

Solutions to some exercises from Walter Rudin's *Functional  
Analysis*

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# Notations and Assumptions

## I Special terms

### i Halmos' iff and definitions

**iff** is a shorthand for “if and only if”. Splitting **iff** into *if-then* clauses shows that it is a rewording of the logical equivalence  $\Leftrightarrow$ . All definitions will use the *if and only if* convention, which is consistent with the fact that every definition expresses an identity.

### ii The assignment operator $\triangleq$

Given  $a$  and  $b$ ,  $\triangleq$  is a specialization of  $=$ . We say that  $a \triangleq b$  **iff**  $a$  and  $b$  are *assumed* to be equal. Typically,  $a \triangleq b$  means that  $a$  is assigned the previously known value  $b$  (some authors write  $a := b$ ) but this is not a strict requirement.

### iii Iverson brackets

Given a boolean expression  $\phi$ ,  $[\phi]$  encodes the truth value of  $\phi$  as a bit:

$$[\phi] \triangleq \begin{cases} 0 & (\phi \text{ is false}) \\ 1 & (\phi \text{ is true}). \end{cases}$$

For example,  $[1 > 0] = 1$  but  $[\sqrt{2} \in \mathbf{Q}] = 0$ . For interpretations of Boolean operators in logic, see [2].

## II Sets

### i Subsets and supersets

$\subseteq$  and  $\supseteq$  are the standard symbols for set ordering. No specific symbol is reserved for strict ordering, every restriction  $X \neq Y$  will be explicitly stated when it applies. Given  $A$  and  $B$ ,  $A \cup B$  is the union of  $A$  and  $B$ . More generally, for any collection  $C$  of set(s)  $S$ , the union  $U$  of  $C$  is expressed as follows:

$$(1) \quad U \triangleq \bigcup C \triangleq \bigcup_{S \in C} S.$$

Similarly,  $\cap$  denotes intersection in the same manner.

### ii Special mappings

The identity  $I$  (or  $\text{id}$ ) is the mapping  $\{(x, x) : x \in X\}$ . Similarly, the projection  $\pi = \{((x, y), x) : x \in X, y \in Y\}$  always exists. Observe that  $I$  is the diagonal of  $X \times X$ .

### iii Equinumerosity

In a metric space context,  $X \equiv Y$  means that  $\phi$  is an onto isometry.

### III Topological vector spaces

#### i Scalar field

$\mathbf{C}$  extends  $\mathbf{R}$ , which implies that a property, *e.g.*, linearity, that holds over  $\mathbf{C}$  also holds over  $\mathbf{R}$ . The complex case is therefore *the stronger one*. This restriction may be significant in some contexts. However, the standard scalar field is  $\mathbf{C}$ . Considering  $\mathbf{R}$  instead of  $\mathbf{C}$  makes no difference, unless stated otherwise, as in [3.1 and Chapter 12] of [4].

#### ii Vector space bases

Given a vector space  $X$ , a subset  $B$  of  $X$  is a basis of  $X$  **iff** the sum

$$(2) \quad \left\{ (z_u)_{u \in B} : z_u \in \mathbf{C}, \{u : z_u \neq 0\} \text{ is finite} \right\} \rightarrow X$$

$$(z_u) \mapsto \sum_{z_u \neq 0} z_u u$$

is a bijection from all *finitely supported* families  $(z_u)$  onto  $X$ . The axiom of choice (AC) forces

1. the existence of such  $B$  (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [4] with  $B$  playing the role of  $\Lambda$ );
2. all bases to have the same cardinality, which is called the *dimension* of  $X$  and is denoted as  $\dim X$ .

We now turn to the finite-dimensional case over the field  $\mathbf{C}$ . The zero-dimensional case is  $B = \emptyset$ , *i.e.*,  $X = \{0\}$ . Our first step is to study  $\mathbf{C}^n$  ( $n > 0$ ) the standard  $n$ -dimensional vector space.

#### iii Finite-dimensional spaces

##### The product topology of $\mathbf{C}^n$

$\mathbf{C}^n$  has a standard basis  $1_{\{0\}}, \dots, 1_{\{n-1\}}$  so that scalar  $z_k$  is the  $k$ -th component of

$$(3) \quad (z_1, \dots, z_n) \triangleq z_1 \cdot \underbrace{(1, 0, \dots, 0)}_{1_{\{0\}}} + \dots + z_n \cdot \underbrace{(0, \dots, 0, 1)}_{1_{\{n-1\}}},$$

as  $z = (z_1, \dots, z_n)$  ranges over  $\mathbf{C}^n$ . Moreover,  $\mathbf{C}^n$  is endowed with the topology generated by all polydiscs

$$(4) \quad \prod_{i=1}^n \underbrace{\{z_i \in \mathbf{C} : |z_i| < r_i\}}_{D_{r_i}} \quad (r_i > 0).$$

Equivalently, we may equip  $\mathbf{C}^n$  with the Euclidean norm

$$(5) \quad \|z\|_2 \triangleq \sqrt{|z_1|^2 + \dots + |z_n|^2},$$

whose open balls centered at the origin are all

$$(6) \quad B_r \triangleq \{z \in \mathbf{C}^n : \|z\|_2 < r\} \quad (r > 0).$$

To show the equivalence, first set  $r_i = r/\sqrt{n}$ . Hence

$$(7) \quad \prod_{i=1}^n D_{r_i} \subseteq B_r.$$

Conversely, put  $r = \min(r_1, \dots, r_n)$  so that

$$(8) \quad B_r \subseteq \prod_{i=1}^n D_{r_i}.$$

### Topology of a finite-dimensional vector space

It is customary to identify any  $n$ -dimensional vector space with  $\mathbf{C}^n$  equipped with the Euclidean norm, see [5]. To show this, choose an isomorphism  $f : \mathbf{C}^n \rightarrow Y$ . For instance, let  $f(e_k)$  be  $u_k$ , as in [1.20] of [4], when  $\{u_k\}$  is a basis of the  $n$ -dimensional  $Y$ , see [ii]. It follows from [1.21] of [4] that  $f$  is a homeomorphism. A striking consequence is that  $\{f(U) : U \text{ open in } \mathbf{C}^n\}$  is the only topological vector space topology for  $Y$ . In other words, there is only one version of  $Y$  as a topological vector space. Thus,  $Y$  is necessarily locally convex and locally bounded, i.e., normable, see [1.39] of [4]. Note that  $\|y\|_Y = \|f^{-1}(y)\|_2$  for all  $y \in Y$  is an example of a norm. Additionally,  $Y$  is locally compact, as the closed unit ball of  $\mathbf{C}^n$  is compact. Choose an  $n$ -dimensional topological vector space  $W$ , and repeat the same reasoning, first with  $g : \mathbf{C}^n \rightarrow W$ , next with  $h = g \circ f^{-1}$ , playing the role of  $f$ . This establishes that the homeomorphism  $h$  maps  $Y$  onto  $W$  and that  $W$  is normable as well. The following assertions are then equivalent in the finite-dimensional context:

1.  $\dim W = \dim Y$ ,
2.  $W$  and  $Y$  are isomorphic to each other,
3.  $W$  and  $Y$  are homeomorphic to each other, they are normable.

Furthermore, the norms on  $W$  and  $Y$  are *equivalent*. That is, for any given norm  $\|\cdot\|_Y$  on  $Y$  and any given norm  $\|\cdot\|_W$  on  $W$ , there exists a positive constant (termed the *modulus of continuity*)  $C = C_h$  such that

$$(9) \quad \|w\|_W \leq C \|y\|_Y \quad ((y, w) \in h),$$

as  $h$  is continuous. When  $W = Y$ , this means that all norms on  $Y$  are equivalent, in the sense that

$$(10) \quad \|h(y)\|_Y \leq C \|y\|_Y.$$

### The standard norms 1, 2, and $\infty$

When  $\mathbf{C}^n$  is equipped with standard norms  $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ , the least  $C_{i,j}$  such that

$$(11) \quad \|z\|_j \leq C_{i,j} \|z\|_i$$

is easily derived from definitions - see [1.19] of [4] - with the exception of  $C_{2,1} = \sqrt{n}$ . Indeed,  $\|z\|_1 \leq \sqrt{n} \|z\|_2$  is a Cauchy-Schwarz inequality, see (1) in [12.2] of [4]. The table below records these sharp  $C_{i,j}$ ,

i \ j	1	2	$\infty$
1	1	1	1
2	$\sqrt{n}$	1	1
$\infty$	$n$	$\sqrt{n}$	1

Table 1. Minimal  $C_{i,j}$  for the standard norms  $i, j = 1, 2, \infty$ .

### iv Continuity and boundedness in normed spaces

A linear mapping  $\Lambda$  is said to be *bounded* **iff**  $\Lambda(E)$  is a bounded set for every bounded set  $E$ ; see [1.31] of [4]. Linearity implies that  $\Lambda : X \rightarrow Y$  is bounded **iff**

$$(12) \quad \|\Lambda\| \triangleq \sup \{\|\Lambda x\| : \|x\| < 1\} < \infty,$$

Given normed spaces  $X$  and  $Y$ , bounded linear maps  $\Lambda$  form a normed space  $B(X, Y)$ , where norm  $\|\Lambda\|$  is given above, see [4.1] of [4]. It is now easy to see that, in the current context, boundedness and continuity coincide. This is a particular case of [2.8] of [4]. When  $Y$  is the scalar field, this equivalence also comes from [1.32] of [4]. Furthermore, we observe, that given a collection of bounded mappings  $\Lambda \in B(X, Y)$ , the property of equicontinuity now reads as

$$(13) \quad \sup_{\Lambda} \|\Lambda\| < \infty.$$

Thus, *uniform boundedness* and uniform continuity coincide as well; cf. [2.4] of [4].

## IV Measure theory

### i Radon measures

Given a locally compact Hausdorff space  $X$ , *e.g.*,  $\mathbf{R}^n$ , a positive Radon measure  $\Lambda$  is a functional that is *positive* on  $C_c(X)$ , in the sense that

$$(i) \quad \phi \geq 0 \Rightarrow \Lambda\phi \geq 0 \quad (\phi \in C_c(X)).$$

Theorem [2.14] of [3] shows that positivity (i) implies the following *continuity* property (ii):

(ii) For each compact  $K \subseteq X$  there exists a modulus of continuity  $M_K$  such that

$$|\Lambda\phi| \leq M_K \|\phi\|_\infty \quad (\phi \in C_c(X) : \text{supp } \phi \subseteq K).$$

Condition (ii) defines Radon measures in a weaker sense, see [5]. Moreover,  $\Lambda$  is bounded on  $C_c(X)$  with respect to the supremum norm **iff** (ii) is satisfied with all  $M_K \leq \|\Lambda\|_\infty < \infty$ . These bounded functionals are then uniformly continuous Radon measures. According to [6.19] of [3], each of them is isometrically identified with a specific regular Borel measure  $\mu$ . Conversely, when the ambient space  $X$  is compact, every Radon measure is  $\|\cdot\|_\infty$ -bounded. For instance, this case is addressed in Exercise 3 of Chapter 2. In measure theory context, by *support* of  $\mu$ , we mean:

$$(14) \quad \text{supp } \mu \triangleq X \setminus \bigcup V,$$

where  $V$  runs through all open sets of measure 0. A very important Radon measure of support  $\{0\}$ , the *Dirac delta function* is described in [A.2].

### ii Lebesgue integration

Theorem [2.14] of [3] states that every positive Radon measure  $\Lambda : C_c(X) \rightarrow \mathbf{C}$  is identified with a specific positive and regular Borel measure  $\beta : X \rightarrow \mathbf{R}$ . Hence

$$(15) \quad \int_X \phi \, d\beta \triangleq \Lambda\phi \quad (\phi \in C_c(X)),$$

where the left side is a Lebesgue integral. The regularity property implies that  $\beta$  has total mass

$$(16) \quad \int_X 1 \, d\beta \triangleq \sup \{ \Lambda\phi : \phi \in C_c(X), \|\phi\|_\infty \leq 1 \}.$$

When we substitute  $f \, d\beta$  for  $d\beta$ , with  $f$   $X$ -measurable, (15) takes the form

$$(17) \quad \int_X f \phi \, d\beta = \Lambda(f\phi).$$

In particular, if  $f$  is positive,

$$(18) \quad \int_X f \, d\beta = \sup \{ \Lambda\phi : \phi \in C_c(X), \|\phi\|_\infty \leq 1 \}.$$

Moreover, given  $N = N(\beta)$  denoting  $\{f : f = 0 \text{ } \beta\text{-a.e.}\}$ , we see that the density  $f$  and the measure  $\Lambda$  are identified modulo  $N$  in (17). Algebraically speaking:

$$(19) \quad g\beta = f\beta \Leftrightarrow g - f \in N.$$

An important issue in calculus is to know whether the modulus integral  $|f|_1$ , namely

$$(20) \quad |f|_1 \triangleq \int_X |f| \, d\beta \quad (f \text{ } X\text{-measurable}),$$

is finite. This justifies the following definitions:

$$(21) \quad \mathcal{L}^1(X, \beta) \triangleq \{f : |f|_1 < \infty\} \subseteq \mathcal{L}_{\text{loc}}^1(X, \beta) \triangleq \bigcap \{ \mathcal{L}^1(K, \beta) : K \subseteq X \text{ is compact.} \}$$

Standard notation  $\mathcal{L}^1(\beta)$  is used to avoid redundancy, or simply  $\mathcal{L}^1$  when the context is clear. Key points include:



- (a)  $\mathcal{L}^1$ , equipped with  $|\cdot|_1$ , is a seminormed space,
- (b)  $N$  is a closed subspace of the seminormed space  $\mathcal{L}^1$ .

Together, these two facts imply that  $L^1 = \mathcal{L}^1/N$ , equipped with the norm

$$(22) \quad \|f + N\|_1 \triangleq |f|_1 \quad (f \in \mathcal{L}^1),$$

is a Banach space. These definitions and properties extend to  $L^p$  spaces ( $p > 1$ ), as follows

$$(23) \quad \mathcal{L}^p \triangleq \{f : |f|^p \in \mathcal{L}^1\},$$

$$(24) \quad L^p \triangleq \mathcal{L}^p/N,$$

$$(25) \quad \|f + N\|_p \triangleq \sqrt[p]{|f|^p}_1 \quad (f \in \mathcal{L}^p).$$

Similar *quasi-Banach*  $L^p$  spaces exist for  $0 < p < 1$ , with the noticeable difference that  $\sqrt[p]{|f|^p}_1$  no longer defines a norm (the triangle inequality is lost). In contrast, the counterpart  $L^\infty = \{f + N : \|f + N\|_\infty < \infty\}$  is equipped with  $\|\cdot\|_\infty$ , where the quotient-norm

$$(26) \quad \|f + N\|_\infty \triangleq \inf \{M : M > 0, |f| < M \text{ } \beta\text{-a.e.}\}$$

over bounded functions  $f$  turns it into a Banach space.



# Chapter 1

## Topological Vector Spaces

### 1 Exercise 1. Basic results

Suppose  $X$  is a vector space. All sets mentioned below are understood to be subsets of  $X$ . Prove the following statements from the axioms as given in section 1.4.

- (a) If  $x, y \in X$  there is a unique  $z \in X$  such that  $x + z = y$ .
- (b)  $0 \cdot x = 0 = \alpha \cdot 0$  ( $\alpha \in \mathbf{C}, x \in X$ ).
- (c)  $2A \subseteq A + A$ .
- (d)  $A$  is convex if and only if  $(s + t)A = sA + tA$  for all positive scalars  $s$  and  $t$ .
- (e) Every union (and intersection) of balanced sets is balanced.
- (f) Every intersection of convex sets is convex.
- (g) If  $\Gamma$  is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of  $\Gamma$  is convex.
- (h) If  $A$  and  $B$  are convex, so is  $A + B$ .
- (i) If  $A$  and  $B$  are balanced, so is  $A + B$ .
- (j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

PROOF. (a) Such property only depends on the group structure of  $X$ : Each  $x$  in  $X$  has an opposite  $-x$ . Let  $x'$  be any opposite of  $x$  so that  $x - x = 0 = x + x'$ . Thus,  $-x + x - x = -x + x + x'$ , which is equivalent to  $-x = x'$ . So is established the uniqueness of  $-x$ . It is now clear that  $x + z = y$  **iff**  $z = -x + y$ , which asserts both the existence and the uniqueness of  $z$ .

(b) Remark that

$$(1.1) \quad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$

$$(1.2) \quad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that  $0 \cdot x = 0$ . So,

$$(1.3) \quad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x.$$

Finally,

$$(1.4) \quad \alpha \cdot 0 \stackrel{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

(c) Remark that

$$(1.5) \quad 2x = (1 + 1)x = x + x$$

for every  $x$  in  $X$ , and so conclude that

$$(1.6) \quad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets  $A$  of  $X$ ; which proves (c).

(d) If  $A$  is convex, then

$$(1.7) \quad A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A;$$

which is

$$(1.8) \quad sA + tA = (s+t)A.$$

Conversely, the special case  $s + t = 1$  is

$$(1.9) \quad sA + (1-s)A = A.$$

The latter extends to  $s = 0$ , since

$$(1.10) \quad 0A + A \stackrel{(b)}{=} \{0\} + A = A.$$

The extension to  $s = 1$  is analogously established (or simply use the fact that  $+$  is commutative!). So ends the proof.

(e) Let  $A$  range over  $B$  a collection of balanced subsets so that

$$(1.11) \quad \alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B$$

for all scalars  $\alpha$  of magnitude  $\leq 1$ . The inclusion  $\alpha \bigcap B \subseteq A$  establishes the first part. Now remark that

$$(1.12) \quad \alpha A \subseteq \bigcup B$$

implies

$$(1.13) \quad \alpha \bigcup B \subseteq \bigcup B;$$

which achieves the proof.

(f) Let  $A$  range over  $C$  a collection of convex subsets so that

$$(1.14) \quad (s+t) \bigcap C \subseteq s \bigcap C + t \bigcap C \subseteq sA + tA \stackrel{(d)}{\subseteq} (s+t)A$$

for all positive scalars  $s, t$ . Inclusions at both extremities force

$$(1.15) \quad s \bigcap C + t \bigcap C = (s+t) \bigcap C.$$

We now conclude from (d) that the intersection of  $C$  is convex. So ends the proof.

(g) Skip all trivial cases  $\Gamma = \emptyset, \{\emptyset\}, \{\{x\}\}, \{\emptyset, \{x\}\}$  then pick  $x_1, x_2$  in  $\bigcup \Gamma$ , so that each  $x_i$  ( $i = 1, 2$ ) lies in some  $C_i \in \Gamma$ . Since  $\Gamma$  is totally ordered by set inclusion, we henceforth assume without loss of generality that  $C_1$  is a subset of  $C_2$ . So,  $x_1, x_2$  are now elements of the convex set  $C_2$ . Every convex combination of our  $x_i$ 's is then in  $C_2 \subseteq \bigcup \Gamma$ . Hence (g).

(h) Simply remark that

$$(1.16) \quad s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B)$$

for all positive scalars  $s$  and  $t$ , then conclude from (d) that  $A+B$  is convex.

- (i) Given any  $\alpha$  from the closed unit disc,

$$(1.17) \quad \alpha(A + B) = \alpha A + \alpha B \subseteq A + B.$$

There is no more to prove:  $A + B$  is balanced.

- (j) Our proof will be based on the following lemma,

*If  $S$  is nonempty, then each of the following three properties*

- (i)  $S$  is a vector subspace of  $X$ ;*
- (ii)  $S$  is convex balanced such that  $S + S = S$ ;*
- (iii)  $S$  is convex balanced such that  $\lambda S = S$  ( $\lambda > 0$ )*

*implies the other two.*

To prove the lemma, let  $S$  run through all nonempty subsets of  $X$ . First, assume that (i) holds: Clearly, every  $S$  is convex balanced. Moreover,  $S + S \subseteq S$ . Conversely,  $S = S + \{0\} \subseteq S + S$ ; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \quad nS = (n-1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). Pick  $\lambda > 0$  then choose  $n$  so large that  $1 < n\lambda < n^2$ . Thus,

$$(1.19) \quad nS \stackrel{(1.18)}{\subseteq} S \subseteq n\lambda S \subseteq n^2S,$$

since  $S$  is balanced. For instance, set  $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$ . Dividing the latter inclusions by  $n$  shows that

$$(1.20) \quad S \subseteq \lambda S \subseteq nS \stackrel{(1.18)}{\subseteq} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$(1.21) \quad \alpha S + \beta S \stackrel{(a)}{=} |\alpha|S + |\beta|S \stackrel{(d)}{=} (|\alpha| + |\beta|)S \stackrel{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as  $S$  is balanced. Moreover (under the sole assumption that  $S$  is balanced), this extends to  $|\alpha| + |\beta| = 0$ , as follows,

$$(1.22) \quad \alpha S + \beta S = 0S + 0S \stackrel{(b)}{=} \{0\} \stackrel{(b)}{=} 0S \subseteq S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let  $V$  be a collection of vector spaces of  $X$ , of intersection  $I$  and union  $U$ . First, remark that every member of  $V$  is convex balanced: So is  $I$  (combine (e) with (f)). Next, let  $Y$  range over  $V$  so that

$$(1.23) \quad I + I \subseteq Y + Y \subseteq Y;$$

which yields

$$(1.24) \quad I + I = I$$

(the fact that  $I = I + \{0\} \subseteq I + I$  was tacitly used). It now follows from the lemma's (ii)  $\Rightarrow$  (i) that  $I$  is a vector subspace of  $X$ . Now temporarily assume that  $S$  is totally ordered by set inclusion: Combining (e) with (g) establishes that  $U$  is convex balanced. To show that  $U$  is more specifically a vector subspace, we first remark that such total order implies that either  $Z \subseteq Y$  or  $Y \subseteq Z$ , as  $Z$  ranges over  $V$ . A straightforward consequence is that

$$(1.25) \quad Y \subseteq Y + Z \subseteq Y \cup Z.$$

Another one is that  $Y \cup Z$  ranges over  $V$  as well. Combined with the latter inclusions, this leads to

$$(1.26) \quad U \subseteq U + U \subseteq U.$$

It then follows from the lemma's (ii)  $\Rightarrow$  (i) that  $U$  is a vector subspace of  $X$ . Finally, let  $A, B$  run through all vector subspaces of  $X$ : Combining (h) with (i) proves that  $A + B$  is convex balanced as well. Furthermore,

$$(1.27) \quad A + B \stackrel{(i) \Rightarrow (ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as  $X$  is an abelian group. We now conclude from (ii) that any  $A + B$  is a vector subspace of  $X$ . So ends the proof.  $\square$

## 2 Exercise 2. Convex hull

The convex hull of a set  $A$  in a vector space  $X$  is the set of all convex combinations of members of  $A$ , that is the set of all sums  $t_1x_1 + \cdots + t_nx_n$  in which  $x_i \in A$ ,  $t_i \geq 0$ ,  $\sum t_i = 1$ ;  $n$  is arbitrary. Prove that the convex hull of a set  $A$  is convex and that is the intersection of all convex sets that contain  $A$ .

PROOF. The convex hull of a set  $S$  will be denoted by  $\text{co}(S)$ . Remark that  $S \supseteq \text{eq co}(S)$  (to see that, take  $t_1 = 1$  for each  $x_1$  in  $S$ ) and that  $\text{co}(A) \supseteq \text{eq co}(B)$  where  $A \supseteq \text{eq} B$  (obvious).

Our proof will directly derive from (i)  $\Rightarrow$  (iv) in the following lemma,

Let  $S$  be a subset of a vector space  $X$ : Its convex hull  $\text{co}(S)$  is convex and the following statements

- (i)  $S$  is convex;
- (ii)  $s_1S + \cdots + s_nS = (s_1 + \cdots + s_n)S$  for all positive scalar variables  $s_1, \dots, s_n$ ;
- (iii)  $t_1S + \cdots + t_nS = S$  for all positive scalar variables  $s_1, \dots, s_n$  such that  $s_1 + \cdots + s_n = 1$ ;
- (iv)  $\text{co}(S) = S$

are equivalent.

From now on, we skip the trivial case  $S = \emptyset$  then only consider nonempty sets. To prove the first part, let  $a, b$  range over  $\text{co}(S)$  so that  $a = t_1x_1 + \cdots + t_nx_n$  and  $b = t_{n+1}x_{n+1} + \cdots + t_{n+p}x_{n+p}$  for some  $(t_i, x_i)$ . Every sum  $sa + (1-s)b$  ( $0 \leq s \leq 1$ ) is then in the convex hull of  $\{x_1, \dots, x_{n+p}\}$ , since

$$(1.28) \quad sa + (1-s)b = \sum_{i=1}^n st_i x_i + \sum_{i=n+1}^{n+p} (1-s)t_i x_i$$

and

$$(1.29) \quad \sum_{i=1}^n st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s \sum_{i=1}^n t_i + (1-s) \sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets  $S$ , this reads as follows,

$$(1.30) \quad s \text{co}(S) + (1-s) \text{co}(S) \subseteq \text{co}(S);$$

which was our first goal. We now aim at the equivalence (i)  $\Rightarrow \cdots \Rightarrow$  (iv)  $\Rightarrow$  (i): An easy proof by induction makes the implication (i)  $\Rightarrow$  (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii)  $\Rightarrow$  (iv) derives from the definition of the convex hull. We now close the chain with (iv)  $\Rightarrow$  (i), by remarking that  $S$  is convex whether  $S = \text{co}(S)$ . The lemma being proved, let us establish the second part.

To do so, we start from the convexity of  $\text{co}(A)$  then set  $F = \{\text{co}(A)\}$ . We may enrich  $F$  as follows,

$$(1.31) \quad B \in F \Rightarrow B \text{ is convex and contains } A.$$

Note that our initial predicate “[ $F$  only encompasses] *all convex sets that contain  $A$* ”, is now the special case

$$(1.32) \quad B \in F \Leftrightarrow B \text{ is convex and contains } A.$$

In any case, the key ingredient is that  $\text{co}(A) \in F$  implies

$$(1.33) \quad \text{co}(A) \supseteq \bigcap_{B \in F} B.$$

Conversely, the next formula

$$(1.34) \quad \text{co}(A) \subseteq \text{co}(B) \stackrel{(i) \Rightarrow (iv)}{=} B \quad (B \in F)$$

is valid and implies

$$(1.35) \quad \text{co}(A) \subseteq \bigcap_{B \in F} B.$$

So ends the proof □

### 3 Exercise 3. Other basic results

Let  $X$  be a topological vector space. All sets mentioned below are understood to be subsets of  $X$ . Prove the following statements:

- (a) The convex hull of every open set is open.
- (b) If  $X$  is locally convex then the convex hull of every bounded set is bounded.
- (c) If  $A$  and  $B$  are bounded, so is  $A+B$ .
- (d) If  $A$  and  $B$  are compact, so is  $A+B$ .
- (e) If  $A$  is compact and  $B$  is closed, then  $A+B$  is closed.
- (f) The sum of two closed sets may fail to be closed.

PROOF. (a) Pick an open set  $A$  then let the variables  $V_i$  ( $i = 1, 2, \dots$ ) run through all open subsets of  $A$  so that

$$(1.36) \quad \text{co}(A) \subseteq \bigcup_{t_i} (t_1 V_1 + \dots + t_i V_i + \dots) \subseteq \text{co}(A)$$

given all convex combinations  $t_1 V_1 + \dots + t_i V_i + \dots$ . We know from [1.7] of [4] that those sums are open; which achieves the proof.

- (b) Provided a bounded set  $E$ , pick  $V$  a neighbourhood of 0: By (b) of Section 1.14 in [14] of [4],  $V$  contains a convex neighbourhood of 0, say  $W$ . There so exists a positive scalar  $s$  such that

$$(1.37) \quad E \subseteq tW \subseteq tV \quad (t > s);$$

which yields

$$(1.38) \quad \text{co}(E) \subseteq \text{co}(tW) = t \text{co}(W) = tW \subseteq tV.$$

So ends the proof.

- (c) At fixed  $V$ , neighbourhood of the origin, we combine the continuity of  $+$  with [1.14] of [4] to conclude that there exists  $U$  a balanced neighborhood of the origin such that

$$(1.39) \quad U + U \subseteq V.$$

Moreover, by the very definition of boundedness,  $A \subseteq rU$  for some positive scalar  $r$ . Similarly,  $B \subseteq sU$  for some positive  $s$ . Finally,

$$(1.40) \quad A + B \subseteq rU + sU \subseteq tU + tU \subseteq tV \quad (t > r, s),$$

since  $U$  is balanced. So ends the proof.

- (d) First,  $A$  and  $B$  are compact: So is  $A \times B$ . Next,  $+$  maps continuously  $A \times B$  onto  $A + B$ . In conclusion,  $A + B$  is compact.
- (e) From now on, we assume that neither  $A$  nor  $B$  is empty, since otherwise the result is trivial. Now pick  $c \in X$  outside  $A + B$ : The result will be established by showing that  $c$  is not in the closure of  $A + B$ .

To do so, we let the variable  $a$  range over  $A$ : Every set  $a + B$  is closed as well, see [1.7] of [4]. Trivially,  $a + B \neq c$ : By Section [1.10] of [4], there so exists  $V = V(a)$  a neighborhood of the origin such that

$$(1.41) \quad (a + B + V) \cap (c + V) = \emptyset.$$

Moreover, there are finitely many  $a + V$ , say  $a_1 + V_1, a_2 + V_2, \dots$ , whose union  $U$  contains the compact set  $A$ . Therefore,

$$(1.42) \quad A + B \subseteq U + B.$$

Now define

$$(1.43) \quad W \triangleq V_1 \cap V_2 \cap \cdots,$$

so that

$$(1.44) \quad (a_i + B + V_i) \cap (c + W) \stackrel{(1.41)}{=} \emptyset \quad (i = 1, 2, \dots).$$

As a conclusion,  $c$  is not in the closure of  $U + B$ . Finally, (1.42) asserts that  $c$  is not in  $\overline{A + B}$  either; which achieves the proof.

**Corollary:** If  $B$  is the closure of a set  $S$ , then

$$(1.45) \quad A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B$$

by [(b) of 1.13] of [4] (since  $A$  is closed; see Section 1.12, from the same source). The special case  $A = \{x\}$ ,  $B = X$  will occur in the proof of Exercise 15 in chapter 2.

- (f) The last proof will consist in exhibiting a counterexample. To do so, let  $f$  be any continuous mapping of the real line such that

- (i)  $f(x) + f(-x) \neq 0 \quad (x \in \mathbf{R})$ ;
- (ii)  $f$  vanishes at infinity.

For instance, we may combine (ii) with  $f$  even and  $f > 0$  by setting  $f(x) = 2^{-|x|}$ ,  $f(x) = e^{-x^2}$ ,  $f(x) = 1/(1 + |x|)$ , ..., and so on.

As a continuous function,  $f$  has closed graph  $G$ , see [2.14] of [4]. Moreover, (i) implies that the origin  $(0, 0) \neq (x - x, f(x) + f(-x))$  is not in  $G + G$ . On the other hand,

$$(1.46) \quad \{(0, f(n) + f(-n)) : n = 1, 2, \dots\} \subseteq G + G.$$

Now the key ingredient is that

$$(1.47) \quad (0, f(n) + f(-n)) \xrightarrow[n \rightarrow \infty]{(ii)} (0, 0).$$

We have so constructed a sequence in  $G + G$  that converges outside  $G + G$ . So ends the proof. □



#### 4 Exercise 4. A nonempty set whose interior is not

Let be  $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$ . Show that  $B$  is balanced but that its interior is not.

PROOF. It is obvious that the nonempty set  $B$  contains the origin  $(0, 0)$ . Additionally, its interior  $B^\circ$  is nonempty as well. Indeed, the following set

$$(1.48) \quad \{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2| < 1/2\} \subseteq B$$

is a neighborhood of  $(1, 2) \in B$ . Moreover,  $B$  is balanced, since

$$(1.49) \quad |\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \quad (|\alpha| \leq 1)$$

for all  $(z_1, z_2)$  in  $B$ . Nevertheless, the nonempty set  $B^\circ$  is not balanced, what we now establish by showing that  $(0, 0) \notin B^\circ$ . To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$(1.50) \quad U \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2| < r\} \subseteq B$$

for some positive  $r$ . Clearly,  $U$  contains  $(r/2, 0)$ , and that special case  $(r/2, 0) \in B$  now contradicts the definition of  $B$ . So ends the proof.  $\square$

## 5 Exercise 5. A first restatement of boundedness

Consider the definition of “bounded set” given in Section 1.6. Would the content of this definition be altered if it merely required that to every neighbourhood  $V$  of  $0$  corresponds some  $t > 0$  such that  $E \subseteq tV$ ?

PROOF. The answer is: No. To prove it, start from (a) of Section 1.14:  $V$  contains  $W$ , a balanced neighbourhood of  $0$ . Assume that  $E$  is bounded in this weaker sense, *i.e.*, there exists a positive  $t$  that satisfies

$$(1.51) \quad E \subseteq tW.$$

Thus,

$$(1.52) \quad E \subseteq tW \subseteq sW \subseteq sV \quad (s > t),$$

since  $W$  is balanced. We so reach the definition given in Section 1.6: The two ones are equivalent.  $\square$

## 6 Exercise 6. A second restatement of boundedness

*Prove that a set  $E$  in a topological vector space is bounded if and only if every countable subset of  $E$  is bounded.*

PROOF. It is clear that every subset of a bounded set is bounded. Conversely, assume that  $E$  is not bounded then pick  $V$  a neighbourhood of the origin: No counting number  $n = 1, 2, \dots$  satisfies  $E \subseteq nV$  (see Exercise 1 in Chapter 1). In other words, there exists a sequence  $\{x_1, \dots, x_n, \dots\} \subseteq E$  such that

$$(1.53) \quad x_n \notin nV.$$

As a consequence,  $x_n/n$  fails to converge to 0 as  $n$  tends to  $\infty$ . In contrast,  $1/n$  succeeds. It then follows from Section 1.30 that  $\{x_1, \dots, x_n, \dots\}$  is not bounded. So ends the proof.  $\square$

## 7 Exercise 7. Metrizable and number theory

Let  $X$  be the vector space of all complex functions on the unit interval  $[0, 1]$ , topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology.

Show that there is a sequence  $\{f_n\}$  in  $X$  such that (a)  $\{f_n\}$  converges to 0 as  $n \rightarrow \infty$ , but (b) if  $\{\gamma_n\}$  is any sequence of scalars such that  $\gamma_n \rightarrow \infty$  then  $\{\gamma_n f_n\}$  does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as  $[0, 1]$ .) This shows that metrizable cannot be omitted in (b) of Theorem 1.28.

### 1 Justifying the terminology

PROOF. The family of seminorms  $p_x$  is separating: The collection  $\mathcal{B}$  of all finite intersections of the sets

$$(1.54) \quad V(x, k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k = 1, 2, 3, \dots)$$

is therefore a local base for a topology  $\tau$  on  $X$ , see Theorem [1.37] of [4]. Hence

$$(1.55) \quad \sum_{n=1}^{\infty} [g_n \notin \cap_{i=1}^m U_i] \leq \sum_{n=1}^{\infty} \sum_{i=1}^m [g_n \notin U_i] = \sum_{i=1}^m \sum_{n=1}^{\infty} [g_n \notin U_i],$$

see [I.iii] for Iverson bracket notation. Now assume that  $\{f_n\}$   $\tau$ -converges to some  $f$ . By definition,

$$(1.56) \quad \sum_{n=1}^{\infty} [f_n - f \notin W] < \infty \quad (W \in \mathcal{B}).$$

The special case  $W = V(x, k)$  implies that, for a fixed  $k$ ,  $|f_n(x) - f(x)| < 2^{-k}$  for all but finitely many  $n$ . In other words,  $\{f_n(x)\}$  converges to  $f(x)$ . Conversely, assume that  $\{f_n\}$  does not  $\tau$ -converge. This implies that, for every  $f \in X$ , there exist finitely many  $V(x_1, k_1), \dots, V(x_m, k_m)$  such that

$$(1.57) \quad \sum_{n=1}^{\infty} [f_n - f \notin \cap_{i=1}^m V(x_i, k_i)] = \infty.$$

Substituting  $f_n - f$  for  $f_n$  in (1.55), with  $U_i = V(x_i, k_i)$ , yields

$$(1.58) \quad \sum_{n=1}^{\infty} [f_n - f \notin \cap_{i=1}^m V(x_i, k_i)] \leq \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n - f \notin V(x_i, k_i)] = \infty.$$

It is now obvious that

$$(1.59) \quad \sum_{n=1}^{\infty} [f_n - f \notin V(x_i, k_i)] = \infty$$

for some  $i$ , which shows that  $\{f_n(x_i)\}$  does not converge to  $f(x_i)$ . Thus,  $\tau$ -convergence coincides with pointwise convergence on  $X$ .  $\square$

### 2 Proof (with the given hint)

We now prove the second part by constructing a specific sequence  $\{f_n\}$  that satisfies both (a) and (b). Indeed, the hint suggests that there exists a bijection

$$(1.60) \quad \begin{aligned} \phi : \{\theta_n : \theta_n \xrightarrow{n \rightarrow \infty} 0\} &\rightarrow [0, 1] \\ (\theta_1, \dots, \theta_n, \dots) &\mapsto x. \end{aligned}$$

PROOF. We set

$$(1.61) \quad f_n(x) \triangleq \theta_n \quad (x = \phi(\theta_1, \dots, \theta_n, \dots))$$

so that  $\{f_n\}$  tends pointwise to 0. Note that, with this construction, the special case

$$(1.62) \quad x_\gamma \triangleq \phi\left(1/\sqrt{1+|\gamma_1|}, \dots, 1/\sqrt{1+|\gamma_n|}, \dots\right)$$

outputs

$$(1.63) \quad \gamma_n f_n(x_\gamma) = \gamma_n / \sqrt{1+|\gamma_n|} \xrightarrow{n \rightarrow \infty} \infty,$$

given a sequence  $\{\gamma_n\}$  diverging to  $\infty$ . This proves (b), since  $\{\gamma_n f_n(x_\gamma)\}$  diverges.  $\square$

### 3 Proof with binary expansions (no hint)

We rely on the following assertion: Every irrational number has a binary expansion that is not eventually periodic.

PROOF. More precisely, there exists a bijective sum

$$(1.64) \quad \sigma : \{\beta \in \{0, 1\}^{\mathbb{N}^+} : \beta \text{ is not eventually periodic}\} \rightarrow [0, 1] \setminus \mathbf{Q}$$

$$(\beta_1, \dots, \beta_n, \dots) \mapsto \sum_{k=1}^{\infty} \beta_k 2^{-k}.$$

A suitable  $\{f_n\}$  can be defined as follows:

$$(1.65) \quad f_n(x) \triangleq \begin{cases} 2^{-(\beta_1 + \dots + \beta_n)} & (x = \sigma(\beta_1, \dots, \beta_n, \dots) \notin \mathbf{Q}) \\ 0 & (x \in \mathbf{Q}). \end{cases}$$

Indeed, we note that every bit stream  $\sigma^{-1}(x)$  has infinitely many 1's, which implies that  $f_n(x) \xrightarrow{n \rightarrow \infty} 0$ . Next, pick an arbitrary  $\gamma_n \rightarrow \infty$ . Thus, for any positive integer  $k$ ,  $\gamma_n > 4^k$  for all sufficiently large  $n$ , say  $n > N_k$ . We select  $n_k > N_k$  so large that

$$(1.66) \quad n_{k+1} - n_k > k.$$

The crucial point is that the sequence  $1_{\{n_1, n_2, \dots\}}$  is not eventually periodic. Moreover, the particular choice

$$(1.67) \quad \beta^\gamma \triangleq 1_{\{n_1, n_2, \dots\}}$$

implies

$$(1.68) \quad \beta_1^\gamma + \dots + \beta_{n_1}^\gamma + \dots + \beta_{n_k}^\gamma = k.$$

Finally, (1.65) and (1.68) together yield

$$(1.69) \quad \gamma_{n_k} f_{n_k}(\sigma(\beta^\gamma)) = \gamma_{n_k} / 2^k > 2^k \xrightarrow{k \rightarrow \infty} \infty.$$

In conclusion, every sequence of scalars  $\gamma_n$  such that  $\gamma_n \rightarrow \infty$  contains a subsequence  $\{\gamma_{n_k}\}$  that causes  $\{\gamma_{n_k} f_{n_k}\}$  to diverge. This is (b).  $\square$

## 9 Exercise 9. Quotient map

Suppose

- (a)  $X$  and  $Y$  are topological vector spaces,
- (b)  $\Lambda : X \rightarrow Y$  is linear.
- (c)  $N$  is a closed subspace of  $X$ ,
- (d)  $\pi : X \rightarrow X/N$  is the quotient map, and
- (e)  $\Lambda x = 0$  for every  $x \in N$ .

Prove that there is a unique  $f : X/N \rightarrow Y$  which satisfies  $\Lambda = f \circ \pi$ , that is,  $\Lambda x = f(\pi(x))$  for all  $x \in X$ . Prove that  $f$  is linear and that  $\Lambda$  is continuous if and only if  $f$  is continuous. Also,  $\Lambda$  is open if and only if  $f$  is open.

PROOF. Bear in mind that  $\pi$  continuously maps  $X$  onto the topological (Hausdorff) space  $X/N$ , since  $N$  is closed (see [1.41] of [4]). Moreover, the equation  $\Lambda = f \circ \pi$  has necessarily a unique solution, which is the binary relation

$$(1.70) \quad f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y.$$

To ensure that  $f$  is actually a mapping, simply remark that the linearity of  $\Lambda$  implies

$$(1.71) \quad \Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'.$$

It straightforwardly derives from (1.70) that  $f$  inherits linearity from  $\pi$  and  $\Lambda$ .

**Remark.** The special case  $N = \{\Lambda = 0\}$ , i.e.,  $\Lambda x = 0$  iff  $x \in N$  (cf. (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$(1.72) \quad f(\pi x) = 0 \stackrel{1.70}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that  $f$  is also one-to-one.

Now assume  $f$  to be continuous. Then so is  $\Lambda = f \circ \pi$ , by [1.41 (a)] of [4]. Conversely, if  $\Lambda$  is continuous, then for each neighborhood  $V$  of  $0_Y$  there exists a neighborhood  $U$  of  $0_X$  such that

$$(1.73) \quad \Lambda(U) = f(\pi(U)) \subseteq V.$$

Since  $\pi$  is open ([1.41 (a)] of [4]),  $\pi(U)$  is a neighborhood of  $N = 0_{X/N}$ : This is sufficient to establish that the linear mapping  $f$  is continuous. If  $f$  is open, so is  $\Lambda = f \circ \pi$ , by [1.41 (a)] of [4]. To prove the converse, remark that every neighborhood  $W$  of  $0_{X/N}$  satisfies

$$(1.74) \quad W = \pi(V)$$

for some neighborhood  $V$  of  $0_X$ . So,

$$(1.75) \quad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if  $\Lambda$  is open, then  $f(W)$  is a neighborhood of  $0_Y$ . So ends the proof.  $\square$

## 10 Exercise 10. An open mapping theorem

Suppose that  $X$  and  $Y$  are topological vector spaces,  $\dim Y < \infty$ ,  $\Lambda : X \rightarrow Y$  is linear, and  $\Lambda(X) = Y$ .

(a) Prove that  $\Lambda$  is an open mapping.

(b) Assume, in addition, that the null space of  $\Lambda$  is closed, and prove that  $\Lambda$  is continuous.

PROOF. Discard the trivial case  $\Lambda = 0$  and assume that  $\dim Y > 0$ . From now on,  $Y$  has a basis  $B = \{e, e', \dots\}$  and  $W$  is an arbitrary neighborhood of 0. Since addition is continuous, there exists a balanced open  $V \subseteq Y$  such that

$$(1.76) \quad \sum_e V \subseteq W.$$

Moreover, the surjectivity of  $\Lambda$  allows us to pick  $x_e$  with  $\Lambda x_e = e$ . This implies that every  $y = \sum_e y_e e$  ( $y_e \in \mathbf{C}$ ) can be expressed as

$$(1.77) \quad y = \sum_e y_e \Lambda x_e.$$

Note that  $\{x_e : e \in B\}$  is bounded (as a finite set). Hence

$$(1.78) \quad \{x_e : e \in B\} \subseteq sV$$

for some  $s > 0$ . Combining this with (1.77) yields

$$(1.79) \quad y \in \sum_e s y_e \Lambda(V).$$

Using (1.76) and the balancedness of  $\Lambda(V)$ , we conclude that

$$(1.80) \quad y \in \sum_e \Lambda(V) \subseteq \Lambda(W) \quad (\|y\|_\infty < 1/s).$$

This proves (a) in the case where  $Y = \mathbf{C}^n$  and  $B$  is the standard basis. The general case now follows from [iii].

To prove (b), assume that the null space  $\{\Lambda = 0\}$  is closed and let  $f, \pi$  be as in Exercise 1.9,  $\{\Lambda = 0\}$  playing the role of  $N$ . Since  $\Lambda$  is onto, the first isomorphism theorem (see Exercise 1.9) asserts that  $f$  is an isomorphism of  $X/N$  onto  $Y$ . We now conclude with the help of [iii] that  $f$  is a homeomorphism of  $X/N$  onto  $Y$ . We have thus established that  $f$  is continuous: So is  $\Lambda = f \circ \pi$ .  $\square$

## 12 Exercise 12. Topology stays, completeness leaves

Suppose  $d_1(x, y) = |x - y|$ ,  $d_2(x, y) = |\phi(x) - \phi(y)|$ , where  $\phi(x) = x/(1 + |x|)$ . Prove that  $d_1$  and  $d_2$  are metrics on  $\mathbf{R}$  which induce the same topology, although  $d_1$  is complete and  $d_2$  is not.

PROOF. First, each  $d_i$  ( $i = 1, 2$ ) induces a topology  $\tau_i$  whose open balls are all

$$(1.81) \quad B_i(a, r) \triangleq \{x \in \mathbf{R} : d_i(a, x) < r\} \quad (a \in \mathbf{R}, r > 0).$$

Next, remark that the monotonically increasing mapping  $\phi : \mathbf{R} \rightarrow ]-1, 1[$  is odd and that

$$(1.82) \quad \phi(x) \xrightarrow{x \rightarrow \infty} 1.$$

$\phi$  is therefore a  $\tau_1$ -homeomorphism of  $\mathbf{R}$  onto  $]-1, 1[$ . A first consequence is that, at fixed  $a \in \mathbf{R}$ , given any positive scalar  $\varepsilon$ , the  $\tau_1$ -continuity of  $\phi$  supplies an open ball  $B_1(a, \eta)$  on which  $|\phi(a) - \phi| < \varepsilon$ . In terms of balls  $B_i$ , this reads as follows,

$$(1.83) \quad B_1(a, \eta) \subseteq B_2(a, \varepsilon).$$

The second consequence is that the  $\tau_1$ -continuity of  $\phi^{-1}$  yields similar inclusions

$$(1.84) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta')$$

provided  $\eta' > 0$ . At arbitrary  $\varepsilon$ , the special case  $\eta' = \eta$  is the concatenation

$$(1.85) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta) \subseteq B_2(a, \varepsilon);$$

which proves that  $\tau_1 = \tau_2$ . Finally, all inequalities  $n < i < j$  over  $\mathbf{N}$  together yield

$$(1.86) \quad d_2(i, j) = |\phi(i) - \phi(j)| \xrightarrow{n \rightarrow \infty} 0.$$

The sequence  $n = 0, 1, 2, \dots$  is therefore  $\tau_2$ -Cauchy. We will nevertheless establish that it  $\tau_2$ -diverges. To do so, we start by offering the  $\tau_2$ -convergence to some  $\lambda$ : The triangle inequality immediately dismisses that assumption, as follows,

$$(1.87) \quad d_2(0, \lambda) \geq d_2(0, n) - d_2(\lambda, n) = \phi(n) - d_2(\lambda, n) \xrightarrow{n \rightarrow \infty} 1.$$

We then conclude that  $d_2$  fails to be complete. □



## 14 Exercise 14. $\mathcal{D}_K$ equipped with other seminorms

Put  $K = [0, 1]$  and define  $\mathcal{D}_K$  as in Section 1.46. Show that the following three families of seminorms (where  $n = 0, 1, 2, \dots$ ) define the same topology on  $\mathcal{D}_K$ . If  $D = d/dx$ :

$$(a) \|D^n f\|_\infty = \sup \{ |D^n f(x)| : 0 < x < 1 \}$$

$$(b) \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$(c) \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

PROOF. Let us equip  $\mathcal{D}_K$  with the inner product  $\langle f|g \rangle = \int_0^1 f \bar{g}$  so that  $\langle f|f \rangle = \|f\|_2^2$ . The following

$$(1.88) \quad \int_0^1 |D^n f|^2 \leq \|1\|_2 \|D^n f\|_2$$

is then a Cauchy-Schwarz inequality, see Theorem [12.2] of [4]FA. We so obtain

$$(1.89) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

since  $K$  has length 1. Obviously, the support of  $D^n f$  lies in  $K$ , hence the below equality

$$(1.90) \quad |D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1.$$

Take the supremum over all  $|D^n f(x)|$ : Combining (1.89) with (1.90) now reads as follows,

$$(1.91) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots < \infty.$$

Finally, put

$$(1.92) \quad V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\},$$

$$(1.93) \quad \mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\},$$

so that (1.91) is mirrored by neighborhood inclusions, provided  $i = 1, 2, \infty$ :

$$(1.94) \quad V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \dots$$

Their subchains  $V_n^{(i)} \supseteq V_{n+1}^{(i)}$  turn  $\mathcal{B}^{(i)}$  into a local base of a topology  $\tau_i$ . The whole chain (1.94) then forces

$$(1.95) \quad \tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1;$$

which achieves the proof. □

## 16 Exercise 16. Uniqueness of topology for test functions

*Prove that the topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Do the same for  $C^\infty(\Omega)$  (Section 1.46).*

**Comment** This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms  $p_n$ , then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [4] The proof consists in combining trivial consequences of the local base definition with a well-known result (for instance, see [2.6] of [3]) about intersection of nonempty compact sets.

**Lemma 1** *Let  $X$  be a topological space with a countable local base  $\{V_n : n = 1, 2, 3, \dots\}$ . If  $\tilde{V}_n = V_1 \cap \dots \cap V_n$ , then every subsequence  $\{\tilde{V}_{\varrho(n)}\}$  is a decreasing (, i.e.,  $\tilde{V}_{\varrho(n)} \supseteq \tilde{V}_{\varrho(n+1)}$ ) local base of  $X$ .*

PROOF. The decreasing property is trivial. Now remark that  $V_n \supseteq \tilde{V}_n$ : This shows that  $\{\tilde{V}_n\}$  is a local base of  $X$ . Then so is  $\{\tilde{V}_{\varrho(n)}\}$ , since  $\tilde{V}_n \supseteq \tilde{V}_{\varrho(n)}$ .  $\square$

The following special case  $V_n = \tilde{V}_n$  is one of the key ingredients:

**Corollary 1 (special case  $V_n = \tilde{V}_n$ )** *Under the same notations of Lemma 1, if  $\{V_n\}$  is a decreasing local base, then so is  $\{V_{\varrho(n)}\}$ .*

**Corollary 2** *If  $\{Q_n\}$  is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence  $\{Q_{\varrho(n)}\}$  also satisfies theses conditions. Furthermore, if  $\tau_Q$  is the  $C(\Omega)$ 's (respectively  $C^\infty(\Omega)$ 's) topology of the seminorms  $p_n$ , as defined in section 1.44 (respectively 1.46), then the seminorms  $p_{\varrho(n)}$  define the same topology  $\tau_Q$ .*

PROOF. Let  $X$  be  $C(\Omega)$  topologized by the seminorms  $p_n$  (the case  $X = C^\infty(\Omega)$  is proved the same way). If  $V_n = \{p_n < 1/n\}$ , then  $\{V_n\}$  is a decreasing local base of  $X$ . Moreover,

$$(1.96) \quad Q_{\varrho(n)} \subseteq \overset{\circ}{Q}_{\varrho(n)+1} \subseteq Q_{\varrho(n)+1} \subseteq Q_{\varrho(n+1)}.$$

Thus,

$$(1.97) \quad Q_{\varrho(n)} \subseteq \overset{\circ}{Q}_{\varrho(n+1)}.$$

In other words,  $Q_{\varrho(n)}$  satisfies the conditions specified in section 1.44.  $\{p_{\varrho(n)}\}$  then defines a topology  $\tau_{Q_\varrho}$  for which  $\{V_{\varrho(n)}\}$  is a local base. So,  $\tau_{Q_\varrho} \subseteq \tau_Q$ . Conversely, the above corollary asserts that  $\{V_{\varrho(n)}\}$  is a local base of  $\tau_Q$ , which yields  $\tau_Q \subseteq \tau_{Q_\varrho}$ .  $\square$

**Lemma 2** *If a sequence of compact sets  $\{Q_n\}$  satisfies the conditions specified in section 1.44, then every compact set  $K$  lies in almost all  $Q_n^\circ$ , i.e., there exists  $m$  such that*

$$(1.98) \quad K \subseteq \overset{\circ}{Q}_m \subseteq \overset{\circ}{Q}_{m+1} \subseteq \overset{\circ}{Q}_{m+2} \subseteq \dots$$

PROOF. The following definition

$$(1.99) \quad C_n \triangleq K \setminus \overset{\circ}{Q}_n$$

shapes  $\{C_n\}$  as a decreasing sequence of compact<sup>1</sup> sets. We now suppose (to reach a contradiction) that no  $C_n$  is empty and so conclude<sup>2</sup> that the  $C_n$ 's intersection contains a point that is not in any  $Q_n^\circ$ . On the other hand, the conditions specified in [1.44] force the  $Q_n^\circ$ 's collection to be an open cover. This contradiction reveals that  $C_m = \emptyset$ , , i.e.,  $K \subseteq Q_m^\circ$ , for some  $m$ . Finally,

$$(1.100) \quad K \subseteq \overset{\circ}{Q}_m \subseteq Q_m \subseteq \overset{\circ}{Q}_{m+1} \subseteq Q_{m+1} \subseteq \overset{\circ}{Q}_{m+2} \subseteq \dots$$

$\square$

We are now in a fair position to establish the following:

<sup>1</sup>See [(b) of 2.5] of [3].

<sup>2</sup>In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of [2.6] of [3].

**Theorem** *The topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of  $C^\infty(\Omega)$ , as long as this sequence satisfies the conditions specified in section 1.44.*

PROOF. With the second corollary's notations,  $\tau_K = \tau_{K_\lambda}$ , for every subsequence  $\{K_{\lambda(n)}\}$ . Similarly, let  $\{L_n\}$  be another sequence of compact subsets of  $\Omega$  that satisfies the condition specified in [1.44], so that  $\tau_L = \tau_{L_\kappa}$  for every subsequence  $\{L_{\kappa(n)}\}$ . Now apply the above Lemma 2 with  $K_i$  ( $i = 1, 2, 3, \dots$ ) and so conclude that  $K_i \subseteq L_{m_i}^\circ \subseteq L_{m_i+1}^\circ \subseteq \dots$  for some  $m_i$ . In particular, the special case  $\kappa_i = m_i + i$  is

$$(1.101) \quad K_i \subseteq \overset{\circ}{L}_{\kappa_i}.$$

Let us reiterate the above proof with  $K_n$  and  $L_n$  in exchanged roles then similarly find a subsequence  $\{\lambda_j : j = 1, 2, 3, \dots\}$  such that

$$(1.102) \quad L_j \subseteq \overset{\circ}{K}_{\lambda_j}$$

Combine (1.101) with (1.102) and so obtain

$$(1.103) \quad K_1 \subseteq \overset{\circ}{L}_{\kappa_1} \subseteq L_{\kappa_1} \subseteq \overset{\circ}{K}_{\lambda_{\kappa_1}} \subseteq K_{\lambda_{\kappa_1}} \subseteq \overset{\circ}{L}_{\kappa_{\lambda_{\kappa_1}}} \subseteq \dots,$$

which means that the sequence  $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$  satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$(1.104) \quad \tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L.$$

So ends the proof □

## 17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that  $f \mapsto D^\alpha f$  is a continuous mapping of  $C^\infty(\Omega)$  into  $C^\infty(\Omega)$  and also of  $\mathcal{D}_K$  into  $\mathcal{D}_K$ , for every multi-index  $\alpha$ .

PROOF. In both cases,  $D^\alpha$  is a linear mapping. It is then sufficient to establish continuity at the origin. We begin with the  $C^\infty(\Omega)$  case.

Let  $U$  be an arbitrary neighborhood of the origin. There so exists  $N$  such that  $U$  contains

$$(1.105) \quad V_N = \{ \phi \in C^\infty(\Omega) : \max \{ |D^\beta \phi(x)| : |\beta| \leq N, x \in K_N \} < 1/N \}.$$

Now pick  $g$  in  $V_{N+|\alpha|}$  so that

$$(1.106) \quad \max \{ |D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N \} < \frac{1}{N + |\alpha|}.$$

(the fact that  $K_N \subseteq K_{N+|\alpha|}$  was tacitly used). The special case  $\gamma = \beta + \alpha$  yields

$$(1.107) \quad \max \{ |D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N \} < \frac{1}{N}.$$

We have just proved that

$$(1.108) \quad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad , \text{ i.e., } \quad D^\alpha \left( V_{N+|\alpha|} \right) \subseteq V_N,$$

which establishes the continuity of  $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$ .

To prove the continuity of the restriction  $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$ , we first remark that the collection of the  $V_N \cap \mathcal{D}_K$  is a local base of the subspace topology of  $\mathcal{D}_K$ .  $V_{N+|\alpha|} \cap \mathcal{D}_K$  is then a neighborhood of 0 in this topology. Furthermore,

$$(1.109) \quad D^\alpha|_{\mathcal{D}_K} \left( V_{N+|\alpha|} \cap \mathcal{D}_K \right) = D^\alpha \left( V_{N+|\alpha|} \cap \mathcal{D}_K \right)$$

$$(1.110) \quad \subseteq D^\alpha \left( V_{N+|\alpha|} \right) \cap D^\alpha(\mathcal{D}_K)$$

$$(1.111) \quad \subseteq V_N \cap \mathcal{D}_K \quad (\text{see (1.108)})$$

So ends the proof. □

## Chapter 2

# Completeness

### 3 Exercise 3. A sequence of Radon measures that does not converge vaguely

Put  $K = [-1, 1]$ ; define  $\mathcal{D}_K$  as in Section 1.46 (with  $\mathbf{R}$  in place of  $\mathbf{R}^n$ ). Suppose  $\{f_n\}$  is a sequence of Lebesgue integrable functions such that  $\Lambda\phi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\phi(t)dt$  exists for every  $\phi \in \mathcal{D}_K$ . Show that  $\Lambda$  is a continuous functional on  $\mathcal{D}_K$ . Show that there is a positive integer  $p$  and a number  $M < \infty$  such that

$$\left| \int_{-1}^1 f_n(t)\phi(t) dt \right| \leq M \|D^p \phi\|_\infty$$

for all  $n$ . For example, if  $f_n(t) = n^3 t$  on  $[-1/n, 1/n]$  and 0 elsewhere, show that this can be done with  $p = 1$ . Construct an example where it can be done with  $p = 2$  but not with  $p = 1$ .

PROOF. Equipped with the supremum norm,  $\mathcal{C}_K = C(\mathbf{R}) \cap \{\phi : \text{supp } \phi \subseteq K\}$  realizes the Banach space  $C_c(K)$  in  $C(\mathbf{R})$ . Each density  $f_n$  is then identified with the Radon measure

$$(2.1) \quad \begin{aligned} \Lambda_n : \mathcal{C}_K &\rightarrow \mathbf{C} \\ \phi &\mapsto \int_{-1}^1 f_n(t)\phi(t) dt. \end{aligned}$$

Every  $\Lambda_n$  is continuous since  $\|\Lambda_n\| = \|f_n\|_1$  is finite; cf. [6.19] of [4]. Note that the dual space  $\mathcal{C}_K^*$  is a Banach space as well, by [4.1] of [4]. Here, we consider only pointwise convergence, which is weaker than norm convergence. We prove that the assumed pointwise convergence

$$(2.2) \quad \Lambda_n \phi \xrightarrow{n \rightarrow \infty} \Lambda \phi \quad (\phi \in \mathcal{D}_K)$$

implies the continuity of  $\Lambda$  in the topology of  $\mathcal{D}_K$ . We also construct a specific sequence  $\{\Lambda_n\}$  whose pointwise limit  $\Lambda$  is not bounded with respect to the supremum norm. By contraposition, Theorem [2.8] of [4] implies that pointwise convergence on  $\mathcal{D}_K$  does not extend to  $\mathcal{C}_K$ . This conclusion also derives from the following bounds:

$$(2.3) \quad |\Lambda_n \phi| \leq M \|\phi'\|_\infty,$$

$$(2.4) \quad |\Lambda_n \phi'| \leq M \|\phi''\|_\infty.$$

Combined with the impossibility of boundedness at order  $p = 0$ , cf. (2.3), the contraposition of Theorem [2.6] of [4] implies that  $\{\Lambda_n\}$  does not converge pointwise on  $\mathcal{C}_K$ . In Radon measure theory, pointwise convergence is known as *vague convergence*. In the next paragraph,  $\phi$  is restricted to  $\mathcal{D}_K$ .

**Continuity of  $\Lambda$ .** We equip  $\mathcal{D}_K$  with derivative norms  $\|N\|\phi \triangleq \|\phi\|_\infty + \|D^1 \phi\|_\infty + \cdots + \|D^N \phi\|_\infty$ . The induced topology  $\tau_K$  of  $\mathcal{D}_K$  is the weakest topology that makes all norms  $\|N\|\cdot$  continuous, cf. [1.46, 6.2] of [4] and Exercise [1.16]. Equivalently, the collection of all convex balanced sets

$$(2.5) \quad V_N \triangleq \left\{ \|N\| < 1/N \right\}$$

forms a local base of  $\tau_K$ , cf. [6.2] of [4]. Note that  $\|N\|\phi < 1$  implies  $\|\phi\|_\infty < 1$ : Every  $\Lambda_n$  is then  $\tau_K$ -continuous by [1.18] of [4], since  $|\Lambda_n|$  is bounded by  $\|\Lambda_n\|$  on  $V_1$ . In summary:

- (a)  $\mathcal{D}_K$ , equipped with the topology  $\tau_K$ , is a Fréchet space; see [1.46] of [4].
- (b) Every functional  $\Lambda_n$  is  $\tau_K$ -continuous.
- (c)  $\Lambda_n \phi \rightarrow \Lambda \phi$  pointwise on  $\mathcal{D}_K$  (our premise).

By [2.6, 2.8] of [4], the equicontinuous sequence  $\{\Lambda_n\}$  converges pointwise to a continuous  $\Lambda$ . Furthermore, the equicontinuity of  $\{\Lambda_n\}$  ensures that all  $|\Lambda_n|$  remain below 1 on a common *balanced* neighborhood  $V_p$ . So,

$$(2.6) \quad \frac{1}{p} \cdot \frac{\phi}{\|\phi\|_p + \varepsilon} \in V_p$$

for all  $\varepsilon > 0$ . This yields  $|\Lambda_n \phi| < p(\|\phi\|_p + \varepsilon)$ , which reduces to  $|\Lambda_n \phi| \leq p \|\phi\|_p$ . Applying Lemma [A.1] with  $\phi, D\phi, \dots, D^p \phi$  outputs

$$(2.7) \quad |\Lambda_n \phi| \leq p(p+1) \|D^p \phi\|_\infty.$$

This completes the first part of the proof, with some  $p$  and a positive constant  $M = M(p)$ .

**The counterexample: A sequence of  $\{\Lambda_n\}$  that does not converge pointwise on  $\mathcal{C}_K$ .** Let  $u$  be a smooth, even mapping that equals 1 on  $[-1/2, 1/2]$ , vanishes outside  $[-1, 1]$ , and satisfies  $0 \leq u \leq 1$  on  $\mathbf{R}$ . The function  $u$  belongs to the general construction in [1.46] of [4]. Alternatively,  $u$  can be the derivative of  $\phi$  from Lemma [A.2], with  $\tau = 1/2$ ,  $\omega = 2$ , and  $A = 1$ . We set

$$(2.8) \quad f_n(t) \triangleq n^3 t [-1/n \leq t \leq 1/n].$$

Under the identification  $C_c(K) \equiv \mathcal{C}_K$ ,  $\Lambda_n$  reads as the difference of two (positive) Radon measures  $\Lambda_n^+$  and  $\Lambda_n^-$ , since

$$(2.9) \quad \Lambda_n \phi = \underbrace{n^3 \int_0^{1/n} t \phi(t) dt}_{\Lambda_n^+ \phi} - \underbrace{n^3 \int_{-1/n}^0 t \phi(t) dt}_{\Lambda_n^- \phi} \quad (\phi \in \mathcal{C}_K).$$

Thus,  $\Lambda_n$  is a signed Radon measure, whose compact support  $[-1/n, 1/n]$  shrinks to  $\{0\}$  as  $n \rightarrow \infty$ . We see that  $\|\Lambda_n\| \leq \|\Lambda_n^+\| + \|\Lambda_n^-\|$ . However, the collection of all  $\Lambda_n$  is not uniformly bounded, since

$$(2.10) \quad \|\Lambda_n\| = \|f_n\|_1 = n = \|\Lambda_n^+\| + \|\Lambda_n^-\|.$$

The *logistic function*  $\sigma_\lambda : t \mapsto 1/(1 + \exp(-\lambda t))$  provides a direct proof of this. It is a standard approximation of the Heaviside step function<sup>1</sup>. Thus, Lebesgue's dominated convergence theorem implies

$$(2.11) \quad \Lambda_n(u \cdot \underbrace{(\sigma_\lambda - 1/2)}_{\text{odd}}) = 2n^3 \underbrace{\int_0^{1/n} t(\sigma_\lambda(t) - 1/2) dt}_{\text{even}} \xrightarrow{\lambda \rightarrow \infty} n \quad (n \geq 2).$$

We refer to [6.19] of [4] for a more general scope. At this step, it is already clear that there is no vague convergence for the current sequence  $\{\Lambda_n\}$ : this would, by Theorem [2.6] of [4], imply that  $\sup_n \|\Lambda_n\| < \infty$ , which would contradict (2.10). However, we investigate further to establish weaker convergence and boundedness in  $\tau_K$ . From now on, we bound  $\phi$  to  $\mathcal{D}_K$ . The mean value theorem implies that

$$(2.12) \quad \phi(1/n) - \phi(-1/n) = \frac{2}{n} \phi'(t_n)$$

for some  $-1/n < t_n < 1/n$ . Moreover, integration by parts yields

$$(2.13) \quad \Lambda_n \phi = \frac{n^3}{2} t^2 \phi(t) \Big|_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt$$

$$(2.14) \quad = \frac{n}{2} (\phi(1/n) - \phi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt$$

$$(2.15) \quad = \phi'(t_n) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt.$$

<sup>1</sup> $\sigma_\lambda$  connects machine learning with statistical mechanics; see [1]

Note that when  $\phi' = 1$  in a neighborhood of 0, *e.g.*, for  $\phi(t) = tu(t)$ , the latter equality reduces to

$$(2.16) \quad \Lambda_n \phi = 1 - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 dt = \frac{2}{3}.$$

This suggests that continuity of  $\phi'$  dictates  $\Lambda_n \phi \rightarrow \frac{2}{3} \phi'(0)$ . We establish this convergence in two steps. First,

$$(2.17) \quad \Lambda_n \phi - \frac{2}{3} \phi'(0) = \phi'(t_n) - \phi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'(t) dt + \underbrace{\phi'(0) \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 dt}_{1/3}$$

$$(2.18) \quad = \phi'(t_n) - \phi'(0) - \frac{n^3}{2} \left( \int_{-1/n}^{1/n} t^2 ((\phi'(t) - \phi'(0))) dt \right).$$

Next, taking absolute values gives

$$(2.19) \quad \left| \Lambda_n \phi - \frac{2}{3} \phi'(0) \right| \leq \left| \phi'(t_n) - \phi'(0) \right| + \frac{1}{3} \max_{[-1/n, 1/n]} \left| \phi' - \phi(0) \right| \xrightarrow{n \rightarrow \infty} 0.$$

As a result,

$$(2.20) \quad \Lambda_n \phi \xrightarrow{n \rightarrow \infty} -\frac{2}{3} \delta' \phi \quad (\phi \in \mathcal{D}_K),$$

where  $\delta' : \phi \mapsto -\phi'(0)$  is the *derivative* of the *Dirac measure*  $\delta : \phi \mapsto \phi(0)$ ; see [6.1, 6.9] of [4] and Section [A.2]. The reasoning from the previous part shows that the limit  $\Lambda = -\frac{2}{3} \delta'$  is  $\tau_K$ -continuous. As a complement, absolute values from (2.15) provide

$$(2.21) \quad \left| \Lambda_n \phi \right| \leq \left| \phi'(t_n) \right| + \frac{1}{3} \max_{[-1/n, 1/n]} \left| \phi' \right|.$$

A simpler bound is

$$(2.22) \quad \left| \Lambda_n \phi \right| \leq \frac{4}{3} \left\| \phi' \right\|_{\infty}.$$

This is a concrete instance of (2.7), with  $p = 1$  and  $M = 4/3$ . To establish it as (2.3), we need to prove that no reduction to order  $p = 0$  is possible. To do so, we first assume, to reach a contradiction, that there exists  $M$  such that

$$(2.23) \quad \left| \Lambda_n \phi \right| \leq M \left\| \phi \right\|_{\infty} \quad (\phi \in \mathcal{D}_K, n = 1, 2, 3, \dots).$$

Next, we choose

$$(2.24) \quad \phi_n \triangleq \tilde{\phi}_n u,$$

where  $\tilde{\phi}_n$  is the function  $\phi$  from Lemma [A.2], with  $\tau = 1/n = 1/\omega = 1/A$ . So,  $\left\| \phi_n \right\|_{\infty} < 2$ . Nevertheless, for all  $n \geq 4$ ,  $\Lambda_n \phi_n$  is now

$$(2.25) \quad \Lambda_n \phi_n = \frac{n}{2} \left( \underbrace{\phi_n(1/n)}_1 - \underbrace{\phi_n(-1/n)}_{-1} \right) - \underbrace{\frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \phi'_n(t) dt}_{\frac{1}{3}n} = \frac{2}{3}n.$$

Hence

$$(2.26) \quad 2M \stackrel{(2.23)}{\geq} \left| \Lambda_n \phi_n \right| \xrightarrow{n \rightarrow \infty} \infty,$$

which brings the desired contradiction. We now combine (2.20) with (2.26) to establish that

$$(2.27) \quad \left| \Lambda \phi_n \right| \geq \left| \Lambda_n \phi_n \right| - \left| \Lambda_n \phi_n - \Lambda \phi_n \right| \xrightarrow{n \rightarrow \infty} \infty.$$

Therefore,  $\Lambda$  is not bounded either. A direct way to see this is to pick  $\phi_{\omega}$  from Lemma [A.2] so that

$$(2.28) \quad \Lambda(u\phi_{\omega}) = \frac{2}{3} \omega \xrightarrow{\omega \rightarrow \infty} \infty$$

contrasts with  $\left\| u\phi_{\omega} \right\|_{\infty} = 1$ . Thus, we have exhibited a sequence of Radon measures  $\Lambda_n$  that

- (a) does not converge vaguely to any Radon measure,
- (b) but converges pointwise on  $\mathcal{D}_K$ , in the specific  $\mathcal{D}_K$ 's topology; see (2.5), (2.20), and (2.22).

We now present a second example, the *derivative*; see [6.1] of [4].

$$(2.29) \quad \begin{aligned} \Lambda'_n : \mathcal{D}_K &\rightarrow \mathbf{C} \\ \phi &\mapsto -\Lambda_n \phi'. \end{aligned}$$

We have proved that every  $\Lambda_n$  is continuous. So is the derivative operator in  $\mathcal{D}_K$  - see Exercise [1.17]. Therefore,  $\Lambda'_n$  is continuous. Now apply (2.20) with  $\phi'$  and so obtain

$$\Lambda'_n \phi \xrightarrow{n \rightarrow \infty} -\frac{2}{3} \phi''(0).$$

Furthermore, Theorem [2.8] of [4] implies that the limit  $-\frac{2}{3} \phi''(0)$  is  $\tau_K$  continuous. Additionally, it follows from (2.22) that the bound (2.4) is

$$(2.30) \quad |\Lambda'_n \phi| \leq \frac{4}{3} \|\phi''\|_\infty.$$

Indeed, we show that 2 is the smallest suitable  $p$ . First, we assume, to reach a contradiction, that

$$(2.31) \quad |\Lambda_n \phi'| \leq M \|\phi'\|_\infty \quad (\phi \in \mathcal{D}_K, n = 1, 2, 3, \dots).$$

Next, let  $\Phi_n$  be the primitive of  $\phi_n$  that vanishes at -1; see (2.24). The oddness of  $\Phi_n$  ( $\phi$  is even) implies that  $\text{supp } \Phi_n \subseteq [-1, 1]$ . So, under our assumption,

$$(2.32) \quad |\Lambda'_n \Phi_n| = |\Lambda_n \Phi'_n| \leq M \|\Phi'_n\|_\infty.$$

Equivalently,

$$(2.33) \quad |\Lambda_n \phi_n| \leq M \|\phi_n\|_\infty,$$

which has already been disproved. Finally, to reach a last contradiction, assume that there exists  $M$  attached to order  $p = 0$  so that

$$(2.34) \quad |\Lambda'_n \phi| \leq M \|\phi\|_\infty \quad (\phi \in \mathcal{D}_K, n = 1, 2, 3, \dots).$$

Lemma [A.2] implies that

$$(2.35) \quad |\Lambda'_n \phi| \leq M \|\phi'\|_\infty.$$

This contradiction concludes the proof. □



## 6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient  $\hat{f}(n)$  of a function  $f \in L^2(T)$  ( $T$  is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all  $n \in \mathbf{Z}$  (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^n \hat{f}(k).$$

Prove that  $\{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$  is a dense subspace of  $L^2(T)$  of the first category.

PROOF. Let  $f(\theta)$  stand for  $f(e^{i\theta})$  so that  $L^2(T)$  is identified with a closed subset of  $L^2([-\pi, \pi])$ , hence the inner product

$$(2.36) \quad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

$$(2.37) \quad \Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well known (and easy to prove) result is

$$(2.38) \quad (e_n, e_{n'}) = [n = n'], \text{ i.e., } \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric ( $\equiv$ ) identification  $L^2 \equiv (L^2)^*$ . So,

$$(2.39) \quad \|\Lambda_n\|^2 \stackrel{2.37}{=} \|e_{-n} + \cdots + e_n\|^2 \stackrel{2.38}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \stackrel{2.38}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

$$(2.40) \quad B \triangleq \{f \in L^2(T) : \sup \{|\Lambda_n f| : n = 1, 2, 3, \dots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem [2.5] of [4] asserts that the sequence  $\{\Lambda_n\}$  is norm-bounded; which is a desired contradiction, since

$$(2.41) \quad \|\Lambda_n\| \stackrel{2.39}{=} \sqrt{2n+1} \xrightarrow{n \rightarrow \infty} \infty.$$

We have just established that  $B$  is actually of the first category; and so is its subset  $L = \{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$ . We now prove that  $L$  is nevertheless dense in  $L^2(T)$ . To do so, we let  $P$  be  $\text{span}\{e_k : k \in \mathbf{Z}\}$ , the collection of the trigonometric polynomials  $p(\theta) = \sum \lambda_k e^{ik\theta}$ . Combining (2.37) with (2.38) shows that  $\Lambda_n(p) = \sum \lambda_k$  for almost all  $n$ . Thus,

$$(2.42) \quad P \subseteq L \subseteq L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that  $P$  is dense in  $L^2(T)$ . We then conclude, with the help of (2.42), that

$$(2.43) \quad L^2(T) = \overline{P} = \overline{L}.$$

So ends the proof □

## 9 Exercise 9. Boundedness without closedness

Suppose  $X, Y, Z$  are Banach spaces and

$$B : X \times Y \rightarrow Z$$

is bilinear and continuous. Prove that there exists  $M < \infty$  such that

$$\|B(x, y)\| \leq M\|x\|\|y\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

PROOF. The answer is: No. To prove this, we only assume that  $X, Y, Z$  are normed spaces. Since  $B$  is continuous at the origin, there exists a positive  $r$  such that

$$(2.44) \quad \|x\| + \|y\| < r \Rightarrow \|B(x, y)\| < 1.$$

Given nonzero  $x, y$ , let  $s$  range over  $]0, r[$  so that the following bound

$$(2.45) \quad \|B(x, y)\| = \frac{4\|x\|\|y\|}{s^2} \left\| B\left(\frac{s}{2\|x\|}x, \frac{s}{2\|y\|}y\right) \right\| \stackrel{(2.44)}{<} \frac{4\|x\|\|y\|}{s^2}$$

is effective. It is now obvious that

$$(2.46) \quad B(x, y) \leq \frac{4}{s^2}\|x\|\|y\| \xrightarrow{s \rightarrow r} \frac{4}{r^2}\|x\|\|y\| \quad ((x, y) \in X \times Y);$$

which achieves the proof.

As a concrete example, choose  $X = Y = Z = C_c(\mathbf{R})$ , topologized by the supremum norm.  $C_c(\mathbf{R})$  is not complete (see [5.4.4] of [5]), nevertheless the bilinear product

$$\begin{aligned} B : C_c(\mathbf{R})^2 &\rightarrow C_c(\mathbf{R}) \\ (f, g) &\mapsto f \cdot g \end{aligned}$$

is bounded (since  $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$ ), and continuous. To show this, pick a positive scalar  $\varepsilon$  smaller than 1, provided any  $(f, g)$ . Next, define

$$(2.47) \quad r \triangleq \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty} < 1.$$

We now restrict  $(u, v)$  to a particular neighborhood of  $(f, g)$ . More specifically,

$$(2.48) \quad \|f - u\|_\infty + \|g - v\|_\infty < r.$$

Next, remark that  $\|u\|_\infty \leq r + \|f\|_\infty$  and so obtain (bear in mind that  $r < 1$ )

$$(2.49) \quad \|fg - uv\|_\infty = \|(f - u) \cdot g + u \cdot (g - v)\|_\infty$$

$$(2.50) \quad \leq \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty$$

$$(2.51) \quad < r \cdot \|g\|_\infty + (r + \|f\|_\infty) \cdot r$$

$$(2.52) \quad < r \cdot (r + \|f\|_\infty + \|g\|_\infty)$$

$$(2.53) \quad < \varepsilon.$$

Since  $\varepsilon$  was arbitrary, it is now established that  $B$  continuous at every  $(f, g)$ . □

## 10 Exercise 10. Continuity of bilinear mappings

*Prove that a bilinear mapping is continuous if it is continuous at the origin  $(0, 0)$ .*

PROOF. Let  $B : X_1 \times X_2 \rightarrow Z$  be a bilinear mapping that is continuous at  $(0, 0)$ , where  $X_i$  and  $Z$  are topological vector spaces. This implies that, for any balanced open set  $W$ ,  $X_i$  contains a balanced open set  $U_i$  such that

$$(2.54) \quad B(U_1 \times U_2) \subseteq W.$$

Pick  $a_i$  from  $X_i$ , and then choose  $r_i > 1/2$  large enough so that  $a_i \in r_i U_i$ . The vector  $b_i - a_i$  is now restricted to  $(r_1 + r_2)^{-1} U_i$ . Hence

$$(2.55) \quad B(b_1, b_2) - B(a_1, a_2) = B(b_1 - a_1, b_2) + B(a_1, b_2) - B(a_1, a_2)$$

$$(2.56) \quad = B(b_1 - a_1, b_2) + B(a_1, b_2 - a_2)$$

$$(2.57) \quad = B(b_1 - a_1, b_2 - a_2) + B(b_1 - a_1, a_2) + B(a_1, b_2 - a_2)$$

$$(2.58) \quad \subseteq \frac{1}{(r_1 + r_2)^2} W + \frac{r_2}{r_1 + r_2} W + \frac{r_1}{r_1 + r_2} W$$

$$(2.59) \quad \subseteq W + W + W$$

We conclude that  $B$  is continuous at every  $(a_1, a_2)$ , since  $W$  was arbitrary. □

## 12 Exercise 12. A bilinear mapping that is not continuous

Let  $X$  be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| \, dt.$$

Put  $B(f, g) = \int_0^1 f(t)g(t)dt$ , and show that  $B$  is a bilinear continuous functional on  $X \times X$  which is separately but not continuous.

PROOF. Let  $f$  denote the first variable,  $g$  the second one. Remark that

$$(2.60) \quad |B(f, g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient ([1.18] of [4]) to assert that any  $f \mapsto B(f, g)$  is continuous. The continuity of all  $g \mapsto B(f, g)$  follows (Put  $C(g, f) = B(f, g)$  and proceed as above). Suppose, to reach a contradiction, that  $B$  is continuous. There so exists a positive  $M$  such that,

$$(2.61) \quad |B(f, g)| < M\|f\|\|g\|.$$

Put

$$(2.62) \quad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \quad (n = 1, 2, 3, \dots),$$

so that

$$(2.63) \quad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand,

$$(2.64) \quad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.64) and (2.61) with (2.63) and so obtain

$$(2.65) \quad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Our continuity assumption is then contradicted. So ends the proof.  $\square$

## 15 Exercise 15. Baire's cut

Suppose  $X$  is an  $F$ -space and  $Y$  is a subspace of  $X$  whose complement is of the first category. Prove that  $Y = X$ .  
Hint:  $Y$  must intersect  $x + Y$  for every  $x \in X$ .

PROOF. Let  $\{E_n\}$  be a sequence of nowhere dense sets such that  $X \setminus Y = \bigcup_{n=1}^{\infty} E_n$ . Note that every  $V_n = X \setminus \overline{E_n}$  is dense in  $X$ . Now pick an arbitrary  $x \in X$ . Since the translation by  $x$  is a self-homeomorphism,  $x + V_n$  is open and dense as well. Alternatively, observe that  $X = x + X \subseteq \overline{x + V_n}$ , which is a special case of [1.3 (b)] of [4]. Next, apply Baire's theorem twice to establish that

(a) every intersection  $W_n = V_n \cap [x + V_n]$  is dense in  $X$ ,

(b) so is the nonempty intersection  $\bigcap_{n=1}^{\infty} W_n$ .

Moreover, the intersection  $\bigcap_{n=1}^{\infty} V_n$  does not cut any  $E_n$ . To sum up,

$$(2.66) \quad w \in \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq Y$$

for some  $w = w(x)$ . Furthermore,  $w$  also lies in every  $x + V_n$ , by (a). Hence

$$(2.67) \quad w - x \in \bigcap_{n=1}^{\infty} V_n \subseteq Y.$$

Finally, (2.66) and (2.67) together yield

$$(2.68) \quad x = w - (w - x) \in Y - Y = Y,$$

as  $Y$  is a subgroup of  $X$ . This establishes that

$$(2.69) \quad X \subseteq Y,$$

since  $x$  was arbitrary. □

## 16 Exercise 16. An elementary closed graph theorem

Suppose that  $X$  and  $K$  are metric spaces, that  $K$  is compact, and that the graph of  $f : X \rightarrow K$  is a closed subset of  $X \times K$ . Prove that  $f$  is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of  $K$  cannot be omitted from the hypotheses, even when  $X$  is compact.

PROOF. Choose a sequence  $\{x_n : n = 1, 2, 3, \dots\}$  whose limit is an arbitrary  $a$ . By compactness of  $K$ , the graph  $G$  of  $f$  contains a subsequence  $\{(x_{p(n)}, f(x_{p(n)}))\}$  of  $\{(x_n, f(x_n))\}$  that converges to some  $(a, b)$  of  $X \times K$ .  $G$  is closed; therefore,  $\{(x_{p(n)}, f(x_{p(n)}))\}$  converges in  $G$ . So,  $b = f(a)$ ; which establishes that  $f$  is sequentially continuous. Since  $X$  is metrizable,  $f$  is also continuous, see [[A6]] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.70) \quad \begin{aligned} f : [0, \infty) &\rightarrow [0, \infty) \\ x &\mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \end{aligned}$$

Clearly,  $f$  has a discontinuity at 0. Nevertheless the graph  $G$  of  $f$  is closed. To see that, first remark that

$$(2.71) \quad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let  $\{(x_n, 1/x_n)\}$  be a sequence in  $G_+ = \{(x, 1/x) : x > 0\}$  that converges to  $(a, b)$ . To be more specific:  $a = 0$  contradicts the boundedness of  $\{(x_n, 1/x_n)\}$ :  $a$  is necessarily positive and  $b = 1/a$ , since  $x \mapsto 1/x$  is continuous on  $\mathbb{R}_+$ . This establishes that  $(a, b) \in G_+$ , hence the closedness  $G_+$ . Finally, we conclude that  $G$  is closed, as a finite union of closed sets.  $\square$

# Chapter 3

## Convexity

### 3 Exercise 3.

Suppose  $X$  is a real vector space (without topology). Call a point  $x_0 \in A \subseteq X$  an internal point of  $A$  if  $A - x_0$  is an absorbing set.

- (a) Suppose  $A$  and  $B$  are disjoint convex sets in  $X$ , and  $A$  has an internal point. Prove that there is a nonconstant functional  $\Lambda$  such that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. (The proof is similar to that of Theorem 3.4)
- (b) Show (with  $X = \mathbf{R}^2$ , for example) that it may not be possible to have  $\Lambda(A)$  and  $\Lambda(B)$  disjoint, under the hypotheses of (a).

PROOF. Take  $A$  and  $B$  as in (a); the trivial case  $B = \emptyset$  is discarded. Since  $A - x_0$  is absorbing, so is its convex superset  $C = A - B - x_0 + b_0$  ( $b_0 \in B$ ). Note that  $C$  contains the origin. Let  $p$  be the Minkowski functional of  $C$ . Since  $A$  and  $B$  are disjoint,  $b_0 - x_0$  is not in  $C$ , hence  $p(b_0 - x_0) \geq 1$ . We now proceed as in the proof of the Hahn-Banach theorem [3.4] of [4] to establish the existence of a functional  $\Lambda : X \rightarrow \mathbf{R}$  such that

$$(3.1) \quad \Lambda \leq p$$

and

$$(3.2) \quad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \quad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

$$(3.4) \quad \Lambda a \leq \Lambda b.$$

We now prove that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. Suppose, to reach a contradiction, that this intersection contains  $y_1$  and  $y_2$ . There so exists  $(a_i, b_i)$  in  $A \times B$  ( $i = 1, 2$ ) such that

$$(3.5) \quad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that  $y_1 < y_2$ . Then,

$$(3.6) \quad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad .$$

Remark that  $a_3 = \frac{1}{2}(a_1 + a_2)$  lies in the convex set  $A$ . This implies

$$(3.7) \quad \Lambda b_1 \stackrel{(3.6)}{<} \Lambda a_3 \stackrel{(3.4)}{\leq} \Lambda b_1 \quad ;$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space  $X$  is  $\mathbf{R}^2$ . Fetch

$$(3.8) \quad S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$

$$(3.9) \quad S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$$

$$(3.10) \quad A \triangleq S_1 \cup S_2,$$

$$(3.11) \quad B \triangleq X \setminus A.$$

Pick  $(x_i, y_i)$  in  $S_i$ . Let  $t$  range over the unit interval, and so obtain

$$(3.12) \quad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A.$$

Thus, every segment that has an extremity in  $S_1$  and the other one in  $S_2$  lies in  $A$ . Moreover, each  $S_i$  is convex. We can now conclude that  $A$  is so. The convexity of  $B$  is proved in the same manner. Furthermore,  $A$  hosts a non degenerate triangle, *i.e.*,  $A^\circ$  is nonempty<sup>1</sup>:  $A$  contains an internal point.

Let  $L$  be a vector line of  $\mathbf{R}^2$ . In other words,  $L$  is the null space of a functional  $\Lambda : \mathbf{R}^2 \rightarrow \mathbf{R}$  (to see this, take some nonzero  $u$  in  $L^\perp$  and set  $\Lambda x = (x, u)$  for all  $x$  in  $\mathbf{R}^2$ ). One easily checks that both  $A$  and  $B$  cut  $L$ . Hence

$$(3.13) \quad \Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad .$$

So ends the proof. □

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<sup>1</sup>For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] of [4] or [2.5] of [3].



## 11 Exercise 11. Meagerness of the polar

Let  $X$  be an infinite-dimensional Fréchet space. Prove that  $X^*$ , with its weak\*-topology, is of the first category in itself.

This is actually a consequence of the lemma below, which we prove first. The proof that  $X^*$  is of the first category in itself comes right after, as a corollary.

**Lemma.** *If  $X$  is an infinite dimensional topological vector space whose dual  $X^*$  separates points on  $X$ , then the polar*

$$(3.14) \quad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\}$$

*of any absorbing subset  $A$  is a weak\*-closed set that has empty interior.*

PROOF. Let  $x$  range over  $X$ . The linear form  $\Lambda \mapsto \Lambda x$  is weak\*-continuous, see [3.14] of [4]. Therefore,  $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$  is weak\*-closed: As the intersection of  $\{P_a : a \in A\}$ ,  $K_A$  is also a weak\*-closed set. We now prove the second half of the statement.

From now on,  $X$  is assumed to be endowed with its weak topology:  $X$  is then locally convex, but its dual space is still  $X^*$  (see [3.11] of [4]). Put

$$(3.15) \quad W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \quad (r_x > 0)$$

where  $F$  runs through the nonempty finite subsets of  $X$ . Clearly, the collection of all such  $W$  is a local base of  $X^*$ . Pick one of those  $W$  and remark that the following subspace

$$(3.16) \quad M \triangleq \text{span}(F)$$

is finite dimensional. Assume, to reach a contradiction, that  $A \subseteq M$ . So, every  $x$  lies in  $t_x M = M$  for some  $t_x > 0$ , since  $A$  is absorbing. As a consequence,  $X$  is the finite dimensional space  $M$ , which is a desired contradiction. We have just established that  $A \not\subseteq M$ : Now pick  $a$  in  $A \setminus M$  and so conclude that

$$(3.17) \quad b \triangleq \frac{a}{t_a} \in A$$

Remark that  $b \notin M$  (otherwise,  $a = t_a b \in t_a M = M$  would hold) and that  $M$ , as a finite dimensional space, is closed (see [1.21 (b)] of [4] for a proof): By the Hahn-Banach theorem [3.5] of [4], there exists  $\Lambda_a$  in  $X^*$  such that

$$(3.18) \quad \Lambda_a b > 2$$

and

$$(3.19) \quad \Lambda_a(M) = \{0\}.$$

The latter equality implies that  $\Lambda_a$  vanishes on  $F$ ; hence  $\Lambda_a$  is an element of  $W$ . On the other hand, given an arbitrary  $\Lambda \in K_A$ , the following inequalities

$$(3.20) \quad |\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that  $\Lambda + \Lambda_a$  is not in  $K_A$ . We have thus proved that

$$(3.21) \quad \Lambda + W \not\subseteq K_A.$$

Since  $W$  and  $\Lambda$  are both arbitrary, this achieves the proof. □

We now give a proof of the original statement.

**Corollary.** *If  $X$  is an infinite-dimensional Fréchet space, then  $X^*$  is meager in itself.*

PROOF. From now on,  $X^*$  is only endowed with its weak\*-topology. Let  $d$  be an invariant distance that is compatible with the topology of  $X$ , so that the following sets

$$(3.22) \quad B_n \triangleq \{x \in X : d(0, x) < 1/n\} \quad (n = 1, 2, 3, \dots)$$

form a local base of  $X$ . If  $\Lambda$  is in  $X^*$ , then

$$(3.23) \quad |\Lambda| \leq m \text{ on } B_n$$

for some  $(n, m) \in \{1, 2, 3, \dots\}^2$ , see [1.18] of [4]. Hence,  $X^*$  is the countable union of all

$$(3.24) \quad m \cdot K_n \quad (m, n = 1, 2, 3, \dots),$$

where  $K_n$  is the polar of  $B_n$ . Clearly, showing that every  $m \cdot K_n$  is nowhere dense is now sufficient. To do so, we use the fact that  $X^*$  separates points; see [3.4] of [4]. As a consequence, the above lemma implies

$$(3.25) \quad (\overline{K_n})^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by  $m$  is a homeomorphism (see [1.7] of [4]), this is equivalent to

$$(3.26) \quad (\overline{m \cdot K_n})^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof. □

# Chapter 4

## Banach Spaces

Throughout this set of exercises,  $X$  and  $Y$  denote Banach spaces, unless the contrary is explicitly stated.

### 1 Exercise 1. Basic results

Let  $\phi$  be the embedding of  $X$  into  $X^{**}$  described in Section 4.5. Let  $\tau$  be the weak topology of  $X$ , and let  $\sigma$  be the weak\*-topology of  $X^{**}$ - the one induced by  $X^*$ .

- (a) Prove that  $\phi$  is a homeomorphism of  $(X, \tau)$  onto a dense subspace of  $(X^{**}, \sigma)$ .
- (b) If  $B$  is the closed unit ball of  $X$ , prove that  $\phi(B)$  is  $\sigma$ -dense in the closed unit ball of  $X^{**}$ . (Use the Hahn-Banach separation theorem.)
- (c) Use (a), (b), and the Banach-Alaoglu theorem to prove that  $X$  is reflexive if and only if  $B$  is weakly compact.
- (d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.
- (e) If  $X$  is reflexive and  $Y$  is a closed subspace of  $X$ , prove that  $X/Y$  is reflexive.
- (f) Prove that  $X$  is reflexive if and only if  $X^*$  is reflexive.  
*Suggestion: One half follows from (c); for the other half, apply (d) to the subspace  $\phi(X)$  of  $X^{**}$ .*

PROOF. Let  $\psi$  be the isometric embedding of  $X^*$  into  $X^{***}$ . The dual space of  $(X^{**}, \sigma)$  is then  $\psi(X^*)$ .

It is sufficient to prove that

$$(4.1) \quad \phi^{-1} : \phi(X) \rightarrow X$$

$$(4.2) \quad \phi(x) \mapsto x$$

is a homeomorphism (with respect to  $\tau$  and  $\sigma$ ). We first consider

$$(4.3) \quad V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}, \phi x^* \rangle| < r\} \quad (x^* \in X^*, r > 0);$$

$$(4.4) \quad U \triangleq \{x \in X : |\langle x, x^* \rangle| < r\} \quad (x^* \in X^*, r > 0).$$

and remark that the so defined  $V$ 's (respectively  $U$ 's) shape a local subbase  $\mathcal{S}_\sigma$  (respectively  $\mathcal{S}_\tau$ ) of  $\sigma$  (respectively  $\tau$ ). We now observe that

$$(4.5) \quad U = \phi^{-1}(V \cap \phi(X)) = \phi^{-1}(V) \cap X \quad (V \in \mathcal{S}_\sigma, U \in \mathcal{S}_\tau) \quad ,$$

since  $\phi^{-1}$  is one-to-one. This remains true whether we enrich each subbase  $\mathcal{S}$  with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak\*-topology that  $\phi^{-1}$  and its inverse  $\phi$  are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$(4.6) \quad \overline{\phi(X)}_\sigma \subseteq X^{**} \quad .$$

and we now assume- to reach a contradiction- that  $(X^{**}, \sigma)$  contains a point  $z^{**}$  outside the  $\sigma$ -closure of  $\phi(X)$ . By [3.5], there so exists  $y^*$  in  $X^*$  such that

$$(4.7) \quad \langle \phi x, \psi y^* \rangle = \langle y^*, \phi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ;$$

$$(4.8) \quad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces  $y^*$  to be a the zero of  $X^*$ . The functional  $\psi y^*$  is then the zero of  $X^{***}$ : (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball  $B^{**}$  of  $X^{**}$  is weak\*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \quad \phi(B) \subseteq B^{**} \quad ,$$

since  $\phi$  is isometric. Hence

$$(4.10) \quad \overline{\phi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} \quad .$$

Now suppose, to reach a contradiction, that  $B^{**} \setminus \overline{\phi(B)}_\sigma$  contains a vector  $z^{**}$ . By [3.7], there exists  $y^*$  in  $X^*$  such that

$$(4.11) \quad |\psi y^*| \leq 1 \quad \text{on } \overline{\phi(B)}_\sigma \quad ;$$

$$(4.12) \quad \langle z^{**}, \psi y^* \rangle > 1 \quad .$$

It follows from (4.11) that

$$(4.13) \quad |\psi y^*| \leq 1 \quad \text{on } \phi(B) \quad , \quad i.e., \quad |y^*| \leq 1 \quad \text{on } B \quad .$$

We have so proved that

$$(4.14) \quad y^* \in B^* \quad .$$

Since  $z^{**}$  lies in  $B^{**}$ , it is now clear that

$$(4.15) \quad |\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ;$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

$$(4.16) \quad B \text{ is weakly compact if and only if } \phi(B) \text{ is weak}^*\text{-compact.}$$

If  $B$  is weakly compact, then  $\phi(B)$  is weak\*-closed. So,

$$(4.17) \quad \phi(B) = \overline{\phi(B)}_\sigma \stackrel{(b)}{=} B^{**} \quad .$$

$\phi$  is therefore onto, *i.e.*,  $X$  is reflexive.

Conversely, keep  $\phi$  as onto: one easily checks that  $\phi(B) = B^{**}$ . The image  $\phi(B)$  is then weak\*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let  $X$  be a reflexive space  $X$ , whose closed unit ball is  $B$ . Let  $Y$  be a norm-closed subspace of  $X$ :  $Y$  is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that  $B$  is weakly compact. We now conclude that the closed unit ball  $B \cap Y$  of  $Y$  is weakly compact. We again use (c) to conclude that  $Y$  is reflexive. (d) is therefore established. Now proceed to (e).

Let  $\equiv$  stand for “isometrically isomorphic” and apply twice [4.9] to obtain, first

$$(4.18) \quad (X/Y)^* \equiv Y^\perp \quad ,$$

next,

$$(4.19) \quad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y \quad .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

*Given a reflexive Banach space  $Z$ , the weak\*-topology of  $Z^*$  is its weak one.*

Assume first that  $X$  is reflexive. Since  $B^*$  is weak\* compact, by (c) of [4.3], (L) implies that  $B^*$  is also weakly compact. Then (c) turns  $X^*$  into a reflexive space.

Conversely, let  $X^*$  be reflexive. What we have just proved that makes  $X^{**}$  reflexive. On the other hand,  $\phi(X)$  is a norm-closed subspace of  $X^{**}$ ; cf. [4.5]. Hence  $\phi(X)$  is reflexive, by (d). It now follows from (c) that  $B^{**} \cap \phi(X)$  is weakly compact, *i.e.*, weak\*-compact (to see this, apply (L) with  $Z = X^*$ ).

By (a),  $B$  is therefore weakly compact, *i.e.*,  $X$  is reflexive, see (c). So ends the proof. □

### 13 Exercise 13. Operator compactness in a Hilbert space

- (a) Suppose  $T \in \mathcal{B}(X, Y)$ ,  $T_n \in \mathcal{B}(X, Y)$  for  $n = 1, 2, 3, \dots$ , each  $T_n$  has finite-dimensional range, and  $\lim \|T - T_n\| = 0$ . Prove that  $T$  is compact.
- (b) Assume  $Y$  is a Hilbert space, and prove the converse of (a): Every compact  $T \in \mathcal{B}(X, Y)$  can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)

PROOF. Since each  $T_n$  is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every  $T_n$  as a compact operator. Let  $\{x_0^i : i \in \mathbf{N}\}$  be in  $U$  the open unit ball of  $X$ . Since  $T_1$  is compact,  $\{x_0^i\}$  contains a subsequence  $\{x_1^i : i \in \mathbf{N}\}$  such that  $\{T_1 x_1^i\}$  converges to a point  $y_1$  of  $Y$ . The same reasoning can be recursively applied to  $T_n$  and  $\{x_{n-1}^i\} \subseteq U$  so that  $\{T_n x_n^i\}$  tends to some  $y_n$  of  $Y$ , as  $\{x_n^i\}$  is a subsequence of  $\{x_{n-1}^i\}$ . Then

$$(4.20) \quad T_n x_p^i \xrightarrow{i \rightarrow \infty} y_n \quad (p = 1, 2, 3, \dots) \quad .$$

Applied with  $\{x_n^i : (n, i) \in \mathbf{N}^2\}$ , a Cantor's diagonal process therefore provides a subsequence  $\{\tilde{x}_j : j \in \mathbf{N}\}$  such that

$$(4.21) \quad T_j \tilde{x}_k \xrightarrow{k \rightarrow \infty} y_j \quad ;$$

$$(4.22) \quad T_j \tilde{x}_j \xrightarrow{j \rightarrow \infty} y_j \quad .$$

We now easily obtain

$$(4.23) \quad \|T_j \tilde{x}_j - T_k \tilde{x}_k\| \leq \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \xrightarrow{k > j \rightarrow \infty} 0 \quad .$$

$\{T_j \tilde{x}_j\}$  is then a Cauchy sequence. So is  $\{T \tilde{x}_j\}$ , since  $\|T - T_j\| \rightarrow 0$ . On the other hand,  $Y$  is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix  $\varepsilon$  as a positive scalar. Since  $T$  is compact,  $Y$  contains a finite set  $C$  such that

$$(4.24) \quad T(U) \subseteq \bigcup_{c \in C} B(c, \varepsilon) \quad .$$

As a Hilbert space,  $Y$  contains a *maximal orthonormal set* (or *Hilbert basis*)  $M$ . This implies that  $\text{span}(M)$  is dense in  $Y$ ; cf. 4.18 & [4.22] of [3]. The finiteness of  $C$  forces  $M$  to enclose a finite set  $S$  so that

$$(4.25) \quad \forall c \in C, \exists s(c) \in \text{span}(S) : \|c - s(c)\| < \varepsilon \quad .$$

Let  $x$  be in  $U$ . It follows from (4.24) that

$$(4.26) \quad \|Tx - c_x\| < \varepsilon$$

for some  $c_x$  of  $C$ . We now combine (4.25) and (4.26) to obtain

$$(4.27) \quad \|Tx - s(c_x)\| \leq \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon$$

As a finite-dimensional subspace,  $\text{span}(S)$  is closed (see footnote 4, Exercise 1.10). We so obtain

$$(4.28) \quad Y = \text{span}(S) \oplus \text{span}(S)^\perp \quad ,$$

by [12.4]. There so exists a unique projection  $\pi = \pi(\varepsilon)$  of  $Y$  onto itself (see [5.6] for the definition) such that

$$(4.29) \quad \pi(Y) = \text{span}(S) \quad , \quad (I - \pi)(Y) = \text{span}(S)^\perp \quad .$$

It is easily checked that  $\pi$  has norm 1. Moreover,

$$(4.30) \quad \pi s = s \quad (s \in \text{span}(S)) \quad .$$

Thus,

$$(4.31) \quad (I - \pi)(Tx) = (I - \pi)(Tx - s(c_x)) \quad (x \in U) \quad .$$

Then,

$$(4.32) \quad \|(I - \pi)(Tx)\| \leq \|I - \pi\| \|Tx - s(c_x)\| < 4\varepsilon \quad (x \in U)$$

(the fact that  $\pi$  has norm 1 is hidden in the right side inequality). We have just so proved that

$$(4.33) \quad \|T - \pi \circ T\| \in O_{\varepsilon \sim 0}(\varepsilon) \quad .$$

That is particularly true when  $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \xrightarrow{n \rightarrow \infty} 0$ . Let so  $T_n$  be  $\pi(\varepsilon_n) \circ T$  and conclude that these (compact) operators approximate  $T$  in the desired fashion, *i.e.*,

$$(4.34) \quad \|T - T_n\| \xrightarrow{n \rightarrow \infty} 0 \quad .$$

□

## 15 Exercise 15. Hilbert-Schmidt operators

Suppose  $\mu$  is a finite (or  $\sigma$ -finite) positive measure on a measure space  $\Omega$ ,  $\mu \times \mu$  is the corresponding product measure on  $\Omega \times \Omega$ , and  $K \in L^2(\mu \times \mu)$ . Define

$$(Tf)(s) = \int_{\Omega} K(s, t) f(t) d\mu(t) \quad [f \in L^2(\mu)].$$

(a) Prove that  $T \in \mathcal{B}(L^2(\mu))$  and that

$$\|T\|^2 \leq \int_{\Omega} \int_{\Omega} |K(s, t)|^2 d\mu(s) d\mu(t).$$

(b) Suppose  $a_i, b_i$  are members of  $L^2(\mu)$ , for  $1 \leq i \leq n$ , put  $K_1 = \sum a_i(s) b_i(t)$  and define  $T_1$  in terms of  $K_1$  as  $T$  was defined in terms of  $K$ . Prove that  $\dim \mathcal{R}(T_1) \leq n$ .

(c) Deduce that  $T$  is a compact operator in  $L^2(\mu)$ . Hint: Use exercise 13.

(d) Suppose  $\lambda \in \mathbf{C}$ ,  $\lambda \neq 0$ . Prove: Either the equation

$$Tf - \lambda f = g$$

has a unique solution  $f \in L^2(\mu)$  for every  $g \in L^2(\mu)$  or there are infinitely many solutions for some  $g$  and none for others. (This is known as the Fredholm alternative.).

(e) Describe the adjoint of  $T$ .

PROOF. Let  $X$  (respectively  $P$ ) be the Banach space  $L^2(\mu)$  (respectively  $L^2(\mu \times \mu)$ ). A consequence of the Radon-Nikodym theorem (cf. [6.16] of [3]) is that there exists a group isomorphism  $\varphi: X \rightarrow X^*$ ,  $f \mapsto f^*$  such that

$$(4.35) \quad \langle u, f^* \rangle = \int_{\Omega} u \cdot f d\mu \quad (u \in X, f \in X) \quad .$$

Define a.e  $K_s, K_t: \Omega \rightarrow \mathbf{C}$  by setting

$$(4.36) \quad K_s(t) \triangleq K_t(s) \triangleq K(s, t) \quad \text{a.e.} \quad ((s, t) \in \Omega) \quad .$$

$T$  is clearly linear. Moreover,

$$(4.37) \quad |(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \quad (\|f\|_X < 1)$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with  $|K|^2$  to obtain

$$(4.38) \quad \|Tf\|_X^2 \leq \int_{\Omega} \|K_s\|_X^2 d\mu(s) = \|K\|_P^2 < \infty \quad (\|f\|_X < 1) \quad .$$

(a) is then proved.

To show (b), remark that

$$(4.39) \quad \int_{\Omega} a_i(s) \cdot b_i \cdot f d\mu \in \mathbf{C} \cdot a_i(s) \quad \text{a.e.} \quad (f \in X, s \in \Omega) \quad .$$

It is now clear that  $T$  maps any  $f$  of  $X$  into  $\mathbf{C} \cdot a_1 + \cdots + \mathbf{C} \cdot a_n$ . We so conclude that  $\dim \mathcal{R}(T_1) \leq n$ .

We now aim at (c). The current part refers to Exercise 4.13.  $X$  is also a Hilbert space and so contains a Hilbert basis  $M$ . Define a.e

$$(4.40) \quad \begin{aligned} a_b: \Omega &\rightarrow \mathbf{C} \\ s &\mapsto (K_s, b) \end{aligned}$$



whenever  $b$  ranges  $M$ . Hence,

$$(4.41) \quad K_s = \sum_{b \in M} a_b(s) \cdot b \quad \text{a.e.} \quad (s \in \Omega) \quad .$$

Provided any positive scalar  $\varepsilon$ , there so exists a finite subset  $S = S(\varepsilon)$  of  $M$  such that

$$(4.42) \quad \|K_s - \sum_{b \in S} a_b(s) \cdot b\|_X < \varepsilon \quad (s \in \Omega) \quad .$$

Remark that  $\sum_{b \in S} a_b \cdot b$  matches the definition of  $K_1$ ; cf. (b): from now on,

$$(4.43) \quad K_1 \triangleq \sum_{b \in S} a_b \cdot b \quad .$$

It follows from (b) that

$$(4.44) \quad \dim R(K_1) < \infty \quad .$$

Now turn back to (a), with  $K - K_1$  playing the role of  $K$ , and so obtain

$$(4.45) \quad \|T - T_1\| < \varepsilon \mu(\Omega) \leq \infty \quad .$$

For if  $\mu$  is finite, use (a) of Exercise 4.13 to conclude that  $T$  is compact. Assume henceforth that  $\mu$  is not (necessarily) finite and pick  $\delta$  in  $\mathbf{R}_+$ . The simple functions (with finite measure support) form a dense family of an  $L^p$  space ( $1 \leq p < \infty$ ); cf. [3.13] of [3]. It then exists a simple function  $K_\delta$  of  $L^2(\mu \times \mu)$  such that

$$(4.46) \quad (\mu \times \mu)(\{K_\delta \neq 0\}) < \infty, \quad \|K - K_\delta\|_p < \delta \quad .$$

Define an operator  $T_\delta$  in terms of  $K_\delta$  as  $T$  was defined in terms of  $K$ , and proceed as in (a) with  $T - T_\delta$  instead of  $T$ . Then

$$(4.47) \quad \|T - T_\delta\| < \delta \quad .$$

The key ingredient is that  $K_\delta$  can be identified with an element of the finite measure space  $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$ . What we have attempted to approximate  $T$  by  $T_1$  can therefore be reiterated (with  $K_\delta$  playing the role of  $K$ ) to achieve an approximation  $T_{\delta,1}$  of  $T_\delta$  so that

$$(4.48) \quad \|T_\delta - T_{\delta,1}\| < \varepsilon \quad .$$

It now follows from (4.47) and (4.48) that

$$(4.49) \quad \|T - T_{\delta,1}\| \leq \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta \quad .$$

Since  $\varepsilon$  and  $\delta$  were arbitrary, the  $\sigma$ -finite case is proved. We now establish (d).

Provided  $g$  of  $X$ , let  $E_g$  be the following equation on  $X$

$$(4.50) \quad Tf - \lambda f = g \quad ,$$

whose solution set is denoted by  $S_g$ . Note that  $S_0$  is  $\ker(T - \lambda)$  and discard the trivial case  $S_0 = X$ <sup>1</sup>: each  $f$  of  $X$  lies in  $S_{Tf - \lambda f}$ , as some  $Tf - \lambda f$ 's are nonzero. Some  $S_g$ 's are then nonempty. Remark that

$$(4.51) \quad S_g = f + S_0 \quad (f \in S_g)$$

in such case. Furthermore, the equality  $\beta = \alpha$  of [4.25] yields

$$(4.52) \quad (T - \lambda I)(X) \neq X, \text{ , i.e., } S_0 \neq \{0\} \quad .$$

---

<sup>1</sup>, e.g.,  $X = L^2(\{0\}, \delta)$  so that  $I = \lambda^{-1}T$  is compact.

So if  $T - \lambda I$  is not onto, not only some  $S_g$ 's are empty, but also  $S_0 \neq \{0\}$ . Every nonempty  $S_g$  (such sets always exist, see above) is then infinite, by (4.51).

Otherwise,  $T - \lambda I$  is bijective and every equation  $E_g$  has then a unique solution  $f$ . The Fredholm alternative is so proved.

Our last step is the description of  $T^*$ . Let  $S : X \rightarrow X$  be such that

$$(4.53) \quad (Sf)(t) \triangleq \int_{\Omega} K_t \cdot f \text{ a.e.} \quad (f \in X, t \in \Omega)$$

Proceed as in (a), with  $S$  instead of  $T$ :  $S$  lies in  $\mathcal{B}(X)$ . Next, we claim that

$$(4.54) \quad \langle u, T^* f^* \rangle = \langle Tu, f^* \rangle$$

$$(4.55) \quad = \int_{\Omega} (Tu) \cdot f \, \mu$$

$$(4.56) \quad = \int_{\Omega^2} K \cdot f \cdot u \, (\mu \times \mu)$$

$$(4.57) \quad = \int_{\Omega} (Sf) \cdot u \, \mu$$

$$(4.58) \quad = \langle u, (Sf)^* \rangle \quad ,$$

whenever  $u$  and  $f$  run through the closed unit ball of  $X$ . Since  $\|T\|$ ,  $\|T^*\|$  are equal and finite, only exactness of (4.56) is possibly in doubt; the justification below dissipates it. In conclusion,

$$(4.59) \quad T^* = \rho S \rho^{-1} \quad .$$

Informally,

$$(4.60) \quad T^* = S \quad .$$

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep  $u$  and  $f$  as above. Let us introduce

$$(4.61) \quad A(s) \triangleq \int_{\Omega} |K_s(t) \cdot u(t)| \, \mu(t) \text{ a.e.} \quad (s \in \Omega) \quad ,$$

to make hold the following Cauchy-Schwarz inequality

$$(4.62) \quad A(s) \leq \|K_s\|_X \quad (s \in \Omega) \quad .$$

Thus,

$$(4.63) \quad \int_{\Omega^2} |K(s, t) u(t) f(s)| \, \mu(s) \mu(t) = \int_{\Omega} |f(s)| A(s) \, \mu(s)$$

$$(4.64) \quad \leq \int_{\Omega} |f(s)| \|K_s\|_X \, \mu(s)$$

$$(4.65) \quad \leq \left[ \int_{\Omega} \|K_s\|_X^2 \, \mu(s) \right]^{\frac{1}{2}} = \|K\|_P < \infty \quad .$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof.  $\square$

# Chapter 6

## Distributions

### 1 Exercise 1. Test functions are almost polynomial

Suppose  $f$  is a complex continuous function in  $\mathbf{R}^n$ , with compact support. Prove that  $\psi P_j \rightarrow f$  uniformly on  $\mathbf{R}^n$ , for some  $\psi \in \mathcal{D}$  and for some sequence  $\{P_j\}$  of polynomials.

PROOF. According to 1.16,  $\Omega$  is union of a compact sets sequence  $\{K_i\}$  and  $\text{supp}(f)$  lies in some  $K = K_i$  so that  $f$  is embedded in  $\mathcal{D}(\Omega)$ . We can apply [1.10] to ensure that  $\Omega$  encloses a compact set  $S = \overline{K + B(\varepsilon)}$  for sufficiently small  $\varepsilon > 0$ .

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra  $\{g \in C(S) : g \text{ is polynomial}\}$  of  $C(S)$ . There so exists a sequence  $\{P_j : j \in \mathbf{N}\}$  of  $\mathbf{R}[X_1, \dots, X_n]$  such that

$$(6.1) \quad \sup_S |f - P_j| \xrightarrow{j \rightarrow \infty} 0 \quad .$$

By [6.20], the open set  $K + B(\varepsilon)$  has a local partition of unity  $\{\psi_j\} \subseteq \mathcal{D}(\Omega)$ . Moreover, there exists an integer  $l$  such that  $\psi = \psi_1 + \dots + \psi_l$  equals 1 on  $K$ . Hence

$$(6.2) \quad \|f - \psi P_j\|_\infty = \|\psi f - \psi P_j\|_\infty = \sup_S |\psi f - \psi P_j|$$

$$(6.3) \quad = \sup_S |f - P_j| \xrightarrow[j \rightarrow \infty]{(6.1)} 0 \quad .$$

□

We will actually prove more by showing that  $\mathcal{D}(\Omega)$  is separable for each nonempty open subset  $\Omega$  of  $\mathbf{R}^n$ .

PROOF. The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the separability of  $\mathcal{D}(\Omega)$ . To do so, we keep  $(\alpha, j)$  in  $\mathbf{N}^n \times \mathbf{N}$ . Remark that  $S$  encloses  $\text{supp}(D^\alpha f)$ : according to the first part, there exists a sequence  $\{P_{\alpha, j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \dots, X_n]$  such that

$$(6.4) \quad \|D^\alpha f - \psi P_{\alpha, j}\|_\infty \xrightarrow{j \rightarrow \infty} 0 \quad .$$

Now let  $m$  range over  $\{1, 2, 3, \dots\}$  and set  $W_{m, j}$  in  $\mathcal{D}(\Omega)$  as follows

$$(6.5) \quad D^{-\alpha} \phi \in \mathcal{D}(\Omega) : D^\alpha D^{-\alpha} \phi = \phi \quad .$$

$$(6.6) \quad W_{m, j}(x) \triangleq D^{-(m, \dots, m)}(\psi P_{(m, \dots, m), j})$$

By (6.4), there exists a natural number  $k(m)$  such that

$$(6.7) \quad \|D^{(m, \dots, m)}(f - W_{m, j})\|_\infty < 1/m \quad (j \geq k(m)) \quad .$$

Assume without loss of generality that  $S$  has diameter 1 so that (6.7) yields

$$(6.8) \quad \|D^\lambda(f - W_{m, k(m)})\|_\infty < 1/m \quad (|\lambda| \leq m) \quad ,$$

by the mean value theorem. In other words (remark that  $\text{supp}(f - W_{m, k(m)}) \subseteq S$ ),

$$(6.9) \quad f - W_{m, k(m)} \in U_m \triangleq \{\phi \in \mathcal{D}_S : \|\phi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \cdots \quad (m = 1, 2, 3, \dots) \quad .$$

Pick  $W$  in  $\beta$  (see (b) of [6.3]):  $W \cap \mathcal{D}_S$  contains a neighbourhood of 0. Hence  $W$  contains some  $U_m$ , for  $m$  sufficiently large. Thus

$$(6.10) \quad W_{m, k(m)} \xrightarrow{m \rightarrow \infty} f \quad (\text{in } \mathcal{D}(\Omega)) \quad .$$

We have so established that the  $W_{m, k(m)}$ 's family is dense in  $\mathcal{D}(\Omega)$ . We now aim to disclose a countable set  $\tilde{W}$  that has the same property.

Choose  $\delta$  in  $\mathbf{R}_+$  and fetch any  $W_{m, k(m)}$ . Let  $X$  be  $(X_1, \dots, X_n)$  and express  $P_{(m, \dots, m), k(m)}$  as

$$(6.11) \quad P(X) = \sum_{|\gamma| \leq d} p_\gamma \cdot X^\gamma \quad .$$

Since  $\bar{Q} = \mathbf{R}$ ,  $Q[X]$  hosts some  $Q(X) = \sum_{|\gamma| \leq d} q_\gamma \cdot X^\gamma$  such that  $|p_\gamma - q_\gamma| < \delta$  for all  $\gamma$ . Thus,

$$(6.12) \quad |P(x) - Q(x)| \leq \sum_{|\gamma| \leq d} |p_\gamma - q_\gamma| |x|^{|\gamma|} \leq \delta \sum_{l \leq d} \binom{1+n-1}{n-1} \|x\|_\infty^l \quad (x \in \mathbf{R}^n) \quad .$$

Since  $S$  is bounded, we so obtain

$$(6.13) \quad \|\phi(P - Q)\|_\infty \in O(\delta) \quad .$$

Now define  $\tilde{W}_m$  in terms of  $Q$  as  $W_{m, k(m)}$  was defined in terms of  $P$ , and consider the integrations made in (6.6): each  $D^\lambda \tilde{W}_m$  ( $|\lambda| \leq m$ ) can be obtained from some of them. So (6.13) yields

$$(6.14) \quad \|D^\lambda(W_{m, k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \quad (|\lambda| \leq m) \quad .$$

To be more specific, these  $\lambda$ 's only exist in finite amount, so the big  $O$  can be assumed to be the same for all them. Since  $\delta$  was arbitrary, combining (6.10) with (6.14) establishes the density of the all  $\tilde{W}_m$ 's family  $\tilde{W}$ .

Furthermore, each member of  $\tilde{W}$  is only made of two ingredients:  $\psi$  and a polynomial of  $Q[X]$ . The mapping  $\psi$  is attached to some  $K_i$  and  $Q[X]$  inherits countableness from  $Q$ . Note that the “integrations packs” of (6.6) only exist in countable amount. Our  $\tilde{W}$  is then countable.  $\square$

## 6 Exercise 6. Around the supports of some distributions

(a) Suppose that  $c_m = \exp\{-(m!)!\}$ ,  $m = 0, 1, 2, \dots$ . Does the series

$$\sum_{m=0}^{\infty} c_m (D^m \phi)(0)$$

converges for every  $\phi \in C^\infty(\mathbf{R})$ ?

(b) Let  $\Omega$  be open in  $\mathbf{R}^n$ , suppose  $\Lambda_i \in \mathcal{D}'(\Omega)$ , and suppose that all  $\Lambda_i$  have their supports in some fixed compact  $K \subseteq \Omega$ . Prove that the sequence  $\{\Lambda_i\}$  cannot converge in  $\mathcal{D}'(\Omega)$  unless the orders of the  $\Lambda_j$  are bounded. Hint: Use the Banach-Steinhaus theorem.

(c) Can the assumption about the supports be dropped in (b)?

PROOF. The answer is: no. Let us establish this assertion. Assume, to reach a contradiction, that the above series converges for every smooth  $\phi : \mathbf{R} \rightarrow \mathbf{C}$ .

The sequence  $\{c_m (D^m \phi)(0)\}$  so converges to 0. Nevertheless, it is proved in [1.46] that  $C^\infty(\Omega)$  is not locally bounded. In other words, it is always possible to excavate a  $\phi$  for which the magnitude of the  $m$ -th derivative at 0

is as large as we please<sup>1</sup>, e.g., greater than  $1/c_m$ . A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume  $\{\Lambda_j\}$  to converge to some  $\Lambda$  of  $\mathcal{D}'(\Omega)$  and we let  $Q$  run through the compact sets of  $\Omega$ . Next, we define

$$(6.15) \quad S(T, Q) \triangleq \{N \in \mathbf{N}, \exists C \in \mathbf{R}_+ : |\Lambda \phi| \leq C \|\phi\|_N \text{ for all } \phi \text{ of } \mathcal{D}_Q\} \quad (T \in \mathcal{D}(\Omega)) \quad .$$

Such subset of  $\mathbf{N}$  has a minimum  $\omega(T, Q)$ . The following value

$$(6.16) \quad \omega(T) \triangleq \max\{\omega(T, Q) : Q \subseteq \Omega, Q \text{ compact}\} \leq \infty$$

is then the order of  $T$ . Assume, to reach a contradiction, that, after passage to a subsequence,

$$(6.17) \quad \omega(\Lambda_j, Q_j) = j \quad (j = 1, 2, 3, \dots)$$

for some compact  $Q = Q_j$ . By (a) of [6.24],  $Q_j$  cuts  $\text{supp } \Lambda_j$ , say in  $p_j$ . Since  $K$  encloses  $\text{supp } \Lambda_j$ ,  $\{p_j\}$  tends, after passage to a subsequence, to some  $p$  of  $K$ . Choose a positive scalar  $r$  so that

$$(6.18) \quad \overline{B}(p, r) \triangleq \{x \in \mathbf{R}^n : |x - p| \leq r\} \subseteq \Omega \quad .$$

Such closed ball  $\overline{B}(p, r)$  is a compact subset of  $\Omega$ . By (b) of [6.5] (which refers to [1.46])  $\mathcal{D}_{\overline{B}(p, r)}$  is then a Fréchet space. It now follows from [2.6] that  $\{\Lambda_j\}$  is equicontinuous on  $\mathcal{D}_{\overline{B}(p, r)}$ . There so exists<sup>2</sup> a nonnegative integer  $N$  such that

$$(6.19) \quad |\Lambda \phi| \leq C \|\phi\|_N \quad (\phi \in \mathcal{D}_{\overline{B}(p, r)})$$

for some positive constant  $C$ . On the other hand,  $\overline{B}(p, r)$  contains almost all the  $p_j$ 's. Hence

$$(6.20) \quad |\Lambda_N \phi| > C \|\phi\|_N$$

for some  $\phi$  of  $\mathcal{D}_{\overline{B}(p, r)}$ . (b) is then established.

To prove (c), we introduce a sequence  $\{x_m : m \in \mathbf{Z}\}$  of  $\Omega$  that has no limit point. Let  $\{\alpha_m : m \in \mathbf{Z}\}$  be in  $\mathbf{N}$  and so define<sup>3</sup>

$$(6.21) \quad \Lambda : \mathcal{D}(\Omega) \rightarrow \mathbf{C} \quad .$$

$$\phi \mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m} \phi)(x_m)$$

$\Lambda$  belongs to  $\mathcal{D}'(\Omega)$ , since  $\{x_m\}$  has no limit point. Next, we easily check that

$$(6.22) \quad \Lambda_j : \mathcal{D}(\Omega) \rightarrow \mathbf{C} \quad (j \in \mathbf{N})$$

$$\phi \mapsto \sum_{|m| \leq j} (D^{\alpha_m} \phi)(x_m)$$

is also a distribution and that  $\{\Lambda_j\}$  tends to  $\Lambda$  in  $\mathcal{D}'(\Omega)$ . Nevertheless, no  $\Lambda_j$ 's can have common support because  $\{x_m\}$  has no limit point. Our assumption can therefore be dropped.  $\square$

## 9 Exercise 9. Convergence in $\mathcal{D}(\Omega)$ vs. convergence in $\mathcal{D}'(\Omega)$

(a) Prove that a set  $E \subseteq \mathcal{D}(\Omega)$  is bounded if and only if

$$\sup\{|\Lambda \phi| : \phi \in E\} < \infty$$

for every  $\Lambda \in \mathcal{D}(\Omega)$ .

<sup>1</sup>indeed [1.46] provides sufficient tools for constructive proof of this, see the  $\phi_j - \tilde{\phi}_j$  involved in (??).

<sup>2</sup>For more details, see Exercise 2.3.

<sup>3</sup>As  $\Omega = \mathbf{R}$ , the case  $\alpha_m = m$  is the “counterpart” of the series of (a) and the case  $(x_m, \alpha_m) = (m, 0)$  is the *Dirac comb*.

- (b) Suppose  $\{\phi_j\}$  is a sequence in  $\mathcal{D}(\Omega)$  such that  $\{\Lambda\phi_j\}$  is a bounded sequence of numbers, for every  $\Lambda \in \mathcal{D}'(\Omega)$ . Prove that some subsequence of  $\{\phi_j\}$  converges, in the topology of  $\mathcal{D}(\Omega)$ .
- (c) Suppose  $\{\Lambda_j\}$  is a sequence in  $\mathcal{D}'(\Omega)$  such that  $\{\Lambda_j\phi\}$  is bounded, for every  $\phi \in \mathcal{D}(\Omega)$ . Prove that some subsequence of  $\{\Lambda_j\}$  converges in  $\mathcal{D}'(\Omega)$  and that the convergence is uniform on every bounded subset of  $\mathcal{D}(\Omega)$ .  
*Hint: By the Banach-Steinhaus theorem, the restrictions of the  $\Lambda_j$  to  $\mathcal{D}_K$  are equicontinuous. Apply Ascoli's theorem.*

PROOF. Since  $\mathcal{D}(\Omega)$  is a locally convex space (see (b) of [6.4]), [3.18] states that  $E$  is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that  $E = \{\phi_j : j \in \mathbf{N}\}$  is bounded: so is  $\bar{E}$ . By (c) of [6.5], there exists some  $\mathcal{D}_K$  that contains  $\bar{E}$ . Since  $\mathcal{D}_K$  has the Heine-Borel property (see [1.46]),  $\bar{E}$  is  $\tau_K$ -compact. Apply [A4] with the metrizable space  $\mathcal{D}_K$  (see [1.46]) to conclude that  $\bar{E}$  has a  $\tau_K$  limit point. It then follows from (b) of [6.5] that (b) holds.  $\square$

# Annex

## A.1 Mean value and bounded derivatives

**Lemma A.1** A mean value inequality for higher-order derivatives.

If  $\phi \in \mathcal{D}_{[a,b]}$ , then

$$(A.1) \quad \|D^k \phi\|_\infty \leq \|D^p \phi\|_\infty \left( \frac{b-a}{2} \right)^{p-k}$$

for all  $k \leq p$  in  $\mathbf{N}$ .

PROOF. Choose  $a < x_0 \leq (a+b)/2$  first. By the mean value theorem, there exists  $a < x_1 < x_0$  such that

$$(A.2) \quad \phi(x_0) = \phi(x_0) - \phi(a) = D\phi(x_1)(x_0 - a).$$

Repeating the same reasoning for  $D\phi, D^2\phi, \dots, D^p\phi \in \mathcal{D}_{[a,b]}$  yields

$$(A.3) \quad \phi(x_0) = D^0\phi(x_0)$$

$$(A.4) \quad = D^1\phi(x_1)(x_0 - a)$$

$$\vdots$$

$$(A.5) \quad = D^p\phi(x_p)(x_{p-1} - a) \cdots (x_0 - a),$$

for some  $a < x_p < \cdots < x_1 < x_0$ . Hence

$$(A.6) \quad |\phi(x_0)| \leq \|D^p\phi\|_\infty \left( \frac{b-a}{2} \right)^p.$$

Similarly, if  $b > x_0 \geq (a+b)/2$  (interchanging the roles of  $a$  and  $b$ ), the same inequality holds. Thus,

$$(A.7) \quad |\phi(x_0)| \leq \|D^p\phi\|_\infty \left( \frac{b-a}{2} \right)^p \quad (a < x_0 < b),$$

which establishes the result when  $k = 0$ . Finally, applying the latter inequality to  $D^k\phi$  in place of  $\phi$  shows that

$$(A.8) \quad \|D^k\phi\|_\infty \leq \|D^p\phi\|_\infty \left( \frac{b-a}{2} \right)^{p-k}$$

for all  $0 \leq k \leq p$ . □

**Lemma A.2** Impossibility of a nontrivial reversed mean value theorem.

There is no general formula to estimate derivatives from the supremum bound.

PROOF. For pulsation, or *angular frequency*  $\omega \rightarrow \infty$ , we consider

$$(A.9) \quad \begin{aligned} (\phi_\omega, \psi_\omega) : \mathbf{R} &\rightarrow [-1, 1] \times [-1, 1] \\ t &\mapsto (\sin(\omega t), \cos(\omega t)) \end{aligned}$$

so that

$$(A.10) \quad \begin{bmatrix} |D^p \phi_\omega(0)| \\ |D^p \psi_\omega(0)| \end{bmatrix} = [p \equiv 0 \pmod{2}] \cdot \begin{bmatrix} 0 \\ \omega^p \end{bmatrix} + [p \equiv 1 \pmod{2}] \cdot \begin{bmatrix} \omega^p \\ 0 \end{bmatrix}$$

In contrast,

$$(A.11) \quad \|\phi_\omega\|_\infty = \|\psi_\omega\|_\infty = 1.$$

This construction already rules out any general reversed inequality. However, the following smooth example  $\phi(t)$  keeps constant derivative around 0, which is more tractable. Let  $p \in C^\infty(\mathbf{R})$  be defined as 1 on  $]-\infty, 0]$ , 0 on  $[1, \infty[$ , and a strict decay on  $]0, 1[$ . A standard choice is  $p = 1 - h$  when

$$(A.12) \quad h(t) \triangleq \frac{e^{-1/t}}{e^{-1/t} + e^{-1/(1-t)}}$$

for all  $0 < t < 1$ . Inspired by signal processing, we choose three positive parameters:  $A$  (maximum amplitude),  $\tau$  (delay), and  $\omega$  (angular frequency). From now on, response  $\phi(t)$  is the solution of

$$(A.13) \quad \begin{cases} \phi(0) &= 0 \\ \frac{d\phi}{dt}(t) &= Ap(\omega(|t| - \tau)), \end{cases}$$

when time  $t$  ranges over the real line. Equivalently, the function  $\phi = \phi(t)$  is odd and, as time turns positive,

$$(A.14) \quad \phi(t) = A \int_0^t p(\omega(s - \tau)) ds = A \min(t, \tau) + A \int_\tau^{\max(t, \tau)} p(\omega(s - \tau)) ds$$

Note that  $|\phi|$  has maximum

$$(A.15) \quad \|\phi\|_\infty = \tau A + A \int_\tau^{\tau+1/\omega} p(\omega(s - \tau)) ds$$

$$(A.16) \quad = \tau A + \frac{A}{\omega} \int_0^1 p(u) du$$

$$(A.17) \quad < \tau A + \frac{A}{\omega}.$$

The special case  $\tau = 1/\omega = 1/A$  is of great interest. In this case, amplitude at  $t = 0$  is  $A \rightarrow \infty$ , as  $\tau \rightarrow 0$ . In contrast,  $\|\phi\|_\infty < 2$ .  $\square$

## A.2 Dirac's impulse, a physicist's detour

Suppose we wish to record the impact of a particle on a surface. To do so, we introduce the detector response  $\phi$ , which vanishes outside the observation segment  $S$ . A given collision occurs at impact time  $t_Q$ , which means the surface absorbs the energy quantum  $Q$ . Thus, the particle contribution is encoded as the Heaviside step function  $H : t \mapsto [t \geq 0]$ , as time  $t$  ranges over the real line. This models the fact that energy

(a) only changes by an infinitely abrupt increase at reference time  $t_Q = 0$ ,

(b) is conserved over time.

We conclude that the derivative of  $H$ , if it could exist, would satisfy the empirical formula

$$(A.18) \quad \int_{\mathbf{R}} dH = 1 : \frac{dH}{dt} = \begin{cases} \infty & (t = 0) \\ 0 & (t \neq 0) \end{cases}$$

The informal measure  $dH$  describes the behavior of the *Dirac  $\delta$  function*. Interpreted as a Borel measure,  $\delta$  has total mass 1 because

$$(A.19) \quad \int_{\mathbf{R}} \delta = \int_{\mathbf{R}} dH = H(\infty) - H(-\infty) = 1.$$



This expresses that integrating  $\delta$  over time recovers the whole energy  $Q$ . Lebesgue's dominated convergence theorem tunes our energy model, as

$$(A.20) \quad \int_S \phi \delta \xrightarrow[\phi \mapsto 1_S]{(b)} \int_S \delta \stackrel{(a)}{=} [0 \in S].$$

The formalism will be complete when the heuristic  $dH$  is discarded from  $\delta$ 's definition. So,

$$(A.21) \quad \int_{\mathbf{R}} \phi \delta = \int_{\mathbf{R}} \phi dH \quad (\text{generalization of (A.19)})$$

$$(A.22) \quad = \left[ H\phi \right]_{-\infty}^{\infty} - \int_{\mathbf{R}} H d\phi \quad (\text{integration by parts})$$

$$(A.23) \quad = - \int_{\mathbf{R}} H d\phi \quad (\text{supp } \phi \text{ is compact})$$

$$(A.24) \quad = \phi(0)$$

The key point is that the right-hand side in (A.23) does not involve  $dH$ . Moreover, we obtain the filtered response  $\phi \mapsto \phi(0)$ . This motivates the following definitions:

$$(A.25) \quad \Lambda_H(\phi) \triangleq \int_{\mathbf{R}} H\phi dt \quad (\text{expresses } H, \text{ the Heaviside step function})$$

$$(A.26) \quad \Lambda'_H(\phi) \triangleq - \int_{\mathbf{R}} H d\phi = \phi(0) \quad (\text{the weak derivative of } H: \text{ impulse at } 0)$$

$$(A.27) \quad \delta(\phi) \triangleq \phi(0) \quad (\delta \text{ now has a rigorous definition}).$$

The functional  $\delta : \phi \mapsto \phi(0)$  is rigorously defined and conveys the idea of energy increasing by a fixed amount at a point in time. Its extension to all  $\phi \in C_c(\mathbf{R})$  turns  $\delta$  into a Radon measure of norm/total variation  $\|\delta\| = 1$  and support  $\{0\}$ . In the sense of distribution theory,  $\delta$  is a (tempered) distribution of order 0; see [Chapters 6 and 7] of [4]. Notably, its Borel-measure counterpart is the *Dirac measure*

$$(A.28) \quad \delta : E \mapsto [0 \in E]$$

restricted to Borel sets in  $\mathbf{R}$ . Hence the instance of (15)

$$(A.29) \quad \int_{\mathbf{R}} \phi \delta = \delta(\phi) = \phi(0).$$

Convolution of  $\delta$  with shifts  $\phi_s : t \mapsto \phi(s - t)$  extends (A.29), since

$$(A.30) \quad [\delta * \phi](s) \triangleq \int_{-\infty}^{\infty} \phi(s - t) d\delta(t) = \int_{\mathbf{R}} \phi_s d\delta = \delta(\phi_s) = \phi(s) \quad (s \in \mathbf{R}).$$

The Radon measure  $\delta$  now serves as the convolution identity, which is of considerable interest.

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