

Solutions to some exercises from Walter Rudin's *Functional Analysis*

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# Notations and Assumptions

## I Logic

### i Propositional logic operators

Given propositional variables  $p, q$ , the boolean operators  $\neg, \vee, \wedge, \Leftrightarrow, \Rightarrow, \Leftarrow$ , are assigned boolean *truth values* (“true” and “false”) as follows,

$\neg \neg p$  and  $p$  have opposite values:

$\vee$  The *disjunction* (“or”)  $p \vee q$  is true unless  $p$  is false and  $q$  is false.

$\wedge$  The *conjunction* (“and”)  $p \wedge q$  is true if and only if  $p$  is true and  $q$  is true.

$\Leftrightarrow$  The *logical equivalence* expresses *tautologies*:  $p \Leftrightarrow q$  is true unless  $p$  and  $q$  have opposite values. It is easily checked that  $(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (p \Leftarrow q))$ , see the definitions below.

$\Rightarrow$  The logical connection  $p$  *implies*  $q$  is supported by  $\Rightarrow$ :  $p \Rightarrow q$  means *if p then q* and is formally defined as  $\neg p \vee q$ . Note that the “reasoning”  $p \Rightarrow q$  is always true unless  $p$  is true and  $q$  is false. Moreover,  $p \wedge (p \Rightarrow q) \Rightarrow q$  is always true. This deductive rule is known as *modus ponens*.

$\Leftarrow$   $q \Leftarrow p$  means that  $q$  is implied by  $p$  and is defined as  $p \Rightarrow q$ . It is commonly read aloud as “ $q$  if  $p$ ” or “ $q$  is a consequence of  $p$ ”.

See Section 1.3 and Subsection 16.1.3 of [1] for further reading.

### ii Iverson notation

Given a boolean expression  $\phi$ ,  $[\phi]$  returns the truth value of  $\phi$ , encoded as follows,

$$[\phi] \triangleq \begin{cases} 0 & \text{if } \phi \text{ is false;} \\ 1 & \text{if } \phi \text{ is true.} \end{cases}$$

For example,  $[1 > 0] = 1$  but  $[\sqrt{2} \in \mathbf{Q}] = 0$ .

## II Special terms

### i Halmos’ $\Leftrightarrow$ (“if and only if”) and definitions

**iff** is a shorthand for “if and only if”. Splitting **iff** into *if-then* clauses shows that it is just a rewording of the logical equivalence  $\Leftrightarrow$ . All definitions will use the **iff** format, which is consistent with the fact that every definition expresses a tautology.

### ii The assignment operator $\triangleq$

Given variables  $a$  and  $b$ ,  $\triangleq$  is a specialization of  $=$ . We say that  $a \triangleq b$  **iff**  $a$  and  $b$  are assumed to be equal. Usually,  $a \triangleq b$  means that  $a$  is assigned the previously known value  $b$  (some authors write  $a := b$ ) but this is not a limitation. Definitions can be redundant and may overlap.

### III Sets

#### i Subsets and supersets

$\subseteq$  and  $\supseteq$  are the standard symbols for set ordering, as follows:

$$(1) \quad X \subseteq Y \text{ iff } x \in X \Rightarrow x \in Y$$

$$(2) \quad Y \supseteq X \text{ iff } X \subseteq Y.$$

No specific symbol is reserved for strict ordering. If necessary,  $X \neq Y$  will be explicitly stated.

#### ii Special mappings

The identity  $I$  is the special mapping  $I_X = \{(x, x) : x \in X\}$ . Similarly, the projection  $\pi = \{((x, y), x) : x \in X, y \in Y\}$  always exists. Note that  $I$  is the diagonal of  $X^2$ .

#### iii Equinumerosity

$X \equiv Y$  means that there exists a bijection  $\phi$  that maps  $X$  to  $Y$ , which lets us identify  $X$  with  $Y$ . In a metric space context,  $X \equiv Y$  means that  $\phi$  is isometric.

## IV Topological vector spaces

#### i Scalar field

$\mathbf{C}$  extends  $\mathbf{R}$ , which implies that a property, *e.g.*, linearity, that is true on  $\mathbf{C}$  is also true on  $\mathbf{R}$ . The complex case is then a *special case* of the real case. Sometimes, this restriction is significant. Nevertheless, the standard scalar field is  $\mathbf{C}$ , which means that considering  $\mathbf{R}$  instead of  $\mathbf{C}$  makes no difference, unless stated otherwise.

#### ii Vector space bases

Given a vector space  $X$  over  $\mathbf{C}$  (or, more generally, over a field), a subset  $B$  of  $X$  is a basis of  $X$  **iff** the sum

$$(3) \quad \left\{ (z_u)_{u \in B} : z_u \in \mathbf{C}, \{u : z_u \neq 0\} \text{ is finite} \right\} \rightarrow X$$

$$(z_u) \mapsto \sum_{z_u \neq 0} z_u u$$

bijectively maps all *finitely supported*  $(z_u)$  onto  $X$ . The axiom of choice (AC) forces

- (a) the existence of such  $B$  (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [4] with  $B$  playing the role of  $\Lambda$ );
- (b) all bases to have the same cardinal, which is called the *dimension* of  $X$  and is denoted as  $\dim(X)$ .

We now turn to the finite-dimensional case. Note that the 0-dimensional case is the degenerate case  $B = \emptyset$ , which is equivalent to  $X = \{0\}$ . Our first step is to study  $\mathbf{C}^n$  ( $n > 0$ ) the standard  $n$ -dimensional vector space.

#### iii Finite-dimensional spaces

##### The product topology of $\mathbf{C}^n$

$\mathbf{C}^n$  has a standard basis  $1_{\{1\}}, \dots, 1_{\{n\}} \in \{0, 1\}^{\{0, \dots, n-1\}}$  so that  $z_k$  ( $1 \leq k \leq n$ ) is the  $k$ -th component of a given  $(z_1, \dots, z_n) = z_1(1, 0, \dots, 0) + \dots + z_n(0, \dots, 0, 1) \in \mathbf{C}^n$ . Furthermore,  $\mathbf{C}^n$  is endowed with the topology generated by all polydiscs

$$(4) \quad \prod_{i=1}^n \underbrace{\{z_i \in \mathbf{C} : |z_i| < r_i\}}_{D_{r_i}} \quad (r_i > 0).$$

Equivalently, we may equip  $\mathbf{C}^n$  with the Euclidean norm

$$(5) \quad \|z\|_2 \triangleq \sqrt{|z_1|^2 + \cdots + |z_n|^2} \quad (z = (z_1, \dots, z_n) \in \mathbf{C}^n),$$

whose open balls centered at the origin are all

$$(6) \quad B_r \triangleq \{z \in \mathbf{C}^n : \|z\|_2 < r\} \quad (r > 0).$$

To show the equivalence, first set  $r_i = r/\sqrt{n}$ . Hence

$$(7) \quad \prod_{i=1}^n D_{r_i} \subseteq B_r.$$

Conversely, put  $r = \min\{r_1, \dots, r_n\}$  so that

$$(8) \quad B_r \subseteq \prod_{i=1}^n D_{r_i}.$$

### Topology of a finite-dimensional vector space

It is customary to identify any  $n$ -dimensional vector space with  $\mathbf{C}^n$  equipped with the Euclidean norm, see [iii]. To show this, pick an  $n$ -dimensional vector space  $Y$  and let  $f$  be an isomorphism of  $\mathbf{C}^n$  onto  $Y$ . For instance, require that  $u_k = f(e_k)$ , like in [1.20] of [4], as  $\{u_k\}$  is a basis of  $Y$ , see [ii]. It follows from [1.21] of [4] that  $f$  is a homeomorphism. The striking consequence is that

$\{f(U) : U \text{ open in } \mathbf{C}^n\}$  is the only vector space topology for  $Y$ .

Thus,  $Y$  is necessarily locally convex and locally bounded, *i.e.*, normable, see [1.39] of [4]. Note that  $\|y\| = \|f^{-1}(y)\|_2$  ( $y \in Y$ ) is an example of a norm. Now pick an  $n$ -dimensional topological vector space  $W$ , then repeat the same reasoning, first with  $g : \mathbf{C}^n \rightarrow W$ , next with  $h = g \circ f^{-1}$  playing the role of  $f$ . This establishes that the homeomorphism  $h$  maps  $Y$  onto  $W$  and that  $W$  is normable as well. It is now clear that the following assertions are equivalent in the finite-dimensional context,

- (a)  $\dim(W) = \dim(Y)$ ,
- (b)  $W$  and  $Y$  are isomorphic to each other,
- (c)  $W$  and  $Y$  are homeomorphic to each other, they are normable.

Furthermore, the norms on  $W$  and  $Y$  are *equivalent*. That is, for any given norm  $\|\cdot\|$  on  $Y$  and any given norm  $\|\cdot\|_W$  on  $W$ , there exists a positive constant (termed as “modulus of continuity”)  $C = C_h$  such that

$$(9) \quad \|w\|_W \leq C\|y\| \quad ((y, w) \in h),$$

as  $h$  is continuous. The special case  $W = Y$  is that all norms on  $Y$  are equivalent, in the sense that

$$(10) \quad \|h(y)\|_{\text{copy of } Y} \leq C\|y\|.$$

### The standard norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$

When  $\mathbf{C}^n$  is equipped with standard norms  $1, 2, \infty$ , the sharp constant, *i.e.*, the smallest  $C_{i,j}$  such that

$$(11) \quad \|z\|_j \leq C_{i,j}\|z\|_i$$

is easily derived from definitions - see [1.19] of [4] - with the noticeable exception of  $C_{2,1} = \sqrt{n}$ . Indeed,  $\|z\|_1 \leq \sqrt{n}\|z\|_2$  is a special Cauchy-Schwarz inequality, see (1) in [12.2] of [4]. The steps of this classical trick are left to the reader.



# Chapter 1

## Topological Vector Spaces

### 1 Exercise 1. Basic results

Suppose  $X$  is a vector space. All sets mentioned below are understood to be subsets of  $X$ . Prove the following statements from the axioms as given as in section 1.4.

- (a) If  $x, y \in X$  there is a unique  $z \in X$  such that  $x + z = y$ .
- (b)  $0 \cdot x = 0 = \alpha \cdot 0$  ( $\alpha \in \mathbf{C}, x \in X$ ).
- (c)  $2A \subseteq A + A$ .
- (d)  $A$  is convex if and only if  $(s+t)A = sA + tA$  for all positive scalars  $s$  and  $t$ .
- (e) Every union (and intersection) of balanced sets is balanced.
- (f) Every intersection of convex sets is convex.
- (g) If  $\Gamma$  is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of  $\Gamma$  is convex.
- (h) If  $A$  and  $B$  are convex, so is  $A + B$ .
- (i) If  $A$  and  $B$  are balanced, so is  $A + B$ .
- (j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

*Proof.* (a) Such property only depends on the group structure of  $X$ : Each  $x$  in  $X$  has an opposite  $-x$ . Let  $x'$  be any opposite of  $x$  so that  $x - x = 0 = x + x'$ . Thus,  $-x + x - x = -x + x + x'$ , which is equivalent to  $-x = x'$ . So is established the uniqueness of  $-x$ . It is now clear that  $x + z = y$  iff  $z = -x + y$ , which asserts both the existence and the uniqueness of  $z$ .

- (b) Remark that

$$(1.1) \quad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$

$$(1.2) \quad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that  $0 \cdot x = 0$ . So,

$$(1.3) \quad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (-1) \cdot x \Rightarrow -1 \cdot x = -x.$$

Finally,

$$(1.4) \quad \alpha \cdot 0 \stackrel{(1.3)}{=} \alpha \cdot (x + (-1 \cdot x)) = \alpha \cdot x + \alpha \cdot (-1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

(c) Remark that

$$(1.5) \quad 2x = (1+1)x = x + x$$

for every  $x$  in  $X$ , and so conclude that

$$(1.6) \quad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets  $A$  of  $X$ ; which proves (c).

(d) If  $A$  is convex, then

$$(1.7) \quad A \subseteq \frac{s}{s+t}A + \frac{t}{s+t}A \subseteq A;$$

which is

$$(1.8) \quad sA + tA = (s+t)A.$$

Conversely, the special case  $s+t=1$  is

$$(1.9) \quad sA + (1-s)A = A.$$

The latter extends to  $s=0$ , since

$$(1.10) \quad 0A + A \stackrel{(b)}{=} \{0\} + A = A.$$

The extension to  $s=1$  is analogously established (or simply use the fact that  $+$  is commutative!). So ends the proof.

(e) Let  $A$  range over  $B$  a collection of balanced subsets so that

$$(1.11) \quad \alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B$$

for all scalars  $\alpha$  of magnitude  $\leq 1$ . The inclusion  $\alpha \bigcap B \subseteq A$  establishes the first part. Now remark that

$$(1.12) \quad \alpha A \subseteq \bigcup B$$

implies

$$(1.13) \quad \alpha \bigcup B \subseteq \bigcup B;$$

which achieves the proof.

(f) Let  $A$  range over  $C$  a collection of convex subsets so that

$$(1.14) \quad (s+t) \bigcap C \subseteq s \bigcap C + t \bigcap C \stackrel{(d)}{\subseteq} sA + tA \subseteq (s+t)A$$

for all positive scalars  $s, t$ . Inclusions at both extremities force

$$(1.15) \quad s \bigcap C + t \bigcap C = (s+t) \bigcap C.$$

We now conclude from (d) that the intersection of  $C$  is convex. So ends the proof.

(g) Skip all trivial cases  $\Gamma = \emptyset, \{\emptyset\}, \{\{x\}\}, \{\emptyset, \{x\}\}$  then pick  $x_1, x_2$  in  $\bigcup \Gamma$ , so that each  $x_i$  ( $i = 1, 2$ ) lies in some  $C_i \in \Gamma$ . Since  $\Gamma$  is totally ordered by set inclusion, we henceforth assume without loss of generality that  $C_1$  is a subset of  $C_2$ . So,  $x_1, x_2$  are now elements of the convex set  $C_2$ . Every convex combination of our  $x_i$ 's is then in  $C_2 \subseteq \bigcup \Gamma$ . Hence (g).

(h) Simply remark that

$$(1.16) \quad s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B)$$

for all positive scalars  $s$  and  $t$ , then conclude from (d) that  $A+B$  is convex.

(i) Given any  $\alpha$  from the closed unit disc,

$$(1.17) \quad \alpha(A + B) = \alpha A + \alpha B \subseteq A + B.$$

There is no more to prove:  $A + B$  is balanced.

(j) Our proof will be based on the following lemma,

*If  $S$  is nonempty, then each of the following three properties*

- (i)  $S$  is a vector subspace of  $X$ ;
- (ii)  $S$  is convex balanced such that  $S + S = S$ ;
- (iii)  $S$  is convex balanced such that  $\lambda S = S$  ( $\lambda > 0$ )

*implies the other two.*

To prove the lemma, let  $S$  run through all nonempty subsets of  $X$ . First, assume that (i) holds: Clearly, every  $S$  is convex balanced. Moreover,  $S + S \subseteq S$ . Conversely,  $S = S + \{0\} \subseteq S + S$ ; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \quad nS = (n - 1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). Pick  $\lambda > 0$  then choose  $n$  so large that  $1 < n\lambda < n^2$ . Thus,

$$(1.19) \quad nS \stackrel{(1.18)}{\subseteq} S \subseteq n\lambda S \subseteq n^2S,$$

since  $S$  is balanced. For instance, set  $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$ . Dividing the latter inclusions by  $n$  shows that

$$(1.20) \quad S \subseteq \lambda S \subseteq nS \stackrel{(1.18)}{\subseteq} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$(1.21) \quad \alpha S + \beta S \stackrel{(a)}{=} |\alpha|S + |\beta|S \stackrel{(d)}{=} (|\alpha| + |\beta|)S \stackrel{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as  $S$  is balanced. Moreover (under the sole assumption that  $S$  is balanced), this extends to  $|\alpha| + |\beta| = 0$ , as follows,

$$(1.22) \quad \alpha S + \beta S = 0S + 0S \stackrel{(b)}{=} \{0\} \stackrel{(b)}{=} 0S \subseteq S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let  $V$  be a collection of vector spaces of  $X$ , of intersection  $I$  and union  $U$ . First, remark that every member of  $V$  is convex balanced: So is  $I$  (combine (e) with (f)). Next, let  $Y$  range over  $V$  so that

$$(1.23) \quad I + I \subseteq Y + Y \subseteq Y;$$

which yields

$$(1.24) \quad I + I = I$$

(the fact that  $I = I + \{0\} \subseteq I + I$  was tacitly used). It now follows from the lemma's (ii)  $\Rightarrow$  (i) that  $I$  is a vector subspace of  $X$ . Now temporarily assume that  $S$  is totally ordered by set inclusion: Combining (e) with (g) establishes that  $U$  is convex balanced. To show that  $U$  is more specifically a vector subspace, we first remark that such total order implies that either  $Z \subseteq Y$  or  $Y \subseteq Z$ , as  $Z$  ranges over  $V$ . A straightforward consequence is that

$$(1.25) \quad Y \subseteq Y + Z \subseteq Y \cup Z.$$

Another one is that  $Y \cup Z$  ranges over  $V$  as well. Combined with the latter inclusions, this leads to

$$(1.26) \quad U \subseteq U + U \subseteq U.$$

It then follows from the lemma's (ii)  $\Rightarrow$  (i) that  $U$  is a vector subspace of  $X$ . Finally, let  $A, B$  run through all vector subspaces of  $X$ : Combining (h) with (i) proves that  $A + B$  is convex balanced as well. Furthermore,

$$(1.27) \quad A + B \stackrel{(i) \Rightarrow (ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as  $X$  is an abelian group. We now conclude from (ii) that any  $A + B$  is a vector subspace of  $X$ . So ends the proof. □

## 2 Exercise 2. Convex hull

The convex hull of a set  $A$  in a vector space  $X$  is the set of all convex combinations of members of  $A$ , that is the set of all sums  $t_1x_1 + \dots + t_nx_n$  in which  $x_i \in A$ ,  $t_i \geq 0$ ,  $\sum t_i = 1$ ;  $n$  is arbitrary. Prove that the convex hull of a set  $A$  is convex and that is the intersection of all convex sets that contain  $A$ .

*Proof.* The convex hull of a set  $S$  will be denoted by  $\text{co}(S)$ . Remark that  $S \supseteq \text{eq co}(S)$  (to see that, take  $t_1 = 1$  for each  $x_1$  in  $S$ ) and that  $\text{co}(A) \supseteq \text{eq co}(B)$  where  $A \supseteq \text{eq}B$  (obvious).

Our proof will directly derive from (i)  $\Rightarrow$  (iv) in the following lemma,

Let  $S$  be a subset of a vector space  $X$ : Its convex hull  $\text{co}(S)$  is convex and the following statements

- (i)  $S$  is convex;
- (ii)  $s_1S + \dots + s_nS = (s_1 + \dots + s_n)S$  for all positive scalar variables  $s_1, \dots, s_n$ ;
- (iii)  $t_1S + \dots + t_nS = S$  for all positive scalar variables  $t_1, \dots, t_n$  such that  $s_1 + \dots + s_n = 1$ ;
- (iv)  $\text{co}(S) = S$

are equivalent.

From now on, we skip the trivial case  $S = \emptyset$  then only consider nonempty sets. To prove the first part, let  $a, b$  range over  $\text{co}(S)$  so that  $a = t_1x_1 + \dots + t_nx_n$  and  $b = t_{n+1}x_{n+1} + \dots + t_{n+p}x_{n+p}$  for some  $(t_i, x_i)$ . Every sum  $sa + (1-s)b$  ( $0 \leq s \leq 1$ ) is then in the convex hull of  $\{x_1, \dots, x_{n+p}\}$ , since

$$(1.28) \quad sa + (1-s)b = \sum_{i=1}^n st_i x_i + \sum_{i=n+1}^{n+p} (1-s)t_i x_i$$

and

$$(1.29) \quad \sum_{i=1}^n st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s \sum_{i=1}^n t_i + (1-s) \sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets  $S$ , this reads as follows,

$$(1.30) \quad s \text{co}(S) + (1-s) \text{co}(S) \subseteq \text{co}(S);$$

which was our fist goal. We now aim at the equivalence (i)  $\Rightarrow \dots \Rightarrow$  (iv)  $\Rightarrow$  (i): An easy proof by induction makes the implication (i)  $\Rightarrow$  (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii)  $\Rightarrow$  (iv) derives from the definition of the convex hull. We now close the chain with (iv)  $\Rightarrow$  (i), by remarking that  $S$  is convex whether  $S = \text{co}(S)$ . The lemma being proved, let us establish the second part.

To do so, we start from the convexity of  $\text{co}(A)$  then set  $F = \{\text{co}(A)\}$ . We may enrich  $F$  as follows,

$$(1.31) \quad B \in F \Rightarrow B \text{ is convex and contains } A.$$

Note that our initial predicate “[ $F$  only encompasses] all convex sets that contain  $A$ ”, is now the special case

$$(1.32) \quad B \in F \Leftrightarrow B \text{ is convex and contains } A.$$

In any case, the key ingredient is that  $\text{co}(A) \in F$  implies

$$(1.33) \quad \text{co}(A) \supseteq \bigcap_{B \in F} B.$$

Conversely, the next formula

$$(1.34) \quad \text{co}(A) \subseteq \text{co}(B) \stackrel{(i) \Rightarrow (iv)}{=} B \quad (B \in F)$$

is valid and implies

$$(1.35) \quad \text{co}(A) \subseteq \bigcap_{B \in F} B.$$

So ends the proof □

### 3 Exercise 3. Other basic results

Let be  $X$  as topological vector space. All sets mentioned below are understood to be the subsets of  $X$ . Prove the following statements:

- (a) The convex hull of every open set is open.
- (b) If  $X$  is locally convex then the convex hull of every bounded set is bounded.
- (c) If  $A$  and  $B$  are bounded, so is  $A+B$ .
- (d) If  $A$  and  $B$  are compact, so is  $A+B$ .
- (e) If  $A$  is compact and  $B$  is closed, then  $A+B$  is closed.
- (f) The sum of two closed sets may fail to be closed.

*Proof.* (a) Pick an open set  $A$  then let the variables  $V_i$  ( $i = 1, 2, \dots$ ) run through all open subsets of  $A$  so that

$$(1.36) \quad \text{co}(A) \subseteq \bigcup_{t_i} (t_1 V_1 + \dots + t_i V_i + \dots) \subseteq \text{co}(A)$$

given all convex combinations  $t_1 V_1 + \dots + t_i V_i + \dots$ . We know from Section 1.7 of [4] that those sums are open; which achieves the proof.

- (b) Provided a bounded set  $E$ , pick  $V$  a neighbourhood of 0: By (b) of Section 1.14 in [4],  $V$  contains a convex neighbourhood of 0, say  $W$ . There so exists a positive scalar  $s$  such that

$$(1.37) \quad E \subseteq tW \subseteq tV \quad (t > s);$$

which yields

$$(1.38) \quad \text{co}(E) \subseteq \text{co}(tW) = t \text{co}(W) = tW \subseteq tV.$$

So ends the proof.

- (c) At fixed  $V$ , neighbourhood of the origin, we combine the continuousness of  $+$  with Section 1.14 of [4] to conclude that there exists  $U$  a balanced neighborhood of the origin such that

$$(1.39) \quad U + U \subseteq V.$$

Moreover, by the very definition of boundedness,  $A \subseteq rU$  for some positive scalar  $r$ . Similarly,  $B \subseteq sU$  for some positive  $s$ . Finally,

$$(1.40) \quad A + B \subseteq rU + sU \subseteq tU + tU \subseteq tV \quad (t > r, s),$$

since  $U$  is balanced. So ends the proof.

- (d) First,  $A$  and  $B$  are compact: So is  $A \times B$ . Next,  $+$  maps continuously  $A \times B$  onto  $A + B$ . In conclusion,  $A + B$  is compact.
- (e) From now on, we assume that neither  $A$  nor  $B$  is empty, since otherwise the result is trivial. Now pick  $c \in X$  outside  $A + B$ : The result will be established by showing that  $c$  is not in the closure of  $A + B$ .

To do so, we let the variable  $a$  range over  $A$ : Every set  $a + B$  is closed as well, see Section 1.7 of [4]. Trivially,  $a + B \neq c$ : By Section 1.10 of [4], there so exists  $V = V(a)$  a neighborhood of the origin such that

$$(1.41) \quad (a + B + V) \cap (c + V) = \emptyset.$$

Moreover, there are finitely many  $a + V$ , say  $a_1 + V_1, a_2 + V_2, \dots$ , whose union  $U$  contains the compact set  $A$ . Therefore,

$$(1.42) \quad A + B \subseteq U + B.$$

Now define

$$(1.43) \quad W \triangleq V_1 \cap V_2 \cap \dots,$$

so that

$$(1.44) \quad (a_i + B + V_i) \cap (c + W) \stackrel{(1.41)}{=} \emptyset \quad (i = 1, 2, \dots).$$

As a conclusion,  $c$  is not in the closure of  $U + B$ . Finally, (1.42) asserts that  $c$  is not in  $\overline{A + B}$  either; which achieves the proof.

**Corollary:** If  $B$  is the closure of a set  $S$ , then

$$(1.45) \quad A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B$$

by (b) of Section 1.13 of [4] (since  $A$  is closed; see Section 1.12, from the same source). The special case  $A = \{x\}$ ,  $B = X$  will occur in the proof of Exercise 15 in chapter 2.

- (f) The last proof will consist in exhibiting a counterexample. To do so, let  $f$  be any continuous mapping of the real line such that

- (i)  $f(x) + f(-x) \neq 0 \quad (x \in \mathbf{R});$
- (ii)  $f$  vanishes at infinity.

For instance, we may combine (ii) with  $f$  even and  $f > 0$  by setting  $f(x) = 2^{-|x|}$ ,  $f(x) = e^{-x^2}$ ,  $f(x) = 1/(1 + |x|)$ , ..., and so on.

As a continuous function,  $f$  has closed graph  $G$ , see [2.14] of [4]. Moreover, (i) implies that the origin  $(0, 0) \neq (x - x, f(x) + f(-x))$  is not in  $G + G$ . On the other hand,

$$(1.46) \quad \{(0, f(n) + f(-n)) : n = 1, 2, \dots\} \subseteq G + G.$$

Now the key ingredient is that

$$(1.47) \quad (0, f(n) + f(-n)) \xrightarrow[n \rightarrow \infty]{(ii)} (0, 0).$$

We have so constructed a sequence in  $G + G$  that converges outside  $G + G$ . So ends the proof. □

#### 4 Exercise 4. A nonempty set whose interior is not

Let be  $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$ . Show that  $B$  is balanced but that its interior is not.

*Proof.* It is obvious that the nonempty set  $B$  contains the origin  $(0, 0)$ . Additionally, its interior  $B^\circ$  is nonempty as well. Indeed, the following set

$$(1.48) \quad \{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2| < 1/2\} \subseteq B$$

is a neighborhood of  $(1, 2) \in B$ . Moreover,  $B$  is balanced, since

$$(1.49) \quad |\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \quad (|\alpha| \leq 1)$$

for all  $(z_1, z_2)$  in  $B$ . Nevertheless, the nonempty set  $B^\circ$  is not balanced, what we now establish by showing that  $(0, 0) \notin B^\circ$ . To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$(1.50) \quad U \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2| < r\} \subseteq B$$

for some positive  $r$ . Clearly,  $U$  contains  $(r/2, 0)$ , and that special case  $(r/2, 0) \in B$  now contradicts the definition of  $B$ . So ends the proof.  $\square$

## 5 Exercise 5. A first restatement of boundedness

Consider the definition of “bounded set” given in Section 1.6. Would the content of this definition be altered if it were required merely required that to every neighbourhood  $V$  of  $0$  corresponds some  $t > 0$  such that  $E \subseteq tV$ ?

*Proof.* The answer is: No. To prove it, start from (a) of Section 1.14:  $V$  contains  $W$ , a balanced neighbourhood of  $0$ . Assume that  $E$  is bounded in this weaker sense, i.e., there exists a positive  $t$  that satisfies

$$(1.51) \quad E \subseteq tW.$$

Thus,

$$(1.52) \quad E \subseteq tW \subseteq sW \subseteq sV \quad (s > t),$$

since  $W$  is balanced. We so reach the definition given in Section 1.6: The two ones are equivalent.  $\square$

## 6 Exercise 6. A second restatement of boundedness

*Prove that a set E in a topological vector space is bounded if and only if every countable subset of E is bounded.*

*Proof.* It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighbourhood of the origin: No counting number  $n = 1, 2, \dots$ , verifies  $E \subseteq nV$  (see Exercise 1 in Chapter 1). In other words, there exists a sequence  $\{x_1, \dots, x_n, \dots\} \subseteq E$  such that

$$(1.53) \quad x_n \notin nV.$$

As a consequence,  $x_n/n$  fails to converge to 0 as n tends to  $\infty$ . In contrast,  $1/n$  succeeds. It then follows from Section 1.30 that  $\{x_1, \dots, x_n, \dots\}$  is not bounded. So ends the proof.  $\square$

## 7 Exercise 7. Metrizability and number theory

Let  $X$  be the vector space of all complex functions on the unit interval  $[0, 1]$ , topologized by the family of seminorms

$$p_x(f) = |f(x)| \quad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology.

Show that there is a sequence  $\{f_n\}$  in  $X$  such that (a)  $\{f_n\}$  converges to 0 as  $n \rightarrow \infty$ , but (b) if  $\{\gamma_n\}$  is any sequence of scalars such that  $\gamma_n \rightarrow \infty$  then  $\{\gamma_n f_n\}$  does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as  $[0, 1]$ .) This shows that metrizability cannot be omitted in (b) of Theorem 1.28.

### 1 Justifying the terminology

*Proof.* The family of seminorms  $p_x$  is separating: The collection  $\mathcal{B}$  of all finite intersections of the sets

$$(1.54) \quad V(x, k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0, 1], k = 1, 2, 3, \dots)$$

is therefore a local base for a topology  $\tau$  on  $X$ , see Section 1.37 of [4]. Thus, given  $\{g_n\} \subseteq X$ , the following inequalities, expressed in Iverson's notation, hold:

$$(1.55) \quad \sum_{n=1}^{\infty} [g_n \notin \cap_{i=1}^m U_i] \leq \sum_{n=1}^{\infty} \sum_{i=1}^m [g_n \notin U_i] = \sum_{i=1}^m \sum_{n=1}^{\infty} [g_n \notin U_i].$$

Now assume that  $\{f_n\}$   $\tau$ -converges to some  $f$ . By definition,

$$(1.56) \quad \sum_{n=1}^{\infty} [f_n - f \notin W] < \infty \quad (W \in \mathcal{B}).$$

The special case  $W = V(x, k)$  implies that, for a fixed  $k$ ,  $|f_n(x) - f(x)| < 2^{-k}$  for all but finitely many  $n$ . In other words,  $\{f_n(x)\}$  converges to  $f(x)$ . Conversely, assume that  $\{f_n\}$  does not  $\tau$ -converge. This implies that, for every  $f \in X$ , there exist finitely many  $V(x_1, k_1), \dots, V(x_m, k_m)$  such that

$$(1.57) \quad \sum_{n=1}^{\infty} [f_n - f \notin \cap_{i=1}^m V(x_i, k_i)] = \infty.$$

In (1.55), the special case  $g_n = f_n - f$  and  $U_i = V(x_i, k_i)$  then gives

$$(1.58) \quad \sum_{n=1}^{\infty} [f_n - f \notin \cap_{i=1}^m V(x_i, k_i)] \stackrel{(1.55)}{\leq} \sum_{i=1}^m \sum_{n=1}^{\infty} [f_n - f \notin V(x_i, k_i)] = \infty.$$

It is now obvious that

$$(1.59) \quad \sum_{n=1}^{\infty} [f_n - f \notin V(x_i, k_i)] = \infty$$

for some  $i$ , which shows that  $\{f_n(x_i)\}$  does not converge to  $f(x_i)$ . Thus,  $\tau$ -convergence coincides with pointwise convergence on  $X$ .  $\square$

### 2 Proof (with the given hint)

We now prove the second part by constructing a specific sequence  $\{f_n\}$  that satisfies both (a) and (b). Indeed, the hint suggests that there exists a bijection

$$(1.60) \quad \begin{aligned} \phi : \{(\theta_n) : \theta_n \xrightarrow{n \rightarrow \infty} 0\} &\rightarrow [0, 1] \\ (\theta_1, \dots, \theta_n, \dots) &\mapsto x. \end{aligned}$$

*Proof.* We set

$$(1.61) \quad f_n(x) \triangleq \theta_n \quad (x = \phi(\theta_1, \dots, \theta_n, \dots))$$

so that  $\{f_n\}$  tends pointwise to 0. Note that, with this construction, the special case

$$(1.62) \quad x_\gamma \triangleq \phi\left(1/\sqrt{1+|\gamma_1|}, \dots, 1/\sqrt{1+|\gamma_n|}, \dots\right)$$

outputs

$$(1.63) \quad \gamma_n f_n(x_\gamma) = \gamma_n / \sqrt{1 + |\gamma_n|} \xrightarrow{n \rightarrow \infty} \infty,$$

given a sequence  $\{\gamma_n\}$  diverging to  $\infty$ . This proves (b), since  $\{\gamma_n f_n(x_\gamma)\}$  diverges.  $\square$

### 3 Proof with binary expansions (no hint)

We rely on the following assertion: Every irrational number has a binary expansion that is not eventually periodic.

*Proof.* More precisely, there exists a bijective sum

$$(1.64) \quad \begin{aligned} \sigma : \{\beta \in \{0, 1\}^{\mathbb{N}_+} : \beta \text{ is not eventually periodic}\} &\rightarrow [0, 1] \setminus \mathbf{Q} \\ (\beta_1, \dots, \beta_n, \dots) &\mapsto \sum_{k=1}^{\infty} \beta_k 2^{-k}. \end{aligned}$$

A suitable  $\{f_n\}$  can be defined as follows:

$$(1.65) \quad f_n(x) \triangleq \begin{cases} 2^{-(\beta_1 + \dots + \beta_n)} & (x = \sigma(\beta_1, \dots, \beta_n, \dots) \notin \mathbf{Q}) \\ 0 & (x \in \mathbf{Q}). \end{cases}$$

Indeed, we note that every bit stream  $\sigma^{-1}(x)$  has infinitely many 1's, which implies that  $f_n(x) \xrightarrow{n \rightarrow \infty} 0$ . Next, pick an arbitrary  $\gamma_n \rightarrow \infty$ . Thus, for any positive integer  $k$ ,  $\gamma_n > 4^k$  for all sufficiently large  $n$ , say  $n > N_k$ . We select  $n_k > N_k$  so large that

$$(1.66) \quad n_{k+1} - n_k > k.$$

The crucial point is that the sequence  $1_{\{n_1, n_2, \dots\}}$  is not eventually periodic. Moreover, the particular choice

$$(1.67) \quad \beta^\gamma \triangleq 1_{\{n_1, n_2, \dots\}}$$

implies

$$(1.68) \quad \beta_1^\gamma + \dots + \beta_{n_1}^\gamma + \dots + \beta_{n_k}^\gamma = k.$$

Finally, (1.65) and (1.68) together yield

$$(1.69) \quad \gamma_{n_k} f_{n_k}(\sigma(\beta^\gamma)) = \gamma_{n_k} / 2^k > 2^k \xrightarrow{k \rightarrow \infty} \infty.$$

In conclusion, every sequence of scalars  $\gamma_n$  such that  $\gamma_n \rightarrow \infty$  contains a subsequence  $\{\gamma_{n_k}\}$  that causes  $\{\gamma_{n_k} f_{n_k}\}$  to diverge. This is (b).  $\square$

## 9 Exercise 9. Quotient map

Suppose

- (a)  $X$  and  $Y$  are topological vector spaces,
- (b)  $\Lambda : X \rightarrow Y$  is linear.
- (c)  $N$  is a closed subspace of  $X$ ,
- (d)  $\pi : X \rightarrow X/N$  is the quotient map, and
- (e)  $\Lambda x = 0$  for every  $x \in N$ .

Prove that there is a unique  $f : X/N \rightarrow Y$  which satisfies  $\Lambda = f \circ \pi$ , that is,  $\Lambda x = f(\pi(x))$  for all  $x \in X$ . Prove that  $f$  is linear and that  $\Lambda$  is continuous if and only if  $f$  is continuous. Also,  $\Lambda$  is open if and only if  $f$  is open.

*Proof.* Bear in mind that  $\pi$  continuously maps  $X$  onto the topological (Hausdorff) space  $X/N$ , since  $N$  is closed (see 1.41 of [4]). Moreover, the equation  $\Lambda = f \circ \pi$  has necessarily a unique solution, which is the binary relation

$$(1.70) \quad f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y.$$

To ensure that  $f$  is actually a mapping, simply remark that the linearity of  $\Lambda$  implies

$$(1.71) \quad \Lambda x \neq \Lambda x' \Rightarrow \pi x \neq \pi x'.$$

It straightforwardly derives from (1.70) that  $f$  inherits linearity from  $\pi$  and  $\Lambda$ .

**Remark.** The special case  $N = \{\Lambda = 0\}$ , i.e.,  $\Lambda x = 0$  iff  $x \in N$  (cf. (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strengthening of (e) yields

$$(1.72) \quad f(\pi x) = 0 \stackrel{(1.70)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that  $f$  is also one-to-one.

Now assume  $f$  to be continuous. Then so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [4]. Conversely, if  $\Lambda$  is continuous, then for each neighborhood  $V$  of  $0_Y$  there exists a neighborhood  $U$  of  $0_X$  such that

$$(1.73) \quad \Lambda(U) = f(\pi(U)) \subseteq V.$$

Since  $\pi$  is open (1.41 (a) of [4]),  $\pi(U)$  is a neighborhood of  $N = 0_{X/N}$ : This is sufficient to establish that the linear mapping  $f$  is continuous. If  $f$  is open, so is  $\Lambda = f \circ \pi$ , by 1.41 (a) of [4]. To prove the converse, remark that every neighborhood  $W$  of  $0_{X/N}$  satisfies

$$(1.74) \quad W = \pi(V)$$

for some neighborhood  $V$  of  $0_X$ . So,

$$(1.75) \quad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if  $\Lambda$  is open, then  $f(W)$  is a neighborhood of  $0_Y$ . So ends the proof.  $\square$

## 10 Exercise 10. An open mapping theorem

Suppose that  $X$  and  $Y$  are topological vector spaces,  $\dim Y < \infty$ ,  $\Lambda : X \rightarrow Y$  is linear, and  $\Lambda(X) = Y$ .

(a) Prove that  $\Lambda$  is an open mapping.

(b) Assume, in addition, that the null space of  $\Lambda$  is closed, and prove that  $\Lambda$  is continuous.

*Proof.* Discard the trivial case  $\Lambda = 0$  and assume that  $\dim Y > 0$ . From now on,  $Y$  has a basis  $B = \{e, e', \dots\}$  and  $W$  is an arbitrary neighborhood of 0. Since addition is continuous, there exists a balanced open  $V \subseteq Y$  such that

$$(1.76) \quad \sum_e V \subseteq W.$$

Moreover, the surjectivity of  $\Lambda$  allows us to pick  $x_e$  with  $\Lambda x_e = e$ . This implies that every  $y = \sum_e y_e e$  ( $y_e \in \mathbf{C}$ ) can be expressed as

$$(1.77) \quad y = \sum_e y_e \Lambda x_e.$$

Note that  $\{x_e : e \in B\}$  is bounded (as a finite set). Hence

$$(1.78) \quad \{x_e : e \in B\} \subseteq sV$$

for some  $s > 0$ . Combining this with (1.77) yields

$$(1.79) \quad y \in \sum_e s y_e \Lambda(V).$$

Using (1.76) and the balancedness of  $\Lambda(V)$ , we conclude that

$$(1.80) \quad y \in \sum_e \Lambda(V) \subseteq \Lambda(W) \quad (\|y\|_\infty < 1/s).$$

This proves (a) in the case where  $Y = \mathbf{C}^n$  and  $B$  is the standard basis. The general case now follows from [iii].

To prove (b), assume that the null space  $\{\Lambda = 0\}$  is closed and let  $f, \pi$  be as in Exercise 1.9,  $\{\Lambda = 0\}$  playing the role of  $N$ . Since  $\Lambda$  is onto, the first isomorphism theorem (see Exercise 1.9) asserts that  $f$  is an isomorphism of  $X/N$  onto  $Y$ . We now conclude with the help of [iii] that  $f$  is a homeomorphism of  $X/N$  onto  $Y$ . We have thus established that  $f$  is continuous: So is  $\Lambda = f \circ \pi$ .  $\square$

## 12 Exercise 12. Topology stays, completeness leaves

Suppose  $d_1(x, y) = |x - y|$ ,  $d_2(x, y) = |\phi(x) - \phi(y)|$ , where  $\phi(x) = x/(1 + |x|)$ . Prove that  $d_1$  and  $d_2$  are metrics on  $\mathbf{R}$  which induce the same topology, although  $d_1$  is complete and  $d_2$  is not.

*Proof.* First, each  $d_i$  ( $i = 1, 2$ ) induces a topology  $\tau_i$  whose open balls are all

$$(1.81) \quad B_i(a, r) \triangleq \{x \in \mathbf{R} : d_i(a, x) < r\} \quad (a \in \mathbf{R}, r > 0).$$

Next, remark that the monotonically increasing mapping  $\phi : \mathbf{R} \rightarrow ]-1, 1[$  is odd and that

$$(1.82) \quad \phi(x) \xrightarrow{x \rightarrow \infty} 1.$$

$\phi$  is therefore a  $\tau_1$ -homeomorphism of  $\mathbf{R}$  onto  $] -1, 1 [$ . A first consequence is that, at fixed  $a \in \mathbf{R}$ , given any positive scalar  $\varepsilon$ , the  $\tau_1$ -continuousness of  $\phi$  supplies an open ball  $B_1(a, \eta)$  on which  $|\phi(a) - \phi| < \varepsilon$ . In terms of balls  $B_i$ , this reads as follows,

$$(1.83) \quad B_1(a, \eta) \subseteq B_2(a, \varepsilon).$$

The second consequence is that the  $\tau_1$ -continuousness of  $\phi^{-1}$  yields similar inclusions

$$(1.84) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta')$$

provided  $\eta' > 0$ . At arbitrary  $\varepsilon$ , the special case  $\eta' = \eta$  is the concatenation

$$(1.85) \quad B_2(a, \varepsilon') \subseteq B_1(a, \eta) \subseteq B_2(a, \varepsilon);$$

which proves that  $\tau_1 = \tau_2$ . Finally, all inequalities  $n < i < j$  over  $\mathbf{N}$  together yield

$$(1.86) \quad d_2(i, j) = |\phi(i) - \phi(j)| \xrightarrow{n \rightarrow \infty} 0.$$

The sequence  $n = 0, 1, 2, \dots$  is therefore  $\tau_2$ -Cauchy. We will nevertheless establish that it  $\tau_2$ -diverges. To do so, we start by offering the  $\tau_2$ -converge to some  $\lambda$ : The triangle inequality immediately dismiss that assumption, as follows,

$$(1.87) \quad d_2(0, \lambda) \geq d_2(0, n) - d_2(\lambda, n) = \phi(n) - d_2(\lambda, n) \xrightarrow{n \rightarrow \infty} 1.$$

We then conclude that  $d_2$  fails to be complete. □

## 14 Exercise 14. $\mathcal{D}_K$ equipped with other seminorms

Put  $K = [0, 1]$  and define  $\mathcal{D}_K$  as in Section 1.46. Show that the following three families of seminorms (where  $n = 0, 1, 2, \dots$ ) define the same topology on  $\mathcal{D}_K$ . If  $D = d/dx$ :

$$(a) \|D^n f\|_\infty = \sup\{|D^n f(x)| : \infty < x < \infty\}$$

$$(b) \|D^n f\|_1 = \int_0^1 |D^n f(x)| dx$$

$$(c) \|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}.$$

*Proof.* Let us equip  $\mathcal{D}_K$  with the inner product  $\langle f|g \rangle = \int_0^1 f \bar{g} dx$  so that  $\langle f|f \rangle = \|f\|_2^2$ . The following

$$(1.88) \quad \int_0^1 1 |D^n f| \leq \|1\|_2 \|D^n f\|_2$$

is then a Cauchy-Schwarz inequality, see Theorem 12.2 of [4]. We so obtain

$$(1.89) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

since  $K$  has length 1. Obviously, the support of  $D^n f$  lies in  $K$ , hence the below equality

$$(1.90) \quad |D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x |D^{n+1} f| \leq \|D^{n+1} f\|_1.$$

Take the supremum over all  $|D^n f(x)|$ : Combining (1.89) with (1.90) now reads as follows,

$$(1.91) \quad \|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \dots < \infty.$$

Finally, put

$$(1.92) \quad V_n^{(i)} \triangleq \{f \in \mathcal{D}_K : \|f\|_i < 2^{-n}\},$$

$$(1.93) \quad \mathcal{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\},$$

so that (1.91) is mirrored by neighborhood inclusions, provided  $i = 1, 2, \infty$ :

$$(1.94) \quad V_n^{(1)} \supseteq V_n^{(2)} \supseteq V_n^{(\infty)} \supseteq V_{n+1}^{(1)} \supseteq \dots$$

Their subchains  $V_n^{(i)} \supseteq V_{n+1}^{(i)}$  turn  $\mathcal{B}^{(i)}$  into a local base of a topology  $\tau_i$ . The whole chain (1.94) then forces

$$(1.95) \quad \tau_1 \subseteq \tau_2 \subseteq \tau_\infty \subseteq \tau_1;$$

which achieves the proof.  $\square$

## 16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Do the same for  $C^\infty(\Omega)$  (Section 1.46).

**Comment** This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms  $p_n$ , then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [4] The proof consists in combining trivial consequences of the local base definition with a well-known result (for instance, see [2.6] in [3]) about intersection of nonempty compact sets.

**Lemma 1** Let  $X$  be a topological space with a countable local base  $\{V_n : n = 1, 2, 3, \dots\}$ . If  $\tilde{V}_n = V_1 \cap \dots \cap V_n$ , then every subsequence  $\{\tilde{V}_{\varphi(n)}\}$  is a decreasing (, i.e.,  $\tilde{V}_{\varphi(n)} \supseteq \tilde{V}_{\varphi(n+1)}$ ) local base of  $X$ .

*Proof.* The decreasing property is trivial. Now remark that  $V_n \supseteq \tilde{V}_n$ : This shows that  $\{\tilde{V}_n\}$  is a local base of  $X$ . Then so is  $\{\tilde{V}_{\varphi(n)}\}$ , since  $\tilde{V}_n \supseteq \tilde{V}_{\varphi(n)}$ .  $\square$

The following special case  $V_n = \tilde{V}_n$  is one of the key ingredients:

**Corollary 1 (special case  $V_n = \tilde{V}_n$ )** Under the same notations of Lemma 1, if  $\{V_n\}$  is a decreasing local base, then so is  $\{V_{\varphi(n)}\}$ .

**Corollary 2** If  $\{Q_n\}$  is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence  $\{Q_{\varphi(n)}\}$  also satisfies these conditions. Furthermore, if  $\tau_Q$  is the  $C(\Omega)$ 's (respectively  $C^\infty(\Omega)$ 's) topology of the seminorms  $p_n$ , as defined in section 1.44 (respectively 1.46), then the seminorms  $p_{\varphi(n)}$  define the same topology  $\tau_Q$ .

*Proof.* Let  $X$  be  $C(\Omega)$  topologized by the seminorms  $p_n$  (the case  $X = C^\infty(\Omega)$  is proved the same way). If  $V_n = \{p_n < 1/n\}$ , then  $\{V_n\}$  is a decreasing local base of  $X$ . Moreover,

$$(1.96) \quad Q_{\varphi(n)} \subseteq \overset{\circ}{Q}_{\varphi(n)+1} \subseteq Q_{\varphi(n)+1} \subseteq Q_{\varphi(n+1)}.$$

Thus,

$$(1.97) \quad Q_{\varphi(n)} \subseteq \overset{\circ}{Q}_{\varphi(n+1)}.$$

In other words,  $Q_{\varphi(n)}$  satisfies the conditions specified in section 1.44.  $\{p_{\varphi(n)}\}$  then defines a topology  $\tau_{Q_{\varphi}}$  for which  $\{V_{\varphi(n)}\}$  is a local base. So,  $\tau_{Q_{\varphi}} \subseteq \tau_Q$ . Conversely, the above corollary asserts that  $\{V_{\varphi(n)}\}$  is a local base of  $\tau_Q$ , which yields  $\tau_Q \subseteq \tau_{Q_{\varphi}}$ .  $\square$

**Lemma 2** If a sequence of compact sets  $\{Q_n\}$  satisfies the conditions specified in section 1.44, then every compact set  $K$  lies in allmost all  $Q_n^\circ$ , i.e., there exists  $m$  such that

$$(1.98) \quad K \subseteq \overset{\circ}{Q}_m \subseteq \overset{\circ}{Q}_{m+1} \subseteq \overset{\circ}{Q}_{m+2} \subseteq \dots$$

*Proof.* The following definition

$$(1.99) \quad C_n \triangleq K \setminus \overset{\circ}{Q}_n$$

shapes  $\{C_n\}$  as a decreasing sequence of compact<sup>1</sup> sets. We now suppose (to reach a contradiction) that no  $C_n$  is empty and so conclude<sup>2</sup> that the  $C_n$ 's intersection contains a point that is not in any  $Q_n^\circ$ . On the other hand, the conditions specified in [1.44] force the  $Q_n^\circ$ 's collection to be an open cover. This contradiction reveals that  $C_m = \emptyset$ , , i.e.,  $K \subseteq Q_m^\circ$ , for some  $m$ . Finally,

$$(1.100) \quad K \subseteq \overset{\circ}{Q}_m \subseteq Q_m \subseteq \overset{\circ}{Q}_{m+1} \subseteq Q_{m+1} \subseteq \overset{\circ}{Q}_{m+2} \subseteq \dots$$

$\square$

We are now in a fair position to establish the following:

<sup>1</sup>See (b) of 2.5 of [3].

<sup>2</sup>In every Hausdorff space, the intersection of a decreasing sequence of nonempty compact sets is nonempty. This is a corollary of 2.6 of [3].

**Theorem** *The topology of  $C(\Omega)$  does not depend on the particular choice of  $\{K_n\}$ , as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of  $C^\infty(\Omega)$ , as long as this sequence satisfies the conditions specified in section 1.44.*

*Proof.* With the second corollary's notations,  $\tau_K = \tau_{K_\lambda}$ , for every subsequence  $\{K_{\lambda(n)}\}$ . Similarly, let  $\{L_n\}$  be another sequence of compact subsets of  $\Omega$  that satisfies the condition specified in [1.44], so that  $\tau_L = \tau_{L_\kappa}$  for every subsequence  $\{L_{\kappa(n)}\}$ . Now apply the above Lemma 2 with  $K_i$  ( $i = 1, 2, 3, \dots$ ) and so conclude that  $K_i \subseteq L_{m_i}^\circ \subseteq L_{m_i+1}^\circ \subseteq \dots$  for some  $m_i$ . In particular, the special case  $\kappa_i = m_i + i$  is

$$(1.101) \quad K_i \subseteq \overset{\circ}{L}_{\kappa_i}.$$

Let us reiterate the above proof with  $K_n$  and  $L_n$  in exchanged roles then similarly find a subsequence  $\{\lambda_j : j = 1, 2, 3, \dots\}$  such that

$$(1.102) \quad L_j \subseteq \overset{\circ}{K}_{\lambda_j}$$

Combine (1.101) with (1.102) and so obtain

$$(1.103) \quad K_1 \subseteq \overset{\circ}{L}_{\kappa_1} \subseteq L_{\kappa_1} \subseteq \overset{\circ}{K}_{\lambda_{\kappa_1}} \subseteq K_{\lambda_{\kappa_1}} \subseteq \overset{\circ}{L}_{\kappa_{\lambda_{\kappa_1}}} \subseteq \dots,$$

which means that the sequence  $Q = (K_1, L_{\kappa_1}, K_{\lambda_{\kappa_1}}, \dots)$  satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$(1.104) \quad \tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\kappa} = \tau_L.$$

So ends the proof □

## 17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that  $f \mapsto D^\alpha f$  is a continuous mapping of  $C^\infty(\Omega)$  into  $C^\infty(\Omega)$  and also of  $\mathcal{D}_K$  into  $\mathcal{D}_K$ , for every multi-index  $\alpha$ .

*Proof.* In both cases,  $D^\alpha$  is a linear mapping. It is then sufficient to establish continuity at the origin. We begin with the  $C^\infty(\Omega)$  case.

Let  $U$  be an arbitrary neighborhood of the origin. There exists  $N$  such that  $U$  contains

$$(1.105) \quad V_N = \{\varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \leq N, x \in K_N\} < 1/N\}.$$

Now pick  $g$  in  $V_{N+|\alpha|}$  so that

$$(1.106) \quad \max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N+|\alpha|}.$$

(the fact that  $K_N \subseteq K_{N+|\alpha|}$  was tacitly used). The special case  $\gamma = \beta + \alpha$  yields

$$(1.107) \quad \max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

$$(1.108) \quad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad , i.e., \quad D^\alpha(V_{N+|\alpha|}) \subseteq V_N,$$

which establishes the continuity of  $D^\alpha : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$ .

To prove the continuity of the restriction  $D^\alpha|_{\mathcal{D}_K} : \mathcal{D}_K \rightarrow \mathcal{D}_K$ , we first remark that the collection of the  $V_N \cap \mathcal{D}_K$  is a local base of the subspace topology of  $\mathcal{D}_K$ .  $V_{N+|\alpha|} \cap \mathcal{D}_K$  is then a neighborhood of 0 in this topology. Furthermore,

$$(1.109) \quad D^\alpha|_{\mathcal{D}_K}(V_{N+|\alpha|} \cap \mathcal{D}_K) = D^\alpha(V_{N+|\alpha|} \cap \mathcal{D}_K)$$

$$(1.110) \quad \subseteq D^\alpha(V_{N+|\alpha|}) \cap D^\alpha(\mathcal{D}_K)$$

$$(1.111) \quad \subseteq V_N \cap \mathcal{D}_K \quad (\text{see (1.108)})$$

So ends the proof.  $\square$

# Chapter 2

## Completeness

### 3 Exercise 3. An equicontinuous sequence of measures

Put  $K = [-1, 1]$ ; define  $\mathcal{D}_K$  as in section 1.46 (with  $\mathbf{R}$  in place of  $\mathbf{R}^n$ ). Suppose  $\{f_n\}$  is a sequence of Lebesgue integrable functions such that  $\Lambda\varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t)\varphi(t)dt$  exists for every  $\varphi \in \mathcal{D}_K$ . Show that  $\Lambda$  is a continuous linear functional on  $\mathcal{D}_K$ . Show that there is a positive integer  $p$  and a number  $M < \infty$  such that

$$\left| \int_{-1}^1 f_n(t)\varphi(t)dt \right| \leq M \|D^p\varphi\|_\infty$$

for all  $n$ . For example, if  $f_n(t) = n^3 t$  on  $[-1/n, 1/n]$  and 0 elsewhere, show that this can be done with  $p = 1$ . Construct an example where it can be done with  $p = 2$  but not with  $p = 1$ .

We will also consider the case  $p = 0$ . Since all supports of  $\varphi, \varphi', \varphi'', \dots$ , are in  $K$ , we make a specialization of the mean value theorem:

**Lemma** If  $\varphi \in \mathcal{D}_{[a,b]}$ , then

$$(2.1) \quad \|D^\alpha \varphi\|_\infty \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (\alpha = 0, 1, \dots, p)$$

at every order  $p = 0, 1, 2, \dots$ ; where  $\lambda$  is the length  $|b - a|$ .

*Proof.* Let  $x_0$  be in  $(a, b)$ . We first consider the case  $x_0 \leq c = (a + b)/2$ : The mean value theorem asserts that there exists  $x_1$  ( $a < x_1 < x_0$ ), such that

$$(2.2) \quad \varphi(x_0) = \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a).$$

Since every  $D^p \varphi$  lies in  $\mathcal{D}_{[a,b]}$ , a straightforward proof by induction shows that there exists a partition  $a < \dots < x_p < \dots < x_0$  such that

$$(2.3) \quad \varphi(x_0) = D^0 \varphi(x_0)$$

$$(2.4) \quad = D^1 \varphi(x_1)(x_0 - a) \\ = \dots$$

$$(2.5) \quad = D^p \varphi(x_p)(x_0 - a) \cdots (x_{p-1} - a),$$

for all  $p$ . More compactly,

$$(2.6) \quad D^\alpha \varphi(x_0) = D^p \varphi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a);$$

which yields,

$$(2.7) \quad |D^\alpha \varphi(x)| \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c])$$

The case  $x_0 \geq c$  outputs a “reversed” result, with  $b > \dots > x_p > \dots > x_0$  and  $x_k - b$  playing the role of  $x_k - a$ : So,

$$(2.8) \quad |D^\alpha \varphi(x)| \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$(2.9) \quad \|D^\alpha \varphi\|_\infty \leq \|D^p \varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}.$$

□

*Proof.* We first consider  $C_0(\mathbf{R})$  topologized by the supremum norm. Given a Lebesgue integrable function  $u$ , we put

$$(2.10) \quad \langle u|\varphi \rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})).$$

The following inequalities

$$(2.11) \quad |\langle u|\varphi \rangle| \leq \int_{\mathbf{R}} |u\varphi| \leq \|u\|_{L^1} \quad (\|\varphi\|_\infty \leq 1)$$

imply that every linear functional

$$(2.12) \quad \begin{aligned} \langle u| : C_0(\mathbf{R}) &\rightarrow \mathbf{C} \\ \varphi &\mapsto \langle u|\varphi \rangle \end{aligned}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [4]. Conversely,  $u$  can be identified with  $\langle u|$ , since  $u$  is determined (a.e) by the integrals  $\langle u|\varphi \rangle$ . In the Banach spaces terminology,  $u$  is then (identified with) a linear *bounded*<sup>1</sup> operator  $\langle u|$ , of norm

$$(2.13) \quad \sup\{|\langle u|\varphi \rangle| : \|\varphi\|_\infty = 1\} = \|u\|_{L^1}.$$

Note that, in the latter equality,  $\leq \|u\|_{L^1}$  follows from (2.11), as the converse follows from the Stone-Weierstrass theorem<sup>2</sup>. We now consider the special cases  $u = g_n$ , where  $g_n$  is

$$(2.14) \quad \begin{aligned} g_n : \mathbf{R} &\rightarrow \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) \end{cases} \end{aligned}$$

First, remark that  $g_n(x) \rightarrow 0$ , as the sequence  $\{g_n\}$  fails to converge in  $C_0(\mathbf{R})$  (since  $g_n(1/n) = n^2 \geq 1$ ), and also in  $L^1$  (since  $\int_{\mathbf{R}} |g_n| = n^2 \rightarrow \infty$ ). Nevertheless, we will show that the  $\langle g_n|$  converge pointwise<sup>3</sup> on  $\mathcal{D}_K$ , i.e., there exists a  $\tau_K$ -continuous linear form  $\Lambda$  such that

$$(2.15) \quad \langle g_n|\varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda\varphi,$$

where  $\varphi$  ranges over  $\mathcal{D}_K$ . We now prove (2.13) in the special cases  $u = g_n$ . To do so, we fetch  $\varphi_1^+, \dots, \varphi_j^+, \dots$ , from  $C_K^\infty(\mathbf{R})$ . More specifically,

- (i)  $\varphi_j^+ = 1$  on  $[e^{-j}, 1 - e^{-j}]$ ;
- (ii)  $\varphi_j^+ = 0$  on  $\mathbf{R} \setminus [-1, 1]$ ;
- (iii)  $0 \leq \varphi_j^+ \leq 1$  on  $\mathbf{R}$ ;

see [1.46] of [4] for a possible construction of those  $\varphi_j^+$ . Let  $\varphi_1^-, \dots, \varphi_j^-, \dots$ , mirror the  $\varphi_j^+$ , in the sense that  $\varphi_j^-(x) = \varphi_j^+(-x)$ , so that

<sup>1</sup>see 1.32, 4.1 of [4]

<sup>2</sup>See 7.26 of [2].

<sup>3</sup>See 3.14 of [4] for a definition of the related topology.

- (iv)  $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$  is odd, as  $g_n$  is;
- (v) every  $\varphi_j$  is in  $C_K^\infty(\mathbf{R})$ ;
- (vi) The sequence  $\{\varphi_j\}$  converges (pointwise) to  $1_{[0,1]} - 1_{[-1,0]}$ , and  $\|\varphi_j\|_\infty = 1$ .

Thus, with the help of the Lebesgue's convergence theorem,

$$(2.16) \quad \langle g_n | \varphi_j \rangle = 2 \int_0^1 g_n(t) \varphi_j^+(t) dt \xrightarrow{j \rightarrow \infty} 2 \int_0^1 g_n(t) dt = \|g_n\|_{L^1} = n.$$

Finally,

$$(2.17) \quad \|g_n\|_{L^1} \stackrel{(2.16)}{\leq} \sup\{|\langle g_n | \varphi \rangle| : \|\varphi\|_\infty = 1\} \stackrel{(2.13)}{\leq} \|g_n\|_{L^1};$$

which is the desired result. So, in terms of boundedness constants: Given  $n$ , there exists  $C_n < \infty$  such that

$$(2.18) \quad |\langle g_n | \varphi \rangle| \leq C_n \quad (\|\varphi\|_\infty = 1);$$

see (2.11). Furthermore,  $\|g_n\|_{L^1}$  is actually the best, *i.e.*, lowest, possible  $C_n$ , see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence  $\{\langle g_n | \varphi_{\rho(n)} \rangle\}$  such that  $\langle g_n | \varphi_{\rho(n)} \rangle$  is greater than, say,  $n - 0.01$ , as  $\|\varphi_{\rho(n)}\|_\infty = 1$ . Consequently, there is no bound  $M$  such that

$$(2.19) \quad |\langle g_n | \varphi \rangle| \leq M \quad (\|\varphi\|_\infty = 1; n = 1, 2, 3, \dots).$$

In other words, the  $g_n$  have no *uniform bound* in  $L^1$ , *i.e.*, the collection of all continuous linear mappings  $\langle g_n |$  is not equicontinuous (see discussion in 2.6 of [4]). As a consequence, the  $\langle g_n |$  do not converge pointwise (or “vaguely”, in Radon measure context): A vague (*i.e.*, pointwise) convergence would be (by definition)

$$(2.20) \quad \langle g_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \Lambda \varphi \quad (\varphi \in C_0(\mathbf{R}))$$

for some  $\Lambda \in C_0(\mathbf{R})^*$ , which would make (2.19) hold, see 2.6, 2.8 of [4]. This by no means says that the  $\langle g_n |$  do not converge pointwise, in a relevant space, to some  $\Lambda$  (see (2.15)).

From now on, unless the contrary is explicitly stated, we assume that  $\varphi$  only denotes an element of  $C_K^\infty(\mathbf{R})$ . Let  $f_n$  be a Lebesgue integrable function such that

$$(2.21) \quad \Lambda \varphi = \lim_{n \rightarrow \infty} \int_K f_n \varphi \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

for some linear form  $\Lambda$ . Since  $\varphi$  vanishes outside  $K$ , we can suppose without loss of generality that the support of  $f_n$  lies in  $K$ . So, (2.21) can be restated as follows,

$$(2.22) \quad \Lambda \varphi = \lim_{n \rightarrow \infty} \langle f_n | \varphi \rangle \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

Let  $K_1, K_2, \dots$  be compact sets that satisfy the conditions specified in 1.44 of [4].  $\mathcal{D}_K$  is  $C_K^\infty(\mathbf{R})$  topologized by the related seminorms  $p_1, p_2, \dots$ , see 1.46, 6.2 of [4] and Exercise 1.16. We know that  $K \subseteq K_m$  for some index  $m$  (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices  $N \geq m$  so that

- (a)  $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha \varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$ , for  $\varphi \in \mathcal{D}_K$ ;
- (b) The collection of the sets  $V_N = \{\varphi \in \mathcal{D}_K : \|\varphi\|_N < 2^{-N}\}$  is a (decreasing) local base of  $\tau_K$ , the subspace topology of  $\mathcal{D}_K$ ; see 6.2 of [4] for a more complete discussion.

Let us specialize (2.11) with  $u = f_n$  and  $\varphi \in V_m$  then conclude that  $\langle f_n |$  is bounded by  $\|f_n\|_{L^1}$  on  $V_m$ : Every linear functional  $\langle f_n |$  is therefore  $\tau_K$ -continuous, see 1.18 of [4].

To sum it up:

- (i)  $\mathcal{D}_K$ , equipped the topology  $\tau_K$ , is a Fréchet space (see section 1.46 of [4]);

(ii) Every linear functional  $\langle f_n | \cdot \rangle$  is continuous with respect to this topology;

(iii)  $\langle f_n | \varphi \rangle \xrightarrow[n \rightarrow \infty]{} \Lambda \varphi$  for all  $\varphi$ , , i.e.,  $\Lambda - \langle f_n | \cdot \rangle \xrightarrow[n \rightarrow \infty]{} 0$ .

With the help of [2.6] and [2.8] of [4], we conclude that  $\Lambda$  is continuous and that the sequence  $\{\langle f_n | \cdot \rangle\}$  is equicontinuous. So is the sequence  $\{\Lambda - \langle f_n | \cdot \rangle\}$ , since addition is continuous. There so exists  $i, j$  such that, for all  $n$ ,

$$(2.23) \quad |\Lambda \varphi| < 1/2 \quad \text{if } \varphi \in V_i,$$

$$(2.24) \quad |\Lambda \varphi - \langle f_n | \varphi \rangle| < 1/2 \quad \text{if } \varphi \in V_j.$$

Choose  $p = \max\{i, j\}$  so that  $V_p = V_i \cap V_j$ : The latter inequalities imply that

$$(2.25) \quad |\langle f_n | \varphi \rangle| \leq |\Lambda \varphi - \langle f_n | \varphi \rangle| + |\Lambda \varphi| < 1 \quad \text{if } \varphi \in V_p.$$

Now remark that every  $\psi = \psi[\mu, \varphi]$ , where

$$(2.26) \quad \psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\varphi\|_p) \varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases}$$

keeps in  $V_p$ . Finally, it is clear that each below statement implies the following one.

$$(2.27) \quad |\langle f_n | \psi \rangle| < 1$$

$$(2.28) \quad |\langle f_n | \varphi \rangle| < 2^p \|\varphi\|_p \cdot \mu$$

$$(2.29) \quad |\langle f_n | \varphi \rangle| \leq 2^p \|\varphi\|_p$$

$$(2.30) \quad |\langle f_n | \varphi \rangle| \leq 2^p \{ \|D^0 \varphi\|_\infty + \dots + \|D^p \varphi\|_\infty \}.$$

Finally, with the help of (2.1),

$$(2.31) \quad |\langle f_n | \varphi \rangle| \leq 2^p (p+1) \|D^p \varphi\|_\infty.$$

The first part is so proved, with *some*  $p$  and  $M = 2^p(p+1)$ .

We now come back to the special case  $f_n = g_n$  (see the first part). From now on,  $f_n(x) = n^3 x$  on  $[-1/n, 1/n]$ , 0 elsewhere. Actually, we will prove that

(a)  $\Lambda \varphi = \lim_{n \rightarrow \infty} \int_{-1}^1 f_n(t) \varphi(t) dt$  exists for every  $\varphi \in \mathcal{D}_K$ ;

(b) A *uniform* bound  $|\langle f_n | \varphi \rangle| \leq M \|D^p \varphi\|_\infty$  ( $n = 1, 2, 3, \dots$ ) exists for all those  $f_n$ , with  $p = 1$  as the smallest possible  $p$ .

Bear in mind that  $K \subseteq K_m$  and shift the  $K_N$ 's indices so that  $K_{m+1}$  becomes  $K_1$ ,  $K_{m+2}$  becomes  $K_2$ , and so on. The resulting topology  $\tau_K$  remains unchanged (see Exercise 1.16). We let  $\varphi$  keep running on  $\mathcal{D}_K$  and so define

$$(2.32) \quad B_n(\varphi) \triangleq \max\{|\varphi(x)| : x \in [-1/n, 1/n]\},$$

$$(2.33) \quad \Delta_n(\varphi) \triangleq \max\{|\varphi(x) - \varphi(0)| : x \in [-1/n, 1/n]\}.$$

The mean value asserts that

$$(2.34) \quad |\varphi(1/n) - \varphi(-1/n)| \leq B_n(\varphi') |1/n - (-1/n)| = \frac{2}{n} B_n(\varphi').$$

Independently, an integration by parts shows that

$$(2.35) \quad \langle f_n | \varphi \rangle = \left[ \frac{n^3 t^2}{2} \varphi(t) \right]_{-1/n}^{1/n} - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt$$

$$(2.36) \quad = \frac{n}{2} (\varphi(1/n) - \varphi(-1/n)) - \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt.$$

Combine (2.34) with (2.36) and so obtain

$$(2.37) \quad |\langle f_n | \varphi \rangle| \leq \frac{n}{2} |\varphi(1/n) - \varphi(-1/n)| + \frac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\phi'(t)| dt$$

$$(2.38) \quad \leq B_n(\phi') + \frac{n^3}{2} B_n(\phi') \int_{-1/n}^{1/n} t^2 dt$$

$$(2.39) \quad \leq \frac{4}{3} B_n(\phi')$$

$$(2.40) \quad \leq \frac{4}{3} \|\phi'\|_\infty.$$

Futhermore, (2.39) gives a hint about the convergence of  $f_n$ : Since  $B_n(\phi')$  tends to  $|\phi'(0)|$ , we may expect that  $f_n$  tends to  $\frac{4}{3}\phi'(0)$ . This is actually true: A straightforward computation shows that

$$(2.41) \quad \langle f_n | \varphi \rangle - \frac{4}{3} \phi'(0) \stackrel{(2.36)}{=} \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \phi'(0) - \frac{n^3}{2} \int_{-1/n}^{1/n} (\phi' - \phi'(0)) t^2 dt.$$

So,

$$(2.42) \quad \left| \langle f_n | \varphi \rangle - \frac{4}{3} \phi'(0) \right| \leq \left| \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \phi'(0) \right| + \frac{1}{3} \Delta_n(\phi') \xrightarrow{n \rightarrow \infty} 0.$$

We have just proved that

$$(2.43) \quad \langle f_n | \varphi \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3} \phi'(0) \quad (\varphi \in \mathcal{D}_K).$$

In other words,

$$(2.44) \quad \langle f_n | \xrightarrow{n \rightarrow \infty} -\frac{4}{3} \delta',$$

where  $\delta$  is the *Dirac measure* and  $\delta', \delta'', \dots$ , its *derivatives*; see 6.1 and 6.9 of [4].

It follows from the previous part that  $-\frac{4}{3}\delta'$  is  $\tau_K$ -continuous, and from (2.40) that

$$(2.45) \quad |\langle f_n | \varphi \rangle| \leq \frac{4}{3} \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$(2.46) \quad \{ \langle f_n | \varphi_{\rho(n)} \rangle : \|\varphi_{\rho(n)}\|_\infty = 1; n = 1, 2, 3, \dots \}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant  $M$  such that

$$(2.47) \quad |\langle f_n | \varphi \rangle| \leq M \|\varphi\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

The previous bound of  $\langle f_n |$  - see (2.40), is therefore the best possible one, *i.e.*,  $p = 1$  is the smallest possible  $p$  and, given  $p = 1$ ,  $M = \frac{4}{3}$  is the smallest possible  $M$  (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give  $f_n$  a *derivative*<sup>4</sup>  $f_n'$ , as follows

$$(2.48) \quad \begin{aligned} f_n' &: \mathcal{D}_K \rightarrow \mathbf{C} \\ \varphi &\mapsto -\langle f_n | \phi' \rangle. \end{aligned}$$

It has been proved that every  $\langle f_n |$  is continuous. So is

$$(2.49) \quad \begin{aligned} D &: \mathcal{D}_K \rightarrow \mathcal{D}_K \\ \varphi &\mapsto \phi'; \end{aligned}$$

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<sup>4</sup>See 6.1 of [4] for a further discussion.

see Exercise 1.17.  $f_n'$  is therefore continuous. Now apply (2.43) with  $\phi'$  and so obtain

$$-\langle f_n | \phi' \rangle \xrightarrow{n \rightarrow \infty} \frac{4}{3} \phi''(0) \quad (\varphi \in \mathcal{D}_K),$$

, i.e.,

$$(2.50) \quad f_n' \xrightarrow{n \rightarrow \infty} \frac{4}{3} \delta''.$$

It follows from (2.40) that,

$$(2.51) \quad |\langle f_n | \phi' \rangle| \leq \frac{4}{3} \|\phi''\|_\infty \quad (n = 1, 2, 3, \dots).$$

It is therefore possible to uniformly bound  $f_n'$  with respect to a norm  $\|D^p\|_\infty$ , namely  $\|D^2\|_\infty$ . Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$(2.52) \quad |\langle f_n | \phi' \rangle| \leq M \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Define

$$(2.53) \quad \Phi_j(x) = \int_{-1}^x \varphi_j.$$

The oddness of  $\varphi_j$  forces  $\Phi_j$  to vanish outside  $[-1, 1]$ :  $\varphi_j$  is therefore in  $\mathcal{D}_K$ . So, under our assumption,

$$(2.54) \quad |\langle f_n | \Phi'_j \rangle| \leq M \|\Phi'_j\|_\infty \quad (n = 1, 2, 3, \dots);$$

which is

$$(2.55) \quad |\langle f_n | \varphi_j \rangle| \leq M \|\varphi_j\|_\infty \quad (n = 1, 2, 3, \dots).$$

We have thus reached a contradiction (again with the sequence  $\{\langle f_n | \varphi_{\rho(n)} \rangle\}$ ) and so conclude that there is no constant M such that

$$(2.56) \quad |\langle f_n | \phi' \rangle| \leq M \|\phi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Finally, assume, to reach a contradiction, that there exists a constant M such that

$$(2.57) \quad |\langle f_n | \phi' \rangle| \leq M \|\phi\|_\infty.$$

The mean value theorem (see (2.1)) asserts that

$$(2.58) \quad |\langle f_n | \phi' \rangle| \leq M \|\phi\|_\infty \leq M \|\phi'\|_\infty;$$

which is, again, a desired contradiction. So ends the proof. □

## 6 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient  $\hat{f}(n)$  of a function  $f \in L^2(T)$  ( $T$  is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all  $n \in \mathbf{Z}$  (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^n \hat{f}(k).$$

Prove that  $\{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$  is a dense subspace of  $L^2(T)$  of the first category.

*Proof.* Let  $f(\theta)$  stand for  $f(e^{i\theta})$  so that  $L^2(T)$  is identified with a closed subset of  $L^2([-π, π])$ , hence the inner product

$$(2.59) \quad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

$$(2.60) \quad \Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well known (and easy to prove) result is

$$(2.61) \quad (e_n, e_{n'}) = [n = n'], \text{ i.e., } \{e_n : n \in \mathbf{Z}\} \text{ is an orthonormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric ( $\equiv$ ) identification  $L^2 \equiv (L^2)^*$ . So,

$$(2.62) \quad \|\Lambda_n\|^2 \stackrel{(2.60)}{=} \|e_{-n} + \cdots + e_n\|^2 \stackrel{(2.61)}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \stackrel{(2.61)}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

$$(2.63) \quad B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \dots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [4] asserts that the sequence  $\{\Lambda_n\}$  is norm-bounded; which is a desired contradiction, since

$$(2.64) \quad \|\Lambda_n\| \stackrel{(2.62)}{=} \sqrt{2n+1} \xrightarrow{n \rightarrow \infty} \infty.$$

We have just established that  $B$  is actually of the first category; and so is its subset  $L = \{f \in L^2(T) : \lim_{n \rightarrow \infty} \Lambda_n f \text{ exists}\}$ . We now prove that  $L$  is nevertheless dense in  $L^2(T)$ . To do so, we let  $P$  be  $\text{span}\{e_k : k \in \mathbf{Z}\}$ , the collection of the trigonometric polynomials  $p(\theta) = \sum \lambda_k e^{ik\theta}$ : Combining (2.60) with (2.61) shows that  $\Lambda_n(p) = \sum \lambda_k$  for almost all  $n$ . Thus,

$$(2.65) \quad P \subseteq L \subseteq L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that  $P$  is dense in  $L^2(T)$ . We then conclude, with the help of (2.65), that

$$(2.66) \quad L^2(T) = \overline{P} = \overline{L}.$$

So ends the proof □

## 9 Exercise 9. Boundedness without closedness

Suppose  $X, Y, Z$  are Banach spaces and

$$B : X \times Y \rightarrow Z$$

is bilinear and continuous. Prove that there exists  $M < \infty$  such that

$$\|B(x, y)\| \leq M \|x\| \|y\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

*Proof.* The answer is: No. To prove this, we only assume that  $X, Y, Z$  are normed spaces. Since  $B$  is continuous at the origin, there exists a positive  $r$  such that

$$(2.67) \quad \|x\| + \|y\| < r \Rightarrow \|B(x, y)\| < 1.$$

Given nonzero  $x, y$ , let  $s$  range over  $]0, r[$  so that the following bound

$$(2.68) \quad \|B(x, y)\| = \frac{4\|x\|\|y\|}{s^2} \left\| B\left(\frac{s}{2\|x\|}x, \frac{s}{2\|y\|}y\right) \right\| \stackrel{(2.67)}{<} \frac{4\|x\|\|y\|}{s^2}$$

is effective. It is now obvious that

$$(2.69) \quad B(x, y) \leq \frac{4}{s^2} \|x\| \|y\| \xrightarrow[s \rightarrow r]{} \frac{4}{r^2} \|x\| \|y\| \quad ((x, y) \in X \times Y);$$

which achieves the proof.

As a concrete example, choose  $X = Y = Z = C_c(\mathbf{R})$ , topologized by the supremum norm.  $C_c(\mathbf{R})$  is not complete (see 5.4.4 of [5]), nevertheless the bilinear product

$$\begin{aligned} B : C_c(\mathbf{R})^2 &\rightarrow C_c(\mathbf{R}) \\ (f, g) &\mapsto f \cdot g \end{aligned}$$

is bounded (since  $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$ ), and continuous. To show this, pick a positive scalar  $\varepsilon$  smaller than 1, provided any  $(f, g)$ . Next, define

$$(2.70) \quad r \triangleq \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty} < 1.$$

We now restrict  $(u, v)$  to a particular neighborhood of  $(f, g)$ . More specifically,

$$(2.71) \quad \|f - u\|_\infty + \|g - v\|_\infty < r.$$

Next, remark that  $\|u\|_\infty \leq r + \|f\|_\infty$  and so obtain (bear in mind that  $r < 1$ )

$$(2.72) \quad \|fg - uv\|_\infty = \|(f - u) \cdot g + u \cdot (g - v)\|_\infty$$

$$(2.73) \quad \leq \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty$$

$$(2.74) \quad < r \cdot \|g\|_\infty + (r + \|f\|_\infty) \cdot r$$

$$(2.75) \quad < r \cdot (r + \|f\|_\infty + \|g\|_\infty)$$

$$(2.76) \quad < \varepsilon.$$

Since  $\varepsilon$  was arbitrary, it is now established that  $B$  is continuous at every  $(f, g)$ .  $\square$

## 10 Exercise 10. Continuousness of bilinear mappings

*Prove that a bilinear mapping is continuous if it is continuous at the origin  $(0, 0)$ .*

*Proof.* Let  $B : X_1 \times X_2 \rightarrow Z$  be a bilinear mapping that is continuous at  $(0, 0)$ , where  $X_i$  and  $Z$  are topological vector spaces. This implies that, for any balanced open set  $W$ ,  $X_i$  contains a balanced open set  $U_i$  such that

$$(2.77) \quad B(U_1 \times U_2) \subseteq W.$$

Pick  $a_i$  from  $X_i$ , and then choose  $r_i > 1/2$  large enough so that  $a_i \in r_i U_i$ . The vector  $b_i - a_i$  is now restricted to  $(r_1 + r_2)^{-1} U_i$ . Hence

$$(2.78) \quad B(b_1, b_2) - B(a_1, a_2) = B(b_1 - a_1, b_2) + B(a_1, b_2) - B(a_1, a_2)$$

$$(2.79) \quad = B(b_1 - a_1, b_2) + B(a_1, b_2 - a_2)$$

$$(2.80) \quad = B(b_1 - a_1, b_2 - a_2) + B(b_1 - a_1, a_2) + B(a_1, b_2 - a_2)$$

$$(2.81) \quad \subseteq \frac{1}{(r_1 + r_2)^2} W + \frac{r_2}{r_1 + r_2} W + \frac{r_1}{r_1 + r_2} W$$

$$(2.82) \quad \subseteq W + W + W$$

We conclude that  $B$  is continuous at every  $(a_1, a_2)$ , since  $W$  was arbitrary. This concludes the proof.  $\square$

## 12 Exercise 12. A bilinear mapping that is not continuous

Let  $X$  be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| dt.$$

Put  $B(f, g) = \int_0^1 f(t)g(t)dt$ , and show that  $B$  is a bilinear continuous functional on  $X \times X$  which is separately but not continuous.

*Proof.* Let  $f$  denote the first variable,  $g$  the second one. Remark that

$$(2.83) \quad |B(f, g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient (1.18 of [4]) to assert that any  $f \mapsto B(f, g)$  is continuous. The continuity of all  $g \mapsto B(f, g)$  follows (Put  $C(g, f) = B(f, g)$  and proceed as above). Suppose, to reach a contradiction, that  $B$  is continuous. There so exists a positive  $M$  such that,

$$(2.84) \quad |B(f, g)| < M\|f\|\|g\|.$$

Put

$$(2.85) \quad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \quad (n = 1, 2, 3, \dots),$$

so that

$$(2.86) \quad \|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow[n \rightarrow \infty]{} 0.$$

On the other hand,

$$(2.87) \quad B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.87) and (2.84) with (2.86) and so obtain

$$(2.88) \quad 1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow[n \rightarrow \infty]{} 0.$$

Our continuous assumption is then contradicted. So ends the proof.  $\square$

## 15 Exercise 15. Baire's cut

Suppose  $X$  is an  $F$ -space and  $Y$  is a subspace of  $X$  whose complement is of the first category. Prove that  $Y = X$ . Hint:  $Y$  must intersect  $x + Y$  for every  $x \in X$ .

*Proof.* Let  $\{E_n\}$  be a sequence of nowhere dense sets such that  $X \setminus Y = \bigcup_{n=1}^{\infty} E_n$ . Note that every  $V_n = X \setminus \overline{E_n}$  is dense in  $X$ . Now pick an arbitrary  $x \in X$ . Since the translation by  $x$  is a self-homeomorphism,  $x + V_n$  is open and dense as well. Alternatively, observe that  $X = x + X \subseteq \overline{x + V_n}$ , which is a special case of 1.3 (b) of [4]. Next, apply Baire's theorem twice to establish that

- (a) every intersection  $W_n = V_n \cap [x + V_n]$  is dense in  $X$ ,
- (b) so is the nonempty intersection  $\bigcap_{n=1}^{\infty} W_n$ .

Moreover, the intersection  $\bigcap_{n=1}^{\infty} V_n$  does not cut any  $E_n$ . To sum up,

$$(2.89) \quad w \in \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq Y$$

for some  $w = w(x)$ . Furthermore,  $w$  also lies in every  $x + V_n$ , by (a). Hence

$$(2.90) \quad w - x \in \bigcap_{n=1}^{\infty} V_n \subseteq Y.$$

Finally, (2.89) and (2.90) together yield

$$(2.91) \quad x = w - (w - x) \in Y - Y = Y,$$

as  $Y$  is a subgroup of  $X$ . This establishes that

$$(2.92) \quad X \subseteq Y,$$

since  $x$  was arbitrary. □

## 16 Exercise 16. An elementary closed graph theorem

Suppose that  $X$  and  $K$  are metric spaces, that  $K$  is compact, and that the graph of  $f : X \rightarrow K$  is a closed subset of  $X \times K$ . Prove that  $f$  is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of  $K$  cannot be omitted from the hypothesis, even when  $X$  is compact.

*Proof.* Choose a sequence  $\{x_n : n = 1, 2, 3, \dots\}$  whose limit is an arbitrary  $a$ . By compactness of  $K$ , the graph  $G$  of  $f$  contains a subsequence  $\{(x_{\varphi(n)}, f(x_{\varphi(n)}))\}$  of  $\{(x_n, f(x_n))\}$  that converges to some  $(a, b)$  of  $X \times K$ .  $G$  is closed; therefore,  $\{(x_{\varphi(n)}, f(x_{\varphi(n)}))\}$  converges in  $G$ . So,  $b = f(a)$ ; which establishes that  $f$  is sequentially continuous. Since  $X$  is metrizable,  $f$  is also continuous, see [A6] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.93) \quad \begin{aligned} f : [0, \infty) &\rightarrow [0, \infty) \\ x &\mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases} \end{aligned}$$

Clearly,  $f$  has a discontinuity at 0. Nevertheless the graph  $G$  of  $f$  is closed. To see that, first remark that

$$(2.94) \quad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let  $\{(x_n, 1/x_n)\}$  be a sequence in  $G_+ = \{(x, 1/x) : x > 0\}$  that converges to  $(a, b)$ . To be more specific:  $a = 0$  contradicts the boundedness of  $\{(x_n, 1/x_n)\}$ :  $a$  is necessarily positive and  $b = 1/a$ , since  $x \mapsto 1/x$  is continuous on  $R_+$ . This establishes that  $(a, b) \in G_+$ , hence the closedness  $G_+$ . Finally, we conclude that  $G$  is closed, as a finite union of closed sets.  $\square$

# Chapter 3

## Convexity

### 3 Exercise 3.

Suppose  $X$  is a real vector space (without topology). Call a point  $x_0 \in A \subseteq X$  an internal point of  $A$  if  $A - x_0$  is an absorbing set.

- (a) Suppose  $A$  and  $B$  are disjoint convex sets in  $X$ , and  $A$  has an internal point. Prove that there is a nonconstant linear functional  $\Lambda$  such that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. (The proof is similar to that of Theorem 3.4)
- (b) Show (with  $X = \mathbf{R}^2$ , for example) that it may not possible to have  $\Lambda(A)$  and  $\Lambda(B)$  disjoint, under the hypotheses of (a).

*Proof.* Take  $A$  and  $B$  as in (a); the trivial case  $B = \emptyset$  is discarded. Since  $A - x_0$  is absorbing, so is its convex superset  $C = A - B - x_0 + b_0$  ( $b_0 \in B$ ). Note that  $C$  contains the origin. Let  $p$  be the Minkowski functional of  $C$ . Since  $A$  and  $B$  are disjoint,  $b_0 - x_0$  is not in  $C$ , hence  $p(b_0 - x_0) \geq 1$ . We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [4] to establish the existence of a linear functional  $\Lambda : X \rightarrow \mathbf{R}$  such that

$$(3.1) \quad \Lambda \leq p$$

and

$$(3.2) \quad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \quad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

$$(3.4) \quad \Lambda a \leq \Lambda b.$$

We now prove that  $\Lambda(A) \cap \Lambda(B)$  contains at most one point. Suppose, to reach a contradiction, that this intersection contains  $y_1$  and  $y_2$ . There so exists  $(a_i, b_i)$  in  $A \times B$  ( $i = 1, 2$ ) such that

$$(3.5) \quad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that  $y_1 < y_2$ . Then,

$$(3.6) \quad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad .$$

Remark that  $a_3 = \frac{1}{2}(a_1 + a_2)$  lies in the convex set  $A$ . This implies

$$(3.7) \quad \Lambda b_1 \stackrel{(3.6)}{<} \Lambda a_3 \stackrel{(3.4)}{\leq} \Lambda b_1 \quad ;$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space  $X$  is  $\mathbf{R}^2$ . Fetch

$$(3.8) \quad S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$

$$(3.9) \quad S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$$

$$(3.10) \quad A \triangleq S_1 \cup S_2,$$

$$(3.11) \quad B \triangleq X \setminus A.$$

Pick  $(x_i, y_i)$  in  $S_i$ . Let  $t$  range over the unit interval, and so obtain

$$(3.12) \quad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A.$$

Thus, every segment that has an extremity in  $S_1$  and the other one in  $S_2$  lies in  $A$ . Moreover, each  $S_i$  is convex. We can now conclude that  $A$  is so. The convexity of  $B$  is proved in the same manner. Furthermore,  $A$  hosts a nondegenerate triangle, *i.e.*,  $A^\circ$  is nonempty<sup>1</sup>:  $A$  contains an internal point.

Let  $L$  be a vector line of  $\mathbf{R}^2$ . In other words,  $L$  is the null space of a linear functional  $\Lambda : \mathbf{R}^2 \rightarrow \mathbf{R}$  (to see this, take some nonzero  $u$  in  $L^\perp$  and set  $\Lambda x = (x, u)$  for all  $x$  in  $\mathbf{R}^2$ ). One easily checks that both  $A$  and  $B$  cut  $L$ . Hence

$$(3.13) \quad \Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset .$$

So ends the proof. □

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<sup>1</sup>For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [3].

## 11 Exercise 11. Meagerness of the polar

Let  $X$  be an infinite-dimensional Fréchet space. Prove that  $X^*$ , with its weak\*-topology, is of the first category in itself.

This is actually a consequence of the lemma below, which we prove first. The proof that  $X^*$  is of the first category in itself comes right after, as a corollary.

**Lemma.** *If  $X$  is an infinite dimensional topological vector space whose dual  $X^*$  separates points on  $X$ , then the polar*

$$(3.14) \quad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \text{ on } A\}$$

*of any absorbing subset  $A$  is a weak\*-closed set that has empty interior.*

*Proof.* Let  $x$  range over  $X$ . The linear form  $\Lambda \mapsto \Lambda x$  is weak\*-continuous, see 3.14 of [4]. Therefore,  $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$  is weak\*-closed: As the intersection of  $\{P_a : a \in A\}$ ,  $K_A$  is also a weak\*-closed set. We now prove the second half of the statement.

From now on,  $X$  is assumed to be endowed with its weak topology:  $X$  is then locally convex, but its dual space is still  $X^*$  (see 3.11 of [4]). Put

$$(3.15) \quad W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \quad (r_x > 0)$$

where  $F$  runs through the nonempty finite subsets of  $X$ . Clearly, the collection of all such  $W$  is a local base of  $X^*$ . Pick one of those  $W$  and remark that the following subspace

$$(3.16) \quad M \triangleq \text{span}(F)$$

is finite dimensional. Assume, to reach a contradiction, that  $A \subseteq M$ . So, every  $x$  lies in  $t_x M = M$  for some  $t_x > 0$ , since  $A$  is absorbing. As a consequence,  $X$  is the finite dimensional space  $M$ , which is a desired contradiction. We have just established that  $A \not\subseteq M$ : Now pick  $a$  in  $A \setminus M$  and so conclude that

$$(3.17) \quad b \triangleq \frac{a}{t_a} \in A$$

Remark that  $b \notin M$  (otherwise,  $a = t_a b \in t_a M = M$  would hold) and that  $M$ , as a finite dimensional space, is closed (see 1.21 (b) of [4] for a proof): By the Hahn-Banach theorem 3.5 of [4], there exists  $\Lambda_a$  in  $X^*$  such that

$$(3.18) \quad \Lambda_a b > 2$$

and

$$(3.19) \quad \Lambda_a(M) = \{0\}.$$

The latter equality implies that  $\Lambda_a$  vanishes on  $F$ ; hence  $\Lambda_a$  is an element of  $W$ . On the other hand, given an arbitrary  $\Lambda \in K_A$ , the following inequalities

$$(3.20) \quad |\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that  $\Lambda + \Lambda_a$  is not in  $K_A$ . We have thus proved that

$$(3.21) \quad \Lambda + W \not\subseteq K_A.$$

Since  $W$  and  $\Lambda$  are both arbitrary, this achieves the proof.  $\square$

We now give a proof of the original statement.

**Corollary.** *If  $X$  is an infinite-dimensional Fréchet space, then  $X^*$  is meager in itself.*

*Proof.* From now on,  $X^*$  is only endowed with its weak\*-topology. Let  $d$  be an invariant distance that is compatible with the topology of  $X$ , so that the following sets

$$(3.22) \quad B_n \triangleq \{x \in X : d(0, x) < 1/n\} \quad (n = 1, 2, 3, \dots)$$

form a local base of  $X$ . If  $\Lambda$  is in  $X^*$ , then

$$(3.23) \quad |\Lambda| \leq m \text{ on } B_n$$

for some  $(n, m) \in \{1, 2, 3, \dots\}^2$ , see 1.18 of [4]. Hence,  $X^*$  is the countable union of all

$$(3.24) \quad m \cdot K_n \quad (m, n = 1, 2, 3, \dots),$$

where  $K_n$  is the polar of  $B_n$ . Clearly, showing that every  $m \cdot K_n$  is nowhere dense is now sufficient. To do so, we use the fact that  $X^*$  separates points; see 3.4 of [4]. As a consequence, the above lemma implies

$$(3.25) \quad (\overline{K_n})^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by  $m$  is a homeomorphism (see 1.7 of [4]), this is equivalent to

$$(3.26) \quad (\overline{m \cdot K_n})^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof.  $\square$

# Chapter 4

## Banach Spaces

Throughout this set of exercises,  $X$  and  $Y$  denote Banach spaces, unless the contrary is explicitly stated.

### 1 Exercise 1. Basic results

Let  $\varphi$  be the embedding of  $X$  into  $X^{**}$  described in Section 4.5. Let  $\tau$  be the weak topology of  $X$ , and let  $\sigma$  be the weak\*-topology of  $X^{**}$ - the one induced by  $X^*$ .

- (a) Prove that  $\varphi$  is a homeomorphism of  $(X, \tau)$  onto a dense subspace of  $(X^{**}, \sigma)$ .
- (b) If  $B$  is the closed unit ball of  $X$ , prove that  $\varphi(B)$  is  $\sigma$ -dense in the closed unit ball of  $X^{**}$ . (Use the Hahn-Banach separation theorem.)
- (c) Use (a), (b), and the Banach-Alaoglu theorem to prove that  $X$  is reflexive if and only if  $B$  is weakly compact.
- (d) Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.
- (e) If  $X$  is reflexive and  $Y$  is a closed subspace of  $X$ , prove that  $X/Y$  is reflexive.
- (f) Prove that  $X$  is reflexive if and only  $X^*$  is reflexive.

Suggestion: One half follows from (c); for the other half, apply (d) to the subspace  $\varphi(X)$  of  $X^{**}$ .

*Proof.* Let  $\psi$  be the isometric embedding of  $X^*$  into  $X^{***}$ . The dual space of  $(X^{**}, \sigma)$  is then  $\psi(X^*)$ .

It is sufficient to prove that

$$(4.1) \quad \varphi^{-1} : \varphi(X) \rightarrow X$$

$$(4.2) \quad \varphi(x) \mapsto x$$

is a homeomorphism (with respect to  $\tau$  and  $\sigma$ ). We first consider

$$(4.3) \quad V \triangleq \{x^{**} \in X^{**} : |\langle x^{**} | \psi x^* \rangle| < r\} \quad (x^* \in X^*, r > 0);$$

$$(4.4) \quad U \triangleq \{x \in X : |\langle x | x^* \rangle| < r\} \quad (x^* \in X^*, r > 0).$$

and remark that the so defined  $V$ 's (respectively  $U$ 's) shape a local subbase  $\mathcal{S}_\sigma$  (respectively  $\mathcal{S}_\tau$ ) of  $\sigma$  (respectively  $\tau$ ). We now observe that

$$(4.5) \quad U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathcal{S}_\sigma, U \in \mathcal{S}_\tau) ,$$

since  $\varphi^{-1}$  is one-to-one. This remains true whether we enrich each subbase  $\mathcal{S}$  with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak\*-topology that  $\varphi^{-1}$  and its inverse  $\varphi$  are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$(4.6) \quad \overline{\varphi(X)}_\sigma \subseteq X^{**} .$$

and we now assume- to reach a contradiction- that  $(X^{**}, \sigma)$  contains a point  $z^{**}$  outside the  $\sigma$ -closure of  $\varphi(X)$ . By [3.5], there so exists  $y^*$  in  $X^*$  such that

$$(4.7) \quad \langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) ;$$

$$(4.8) \quad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces  $y^*$  to be a the zero of  $X^*$ . The functional  $\psi y^*$  is then the zero of  $X^{***}$ : (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball  $B^{**}$  of  $X^{**}$  is weak\*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \quad \varphi(B) \subseteq B^{**} ,$$

since  $\varphi$  is isometric. Hence

$$(4.10) \quad \overline{\varphi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**} .$$

Now suppose, to reach a contradiction, that  $B^{**} \setminus \overline{\varphi(B)}_\sigma$  contains a vector  $z^{**}$ . By [3.7], there exists  $y^*$  in  $X^*$  such that

$$(4.11) \quad |\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_\sigma ;$$

$$(4.12) \quad \langle z^{**}, \psi y^* \rangle > 1 .$$

It follows from (4.11) that

$$(4.13) \quad |\psi y^*| \leq 1 \text{ on } \varphi(B) , \text{ i.e., } |y^*| \leq 1 \text{ on } B .$$

We have so proved that

$$(4.14) \quad y^* \in B^* .$$

Since  $z^{**}$  lies in  $B^{**}$ , it is now clear that

$$(4.15) \quad |\langle z^{**}, \psi y^* \rangle| \leq 1 ;$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

$$(4.16) \quad B \text{ is weakly compact if and only if } \varphi(B) \text{ is weak } ^* \text{-compact.}$$

If  $B$  is weakly compact, then  $\varphi(B)$  is weak\*-closed. So,

$$(4.17) \quad \varphi(B) = \overline{\varphi(B)}_\sigma \stackrel{(b)}{=} B^{**} .$$

$\varphi$  is therefore onto, i.e.,  $X$  is reflexive.

Conversely, keep  $\varphi$  as onto: one easily checks that  $\varphi(B) = B^{**}$ . The image  $\varphi(B)$  is then weak\*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let  $X$  be a reflexive space  $X$ , whose closed unit ball is  $B$ . Let  $Y$  be a norm-closed subspace of  $X$ :  $Y$  is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that  $B$  is weakly compact. We now conclude that the closed unit ball  $B \cap Y$  of  $Y$  is weakly compact. We again use (c) to conclude that  $Y$  is reflexive. (d) is therefore established. Now proceed to (e).

Let  $\equiv$  stand for “isometrically isomorphic” and apply twice [4.9] to obtain, first

$$(4.18) \quad (X/Y)^* \equiv Y^\perp ,$$

next,

$$(4.19) \quad (X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y .$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space  $Z$ , the weak\*-topology of  $Z^*$  is its weak one.

Assume first that  $X$  is reflexive. Since  $B^*$  is weak\* compact, by (c) of [4.3], (L) implies that  $B^*$  is also weakly compact. Then (c) turns  $X^*$  into a reflexive space.

Conversely, let  $X^*$  be reflexive. What we have just proved that makes  $X^{**}$  reflexive. On the other hand,  $\varphi(X)$  is a norm-closed subspace of  $X^{**}$ ; cf. [4.5]. Hence  $\varphi(X)$  is reflexive, by (d). It now follows from (c) that  $B^{**} \cap \varphi(X)$  is weakly compact, *i.e.*, weak\*-compact (to see this, apply (L) with  $Z = X^*$ ).

By (a),  $B$  is therefore weakly compact, *i.e.*,  $X$  is reflexive, see (c). So ends the proof.  $\square$

### 13 Exercise 13. Operator compactness in a Hilbert space

- (a) Suppose  $T \in \mathcal{B}(X, Y)$ ,  $T_n \in \mathcal{B}(X, Y)$  for  $n = 1, 2, 3, \dots$ , each  $T_n$  has finite-dimensional range, and  $\lim \|T - T_n\| = 0$ . Prove that  $T$  is compact.
- (b) Assume  $Y$  is a Hilbert space, and prove the converse of (a): Every compact  $T \in \mathcal{B}(X, Y)$  can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)

*Proof.* Since each  $T_n$  is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every  $T_n$  as a compact operator. Let  $\{x_0^i : i \in \mathbf{N}\}$  be in  $U$  the open unit ball of  $X$ . Since  $T_1$  is compact,  $\{x_0^i\}$  contains a subsequence  $\{x_1^i : i \in \mathbf{N}\}$  such that  $\{T_1 x_1^i\}$  converges to a point  $y_1$  of  $Y$ . The same reasoning can be recursively applied to  $T_n$  and  $\{x_{n-1}^i\} \subseteq U$  so that  $\{T_n x_n^i\}$  tends to some  $y_n$  of  $Y$ , as  $\{x_n^i\}$  is a subsequence of  $\{x_{n-1}^i\}$ . Then

$$(4.20) \quad T_n x_p^i \xrightarrow[i \rightarrow \infty]{} y_n \quad (p n = 1, 2, 3, \dots) .$$

Applied with  $\{x_n^i : (n, i) \in \mathbf{N}^2\}$ , a Cantor's diagonal process therefore provides a subsequence  $\{\tilde{x}_j : j \in \mathbf{N}\}$  such that

$$(4.21) \quad T_j \tilde{x}_k \xrightarrow[k \rightarrow \infty]{} y_j ;$$

$$(4.22) \quad T_j \tilde{x}_j \xrightarrow[j \rightarrow \infty]{} y_j .$$

We now easily obtain

$$(4.23) \quad \|T_j \tilde{x}_j - T_k \tilde{x}_k\| \leq \|T_j \tilde{x}_j - y_j\| + \|y_j - T_k \tilde{x}_k\| + \|T_j - T_k\| \xrightarrow[k > j \rightarrow \infty]{} 0 .$$

$\{T_j \tilde{x}_j\}$  is then a Cauchy sequence. So is  $\{T \tilde{x}_j\}$ , since  $\|T - T_j\| \rightarrow 0$ . On the other hand,  $Y$  is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix  $\varepsilon$  as a positive scalar. Since  $T$  is compact,  $Y$  contains a finite set  $C$  such that

$$(4.24) \quad T(U) \subseteq \bigcup_{c \in C} B(c, \varepsilon) .$$

As a Hilbert space,  $Y$  contains a *maximal orthonormal set* (or *Hilbert basis*)  $M$ . This implies that  $\text{span}(M)$  is dense in  $Y$ ; cf. 4.18 & 4.22 of [3]. The finiteness of  $C$  forces  $M$  to enclose a finite set  $S$  so that

$$(4.25) \quad \forall c \in C, \exists s(c) \in \text{span}(S) : \|c - s(c)\| < \varepsilon .$$

Let  $x$  be in  $U$ . It follows from (4.24) that

$$(4.26) \quad \|Tx - c_x\| < \varepsilon$$

for some  $c_x$  of  $C$ . We now combine (4.25) and (4.26) to obtain

$$(4.27) \quad \|Tx - s(c_x)\| \leq \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon$$

As a finite-dimensional subspace,  $\text{span}(S)$  is closed (see footnote 4, Exercise 1.10). We so obtain

$$(4.28) \quad Y = \text{span}(S) \oplus \text{span}(S)^\perp ,$$

by [12.4]. There so exists a unique projection  $\pi = \pi(\varepsilon)$  of  $Y$  onto itself (see [5.6] for the definition) such that

$$(4.29) \quad \pi(Y) = \text{span}(S), \quad (I - \pi)(Y) = \text{span}(S)^\perp .$$

It is easily checked that  $\pi$  has norm 1. Moreover,

$$(4.30) \quad \pi s = s \quad (s \in \text{span}(S)) \quad .$$

Thus,

$$(4.31) \quad (I - \pi)(Tx) = (I - \pi)(Tx - s(c_x)) \quad (x \in U) \quad .$$

Then,

$$(4.32) \quad \|(I - \pi)(Tx)\| \leq \|I - \pi\| \|Tx - s(c_x)\| < 4\varepsilon \quad (x \in U)$$

(the fact that  $\pi$  has norm 1 is hidden in the right side inequality). We have just so proved that

$$(4.33) \quad \|T - \pi \circ T\| \in O_{\varepsilon \sim 0}(\varepsilon) \quad .$$

That is particularly true whether  $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \xrightarrow{n \rightarrow \infty} 0$ . Let so  $T_n$  be  $\pi(\varepsilon_n) \circ T$  and conclude that these (compact) operators approximate  $T$  in the desired fashion, *i.e.*,

$$(4.34) \quad \|T - T_n\| \xrightarrow{n \rightarrow \infty} 0 \quad .$$

□

## 15 Exercise 15. Hilbert-Schmidt operators

Suppose  $\mu$  is a finite (or  $\sigma$ -finite) positive measure on a measure space  $\Omega$ ,  $\mu \times \mu$  is the corresponding product measure on  $\Omega \times \Omega$ , and  $K \in L^2(\mu \times \mu)$ . Define

$$(Tf)(s) = \int_{\Omega} K(s, t) f(t) d\mu(t) \quad [f \in L^2(\mu)].$$

(a) Prove that  $T \in \mathcal{B}(L^2(\mu))$  and that

$$\|T\|^2 \leq \int_{\Omega} \int_{\Omega} |K(s, t)|^2 d\mu(s) d\mu(t).$$

(b) Suppose  $a_i, b_i$  are members of  $L^2(\mu)$ , for  $1 \leq i \leq n$ , put  $K_1 = \sum a_i(s)b_i(t)$  and define  $T_1$  in terms of  $K_1$  as  $T$  was defined in terms of  $K$ . Prove that  $\dim \mathcal{R}(T_1) \leq n$ .

(c) Deduce that  $T$  is a compact operator in  $L^2(\mu)$ . Hint: Use exercise 13.

(d) Suppose  $\lambda \in \mathbf{C}$ ,  $\lambda \neq 0$ . Prove: Either the equation

$$Tf - \lambda f = g$$

has a unique solution  $f \in L^2(\mu)$  for every  $g \in L^2(\mu)$  or there are infinitely many solutions for some  $g$  and none for others. (This is known as the Fredholm alternative.).

(e) Describe the adjoint of  $T$ .

*Proof.* Let  $X$  (respectively  $P$ ) be the Banach space  $L^2(\mu)$  (respectively  $L^2(\mu \times \mu)$ ). A consequence of the Radon-Nikodym theorem (cf. 6.16 of [3]) is that there exists a group isomorphism  $\varphi : X \rightarrow X^*$ ,  $f \mapsto f^*$  such that

$$(4.35) \quad \langle u, f^* \rangle = \int_{\Omega} u \cdot f \mu \quad (u \in X, f \in X) .$$

Define a.e  $K_s, K_t : \Omega \rightarrow \mathbf{C}$  by setting

$$(4.36) \quad K_s(t) \triangleq K_t(s) \triangleq K(s, t) \quad \text{a.e} \quad ((s, t) \in \Omega) .$$

$T$  is clearly linear. Moreover,

$$(4.37) \quad |(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \quad (\|f\|_X < 1)$$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with  $|K|^2$  to obtain

$$(4.38) \quad \|Tf\|_X^2 \leq \int_{\Omega} \|K_s\|_X^2 \mu(s) = \|K\|_P^2 < \infty \quad (\|f\|_X < 1) .$$

(a) is then proved.

To show (b), remark that

$$(4.39) \quad \int_{\Omega} a_i(s) \cdot b_i \cdot f \mu \in \mathbf{C} \cdot a_i(s) \quad \text{a.e} \quad (f \in X, s \in \Omega) .$$

It is now clear that  $T$  maps any  $f$  of  $X$  into  $\mathbf{C} \cdot a_1 + \dots + \mathbf{C} \cdot a_n$ . We so conclude that  $\dim \mathcal{R}(T_1) \leq n$ .

We now aim at (c). The current part refers to Exercise 4.13.  $X$  is also a Hilbert space and so contains a Hilbert basis  $M$ . Define a.e

$$(4.40) \quad \begin{aligned} a_b : \Omega &\rightarrow \mathbf{C} \\ s &\mapsto (K_s, b) \end{aligned}$$

whenever  $b$  ranges  $M$ . Hence,

$$(4.41) \quad K_s = \sum_{b \in M} a_b(s) \cdot b \text{ a.e } (s \in \Omega) .$$

Provided any positive scalar  $\varepsilon$ , there so exists a finite subset  $S = S(\varepsilon)$  of  $M$  such that

$$(4.42) \quad \|K_s - \sum_{b \in S} a_b(s) \cdot b\|_X < \varepsilon \quad (s \in \Omega) .$$

Remark that  $\sum_{b \in S} a_b \cdot b$  matches the definition of  $K_1$ ; cf. (b): from now on,

$$(4.43) \quad K_1 \triangleq \sum_{b \in S} a_b \cdot b .$$

It follows from (b) that

$$(4.44) \quad \dim R(K_1) < \infty .$$

Now turn back to (a), with  $K - K_1$  playing the role of  $K$ , and so obtain

$$(4.45) \quad \|T - T_1\| < \varepsilon \mu(\Omega) \leq \infty .$$

For if  $\mu$  is finite, use (a) of Exercise 4.13 to conclude that  $T$  is compact. Assume henceforth that  $\mu$  is not (necessarily) finite and pick  $\delta$  in  $\mathbf{R}_+$ . The simple functions (with finite measure support) form a dense family of an  $L^p$  space ( $1 \leq p < \infty$ ); cf. 3.13 of [3]. It then exists a simple function  $K_\delta$  of  $L^2(\mu \times \mu)$  such that

$$(4.46) \quad (\mu \times \mu)(\{K_\delta \neq 0\}) < \infty, \|K - K_\delta\|_p < \delta .$$

Define an operator  $T_\delta$  in terms of  $K_\delta$  as  $T$  was defined in terms of  $K$ , and proceed as in (a) with  $T - T_\delta$  instead of  $T$ . Then

$$(4.47) \quad \|T - T_\delta\| < \delta .$$

The key ingredient is that  $K_\delta$  can be identified with an element of the finite measure space  $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$ . What we have attempted to approximate  $T$  by  $T_1$  can therefore be reiterated (with  $K_\delta$  playing the role of  $K$ ) to achieve an approximation  $T_{\delta,1}$  of  $T_\delta$  so that

$$(4.48) \quad \|T_\delta - T_{\delta,1}\| < \varepsilon .$$

It now follows from (4.47) and (4.48) that

$$(4.49) \quad \|T - T_{\delta,1}\| \leq \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta .$$

Since  $\varepsilon$  and  $\delta$  were arbitrary, the  $\sigma$ -finite case is proved. We now establish (d).

Provided  $g$  of  $X$ , let  $E_g$  be the following equation on  $X$

$$(4.50) \quad Tf - \lambda f = g ,$$

whose solution set is denoted by  $S_g$ . Note that  $S_0$  is  $\ker(T - \lambda)$  and discard the trivial case  $S_0 = X$ <sup>1</sup>: each  $f$  of  $X$  lies in  $S_{Tf - \lambda f}$ , as some  $Tf - \lambda f$ 's are nonzero. Some  $S_g$ 's are then nonempty. Remark that

$$(4.51) \quad S_g = f + S_0 \quad (f \in S_g)$$

in such case. Furthermore, the equality  $\beta = \alpha$  of [4.25] yields

$$(4.52) \quad (T - \lambda I)(X) \neq X, \text{ i.e., } S_0 \neq \{0\} .$$

---

<sup>1</sup>, e.g.,  $X = L^2(\{0\}, \delta)$  so that  $I = \lambda^{-1}T$  is compact.

So if  $T - \lambda I$  is not onto, not only some  $S_g$ 's are empty, but also  $S_0 \neq \{0\}$ . Every nonempty  $S_g$  (such sets always exist, see above) is then infinite, by (4.51).

Otherwise,  $T - \lambda I$  is bijective and every equation  $E_g$  has then a unique solution  $f$ . The Fredholm alternative is so proved.

Our last step is the description of  $T^*$ . Let  $S : X \rightarrow X$  be such that

$$(4.53) \quad (Sf)(t) \triangleq \int_{\Omega} K_t \cdot f \text{ a.e. } (f \in X, t \in \Omega)$$

Proceed as in (a), with  $S$  instead of  $T$ :  $S$  lies in  $\mathcal{B}(X)$ . Next, we claim that

$$(4.54) \quad \langle u, T^* f^* \rangle = \langle Tu, f^* \rangle$$

$$(4.55) \quad = \int_{\Omega} (Tu) \cdot f \mu$$

$$(4.56) \quad = \int_{\Omega^2} K \cdot f \cdot u (\mu \times \mu)$$

$$(4.57) \quad = \int_{\Omega} (Sf) \cdot u \mu$$

$$(4.58) \quad = \langle u, (Sf)^* \rangle ,$$

whenever  $u$  and  $f$  run through the closed unit ball of  $X$ . Since  $\|T\|, \|T^*\|$  are equal and finite, only exactness of (4.56) is possibly in doubt; the justification below dissipates it. In conclusion,

$$(4.59) \quad T^* = \rho S \rho^{-1} .$$

Informally,

$$(4.60) \quad T^* = S .$$

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep  $u$  and  $f$  as above. Let us introduce

$$(4.61) \quad A(s) \triangleq \int_{\Omega} |K_s(t) \cdot u(t)| \mu(t) \text{ a.e. } (s \in \Omega) ,$$

to make hold the following Cauchy-Schwarz inequality

$$(4.62) \quad A(s) \leq \|K_s\|_X \quad (s \in \Omega) .$$

Thus,

$$(4.63) \quad \int_{\Omega^2} |K(s, t) u(t) f(s)| \mu(s) \mu(t) = \int_{\Omega} |f(s)| A(s) \mu(s)$$

$$(4.64) \quad \leq \int_{\Omega} |f(s)| \|K_s\|_X \mu(s)$$

$$(4.65) \quad \leq \left[ \int_{\Omega} \|K_s\|_X^2 \mu(s) \right]^{\frac{1}{2}} = \|K\|_P < \infty .$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof.  $\square$

# Chapter 6

## Distributions

### 1 Exercise 1. Test functions are almost polynomial

Suppose  $f$  is a complex continuous function in  $\mathbf{R}^n$ , with compact support. Prove that  $\psi P_j \rightarrow f$  uniformly on  $\mathbf{R}^n$ , for some  $\psi \in \mathcal{D}$  and for some sequence  $\{P_j\}$  of polynomials.

*Proof.* According to 1.16,  $\Omega$  is union of a compact sets sequence  $\{K_i\}$  and  $\text{supp}(f)$  lies in some  $K = K_i$  so that  $f$  is embedded in  $\mathcal{D}(\Omega)$ . We can apply [1.10] to ensure that  $\Omega$  encloses a compact set  $S = K + B(\varepsilon)$  for sufficiently small  $\varepsilon > 0$ .

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra  $\{g \in C(S) : g \text{ is polynomial}\}$  of  $C(S)$ . There so exists a sequence  $\{P_j : j \in \mathbf{N}\}$  of  $\mathbf{R}[X_1, \dots, X_n]$  such that

$$(6.1) \quad \sup_S |f - P_j| \xrightarrow{j \infty} 0 .$$

By [6.20], the open set  $K + B(\varepsilon)$  has a local partition of unity  $\{\psi_i\} \subseteq \mathcal{D}(\Omega)$ . Moreover, there exists an integer  $l$  such that  $\psi = \psi_1 + \dots + \psi_l$  equals 1 on  $K$ . Hence

$$(6.2) \quad \|f - \psi P_j\|_\infty = \|\psi f - \psi P_j\|_\infty = \sup_S |\psi f - \psi P_j|$$

$$(6.3) \quad = \sup_S |f - P_j| \xrightarrow{j \infty} 0 .$$

□

We will actually prove more by showing that  $\mathcal{D}(\Omega)$  is separable for each nonempty open subset  $\Omega$  of  $\mathbf{R}^n$ .

*Proof.* The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by proving the separability of  $\mathcal{D}(\Omega)$ . To do so, we keep  $(\alpha, j)$  in  $\mathbf{N}^n \times \mathbf{N}$ . Remark that  $S$  encloses  $\text{supp}(D^\alpha f)$ : according to the first part, there exists a sequence  $\{P_{\alpha, j} : j \in \mathbf{N}\} \subseteq \mathbf{R}[X_1, \dots, X_n]$  such that

$$(6.4) \quad \|D^\alpha f - \psi P_{\alpha, j}\|_\infty \xrightarrow{j \infty} 0 .$$

Now let  $m$  range over  $\{1, 2, 3, \dots\}$  and set  $W_{m, j}$  in  $\mathcal{D}(\Omega)$  as follows

$$(6.5) \quad D^{-\alpha} \varphi \in \mathcal{D}(\Omega) : D^\alpha D^{-\alpha} \varphi = \varphi .$$

$$(6.6) \quad W_{m, j}(x) \triangleq D^{-(m, \dots, m)}(\psi P_{(m, \dots, m), j})$$

By (6.4), there exists a natural number  $k(m)$  such that

$$(6.7) \quad \|D^{(m, \dots, m)}(f - W_{m, j})\|_\infty < 1/m \quad (j \geq k(m)) .$$

Assume without loss of generality that  $S$  has diameter 1 so that (6.7) yields

$$(6.8) \quad \|D^\lambda(f - W_{m, k(m)})\|_\infty < 1/m \quad (|\lambda| \leq m) ,$$

by the mean value theorem. In other words (remark that  $\text{supp}(f - W_{m,k(m)}) \subseteq S$ ),

$$(6.9) \quad f - W_{m,k(m)} \in U_m \triangleq \{\varphi \in \mathcal{D}_S : \|\varphi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \dots \quad (m = 1, 2, 3, \dots)$$

Pick  $W$  in  $\beta$  (see (b) of [6.3]):  $W \cap \mathcal{D}_S$  contains a neighbourhood of 0. Hence  $W$  contains some  $U_m$ , for  $m$  sufficiently large. Thus

$$(6.10) \quad W_{m,k(m)} \xrightarrow[m \rightarrow \infty]{} f \quad (\text{in } \mathcal{D}(\Omega))$$

We have so established that the  $W_{m,k(m)}$ 's family is dense in  $\mathcal{D}(\Omega)$ . We now aim to disclose a countable set  $\tilde{W}$  that has the same property.

Choose  $\delta$  in  $\mathbf{R}_+$  and fetch any  $W_{m,k(m)}$ . Let  $X$  be  $(X_1, \dots, X_n)$  and express  $P_{(m, \dots, m), k(m)}$  as

$$(6.11) \quad P(X) = \sum_{|\gamma| \leq d} p_\gamma \cdot X^\gamma$$

Since  $\bar{Q} = \mathbf{R}$ ,  $Q[X]$  hosts some  $Q(X) = \sum_{|\gamma| \leq d} q_\gamma \cdot X^\gamma$  such that  $|p_\gamma - q_\gamma| < \delta$  for all  $\gamma$ . Thus,

$$(6.12) \quad |P(x) - Q(x)| \leq \sum_{|\gamma| \leq d} |p_\gamma - q_\gamma| |x|^{\gamma} \leq \delta \sum_{l \leq d} \binom{1+n-1}{n-1} \|x\|_\infty^l \quad (x \in \mathbf{R}^n)$$

Since  $S$  is bounded, we so obtain

$$(6.13) \quad \|\psi(P - Q)\|_\infty \in O(\delta)$$

Now define  $\tilde{W}_m$  in terms of  $Q$  as  $W_{m,k(m)}$  was defined in terms of  $P$ , and consider the integrations made in (6.6): each  $D^\lambda \tilde{W}_m$  ( $|\lambda| \leq m$ ) can be obtained from some of them. So (6.13) yields

$$(6.14) \quad \|D^\lambda (W_{m,k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \quad (|\lambda| \leq m)$$

To be more specific, these  $\lambda$ 's only exist in finite amount, so the big  $O$  can be assumed to be the same for all them. Since  $\delta$  was arbitrary, combining (6.10) with (6.14) establishes the density of the all  $\tilde{W}_m$ 's family  $\tilde{W}$ .

Furthermore, each member of  $\tilde{W}$  is only made of two ingredients:  $\psi$  and a polynomial of  $Q[X]$ . The mapping  $\psi$  is attached to some  $K_i$  and  $Q[X]$  inherits countableness from  $Q$ . Note that the “integrations packs” of (6.6) only exist in countable amount. Our  $\tilde{W}$  is then countable.  $\square$

## 6 Exercise 6. Around the supports of some distributions

(a) Suppose that  $c_m = \exp\{-(m!)^2\}$ ,  $m = 0, 1, 2, \dots$ . Does the series

$$\sum_{m=0}^{\infty} c_m (D^m \varphi)(0)$$

converges for every  $\varphi \in C^\infty(\mathbf{R})$ ?

(b) Let  $\Omega$  be open in  $\mathbf{R}^n$ , suppose  $\Lambda_i \in \mathcal{D}'(\Omega)$ , and suppose that all  $\Lambda_i$  have their supports in some fixed compact  $K \subseteq \Omega$ . Prove that the sequence  $\{\Lambda_i\}$  cannot converge in  $\mathcal{D}'(\Omega)$  unless the orders of the  $\Lambda_j$  are bounded. Hint: Use the Banach-Steinhaus theorem.

(c) Can the assumption about the supports be dropped in (b)?

*Proof.* The answer is: no. Let us establish this assertion. Assume, to reach a contradiction, that the above series converges for every smooth  $\varphi : \mathbf{R} \rightarrow \mathbf{C}$ .

The sequence  $\{c_m (D^m \varphi)(0)\}$  so converges to 0. Nevertheless, it is proved in [1.46] that  $C^\infty(\Omega)$  is not locally bounded. In other words, it is always possible to excavate a  $\varphi$  for which the magnitude of the  $m$ -th derivative at 0

is as large as we please<sup>1</sup>, e.g., greater than  $1/c_m$ . A desired contradiction is then reached. We now prove (b), again by contradiction.

To do so we assume  $\{\Lambda_j\}$  to converge to some  $\Lambda$  of  $\mathcal{D}'(\Omega)$  and we let  $Q$  run through the compact sets of  $\Omega$ . Next, we define

$$(6.15) \quad S(T, Q) \triangleq \{N \in \mathbf{N}, \exists C \in \mathbf{R}_+ : |T\varphi| \leq C \|\varphi\|_N \text{ for all } \varphi \text{ of } \mathcal{D}_Q\} \quad (T \in \mathcal{D}(\Omega)) .$$

Such subset of  $\mathbf{N}$  has a minimum  $\omega(T, Q)$ . The following value

$$(6.16) \quad \omega(T) \triangleq \max\{\omega(T, Q) : Q \subseteq \Omega, Q \text{ compact}\} \leq \infty$$

is then the order of  $T$ . Assume, to reach a contradiction, that, after passage to a subsequence,

$$(6.17) \quad \omega(\Lambda_j, Q_j) = j \quad (j = 1, 2, 3, \dots)$$

for some compact  $Q = Q_j$ . By (a) of [6.24],  $Q_j$  cuts  $\text{supp}\Lambda_j$ , say in  $p_j$ . Since  $K$  encloses  $\text{supp}\Lambda_j$ ,  $\{p_j\}$  tends, after passage to a subsequence, to some  $p$  of  $K$ . Choose a positive scalar  $r$  so that

$$(6.18) \quad \bar{B}(p, r) \triangleq \{x \in \mathbf{R}^n : |x - p| \leq r\} \subseteq \Omega .$$

Such closed ball  $\bar{B}(p, r)$  is a compact subset of  $\Omega$ . By (b) of [6.5] (which refers to [1.46])  $\mathcal{D}_{\bar{B}(p, r)}$  is then a Fréchet space. It now follows from [2.6] that  $\{\Lambda_j\}$  is equicontinuous on  $\mathcal{D}_{\bar{B}(p, r)}$ . There so exists<sup>2</sup> a nonnegative integer  $N$  such that

$$(6.19) \quad |\Lambda\varphi| \leq C \|\varphi\|_N \quad (\varphi \in \mathcal{D}_{\bar{B}(p, r)})$$

for some positive constant  $C$ . On the other hand,  $\bar{B}(p, r)$  contains almost all the  $p_j$ 's. Hence

$$(6.20) \quad |\Lambda_N \varphi| > C \|\varphi\|_N$$

for some  $\varphi$  of  $\mathcal{D}_{\bar{B}(p, r)}$ . (b) is then established.

To prove (c), we introduce a sequence  $\{x_m : m \in \mathbf{Z}\}$  of  $\Omega$  that has no limit point. Let  $\{\alpha_m : m \in \mathbf{Z}\}$  be in  $\mathbf{N}$  and so define<sup>3</sup>

$$(6.21) \quad \begin{aligned} \Lambda : \mathcal{D}(\Omega) &\rightarrow \mathbf{C} \\ \varphi &\mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m} \varphi)(x_m) \end{aligned} .$$

$\Lambda$  belongs to  $\mathcal{D}'(\Omega)$ , since  $\{x_m\}$  has no limit point. Next, we easily check that

$$(6.22) \quad \begin{aligned} \Lambda_j : \mathcal{D}(\Omega) &\rightarrow \mathbf{C} & (j \in \mathbf{N}) \\ \varphi &\mapsto \sum_{|m| \leq j} (D^{\alpha_m} \varphi)(x_m) \end{aligned}$$

is also a distribution and that  $\{\Lambda_j\}$  tends to  $\Lambda$  in  $\mathcal{D}'(\Omega)$ . Nevertheless, no  $\Lambda_j$ 's can have common support because  $\{x_m\}$  has no limit point. Our assumption can therefore be dropped.  $\square$

## 9 Exercise 9. Convergence in $\mathcal{D}(\Omega)$ vs. convergence in $\mathcal{D}'(\Omega)$

(a) Prove that a set  $E \subseteq \mathcal{D}(\Omega)$  is bounded if and only if

$$\sup\{|\Lambda\varphi| : \varphi \in E\} < \infty$$

for every  $\Lambda \in \mathcal{D}(\Omega)$ .

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<sup>1</sup>indeed [1.46] provides sufficient tools for constructive proof of this, see the  $\varphi_j - \check{\varphi}_j$  involved in (??).

<sup>2</sup>For more details, see Exercise 2.3.

<sup>3</sup>As  $\Omega = \mathbf{R}$ , the case  $\alpha_m = m$  is the ‘‘counterpart’’ of the series of (a) and the case  $(x_m, \alpha_m) = (m, 0)$  is the *Dirac comb*.

- (b) Suppose  $\{\varphi_j\}$  is a sequence in  $\mathcal{D}(\Omega)$  such that  $\{\Lambda \varphi_j\}$  is a bounded sequence of numbers, for every  $\Lambda \in \mathcal{D}'(\Omega)$ . Prove that some subsequence of  $\{\varphi_j\}$  converges, in the topology of  $\mathcal{D}(\Omega)$ .
- (c) Suppose  $\{\Lambda_j\}$  is a sequence in  $\mathcal{D}'(\Omega)$  such that  $\{\Lambda_j \varphi\}$  is bounded, for every  $\varphi \in \mathcal{D}(\Omega)$ . Prove that some subsequence of  $\{\Lambda_j\}$  converges in  $\mathcal{D}'(\Omega)$  and that the convergence is uniform on every bounded subset of  $\mathcal{D}(\Omega)$ . Hint: By the Banach-Steinhaus theorem, the restrictions of the  $\Lambda_j$  to  $\mathcal{D}_K$  are equicontinuous. Apply Ascoli's theorem.

*Proof.* Since  $\mathcal{D}(\Omega)$  locally convex space (see (b) of [6.4]), [3.18] states that  $E$  is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that  $E = \{\varphi_j : j \in \mathbf{N}\}$  is bounded: so is  $\overline{E}$ . By (c) of [6.5], there exists some  $\mathcal{D}_K$  that contains  $\overline{E}$ . Since  $\mathcal{D}_K$  has the Heine-Borel property (see [1.46]),  $\overline{E}$  is  $\tau_K$ -compact. Apply [A4] with the metrizable space  $\mathcal{D}_K$  (see [1.46]) to conclude that  $\overline{E}$  has a  $\tau_K$  limit point. It then follows from (b) of [6.5] that (b) holds.  $\square$

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