

Solutions to some exercises from Walter Rudin's  
*Functional Analysis*

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# Contents

<b>1</b>	<b>Topological Vector Spaces</b>	<b>1</b>
1.1	Exercise 7. Metrizable & number theory . . . . .	2
1.2	Exercise 9. Quotient map . . . . .	4
1.3	Exercise 10. An open mapping theorem . . . . .	5
1.4	Exercise 16. Uniqueness of topology for test functions . . . . .	6
	<b>bibliography</b>	<b>8</b>



## Chapter 1

# Topological Vector Spaces

## 1.1 Exercise 7. Metrizable & number theory

Let be  $X$  the vector space of all complex functions on the unit interval  $[0, 1]$ , topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence  $\{f_n\}$  in  $X$  such that (a)  $\{f_n\}$  converges to 0 as  $n \rightarrow \infty$ , but (b) if  $\{\gamma_n\}$  is any sequence of scalars such that  $\gamma_n \rightarrow \infty$  then  $\{\gamma_n f_n\}$  does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as  $[0, 1]$ .) This shows that metrizable cannot be omitted in (b) of Theorem 1.28.

*Proof.* Our justification consists in proving that  $\tau$ -convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms  $p_x$  is separating. By [1.37], the collection  $\mathcal{B}$  of all finite intersections of the sets

$$V^{(x,k)} \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k \in \mathbf{N}) \quad (1.1)$$

is then a local base for a topology  $\tau$  on  $X$ . Given  $\{f_n : n = 1, 2, 3, \dots\}$ , we set

$$\text{off}(U) \triangleq \sum_{n=1}^{\infty} [f_n \notin U] \quad (U \in \tau), \quad (1.2)$$

with the convention  $\text{off}(U) = \infty$  whether the sum has no finite support. So,

$$\sum_{i=1}^m \text{off}(U^{(i)}) = \sum_{n=1}^{\infty} \sum_{i=1}^m [f_n \notin U^{(i)}] \geq \text{off}(U^{(1)} \cap \dots \cap U^{(m)}) \quad (1.3)$$

We first assume that  $\{f_n\}$   $\tau$ -converges to some  $f$  in  $X$ , *i.e.*

$$\text{off}(f + V) < \infty \quad (V \in \mathcal{B}). \quad (1.4)$$

The special cases  $V = V^{(x,k)}$  mean the pointwise convergence of  $\{f_n\}$ . Conversely, assume that  $\{f_n\}$  does not  $\tau$ -converges to any  $g$  in  $X$ , *i.e.*

$$\forall g \in X, \exists V^{(g)} \in \mathcal{B} : \text{off}(g + V^{(g)}) = \infty. \quad (1.5)$$

Given  $g$ ,  $V^{(g)}$  is then an intersection  $V^{(x^{(1)}, k^{(1)})} \cap \dots \cap V^{(x^{(m)}, k^{(m)})}$ . Thus

$$\sum_{i=1}^m \text{off}(g + V^{(x^{(i)}, k^{(i)})}) \stackrel{(1.3)}{\geq} \text{off}(g + V^{(g)}) \stackrel{(1.5)}{=} \infty. \quad (1.6)$$

One of the sum  $\text{off}(g + V^{(x^{(i)}, k^{(i)})})$  must then be  $\infty$ . This implies that convergence of  $f_n$  to  $g$  fails at point  $x_i$ .  $g$  being arbitrary, we so conclude that  $f_n$  does not converge pointwise. We have just proved that  $\tau$ -convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on,  $k$ ,  $n$  and  $p$  run on  $\mathbf{N}_+$ . Let  $\text{dyadic}(x)$  be the usual dyadic expansion of a real number  $x$ , so that  $\text{dyadic}(x)$  is an aperiodic binary sequence **iff**  $x$  is irrational. Define

$$f_n(x) \triangleq \begin{cases} 2^{-\sum_{k=1}^n \text{dyadic}(x)_k} & (x \in [0, 1] \setminus \mathbf{Q}) \\ 0 & (x \in [0, 1] \cap \mathbf{Q}) \end{cases} \quad (1.7)$$

so that  $f_n(x) \xrightarrow{n \rightarrow \infty} 0$  and take scalars  $\gamma_n$  such that  $\xrightarrow{n \rightarrow \infty} \infty$ , *i.e.* at fixed  $p$ ,  $\gamma_n$  is greater than  $2^p$  for almost all  $n$ . Next, choose  $n^{(p)}$  among those *almost all*  $n$  that are large enough to satisfy

$$n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)} \quad (1.8)$$

(start with  $n^{(-1)} = n^{(0)} = 0$ ) and so obtain

$$2^p < \gamma_{n^{(p)}} : 0 < n^{(p)} - n^{(p-1)} \xrightarrow{p \rightarrow \infty} \infty. \quad (1.9)$$

The indicator  $\chi$  of  $\{n^{(1)}, n^{(2)}, \dots\}$  is then aperiodic, *i.e.*

$$x^{(\gamma)} \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \quad (1.10)$$

is irrational. Consequently,

$$\text{dyadic}(x^{(\gamma)})_{-k} = \chi_k. \quad (1.11)$$

We now easily see that

$$\chi_1 + \dots + \chi_{n^{(p)}} = p, \quad (1.12)$$

which, combined with (1.7), yields

$$f_{n^{(p)}}(x^{(\gamma)}) = 2^{-p}. \quad (1.13)$$

Finally,

$$\gamma_{n^{(p)}} f_{n^{(p)}}(x^{(\gamma)}) > 1. \quad (1.14)$$

We have so established that the subsequence  $\{\gamma_{n^{(p)}} f_{n^{(p)}}\}$  does not tend pointwise to 0, hence neither does the whole sequence  $\{\gamma_n f_n\}$ . In other words, (b) holds, which is in violent contrast with [1.28]:  $X$  is then not metrizable. So ends the proof.  $\square$

## 1.2 Exercise 9. Quotient map

Suppose

(a)  $X$  and  $Y$  are topological vector spaces,

(b)  $\Lambda : X \rightarrow Y$  is linear.

(c)  $N$  is a closed subspace of  $X$ ,

(d)  $\pi : X \rightarrow X/N$  is the quotient map, and

(e)  $\Lambda x = 0$  for every  $x \in N$ .

Prove that there is a unique  $f : X/N \rightarrow Y$  which satisfies  $\Lambda = f \circ \pi$ , that is,  $\Lambda x = f(\pi(x))$  for all  $x \in X$ . Prove that  $f$  is linear and that  $\Lambda$  is continuous if and only if  $f$  is continuous. Also,  $\Lambda$  is open if and only if  $f$  is open.

*Proof.* The equation  $\Lambda = f \circ \pi$  has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \quad (1.15)$$

To ensure that  $f$  is actually a mapping, simply remark that the linearity of  $\Lambda$  implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x. \quad (1.16)$$

It straightforwardly derives from (1.15) that  $f$  inherits linearity from  $\pi$  and  $\Lambda$ . Now remark that

$$\pi x = N \stackrel{(f \text{ linear})}{\Rightarrow} f(\pi x) = 0 \stackrel{(1.15)}{\Rightarrow} \Lambda x = 0 \Rightarrow \pi x = N \quad (1.17)$$

and so conclude that  $f$  is also one-to-one. Now assume  $f$  to be continuous. Then so is  $\Lambda = f \circ \pi$ , by (a) of [1.41]. Conversely, if  $\Lambda$  is continuous, then for each neighborhood  $V$  of  $0_Y$  there exists a neighborhood  $U$  of  $0_X$  such that

$$\Lambda(U) = f(\pi(U)) \subset V. \quad (1.18)$$

Since  $\pi$  is open (see (a) of [1.41]),  $\pi(U)$  is a neighborhood of  $N = 0_{X/N}$ . This is sufficient to establish that the linear mapping  $f$  is continuous. If  $f$  is open, so is  $\Lambda = f \circ \pi$ , by (a) of [1.41]. Conversely, let

$$W \triangleq \pi(V) \subset X/N \quad (V \text{ neighborhood of } 0_X) \quad (1.19)$$

range over all neighborhoods of  $N$ , as  $\Lambda$  is kept open: So is

$$\Lambda(V) = f(\pi(V)) = f(W). \quad (1.20)$$

The linear mapping  $f$  is then open. □



### 1.3 Exercise 10. An open mapping theorem

Suppose that  $X$  and  $Y$  are topological vector spaces,  $\dim Y < \infty$ ,  $\Lambda : X \rightarrow Y$  is linear, and  $\Lambda(X) = Y$ .

- (a) Prove that  $\Lambda$  is an open mapping.
- (b) Assume, in addition, that the null space of  $\Lambda$  is closed, and prove that  $\Lambda$  is continuous.

*Proof.* (a) Let  $e$  range over a base of  $Y$ : For each  $e$ , there exists  $x_e$  in  $X$  such that  $\Lambda(x_e) = e$ , since  $\Lambda$  is onto. So,

$$y = \sum_e y_e \Lambda x_e \quad (y \in Y). \quad (1.21)$$

The sequence  $\{x_e\}$  is finite hence bounded: Given  $V$  a balanced neighborhood of the origin, there exists a positive scalar  $s$  such that

$$x_e \in sV \quad (1.22)$$

for all  $x_e$ . Combining this with (1.21) shows that

$$y \in \sum_e \Lambda(V) \quad (y \in Y : |y_e| < s^{-1}). \quad (1.23)$$

- (b) Since  $N$  is closed,  $\pi$  continuously maps  $X$  onto  $X/N$ , another topological (Hausdorff) vector space, see [1.41]. Now take  $f$  as in Exercise 9: Since  $\Lambda$  is onto,  $f$  is an isomorphism of  $X/N$  onto  $Y$ .  $X/N$  has then dimension  $\dim Y$ . The inverse isomorphism  $f^{-1}$  is then a homeomorphism of  $Y \equiv \mathbf{C}^{\dim Y}$  onto  $X/N$ ; see [1.21]. Consequently, both  $f$  and  $f^{-1}$  are continuous: So is  $\Lambda = f \circ \pi$ .

□

## 1.4 Exercise 16. Uniqueness of topology for test functions

*Prove that the topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Do the same for  $C^\infty(\Omega)$  (Section 1.46).*

**Comment** This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms  $|f|_N = \max |f|$ , then, eventually, only on the ambient space itself. This should then be regarded as a very part of the textbook [2] The proof consists in combining trivial consequences of the local base definition with the well-known result (e.g. [2.6] in [1]) about intersection of nonempty compact sets.

**Lemma** Let  $X$  be a topological space with a countable local base  $\{V_N : N = 1, 2, 3, \dots\}$ . If  $\tilde{V}_N = V_1 \cap \dots \cap V_N$ , then every subsequence  $\{\tilde{V}_{\rho(N)}\}$  is also a decreasing (i.e.  $\tilde{V}_{\rho(N)} \supset \tilde{V}_{\rho(N+1)}$ ) local base of  $X$ .

*Proof.* The decreasing property is trivial. Now remark that  $V_N \supset \tilde{V}_N \supset \tilde{V}_{N+1}$ : The left inclusion shows that  $\{\tilde{V}_N\}$  is a local base of  $X$ . Then so is  $\{\tilde{V}_{\rho(N)}\}$ , since  $\tilde{V}_N \supset \tilde{V}_{\rho(N)}$ .  $\square$

The following special case  $V_N = \tilde{V}_N$  is one of the key ingredients:

**Corollary 1 (special case)** With the same notations, if  $\{V_N\}$  is a decreasing local base, then so is  $\{V_{\rho(N)}\}$ .

**Corollary 2** If  $\{Q_N\}$  is a sequence of compacts that satisfies the conditions specified in section 1.44, then every subsequence  $\{Q_{\rho(N)}\}$  also satisfies these conditions. Furthermore, if  $\tau_Q$  is the  $C(\Omega)$ 's (respectively  $C^\infty(\Omega)$ ) topology of the seminorms  $p_N$ , as defined in section 1.44 (respectively 1.46), then the seminorms  $p_{\rho(N)}$  define the same topology  $\tau_Q$ .

*Proof.* Let  $X$  be  $C(\Omega)$  topologized with the seminorms  $p_N$  (the case  $X = C^\infty(\Omega)$  is proved the same way). If  $V_N = \{p_N < 1/N\}$ , then  $\{V_N\}$  is a decreasing local base of  $X$ . Moreover,

$$Q_{\rho(N)} \subset \overset{\circ}{Q}_{\rho(N)+1} \subset Q_{\rho(N)+1} \subset Q_{\rho(N+1)}, \quad (1.24)$$

and this yields

$$Q_{\rho(N)} \subset \overset{\circ}{Q}_{\rho(N+1)}. \quad (1.25)$$

In other words,  $Q_{\rho(N)}$  satisfies the conditions specified in section 1.44.  $\{p_{\rho(N)}\}$  then defines a topology  $\tau_{Q_\rho}$  for which  $\{V_{\rho(N)}\}$  is a local base. So,  $\tau_{Q_\rho} \subset \tau_Q$ . Conversely, the above corollary turns  $\{V_{\rho(N)}\}$  into a local base of  $\tau_Q$ . Hence  $\tau_Q \subset \tau_{Q_\rho}$ .  $\square$

We are now in a fair position to establish the following:

**Theorem** The topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of  $C^\infty(\Omega)$ , as long as this sequence satisfies the conditions specified in section 1.46.

*Proof.* With the second corollary's notations,  $\tau_K = \tau_{K_\kappa}$ , for every subsequence  $\{K_{\kappa(n)}\}$ . Similarly, let  $\{L_n\}$  be a sequence of compact subsets of  $\Omega$  that satisfies the condition specified in [1.44], so that  $\tau_L = \tau_{L_\lambda}$  for every subsequence  $\{L_{\lambda(n)}\}$ . The following definition

$$C_{i,j} \triangleq K_i \setminus \overset{\circ}{L}_j \quad (i, j = 1, 2, 3, \dots) \quad (1.26)$$

shapes  $\{C_{i,j} : j = 1, 2, 3, \dots\}$  as a decreasing sequence of compacts. We now suppose (to reach a contradiction) that no  $C_{i,j}$  is empty and so conclude that  $C_{i,1} \cap C_{i,2} \cap \dots$  contains a point that is not in any  $\text{int}(L_j)$ . But the conditions specified in [1.44] force  $\{\text{int}(L_j) : j = 1, 2, 3, \dots\}$  to be an open cover. This contradiction reveals that  $C_{i,j} = C_{i,j+1} = C_{i,j+2} = \dots = \emptyset$  for some  $j = j^{(i)}$ . We then define  $\lambda_i = i + j^{(i)}$ , so that

$$K_i \setminus \overset{\circ}{L}_{\lambda_i} = \emptyset, \quad i.e. \quad K_i \subset \overset{\circ}{L}_{\lambda_i}. \quad (1.27)$$

Let us reiterate the above proof with  $K_n$  and  $L_n$  in exchanged roles then similarly find a subsequence  $\{\kappa_j : j = 1, 2, 3, \dots\}$  such that

$$L_j \subset \overset{\circ}{K}_{\kappa_j} \quad (1.28)$$

Combine (1.27) with (1.28) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\lambda_1} \subset L_{\lambda_1} \subset \overset{\circ}{K}_{\kappa \circ \lambda_1} \subset K_{\kappa \circ \lambda_1} \subset \overset{\circ}{L}_{\lambda_{\kappa \circ \lambda_1}} \subset \dots \quad (1.29)$$

Thus the sequence  $Q = (K_1, L_{\lambda_1}, K_{\kappa \circ \lambda_1}, L_{\lambda_{\kappa \circ \lambda_1}}, \dots)$  satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\kappa} = \tau_Q = \tau_{L_\lambda} = \tau_L. \quad (1.30)$$

So ends the proof □

# Bibliography

- [1] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, 1986.
- [2] Walter Rudin. *Functional Analysis*. McGraw-Hill, 1991.