Solutions to some exercises from Walter Rudin's *Functional Analysis*

gitcordier

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Contents

Notations and Conventions

I Logic

i Propositional logic

Given propositional variables p, q, the boolean operators $\neg$, $\vee$, $\wedge$, $\Leftrightarrow$, $\Rightarrow$, $\Leftarrow$, assign boolean *truth values* as follows,

$\neg$ $\neg$p has not the truth value of p.

$\vee$ The *conjonction* p $\vee$ q is true, unless: p false, q false.

$\wedge$ The *disjunction* is false, unless: p true, q true.

$\Leftrightarrow$ The *logical equivalence* expresses *tautologies*: p $\Leftrightarrow$ q is true, unless: p has not the truth value of q. It is easily checked that $(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (p \Leftarrow q))$; see the below definitions.

$\Rightarrow$ The logical implication is denoted by $\Rightarrow$: p $\Rightarrow$ q means *if (criterion/premise)* p *then (conclusion)* q, or, alternatively, p *implies* q. p $\Rightarrow$ q is formally defined as $\neg$p $\vee$ q. Remark that the "reasoning" p $\Rightarrow$ q is always valid, unless: p true, q false. Moreover, $p \wedge (p \Rightarrow q) \Rightarrow q$ is always true.

$\Leftarrow$ q $\Leftarrow$ p is p $\Rightarrow$ q read backward. A common pronunciation is q *since* p.

For a subtle introduction to proposition logic, see Section 1.3 and Subsection 16.1.3 of [1].

ii Iverson notation

Given a boolean expression ϕ, $[\,\phi\,]$ returns the truth value of ϕ, encoded as follows,

$$[\,\phi\,] \triangleq \begin{cases} 0 & \text{if } \phi \text{ is false;} \\ 1 & \text{if } \phi \text{ is true.} \end{cases}$$

For example, $[\,1 > 0\,] = 1$ but $\left[\,\sqrt{2} \in \mathbf{Q}\,\right] = 0$.

II Special terms

i Halmos' iff and definitions

iff is a short for "if and only if". Splitting **iff** into *if-then* clauses shows that it is just a rewording of the logical equivalence $\Leftrightarrow$. All definitions will use the **iff** format; which is consistent with the fact that every definition expresses a tautology.

ii   Assigning values

Given variables a and b, $\triangleq$ is a specialization of $=$. We say that x $\triangleq$ y **iff** x and y are assumed to be equal. Usually, x $\triangleq$ y means that x is assigned the previously known value y (some authors write x := y) but this is not a limitation. Definitions can be redundant and may overlap. The only restriction is that x $\triangleq$ y is inconsistent whether x $\neq$ y.

iii   Equinumerosity

a $\equiv$ b means that there exists a bijection $\rightarrow$ that maps a to b; which let us identify a with b. In a metric space context, a $\equiv$ b means that $\rightarrow$ is isometric.

III   Topological vector spaces

i   Scalar field

The usual (complete) scalar field is $\mathbf{C}$. A property, *e.g.* linearity, that is true on $\mathbf{C}$ is also true on $\mathbf{R}$. The complex case is then a *special case* of the real one. Sometimes, this specialization is not harmless. For example, theorem 12.7 of [4] asserts that, in a Hilbert space H equipped with the inner product $\langle \cdot \,|\, \cdot \rangle$, every nonzero linear continuous operator T "breaks orthogonality", in the sense that there always exists x = x(T) in H that satisfies $\langle Tx|x \rangle \neq 0$. The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exists. To see that, consider the 90° rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitly mentioned*, the exension to the real case will always be obvious. So, taking $\mathbf{C}$ as the scalar field shall mean

> *Instead of letting the scalar field undefined, we choose* $\mathbf{C}$ *for the sake of expessivity. But considering* $\mathbf{R}$ *instead of* $\mathbf{C}$ *would actually make no difference here.*

ii   Vector space bases

Given a vector space X over $\mathbf{C}$ (or, more generally, over a field), a subset B of X is a basis of X **iff** $(z_u) \mapsto \sum_{u \in B} z_u u$ bijectively maps all *almost null* $z : B \rightarrow \mathbf{K}, u \mapsto z_u$ onto X. The axiom of choice (AC) forces

 (*a*) the existence of such B (the proof is similar to the second part of the Hahn-Banach theorem [3.1] of [4] with B playing the role of Λ);

 (*b*) all bases to have the same cardinal, which is called the *dimension* of X and is denoted as dim(X).

We now come to the *finite-dimensional* case, *i.e.* dim(X) is a nonnegative integer n. Remark that n = 0, *i.e.* B = $\emptyset$, means that X is a singleton. Our first step consists in studying $\mathbf{C}^n$, which is the standard n-dimensional vector space.

iii   Finite-dimensional spaces

From now on, the zero-dimensional case, which is trivial, shall be skipped.

The product topology of $\mathbf{C}^n$

As the n-th power of $\mathbf{C}$, $\mathbf{C}^n$ has a standard base $\{e_k : k = 1, \ldots, n\}$ (where $e_k = 1_{\{k\}}$). Furthermore, it is topologized by the *polydiscs*

$$(1) \qquad \prod_{i=1}^{n} D_{r_i} \quad (D_{r_i} \triangleq \{z_i \in C : |z_i| < r_i\}),$$

Equivalently, we may equipp $\mathbf{C}^n$ with the euclidian norm

$$(2) \qquad \|z\|_2 \triangleq \sqrt{|z_1|^2 + \cdots + |z_n|^2} \quad (z = (z_1, \ldots, z_n) \in \mathbf{C}^n),$$

whose open balls centered at the origin are all nonempty

$$(3) \qquad B_r \triangleq \{z \in \mathbf{C}^n : \|z\|_2 < r\}.$$

To see such equivalence, first pick a positive r then set $r_i = r/\sqrt{n}$, so that

$$(4) \qquad \prod_{i=1}^{n} D_{r_i} \subseteq B_r.$$

Next, conversely choosing $r = \min\{r_1, \ldots, r_n\}$ yields

$$(5) \qquad B_r \subseteq \prod_{i=1}^{n} D_{r_i}.$$

Topology of a finite-dimensional vector space

It is customary to identify any n-dimensional vector space with $\mathbf{C}^n$ topologized by the euclidian norm; see [iii]. To see this, pick a n-dimensional vector space Y, of basis $\{u_1, \ldots, u_n\}$; see [ii]. Setting $u_k = f(e_k)$ means a special case of [1.20] of [4], where f is an isomorphism of $\mathbf{C}^n$ onto Y. Actually, Y is endowed with the topology $\{f(U) : U \text{ open}\}$, and [1.21] of [4] states that f is an homeomorphism, which implies that

$\{f(U) : U \text{ open}\}$ *is the only vector space topology for* Y.

As a consequence, Y is necessarily locally convex and bounded; *i.e.* normable; see [1.39] of [4]. Moreover, provided a norm $\|\cdot\|$ on Y, there exists a positive *modulus of continuity* $C = C_f$ such that

$$(6) \qquad \|y\| \leq C\|z\|_2 \quad ((z, y) \in f),$$

since f is continuous. Now pick a n-dimensional topological vector space W then repeat the same reasoning, first with $g : \mathbf{C}^n \to W$, next with $h = g \circ f^{-1}$, in the role of f, and so conclude that the homeomorphism h maps Y's topology onto Y's topology and that W is normable. To sum it up,

$\dim(Y) = \dim(W)$, *i.e.* Y *and* W *are isomorphic each other,
means that* Y *and* W *are two normable spaces that are home-
omorphic each other.*

We then equip W with a norm $|||\cdot|||$, so that

$$(7) \qquad |||w||| \leq C_h \|y\| \quad ((y,w) \in h)$$

for some positive C_h. The special case $g = f$ means that Y's norms are
equivalent, in the sense that there exists a positive C_{id} such that

$$(8) \qquad |||y||| \leq C_{id}\|y\|.$$

The standard norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$

In all special cases $Y = \mathbf{C}^n$ topologized by the standard norms $1, 2, \infty$,
the optimal modulus, *i.e.* the smallest $C = C_{i,j}$ such that

$$(9) \qquad \|z\|_j \leq C_{i,j}\|z\|_i \quad (z \in \mathbf{C}^n)),$$

is easily derived from definitions - see [1.19] of [4] - with the noticeable
exception of $C_{2,1} = \sqrt{n}$, which is usually seen as a special *Cauchy-
Schwarz inequality*; see (1) in [12.2] of [4]. The very steps of this
classical hack are left to the reader.

Chapter 1

Topological Vector Spaces

1   Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X. *Prove the following statements from the axioms as given as in section 1.4.*

(a) *If* x, y $\in$ X *there is a unique* z $\in$ X *such that* x + z = y.

(b) $0 \cdot x = 0 = \alpha \cdot 0$ $(\alpha \in \mathbf{C}, x \in X).$

(c) $2A \subseteq A + A.$

(d) A *is convex if and only if* (s+t)A = sA+tA *for all positive scalars* s *and* t.

(e) *Every union (and intersection) of balanced sets is balanced.*

(f) *Every intersection of convex sets is convex.*

(g) *If* Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

(h) *If* A *and* B *are convex, so is* A + B.

(i) *If* A *and* B *are balanced, so is* A + B.

(j) *Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.*

Proof. (a) Such property only depends on the group structure of X: Each x in X has an opposite -x. Let x′ be any opposite of x, so that x − x = 0 = x + x′. Thus, -x + x − x = -x + x + x′, which is equivalent to -x = x′. So is established the uniqueness of -x. It is now clear that x + z = y **iff** z = -x + y, which asserts both the existence and the uniqueness of z.

(b) Remark that

$$(1.1) \qquad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$
$$(1.2) \qquad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that $0 \cdot x = 0$. So,

$$(1.3) \qquad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (\text{-}1) \cdot x \Rightarrow \text{-}1 \cdot x = \text{-}x.$$

Finally,

(1.4)
$$\alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (x + (\text{-}1 \cdot x)) = \alpha \cdot x + \alpha \cdot (\text{-}1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

(c) Remark that

$$(1.5) \qquad\qquad 2x = (1 + 1)x = x + x$$

for every x in X, and so conclude that

(1.6)
$$2A = \{2x : x \in A\} = \{x + x : x \in A\} \subseteq \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets A of X; which proves (c).

(d) If A is convex, then

$$(1.7) \qquad\qquad A \subseteq \frac{s}{s + t} A + \frac{t}{s + t} A \subseteq A;$$

which is

$$(1.8) \qquad\qquad sA + tA = (s + t)A.$$

Conversely, the special case $s + t = 1$ is

$$(1.9) \qquad\qquad sA + (1 - s)A = A.$$

The latter extends to $s = 0$, since

$$(1.10) \qquad\qquad 0A + A \overset{(b)}{=} \{0\} + A = A.$$

The extension to $s = 1$ is analogously established (or simply use the fact that $+$ is commutative!). So ends the proof.

(e) Let A range over B a collection of balanced subsets, so that

$$(1.11) \qquad\qquad \alpha \bigcap B \subseteq \alpha A \subseteq A \subseteq \bigcup B$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap B \subseteq A$ establishes the first part. Now remark that

$$(1.12) \qquad\qquad \alpha A \subseteq \bigcup B$$

implies

$$(1.13) \qquad\qquad \alpha \bigcup B \subseteq \bigcup B;$$

which achieves the proof.

(f) Let A range over C a collection of convex subsets, so that

$$(1.14) \qquad (s+t)\bigcap C \subseteq s\bigcap C + t\bigcap C \subseteq sA + tA \overset{(d)}{\subseteq} (s+t)A$$

for all positives scalars s, t. Inclusions at both extremities force

$$(1.15) \qquad s\bigcap C + t\bigcap C = (s+t)\bigcap C.$$

We now conclude from (d) that the intersection of C is convex. So ends the proof.

(g) Skip all trivial cases $\Gamma = \emptyset$, $\{\emptyset\}$, $\{\{x\}\}$, $\{\emptyset, \{x\}\}$ then pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i ($i = 1, 2$) lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_i's is then in $C_2 \subseteq \bigcup \Gamma$. Hence (g).

(h) Simply remark that

$$(1.16)$$
$$s(A+B) + t(A+B) = sA + tA + sB + tB = (s+t)(A+B)$$

for all positive scalars s and t, then conclude from (d) that $A+B$ is convex.

(i) Given any α from the closed unit disc,

$$(1.17) \qquad \alpha(A+B) = \alpha A + \alpha B \subseteq A + B.$$

There is no more to prove: $A + B$ is balanced.

(j) Our proof will be based on the following lemma,

> *If* S *is nonempty, then each of the following three properties*
> *(i)* S *is a vector subspace of* X*;*
> *(ii)* S *is convex balanced such that* S + S = S*;*
> *(iii)* S *is convex balanced such that* λS = S $\quad (\lambda > 0)$
> *implies the other two.*

To prove the lemma, let S run through all nonempty subsets of X. First, assume that (i) holds: Clearly, every S is convex balanced. Moreover, $S + S \subseteq S$. Conversely, $S = S + \{0\} \subseteq S + S$; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \qquad nS = (n-1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). Pick $\lambda > 0$ then choose n so large that $1 < n\lambda < n^2$. Thus,

$$(1.19) \qquad nS \overset{(1.18)}{\subseteq} S \subseteq n\lambda S \subseteq n^2 S,$$

since S is balanced. For instance, set $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$. Dividing the latter inclusions by n shows that

$$(1.20) \qquad\qquad S \subseteq \lambda S \subseteq nS \overset{(1.18)}{\subseteq} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

(1.21)

$$\alpha S + \beta S \overset{(a)}{=} |\alpha|S + |\beta|S \overset{(d)}{=} (|\alpha| + |\beta|)S \overset{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as S is balanced. Moreover (under the sole assumption that S is balanced), this extends to $|\alpha| + |\beta| = 0$, as follows,

$$(1.22) \qquad\quad \alpha S + \beta S = 0S + 0S \overset{(b)}{=} \{0\} \overset{(b)}{=} 0S \subseteq S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let V be a collection of vector spaces of X, of intersection I and union U. First, remark that every member of V is convex balanced: So is I (combine (e) with (f)). Next, let Y range over V, so that

$$(1.23) \qquad\qquad\qquad I + I \subseteq Y + Y \subseteq Y;$$

which yields

$$(1.24) \qquad\qquad\qquad\qquad I + I = I$$

(the fact that $I = I + \{0\} \subseteq I + I$ was tacitly used). It now follows from the lemma's (ii) $\Rightarrow$ (i) that I is a vector subspace of X. Now temporarily assume that S is totally ordered by set inclusion: Combining (e) with (g) establishes that U is convex balanced. To show that U is more specifically a vector subspace, we first remark that such total order implies that either $Z \subseteq Y$ or $Y \subseteq Z$, as Z ranges over V. A straightforward consequence is that

$$(1.25) \qquad\qquad\qquad Y \subseteq Y + Z \subseteq Y \cup Z.$$

Another one is that $Y \cup Z$ ranges over V as well. Combined with the latter inclusions, this leads to

$$(1.26) \qquad\qquad\qquad U \subseteq U + U \subseteq U.$$

It then follows from the lemma's (ii) $\Rightarrow$ (i) that U is a vector subspace of X. Finally, let A, B run through all vector subspaces of X: Combining (h) with (i) proves that A+B is convex balanced as well. Furthermore,

$$(1.27) \quad A + B \overset{(i)\Rightarrow(ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as X is an abelian group. We now conclude from (ii) that any $A + B$ is a vector subspace of X. So ends the proof.

$$\square$$

2 Exercise 2. Convex hull

The convex hull of a set A *in a vector space* X *is the set of all convex combinations of members of* A, *that is the set of all sums* $t_1x_1 + \cdots + t_nx_n$ *in which* $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$; n *is arbitrary. Prove that the convex hull of a set* A *is convex and that is the intersection of all convex sets that contain* A.

Proof. The convex hull of a set S will be denoted by co(S). Remark that $S \supseteq$ eq co(S) (to see that, take $t_1 = 1$ for each x_1 in S) and that co(A) $\supseteq$ eq co(B) where A $\supseteq$ eqB (obvious).

Our proof will directly derive from (i) $\Rightarrow$ (iv) in the following lemma,

> Let S *be a subset of a vector space* X: *Its convex hull* co(S) *is convex and the following statements*
>
> *(i)* S *is convex;*
>
> *(ii)* $s_1S + \cdots + s_nS = (s_1 + \cdots + s_n)S$ *for all positive scalar variables* $s_1, \ldots, s_n$;
>
> *(iii)* $t_1S + \cdots + t_nS = S$ *for all positive scalar variables* $s_1, \ldots, s_n$ *such that* $s_1 + \cdots + s_n = 1$;
>
> *(iv)* co(S) $=$ S
>
> *are equivalent.*

From now on, we skip the trivial case $S = \emptyset$ then only consider non-empty sets. To prove the first part, let *a*, *b* range over co(S), so that $a = t_1x_1 + \cdots + t_nx_n$ and $b = t_{n+1}x_{n+1} + \cdots + t_{n+p}x_{n+p}$ for some (t_i, x_i). Every sum $sa + (1-s)b$ $(0 \leq s \leq 1)$ is then in the convex hull of $\{x_1, \ldots, x_{n+p}\}$, since

$$(1.28) \qquad sa + (1-s)b = \sum_{i=1}^{n} st_ix_i + \sum_{i=n+1}^{n+p} (1-s)t_ix_i$$

and

$$(1.29) \qquad \sum_{i=1}^{n} st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s\sum_{i=1}^{n} t_i + (1-s)\sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets S, this reads as follows,

$$(1.30) \qquad s\,\mathrm{co}(S) + (1-s)\,\mathrm{co}(S) \subseteq \mathrm{co}(S);$$

which was our fist goal. We now aim at the equivalence (i) $\Rightarrow \cdots \Rightarrow$ (iv) $\Rightarrow$ (i): An easy proof by induction makes the implication (i) $\Rightarrow$ (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii) $\Rightarrow$ (iv) derives from the definition of the convex hull. We now close the chain with (iv) $\Rightarrow$ (i), by remarking that S is convex whether $S = \mathrm{co}(S)$. The lemma being

proved, let us establish the second part.

To do so, we start from the convexity of co(A) then set $F = \{co(A)\}$. We may enrich F as follows,

(1.31) $B \in F \Rightarrow B$ is convex and contains A.

Note that our initial predicate "[F only encompasses] *all convex sets that contain A*", is now the special case

(1.32) $B \in F \Leftrightarrow B$ is convex and contains A.

In any case, the key ingredient is that $co(A) \in F$ implies

$$(1.33) \qquad co(A) \supseteq \bigcap_{B \in F} B.$$

Conversely, the next formula

$$(1.34) \qquad co(A) \subseteq co(B) \stackrel{(i)\Rightarrow(iv)}{=} B \quad (B \in F)$$

is valid and implies

$$(1.35) \qquad co(A) \subseteq \bigcap_{B \in F} B.$$

So ends the proof $\square$

3 Exercise 3. Other basic results

Let be X as topological vector space. All sets mentioned below are understood to be the subsets of X. Prove the following statements:

(a) *The convex hull of every open set is open.*

(b) *If X is locally convex then the convex hull of every bounded set is bounded.*

(c) *If A and B are bounded, so is A+B.*

(d) *If A and B are compact, so is A+B.*

(e) *If A is compact and B is closed, then A+B is closed.*

(f) *The sum of two closed sets may fail to be closed.*

Proof. (a) Pick an open set A then let the variables V_i (i $= 1, 2, \dots$) run through all open subsets of A, so that

$$(1.36) \qquad \mathrm{co}(A) \subseteq \bigcup_{t_i} (t_1 V_1 + \dots + t_i V_i + \dots) \subseteq \mathrm{co}(A)$$

given all convex combinations $t_1 V_1 + \dots + t_i V_i + \dots$. We know from Section 1.7 of [4] that those sums are open; which achieves the proof.

(b) Provided a bounded set E, pick V a neighbourhood of 0: By (b) of Section 1.14 in [4], V contains a convex neighbourhood of 0, say W. There so exists a positive scalar s such that

$$(1.37) \qquad E \subseteq tW \subseteq tV \quad (t > s);$$

which yields

$$(1.38) \qquad \mathrm{co}(E) \subseteq \mathrm{co}(tW) = t\,\mathrm{co}(W) = tW \subseteq tV.$$

So ends the proof.

(c) At fixed V, neighbourhood of the origin, we combine the continuousness of $+$ with Section 1.14 of [4] to conclude that there exists U a balanced neighborhood of the origin such that

$$(1.39) \qquad\qquad U + U \subseteq V.$$

Moreover, by the very definition of boundedness, $A \subseteq rU$ for some positive scalar r. Similarly, $B \subseteq sU$ for some positive s. Finally,

$$(1.40) \qquad A + B \subseteq rU + sU \subseteq tU + tU \subseteq tV \quad (t > r, s),$$

since U is balanced. So ends the proof.

(d) First, A and B are compact: So is $A \times B$. Next, $+$ maps continuously $A \times B$ onto $A + B$. In conclusion, $A + B$ is compact.

(e) From now on, we assume that neither A nor B is empty, since otherwise the result is trivial. Now pick $c \in X$ outside $A + B$: The result will be established by showing that c is not in the closure of $A + B$.

To do so, we let the variable *a* range over A: Every set $a + B$ is closed as well; see Section 1.7 of [4]. Trivially, $a + B \neq c$: By Section 1.10 of [4], there so exists $V = V(a)$ a neighborhood of the origin such that

(1.41) $(a + B + V) \cap (c + V) = \emptyset.$

Moreover, there are finitely many $a + V$, say $a_1 + V_1, a_2 + V_2, \ldots$, whose union U contains the compact set A. Therefore,

(1.42) $A + B \subseteq U + B.$

Now define

(1.43) $W \triangleq V_1 \cap V_2 \cap \cdots,$

so that

(1.44) $(a_i + B + V_i) \cap (c + W) \overset{(1.41)}{=} \emptyset \quad (i = 1, 2, \ldots).$

As a conclusion, c is not in the closure of $U + B$. Finally, (1.42) asserts that c is not in $\overline{A + B}$ either; which achieves the proof.

Corollary: If B is the closure of a set S, then

(1.45) $A + B \subseteq \overline{A + S} \subseteq \overline{A + B} = A + B$

by (b) of Section 1.13 of [4] (since A is closed; see Section 1.12, from the same source). The special case $A = \{x\}$, $B = X$ will occur in the proof of Exercise 15 in chapter 2.

(f) The last proof will consist in exibhiting a counterexample. To do so, let f be any continuous mapping of the real line such that
  (i) $f(x) + f(-x) \neq 0 \quad (x \in \mathbf{R})$;
  (ii) f vanishes at infinity.

For instance, we may combine (ii) with f even and $f > 0$ by setting $f(x) = 2^{-|x|}$, $f(x) = e^{-x^2}$, $f(x) = 1/(1 + |x|)$, ..., and so on.

As a continous function, f has closed graph G; see [2.14] of [4]. Moreover, (i) implies that the origin $(0,0) \neq (x - x, f(x) + f(-x))$ is not in $G + G$. On the other hand,

(1.46) $\{(0, f(n) + f(-n)) : n = 1, 2, \ldots\} \subseteq G + G.$

Now the key ingredient is that

(1.47) $(0, f(n) + f(-n)) \xrightarrow[n\to\infty]{(ii)} (0,0).$

We have so constructed a sequence in G+G that converges outside $G + G$. So ends the proof.

□

4   Exercise 4. A nonempty set whose interior is not

Let be $B = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \leq |z_2|\}$. *Show that* B *is balanced but that its interior is not.*

Proof. It is obvious that the nonempty set B contains the origin $(0,0)$. Additionally, its interior B° is nonempty as well. Indeed, the following set

$$(1.48) \qquad \{(z_1, z_2) \in \mathbf{C}^2 : |1 - z_1| + |2 - z_2] < 1/2\} \subseteq B$$

is a neighborhood of $(1, 2) \in B$. Moreover, B is balanced, since

$$(1.49) \qquad |\alpha z_1| = |\alpha||z_1| \leq |\alpha||z_2| = |\alpha z_2| \quad (|\alpha| \leq 1)$$

for all (z_1, z_2) in B. Nevertheless, the nonempty set B° is not balanced, what we now establish by showing that $(0,0) \notin B^\circ$. To do so, assume, to reach a contradiction, that the origin has a neighborhood

$$(1.50) \qquad U \triangleq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| + |z_2] < r\} \subseteq B$$

for some positive r. Clearly, U contains $(r/2, 0)$, and that special case $(r/2, 0) \in B$ now contradicts the definition of B. So ends the proof. $\square$

5   Exercise 5. A first restatement of bound-edness

Consider the definition of "bounded set" given in Section 1.6. Would the content of this definition be altered if it were required merely required that to every neighbourhood V *of 0 corresponds* some t > 0 *such that* E $\subseteq$ tV*?*

Proof. The answer is: No. To prove it, start from (a) of Section 1.14: V contains W, a balanced neighbourhood of 0. Assume that E is bounded in this weaker sense, *i.e.* there exists a positive t that satisfies

$$(1.51) \qquad\qquad\qquad E \subseteq tW.$$

Thus,

$$(1.52) \qquad\qquad E \subseteq tW \subseteq sW \subseteq sV \quad (s > t),$$

since W is balanced. We so reach the definition given in Section 1.6: The two ones are equivalent. $\square$

6   Exercise 6.   A second restatement of boundedness

Prove that a set E *in a topological vector space is bounded if and only if every countable subset of E is bounded.*

Proof. It is clear that every subset of a bounded set is bounded. Conversely, assume that E is not bounded then pick V a neighbourhood of the origin: No counting number $n = 1, 2, \ldots$, verifies $E \subseteq nV$ (see Exercise 1 in Chapter 1). In other words, there exists a sequence $\{x_1, \ldots, x_n, \ldots\} \subseteq E$ such that

$$(1.53) \qquad\qquad x_n \notin nV.$$

As a consequence, x_n/n fails to converge to 0 as n tends to ∞. In contrast, $1/n$ succeeds. It then follows from Section 1.30 that $\{x_1, \ldots, x_n, \ldots\}$ is not bounded. So ends the proof. $\square$

7   Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$, topologized by the family of seminorms

$$p_x(f) = |f(x) \qquad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \to \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \to \infty$ then $\{\gamma_n f_n\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0,1]$.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.

Proof. The family of the seminorms p_x is separating: The collection $\mathscr{B}$ of all finite intersections of the sets

$$(1.54) \qquad V(x,k) \triangleq \{p_x < 2^{-k}\} \quad (x \in [0,1], k = 1, 2, 3, \dots)$$

is therefore a local base for a topology τ on X; see Section 1.37 of [4]. So,

(1.55)
$$\sum_{n=1}^{\infty}[f_n \notin \cap_{i=1}^{m} U_i] \le \sum_{n=1}^{\infty}\sum_{i=1}^{m}[f_n \notin U_i] = \sum_{i=1}^{m}\sum_{n=1}^{\infty}[f_n \notin U_i] \quad (f_n \in X, U_i \in \tau).$$

Now assume that $\{f_n\}$ τ-converges to some f, *i.e.*

$$(1.56) \qquad \sum_{n=1}^{\infty}[f_n \notin f + W] < \infty \quad (W \in \mathscr{B}).$$

The special case $W = V(x,k)$ means that, given k, $|f_n(x) - f(x)| < 2^{-k}$ for almost all n. In other words, $\{f_n(x)\}$ converges to $f(x)$. Conversely, assume that $\{f_n\}$ does not τ-converges in X, *i.e.*

$$(1.57) \qquad \forall f \in X, \exists W \in \mathscr{B} : \sum_{n=1}^{\infty}[f_n \notin f + W] = \infty.$$

W is now the (nonempty) intersection of finitely many $V(x,k)$, say $V(x_1, k_1), \dots, V(x_m, k_m)$. Thus,

$$(1.58) \qquad \sum_{i=1}^{m}\sum_{n=1}^{\infty}[f_n \notin f + V(x_i, k_i)] \overset{(1.55)}{\ge} \sum_{n=1}^{\infty}[f_n \notin f + W] \overset{(1.57)}{=} \infty.$$

We can now conclude that, for some index i,

$$(1.59) \qquad \sum_{n=1}^{\infty}[f_n \notin f + V(x_i, k_i)] = \infty.$$

In other words, $\{f_n(x_i)\}$ fails to converge to $f(x_i)$. We have so proved that τ-convergence is a rewording of pointwise convergence. We now establish the second part by constructing a specific sequence $\{f_n\}$ that satisfies both (a) and (b).

The proof will be based on the following well-known result: Each irrational number α has a *unique* binary expansion. More precisely, there exists a bijection

$$(1.60) \quad b : [0,1] \setminus \mathbf{Q} \to \{\beta \in \{0,1\}^{\mathbf{N}_+} : \beta \text{ is not eventually periodic}\}$$
$$\alpha \mapsto (\beta_1, \beta_2, \dots)$$

where $(\beta_1, \beta_2, \dots)$ is the only bit stream such that

$$(1.61) \qquad \alpha = \sum_{k=1}^{\infty} \beta_k 2^{-k}.$$

First, remark that $\beta_1 + \beta_2 + \cdots = \infty$ then define

$$(1.62) \qquad f_n(\alpha) \triangleq \frac{1}{1 + \beta_1 + \cdots + \beta_n}, \quad f_n = 0 \text{ on } \mathbf{Q},$$

so that f_n tends to 0 (pointwise). Independently, take an arbitrary $\gamma_n \longrightarrow \infty$: Given $p = 1, 2, 3, \dots$, γ_n is greater than 2^p for all but finitely many n. Next, we choose n_p among those *almost all* n that are large enough to additionally satisfy

$$(1.63) \qquad n - n_{p-1} > p,$$

provided $n_0 = 0$. This way, the distribution of $n_1, n_2, \dots$, *displays no periodic pattern*: Setting $\beta_{\gamma,k} = [k \in \{n_1, n_2, \dots\}]$ yields

$$(1.64) \qquad \alpha_\gamma \triangleq \sum_{k=1}^{\infty} \beta_{\gamma,k} 2^{-k} \notin \mathbf{Q}.$$

Moreover,

$$(1.65) \qquad p = \beta_{\gamma,1} + \cdots + \beta_{\gamma,n_1} + \cdots + \beta_{\gamma,n_p}.$$

Hence

$$(1.66) \qquad \gamma_{n_p} f_{n_p}(\alpha_\gamma) = \frac{\gamma_{n_p}}{p+1} > \frac{2^p}{p+1} \xrightarrow[p\to\infty]{} \infty.$$

There so exists a subsequence $\{\gamma_{n_p}\}$ that prevents $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ to converge pointwise to 0. Since $\{\gamma_n\}$ was arbitrary, this proves (b). $\qquad\square$

9   Exercise 9. Quotient map

Suppose

  (a) X *and* Y *are topological vector spaces,*

  (b) $\Lambda : X \to Y$ *is linear.*

  (c) N *is a closed subspace of* X,

  (d) $\pi : X \to X/N$ *is the quotient map, and*

  (e) $\Lambda x = 0$ *for every* $x \in N$.

Prove that there is a unique f $: X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$, *that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$. *Prove that f is linear and that* Λ *is continuous if and only if f is continuous. Also,* Λ *is open if and only if f is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [4]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

(1.67) $f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subseteq X/N \times Y.$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

(1.68) $\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'.$

It straightforwardly derives from (1.67) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

(1.69) $f(\pi x) = 0 \overset{(1.67)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [4]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

(1.70) $\Lambda(U) = f(\pi(U)) \subseteq V.$

Since π is open (1.41 (a) of [4]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [4]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

(1.71) $W = \pi(V)$

for some neighborhood V of 0_X. So,

$$(1.72) \qquad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if Λ is open, then f(W) is a neighborhood of 0_Y. So ends the proof. $\qquad \square$

10   Exercise 10.   An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \to Y$ *is linear, and* $\Lambda(X) = Y$.

(a) *Prove that* Λ *is an open mapping.*

(b) *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. Discard the trivial case $\Lambda = 0$ and assume that $\dim Y = n$ for some positive n. Let e range over a basis of B of Y then pick in X an arbitrary neighborhood W of the origin: There so exists V a balanced neighborhood of the origin of X such that

$$(1.73) \qquad\qquad \sum_e V \subseteq W,$$

since addition is continuous. Moreover, for each e, there exists x_e in X such that $\Lambda(x_e) = e$, simply because Λ is onto: Given y in Y, of e-component(s) y_e, we now obtain

$$(1.74) \qquad\qquad y = \sum_e y_e \Lambda(x_e).$$

As a finite set, $\{x_e : e \in B\}$ is bounded: There so exists a positive scalar s such that

$$(1.75) \qquad\qquad \forall e \in B, x_e \in sV.$$

Combining this with (1.74) shows that

$$(1.76) \qquad\qquad y \in \sum_e y_e \, s\Lambda(V).$$

We now come back to (1.73) and so conclude that

$$(1.77) \qquad\qquad y \in \sum_e \Lambda(V) \subseteq \Lambda(W)$$

for if $|y_e| < 1/s$; which proves (a) whether B is the standard basis of $Y = \mathbf{C}^n$ equipped with $\|\cdot\|_\infty$. The general case is now provided for free by [iii].

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. We now conclude with the help of [iii] that f is an homeomorphism of X/N onto Y. We have thus established that f is continuous: So is $\Lambda = f \circ \pi$. $\square$

12   Exercise 12. Topology stays, completeness leaves

Suppose $d_1(x, y) = |x - y|, d_2(x, y) = |\phi(x) - \phi(y)|$, *where* $\phi(x) = x/(1 + |x|)$. *Prove that* d_1 *and* d_2 *are metrics on* $\mathbf{R}$ *which induce the same topology, although* d_1 *is complete and* d_2 *is not.*

Proof. First, each d_i $(i = 1, 2)$ induces a topology τ_i whose open balls are all

(1.78) $B_i(a, r) \triangleq \{x \in \mathbf{R} : d_i(a, x) < r\}$ $(a \in \mathbf{R}, r > 0)$.

Next, remark that the monotonically increasing mapping $\phi : \mathbf{R} \to]\text{-}1, 1[$ is odd and that

(1.79) $\phi(x) \xrightarrow[x \to \infty]{} 1.$

ϕ is therefore a τ_1-homeomorphsim of $\mathbf{R}$ onto $]\text{-}1, 1[$. A first consequence is that, at fixed $a \in \mathbf{R}$, given any positive scalar ε, the τ_1-continuousness of ϕ supplies an open ball $B_1(a, \eta)$ on which $|\phi(a) - \phi| < \varepsilon$. In terms of balls B_i, this reads as follows,

(1.80) $B_1(a, \eta) \subseteq B_2(a, \varepsilon).$

The second consequence is that the τ_1-continuousness of ϕ^{-1} yields similar inclusions

(1.81) $B_2(a, \varepsilon') \subseteq B_1(a, \eta')$

provided $\eta' > 0$. At arbitrary ε, the special case $\eta' = \eta$ is the concatenation

(1.82) $B_2(a, \varepsilon') \subseteq B_1(a, \eta) \subseteq B_2(a, \varepsilon);$

which proves that $\tau_1 = \tau_2$. Finally, all inequalities $n < i < j$ over $\mathbf{N}$ together yield

(1.83) $d_2(i, j) = |\phi(i) - \phi(j)| \xrightarrow[n \to \infty]{} 0.$

The sequence $n = 0, 1, 2, \dots$ is therefore τ_2-Cauchy. We will nevertheless establish that it τ_2-diverges. To do so, we start by offering the τ_2-converge to some λ: The triangle inequality immediately dismiss that assumption, as follows,

(1.84) $d_2(0, \lambda) \geq d_2(0, n) - d_2(\lambda, n) = \phi(n) - d_2(\lambda, n) \xrightarrow[n \to \infty]{} 1.$

We then conclude that d_2 fails to be complete. $\square$

14   Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms

Put $\mathrm{K} = [0, 1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0, 1, 2, \dots$ *) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

(a) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{\infty} = \sup\{|\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})| : \infty < \mathrm{x} < \infty\}$

(b) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{1} = \int_{0}^{1} |\mathrm{D}^{\mathrm{n}}\mathrm{f}(x)|\, \mathrm{d}x$

(c) $\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{2} = \left\{\int_{0}^{1} |\mathrm{D}^{\mathrm{n}}\mathrm{f}(x)|^{2}\mathrm{d}x\right\}^{1/2}$.

Proof. Let us equipp $\mathscr{D}_{\mathrm{K}}$ with the inner product $\langle \mathrm{f}|\mathrm{g}\rangle = \int_{0}^{1} \mathrm{f}\,\bar{\mathrm{g}}$, so that $\langle \mathrm{f}|\mathrm{f}\rangle = \|\mathrm{f}\|_{2}^{2}$. The following

$$(1.85) \qquad \int_{0}^{1} 1\,|\mathrm{D}^{\mathrm{n}}\mathrm{f}| \leq \|1\|_{2}\|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{2}$$

is then a Cauchy-Schwarz inequality; see Theorem 12.2 of [4]. We so obtain

$$(1.86) \qquad \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{1} \leq \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{2} \leq \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{\infty} < \infty$$

since K has length 1. Obviously, the support of $\mathrm{D}^{\mathrm{n}}\mathrm{f}$ lies in K, hence the below equality

$$(1.87) \qquad |\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})| = \left|\int_{0}^{\mathrm{x}} \mathrm{D}^{\mathrm{n}+1}\mathrm{f}\right| \leq \int_{0}^{\mathrm{x}} |\mathrm{D}^{\mathrm{n}+1}\mathrm{f}| \leq \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_{1}.$$

Take the supremum over all $|\mathrm{D}^{\mathrm{n}}\mathrm{f}(\mathrm{x})|$: Combining (1.86) with (1.87) now reads as follows,

$$(1.88) \qquad \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{1} \leq \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{2} \leq \|\mathrm{D}^{\mathrm{n}}\mathrm{f}\|_{\infty} \leq \|\mathrm{D}^{\mathrm{n}+1}\mathrm{f}\|_{1} \leq \cdots < \infty.$$

Finally, put

$$(1.89) \qquad \mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \triangleq \{\mathrm{f} \in \mathscr{D}_{\mathrm{K}} : \|\mathrm{f}\|_{\mathrm{i}} < 2^{-\mathrm{n}}\},$$
$$(1.90) \qquad \mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} : \mathrm{n} = 0, 1, 2, \dots\},$$

so that (1.88) is mirrored by neighborhood inclusions, provided $\mathrm{i} = 1, 2, \infty$:

$$(1.91) \qquad \mathrm{V}_{\mathrm{n}}^{(1)} \supseteq \mathrm{V}_{\mathrm{n}}^{(2)} \supseteq \mathrm{V}_{\mathrm{n}}^{(\infty)} \supseteq \mathrm{V}_{\mathrm{n}+1}^{(1)} \supseteq \cdots.$$

Their subchains $\mathrm{V}_{\mathrm{n}}^{(\mathrm{i})} \supseteq \mathrm{V}_{\mathrm{n}+1}^{(\mathrm{i})}$ turn $\mathscr{B}^{(\mathrm{i})}$ into a local base of a topology τ_{i}. The whole chain (1.91) then forces

$$(1.92) \qquad \tau_{1} \subseteq \tau_{2} \subseteq \tau_{\infty} \subseteq \tau_{1};$$

which achieves the proof. $\square$

16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for $C^\infty(\Omega)$ (Section 1.46).

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms p_n, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [4] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [3]) about intersection of nonempty compact sets.

Lemma 1 *Let X be a topological space with a countable local base $\{V_n : n = 1, 2, 3, \dots\}$. If $\tilde{V}_n = V_1 \cap \cdots \cap V_n$, then every subsequence $\{\tilde{V}_{\varrho(n)}\}$ is a decreasing (i.e. $\tilde{V}_{\varrho(n)} \supseteq \tilde{V}_{\varrho(n+1)}$) local base of X.*

Proof. The decreasing property is trivial. Now remark that $V_n \supseteq \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X. Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supseteq \tilde{V}_{\rho(n)}$. $\qquad\square$

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$) *Under the same notations of Lemma 1, if $\{V_n\}$ is a decreasing local base, then so is $\{V_{\varrho(n)}\}$.*

Corollary 2 *If $\{Q_n\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{Q_{\varrho(n)}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $C(\Omega)$'s (respectively $C^\infty(\Omega)$'s) topology of the seminorms p_n, as defined in section 1.44 (respectively 1.46), then the seminorms $p_{\varrho(n)}$ define the same topology τ_Q.*

Proof. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X. Moreover,

$$(1.93) \qquad Q_{\rho(n)} \subseteq \mathring{Q}_{\rho(n)+1} \subseteq Q_{\rho(n)+1} \subseteq Q_{\rho(n+1)}.$$

Thus,

$$(1.94) \qquad Q_{\rho(n)} \subseteq \mathring{Q}_{\rho(n+1)}.$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(n)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(n)}\}$ is a local base. So, $\tau_{Q_\rho} \subseteq \tau_Q$. Conversely, the above corollary asserts that $\{V_{\rho(n)}\}$ is a local base of τ_Q, which yields $\tau_Q \subseteq \tau_{Q_\rho}$. $\qquad\square$

Lemma 2   *If a sequence of compact sets $\{Q_n\}$ satisfies the conditions specified in section 1.44, then every compact set* K *lies in allmost all Q_n°, i.e. there exists m such that*

$$(1.95) \qquad\qquad K \subseteq \overset{\circ}{Q}_m \subseteq \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots .$$

Proof. The following definition

$$(1.96) \qquad\qquad C_n \triangleq K \setminus \overset{\circ}{Q}_n$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subseteq Q_m^\circ$, for some m. Finally,

$$(1.97) \qquad K \subseteq \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots .$$

$\square$

We are now in a fair position to establish the following:

Theorem   *The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.*

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with K_i (i = 1, 2, 3, ...) and so conclude that $K_i \subseteq L_{m_i}^\circ \subseteq L_{m_i+1}^\circ \subset \cdots$ for some m_i. In particular, the special case $\varkappa_i = m_i + i$ is

$$(1.98) \qquad\qquad\qquad K_i \subseteq \overset{\circ}{L}_{\varkappa_i}.$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$(1.99) \qquad\qquad\qquad L_j \subseteq \overset{\circ}{K}_{\lambda_j}$$

Combine (1.98) with (1.99) and so obtain

$$(1.100) \qquad K_1 \subseteq \overset{\circ}{L}_{\varkappa_1} \subseteq L_{\varkappa_1} \subseteq \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subseteq K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots ,$$

---

[1]See (b) of 2.5 of [3].

[2]In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [3].

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

(1.101)
$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L.$$

So ends the proof $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

17   Exercise 17.   Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$, *for every multi-index* α.

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

(1.102)
$$V_N = \left\{ \varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \le N, x \in K_N\} < 1/N \right\}.$$

Now pick g in $V_{N+|\alpha|}$, so that

(1.103) $$\max\{|D^\gamma g(x)| : |\gamma| \le N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}.$$

(the fact that $K_N \subseteq K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

(1.104) $$\max\{|D^\beta D^\alpha g(x)| : |\beta| \le N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

(1.105) $$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad i.e. \quad D^\alpha\left(V_{N+|\alpha|}\right) \subseteq V_N,$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^\alpha|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

(1.106) $$D^\alpha|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) = D^\alpha\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right)$$
(1.107) $$\subset D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha\left(\mathscr{D}_K\right)$$
(1.108) $$\subseteq V_N \cap \mathscr{D}_K \quad \text{(see (1.105))}$$

So ends the proof.   □

Chapter 2

Completeness

3   Exercise 3. An equicontinous sequence of measures

Put $K = [-1, 1]$; define $\mathscr{D}_K$ as in section 1.46 (with $\mathbf{R}$ in place of $\mathbf{R}^n$). Suppose $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\Lambda\varphi = \lim\limits_{n \to \infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathscr{D}_K$. Show that Λ is a continuous linear functional on $\mathscr{D}_K$. Show that there is a positive integer p and a number $M < \infty$ such that

$$\left| \int_{-1}^{1} f_n(t)\varphi(t)dt \right| \leq M\|D^p\|_\infty$$

for all n. For example, if $f_n(t) = n^3 t$ on $[-1/n, 1/n]$ and 0 elsewhere, show that this can be done with $p = 1$. Construct an example where it can be done with $p = 2$ but not with $p = 1$.

We will also consider the case $p = 0$. Since all supports of $\varphi, \varphi', \varphi'', \ldots,$ are in K, we make a specialization of the mean value theorem:

Lemma   If $\varphi \in \mathscr{D}_{[a,b]}$, then

(2.1) $\|D^\alpha\varphi\|_\infty \leq \|D^p\varphi\|_\infty \left(\dfrac{\lambda}{2}\right)^{p-\alpha}$ $(\alpha = 0, 1, \ldots, p)$

at every order $p = 0, 1, 2, \ldots$; where λ is the length $|b - a|$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a+b)/2$: The mean value theorem asserts that there exists x_1 $(a < x_1 < x_0)$, such that

(2.2) $\varphi(x_0) = \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a).$

Since every $D^p\varphi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$(2.3) \qquad \varphi(x_0) = D^0\varphi(x_0)$$
$$(2.4) \qquad = D^1\varphi(x_1)(x_0 - a)$$
$$= \cdots$$
$$(2.5) \qquad = D^p\varphi(x_p)(x_0 - a) \cdots (x_{p-1} - a),$$

for all p. More compactly,

$$(2.6) \qquad D^\alpha\varphi(x_0) = D^p\varphi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a);$$

which yields,

$$(2.7) \qquad |D^\alpha\varphi(x)| \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (x \in [a, c])$$

The case $x_0 \geq c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$(2.8) \qquad |D^\alpha\varphi(x)| \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$(2.9) \qquad \|D^\alpha\varphi\|_\infty \leq \|D^p\varphi\|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}.$$

$\square$

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$(2.10) \qquad \langle u | \varphi \rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})).$$

The following inequalities

$$(2.11) \qquad |\langle u | \varphi \rangle| \leq \int_{\mathbf{R}} |u\varphi| \leq \|u\|_{L^1} \quad (\|\varphi\|_\infty \leq 1)$$

imply that every linear functional

$$(2.12) \qquad \langle u | : C_0(\mathbf{R}) \to \mathbf{C}$$
$$\varphi \mapsto \langle u | \varphi \rangle$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [4]. Conversely, u can be identified with $\langle u|$, since u is determined

(a.e) by the integrals $\langle u | \varphi \rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u |$, of norm

$$(2.13) \qquad \sup\{|\langle u | \varphi \rangle| : \|\varphi\|_\infty = 1\} = \|u\|_{L^1}.$$

Note that, in the latter equality, $\leq \|u\|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$(2.14) \qquad g_n : \mathbf{R} \to \mathbf{R}$$

$$x \mapsto \begin{cases} n^3 x & \left(x \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right) \\ 0 & \left(x \notin \left[-\frac{1}{n}, \frac{1}{n}\right]\right). \end{cases}$$

First, remark that $g_n(x) \longrightarrow 0$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n |$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$(2.15) \qquad \langle g_n | \varphi \rangle \underset{n \to \infty}{\longrightarrow} \Lambda \varphi,$$

where φ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\varphi_1^+, \ldots, \varphi_j^+, \ldots$, from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\varphi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

(ii) $\varphi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

(iii) $0 \leq \varphi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [4] for a possible construction of those φ_j^+. Let $\varphi_1^-, \ldots, \varphi_j^-, \ldots$, mirror the φ_j^+, in the sense that $\varphi_j^-(x) = \varphi_j^+(-x)$, so that

(iv) $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$ is odd, as g_n is;

(v) every φ_j is in $C_K^\infty(\mathbf{R})$;

(vi) The sequence $\{\varphi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\left\|\varphi_j\right\|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$(2.16) \quad \langle g_n | \varphi_j \rangle = 2 \int_0^1 g_n(t) \varphi_j^+(t) dt \underset{j \to \infty}{\longrightarrow} 2 \int_0^1 g_n(t) dt = \|g_n\|_{L^1} = n.$$

Finally,

$$(2.17) \qquad \|g_n\|_{L^1} \overset{(2.16)}{\leq} \sup\{|\langle g_n | \varphi \rangle| : \|\varphi\|_\infty = 1\} \overset{(2.13)}{\leq} \|g_n\|_{L^1};$$

---

[1] see 1.32, 4.1 of [4]

[2] See 7.26 of [2].

[3] See 3.14 of [4] for a definition of the related topology.

which is the desired result. So, in terms of boundedness constants:
Given n, there exists $C_n < \infty$ such that

(2.18) $|\langle g_n | \varphi \rangle| \leq C_n \quad (\|\varphi\|_\infty = 1);$

see (2.11). Furthermore, $\|g_n\|_{L^1}$ is actually the best, *i.e.* lowest, pos-
sible C_n; see (2.17). But, on the other hand, (2.16) shows that there
exists a subsequence $\{\langle g_n | \varphi_{\rho(n)} \rangle\}$ such that $\langle g_n | \varphi_{\rho(n)} \rangle$ is greater than,
say, $n - 0.01$, as $\left\| \varphi_{\rho(n)} \right\|_\infty = 1$. Consequently, there is no bound M
such that

(2.19) $|\langle g_n | \varphi \rangle| \leq M \quad (\|\varphi\|_\infty = 1; n = 1, 2, 3, \dots).$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of
all continous linear mappings $\langle g_n |$ is not equicontinous (see discussion
in 2.6 of [4]). As a consequence, the $\langle g_n |$ do not converge pointwise
(or "vaguely", in Radon measure context): A vague (*i.e.* pointwise)
convergence would be (by definition)

(2.20) $\langle g_n | \varphi \rangle \xrightarrow[n \to \infty]{} \Lambda \varphi \quad (\varphi \in C_0(\mathbf{R}))$

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of
[4]. This by no means says that the $\langle g_n |$ do not converge pointwise, in
a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitly stated, we asume that
φ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable
function such that

(2.21) $\Lambda \varphi = \lim_{n \to \infty} \int_K f_n \varphi \quad (\varphi \in C_K^\infty(\mathbf{R})).$

for some linear form Λ. Since φ vanishes outside K, we can suppose
without loss of generality that the support of f_n lies in K. So, (2.21)
can be restated as follows,

(2.22) $\Lambda \varphi = \lim_{n \to \infty} \langle f_n | \varphi \rangle \quad (\varphi \in C_K^\infty(\mathbf{R})).$

Let $K_1, K_2, \dots$, be compact sets that satisfy the conditions specified
in 1.44 of [4]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms
$p_1, p_2, \dots$; see 1.46, 6.2 of [4] and Exercise 1.16. We know that $K \subseteq K_m$
for some index m (see Lemma 2 of Exercise 1.16): From now on, we
only consider the indices $N \geq m$, so that

(a) $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha \varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\varphi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\varphi \in \mathscr{D}_K : \|\varphi\|_N < 2^{-N}\}$ is a
     (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see
     6.2 of [4] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\varphi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|f_n\|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [4].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [4]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi$ for all φ, $i.e.$ $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [4], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$(2.23) \qquad\qquad |\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i,$$
$$(2.24) \qquad\qquad |\Lambda\varphi - \langle f_n|\varphi\rangle| < 1/2 \quad \text{if } \varphi \in V_j.$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$(2.25) \qquad |\langle f_n|\varphi\rangle| \le |\Lambda\varphi - \langle f_n|\varphi\rangle| + |\Lambda\varphi| < 1 \quad \text{if } \varphi \in V_p.$$

Now remark that every $\psi = \psi[\mu, \varphi]$, where

$$(2.26) \qquad \psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p\|\varphi\|_p)\varphi & (\varphi \ne 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$(2.27) \qquad |\langle f_n|\psi\rangle| < 1$$
$$(2.28) \qquad |\langle f_n|\varphi\rangle| < 2^p\|\varphi\|_p \cdot \mu$$
$$(2.29) \qquad |\langle f_n|\varphi\rangle| \le 2^p\|\varphi\|_p$$
$$(2.30) \qquad |\langle f_n|\varphi\rangle| \le 2^p\{\|D^0\varphi\|_\infty + \cdots + \|D^p\varphi\|_\infty\}.$$

Finally, with the help of (2.1),

$$(2.31) \qquad\qquad |\langle f_n|\varphi\rangle| \le 2^p(p+1)\|D^p\varphi\|_\infty.$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3 x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\varphi = \lim\limits_{n\to\infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\varphi\rangle| \leq M\|D^p\varphi\|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subseteq K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let φ keep running on $\mathscr{D}_K$ and so define

$$(2.32) \qquad B_n(\varphi) \triangleq \max\{|\varphi(x)| : x \in [\text{-}1/n, 1/n]\},$$

$$(2.33) \qquad \Delta_n(\varphi) \triangleq \max\{|\varphi(x) - \varphi(0)| : x \in [\text{-}1/n, 1/n]\}.$$

The mean value asserts that

$$(2.34) \qquad |\varphi(1/n) - \varphi(\text{-}1/n)| \leq B_n(\varphi')|1/n - (\text{-}1/n)| = \frac{2}{n}B_n(\varphi').$$

Independently, an integration by parts shows that

$$(2.35) \qquad \langle f_n|\varphi\rangle = \left[\frac{n^3 t^2}{2}\varphi(t)\right]_{\text{-}1/n}^{1/n} - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2\varphi'(t)dt$$

$$(2.36) \qquad\qquad = \frac{n}{2}\left(\varphi(1/n) - \varphi(\text{-}1/n)\right) - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2\varphi'(t)dt.$$

Combine (2.34) with (2.36) and so obtain

$$(2.37) \qquad |\langle f_n|\varphi\rangle| \leq \frac{n}{2}|\varphi(1/n) - \varphi(\text{-}1/n)| + \frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2|\varphi'(t)|dt$$

$$(2.38) \qquad\qquad \leq B_n(\varphi') + \frac{n^3}{2}B_n(\varphi')\int_{\text{-}1/n}^{1/n} t^2 dt$$

$$(2.39) \qquad\qquad \leq \frac{4}{3}B_n(\varphi')$$

$$(2.40) \qquad\qquad \leq \frac{4}{3}\|\varphi'\|_\infty.$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\varphi')$ tends to $|\varphi'(0)|$, we may expect that f_n tends to $\frac{4}{3}\varphi'(0)$. This is actually true: A straightforward computation shows that

(2.41)
$$\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0) \overset{(2.36)}{=} \frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0) - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n}(\varphi' - \varphi'(0))t^2 dt.$$

So,

(2.42)
$$\left|\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0)\right| \leq \left|\frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0)\right| + \frac{1}{3}\Delta_n(\varphi') \xrightarrow[n\to\infty]{} 0.$$

We have just proved that

$$(2.43) \qquad\qquad \langle f_n|\varphi\rangle \xrightarrow[n\to\infty]{} \frac{4}{3}\varphi'(0) \quad (\varphi \in \mathscr{D}_K).$$

In other words,

$$(2.44) \qquad \langle f_n| \xrightarrow[n\to\infty]{} -\frac{4}{3}\delta',$$

where δ is the *Dirac measure* and $\delta', \delta'', \ldots$, its *derivatives*; see 6.1 and 6.9 of [4].

It follows from the previous part that $-\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$(2.45) \qquad |\langle f_n|\varphi\rangle| \leq \frac{4}{3}\|\varphi'\|_\infty \quad (n = 1, 2, 3, \ldots)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$(2.46) \qquad \{\langle f_n|\varphi_{\rho(n)}\rangle : \left\|\varphi_{\rho(n)}\right\|_\infty = 1; n = 1, 2, 3, \ldots\}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$(2.47) \qquad |\langle f_n|\varphi\rangle| \leq M\|\varphi\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* p = 1 is the smallest possible p and, given p = 1, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*[4] f_n', as follows

$$(2.48) \qquad f_n' : \mathscr{D}_K \to \mathbf{C}$$
$$\varphi \mapsto -\langle f_n|\varphi'\rangle.$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$(2.49) \qquad D : \mathscr{D}_K \to \mathscr{D}_K$$
$$\varphi \mapsto \varphi';$$

see Exercise 1.17. f_n' is therefore continuous. Now apply (2.43) with φ' and so obtain

$$-\langle f_n|\varphi'\rangle \xrightarrow[n\to\infty]{} \frac{4}{3}\varphi''(0) \quad (\varphi \in \mathscr{D}_K),$$

i.e.

$$(2.50) \qquad f_n' \xrightarrow[n\to\infty]{} \frac{4}{3}\delta''.$$

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[4]See 6.1 of [4] for a further discussion.

It follows from (2.40) that,

$$(2.51) \qquad |\langle f_n|\varphi'\rangle| \leq \frac{4}{3}\|\varphi''\|_\infty \quad (n = 1, 2, 3, \dots).$$

It is therefore possible to uniformly bound $f_n{}'$ with respect to a norm $\|D^p\cdot\|_\infty$, namely $\|D^2\cdot\|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$(2.52) \qquad |\langle f_n|\varphi'\rangle| \leq M\|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Define

$$(2.53) \qquad \Phi_j(x) = \int_{-1}^{x} \varphi_j.$$

The oddness of φ_j forces Φ_j to vanish outside $[-1, 1]$: φ_j is therefore in $\mathscr{D}_K$. So, under our assumption,

$$(2.54) \qquad |\langle f_n|\Phi_j'\rangle| \leq M\|\Phi_j'\|_\infty \quad (n = 1, 2, 3, \dots);$$

which is

$$(2.55) \qquad |\langle f_n|\varphi_j\rangle| \leq M \quad (n = 1, 2, 3, \dots).$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n|\varphi_{\rho(n)}\rangle\}$) and so conclude that there is no constant M such that

$$(2.56) \qquad |\langle|f_n\varphi'\rangle| \leq M\|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots).$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$(2.57) \qquad |\langle f_n|\varphi'\rangle| \leq M\|\varphi\|_\infty.$$

The mean value theorem (see (2.1)) asserts that

$$(2.58) \qquad |\langle f_n|\varphi'\rangle| \leq M\|\varphi\|_\infty \leq M\|\varphi'\|_\infty;$$

which is, again, a desired contradiction. So ends the proof.

$\square$

6   Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ of a function $f \in L^2(T)$ (T is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all $n \in \mathbf{Z}$ (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f \text{ exists}\}$ is a dense subspace of $L^2(T)$ of the first category.

Proof. Let $f(\theta)$ stand for $f(e^{i\theta})$, so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$(2.59) \qquad \hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

$$(2.60) \qquad \Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well known (and easy to prove) result is

(2.61)
$(e_n, e_{n'}) = [n = n']$, *i.e.* $\{e_n : n \in \mathbf{Z}\}$ is an orthormal subset of $L^2(T)$.

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

(2.62)
$$\|\Lambda_n\|^2 \overset{(2.60)}{=} \|e_{-n} + \cdots + e_n\|^2 \overset{(2.61)}{=} \|e_{-n}\|^2 + \cdots + \|e_n\|^2 \overset{(2.61)}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

$$(2.63) \qquad B \triangleq \{f \in L^2(T) : \sup\{|\Lambda_n f| : n = 1, 2, 3, \ldots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [4] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$(2.64) \qquad \|\Lambda_n\| \overset{(2.62)}{=} \sqrt{2n + 1} \underset{n\to\infty}{\longrightarrow} \infty.$$

We have just established that B is actually of the first category; and so is its subset L=$\{f \in L^2(T) : \lim_{n\longrightarrow\infty} \Lambda_n f \text{ exists}\}$. We now prove

that L is nevertheless dense in $L^2(T)$. To do so, we let P be span$\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$(2.65) \qquad\qquad P \subseteq L \subseteq L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.65), that

$$(2.66) \qquad\qquad L^2(T) = \overline{P} = \overline{L}.$$

So ends the proof $\square$

9   Exercise 9. Boundedness without closedness

Suppose X, Y, Z *are Banach spaces and*

$$B : X \times Y \to Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ *such that*

$$\|B(x, y)\| \leq M\|x\|\|x\| \quad (x \in X, y \in Y).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Since B is continous at the origin, there exists a positive r such that

$$(2.67) \qquad \qquad \|x\| + \|y\| < r \Rightarrow \|B(x, y)\| < 1.$$

Given nonzero x, y, let s range over $]0, r[$, so that the folllowing bound

$$(2.68) \quad \|B(x, y)\| = \frac{4\|x\|\|y\|}{s^2} \left\| B\left(\frac{s}{2\|x\|}x, \frac{s}{2\|y\|}y \right) \right\| \overset{(2.67)}{<} \frac{4\|x\|\|y\|}{s^2}$$

is effective. It is now obvious that

$$(2.69) \qquad B(x, y) \leq \frac{4}{s^2}\|x\|\|y\| \underset{s \to r}{\longrightarrow} \frac{4}{r^2}\|x\|\|y\| \quad ((x, y) \in X \times Y);$$

which achieves the proof.

As a concrete example, choose $X = Y = Z = C_c(\mathbf{R})$, topologized by the supremum norm. $C_c(\mathbf{R})$ is not complete (see 5.4.4 of [5]), nevertheless the bilinear product

$$\begin{aligned} B : \quad C_c(\mathbf{R})^2 &\to \quad C_c(\mathbf{R}) \\ (f, g) &\mapsto \quad f \cdot g \end{aligned}$$

is bounded (since $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$), and continuous. To show this, pick a positive scalar ε smaller than 1, provided any (f, g). Next, define

$$(2.70) \qquad \qquad r \triangleq \frac{\varepsilon}{1 + \|f\|_\infty + \|g\|_\infty} < 1.$$

We now restrict (u, v) to a particular neighborhood of (f, g). More specifically,

$$(2.71) \qquad \qquad \|f - u\|_\infty + \|g - v\|_\infty < r.$$

Next, remark that $\|u\|_\infty \le r + \|f\|_\infty$ and so obtain (bear in mind that $r < 1$)

$$(2.72) \qquad \|fg - uv\|_\infty = \|(f - u) \cdot g + u \cdot (g - v)\|_\infty$$

$$(2.73) \qquad \le \|f - u\|_\infty \cdot \|g\|_\infty + \|u\|_\infty \cdot \|g - v\|_\infty$$

$$(2.74) \qquad < r \cdot \|g\|_\infty + (r + \|f\|_\infty) \cdot r$$

$$(2.75) \qquad < r \cdot (r + \|f\|_\infty + \|g\|_\infty)$$

$$(2.76) \qquad < \varepsilon.$$

Since ε was arbitrary, it is now established that B continuous at every (f, g). $\qquad\qquad\square$

10 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let (X_1, X_2, Z) be topological spaces and B a bilinear mapping

$$(2.77) \qquad B : X_1 \times X_2 \to Z.$$

From now on, $x = (x_1, x_2)$ denotes an arbitrary element of $X_1 \times X_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $X_1 \times X_2$, *i.e.* given an arbitrary **balanced** open subset W of Z, there exists in X_i $(i = 1, 2)$ a **balanced** open subset U_i such that

$$(2.78) \qquad B(U_1 \times U_2) \subseteq W.$$

In such context, $\lambda_i(x)$ is chosen greater than $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$; see [1.33] of [4] for further reading about the *Minkowski functionals* μ. In other words, x_i lies in $\lambda_i(x)U_i$, since U_i is balanced. Thus,

$$(2.79) \qquad B(x_1, x_2) = \lambda_1(x)\lambda_2(x) \cdot B(x_1/\lambda_1(x), x_2/\lambda_2(x))$$
$$(2.80) \qquad \qquad \in \lambda_1(x)\lambda_2(x) \cdot B(U_1 \times U_2)$$
$$(2.81) \qquad \qquad \subseteq \lambda_1(x)\lambda_2(x) \cdot W.$$

Pick $p = (p_1, p_2)$ in $X_1 \times X_2$, and let $q = (q_1, q_2)$ range over $X \times Y$, as a first step: It directly follows from (2.81) that

$$(2.82)$$
$$B(p) - B(q) = B(p_1, p_2 - q_2) + B(p_1, q_2) - B(q_1, q_2)$$
$$(2.83)$$
$$= B(p_1, p_2 - q_2) + B(p_1 - q_1, q_2)$$
$$(2.84)$$
$$= B(p_1, p_2 - q_2) + B(p_1 - q_1, q_2 - p_2) + B(p_1 - q_1, p_2)$$
$$(2.85)$$
$$\in \lambda_1(p)\lambda_2(p - q)W + \lambda_1(p - q)\lambda_2(q - p)W + \lambda_1(p - q)\lambda_2(p)W.$$

We now restrict q to a particular neighborhood of p. More specifically,

$$(2.86) \qquad p_i - q_i \in \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 2} U_i;$$

which implies

$$(2.87) \qquad \mu_i(q_i - p_i) = \mu_i(p_i - q_i) \le \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 2}$$

(the equality at the left is valid, since $U_i = -U_i$). The special case

$$(2.88) \qquad \lambda_i(p) \triangleq \mu_1(p_1) + \mu_2(p_2) + 1,$$

$$(2.89) \qquad \lambda_i(p - q) \triangleq \frac{1}{\mu_1(p_1) + \mu_2(p_2) + 1} \triangleq \lambda_i(q - p)$$

implies that

(2.90) $\mathrm{B(p) - B(q) \in W + W + W,}$

since W is balanced. W being arbitrary, we have so established the
continuousness of B at arbitrary p; which achieves the proof. $\square$

12 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\|f\| = \int_0^1 |f(t)| \ dt.$$

Put $B(f, g) = \int_0^1 f(t)g(t)dt$, *and show that* B *is a bilinear continuous functional on* $X \times X$ *which is separately but not continuous.*

Proof. Let f denote the first variable, g the second one. Remark that

(2.91) $$|B(f, g)| < \|f\| \cdot \max_{[0,1]} |g|;$$

which is sufficient (1.18 of [4]) to assert that any $f \mapsto B(f, g)$ is continuous. The continuity of all $g \mapsto B(f, g)$ follows (Put $C(g, f) = B(f, g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

(2.92) $$|B(f, g)| < M\|f\|\|g\|.$$

Put

(2.93) $$f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \qquad (n = 1, 2, 3, \dots),$$

so that

(2.94) $$\|f_n\| = \frac{2\sqrt{n}}{n+1} \xrightarrow[n \to \infty]{} 0.$$

On the other hand,

(2.95) $$B(f_n, f_n) = \frac{4n}{2n+1} > 1.$$

Finally, we combine (2.95) and (2.92) with (2.94) and so obtain

(2.96) $$1 < B(f_n, f_n) < M\|f_n\|^2 \xrightarrow[n \to \infty]{} 0.$$

Our continuousness assumption is then contradicted. So ends the proof.
$\square$

15   Exercise 15. Baire's cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* Y = X. Hint: Y *must intersect* x + Y *for every* x ∈ X.

Proof. Assume Y is a subgroup of X. Under our assumptions, there exists a sequence $\{E_n : n = 1, 2, 3, \ldots\}$ of X such that

(i) $(\overline{E}_n)^{\circ} = \emptyset$;

(ii) $X \setminus Y = \bigcup_{n=1}^{\infty} E_n$.

By (i), the complement V_n of $\overline{E}_n$ is a dense open set. Now pick x in X: $x + V_n$ is dense open as well, since the translation by x is an homeomorphism of X. Note that the density is also a special case of 1.3 (b) of [4], as follows,

$$(2.97) \qquad\qquad X = x + X \subseteq \overline{x + V_n}.$$

We now apply the Baire's theorem twice to establish that

(i) every intersection $W_n = V_n \cap (x + V_n)$ is dense in X;

(ii) so is the intersection $\bigcap_{n=1}^{\infty} W_n$.

Bear in mind that every dense subset of X is nonempty and remark that

$$(2.98) \qquad\qquad \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq Y$$

holds, since $X \setminus Y \subseteq \bigcup_{n=1}^{\infty} \overline{E}_n$. Now pick w in $\bigcap_{n=1}^{\infty} W_n$, so that w is an element of Y (by (2.98)) that lies in every $x + V_n$. Hence

$$(2.99) \qquad\qquad w - x \in \bigcap_{n=1}^{\infty} V_n \subseteq Y;$$

again with (2.98). It is now clear that $x + Y$ cuts Y in $x + (w - x) = w$, which establishes that x lies is $Y - Y$. As a conclusion,

$$(2.100) \qquad\qquad\qquad X \subseteq Y,$$

since x was arbitrary and that Y is a subgroup. So ends the proof. □

16 Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $f : X \to K$ *is a closed subset of* $X \times K$. *Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

Proof. Choose a sequence $\{x_n : n = 1, 2, 3, \dots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of $X \times K$. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [4]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.101) \qquad f : [0, \infty) \to [0, \infty)$$

$$x \mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases}$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$(2.102) \qquad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b). To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on $\mathbb{R}_+$. This establishes that $(a, b) \in G_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. $\square$

Chapter 3

Convexity

3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subseteq X$ an internal point of A if $A - x_0$ is an absorbing set.

(a) *Suppose A and B are disjoint convex sets in X, and A has an internal point. Prove that there is a nonconstant linear functional Λ such that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) *Show (with $X = \mathbf{R}^2$, for example) that it may not possible to have $\Lambda(A)$ and $\Lambda(B)$ disjoint, under the hypotheses of (a).*

Proof. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [4] to establish the existence of a linear functional $\Lambda : X \to \mathbf{R}$ such that

(3.1) $$\Lambda \leq p$$

and

(3.2) $$\Lambda(b_0 - x_0) = 1.$$

Then

(3.3)
$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

(3.4) $$\Lambda a \leq \Lambda b.$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There so exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

(3.5) $$\Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that $y_1 < y_2$. Then,

(3.6) $2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2)$.

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

(3.7) $\Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1$;

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

(3.8) $S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$

(3.9) $S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$

(3.10) $A \triangleq S_1 \cup S_2,$

(3.11) $B \triangleq X \setminus A.$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

(3.12)
$$t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1 - t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1 - t) \cdot x_2 \\ t \cdot y_1 + (1 - t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subseteq A.$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.* A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a linear functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

(3.13) $\Lambda(L) = \{0\} \subseteq \Lambda(A) \cap \Lambda(B) \neq \emptyset$.

So ends the proof.   □

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[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [3].

11   Exercise 11. Meagerness of the polar

Let X *be an infinite-dimensional Fréchet space. Prove that* X^**, with its weak*-topology, is of the first category in itself.*

This is actually a consequence of the below lemma, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual* X^* *separates points on* X*, then the polar*

$$(3.14) \qquad\qquad K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \ on \ A\}$$

of any absorbing subset A *is a weak*-closed set that has empty interior.*

Proof. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous; see 3.14 of [4]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see 3.11 of [4]). Put

$$(3.15) \qquad W_{F,x} \triangleq \bigcap_{x \in F}\{\Lambda \in X^* : |\Lambda x| < r_x\} \qquad (r_x > 0)$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$(3.16) \qquad\qquad M \triangleq \mathrm{span}(F)$$

is finite dimensional. Assume, to reach a contradiction, that $A \subseteq M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subseteq M$: Now pick *a* in $A \setminus M$ and so conclude that

$$(3.17) \qquad\qquad b \triangleq \frac{a}{t_a} \in A$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see 1.21 (b) of [4] for a proof): By the Hahn-Banach theorem 3.5 of [4], there exists Λ_a in X^* such that

$$(3.18) \qquad\qquad \Lambda_a b > 2$$

and

(3.19) $$\Lambda_a(M) = \{0\}.$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

(3.20) $$|\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

(3.21) $$\Lambda + W \not\subseteq K_A.$$

Since W and Λ are both arbitrary, this achieves the proof. $\square$

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

(3.22) $$B_n \triangleq \{x \in X : d(0,x) < 1/n\} \qquad (n = 1, 2, 3, \dots)$$

form a local base of X. If Λ is in X^*, then

(3.23) $$|\Lambda| \leq m \text{ on } B_n$$

for some $(n, m) \in \{1, 2, 3, \dots\}^2$; see 1.18 of [4]. Hence, X^* is the countable union of all

(3.24) $$m \cdot K_n \qquad (m, n = 1, 2, 3, \dots),$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [4]. As a consequence, the above lemma implies

(3.25) $$\left(\overline{K_n}\right)^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by m is an homeomorphism (see 1.7 of [4]), this is equivalent to

(3.26) $$\left(\overline{m \cdot K_n}\right)^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof. $\square$

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

1   Exercise 1. Basic results

Let φ be the embedding of X into X^{**} decribed in Section 4.5. Let τ be the weak topology of X, and let σ be the weak*- topology of X^{**}- the one induced by X^*.

(a) *Prove that φ is an homeomorphism of (X, τ) onto a dense subspace of (X^{**}, σ).*

(b) *If B is the closed unit ball of X, prove that $\varphi(B)$ is σ-dense in the closed unit ball of X^{**}. (Use the Hahn-Banach separation theorem.)*

(c) *Use (a), (b), and the Banach-Alaoglu theorem to prove that X is reflexive if and only if B is weakly compact.*

(d) *Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.*

(e) *If X is reflexive and Y is a closed subspace of X, prove that X/Y is reflexive.*

(f) *Prove that X is reflexive if and only X^* if reflexive.*
    Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\varphi(X)$ of X^{**}.

Proof. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

(4.1) $$\varphi^{-1} : \varphi(X) \to X$$
(4.2) $$\varphi(x) \mapsto x$$

is an homeomorphism (with respect to τ and σ). We first consider

(4.3) $V \triangleq \{x^{**} \in X^{**} : |\langle x^{**} | \psi x^* \rangle| < r\}$ $(x^* \in X^*, r > 0)$;

(4.4) $U \triangleq \{x \in X : |\langle x | x^* \rangle| < r\}$ $(x^* \in X^*, r > 0)$.

and remark that the so defined V's (respectively U's) shape a local subbase $\mathscr{S}_\sigma$ (respectively $\mathscr{S}_\tau$) of σ (respectively τ). We now observe that

(4.5) $U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X$ $(V \in \mathscr{S}_\sigma,\ U \in \mathscr{S}_\tau)$,

since φ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathscr{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that φ^{-1} and its inverse φ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

(4.6) $$\overline{\varphi(X)}_\sigma \subseteq X^{**}   .$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\varphi(X)$. By [3.5], there so exists y^* in X^* such that

(4.7) $\langle \varphi x,\ \psi y^* \rangle = \langle y^*,\ \varphi x \rangle = \langle x,\ y^* \rangle = 0$ $(x \in X)$;

(4.8) $\langle z^{**},\ \psi y^* \rangle = 1$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

(4.9) $$\varphi(B) \subseteq B^{**}   ,$$

since φ is isometric. Hence

(4.10) $$\overline{\varphi(B)}_\sigma \subseteq \overline{(B^{**})}_\sigma = B^{**}   .$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\varphi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

(4.11) $$|\psi y^*| \leq 1   \text{on } \overline{\varphi(B)}_\sigma   ;$$

(4.12) $$\langle z^{**},\ \psi y^* \rangle > 1   .$$

It follows from (4.11) that

(4.13) $|\psi y^*| \leq 1 \text{ on } \varphi(B),\ i.e. |y^*| \leq 1 \text{ on } B$.

We have so proved that

(4.14) $y^* \in B^*$.

Since z^{**} lies in B^{**}, it is now clear that

(4.15) $|\langle z^{**}, \psi y^* \rangle| \leq 1$;

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

(4.16)   B is weakly compact if and only if $\varphi(B)$ is weak*-compact.

If B is weakly compact, then $\varphi(B)$ is weak*-closed. So,

(4.17) $\varphi(B) = \overline{\varphi(B)}_\sigma \overset{(b)}{=} B^{**}$.

φ is therefore onto, *i.e.* X is reflexive.
Conversely, keep φ as onto: one easily checks that $\varphi(B) = B^{**}$. The
image $\varphi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now
follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y
be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]).
On the other hand, it follows from (c) that B is weakly compact. We
now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact.
We again use (c) to conclude that Y is reflexive. (d) is therefore estab-
lished. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to
obtain, first

(4.18) $(X/Y)^* \equiv Y^\perp$,

next,

(4.19) $(X/Y)^{**} \equiv (Y^\perp)^* \equiv X^{**}/(Y^\perp)^\perp \equiv X/Y$.

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma
(L)

> *Given a reflexive Banach space* Z, *the weak*-topology of* Z^*
> *is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of
[4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^*
into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes

X^{**} reflexive. On the other hand, $\varphi(X)$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\varphi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \varphi(X)$ is weakly compact, *i.e.* weak*-compact (to see this, apply (L) with $Z = X^*$).

By (a), B is therefore weakly compact, *i.e.* X is reflexive; see (c). So ends the proof. $\square$

13   Exercise 13. Operator compactness in a Hilbert space

(a) *Suppose* $T \in \mathscr{B}(X, Y), T_n \in \mathscr{B}(X, Y)$ *for* $n = 1, 2, 3, \ldots$, *each* T_n *has finite-dimensional range, and* $\lim \|T - T_n\| = 0$. *Prove that* T *is compact.*

(b) *Assume* Y *is a Hilbert space, and prove the converse of (a): Every compact* $T \in \mathscr{B}(X, Y)$ *can be approximated in the operator norm by operators with finite-dimensional ranges. Hint: In a Hilbert space there are linear projections of norm 1 onto any closed subspace. (See theorems 5.16, 12.4.)*

Proof. Since each T_n is compact, (a) follows from (c) of [4.18]. Besides, we take the opportunity to alternatively prove that the compact operators subspace is norm closed.

Reset every T_n as a compact operator. Let $\{x_0^i : i \in \mathbf{N}\}$ be in U the open unit ball of X. Since T_1 is compact, $\{x_0^i\}$ contains a subsequence $\{x_1^i : i \in \mathbf{N}\}$ such that $\{T_1 x_1^i\}$ converges to a point y_1 of Y. The same reasoning can be recursively applied to T_n and $\{x_{n-1}^i\} \subseteq U$ so that $\{T_n x_n^i\}$ tends to some y_n of Y, as $\{x_n^i\}$ is a subsequence of $\{x_{n-1}^i\}$. Then

$$(4.20) \qquad T_n x_p^i \xrightarrow[i \to \infty]{} y_n \quad (\text{p } n = 1, 2, 3, \cdots) \quad .$$

Applied with $\{x_n^i : (n, i) \in \mathbf{N}^2\}$, a Cantor's diagonal process therefore provides a subsequence $\{\tilde{x}_j : j \in \mathbf{N}\}$ such that

$$(4.21) \qquad T_j \tilde{x}_k \xrightarrow[k \to \infty]{} y_j \quad ;$$

$$(4.22) \qquad T_j \tilde{x}_j \xrightarrow[j \to \infty]{} y_j \quad .$$

We now easily obtain

(4.23)
$$\|T_j \tilde{x}_j - T_k \tilde{x}_k\| \leq \|T_j \tilde{x}_j - y_j\| + \|y_j - T_j \tilde{x}_k\| + \|T_j - T_k\| \xrightarrow[k > j \to \infty]{} 0 \quad .$$

$\{T_j \tilde{x}_j\}$ is then a Cauchy sequence. So is $\{T \tilde{x}_j\}$, since $\|T - T_j\| \to 0$. On the other hand, Y is complete: (a) is then proved and we now establish the counterpart in a Hilbert space.

Fix ε as a positive scalar. Since T is compact, Y contains a finite set C such that

$$(4.24) \qquad T(U) \subseteq \bigcup_{c \in C} B(c, \varepsilon) \quad .$$

As a Hilbert space, Y contains a *maximal orthonormal set* (or *Hilbert basis*) M. This implies that span(M) is dense in Y; *cf.* 4.18 & 4.22 of [3]. The finiteness of C forces M to enclose a finite set S so that

$$(4.25) \qquad \forall c \in C, \exists s(c) \in \text{span}(S) : \|c - s(c)\| < \varepsilon \quad .$$

Let x be in U. It follows from (4.24) that

$$(4.26) \qquad \|Tx - c_x\| < \varepsilon$$

for some c_x of C. We now combine (4.25) and (4.26) to obtain

$$(4.27) \qquad \|Tx - s(c_x)\| \leq \|Tx - c_x\| + \|c_x - s(c_x)\| < 2\varepsilon$$

As a finite-dimensional subspace, span(S) is closed (see footnote 4, Exercise 1.10). We so obtain

$$(4.28) \qquad Y = \mathrm{span}(S) \oplus \mathrm{span}(S)^\perp \quad ,$$

by [12.4]. There so exists a unique projection projection $\pi = \pi(\varepsilon)$ of Y onto itself (see [5.6] for the definition) such that

$$(4.29) \qquad \pi(Y) = \mathrm{span}(S), \ (I - \pi)(Y) = \mathrm{span}(S)^\perp \quad .$$

It is easily checked that π has norm 1. Moreover,

$$(4.30) \qquad \pi s = s \quad (s \in \mathrm{span}(S)) \quad .$$

Thus,

$$(4.31) \qquad (I - \pi)(Tx) = (I - \pi)(Tx - s(c_x)) \quad (x \in U) \quad .$$

Then,

$$(4.32) \qquad \|(I - \pi)(Tx)\| \leq \|I - \pi\| \, \|Tx - s(c_x)\| < 4\varepsilon \quad (x \in U)$$

(the fact that π has norm 1 is hidden in the right side inequality). We have just so proved that

$$(4.33) \qquad \|T - \pi \circ T\| \in \mathop{O}_{\varepsilon \sim 0}(\varepsilon) \quad .$$

That is particularly true whether $\varepsilon = \varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \xrightarrow[n \to \infty]{} 0$. Let so T_n be $\pi(\varepsilon_n) \circ T$ and conclude that these (compact) operators approximate T in the desired fashion, *i.e.*

$$(4.34) \qquad \|T - T_n\| \xrightarrow[n \to \infty]{} 0 \quad .$$

$\square$

15   Exercise 15.   Hilbert-Schmidt operators

Suppose μ is a finite (or σ-finite) positive measure on a measure space Ω, $\mu \times \mu$ is the corresponding product measure on $\Omega \times \Omega$, and $K \in L^2(\mu \times \mu)$. Define

$$(Tf)(s) = \int_\Omega K(s, t) f(t) d\mu(t) \quad [f \in L^2(\mu)].$$

(a) *Prove that $T \in \mathscr{B}(L^2(\mu))$ and that*

$$\|T\|^2 \leq \int_\Omega \int_\Omega |K(s, t)|^2 d\mu(s) d\mu(t).$$

(b) *Suppose a_i, b_i are members of $L^2(\mu)$, for $1 \leq i \leq n$, put $K_1 = \sum a_i(s) b_i(t)$ and define T_1 in terms of K_1 a T was defined in terms of K. Prove that $\dim \mathscr{R}(T_1) \leq n$.*

(c) *Deduce that T is a compact operator in $L^2(\mu)$. Hint: Use exercise 13.*

(d) *Suppose $\lambda \in \mathbf{C}$, $\lambda \neq 0$. Prove: Either the equation*

$$Tf - \lambda f = g$$

has a unique solution $f \in L^2(\mu)$ for every $g \in L^2(\mu)$ or there are infinitely many solutions for some g and none for others. (This is known as the Fredholm alternative.).

(e) *Describe the adjoint of T.*

Proof. Let X (respectively P) be the Banach space $L^2(\mu)$ (respectively $L^2(\mu \times \mu)$). A consequence of the Radon-Nikodym theorem (*cf.* 6.16 of [3]) is that there exists a group isomorphism $\rho : X \to X^*$, $f \mapsto f^*$ such that

(4.35) $\langle u, f^* \rangle = \int_\Omega u \cdot f \mu \quad (u \in X, f \in X)$.

Define a.e K_s, $K_t : \Omega \to \mathbf{C}$ by setting

(4.36) $K_s(t) \triangleq K_t(s) \triangleq K(s, t)$ a.e $((s, t) \in \Omega)$.

T is clearly linear. Moreover,

(4.37) $|(Tf)(s)| = |\langle K_s, f^* \rangle| \leq \|K_s\|_X \quad (\|f\|_X < 1)$

(the latter inequality is a Cauchy-Schwarz one). Now apply the Fubini's theorem with $|K|^2$ to obtain

(4.38) $\|Tf\|_X^2 \leq \int_\Omega \|K_s\|_X^2 \, \mu(s) = \|K\|_P^2 < \infty \quad (\|f\|_X < 1)$.

(a) is then proved.

To show (b), remark that

$$(4.39) \qquad \int_\Omega a_i(s) \cdot b_i \cdot f \, \mu \ \in \mathbf{C} \cdot a_i(s) \ \text{ a.e} \qquad (f \in X, \, s \in \Omega) \quad .$$

It is now clear that T maps any f of X into $\mathbf{C} \cdot a_1 + \cdots + \mathbf{C} \cdot a_n$. We so conclude that $\dim R(T_1) \leq n$.

We now aim at (c). The current part refers to Exercise 4.13. X is also a Hilbert space and so contains a Hilbert basis M. Define a.e

$$(4.40) \qquad\qquad a_b : \Omega \to \mathbf{C}$$
$$s \mapsto (K_s, b)$$

whenever b ranges M. Hence,

$$(4.41) \qquad\qquad K_s = \sum_{b \in M} a_b(s) \cdot b \ \text{ a.e} \quad (s \in \Omega) \quad .$$

Provided any positive scalar ε, there so exists a finite subset $S = S(\varepsilon)$ of M such that

$$(4.42) \qquad\qquad \Big\| K_s - \sum_{b \in S} a_b(s) \cdot b \Big\|_X < \varepsilon \quad (s \in \Omega) \quad .$$

Remark that $\sum_{b \in S} a_b \cdot b$ matches the definition of K_1; *cf.* (b): from now on,

$$(4.43) \qquad\qquad K_1 \triangleq \sum_{b \in S} a_b \cdot b \quad .$$

It follows from (b) that

$$(4.44) \qquad\qquad \dim R(K_1) < \infty \quad .$$

Now turn back to (a), with $K - K_1$ playing the role of K, and so obtain

$$(4.45) \qquad\qquad \| T - T_1 \| < \varepsilon \mu(\Omega) \leq \infty \quad .$$

For if μ is finite, use (a) of Exercise 4.13 to conclude that T is compact. Assume henceforth that μ is not (necessarily) finite and pick δ in $\mathbf{R}_+$. The simple functions (with finite measure support) form a dense family of an L^p space ($1 \leq p < \infty$); *cf.* 3.13 of [3]. It then exists a simple function K_δ of $L^2(\mu \times \mu)$ such that

$$(4.46) \qquad (\mu \times \mu)(\{K_\delta \neq 0\}) < \infty, \ \| K - K_\delta \|_P < \delta \quad .$$

Define an operator T_δ in terms of K_δ as T was defined in terms of K, and proceed as in (a) with $T - T_\delta$ instead of T. Then

$$(4.47) \qquad\qquad \| T - T_\delta \| < \delta \quad .$$

The key ingredient is that K_δ can be identified with an element of the finite measure space $L^2(\{K_\delta \neq 0\}, \mu \times \mu)$. What we have attempted to approximate T by T_1 can therefore be reiterated (with K_δ playing the role of K) to achieve an approximation $T_{\delta,1}$ of T_δ so that

$$(4.48) \qquad\qquad \|T_\delta - T_{\delta,1}\| < \varepsilon \quad .$$

It now follows from (4.47) and (4.48) that

$$(4.49) \qquad \|T - T_{\delta,1}\| \leq \|T - T_\delta\| + \|T_\delta - T_{\delta,1}\| < \varepsilon + \delta \quad .$$

Since ε and δ were arbitrary, the σ-finite case is proved. We now establish (d).

Provided g of X, let E_g be the following equation on X

$$(4.50) \qquad\qquad Tf - \lambda f = g \quad ,$$

whose solution set is denoted by S_g. Note that S_0 is $\ker(T - \lambda)$ and discard the trivial case $S_0 = X$[1]: each f of X lies in $S_{Tf-\lambda f}$, as some $Tf - \lambda f$'s are nonzero. Some S_g's are then nonempty. Remark that

$$(4.51) \qquad\qquad S_g = f + S_0 \quad (f \in S_g)$$

in such case. Furthermore, the equality $\beta = \alpha$ of [4.25] yields

$$(4.52) \qquad\qquad (T - \lambda I)(X) \neq X, i.e. \ S_0 \neq \{0\} \quad .$$

So if $T - \lambda I$ is not onto, not only some S_g's are empty, but also $S_0 \neq \{0\}$. Every nonempty S_g (such sets always exist, see above) is then infinite, by (4.51).
Otherwise, $T - \lambda I$ is bijective and every equation E_g has then a unique solution f. The Fredholm alternative is so proved.

Our last step is the description of T^*. Let $S : X \rightarrow X$ be such that

$$(4.53) \qquad (Sf)(t) \triangleq \int_\Omega K_t \cdot f \ a.e \qquad (f \in X, t \in \Omega)$$

Proceed as in (a), with S instead of T: S lies in $\mathscr{B}(X)$. Next, we claim that

$$(4.54) \qquad\qquad \langle u, T^*f^* \rangle = \langle Tu, f^* \rangle$$

$$(4.55) \qquad\qquad\qquad = \int_\Omega (Tu) \cdot f \ \mu$$

$$(4.56) \qquad\qquad\qquad = \int_{\Omega^2} K \cdot f \cdot u \ (\mu \times \mu)$$

$$(4.57) \qquad\qquad\qquad = \int_\Omega (Sf) \cdot u \ \mu$$

$$(4.58) \qquad\qquad\qquad = \langle u, (Sf)^* \rangle \quad ,$$

---

[1] *e.g.* $X = L^2(\{0\}, \delta)$, so that $I = \lambda^{-1}T$ is compact.

whenever u and f run through the closed unit ball of X. Since $\|T\|$, $\|T^*\|$ are equal and finite, only exactness of (4.56) is possibly in doubt; the below justification dissipates it. In conclusion,

$$(4.59) \qquad\qquad T^* = \rho S \rho^{-1} \quad .$$

Informally,

$$(4.60) \qquad\qquad T^* = S \quad .$$

Justification of (4.56). The current proof shall be complete once we have justified (4.56). To do so, keep u and f as above. Let us introduce

$$(4.61) \qquad A(s) \triangleq \int_\Omega |K_s(t) \cdot u(t)| \, \mu(t) \quad \text{a.e} \quad (s \in \Omega) \quad ,$$

to make hold the following Cauchy-Schwarz inequality

$$(4.62) \qquad\qquad A(s) \leq \|K_s\|_X \quad (s \in \Omega) \quad .$$

Thus,

(4.63)

$$\int_{\Omega^2} |K(s,\, t)\, u(t)\, f(s)|\, \mu(s)\mu(t) = \int_\Omega |f(s)|\, A(s)\, \mu(s)$$

$$(4.64) \qquad\qquad\qquad\qquad \leq \int_\Omega |f(s)|\, \|K_s\|_X\, \mu(s)$$

$$(4.65) \qquad\qquad\qquad\qquad \leq \left[\int_\Omega \|K_s\|_X^2 \, \mu(s) \right]^{\frac{1}{2}} = \|K\|_P < \infty \quad .$$

The inequality in (4.65) is a Cauchy-Schwarz one, the following equality follows from the Fubini's theorem. This achieves the proof. $\qquad\square$

Chapter 6

Distributions

1 Exercise 1. Test functions are almost polynomial

Suppose f *is a complex continuous function in* $\mathbf{R}^n$, *with compact support. Prove that* $\psi P_j \to f$ *uniformly on* $\mathbf{R}^n$, *for some* $\psi \in \mathscr{D}$ *and for some sequence* $\{P_j\}$ *of polynomials.*

Proof. According to 1.16, Ω is union of a compact sets sequence $\{K_i\}$ and supp(f) lies in some $K = K_i$, so that f is embedded in $\mathscr{D}(\Omega)$. We can apply [1.10] to ensure that Ω encloses a compact set $S = \overline{K + B(\varepsilon)}$ for sufficiently small $\varepsilon > 0$.

One easily checks that the Stone-Weierstraß theorem [5.7] can be applied with the subalgebra $\{g \in C(S) : g \text{ is polynomial}\}$ of $C(S)$. There so exists a sequence $\{P_j : j \in \mathbf{N}\}$ of $\mathbf{R}[X_1, \dots, X_n]$ such that

$$(6.1) \qquad \sup_S |f - P_j| \underset{j\infty}{\longrightarrow} 0 \quad .$$

By [6.20], the open set $K + B(\varepsilon)$ has a local partition of unity $\{\psi_i\} \subseteq \mathscr{D}(\Omega)$. Moreover, there exists an integer l such that $\psi = \psi_1 + \cdots + \psi_l$ equals 1 on K. Hence

$$(6.2) \qquad \| f - \psi P_j \|_\infty = \| \psi f - \psi P_j \|_\infty = \sup_S | \psi f - \psi P_j |$$

$$(6.3) \qquad \qquad \qquad = \sup_S |f - P_j| \xrightarrow[j\infty]{(6.1)} 0 \quad .$$

$\square$

We will actually prove more by showing that $\mathscr{D}(\Omega)$ is separable for each nonempty open subset Ω of $\mathbf{R}^n$.

Proof. The following is split in three parts. The first one is about the above requested result: That was our first part. We now go further by

proving the separability of $\mathscr{D}(\Omega)$. To do so, we keep (α, j) in $\mathbf{N}^n \times \mathbf{N}$. Remark that S encloses $\mathrm{supp}(D^\alpha f)$: according to the first part, there exists a sequence $\{P_{\alpha, j} : j \in \mathbf{N}\} \subset \mathbf{R}[X_1, \ldots, X_n]$ such that

$$(6.4) \qquad \|D^\alpha f - \psi P_{\alpha, j}\|_\infty \xrightarrow[j\infty]{} 0 \quad .$$

Now let m range over $\{1, 2, 3, \ldots\}$ and set $W_{m, j}$ in $\mathscr{D}(\Omega)$ as follows

$$(6.5) \qquad D^{-\alpha}\varphi \in \mathscr{D}(\Omega) : D^\alpha D^{-\alpha}\varphi = \varphi \quad .$$

$$(6.6) \qquad W_{m, j}(x) \triangleq D^{-(m,\ldots, m)}(\psi P_{(m,\ldots, m), j})$$

By (6.4), there exists a natural number k(m) such that

$$(6.7) \qquad \|D^{(m,\ldots, m)}(f - W_{m, j})\|_\infty < 1/m \quad (j \geq k(m)) \quad .$$

Assume without loss of generality that S has diameter 1, so that (6.7) yields

$$(6.8) \qquad \|D^\lambda(f - W_{m, k(m)})\|_\infty < 1/m \quad (|\lambda| \leq m) \quad ,$$

by the mean value theorem. In other words (remark that $\mathrm{supp}(f - W_{m, k(m)}) \subseteq S$),

$$(6.9)$$
$$f - W_{m, k(m)} \in U_m \triangleq \{\varphi \in \mathscr{D}_S : \|\varphi\|_m < 1/m\} \supseteq U_{m+1} \supseteq \cdots \quad (m = 1, 2, 3, \ldots) \quad .$$

Pick W in β (see (b) of [6.3]): $W \cap \mathscr{D}_S$ contains a neighbourhood of 0. Hence W contains some U_m, for m sufficiently large. Thus

$$(6.10) \qquad W_{m, k(m)} \xrightarrow[m\infty]{} f \quad (\text{in } \mathscr{D}(\Omega)) \quad .$$

We have so established that the $W_{m, k(m)}$'s family is dense in $\mathscr{D}(\Omega)$. We now aim to disclose a countable set $\widetilde{W}$ that has the same property.

Choose δ in $\mathbf{R}_+$ and fetch any $W_{m, k(m)}$. Let X be $(X_1, \ldots, X_n)$ and express $P_{(m,\ldots, m), k(m)}$ as

$$(6.11) \qquad P(X) = \sum_{|\gamma| \leq d} p_\gamma \cdot X^\gamma \quad .$$

Since $\bar{\mathbf{Q}} = \mathbf{R}$, $\mathbf{Q}[X]$ hosts some $\mathbf{Q}(X) = \sum_{|\gamma| \leq d} q_\gamma \cdot X^\gamma$ such that $|p_\gamma - q_\gamma| < \delta$ for all γ. Thus,

$$(6.12)$$
$$|P(x) - Q(x)| \leq \sum_{|\gamma| \leq d} |p_\gamma - q_\gamma| \, |x|^{|\gamma|} \leq \delta \sum_{l \leq d} \binom{1 + n - 1}{n - 1} \|x\|_\infty^l \quad (x \in \mathbf{R}^n) \quad .$$

Since S is bounded, we so obtain

$$(6.13) \qquad \|\psi(P - Q)\|_\infty \in O(\delta) \quad .$$

Now define $\tilde{W}_m$ in terms of Q as $W_{m,k(m)}$ was defined in terms of P, and consider the integrations made in (6.6): each $D^\lambda \tilde{W}_m$ ($|\lambda| \leq m$) can be obtained from some of them. So (6.13) yields

$$(6.14) \qquad \|D^\lambda(W_{m,k(m)} - \tilde{W}_m)\|_\infty \in O(\delta) \quad (|\lambda| \leq m) \quad .$$

To be more specific, these λ's only exist in finite amount, so the big O can be assumed to be the same for all them. Since δ was arbitrary, combining (6.10) with (6.14) establishes the density of the all $\tilde{W}_m$'s family $\tilde{W}$.

Furthermore, each member of $\tilde{W}$ is only made of two ingredients: ψ and a polynomial of $\mathbf{Q}[X]$. The mapping ψ is attached to some K_i and $\mathbf{Q}[X]$ inherits countableness from $\mathbf{Q}$. Note that the "integrations packs" of (6.6) only exist in countable amount. Our $\tilde{W}$ is then countable. $\square$

6   Exercise 6. Around the supports of some distributions

(a) *Suppose that* $c_m = \exp\{-(m!)!\}$, $m = 0, 1, 2, \ldots$. *Does the series*

$$\sum_{m=0}^{\infty} c_m (D^m \varphi)(0)$$

converges for every $\varphi \in C^\infty(\mathrm{R})$?

(b) *Let* Ω *be open in* $\mathbf{R}^n$, *suppose* $\Lambda_i \in \mathscr{D}'(\Omega)$, *and suppose that all* Λ_i *have their supports in some fixed compact* $K \subseteq \Omega$. *Prove that the sequence* $\{\Lambda_i\}$ *cannot converge in* $\mathscr{D}'(\Omega)$ *unless the orders of the* Λ_j *are bounded. Hint: Use the Banach-Steinhaus theorem.*

(c) *Can the assumption about the supports be dropped in (b)?*

Proof. The answer is: no. Let us establish this assertion. Assume, to reach a contradiction, that the above series converges for every smooth $\varphi : \mathbf{R} \to \mathbf{C}$.

The sequence $\{c_m (D^m \varphi)(0)\}$ so converges to 0. Nevertheless, it is proved in [1.46] that $C^\infty(\Omega)$ is not locally bounded. In other words, it is always possible to excavate a φ for which the magnitude of the m-th derivative at 0 is as large as we please[1], *e.g.* greater than $1/c_m$. A desired contradiction is then reached. We now prove (b), again by contradiction.

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[1]indeed [1.46] provides sufficient tools for constructive proof of this; see the $\varphi_j - \check{\varphi}_j$ involved in (??).

To do so we assume $\{\Lambda_j\}$ to converge to some Λ of $\mathscr{D}'(\Omega)$ and we let Q run through the compact sets of Ω. Next, we define

(6.15)
$$S(T, Q) \triangleq \{N \in \mathbf{N}, \exists C \in \mathbf{R}_+ : |T\varphi| \leq C\|\varphi\|_N \text{ for all } \varphi \text{ of } \mathscr{D}_Q\} \quad (T \in \mathscr{D}(\Omega)) \quad .$$

Such subset of $\mathbf{N}$ has a minimum $\omega(T, Q)$. The following value

(6.16) $\qquad \omega(T) \triangleq \max\{\omega(T, Q) : Q \subseteq \Omega, \ Q \text{ compact }\} \leq \infty$

is then the order of T. Assume, to reach a contradiction, that, after passage to a subsequence,

(6.17) $\qquad \omega(\Lambda_j, Q_j) = j \quad (j = 1, 2, 3, \dots)$

for some compact $Q = Q_j$. By (a) of [6.24], Q_j cuts $\operatorname{supp}\Lambda_j$, say in p_j. Since K encloses $\operatorname{supp}\Lambda_j$, $\{p_j\}$ tends, after passage to a subsequence, to some p of K. Choose a positive scalar r so that

(6.18) $\qquad \overline{B}(p, r) \triangleq \{x \in \mathbf{R}^n : |x - p| \leq r\} \subseteq \Omega \quad .$

Such closed ball $\overline{B}(p, r)$ is a compact subset of Ω. By (b) of [6.5] (which refers to [1.46]) $\mathscr{D}_{\overline{B}(p, r)}$ is then a Fréchet space. It now follows from [2.6] that $\{\Lambda_j\}$ is equicontinuous on $\mathscr{D}_{\overline{B}(p, r)}$. There so exists[2] a nonnegative integer N such that

(6.19) $\qquad |\Lambda\varphi| \leq C\|\varphi\|_N \quad (\varphi \in \mathscr{D}_{\overline{B}(p, r)})$

for some positive constant C. On the other hand, $\overline{B}(p, r)$ contains almost all the p_j's. Hence

(6.20) $\qquad |\Lambda_N \varphi| > C\|\varphi\|_N$

for some φ of $\mathscr{D}_{\overline{B}(p,r)}$. (b) is then established.

To prove (c), we introduce a sequence $\{x_m : m \in \mathbf{Z}\}$ of Ω that has no limit point. Let $\{\alpha_m : m \in \mathbf{Z}\}$ be in $\mathbf{N}$ and so define[3]

(6.21) $\qquad \Lambda : \mathscr{D}(\Omega) \to \mathbf{C} \qquad\qquad .$

$$\varphi \mapsto \sum_{m=-\infty}^{\infty} (D^{\alpha_m}\varphi)(x_m)$$

Λ belongs to $\mathscr{D}'(\Omega)$, since $\{x_m\}$ has no limit point. Next, we easily check that

(6.22) $\qquad \Lambda_j : \mathscr{D}(\Omega) \to \mathbf{C} \qquad\qquad\qquad (j \in \mathbf{N})$

$$\varphi \mapsto \sum_{|m| \leq j} (D^{\alpha_m}\varphi)(x_m)$$

is also a distribution and that $\{\Lambda_j\}$ tends to Λ in $\mathscr{D}'(\Omega)$. Nevertheless, no Λ_j's can have common support because $\{x_m\}$ has no limit point. Our assumption can therefore be dropped. $\qquad\qquad \square$

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[2]For more details, see Exercise 2.3.

[3]As $\Omega = \mathbf{R}$, the case $\alpha_m = m$ is the "counterpart" of the series of (a) and the case $(x_m, \alpha_m) = (m, 0)$ is the *Dirac comb*.

9   Exercise 9.   Convergence in $\mathscr{D}(\Omega)$ vs. convergence in $\mathscr{D}'(\Omega)$

(a) *Prove that a set* $\mathrm{E} \subseteq \mathscr{D}(\Omega)$ *is bounded if and only if*

$$\sup\{|\Lambda\varphi| : \varphi \in \mathrm{E}\,\} < \infty$$

for every $\Lambda \in \mathscr{D}(\Omega)$.

(b) *Suppose* $\{\varphi_j\}$ *is a sequence in* $\mathscr{D}(\Omega)$ *such that* $\{\Lambda\varphi_j\}$ *is a bounded sequence of numbers, for every* $\Lambda \in \mathscr{D}'(\Omega)$. *Prove that some subsequence of* $\{\varphi_j\}$ *converges, in the topology of* $\mathscr{D}(\Omega)$.

(c) *Suppose* $\{\Lambda_j\}$ *is a sequence in* $\mathscr{D}'(\Omega)$ *such that* $\{\Lambda_j\varphi\}$ *is bounded, for every* $\varphi \in \mathscr{D}(\Omega)$. *Prove that some subsequence of* $\{\Lambda_j\}$ *converges in* $\mathscr{D}'(\Omega)$ *and that the convergence is uniform on every bounded subset of* $\mathscr{D}(\Omega)$. *Hint: By the Banach-Steinhaus theorem, the restrictions of the* Λ_j *to* $\mathscr{D}_{\mathrm{K}}$ *are equicontinuous. Apply Ascoli's theorem.*

Proof. Since $\mathscr{D}(\Omega)$ locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $\mathrm{E} = \{\varphi_j : j \in \mathbf{N}\}$ is bounded: so is $\overline{\mathrm{E}}$. By (c) of [6.5], there exists some $\mathscr{D}_{\mathrm{K}}$ that contains $\overline{\mathrm{E}}$. Since $\mathscr{D}_{\mathrm{K}}$ has the Heine-Borel property (see [1.46]), $\overline{\mathrm{E}}$ is τ_{K}-compact. Apply [A4] with the metrizable space $\mathscr{D}_{\mathrm{K}}$ (see [1.46]) to conclude that $\overline{\overline{\mathrm{E}}}$ has a τ_{K} limit point. It then follows from (b) of [6.5] that (b) holds. $\square$

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