Solutions to some exercises from Walter Rudin's *Functional Analysis*

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August 19, 2020

Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$*, topologized by the family of seminorms*

$$\mathrm{p_x(f) = |f(x)| \quad (0 \le x \le 1).}$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{\mathrm{f_n}\}$ *in X such that (a)* $\{\mathrm{f_n}\}$ *converges to* 0 *as* $\mathrm{n} \to \infty$*, but (b) if* $\{\gamma_\mathrm{n}\}$ *is any sequence of scalars such that* $\gamma_\mathrm{n} \to \infty$ *then* $\{\gamma_\mathrm{n}\mathrm{f_n}\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms $\mathrm{p_x}$ is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$\mathrm{V^{((x,k)} \triangleq \{p_x < 2^{\text{-}k}\} \quad (x \in [0,1], k \in \mathbf{N})} \tag{1.1}$$

is then a local base for a topology τ on X. Given $\{\mathrm{f_n : n = 1, 2, 3, \dots}\}$, we set

$$\mathtt{off}(\mathrm{U}) \triangleq \sum_{\mathrm{n}=1}^{\infty} [\mathrm{f_n} \notin \mathrm{U}] \quad (\mathrm{U} \in \tau), \tag{1.2}$$

with the convention $\mathtt{off}(\mathrm{U}) = \infty$ whether the sum has no finite support. So,

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{U^{(i)}}) = \sum_{\mathrm{n}=1}^{\infty} \sum_{\mathrm{i}=1}^{\mathrm{m}} [\mathrm{f_n} \notin \mathrm{U^{(i)}}] \ge \mathtt{off}(\mathrm{U^{(1)}} \cap \dots \cap \mathrm{U^{(m)}}) \tag{1.3}$$

We first assume that $\{\mathrm{f_n}\}$ τ-converges to some f in X, *i.e.*

$$\mathtt{off}(\mathrm{f} + \mathrm{V}) < \infty \quad (\mathrm{V} \in \mathscr{B}). \tag{1.4}$$

The special cases $\mathrm{V = V^{(x,k)}}$ mean the pointwise convergence of $\{\mathrm{f_n}\}$. Conversely, assume that $\{\mathrm{f_n}\}$ does not τ-converges to any g in X, *i.e.*

$$\forall \mathrm{g} \in \mathrm{X}, \exists \mathrm{V^{(g)}} \in \mathscr{B} : \mathtt{off}(\mathrm{g} + \mathrm{V^{(g)}}) = \infty. \tag{1.5}$$

Given g, $\mathrm{V^{(g)}}$ is then an intersection $\mathrm{V^{(x^{(1)},k^{(1)})}} \cap \dots \cap \mathrm{V^{(x^{(m)},k^{(m)})}}$. Thus

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \mathtt{off}(\mathrm{g} + \mathrm{V^{(x^{(i)},k^{(i)})}}) \overset{(1.3)}{\ge} \mathtt{off}(\mathrm{g} + \mathrm{V^{(g)}}) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\mathtt{off}(\mathrm{g} + \mathrm{V^{(x^{(i)},k^{(i)})}})$ must then be ∞. This implies that convergence of $\mathrm{f_n}$ to g fails at point $\mathrm{x_i}$. g being arbitrary, we so conclude that $\mathrm{f_n}$ does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on, k, n and p run on $\mathbf{N}_+$. Let $\mathtt{dyadic}(\mathrm{x})$ be the usual dyadic expansion of a real number x, so that $\mathtt{dyadic}(\mathrm{x})$ is an aperiodic binary sequence **iff** x is irrational. Define

$$\mathrm{f_n(x)} \triangleq \begin{cases} 2^{-\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathtt{dyadic}(\mathrm{x})_{-\mathrm{k}}} & (\mathrm{x} \in [0,1] \setminus \mathbf{Q}) \\ 0 & (\mathrm{x} \in [0,1] \cap \mathbf{Q}) \end{cases} \tag{1.7}$$

so that $f_n(x) \underset{n\to\infty}{\longrightarrow} 0$, and take scalars γ_n such that $\underset{n\to\infty}{\longrightarrow} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose $n^{(p)}$ among those *almost all* n that are large enough to satisfy

$$n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)} \tag{1.8}$$

(start with $n^{(-1)} = n^{(0)} = 0$) and so obtain

$$2^p < \gamma_{n^{(p)}} : 0 < n^{(p)} - n^{(p-1)} \underset{p\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator χ of $\{n^{(1)}, n^{(2)}, \dots\}$ is then aperiodic, *i.e.*

$$x^{(\gamma)} \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \tag{1.10}$$

is irrational. Consequently,

$$\mathtt{dyadic}(x^{(\gamma)})_{-k} = \chi_k. \tag{1.11}$$

We now easily see that

$$\chi_1 + \cdots + \chi_{n^{(p)}} = p, \tag{1.12}$$

which, combined with (1.7), yields

$$f_{n^{(p)}}(x^{(\gamma)}) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n^{(p)}} f_{n^{(p)}}(x^{(\gamma)}) > 1. \tag{1.14}$$

We have so established that the subsequence $\{\gamma_{n^{(p)}} f_{n^{(p)}}\}$ does not tend pointwise to 0, hence neither does the whole sequence $\{\gamma_n f_n\}$. In other words, (b) holds, which is in violent contrast with [1.28]: X is then not metrizable. So ends the proof. $\square$

1.2 Exercise 9. Quotient map

Suppose

(*a*) X *and* Y *are topological vector spaces,*

(*b*) $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear.*

(*c*) N *is a closed subspace of* X*,*

(*d*) $\pi : \mathrm{X} \to \mathrm{X}/\mathrm{N}$ *is the quotient map, and*

(*e*) $\Lambda \mathrm{x} = 0$ *for every* $\mathrm{x} \in \mathrm{N}$.

Prove that there is a unique $\mathrm{f} : \mathrm{X}/\mathrm{N} \to \mathrm{Y}$ *which satisfies* $\Lambda = \mathrm{f} \circ \pi$*, that is,* $\Lambda \mathrm{x} = \mathrm{f}(\pi(\mathrm{x}))$ *for all* $\mathrm{x} \in \mathrm{X}$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = \mathrm{f} \circ \pi$ has necessarily a unique solution, which is the binary relation

$$\mathrm{f} \triangleq \{(\pi \mathrm{x}, \Lambda \mathrm{x}) : \mathrm{x} \in \mathrm{X}\} \subset \mathrm{X}/\mathrm{N} \times \mathrm{Y}. \tag{1.15}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda \mathrm{x} \neq \Lambda \mathrm{x}' \Rightarrow \pi \mathrm{x}' \neq \pi \mathrm{x}'. \tag{1.16}$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ.

Remark. The special case $\mathrm{N} = \{\Lambda = 0\}$, *i.e.* $\Lambda \mathrm{x} = 0$ **iff** $\mathrm{x} \in \mathrm{N}$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$\mathrm{f}(\pi \mathrm{x}) = 0 \overset{(1.15)}{\Rightarrow} \Lambda \mathrm{x} = 0 \Rightarrow \mathrm{x} \in \mathrm{N} \Rightarrow \pi \mathrm{x} = \mathrm{N} \tag{1.17}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = \mathrm{f} \circ \pi$, by (a) of [1.41]. Conversely, if Λ is continuous, then for each neighborhood V of 0_{Y} there exists a neighborhood U of 0_{X} such that

$$\Lambda(\mathrm{U}) = \mathrm{f}(\pi(\mathrm{U})) \subset \mathrm{V}. \tag{1.18}$$

Since π is open (see (a) of [1.41]), $\pi(\mathrm{U})$ is a neighborhood of $\mathrm{N} = 0_{\mathrm{X}/\mathrm{N}}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = \mathrm{f} \circ \pi$, by (a) of [1.41]. To prove the converse, remark that every neighborhood W of $0_{\mathrm{X}/\mathrm{N}}$ satisfies

$$\mathrm{W} = \pi(\mathrm{V}) \tag{1.19}$$

for some neighborhood V of 0_{X}. So,

$$\mathrm{f}(\mathrm{W}) = \mathrm{f}(\pi(\mathrm{V})) = \Lambda(\mathrm{V}). \tag{1.20}$$

As a consequence, if Λ is open, then $\mathrm{f}(\mathrm{W})$ is a neighborhood of 0_{Y}. So ends the proof. □

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$, $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$.

(*a*) *Prove that* Λ *is an open mapping.*

(*b*) *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. We discard the trivial case $\dim \mathrm{Y} = 0$ then henceforth assume that $\dim \mathrm{Y}$ has positive dimension n.

Let e range over a base of Y: For each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, since Λ is onto. So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \Lambda \mathrm{x_e} \quad (\mathrm{y} \in \mathrm{Y}). \tag{1.21}$$

The sequence $\{\mathrm{x_e}\}$ is finite hence bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$\mathrm{x_e} \in \mathrm{sV} \tag{1.22}$$

for all $\mathrm{x_e}$. Combining this with (1.21) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \quad (\mathrm{y} \in \mathrm{Y} : |\mathrm{y_e}| < \mathrm{s}^{-1}), \tag{1.23}$$

which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, with $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$\dim \mathrm{X/N} = \mathrm{n}. \tag{1.24}$$

f is then an homeomorphism of $\mathrm{X/N} \equiv \mathbf{C}^{\mathrm{n}}$ onto Y; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$. □

1.4 Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms

Put $\mathrm{K} = [0,1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0, 1, 2, \dots$*) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

(*a*) $\|\mathrm{D^n f}\|_\infty = \sup\{|\mathrm{D^n f(x)}| : \infty < \mathrm{x} < \infty\}$

(*b*) $\|\mathrm{D^n f}\|_1 = \int_0^1 |\mathrm{D^n f}(x)|\,\mathrm{d}x$

(*c*) $\|\mathrm{D^n f}\|_2 = \left\{\int_0^1 |\mathrm{D^n f}(x)|^2 \mathrm{d}x\right\}^{1/2}$.

Proof. First, remark that

$$\|\mathrm{D^n f}\|_1 \leq \|\mathrm{D^n f}\|_2 \leq \|\mathrm{D^n f}\|_\infty < \infty \tag{1.25}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, start from

$$\mathrm{D^n f(x)} = \int_{-\infty}^{\mathrm{x}} \mathrm{D^{n+1} f} \tag{1.26}$$

(which is true, since f has a bounded support) to obtain

$$|\mathrm{D^n f(x)}| \leq \int_{-\infty}^{\mathrm{x}} \left|\mathrm{D^{n+1} f}\right| \leq \|\mathrm{D^{n+1} f}\|_1 \tag{1.27}$$

hence

$$\|\mathrm{D^n f}\|_\infty \leq \|\mathrm{D^{n+1} f}\|_1. \tag{1.28}$$

Combining (1.25) with (1.28) yields

$$\|\mathrm{D^0 f}\|_1 \leq \cdots \leq \|\mathrm{D^n f}\|_1 \leq \|\mathrm{D^n f}\|_2 \leq \|\mathrm{D^n f}\|_\infty \leq \|\mathrm{D^{n+1} f}\|_1 \leq \cdots. \tag{1.29}$$

We now define

$$\mathrm{V_n^{(i)}} \triangleq \{\mathrm{f} \in \mathscr{D}_{\mathrm{K}} : \|\mathrm{f}\|_{\mathrm{i}} < 1/\mathrm{n}\} \quad (\mathrm{i} = 1, 2, \infty) \tag{1.30}$$

$$\mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V_n^{(i)}} : \mathrm{n} = 1, 2, 3, \dots\} \tag{1.31}$$

so that (1.29) is mirrored in terms of neighborhood inclusions, as follows,

$$\mathrm{V_1^{(1)}} \supset \cdots \supset \mathrm{V_n^{(1)}} \supset \mathrm{V_n^{(2)}} \supset \mathrm{V_n^{(\infty)}} \supset \mathrm{V_{n+1}^{(1)}} \supset \cdots. \tag{1.32}$$

Since $\mathrm{V_n^{(i)}} \supset \mathrm{V_{n+1}^{(i)}}$, $\mathscr{B}^{(\mathrm{i})}$ is the local base of a topology τ_{i}. But the chain (1.32) forces the τ_{i}'s to be equals. To see that, choose a set S that is τ_1-open at, say a, *i.e.* $\mathrm{V_n^{(1)}} \subset \mathrm{S} - \mathrm{a}$ for some n. Next, concatenate this with $\mathrm{V_n^{(2)}} \subset \mathrm{V_n^{(1)}}$ (see (1.32)) and so obtain $\mathrm{V_n^{(2)}} \subset \mathrm{S} - \mathrm{a}$, which implies that S is τ_2-open at a. Similarly, we deduce, still from (1.32), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.33}$$

So ends the proof. □

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $\mathrm{p_n}$, then, eventually, only on the ambient space itself. This should then be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1 Let X be a topological space with a countable local base $\{\mathrm{V_n} : \mathrm{n} = 1, 2, 3, \dots\}$. If $\tilde{\mathrm{V}}_\mathrm{n} = \mathrm{V}_1 \cap \dots \cap \mathrm{V_n}$, then every subsequence $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$ is a decreasing (*i.e.* $\tilde{\mathrm{V}}_{\rho(\mathrm{n})} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n}+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_n} \supset \tilde{\mathrm{V}}_\mathrm{n}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{n}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$, since $\tilde{\mathrm{V}}_\mathrm{n} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n})}$. □

The following special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$ is one of the key ingredients:

Corollary 1 (special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$) Under the same notations of Lemma 1, if $\{\mathrm{V_n}\}$ is a decreasing local base, then so is $\{\mathrm{V}_{\rho(\mathrm{n})}\}$.

Corollary 2 If $\{\mathrm{Q_n}\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{\mathrm{Q}_{\rho(\mathrm{n})}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $\mathrm{C}(\Omega)$'s (respectively $\mathrm{C}^\infty(\Omega)$'s) topology of the seminorms $\mathrm{p_n}$, as defined in section 1.44 (respectively 1.46), then the seminorms $\mathrm{p}_{\rho(\mathrm{n})}$ define the same topology τ_Q.

Proof. Let X be $\mathrm{C}(\Omega)$ topologized by the seminorms $\mathrm{p_n}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_n} = \{\mathrm{p_n} < 1/\mathrm{n}\}$, then $\{\mathrm{V_n}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n}+1)}. \tag{1.34}$$

Thus,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n}+1)}. \tag{1.35}$$

In other words, $\mathrm{Q}_{\rho(\mathrm{n})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\rho(\mathrm{n})}\}$ then defines a topology τ_{Q_ρ} for which $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\rho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base of τ_Q, which yields $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\rho}$. □

Lemma 2 If a sequence of compact sets $\{\mathrm{Q_n}\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in allmost all $\mathrm{Q}_\mathrm{n}^\circ$, *i.e.* there exists m such that

$$\mathrm{K} \subset \mathring{\mathrm{Q}}_\mathrm{m} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+1} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+2} \subset \cdots . \tag{1.36}$$

Proof. The following definition

$$\mathrm{C_n} \triangleq \mathrm{K} \setminus \overset{\circ}{\mathrm{Q}}_{\mathrm{n}} \quad (\mathrm{n} = 1, 2, 3, \dots) \tag{1.37}$$

shapes $\{\mathrm{C_n}\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no $\mathrm{C_n}$ is empty and so conclude[2] that the $\mathrm{C_n}$'s intersection contains a point that is not in any $\mathrm{Q_n^\circ}$. On the other hand, the conditions specified in [1.44] force the $\mathrm{Q_n^\circ}$'s collection to be an open cover. This contradiction reveals that $\mathrm{C_m} = \emptyset$, *i.e.* $\mathrm{K} \subset \mathrm{Q_m^\circ}$, for some m. Finally,

$$\mathrm{K} \subset \overset{\circ}{\mathrm{Q}}_{\mathrm{m}} \subset \mathrm{Q_m} \subset \overset{\circ}{\mathrm{Q}}_{\mathrm{m+1}} \subset \mathrm{Q_{m+1}} \subset \overset{\circ}{\mathrm{Q}}_{\mathrm{m+2}} \subset \cdots . \tag{1.38}$$

□

We are now in a fair position to establish the following:

Theorem The topology of $\mathrm{C}(\Omega)$ does not depend on the particular choice of $\{\mathrm{K_n}\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $\mathrm{C}^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

Proof. With the second corollary's notations, $\tau_{\mathrm{K}} = \tau_{\mathrm{K}_\lambda}$, for every subsequence $\{\mathrm{K}_{\lambda(\mathrm{n})}\}$. Similarly, let $\{\mathrm{L_n}\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_{\mathrm{L}} = \tau_{\mathrm{L}_\varkappa}$ for every subsequence $\{\mathrm{L}_{\varkappa(\mathrm{n})}\}$. Now apply the above Lemma 2 with $\mathrm{K_i}$ $(\mathrm{i} = 1, 2, 3, \dots)$ and so conclude that $\mathrm{K_i} \subset \mathrm{L}^\circ_{\mathrm{m_i}} \subset \mathrm{L}^\circ_{\mathrm{m_i}+1} \subset \cdots$ for some $\mathrm{m_i}$. In particular, the special case $\varkappa_{\mathrm{i}} = \mathrm{m_i} + \mathrm{i}$ is

$$\mathrm{K_i} \subset \overset{\circ}{\mathrm{L}}_{\varkappa_{\mathrm{i}}}. \tag{1.39}$$

Let us reiterate the above proof with $\mathrm{K_n}$ and $\mathrm{L_n}$ in exchanged roles then similarly find a subsequence $\{\lambda_{\mathrm{j}} : \mathrm{j} = 1, 2, 3, \dots\}$ such that

$$\mathrm{L_j} \subset \overset{\circ}{\mathrm{K}}_{\lambda_{\mathrm{j}}} \tag{1.40}$$

Combine (1.39) with (1.40) and so obtain

$$\mathrm{K_1} \subset \overset{\circ}{\mathrm{L}}_{\varkappa_1} \subset \mathrm{L}_{\varkappa_1} \subset \overset{\circ}{\mathrm{K}}_{\lambda_{\varkappa_1}} \subset \mathrm{K}_{\lambda_{\varkappa_1}} \subset \overset{\circ}{\mathrm{L}}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots , \tag{1.41}$$

which means that the sequence $\mathrm{Q} = (\mathrm{K_1}, \mathrm{L}_{\varkappa_1}, \mathrm{K}_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_{\mathrm{K}} = \tau_{\mathrm{K}_\lambda} = \tau_{\mathrm{Q}} = \tau_{\mathrm{L}_\varkappa} = \tau_{\mathrm{L}}. \tag{1.42}$$

So ends the proof □

[1] See (b) of 2.5 of [2].

[2] The intersection of a decreasing sequence of nomempty Hausdorff compact sets is nonempty. This is a corollary of 2.6 of [2].

1.6 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $\mathrm{f} \mapsto \mathrm{D}^\alpha \mathrm{f}$ *is a continuous mapping of* $\mathrm{C}^\infty(\Omega)$ *into* $\mathrm{C}^\infty(\Omega)$ *and also of* $\mathscr{D}_\mathrm{K}$ *into* $\mathscr{D}_\mathrm{K}$*, for every multi-index* α.

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $\mathrm{C}^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$\mathrm{V_N} = \left\{\phi \in \mathrm{C}^\infty(\Omega) : \max\{|\mathrm{D}^\beta \phi(\mathrm{x})| : |\beta| \le \mathrm{N}, \mathrm{x} \in \mathrm{K_N}\} < 1/\mathrm{N}\right\}. \tag{1.43}$$

Now pick g in $\mathrm{V}_{\mathrm{N}+|\alpha|}$, so that

$$\max\{|\mathrm{D}^\gamma \mathrm{g}(\mathrm{x})| : |\gamma| \le \mathrm{N} + |\alpha|, \mathrm{x} \in \mathrm{K_N}\} < \frac{1}{\mathrm{N}+|\alpha|}. \tag{1.44}$$

(the fact that $\mathrm{K_N} \subset \mathrm{K}_{\mathrm{N}+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|\mathrm{D}^\beta \mathrm{D}^\alpha \mathrm{g}(\mathrm{x})| : |\beta| \le \mathrm{N}, \mathrm{x} \in \mathrm{K_N}\} < \frac{1}{\mathrm{N}}. \tag{1.45}$$

We have just proved that

$$\mathrm{g} \in \mathrm{V}_{\mathrm{N}+|\alpha|} \Rightarrow \mathrm{D}^\alpha \mathrm{g} \in \mathrm{V_N}, \quad \textit{i.e.} \quad \mathrm{D}^\alpha\left(\mathrm{V}_{\mathrm{N}+|\alpha|}\right) \subset \mathrm{V_N}, \tag{1.46}$$

which establishes the continuity of $\mathrm{D}^\alpha : \mathrm{C}^\infty(\Omega) \to \mathrm{C}^\infty(\Omega)$.

To prove the continuousness of the restriction $\mathrm{D}^\alpha|_{\mathscr{D}_\mathrm{K}} : \mathscr{D}_\mathrm{K} \to \mathscr{D}_\mathrm{K}$, we first remark that the collection of the $\mathrm{V_N} \cap \mathscr{D}_\mathrm{K}$ is a local base of the subspace topology of $\mathscr{D}_\mathrm{K}$. $\mathrm{V}_{\mathrm{N}+|\alpha|} \cap \mathscr{D}_\mathrm{K}$ is then a neighborhood of 0 in this topology. Furthermore,

$$\begin{aligned} \mathrm{D}^\alpha|_{\mathscr{D}_\mathrm{K}}\left(\mathrm{V}_{\mathrm{N}+|\alpha|} \cap \mathscr{D}_\mathrm{K}\right) &= \mathrm{D}^\alpha\left(\mathrm{V}_{\mathrm{N}+|\alpha|} \cap \mathscr{D}_\mathrm{K}\right) && (1.47)\\ &\subset \mathrm{D}^\alpha\left(\mathrm{V}_{\mathrm{N}+|\alpha|}\right) \cap \mathrm{D}^\alpha\left(\mathscr{D}_\mathrm{K}\right) && (1.48)\\ &\subset \mathrm{V_N} \cap \mathscr{D}_\mathrm{K} \quad \text{(see (1.46))} && (1.49) \end{aligned}$$

So ends the proof. □

Chapter 2

Completeness

2.1 Exercise 3. An equicontinous sequence of measures

Put $K = [-1, 1]$*; define* $\mathscr{D}_K$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^n$*). Supose* $\{f_n\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\phi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ *exists for every* $\phi \in \mathscr{D}_K$*. Show that* Λ *is a continuous linear functional on* $\mathscr{D}_K$*. Show that there is a positive integer* p *and a number* $M < \infty$ *such that*

$$\left| \int_{-1}^{1} f_n(t)\phi(t)dt \right| \leq M\|D^p\phi\|_\infty$$

for all n*. For example, if* $f_n(t) = n^3 t$ *on* $[-1/n, 1/n]$ *and* 0 *elsewhere, show that this can be done with* $p = 1$*. Construct an example where it can be done with* $p = 2$ *but not with* $p = 1$*.*

We will also consider the case $p = 0$. The following version of the mean value theorem will be of a great deal of help.

Lemma If $\phi \in \mathscr{D}_{[a,b]}$, then

$$\|D^\alpha\phi\|_\infty \leq \|D^p\phi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (\alpha = 0, 1, \ldots, p) \tag{2.1}$$

at every order $p = 0, 1, 2, \ldots$; where λ is the length $|b - a|$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 $(a < x_1 < x_0)$, such that

$$\phi(x_0) = \phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a). \tag{2.2}$$

Since every $D^p\phi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$\begin{aligned} \phi(x_0) &= D^0\phi(x_0) && (2.3)\\ &= D^1\phi(x_1)(x_0 - a) && (2.4)\\ &= \cdots \\ &= D^p\phi(x_p)(x_0 - a)\cdots(x_{p-1} - a), && (2.5) \end{aligned}$$

for all p. More compactly,

$$D^{\alpha}\phi(x_0) = D^{p}\phi(x_p)\prod_{k=\alpha}^{p-1}(x_k - a); \tag{2.6}$$

which yields,

$$|D^{\alpha}\phi(x)| \leq \|D^{p}\phi\|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c]) \tag{2.7}$$

The case $x_0 \geq c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^{\alpha}\phi(x)| \leq \|D^{p}\phi\|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [c, b]). \tag{2.8}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\|D^{\alpha}\phi\|_{\infty} \leq \|D^{p}\phi\|_{\infty}\left(\frac{\lambda}{2}\right)^{p-\alpha}. \tag{2.9}$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\phi\rangle \triangleq \int_{\mathbf{R}} u\phi \quad (\phi \in C_0(\mathbf{R})). \tag{2.10}$$

The following inequalities

$$|\langle u|\phi\rangle| \leq \int_{\mathbf{R}} |u\phi| \leq \|u\|_{L^1} \quad (\|\phi\|_{\infty} \leq 1) \tag{2.11}$$

imply that every linear functional

$$\begin{aligned} \langle u| : C_0(\mathbf{R}) &\to \mathbf{C} \\ \phi &\mapsto \langle u|\phi\rangle \end{aligned} \tag{2.12}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u|$, since $C_c(\mathbf{R})$ is sufficiently large to assure that u is determined (a.e) by the integrals $\langle u|\phi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\|u\| \triangleq \sup\{|\langle u|\phi\rangle| : \|\phi\|_{\infty} \leq 1\} = \|u\|_{L^1}. \tag{2.13}$$

Note that, in the latter equality, $\|u\| \leq \|u\|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$\begin{aligned} g_n : \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & \left(x \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right) \\ 0 & \left(x \notin \left[-\frac{1}{n}, \frac{1}{n}\right]\right). \end{cases} \end{aligned} \tag{2.14}$$

[1] see 1.32, 4.1 of [3]

[2] See 7.26 of [1].

First, remark that $g_n(x) \underset{n\to\infty}{\longrightarrow} 0$ $(x \in \mathbf{R})$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$\langle g_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi, \tag{2.15}$$

where ϕ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\phi_1^+, \dots, \phi_j^+, \dots,$ from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\phi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

(ii) $\phi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

(iii) $0 \leq \phi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those ϕ_j^+. Let $\phi_1^-, \dots, \phi_j^-, \dots,$ mirror the ϕ_j^+, in the sense that $\phi_j^-(x) = \phi_j^+(-x)$, so that

(iv) $\phi_j \triangleq \phi_j^+ - \phi_j^-$ is odd, as g_n is;

(v) every ϕ_j is in $C_K^\infty(\mathbf{R})$;

(vi) The sequence $\{\phi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $|\phi_j| \leq 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n|\phi_j\rangle = 2\int_0^1 g_n(t)\phi_j^+(t)dt \underset{j\to\infty}{\longrightarrow} 2\int_0^1 g_n(t)dt = \|g_n\|_{L^1} = n. \tag{2.16}$$

Finally,

$$\|g_n\|_{L^1} \overset{(2.13)}{\geq} \sup\{|\langle g_n|\phi\rangle| : \|\phi\| \leq 1\} \overset{(2.16)}{\geq} \|g_n\|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exist constants C_n such that

$$|\langle g_n|\phi\rangle| \leq C_n \quad (\|\phi\|_\infty \leq 1); \tag{2.18}$$

see (2.11). Furthermore, $\|g_n\|_{L^1} = n$ is actually the best (the lowest) possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\phi_{\rho(n)}\rangle\}$ where $\langle g_n|\phi_{\rho(n)}\rangle$ is greater than, say, $n/2$. Consequently, there is no common bound M such that

$$|\langle g_n|\phi\rangle| \leq M \quad (\|\phi\|_\infty \leq 1; n = 1, 2, 3, \dots). \tag{2.19}$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (say "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi \quad (\phi \in C_0(\mathbf{R})) \tag{2.20}$$

[3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise to some Λ, in a relevant space (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that ϕ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$\Lambda\phi = \lim_{n\to\infty} \int_K f_n\phi \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.21}$$

for some linear form Λ. Since ϕ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$\Lambda\phi = \lim_{n\to\infty} \langle f_n|\phi\rangle \quad (\phi \in C_K^\infty(\mathbf{R})). \tag{2.22}$$

Let $K_1, K_2, \ldots,$ be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\phi) = \|\phi\|_N \triangleq \max\{|D^\alpha\phi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\phi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\phi \in \mathscr{D}_K : \|\phi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

$\langle f_n|$ is bounded on V_m, this is a special case of (2.11). Every functional $\langle f_n|$ is therefore τ_K-continuous, see 1.18 of [3]. To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi$ for all ϕ, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that

$$|\Lambda\phi| < 1/2 \quad \text{if } \phi \in V_i, \tag{2.23}$$
$$|\Lambda\phi - \langle f_n|\phi\rangle| < 1/2 \quad \text{if } \phi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\phi\rangle| \leq |\Lambda\phi - \langle f_n|\phi\rangle| + |\Lambda\phi| < 1 \quad \text{if } \phi \in V_p. \tag{2.25}$$

Now remark that every

$$\phi_\mu \triangleq \begin{cases} \frac{1}{\mu\cdot 2^p\|\phi\|_p}\phi & (\phi \neq 0, \mu > 1) \\ 0 & (\phi = 0, \mu > 1) \end{cases} \tag{2.26}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\phi_\mu\rangle| < 1 \tag{2.27}$$
$$|\langle f_n|\phi\rangle| < 2^p\|\phi\|_p \cdot \mu \tag{2.28}$$
$$|\langle f_n|\phi\rangle| \leq 2^p\|\phi\|_p \tag{2.29}$$
$$|\langle f_n|\phi\rangle| \leq 2^p\left(\|D^0\phi\|_\infty + \cdots + \|D^p\phi\|_\infty\right) \tag{2.30}$$
$$|\langle f_n|\phi\rangle| \leq 2^p(p+1)\|D^p\phi\|_\infty \quad \text{(see 2.1)} \tag{2.31}$$

The first part is so proved, with $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\phi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\phi(t)dt$ exists for every $\phi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\phi\rangle| \leq M\|D^p\phi\|$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the K_N's indices by -m, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let ϕ keep running on $\mathscr{D}_K$ and so define

$$B_n(\phi) \triangleq \max\{|\phi(x)| : x \in [-1/n, 1/n]\}, \tag{2.32}$$
$$\Delta_n(\phi) \triangleq \max\{|\phi(x) - \phi(0)| : x \in [-1/n, 1/n]\}. \tag{2.33}$$

The mean value asserts that

$$|\phi(1/n) - \phi(-1/n)| \leq B_n(\phi')\,|1/n - (-1/n)| = \frac{2}{n}B_n(\phi') \tag{2.34}$$

. An integration by parts shows that

$$\langle f_n|\phi\rangle = \left[\frac{n^3t^2}{2}\phi(t)\right]_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)dt \tag{2.35}$$
$$= \frac{n}{2}\left(\phi(1/n) - \phi(-1/n)\right) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\phi'(t)dt. \tag{2.36}$$

Combine (2.34) with (2.35) and so obtain

$$|\langle f_n|\phi\rangle| \leq \frac{n}{2}\,|\phi(1/n) - \phi(-1/n)| + \frac{n^3}{2}\int_{-1/n}^{1/n} t^2|\phi'(t)|dt \tag{2.37}$$
$$\leq B_n(\phi') + \frac{n^3}{2}B_n(\phi')\int_{-1/n}^{1/n} t^2dt \tag{2.38}$$
$$\leq \frac{4}{3}B_n(\phi') \tag{2.39}$$
$$\leq \frac{4}{3}\|\phi'\|_\infty; \tag{2.40}$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\phi')$ tends to $\phi'(0)$, we may expect that f_n tends to $\frac{4}{3}\phi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0) \overset{(2.36)}{=} \frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n}(\phi' - \phi'(0))t^2dt. \tag{2.41}$$

So,

$$\left|\langle f_n|\phi\rangle - \frac{4}{3}\phi'(0)\right| \leq \left|\frac{\phi(1/n) - \phi(-1/n)}{1/n - (-1/n)} - \phi'(0)\right| + \frac{1}{3}\Delta_n(\phi') \underset{n\to\infty}{\longrightarrow} 0. \tag{2.42}$$

We have just proved that

$$\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi = \frac{4}{3}\phi'(0) \quad (\phi \in \mathscr{D}_K). \tag{2.43}$$

In other words,

$$\langle f_n| \underset{n\to\infty}{\longrightarrow} \Lambda = -\frac{4}{3}\delta', \tag{2.44}$$

where δ is the *Dirac measure* and $\delta', \delta'', \dots,$ its derivatives ; see 6.1 and 6.9 of [3].

It follows from the previous part that Λ is τ_K-continuous. Moreover, we have a bound

$$|\langle f_n|\phi\rangle| \le M\|D^p\phi\|_\infty \quad (n = 1, 2, 3, \dots), \tag{2.45}$$

with $p = 1$ and $M = \frac{4}{3}$ (which is a concrete version of (2.40)). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n|\phi_{\rho(n)}\rangle : \|\phi_{\rho(n)}\|_\infty \le 1, n = 1, 2, 3, \dots\} \tag{2.46}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle f_n|\phi\rangle| \le M \quad (\phi \in C_K^\infty(\mathbf{R}), \|\phi\|_\infty \le 1). \tag{2.47}$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

f_n is not differentiable on $\mathbf{R}$, so we give $\langle f_n|$ a *derivative* [4], as follows

$$\begin{aligned} \langle f_n|' : \mathscr{D}_K &\to \mathbf{C} \\ \phi &\mapsto -\left\langle f_n\middle|\phi'\right\rangle ; \end{aligned} \tag{2.48}$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$\begin{aligned} D : \mathscr{D}_K &\to \mathscr{D}_K \\ \phi &\mapsto \phi'; \end{aligned} \tag{2.49}$$

see Exercise 1.17. $\langle f_n|'$ is therefore continuous. Now apply (2.43). with ϕ' and so obtain

$$-\left\langle f_n\middle|\phi'\right\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\phi''(0) \quad (\phi \in \mathscr{D}_K),$$

i.e.

$$\langle f_n|' \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\delta''. \tag{2.50}$$

It follows from (2.40) that, for some positive constant C,

$$\left|\left\langle f_n\middle|\phi'\right\rangle\right| \le C\|\phi''\|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.51}$$

[4] See 6.1 of [3] for a further discussion.

It is therefore possible to uniformly bound $\langle \mathrm{f_n}|'$ with respect to a norm $\|\mathrm{D^p}\cdot\|_\infty$, namely $\|\mathrm{D^2}\cdot\|$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle \mathrm{f_n}|\phi'\rangle| \le \mathrm{C}\|\phi'\|_\infty \quad (\mathrm{n} = 1, 2, 3, \dots). \tag{2.52}$$

Define

$$\Phi_\mathrm{j}(\mathrm{x}) = \int_{-1}^{\mathrm{x}} \phi_\mathrm{j}. \tag{2.53}$$

The oddness of ϕ_j forces Φ_j to vanish outside [-1, 1]. ϕ_j is then in $\mathscr{D}_\mathrm{K}$. So, under our assumption,

$$|\langle \mathrm{f_n}|\Phi_\mathrm{j}'\rangle| \le \mathrm{M}\|\Phi_\mathrm{j}'\| \quad (\mathrm{n} = 1, 2, 3, \dots); \tag{2.54}$$

which is

$$|\langle \mathrm{f_n}|\phi_\mathrm{j}\rangle| \le \mathrm{M} \quad (\mathrm{n} = 1, 2, 3, \dots). \tag{2.55}$$

We have thus reached a contradiction (again with the sequence $\{\langle \mathrm{f_n}|\phi_{\rho(\mathrm{n})}\rangle\}$) and so conclude that there is no constant M such that

$$|\langle|\mathrm{f_n}\phi'\rangle| \le \mathrm{M}\|\phi'\|_\infty \quad (\mathrm{n} = 1, 2, 3, \dots). \tag{2.56}$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$|\langle \mathrm{f_n}|\phi'\rangle| \le \mathrm{M}\|\phi\|_\infty. \tag{2.57}$$

The mean value theorem (see (2.1)) asserts that

$$|\langle \mathrm{f_n}|\phi'\rangle| \le \mathrm{M}\|\phi\|_\infty \le \mathrm{C}\frac{\lambda}{2}\|\phi'\|; \tag{2.58}$$

which is, again, a desired contradiction. So ends the proof.

□

Bibliography

[1] Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 1976.

[2] Walter Rudin. *Real and Complex Analysis.* McGraw-Hill, 1986.

[3] Walter Rudin. *Functional Analysis.* McGraw-Hill, 1991.

[4] Laurent Schwartz. *Analyse*, volume III (in French). Hermann, 1997.