Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0, 1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ *in X such that (a)* $\{f_n\}$ *converges to* 0 *as* $n \to \infty$*, but (b) if* $\{\gamma_n\}$ *is any sequence of scalars such that* $\gamma_n \to \infty$ *then* $\{\gamma_n f_n\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0, 1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms p_x is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$V^{((x,k)} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k \in \mathbf{N}) \tag{1.1}$$

is then a local base for a topology τ on X. Given $\{f_n : n = 1, 2, 3, \dots\}$, we set

$$\texttt{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \tag{1.2}$$

with the convention $\texttt{off}(U) = \infty$ whether the sum has no finite support. So,

$$\sum_{i=1}^{m} \texttt{off}(U^{(i)}) = \sum_{n=1}^{\infty} \sum_{i=1}^{m} [f_n \notin U^{(i)}] \geq \texttt{off}(U^{(1)} \cap \dots \cap U^{(m)}) \tag{1.3}$$

We first assume that $\{f_n\}$ τ-converges to some f in X, *i.e.*

$$\texttt{off}(f + V) < \infty \quad (V \in \mathscr{B}). \tag{1.4}$$

The special cases $V = V^{(x,k)}$ mean the pointwise convergence of $\{f_n\}$. Conversely, assume that $\{f_n\}$ does not τ-converges to any g in X, *i.e.*

$$\forall g \in X, \exists V^{(g)} \in \mathscr{B} : \texttt{off}(g + V^{(g)}) = \infty. \tag{1.5}$$

Given g, $V^{(g)}$ is then an intersection $V^{(x^{(1)},k^{(1)})} \cap \dots \cap V^{(x^{(m)},k^{(m)})}$. Thus

$$\sum_{i=1}^{m} \texttt{off}(g + V^{(x^{(i)},k^{(i)})}) \overset{(1.3)}{\geq} \texttt{off}(g + V^{(g)}) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\texttt{off}(g + V^{(x^{(i)},k^{(i)})})$ must then be ∞. This implies that convergence of f_n to g fails at point x_i. g being arbitrary, we so conclude that f_n does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on, *k*, *n* and *p* run on $\mathbf{N}_+$. Let $\texttt{dyadic}(x)$ be the usual dyadic expansion of a real number x, so that $\texttt{dyadic}(x)$ is an aperiodic binary sequence **iff** x is irrational. Define

$$f_n(x) \triangleq \begin{cases} 2^{-\sum_{k=1}^{n} \texttt{dyadic}(x)_{-k}} & (x \in [0, 1] \setminus \mathbf{Q}) \\ 0 & (x \in [0, 1] \cap \mathbf{Q}) \end{cases} \tag{1.7}$$

so that $f_n(x) \underset{n\to\infty}{\longrightarrow} 0$ and take scalars γ_n such that $\underset{n\to\infty}{\longrightarrow} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose $n^{(p)}$ among those *almost all* n that are large enough to satisfy

$$n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)} \tag{1.8}$$

(start with $n^{(-1)} = n^{(0)} = 0$) and so obtain

$$2^p < \gamma_{n^{(p)}} : 0 < n^{(p)} - n^{(p-1)} \underset{p\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator χ of $\{n^{(1)}, n^{(2)}, \dots\}$ is then aperiodic, *i.e.*

$$x^{(\gamma)} \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \tag{1.10}$$

is irrational. Consequently,

$$\texttt{dyadic}(x^{(\gamma)})_{-k} = \chi_k. \tag{1.11}$$

We now easily see that

$$\chi_1 + \cdots + \chi_{n^{(p)}} = p, \tag{1.12}$$

which, combined with (1.7), yields

$$f_{n^{(p)}}(x^{(\gamma)}) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n^{(p)}} f_{n^{(p)}}(x^{(\gamma)}) > 1. \tag{1.14}$$

We have so established that the subsequence $\{\gamma_{n^{(p)}} f_{n^{(p)}}\}$ does not tend pointwise to 0, hence neither does the whole sequence $\{\gamma_n f_n\}$. In other words, (b) holds, which is in violent contrast with [1.28]: X is then not metrizable. So ends the proof. □

1.2 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ *does not depend on the particular choice of* $\{K_n\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $C^\infty(\Omega)$ *(Section 1.46).*

Lemma Let X be a topological space with a countable local base $\{V_N : N = 1, 2, 3, \dots\}$. If $\tilde{V}_N = V_1 \cap \dots \cap V_N$, then every subsequence $\{\tilde{V}_{\rho(N)}\}$ is a also a decreasing (*i.e.* $\tilde{V}_{\rho(N)} \supset \tilde{V}_{\rho(N+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $V_N \supset \tilde{V}_N \supset \tilde{V}_{N+1}$: The left inclusion shows that $\{\tilde{V}_N\}$ is a local base of X. Then so is $\{\tilde{V}_{\rho(N)}\}$, since $\tilde{V}_N \supset \tilde{V}_{\rho(N)}$. □

Corollary If $\{Q_N\}$ is a sequence of compacts that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{\rho(N)}\}$ also satisfies theses conditions. Furthermore, if τ^Q is the $C(\Omega)$'s (respectively $C^\infty(\Omega)$) topology of the seminorms $p^Q_N = p_N$, as defined in section 1.44 (respectively 1.46), then the seminorms $p^Q_{\rho(N)}$ define the same topology τ^Q.

Proof. Let X be $C(\Omega)$ topologized with the seminorms p^Q_N (the case $X = C^\infty(\Omega)$ is proved the same way). If $V^Q_N = \{p^Q_N < 1/N\}$, then $\{V^Q_N\}$ is a decreasing local base of X. Moreover,

$$Q_{\rho(N)} \subset \overset{\circ}{Q}_{\rho(N)+1} \subset Q_{\rho(N)+1} \subset Q_{\rho(N+1)}, \tag{1.15}$$

and this yields

$$Q_{\rho(N)} \subset \overset{\circ}{Q}_{\rho(N+1)}. \tag{1.16}$$

In other words, $Q_{\rho(N)}$ satisfies the conditions specified in section 1.44. $\{p^Q_{\rho(N)}\}$ then defines a topology τ^{Q_ρ} for which $\{V^Q_{\rho(N)}\}$ is a local base. So, $\tau^{Q_\rho} \subset \tau^Q$. Conversely, the Lemma turns $\{V^Q_{\rho(N)}\}$ into a local base of τ^Q. Hence $\tau^Q \subset \tau^{Q_\rho}$. □

Theorem The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.46.

Proof. With the Corollary's notations, $\tau^K = \tau^{K_\varkappa}$, for every subsequence $\{K_{\varkappa(n)}\}$. Similarly, let $\{L_n\}$ be a sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau^L = \tau^{L_\lambda}$ for every subsequence $\{L_{\lambda(n)}\}$. The following definition

$$C_{i,j} \triangleq K_i \setminus \overset{\circ}{L}_j \quad (i, j = 1, 2, 3, \dots) \tag{1.17}$$

turns $\{C_{i,j} : j = 1, 2, 3, \dots\}$ into a decreasing sequence of compacts. We now suppose (to reach a contradiction) that no $C_{i,j}$ is empty and so conclude that $\bigcap_{j=1}^\infty C_{i,j}$ contains a point that is not in any L_j. But the conditions specified in [1.44] force $\{\overset{\circ}{L}_j\}$ to be an open cover. This contradiction reveals that $C_{i,j}, C_{i,j+1}, C_{i,j+2}, \dots,$ are actually empty for some $j = j^{(i)}$. We then define $\lambda(i) = i + j^{(i)}$, so that

$$K_i \subset \overset{\circ}{L}_{\lambda(i)}. \tag{1.18}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\varkappa(j) : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\varkappa(j)} \tag{1.19}$$

Combine (1.18) with (1.19) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\lambda(1)} \subset L_{\lambda(1)} \subset \overset{\circ}{K}_{\varkappa\circ\lambda(1)} \subset K_{\varkappa\circ\lambda(1)} \subset \overset{\circ}{L}_{\lambda\circ\varkappa\circ\lambda(1)} \subset \cdots \tag{1.20}$$

Thus the sequence $Q = (K_1, L_{\lambda(1)}, K_{\varkappa\circ\lambda(1)}, L_{\lambda\circ\varkappa\circ\lambda(1)}, \dots)$
satisfies the conditions specified in section 1.44. It now follows from the Corollary that

$$\tau^K = \tau^{K_\varkappa} = \tau^Q = \tau^{L_\lambda} = \tau^L. \tag{1.21}$$

So ends the proof $\square$