrow{j \rightarrow \infty} y_j \quad .$$

We now easily obtain

$$(4.23) \quad \|T_j \tilde{x}_j - T_k \tilde{x}_k\| \leq \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \xrightarrow{k \rightarrow \infty} 0 \quad .$$

$\{T_j \tilde{x}_j\}$ is then a Cauchy sequence. So is $\{T \tilde{x}_j\}$, since $\|T - T_j\| \rightarrow 0$. On the other hand, Y is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix ε as a positive scalar. Since T is compact, Y contains a finite set C such that

$$(4.24) \quad T(U) \subseteq \bigcup_{c \in C} B(c, \varepsilon) \quad .$$

As a Hilbert space, Y contains a maximal orthonormal set (or Hilbert basis) M . This implies that $\text{span}(M)$ is dense in Y ; cf. 4.18 & [4.22] of [3]. The finiteness of C forces M to enclose a finite set S so that

$$(4.25) \quad \forall c \in C, \exists s(c) \in \text{span}(S) : \|c - s(c)\| < \varepsilon \quad .$$

Let x be in U . It follows from (4.24) that

$$(4.26) \quad \|Tx - c_x\| < \varepsilon$$

for some c_x of C . We now combine (4.25) and (4.26) to obtain

$$(4.27) \quad \|Tx - s(c_x)\| \leq \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon$$

As a finite-dimensional subspace, $\text{span}(S)$ is closed (see footnote 4, Exercise 1.10). We so obtain

$$(4.28) \quad Y = \text{span}(S) \oplus \text{span}(S)^\perp ,$$

by [12.4]. There so exists a unique projection $\pi = \pi(\varepsilon)$ of Y onto itself (see [5.6] for the definition) such that

$$(4.29) \quad \pi(Y) = \text{span}(S) , \quad (I - \pi)(Y) = \text{span}(S)^\perp .$$

It is easily checked that π has norm 1. Moreover,

$$(4.30) \quad \pi s = s \quad (s \in \text{span}(S)) .$$

Thus,

$$(4.31) \quad (I - \pi)(Tx) = (I - \pi)(Tx - s(c_x)) \quad (x \in U) .$$

Then,

$$(4.32) \quad \|(I - \pi)(Tx)\| \leq \|I - \pi\| \|Tx - s(c_x)\| < 4\varepsilon \quad (x \in U)$$

(the fact that π has norm 1 is hidden in the right side inequality). We have just so proved that

$$(4.33) \quad \|T - \pi \circ T\| \in O_{\varepsilon \sim 0}(\varepsilon) .$$

That is particularly true when $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \xrightarrow{n \rightarrow \infty} 0$. Let so T_n be $\pi(\varepsilon_n) \circ T$ and conclude that these (compact) operators approximate T in the desired fashion, *i.e.*,

$$(4.34) \quad \|T - T_n\| \xrightarrow{n \rightarrow \infty} 0 .$$

□

15 Exercise 15. Hilbert-Schmidt operators

Suppose μ is a finite (or σ -finite) positive measure on a measure space Ω , $\mu \times \mu$ is the corresponding product measure on $\Omega \times \Omega$, and $K \in L^2(\mu \times \mu)$. Define

$$(Tf)(s) = \int_{\Omega} K(s, t) f(t) d\mu(t) \quad [f \in L^2(\mu)] .$$

(a) Prove that $T \in \mathcal{B}(L^2(\mu))$ and that

$$\|T\|^2 \leq \int_{\Omega} \int_{\Omega} |K(s, t)|^2 d\mu(s) d\mu(t) .$$

(b) Suppose a_i, b_i are members of $L^2(\mu)$, for $1 \leq i \leq n$, put $K_1 = \sum a_i(s) b_i(t)$ and define T_1 in terms of K_1 as T was defined in terms of K . Prove that $\dim \mathcal{R}(T_1) \leq n$.

(c) Deduce that T is a compact operator in $L^2(\mu)$. Hint: Use exercise 13.

(d) Suppose $\lambda \in \mathbb{C}, \lambda \neq 0$. Prove: Either the equation

$$Tf - \lambda f = g$$

has a unique solution $f \in L^2(\mu)$ for every $g \in L^2(\mu)$ or there are infinitely many solutions for some g and none for others. (This is known as the Fredholm alternative.).

(e) Describe the adjoint of T .

PROOF. Let X (respectively P) be the Banach space $L^2(\mu)$ (respectively $L^2(\mu \times \mu)$). A consequence of the Radon-Nikodym theorem (cf. [6.16] of [3]) is that there exists a group isomorphism $\rho : X \rightarrow X^*$, $f \mapsto f^*$ such that

$$(4.35) \quad \langle u, f^* \rangle = \int_{\Omega} u \cdot f \, d\mu \quad (u \in X, f \in X) \quad .$$

Define a.e $K_s, K_t : \Omega \rightarrow \mathbf{C}$ by setting

$$(4.36) \quad K_s(t) \triangleq K_t(s) \triangleq K(s, t) \quad \text{a.e.} \quad ((s, t) \in \Omega) \quad .$$

T is clearly linear. Moreover,

$$(4.37) \quad |(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \quad (\|f\|_X < 1)$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with $|K|^2$ to obtain

$$(4.38) \quad \|Tf\|_X^2 \leq \int_{\Omega} \|K_s\|_X^2 \, \mu(s) = \|K\|_P^2 < \infty \quad (\|f\|_X < 1) \quad .$$

(a) is then proved.

To show (b), remark that

$$(4.39) \quad \int_{\Omega} a_i(s) \cdot b_i \cdot f \, d\mu \in \mathbf{C} \cdot a_i(s) \quad \text{a.e.} \quad (f \in X, s \in \Omega) \quad .$$

It is now clear that T maps any f of X into $\mathbf{C} \cdot a_1 + \dots + \mathbf{C} \cdot a_n$. We so conclude that $\dim R(T_1) \leq n$.

We now prove (c). The current part refers to Exercise 4.13. X is also a Hilbert space and so contains a Hilbert basis M . Define a.e

$$(4.40) \quad \begin{aligned} a_b : \Omega &\rightarrow \mathbf{C} \\ s &\mapsto (K_s, b) \end{aligned}$$

whenever b ranges M . Hence,

$$(4.41) \quad K_s = \sum_{b \in M} a_b(s) \cdot b \quad \text{a.e.} \quad (s \in \Omega) \quad .$$

Provided any positive scalar ε , there so exists a finite subset $S = S(\varepsilon)$ of M such that

$$(4.42) \quad \|K_s - \sum_{b \in S} a_b(s) \cdot b\|_X < \varepsilon \quad (s \in \Omega) \quad .$$

Remark that $\sum_{b \in S} a_b \cdot b$ matches the definition of K_1 ; cf. (b): from now on,

$$(4.43) \quad K_1 \triangleq \sum_{b \in S} a_b \cdot b \quad .$$

It follows from (b) that

$$(4.44) \quad \dim R(K_1) < \infty \quad .$$

Now turn back to (a), with $K - K_1$ playing the role of K , and so obtain

$$(4.45) \quad \|T - T_1\| < \varepsilon \mu(\Omega) \leq \infty \quad .$$

For if μ is finite, use (a) of Exercise 4.13 to conclude that T is compact. Assume henceforth that μ is not (necessarily) finite and pick δ in \mathbf{R}_+ . The simple functions (with finite measure support) form a dense family of an L^p space ($1 \leq p < \infty$); cf. [3.13] of [3]. It then exists a simple function K_{δ} of $L^2(\mu \times \mu)$ such that

$$(4.46) \quad (\mu \times \mu)(\{K_{\delta} \neq 0\}) < \infty, \quad \|K - K_{\delta}\|_P < \delta \quad .$$

Define an operator T_δ in terms of K_δ as T was defined in terms of K , and proceed as in (a) with $T - T_\delta$ instead of T . Then

$$(4.47) \quad \|T - T_\delta\| < \delta \quad .$$

The key ingredient is that K_δ can be identified with an element of the finite measure space $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$. What we have attempted to approximate T by T_1 can therefore be reiterated (with K_δ playing the role of K) to achieve an approximation $T_{\delta,1}$ of T_δ so that

$$(4.48) \quad \|T_\delta - T_{\delta,1}\| < \varepsilon \quad .$$

It now follows from (4.47) and (4.48) that

$$(4.49) \quad \|T - T_{\delta,1}\| \leq \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta \quad .$$

Since ε and δ were arbitrary, the σ -finite case is proved. We now establish (d).

Provided g of X , let E_g be the following equation on X

$$(4.50) \quad Tf - \lambda f = g \quad ,$$

whose solution set is denoted by S_g . Note that S_0 is $\ker(T - \lambda)$ and discard the trivial case $S_0 = X^1$: each f of X lies in $S_{Tf - \lambda f}$, as some $Tf - \lambda f$'s are nonzero. Some S_g 's are then nonempty. Remark that

$$(4.51) \quad S_g = f + S_0 \quad (f \in S_g)$$

in such case. Furthermore, the equality $\beta = \alpha$ of [4.25] yields

$$(4.52) \quad (T - \lambda I)(X) \neq X, \text{ , i.e., } S_0 \neq \{0\} \quad .$$

So if $T - \lambda I$ is not onto, not only some S_g 's are empty, but also $S_0 \neq \{0\}$. Every nonempty S_g (such sets always exist, see above) is then infinite, by (4.51).

Otherwise, $T - \lambda I$ is bijective and every equation E_g has then a unique solution f . The Fredholm alternative is so proved.

Our last step is the description of T^* . Let $S : X \rightarrow X$ be such that

$$(4.53) \quad (Sf)(t) \triangleq \int_{\Omega} K_t \cdot f \text{ a.e.} \quad (f \in X, t \in \Omega)$$

Proceed as in (a), with S instead of T : S lies in $\mathcal{B}(X)$. Next, we claim that

$$(4.54) \quad \langle u, T^*f^* \rangle = \langle Tu, f^* \rangle$$

$$(4.55) \quad = \int_{\Omega} (Tu) \cdot f \, d\mu$$

$$(4.56) \quad = \int_{\Omega^2} K \cdot f \cdot u \, d(\mu \times \mu)$$

$$(4.57) \quad = \int_{\Omega} (Sf) \cdot u \, d\mu$$

$$(4.58) \quad = \langle u, (Sf)^* \rangle \quad ,$$

whenever u and f run through the closed unit ball of X . Since $\|T\|, \|T^*\|$ are equal and finite, only exactness of (4.56) is possibly in doubt; the justification below dissipates it. In conclusion,

$$(4.59) \quad T^* = \rho S \rho^{-1} \quad .$$

Informally,

$$(4.60) \quad T^* = S \quad .$$

¹, e.g., $X = L^2(\{0\}, \delta)$ so that $I = \lambda^{-1}T$ is compact.

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep u and f as above. Let us introduce

$$(4.61) \quad A(s) \triangleq \int_{\Omega} |K_s(t) \cdot u(t)| \, d\mu(t) \quad \text{a.e. } (s \in \Omega) \quad ,$$

to make hold the following Cauchy-Schwarz inequality

$$(4.62) \quad A(s) \leq \|K_s\|_X \quad (s \in \Omega) \quad .$$

Thus,

$$(4.63) \quad \int_{\Omega^2} |K(s, t) u(t) f(s)| \, d\mu(s) \, d\mu(t) = \int_{\Omega} |f(s)| A(s) \, d\mu(s)$$

$$(4.64) \quad \leq \int_{\Omega} |f(s)| \|K_s\|_X \, d\mu(s)$$

$$(4.65) \quad \leq \left[\int_{\Omega} \|K_s\|_X^2 \, d\mu(s) \right]^{\frac{1}{2}} = \|K\|_P < \infty \quad .$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof. \square

Chapter 6

Distributions

1 Exercise 1. Test functions are almost polynomial

Suppose f is a complex continuous function in \mathbf{R}^n , with compact support. Prove that $\psi P_j \rightarrow f$ uniformly on \mathbf{R}^n , for some $\psi \in \mathcal{D}$ and for some sequence $\{P_j\}$ of polynomials.

PROOF. According to 1.16, Ω is union of a compact sets sequence $\{K_i\}$ and $\text{supp}(f)$ lies in some $K = K_i$ so that f is embedded in $\mathcal{D}(\Omega)$. We can apply [1.10] to ensure that Ω encloses a compact set $S = \overline{K + B(\varepsilon)}$ for sufficiently small $\varepsilon > 0$.

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra $\{g \in C(S) : g \text{ is polynomial}\}$ of $C(S)$. There so exists a sequence $\{P_j : j \in \mathbf{N}\}$ of $\mathbf{R}[X_1, \dots, X_n]$ such that

$$(6.1) \quad \sup_S |f - P_j| \xrightarrow{j \rightarrow \infty} 0 \quad .$$

By [6.20], the open set $K + B(\varepsilon)$ has a local partition of unity $\{\psi_i\} \subseteq \mathcal{D}(\Omega)$. Moreover, there exists an integer l such that $\psi = \psi_1 + \dots + \psi_l$ equals 1 on K . Hence

$$(6.2) \quad \|f - \psi P_j\|_\infty = \|\psi f - \psi P_j\|_\infty = \sup_S |\psi f - \psi P_j|$$

$$(6.3) \quad = \sup_S |f - P_j| \xrightarrow[j \rightarrow \infty]{(6.1)} 0 \quad .$$

□

We will actually prove more by showing that $\mathcal{D}(\Omega)$ is separable for each nonempty open subset Ω of \mathbf{R}^n .

PROOF. The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the separability of $\mathcal{D}(\Omega)$. To do so, we keep (α, j) in $\mathbf{N}^n \times \mathbf{N}$. Remark that S encloses $\text{supp}(D^\alpha f)$: according to the first part, there exists a sequence $\{P_{\alpha, j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \dots, X_n]$ such that

$$(6.4) \quad \|D^\alpha f - \psi P_{\alpha, j}\|_\infty \xrightarrow{j \rightarrow \infty} 0 \quad .$$

Now let m range over $\{1, 2, 3, \dots\}$ and set $W_{m, j}$ in $\mathcal{D}(\Omega)$ as follows

$$(6.5) \quad D^{-\alpha} \phi \in \mathcal{D}(\Omega) : D^\alpha D^{-\alpha} \phi = \phi \quad .$$

$$(6.6) \quad W_{m, j}(x) \triangleq D^{-(m, \dots, m)}(\psi P_{(m, \dots, m), j})$$

By (6.4), there exists a natural number $k(m)$ such that

$$(6.7) \quad \|D^{(m, \dots, m)}(f - W_{m, j})\|_\infty < 1/m \quad (j \geq k(m)) \quad .$$

Assume without loss of generality that S has diameter 1 so that (6.7) yields

$$(6.8) \quad \|D^\lambda(f - W_{m, k(m)})\|_\infty < 1/m \quad (|\lambda| \leq m) \quad ,$$

by the mean value theorem. In other words (remark that $\text{supp}(f - W_{m,k(m)}) \subseteq S$),

$$(6.9) \quad f - W_{m,k(m)} \in U_m \triangleq \{\phi \in \mathcal{D}_S : \|\phi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \dots \quad (m = 1, 2, 3, \dots) \quad .$$

Pick W in β (see (b) of [6.3]): $W \cap \mathcal{D}_S$ contains a neighborhood of 0. Hence W contains some U_m , for m sufficiently large. Thus

$$(6.10) \quad W_{m,k(m)} \xrightarrow{m \rightarrow \infty} f \quad (\text{in } \mathcal{D}(\Omega)) \quad .$$

We have so established that the $W_{m,k(m)}$'s family is dense in $\mathcal{D}(\Omega)$. We now aim to disclose a countable set \tilde{W} that has the same property.

Choose δ in \mathbf{R}_+ and fetch any $W_{m,k(m)}$. Let X be (X_1, \dots, X_n) and express $P_{(m, \dots, m), k(m)}$ as

$$(6.11) \quad P(X) = \sum_{|\gamma| \leq d} p_\gamma \cdot X^\gamma \quad .$$

Since $\bar{Q} = \mathbf{R}$, $Q[X]$ hosts some $Q(X) = \sum_{|\gamma| \leq d} q_\gamma \cdot X^\gamma$ such that $|p_\gamma - q_\gamma| < \delta$ for all γ . Thus,

$$(6.12) \quad |P(x) - Q(x)| \leq \sum_{|\gamma| \leq d} |p_\gamma - q_\gamma| |x|^{|\gamma|} \leq \delta \sum_{l \leq d} \binom{1+n-1}{n-1} \|x\|_\infty^l \quad (x \in \mathbf{R}^n) \quad .$$

Since S is bounded, we so obtain

$$(6.13) \quad \|\psi(P - Q)\|_\infty \in O(\delta) \quad .$$

Now define \tilde{W}_m in terms of Q as $W_{m,k(m)}$ was defined in terms of P , and consider the integrations made in (6.6): each $D^\lambda \tilde{W}_m$ ($|\lambda| \leq m$) can be obtained from some of them. So (6.13) yields

$$(6.14) \quad \|D^\lambda(W_{m,k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \quad (|\lambda| \leq m) \quad .$$

To be more specific, these λ 's only exist in finite amount, so the big O can be assumed to be the same for all them. Since δ was arbitrary, combining (6.10) with (6.14) establishes the density of the all \tilde{W}_m 's family \tilde{W} .

Furthermore, each member of \tilde{W} is only made of two ingredients: ψ and a polynomial of $Q[X]$. The mapping ψ is attached to some K_i and $Q[X]$ inherits countableness from \bar{Q} . Note that the “integrations packs” of (6.6) only exist in countable amount. Our \tilde{W} is then countable. \square

6 Exercise 6. Around the supports of some distributions

(a) Suppose that $c_m = \exp\{-(m!)!\}$, $m = 0, 1, 2, \dots$. Does the series

$$\sum_{m=0}^{\infty} c_m (D^m \phi)(0)$$

converges for every $\phi \in C^\infty(\mathbf{R})$?

(b) Let Ω be open in \mathbf{R}^n , suppose $\Lambda_i \in \mathcal{D}'(\Omega)$, and suppose that all Λ_i have their supports in some fixed compact $K \subseteq \Omega$. Prove that the sequence $\{\Lambda_i\}$ cannot converge in $\mathcal{D}'(\Omega)$ unless the orders of the Λ_j are bounded. Hint: Use the Banach-Steinhaus theorem.

(c) Can the assumption about the supports be dropped in (b)?

PROOF. The answer is: no. Let us establish this assertion. Assume, to reach a contradiction, that the above series converges for every smooth $\phi : \mathbf{R} \rightarrow \mathbf{C}$.

The sequence $\{c_m (D^m \phi)(0)\}$ so converges to 0. Nevertheless, it is proved in [1.46] that $C^\infty(\Omega)$ is not locally bounded. In other words, it is always possible to excavate a ϕ for which the magnitude of the m -th derivative at 0

is as large as we please¹, *e.g.*, greater than $1/c_m$. A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume $\{\Lambda_j\}$ to converge to some Λ of $\mathcal{D}'(\Omega)$ and we let Q run through the compact sets of Ω . Next, we define

$$(6.15) \quad S(T, Q) \triangleq \{N \in \mathbf{N}, \exists C \in \mathbf{R}_+ : |T\phi| \leq C \|\phi\|_N \text{ for all } \phi \text{ of } \mathcal{D}_Q\} \quad (T \in \mathcal{D}'(\Omega)) \quad .$$

Such subset of \mathbf{N} has a minimum $\omega(T, Q)$. The following value

$$(6.16) \quad \omega(T) \triangleq \max\{\omega(T, Q) : Q \subseteq \Omega, Q \text{ compact}\} \leq \infty$$

is then the order of T . Assume, to reach a contradiction, that, after passage to a subsequence,

$$(6.17) \quad \omega(\Lambda_j, Q_j) = j \quad (j = 1, 2, 3, \dots)$$

for some compact $Q = Q_j$. By (a) of [6.24], Q_j cuts $\text{supp } \Lambda_j$, say in p_j . Since K encloses $\text{supp } \Lambda_j$, $\{p_j\}$ tends, after passage to a subsequence, to some p of K . Choose a positive scalar r so that

$$(6.18) \quad \bar{B}(p, r) \triangleq \{x \in \mathbf{R}^n : |x - p| \leq r\} \subseteq \Omega \quad .$$

Such closed ball $\bar{B}(p, r)$ is a compact subset of Ω . By (b) of [6.5] (which refers to [1.46]) $\mathcal{D}_{\bar{B}(p, r)}$ is then a Fréchet space. It now follows from [2.6] that $\{\Lambda_j\}$ is equicontinuous on $\mathcal{D}_{\bar{B}(p, r)}$. There so exists² a nonnegative integer N such that

$$(6.19) \quad |\Lambda\phi| \leq C \|\phi\|_N \quad (\phi \in \mathcal{D}_{\bar{B}(p, r)})$$

for some positive constant C . On the other hand, $\bar{B}(p, r)$ contains almost all the p_j 's. Hence

$$(6.20) \quad |\Lambda_N \phi| > C \|\phi\|_N$$

for some ϕ of $\mathcal{D}_{\bar{B}(p, r)}$. (b) is then established.

To prove (c), we introduce a sequence $\{x_m : m \in \mathbf{Z}\}$ of Ω that has no limit point. Let $\{\alpha_m : m \in \mathbf{Z}\}$ be in \mathbf{N} and so define³

$$(6.21) \quad \begin{aligned} \Lambda : \mathcal{D}(\Omega) &\rightarrow \mathbf{C} \\ \phi &\mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m} \phi)(x_m) \end{aligned} \quad .$$

Λ belongs to $\mathcal{D}'(\Omega)$, since $\{x_m\}$ has no limit point. Next, we easily check that

$$(6.22) \quad \begin{aligned} \Lambda_j : \mathcal{D}(\Omega) &\rightarrow \mathbf{C} \\ \phi &\mapsto \sum_{|m| \leq j} (D^{\alpha_m} \phi)(x_m) \end{aligned} \quad (j \in \mathbf{N})$$

is also a distribution and that $\{\Lambda_j\}$ tends to Λ in $\mathcal{D}'(\Omega)$. Nevertheless, no Λ_j 's can have common support because $\{x_m\}$ has no limit point. Our assumption can therefore be dropped. \square

9 Exercise 9. Convergence in \mathcal{D} vs. convergence in \mathcal{D}'

(a) Prove that a set $E \subseteq \mathcal{D}(\Omega)$ is bounded if and only if

$$\sup\{|\Lambda\phi| : \phi \in E\} < \infty$$

for every $\Lambda \in \mathcal{D}'(\Omega)$.

¹indeed [1.46] provides sufficient tools for constructive proof of this, see the $\phi_j - \check{\phi}_j$ involved in (??).

²For more details, see Exercise 2.3.

³As $\Omega = \mathbf{R}$, the case $\alpha_m = m$ is the “counterpart” of the series of (a) and the case $(x_m, \alpha_m) = (m, 0)$ is the Dirac comb.

- (b) Suppose $\{\phi_j\}$ is a sequence in $\mathcal{D}(\Omega)$ such that $\{\Lambda\phi_j\}$ is a bounded sequence of numbers, for every $\Lambda \in \mathcal{D}'(\Omega)$. Prove that some subsequence of $\{\phi_j\}$ converges, in the topology of $\mathcal{D}(\Omega)$.
- (c) Suppose $\{\Lambda_j\}$ is a sequence in $\mathcal{D}'(\Omega)$ such that $\{\Lambda_j\phi\}$ is bounded, for every $\phi \in \mathcal{D}(\Omega)$. Prove that some subsequence of $\{\Lambda_j\}$ converges in $\mathcal{D}'(\Omega)$ and that the convergence is uniform on every bounded subset of $\mathcal{D}(\Omega)$.
Hint: By the Banach-Steinhaus theorem, the restrictions of the Λ_j to \mathcal{D}_K are equicontinuous. Apply Ascoli's theorem.

PROOF. Since $\mathcal{D}(\Omega)$ is a locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $E = \{\phi_j : j \in \mathbf{N}\}$ is bounded: so is \bar{E} . By (c) of [6.5], there exists some \mathcal{D}_K that contains \bar{E} . Since \mathcal{D}_K has the Heine-Borel property (see [1.46]), \bar{E} is τ_K -compact. Apply [A4] with the metrizable space \mathcal{D}_K (see [1.46]) to conclude that \bar{E} has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds. \square

Annex

A.1 Mean value and bounded derivatives

Lemma A.1 A mean value inequality for higher-order derivatives.

If $\phi \in \mathcal{D}_{[a,b]}$, then

$$(A.1) \quad \|D^k \phi\|_{\infty} \leq \|D^p \phi\|_{\infty} \left(\frac{b-a}{2} \right)^{p-k}$$

for all $k \leq p$ in \mathbf{N} .

PROOF. First, consider $a < x_0 \leq (a+b)/2$. By the mean value theorem, there exists $a < x_1 < x_0$ such that

$$(A.2) \quad \phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a).$$

Repeating the same reasoning for $D\phi, D^2\phi, \dots, D^p\phi \in \mathcal{D}_{[a,b]}$ gives

$$(A.3) \quad \phi(x_0) = D^0\phi(x_0)$$

$$(A.4) \quad = D^1\phi(x_1)(x_0 - a)$$

$$\vdots$$

$$(A.5) \quad = D^p\phi(x_p)(x_{p-1} - a) \cdots (x_0 - a),$$

for some $a < x_p < \cdots < x_1 < x_0$. Hence

$$(A.6) \quad |\phi(x_0)| \leq \|D^p\phi\|_{\infty} \left(\frac{b-a}{2} \right)^p.$$

Similarly, if $b > x_0 \geq (a+b)/2$ (with b playing the role of a), the same inequality holds. Thus,

$$(A.7) \quad |\phi(x_0)| \leq \|D^p\phi\|_{\infty} \left(\frac{b-a}{2} \right)^p \quad (a < x_0 < b),$$

which establishes the result when $k = 0$. Finally, applying the latter inequality to $D^k\phi$ in place of ϕ shows that

$$(A.8) \quad \|D^k\phi\|_{\infty} \leq \|D^p\phi\|_{\infty} \left(\frac{b-a}{2} \right)^{p-k}$$

for all $0 \leq k \leq p$. □

Lemma A.2 Higher derivatives cannot be bounded by lower derivatives.

There is no general formula to estimate higher-order derivatives from lower-order derivatives. This immediately implies that no reversed mean value theorem exists.

PROOF. For *angular frequency* $\omega > 1$, we consider

$$(A.9) \quad \begin{aligned} \phi_{\omega} : \mathbf{R} &\rightarrow [-1, 1] \\ t &\mapsto \sin(\omega t) \end{aligned}$$

so that

$$(A.10) \quad \frac{\|D^p \phi_\omega\|_\infty}{\|D^k \phi_\omega\|_\infty} = |\omega|^{p-k} \quad (0 \leq k < p)$$

is unbounded as p or ω tends to ∞ . Note that no pointwise estimation holds either. Indeed, when it exists, the quotient $Q_\omega(t) = \left| \frac{D^p \phi_\omega}{D^k \phi_\omega}(t) \right|$ is ω^{p-k} if p and k have same parity. Otherwise, $Q_\omega(t)$ is either $\omega^{p-k} \tan(\omega t)$ or $\omega^{p-k} \cot(\omega t)$. In all cases, $Q_\omega(t)$ can be arbitrarily large when $\omega \rightarrow \infty$, which defeats any pointwise bound $\sup_\omega Q_\omega(t) \leq M(t)$. This example rules out any general inequality in the reversed direction. However, the following smooth example ϕ keeps constant derivative ϕ' around 0, which is the simplest possible behavior.

Let $\rho \in C^\infty(\mathbf{R})$ be defined as 1 on $(-\infty, 0]$, 0 on $[1, \infty)$, and a strict decay on $]0, 1[$. A standard choice is $\rho = 1 - h$ with

$$(A.11) \quad h(t) \triangleq \frac{e^{-1/t}}{e^{-1/t} + e^{-1/(1-t)}}$$

for all $0 < t < 1$. Inspired by signal processing, we choose two more positive parameters: maximum amplitude (A) and delay (τ). We now define ϕ as the time-dependent solution of

$$(A.12) \quad \begin{cases} \phi(0) &= 0 \\ \phi'(t) &= A\rho(\omega(|t| - \tau)) \end{cases} \quad (t \in \mathbf{R})$$

Equivalently, ϕ is odd and, for $t > 0$,

$$(A.13) \quad \phi(t) = A \int_0^t \rho(\omega(s - \tau)) ds = A \min(t, \tau) + A \int_\tau^{\max(t, \tau)} \rho(\omega(s - \tau)) ds$$

Note that the maximum of $|\phi|$ is

$$(A.14) \quad \|\phi\|_\infty = \tau A + A \int_\tau^{\tau+1/\omega} \rho(\omega(s - \tau)) ds$$

$$(A.15) \quad = \tau A + \frac{A}{\omega} \int_0^1 \rho(u) du$$

$$(A.16) \quad < \tau A + \frac{A}{\omega}.$$

The special case $\tau = 1/\omega = 1/A$ is of great interest. In this case, $\phi'(0) = A \rightarrow \infty$ as $\tau \rightarrow 0$. In contrast, the maximum $\|\phi\|_\infty$ remains less than 2. \square

A.2 Dirac's impulse, a physicist's detour

Consider a physical example: a particle colliding with a surface, which absorbs a unit of energy at impact time $t = 0$. We start with H the *Heaviside step function* $t \mapsto [t \geq 0]$, so that $H(t)$ indicates whether the particle has contributed its energy by time t . This formalism expresses that

- (a) The energy is transferred by an instantaneous jump at time $t = 0$.
- (b) The energy is conserved over time.

Heuristically, we *write*,

$$(A.17) \quad \int_{\mathbf{R}} d\mathcal{H} = 1, \quad \frac{d\mathcal{H}}{dt} = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

These properties cannot coexist in standard calculus. Nevertheless the informal measure $d\mathcal{H}$ describes the *Dirac δ function*. When identified with a positive Borel measure, δ has total mass 1 because

$$(A.18) \quad \int_{\mathbf{R}} d\delta = \int_{\mathbf{R}} d\mathcal{H} = [H]_{-\infty}^{\infty} = 1.$$

Physically, integrating δ over time recaptures all the energy. Let W be the observation window, which is adjusted so that either $0 \in W^\circ$ or $0 \notin \overline{W}$. Next, consider any smooth real function ϕ with compact support in W° as a test signal. In this formalism, the integral $\int_{\mathbf{R}} \phi d\delta$ represents the detector's response to the collision when ϕ is not identically zero. If $\max |\phi| = 1$, then Lebesgue's dominated convergence theorem ensures

$$(A.19) \quad \sup_{\phi} \int_{W^\circ} |\phi| d\delta = \int_{W^\circ} d\delta = [0 \in W^\circ].$$

We now make the model rigorous by eliminating the heuristic \mathcal{dH} , as follows:

$$(A.20) \quad \int_{\mathbf{R}} \phi d\delta = \int_{\mathbf{R}} \phi d\mathcal{H} \quad (\text{generalization of (A.18)})$$

$$(A.21) \quad = \left[H\phi \right]_{-\infty}^{\infty} - \int_{\mathbf{R}} H d\phi \quad (\text{integration by parts})$$

$$(A.22) \quad = - \int_{\mathbf{R}} H d\phi \quad (\text{supp } \phi \text{ is compact})$$

$$(A.23) \quad = \phi(0)$$

The key point is that the right-hand side in (A.22) is valid in standard calculus. Nevertheless, we obtain all filtered responses as the evaluation functional $\phi \mapsto \phi(0)$. This motivates the following definitions:

$$(A.24) \quad \Lambda_H(\phi) \triangleq \int_{\mathbf{R}} H\phi dt \quad (\text{expresses } H)$$

$$(A.25) \quad \Lambda'_H(\phi) \triangleq - \int_{\mathbf{R}} H d\phi = \phi(0) \quad (\text{the weak derivative of } H: \text{ impulse at } 0)$$

$$(A.26) \quad \delta(\phi) \triangleq \phi(0) \quad (\delta \text{ now has a rigorous definition}).$$

The functional $\delta : \phi \mapsto \phi(0)$ is rigorously defined and so represents an instantaneous energy injection at $t = 0$. Its extension to all $\phi \in C_c(\mathbf{R})$ turns δ into a positive Radon measure of norm/total variation $\|\delta\| = 1$ and support $\{0\}$. In the sense of distribution theory, δ is a (tempered) distribution of order 0; see [Chapters 6 and 7] of [4]. Notably, its Borel-measure counterpart is the *Dirac measure*

$$(A.27) \quad \delta : E \mapsto [0 \in E]$$

restricted to Borel sets in \mathbf{R} ; hence the special case of (15)

$$(A.28) \quad \int_{\mathbf{R}} \phi d\delta = \delta(\phi) = \phi(0).$$

Convolution of δ with translated signal $\phi_t : s \mapsto \phi(t - s)$ extends (A.28), as follows:

$$(A.29) \quad [\delta * \phi](t) \triangleq \int_{\mathbf{R}} \phi_t d\delta = \delta(\phi_t) = \phi(t).$$

The Radon measure δ now serves as the convolution identity.

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