# Solutions to some exercises from Walter Rudin's *Functional Analysis*

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# Contents

# Chapter 1

# Topological Vector Spaces

## 1.1 Exercise 7. Metrizability & number theory

*Let be X the vector space of all complex functions on the unit interval* $[0,1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \quad (0 \le x \le 1).$$

*This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence* $\{f_n\}$ *in X such that (a)* $\{f_n\}$ *converges to* $0$ *as* $n \to \infty$*, but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to* $0$*. (Use the fact that the collection of all complex sequences converging to* $0$ *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

*Proof.* Our justification consists in proving that $\tau$-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms $p_x$ is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$V_{x,k} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0,1], k \in \mathbf{N}) \tag{1.1}$$

is therefore a local base for a topology $\tau$ on X. Given $\{f_n : n = 1, 2, 3, \dots\}$, we put

$$\mathtt{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \tag{1.2}$$

with the convention "$\Sigma = \infty$" whether the sum has no finite support. So,

$$\sum_{i=1}^{m} \mathtt{off}(U_i) = \sum_{n=1}^{\infty} \sum_{i=1}^{m} [f_n \notin U_i] \ge \mathtt{off}(U_1 \cap \dots \cap U_m). \tag{1.3}$$

We first assume that $\{f_n\}$ $\tau$-converges to some f in X, *i.e.*

$$\mathtt{off}(f + V) < \infty \quad (V \in \mathscr{B}). \tag{1.4}$$

The special cases $V_{x,1}, V_{x,2}, \dots,$ mean the pointwise convergence $f_n(x) \overset{n\infty}{\longrightarrow} f(x)$. Conversely, assume that $\{f_n\}$ does not $\tau$-converges to any g in X, *i.e.*

$$\forall g \in X, \exists W \in \mathscr{B} : \mathtt{off}(g + W) = \infty. \tag{1.5}$$

Given g, such W is, by definition, a finite intersection $V_{x_1,k_1} \cap \dots \cap V_{x_m,k_m}$. Thus,

$$\sum_{i=1}^{m} \mathtt{off}(g + V_{x_i,k_i}) \overset{(1.3)}{\ge} \mathtt{off}(g + W) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\mathtt{off}(g + V_{x_i,k_i})$ must then be $\infty$. In other words, there exists a point $x_i$ for which $\{f_n(x_i)\}$ does not converge to $g(x_i)$. g being arbitrary, we so conclude that $f_n$ does not converge pointwise. We have just proved that $\tau$-convergence is a rewording of pointwise convergence. We now prove the second part. From now on, we let *k*, *n* and *p* run on $\mathbf{N}_+$, as $\mathtt{dyadic}(x)$ denotes the usual dyadic expansion of x, so that $\mathtt{dyadic}(x)$ is an aperiodic binary sequence **iff** x is irrational. Define

$$f_n(x) \triangleq \begin{cases} \exp_2\left(-\sum_{k=1}^{n} \mathtt{dyadic}(x)_{-k}\right) & (x \in [0,1] \setminus \mathbf{Q}) \\ 0 & (x \in [0,1] \cap \mathbf{Q}), \end{cases} \tag{1.7}$$

so that $f_n(x) \xrightarrow{n\infty} 0$, and take $\gamma_n \xrightarrow{n\infty} \infty$, *i.e.* at fixed p, $\gamma_n$ is greater than $2^p$ for almost all n. Next, choose $n_p$ among those *almost all* n that are large enough to satisfy

$$n_{p-1} - n_{p-2} < n_p - n_{p-1} \tag{1.8}$$

(start with $n_{-1} = n_0 = 0$) and so obtain

$$2^p < \gamma_{n_p} : 0 < n_p - n_{p-1} \underset{p\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator $\chi$ of $\{n_1, n_2, \dots\}$ in $\mathbf{Z}$ is then aperiodic, *i.e.*

$$\alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \in [0,1] \setminus \mathbf{Q}. \tag{1.10}$$

Hence, $\chi$ is not a the infinite-support expansion of a rational number; which forces

$$\mathtt{dyadic}(\alpha_\gamma)_{-k} = \chi_k. \tag{1.11}$$

The key ingredient is that

$$\chi_1 + \cdots + \chi_{n_p} = p\,. \tag{1.12}$$

Combined with (1.7), it yields

$$f_{n_p}(\alpha_\gamma) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n_p} f_{n_p}(\alpha_\gamma) > 1. \tag{1.14}$$

There so exists $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. In other words, (b) holds, which is in violent contrast with 1.28 of [3]: X is therefore not metrizable. So ends the proof. □

## 1.2 Exercise 9. Quotient map

*Suppose*

(*a*) $X$ *and* $Y$ *are topological vector spaces,*

(*b*) $\Lambda : X \to Y$ *is linear.*

(*c*) $N$ *is a closed subspace of* $X$*,*

(*d*) $\pi : X \to X/N$ *is the quotient map, and*

(*e*) $\Lambda x = 0$ *for every* $x \in N$*.*

*Prove that there is a unique* $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$*. Prove that* $f$ *is linear and that* $\Lambda$ *is continuous if and only if* $f$ *is continuous. Also,* $\Lambda$ *is open if and only if* $f$ *is open.*

*Proof.* Bear in mind that $\pi$ continously maps $X$ onto the topological (Hausdorff) space $X/N$, since $N$ is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \tag{1.15}$$

To ensure that f is actually a mapping, simply remark that the linearity of $\Lambda$ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'. \tag{1.16}$$

It straightforwardly derives from (1.15) that f inherits linearity from $\pi$ and $\Lambda$.

**Remark.** The special case $N = \{\Lambda = 0\}$ , *i.e.* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$f(\pi x) = 0 \overset{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \tag{1.17}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. Conversely, if $\Lambda$ is continuous, then for each neighborhood V of $0_Y$ there exists a neighborhood U of $0_X$ such that

$$\Lambda(U) = f(\pi(U)) \subset V. \tag{1.18}$$

Since $\pi$ is open (1.41 (a) of [3]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \tag{1.19}$$

for some neighborhood V of $0_X$. So,

$$f(W) = f(\pi(V)) = \Lambda(V). \tag{1.20}$$

As a consequence, if $\Lambda$ is open, then f(W) is a neighborhood of $0_Y$. So ends the proof. $\square$

## 1.3 Exercise 10. An open mapping theorem

*Suppose that X and Y are topological vector spaces,* $\dim \mathrm{Y} < \infty$, $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$.

(*a*) *Prove that* $\Lambda$ *is an open mapping.*

(*b*) *Assume, in addition, that the null space of* $\Lambda$ *is closed, and prove that* $\Lambda$ *is continuous.*

*Proof.* We discard the trivial case $\dim \mathrm{Y} = 0$ then henceforth assume that $\dim \mathrm{Y}$ has positive dimension n.

Let e range over a base of Y: For each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, since $\Lambda$ is onto. So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \Lambda \mathrm{x_e} \quad (\mathrm{y} \in \mathrm{Y}). \tag{1.21}$$

The sequence $\{\mathrm{x_e}\}$ is finite; therefore it is bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$\mathrm{x_e} \in \mathrm{sV} \text{ for all } \mathrm{x_e}. \tag{1.22}$$

Combining this with (1.21) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \quad (\mathrm{y} \in \mathrm{Y} : |\mathrm{y_e}| < \mathrm{s}^{-1}), \tag{1.23}$$

which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let $\mathrm{f}, \pi$ be as in Exercise 1.9, with $\{\Lambda = 0\}$ playing the role of N. Since $\Lambda$ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$\dim \mathrm{X/N} = \mathrm{n}. \tag{1.24}$$

f is then an homeomorphism of $\mathrm{X/N} \equiv \mathbf{C}^{\mathrm{n}}$ onto Y; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$. □

## 1.4 Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms

*Put* $\mathrm{K} = [0,1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0,1,2,\dots$*) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

(*a*) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty = \sup\{|\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})| : \infty < \mathrm{x} < \infty\}$

(*b*) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_1 = \int_0^1 |\mathrm{D}^{\mathrm{n}}\mathrm{f}(x)|\,\mathrm{d}x$

(*c*) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_2 = \left\{\int_0^1 |\mathrm{D}^{\mathrm{n}}\mathrm{f}(x)|^2\mathrm{d}x\right\}^{1/2}$.

*Proof.* First, remark that

$$\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_1 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_2 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty < \infty \tag{1.25}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of $\mathrm{D}^{\mathrm{n}}\mathrm{f}$ lies in K; which yields

$$|\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})\,| = \left|\int_0^{\mathrm{x}} \mathrm{D}^{\mathrm{n}+1}\mathrm{f}\right| \le \int_0^{\mathrm{x}} \left|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\,\right| \le \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_1. \tag{1.26}$$

So,

$$\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty \le \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_1. \tag{1.27}$$

We now combine (1.25) with (1.27) and so obtain

$$\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_1 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_2 \le \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_\infty \le \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_1 \le \cdots \quad (\mathrm{n} = 0,1,2,\dots). \tag{1.28}$$

Put

$$\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \triangleq \{\mathrm{f} \in \mathscr{D}_{\mathrm{K}} : \|\,\mathrm{f}\,\|_{\mathrm{i}} < 2^{-\mathrm{n}}\} \quad (\mathrm{i} = 1,2,\infty) \tag{1.29}$$

$$\mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} : \mathrm{n} = 0,1,2,\dots\}, \tag{1.30}$$

so that (1.28) is mirrored in terms of neighborhood inclusions, as follows,

$$\mathrm{V}_{\mathrm{n}}^{(1)} \supset \mathrm{V}_{\mathrm{n}}^{(2)} \supset \mathrm{V}_{\mathrm{n}}^{(\infty)} \supset \mathrm{V}_{\mathrm{n}+1}^{(1)} \supset \cdots. \tag{1.31}$$

Since $\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \supset \mathrm{V}_{\mathrm{n}+1}^{(\mathrm{i})}$, $\mathscr{B}^{(\mathrm{i})}$ is a local base of a topology $\tau_{\mathrm{i}}$. But the chain (1.31) forces

$$\tau_1 = \tau_2 = \tau_\infty. \tag{1.32}$$

To see that, choose a set S that is $\tau_1$-open at f, *i.e.* $\mathrm{V}_{\mathrm{n}}^{(1)} \subset \mathrm{S} - \mathrm{f}$ for some n. Next, concatenate this with $\mathrm{V}_{\mathrm{n}}^{(2)} \subset \mathrm{V}_{\mathrm{n}}^{(1)}$ (see (1.31)) and so obtain $\mathrm{V}_{\mathrm{n}}^{(2)} \subset \mathrm{S} - \mathrm{f}$; which implies that S is $\tau_2$-open at f. Similarly, we deduce, still from (1.31), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.33}$$

So ends the proof. □

## 1.5 Exercise 16. Uniqueness of topology for test functions

*Prove that the topology of* $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

**Comment** This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $\mathrm{p_n}$, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

**Lemma 1** Let X be a topological space with a countable local base $\{\mathrm{V_n} : \mathrm{n} = 1, 2, 3, \dots\}$. If $\tilde{\mathrm{V}}_\mathrm{n} = \mathrm{V_1} \cap \cdots \cap \mathrm{V_n}$, then every subsequence $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$ is a decreasing (*i.e.* $\tilde{\mathrm{V}}_{\rho(\mathrm{n})} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n}+1)}$) local base of X.

*Proof.* The decreasing property is trivial. Now remark that $\mathrm{V_n} \supset \tilde{\mathrm{V}}_\mathrm{n}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{n}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$, since $\tilde{\mathrm{V}}_\mathrm{n} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n})}$. □

The following special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$ is one of the key ingredients:

**Corollary 1 (special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$)** Under the same notations of Lemma 1, if $\{\mathrm{V_n}\}$ is a decreasing local base, then so is $\{\mathrm{V}_{\rho(\mathrm{n})}\}$.

**Corollary 2** If $\{\mathrm{Q_n}\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{\mathrm{Q}_{\rho(\mathrm{n})}\}$ also satisfies theses conditions. Furthermore, if $\tau_\mathrm{Q}$ is the $\mathrm{C}(\Omega)$'s (respectively $\mathrm{C}^\infty(\Omega)$'s) topology of the seminorms $\mathrm{p_n}$, as defined in section 1.44 (respectively 1.46), then the seminorms $\mathrm{p}_{\rho(\mathrm{n})}$ define the same topology $\tau_\mathrm{Q}$.

*Proof.* Let X be $\mathrm{C}(\Omega)$ topologized by the seminorms $\mathrm{p_n}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_n} = \{\mathrm{p_n} < 1/\mathrm{n}\}$, then $\{\mathrm{V_n}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n}+1)}. \tag{1.34}$$

Thus,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n}+1)}. \tag{1.35}$$

In other words, $\mathrm{Q}_{\rho(\mathrm{n})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\rho(\mathrm{n})}\}$ then defines a topology $\tau_{\mathrm{Q}_\rho}$ for which $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\rho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base of $\tau_\mathrm{Q}$, which yields $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\rho}$. □

**Lemma 2** If a sequence of compact sets $\{\mathrm{Q_n}\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in allmost all $\mathrm{Q}^\circ_\mathrm{n}$, *i.e.* there exists m such that

$$\mathrm{K} \subset \mathring{\mathrm{Q}}_\mathrm{m} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+1} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+2} \subset \cdots. \tag{1.36}$$

*Proof.* The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \quad (n = 1, 2, 3, \dots) \tag{1.37}$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no $C_n$ is empty and so conclude[2] that the $C_n$'s intersection contains a point that is not in any $Q_n^\circ$. On the other hand, the conditions specified in [1.44] force the $Q_n^\circ$'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset Q_m^\circ$, for some m. Finally,

$$K \subset \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots . \tag{1.38}$$

□

We are now in a fair position to establish the following:

**Theorem** The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

*Proof.* With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of $\Omega$ that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with $K_i$ $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subset L_{m_i}^\circ \subset L_{m_i+1}^\circ \subset \cdots$ for some $m_i$. In particular, the special case $\varkappa_i = m_i + i$ is

$$K_i \subset \overset{\circ}{L}_{\varkappa_i}. \tag{1.39}$$

Let us reiterate the above proof with $K_n$ and $L_n$ in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \tag{1.40}$$

Combine (1.39) with (1.40) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\varkappa_1} \subset L_{\varkappa_1} \subset \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subset K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots , \tag{1.41}$$

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L. \tag{1.42}$$

So ends the proof □

[1] See (b) of 2.5 of [2].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [2].

## 1.6 Exercise 17. Derivation in some non normed space

*In the setting of Section 1.46, prove that* $f \mapsto D^{\alpha} f$ *is a continuous mapping of* $C^{\infty}(\Omega)$ *into* $C^{\infty}(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$*, for every multi-index* $\alpha$.

*Proof.* In both cases, $D^{\alpha}$ is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^{\infty}(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{ \phi \in C^{\infty}(\Omega) : \max\{|D^{\beta}\phi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}. \tag{1.43}$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^{\gamma} g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}. \tag{1.44}$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^{\beta} D^{\alpha} g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}. \tag{1.45}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^{\alpha} g \in V_N, \quad \textit{i.e.} \quad D^{\alpha}\left(V_{N+|\alpha|}\right) \subset V_N, \tag{1.46}$$

which establishes the continuity of $D^{\alpha} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$.

To prove the continuousness of the restriction $D^{\alpha}|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^{\alpha}|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) = D^{\alpha}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) \tag{1.47}$$

$$\subset D^{\alpha}\left(V_{N+|\alpha|}\right) \cap D^{\alpha}\left(\mathscr{D}_K\right) \tag{1.48}$$

$$\subset V_N \cap \mathscr{D}_K \quad (\text{see } (1.46)) \tag{1.49}$$

So ends the proof. $\square$

# Chapter 2

# Completeness

## 2.1 Exercise 3. An equicontinous sequence of measures

*Put* $K = [-1, 1]$*; define* $\mathscr{D}_K$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^n$*). Supose* $\{f_n\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\phi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ *exists for every* $\phi \in \mathscr{D}_K$*. Show that* $\Lambda$ *is a continuous linear functional on* $\mathscr{D}_K$*. Show that there is a positive integer* p *and a number* $M < \infty$ *such that*

$$\left| \int_{-1}^{1} f_n(t)\phi(t)dt \right| \leq M\| D^p \|_\infty$$

*for all* n*. For example, if* $f_n(t) = n^3 t$ *on* [-1/n, 1/n] *and* 0 *elsewhere, show that this can be done with* $p = 1$*. Construct an example where it can be done with* $p = 2$ *but not with* $p = 1$*.*

We will also consider the case $p = 0$. Since all supports of $\phi, \phi', \phi'', \ldots,$ are in K, we make a specialization of the mean value theorem:

**Lemma** If $\phi \in \mathscr{D}_{[a,b]}$, then

$$\| D^\alpha \phi \|_\infty \leq \| D^p \phi \|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (\alpha = 0, 1, \ldots, p) \tag{2.1}$$

at every order $p = 0, 1, 2, \ldots$; where $\lambda$ is the length $|b - a|$.

*Proof.* Let $x_0$ be in $(a, b)$ . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists $x_1$ $(a < x_1 < x_0)$, such that

$$\phi(x_0) = \phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a). \tag{2.2}$$

Since every $D^p\phi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$\begin{aligned}
\phi(x_0) &= D^0\phi(x_0) && (2.3)\\
&= D^1\phi(x_1)(x_0 - a) && (2.4)\\
&= \cdots \\
&= D^p\phi(x_p)(x_0 - a)\cdots(x_{p-1} - a), && (2.5)
\end{aligned}$$

for all p. More compactly,

$$D^{\alpha}\phi(x_0) = D^{p}\phi(x_p)\prod_{k=\alpha}^{p-1}(x_k - a); \tag{2.6}$$

which yields,

$$|D^{\alpha}\phi(x)| \leq \| D^{p}\phi \|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (x \in [a, c]) \tag{2.7}$$

The case $x_0 \geq c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^{\alpha}\phi(x)| \leq \| D^{p}\phi \|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha} \tag{2.8}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\| D^{\alpha}\phi \|_{\infty} \leq \| D^{p}\phi \|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha}. \tag{2.9}$$

□

*Proof.* We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\phi\rangle \triangleq \int_{\mathbf{R}} u\phi \quad (\phi \in C_0(\mathbf{R})). \tag{2.10}$$

The following inequalities

$$|\langle u|\phi\rangle| \leq \int_{\mathbf{R}} |u\phi| \leq \| u \|_{L^1} \quad (\| \phi \|_{\infty} \leq 1) \tag{2.11}$$

imply that every linear functional

$$\begin{aligned} \langle u| : C_0(\mathbf{R}) &\to \mathbf{C} \\ \phi &\mapsto \langle u|\phi\rangle \end{aligned} \tag{2.12}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u|$, since u is determined (a.e) by the integrals $\langle u|\phi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\sup\{|\langle u|\phi\rangle| : \| \phi \|_{\infty} = 1\} = \| u \|_{L^1}. \tag{2.13}$$

Note that, in the latter equality, $\leq \| u \|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$ , where $g_n$ is

$$\begin{aligned} g_n : \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \begin{cases} n^3x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) . \end{cases} \end{aligned} \tag{2.14}$$

[1] see 1.32, 4.1 of [3]

[2] See 7.26 of [1].

First, remark that $g_n(x) \underset{n\to\infty}{\longrightarrow} 0$ ($x \in \mathbf{R}$), as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in $L^1$ (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a $\tau_K$-continuous linear form $\Lambda$ such that

$$\langle g_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi, \tag{2.15}$$

where $\phi$ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\phi_1^+, \dots, \phi_j^+, \dots,$ from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\phi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

(ii) $\phi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

(iii) $0 \leq \phi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those $\phi_j^+$. Let $\phi_1^-, \dots, \phi_j^-, \dots,$ mirror the $\phi_j^+$, in the sense that $\phi_j^-(x) = \phi_j^+(-x)$, so that

(iv) $\phi_j \triangleq \phi_j^+ - \phi_j^-$ is odd, as $g_n$ is;

(v) every $\phi_j$ is in $C_K^\infty(\mathbf{R})$;

(vi) The sequence $\{\phi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\| \phi_j \|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n|\phi_j\rangle = 2\int_0^1 g_n(t)\phi_j^+(t)dt \underset{j\to\infty}{\longrightarrow} 2\int_0^1 g_n(t)dt = \| g_n \|_{L^1} = n. \tag{2.16}$$

Finally,

$$\| g_n \|_{L^1} \overset{(2.16)}{\leq} \sup\{|\langle g_n|\phi\rangle| : \| \phi \|_\infty = 1\} \overset{(2.13)}{\leq} \| g_n \|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$|\langle g_n|\phi\rangle| \leq C_n \quad (\| \phi \|_\infty = 1); \tag{2.18}$$

see (2.11). Furthermore, $\| g_n \|_{L^1}$ is actually the best, *i.e.* lowest, possible $C_n$; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\phi_{\rho(n)}\rangle\}$ such that $\langle g_n|\phi_{\rho(n)}\rangle$ is greater than, say, $n - 0.01$, as $\| \phi_{\rho(n)} \|_\infty = 1$. Consequently, there is no bound M such that

$$|\langle g_n|\phi\rangle| \leq M \quad (\| \phi \|_\infty = 1; n = 1, 2, 3, \dots). \tag{2.19}$$

In other words, the $g_n$ have no *uniform bound* in $L^1$, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi \quad (\phi \in C_0(\mathbf{R})) \tag{2.20}$$

[3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some $\Lambda$ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that $\phi$ only denotes an element of $C_K^\infty(\mathbf{R})$. Let $f_n$ be a Lebesgue integrable function such that

$$\Lambda\phi = \lim_{n\to\infty} \int_K f_n\phi \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.21}$$

for some linear form $\Lambda$. Since $\phi$ vanishes outside K, we can suppose without loss of generality that the support of $f_n$ lies in K. So, (2.21) can be restated as follows,

$$\Lambda\phi = \lim_{n\to\infty} \langle f_n|\phi\rangle \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.22}$$

Let $K_1, K_2, \ldots,$ be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\phi) = \|\,\phi\,\|_N \triangleq \max\{|D^\alpha\phi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\phi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\phi \in \mathscr{D}_K : \|\,\phi\,\|_N < 2^{-N}\}$ is a (decreasing) local base of $\tau_K$, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\phi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|\,f_n\,\|_{L^1}$ on $V_m$: Every linear functional $\langle f_n|$ is therefore $\tau_K$-continuous; see 1.18 of [3].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology $\tau_K$, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi$ for all $\phi$, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that $\Lambda$ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$|\Lambda\phi| < 1/2 \quad \text{if } \phi \in V_i, \tag{2.23}$$

$$|\Lambda\phi - \langle f_n|\phi\rangle| < 1/2 \quad \text{if } \phi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\phi\rangle| \leq |\Lambda\phi - \langle f_n|\phi\rangle\,| + |\Lambda\phi\,| < 1 \quad \text{if } \phi \in V_p. \tag{2.25}$$

Now remark that every $\psi = \psi[\mu, \phi]$, where

$$\psi[\mu,\phi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\,\phi\,\|_p)\phi & (\phi \neq 0, \mu > 1) \\ 0 & (\phi = 0, \mu > 1), \end{cases} \tag{2.26}$$

keeps in $V_p$. Finally, it is clear that each below statement implies the following one.

$$
\begin{aligned}
&|\langle f_n|\psi\rangle| < 1 && (2.27)\\
&|\langle f_n|\phi\rangle| < 2^p \| \phi \|_p \cdot \mu && (2.28)\\
&|\langle f_n|\phi\rangle| \leq 2^p \| \phi \|_p && (2.29)\\
&|\langle f_n|\phi\rangle| \leq 2^p \{ \| D^0 \phi \|_\infty + \cdots + \| D^p \phi \|_\infty \}. && (2.30)
\end{aligned}
$$

Finally, with the help of (2.1),

$$|\langle f_n|\phi\rangle| \leq 2^p (p+1) \| D^p \phi \|_\infty. \tag{2.31}$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3 x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\phi = \lim\limits_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ exists for every $\phi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\phi\rangle| \leq M\| D^p \phi \|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those $f_n$, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the $K_N$'s indices, so that $K_{m+1}$ becomes $K_1$, $K_{m+2}$ becomes $K_2$, and so on. The resulting topology $\tau_K$ remains unchanged (see Exercise 1.16). We let $\phi$ keep running on $\mathscr{D}_K$ and so define

$$
\begin{aligned}
B_n(\phi) &\triangleq \max\{ |\phi(x)| : x \in [-1/n, 1/n] \}, && (2.32)\\
\Delta_n(\phi) &\triangleq \max\{ |\phi(x) - \phi(0)| : x \in [-1/n, 1/n] \}. && (2.33)
\end{aligned}
$$

The mean value asserts that

$$|\phi(1/n) - \phi(-1/n)| \leq B_n(\phi')\,|1/n - (-1/n)| = \frac{2}{n} B_n(\phi'). \tag{2.34}$$

Independently, an integration by parts shows that

$$
\begin{aligned}
\langle f_n|\phi\rangle &= \left[ \frac{n^3 t^2}{2} \phi(t) \right]_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t)dt && (2.35)\\
&= \frac{n}{2} \left( \phi(1/n) - \phi(-1/n) \right) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t)dt. && (2.36)
\end{aligned}
$$

Combine (2.34) with (2.36) and so obtain

$$
\begin{aligned}
|\langle f_n|\phi\rangle| &\leq \frac{n}{2} |\phi(1/n) - \phi(-1/n)| + \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\phi'(t)| dt && (2.37)\\
&\leq B_n(\phi') + \frac{n^3}{2} B_n(\phi') \int_{-1/n}^{1/n} t^2 dt && (2.38)\\
&\leq \frac{4}{3} B_n(\phi') && (2.39)\\
&\leq \frac{4}{3} \| \phi' \|_\infty. && (2.40)
\end{aligned}
$$

Futhermore, (2.39) gives a hint about the convergence of $f_n$: Since $B_n(\phi')$ tends to $|\phi'(0)|$, we may expect that $f_n$ tends to $\frac{4}{3}\phi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0) \overset{(2.36)}{=} \frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n}(\phi' - \phi'(0))t^2 dt. \tag{2.41}$$

So,

$$\left|\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0)\right| \leq \left|\frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0)\right| + \frac{1}{3}\Delta_n(\phi') \underset{n\to\infty}{\longrightarrow} 0. \tag{2.42}$$

We have just proved that

$$\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\phi'(0) \quad (\phi \in \mathscr{D}_K). \tag{2.43}$$

In other words,

$$\langle f_n| \underset{n\to\infty}{\longrightarrow} -\frac{4}{3}\delta', \tag{2.44}$$

where $\delta$ is the *Dirac measure* and $\delta', \delta'', \ldots,$ its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $-\frac{4}{3}\delta'$ is $\tau_K$-continuous, and from (2.40) that

$$|\langle f_n|\phi\rangle| \leq \frac{4}{3}\|\,\phi'\,\|_\infty \quad (n = 1, 2, 3, \ldots) \tag{2.45}$$

(which is a constructive version of (2.31) ). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n|\phi_{\rho(n)}\rangle : \|\,\phi_{\rho(n)}\,\|_\infty = 1; n = 1, 2, 3, \ldots\} \tag{2.46}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle f_n|\phi\rangle| \leq M\|\,\phi\,\|_\infty \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.47}$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give $f_n$ a *derivative*[4] ${f_n}'$, as follows

$$\begin{aligned} {f_n}' : \mathscr{D}_K &\to \mathbf{C} \\ \phi &\mapsto -\left\langle f_n|\phi'\right\rangle. \end{aligned} \tag{2.48}$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$\begin{aligned} D : \mathscr{D}_K &\to \mathscr{D}_K \\ \phi &\mapsto \phi'; \end{aligned} \tag{2.49}$$

[4] See 6.1 of [3] for a further discussion.

see Exercise 1.17. $f_n{}'$ is therefore continuous. Now apply (2.43) with $\phi'$ and so obtain

$$-\langle f_n|\phi'\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\phi''(0) \quad (\phi \in \mathscr{D}_K),$$

*i.e.*

$$f_n{}' \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\delta''. \tag{2.50}$$

It follows from (2.40) that,

$$|\langle f_n|\phi'\rangle| \leq \frac{4}{3}\| \phi'' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.51}$$

It is therefore possible to uniformly bound $f_n{}'$ with respect to a norm $\| D^p \cdot \|_\infty$, namely $\| D^2 \cdot \|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle f_n|\phi'\rangle| \leq M\| \phi' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.52}$$

Define

$$\Phi_j(x) = \int_{-1}^{x} \phi_j. \tag{2.53}$$

The oddness of $\phi_j$ forces $\Phi_j$ to vanish outside [-1, 1]: $\phi_j$ is therefore in $\mathscr{D}_K$. So, under our assumption,

$$|\langle f_n|\Phi_j'\rangle| \leq M\| \Phi_j' \|_\infty \quad (n = 1, 2, 3, \dots); \tag{2.54}$$

which is

$$|\langle f_n|\phi_j\rangle| \leq M \quad (n = 1, 2, 3, \dots). \tag{2.55}$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n|\phi_{\rho(n)}\rangle\}$) and so conclude that there is no constant M such that

$$|\langle|f_n\phi'\rangle| \leq M\| \phi' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.56}$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$|\langle f_n|\phi'\rangle| \leq M\| \phi \|_\infty. \tag{2.57}$$

The mean value theorem (see (2.1)) asserts that

$$|\langle f_n|\phi'\rangle| \leq M\| \phi \|_\infty \leq M\| \phi' \|_\infty; \tag{2.58}$$

which is, again, a desired contradiction. So ends the proof.

□

## 2.2 Exercise 6. Fourier series may diverge at 0

*Define the Fourier coefficient* $\hat{\mathrm{f}}(\mathrm{n})$ *of a function* $\mathrm{f} \in \mathrm{L}^2(\mathrm{T})$ *(*$\mathrm{T}$ *is the unit circle) by*

$$\hat{\mathrm{f}}(\mathrm{n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{f}(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}\mathrm{n}\theta} \mathrm{d}\theta$$

*for all* $\mathrm{n} \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_\mathrm{n}\mathrm{f} = \sum_{\mathrm{k}=-\mathrm{n}}^{\mathrm{n}} \hat{\mathrm{f}}(\mathrm{k}).$$

*Prove that* $\{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \lim_{\mathrm{n}\infty} \Lambda_\mathrm{n}\mathrm{f}$ *exists*$\}$ *is a dense subspace of* $\mathrm{L}^2(\mathrm{T})$ *of the first category.*

*Proof.* Let $\mathrm{f}(\theta)$ stand for $\mathrm{f}(\mathrm{e}^{\mathrm{i}\theta})$, so that $\mathrm{L}^2(\mathrm{T})$ is identified with a closed subset of $\mathrm{L}^2([-\pi, \pi])$, hence the inner product

$$\hat{\mathrm{f}}(\mathrm{n}) = (\mathrm{f}, \mathrm{e}_\mathrm{n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{f}(\theta) \mathrm{e}^{-\mathrm{i}\mathrm{n}\theta} \mathrm{d}\theta. \tag{2.59}$$

We believe it is customary to write

$$\Lambda_\mathrm{n}(\mathrm{f}) = (\mathrm{f}, \mathrm{e}_{-\mathrm{n}}) + \cdots + (\mathrm{f}, \mathrm{e}_\mathrm{n}). \tag{2.60}$$

Moreover, a well known (and easy to prove) result is

$$(\mathrm{e}_\mathrm{n}, \mathrm{e}_{\mathrm{n}'}) = [\mathrm{n} = \mathrm{n}'], \textit{i.e.} \ \{\mathrm{e}_\mathrm{n} : \mathrm{n} \in \mathbf{Z}\} \text{ is an orthormal subset of } \mathrm{L}^2(\mathrm{T}). \tag{2.61}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $\mathrm{L}^2 \equiv (\mathrm{L}^2)^*$. So,

$$\| \Lambda_\mathrm{n} \|^2 \stackrel{(2.60)}{=} \| \mathrm{e}_{-\mathrm{n}} + \cdots + \mathrm{e}_\mathrm{n} \|^2 \stackrel{(2.61)}{=} \| \mathrm{e}_{-\mathrm{n}} \|^2 + \cdots + \| \mathrm{e}_\mathrm{n} \|^2 \stackrel{(2.61)}{=} 2\mathrm{n} + 1. \tag{2.62}$$

We now assume, to reach a contradiction, that

$$\mathrm{B} \triangleq \{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \sup\{|\Lambda_\mathrm{n}\, \mathrm{f}\,| : \mathrm{n} = 1, 2, 3, \ldots\} < \infty\} \tag{2.63}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_\mathrm{n}\}$ is norm-bounded; which is a desired contradiction, since

$$\| \Lambda_\mathrm{n} \| \stackrel{(2.62)}{=} \sqrt{2\mathrm{n} + 1} \underset{\mathrm{n}\to\infty}{\longrightarrow} \infty. \tag{2.64}$$

We have just established that B is actually of the first category; and so is its subset $\mathrm{L}=\{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \lim_{\mathrm{n}\longrightarrow\infty} \Lambda_\mathrm{n}\mathrm{f} \text{ exists}\}$. We now prove that L is nevertheless dense in $\mathrm{L}^2(\mathrm{T})$. To do so, we let P be span$\{\mathrm{e}_\mathrm{k} : \mathrm{k} \in \mathrm{Z}\}$, the collection of the trignometric polynomials $\mathrm{p}(\theta) = \sum \lambda_\mathrm{k} \mathrm{e}^{\mathrm{i}\mathrm{k}\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_\mathrm{n}(\mathrm{p}) = \sum \lambda_\mathrm{k}$ for almost all n. Thus,

$$\mathrm{P} \subset \mathrm{L} \subset \mathrm{L}^2(\mathrm{T}). \tag{2.65}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $\mathrm{L}^2(\mathrm{T})$. We then conclude, with the help of (2.65), that

$$\mathrm{L}^2(\mathrm{T}) = \overline{\mathrm{P}} = \overline{\mathrm{L}}. \tag{2.66}$$

So ends the proof ☐

## 2.3 Exercise 9. Boundedness without closedness

*Suppose* $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ *are Banach spaces and*

$$\mathrm{B} : \mathrm{X} \times \mathrm{Y} \to \mathrm{Z}$$

*is bilinear and continuous. Prove that there exists* $\mathrm{M} < \infty$ *such that*

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y} \| \leq \mathrm{M} \| \mathrm{x} \| \| \mathrm{y} \| \quad (\mathrm{x} \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}).$$

*Is completeness needed here?*

*Proof.* The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Let $(\mathrm{x}, \mathrm{y})$ range over $\mathrm{X} \times \mathrm{Y}$. B is continous at the origin; thus, there exists a positive r such that

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| < 1 \quad (\max\{\| \mathrm{x} \|, \| \mathrm{y} \|\} < \mathrm{r}). \tag{2.67}$$

Given $(\mathrm{x}, \mathrm{y})$, we choose two scalars $\alpha$, $\beta$ such that $\mathrm{r}\alpha > \| \mathrm{x} \|$ and $\mathrm{r}\beta > \| \mathrm{y} \|$. Thus,

$$\begin{aligned} \| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| &= \alpha\beta \| \mathrm{B}\left(\alpha^{-1}\mathrm{x}, \beta^{-1}\mathrm{y}\right) \| && (2.68) \\ &< \alpha\beta. && (2.69) \end{aligned}$$

We now conclude that

$$\mathrm{B}(\mathrm{x}, \mathrm{y}) \leq \mathrm{r}^{-2} \| \mathrm{x} \| \| \mathrm{y} \|. \tag{2.70}$$

So ends the proof.

As a concrete example, choose $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = \mathrm{C_c}(\mathbf{R})$, topologized by the supremum norm. $\mathrm{C_c}(\mathbf{R})$ is not complete[5], nevertheless the bilinear product

$$\mathrm{B}(\mathrm{f}, \mathrm{g}) = \mathrm{f} \times \mathrm{g} \quad ((\mathrm{f}, \mathrm{g}) \in \mathrm{C_c}(\mathbf{R})^2) \tag{2.71}$$

is bounded, (since $\| \mathrm{B}(\mathrm{f}, \mathrm{g}) \|_\infty = \| \mathrm{f} \|_\infty \| \mathrm{g} \|_\infty$) and continuous. To see that, pick $(\mathrm{u}, \mathrm{v})$ in $\mathrm{C_c}(\mathbf{R})^2$: Given any positive scalar $\varepsilon$, there exists another positive scalar r such that $\mathrm{r}(\mathrm{r} + \| \mathrm{u} \| + \| \mathrm{v} \|) < \varepsilon$. So, under the following assumption

$$\max\{\| \mathrm{f} - \mathrm{u} \|_\infty, \| \mathrm{g} - \mathrm{v} \|_\infty\} < \mathrm{r}, \tag{2.72}$$

we reach

$$\begin{aligned} \| \mathrm{fg} - \mathrm{uv} \|_\infty &\leq \| \mathrm{f} - \mathrm{u} \|_\infty \cdot \| \mathrm{g} \|_\infty + \| \mathrm{u} \|_\infty \cdot \| \mathrm{g} - \mathrm{v} \|_\infty && (2.73) \\ &< \mathrm{r}(\mathrm{r} + \| \mathrm{v} \|) + \| \mathrm{u} \| \mathrm{r} && (2.74) \\ &< \mathrm{r}(\mathrm{r} + \| \mathrm{u} \| + \| \mathrm{v} \|) && (2.75) \\ &< \varepsilon; && (2.76) \end{aligned}$$

which establishes the continuousness of B. □

[5] See 5.4.4 [4]

## 2.4 Exercise 10. Continuousness of bilinear mappings

*Prove that a bilinear mapping is continuous if it is continuous at the origin* $(0,0)$.

*Proof.* Let $(X_1, X_2, Z)$ be topological spaces and B a bilinear mapping

$$B : X_1 \times X_2 \to Z \tag{2.77}$$

From now on, $x = (x_1, x_2)$ denotes an arbitrary element of $X_1 \times X_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $X_1 \times X_2$, *i.e.* given an arbitrary balanced open subset W of Z, there exists in $X_i$ $(i = 1,2)$ a balanced open subset $U_i$ such that

$$B(U_1 \times U_2) \subset W. \tag{2.78}$$

Let $\nu_i(x)$ denote any scalar that is greater than $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$. So,

$$\begin{aligned} B(x_1, x_2) &= \nu_1(x)\nu_2(x) \cdot B\left(\nu_1(x)^{-1}x_1, \nu_2(x)^{-1}x_2\right) && (2.79)\\ &\in \nu_1(x)\nu_2(x) \cdot B(U_1 \times U_2) && (2.80)\\ &\subset \nu_1(x)\nu_2(x) \cdot W. && (2.81) \end{aligned}$$

Now pick $p = (p_1, p_2)$ in $X_1 \times X_2$: It directly follows from (2.81) that

$$\begin{aligned} B(p_1,p_2) - B(x_1,x_2) &= B(p_1, p_2 - x_2) + B(p_1 - x_1, x_2 - p_2) + B(p_1 - x_1, p_2) && (2.82)\\ &\in \nu_1(p)\nu_2(p-x) \cdot W + \nu_1(p-x)\nu_2(x-p) \cdot W + \nu_1(p-x)\nu_2(p) \cdot W. && (2.83) \end{aligned}$$

Let us henceforth assume that

$$p_i - x_i \in [\mu_1(p) + \mu_2(p) + 2]^{-1} \cdot U_i; \tag{2.84}$$

which yields

$$\mu_i(p_i - x_i) \le [\mu_1(p) + \mu_2(p) + 2]^{-1}. \tag{2.85}$$

Finally, combine the special case

$$\begin{aligned} \nu_i(p - x) &= [\mu_1(p) + \mu_2(p) + 1]^{-1}, && (2.86)\\ \nu_i(p) &= \mu_1(p) + \mu_2(p) + 1 && (2.87) \end{aligned}$$

with (2.83) and so obtain

$$B(p_1, p_2) - B(x_1, x_2) \in W + W + W. \tag{2.88}$$

W being arbitrary, we have so established the continuousness of B at $(p_1, p_2)$. Since $(p_1, p_2)$ is also arbitrary, the proof is complete. $\square$

## 2.5 Exercise 12. A bilinear mapping that is not continuous

*Let X be the normed space of all real polynomials in one variable, with*

$$\| \mathrm{f} \| = \int_0^1 |\mathrm{f}(\mathrm{t})| \, \mathrm{dt}.$$

*Put* $\mathrm{B}(\mathrm{f}, \mathrm{g}) = \int_0^1 \mathrm{f}(\mathrm{t})\mathrm{g}(\mathrm{t})\mathrm{dt}$, *and show that* B *is a bilinear continuous functional on* $\mathrm{X} \times \mathrm{X}$ *which is separately but not continuous.*

*Proof.* Let f denote the first variable, g the second one. Remark that

$$|\mathrm{B}(\mathrm{f}, \mathrm{g})\,| < \| \mathrm{f} \| \cdot \max_{[0,1]} |\mathrm{g}\,| \,; \tag{2.89}$$

which is sufficient (1.18 of [3]) to assert that any $\mathrm{f} \mapsto \mathrm{B}(\mathrm{f}, \mathrm{g})$ is continuous. The continuity of all $\mathrm{g} \mapsto \mathrm{B}(\mathrm{f}, \mathrm{g})$ follows (Put $\mathrm{C}(\mathrm{g}, \mathrm{f}) = \mathrm{B}(\mathrm{f}, \mathrm{g})$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$|\mathrm{B}(\mathrm{f}, \mathrm{g})\,| \leq \mathrm{M}\| \mathrm{f} \| \| \mathrm{g} \|. \tag{2.90}$$

Put

$$\mathrm{f_n}(\mathrm{X}) \triangleq 2\sqrt{\mathrm{n}} \cdot \mathrm{X^n} \in \mathbf{R}[\mathrm{X}] \quad (\mathrm{n} = 1, 2, 3, \dots), \tag{2.91}$$

so that

$$\| \mathrm{f_n} \| = \frac{2\sqrt{\mathrm{n}}}{\mathrm{n}+1} \underset{\mathrm{n}\to\infty}{\longrightarrow} 0. \tag{2.92}$$

On the other hand,

$$\mathrm{B}(\mathrm{f_n}, \mathrm{f_n}) = \frac{4\mathrm{n}}{2\mathrm{n}+1} > 1. \tag{2.93}$$

Finally, we combine (2.92) and (2.93) with (2.90) and so obtain

$$1 < \mathrm{B}(\mathrm{f_n}, \mathrm{f_n}) \leq \mathrm{M}\| \mathrm{f_n} \|^2 \underset{\mathrm{n}\to\infty}{\longrightarrow} 0. \tag{2.94}$$

Our continuousness assumption is then contradicted. So ends the proof. □

## 2.6 Exercise 15. Baire cut

*Suppose* $\mathrm{X}$ *is an F-space and* $\mathrm{Y}$ *is a subspace of* $\mathrm{X}$ *whose complement is of the first category. Prove that* $\mathrm{Y} = \mathrm{X}$*.* Hint: $\mathrm{Y}$ *must intersect* $\mathrm{x} + \mathrm{Y}$ *for every* $\mathrm{x} \in \mathrm{X}$*.*

*Proof.* Assume $\mathrm{Y}$ is a subgroup of $\mathrm{X}$. Under our assumptions, there exists a sequence $\{\mathrm{E_n} : \mathrm{n} = 1, 2, 3, \dots\}$ of $\mathrm{X}$ such that

(i) $(\overline{\mathrm{E}}_\mathrm{n})^\circ = \emptyset$;

(ii) $\mathrm{X} \setminus \mathrm{Y} = \bigcup_{\mathrm{n}=1}^{\infty} \mathrm{E_n}$.

By (i), the complement $\mathrm{V_n}$ of $\overline{\mathrm{E}}_\mathrm{n}$ is a dense open set. Since $\mathrm{X}$ is an F-space, it follows from the Baire's theorem that the intersection $\mathrm{S}$ of the $\mathrm{V_n}$'s is dense in $\mathrm{X}$: So is $\mathrm{x} + \mathrm{S}$ $(\mathrm{x} \in \mathrm{X})$. To see that, remark that

$$\mathrm{X} = \mathrm{x} + \overline{\mathrm{S}} \subset \overline{\mathrm{x} + \mathrm{S}} \tag{2.95}$$

follows from 1.3 (b) of [3]. Since $\mathrm{S}$ and $\mathrm{x} + \mathrm{S}$ are both dense open subsets of $\mathrm{X}$, the Baire's theorem asserts that

$$\overline{(\mathrm{x} + \mathrm{S}) \cap \mathrm{S}} = \mathrm{X}. \tag{2.96}$$

Thus,

$$(\mathrm{x} + \mathrm{S}) \cap \mathrm{S} \neq \emptyset. \tag{2.97}$$

Moreover, it follows from (ii) that $\mathrm{X} \setminus \mathrm{Y} \subset \bigcup_\mathrm{n} \overline{\mathrm{E}}_\mathrm{n}$, *i.e.* $\mathrm{Y} \supset \mathrm{S}$. Combined with (2.97), this shows that $\mathrm{x} + \mathrm{Y}$ cuts $\mathrm{Y}$. Therefore, our arbitrary $\mathrm{x}$ is an element of the subgroup $\mathrm{Y}$. We have thus established that $\mathrm{X} \subset \mathrm{Y}$, which achieves the proof. □

## 2.7 Exercise 16. An elementary closed graph theorem

$\{(x, 1/x) : x > 0\}$*uppose that* X *and* K *are metric spaces, that* K *is compact, and that the graph of* $f : X \to K$ *is a closed subset of* $X \times K$. *Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.)* $\{(x, 1/x) : x > 0\}$*how that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

*Proof.* Choose a sequence $\{x_n : n = 1, 2, 3, \dots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some $(a, b)$ of $X \times K$. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [3].

To prove that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$f : [0, \infty) \to [0, \infty) \tag{2.98}$$

$$x \mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \tag{2.99}$$

Clearly, the graph of f is the following set

$$\{(x, 1/x) : x > 0\} \cup \{(0, 0)\}. \tag{2.100}$$

Let $\{(x_n, f(x_n))\}$ be a sequence in $\{(x, f(x)) : x > 0\}$ that converges to, say, $(a, b)$. Clearly, $a \neq 0$, since $\{(x_n, f(x_n))\}$ is bounded. So, $a > 0$ and $b = 1/a$ (since f is continuous on $R_+$). This establishes that the subset $\{(x, 1/x) : x > 0\}$ is closed. As a finite union of closed sets, the graph of f is closed. Nevertheless, f is not continuous on $[0, \infty)$. □

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