Solutions to some exercises from Walter Rudin's *Functional Analysis*

gitcordier

August 18, 2023

Contents

Notations and Conventions

0.1 Logic

1. **Halmos' iff. iff** is a short for "if and only if".

2. **Definitions (of values) with $\triangleq$.** Given variables a and b, a $\triangleq$ b means that a is defined as equal to b.

3. **$\equiv$.** a $\equiv$ b means that there exists a "natural" bijection $\rightarrow$ that maps a to b; which let us identify a with b. In a metric space context, a $\equiv$ b means that $\rightarrow$ is isometric.

4. **Definitions (formulæ).** Definitions use the **iff** format. In other words, every definition has a "only if".

5. **Iverson notation.** Given a boolean expression Φ, $[\,\Phi\,]$ returns the truth value of Φ, encoded as follows,

$$[\,\Phi\,] \triangleq \begin{cases} 0 & \text{if } \Phi \text{ is false;} \\ 1 & \text{if } \Phi \text{ is true.} \end{cases}$$

For example, $[\,1 > 0\,] = 1$ but $[\,\sqrt{2} \in \mathbf{Q}\,] = 0$.

0.2 Topological vector spaces

1. **Product space**

2. **Scalar field.** The usual (complete) scalar field is $\mathbf{C}$. A property, *e.g.* linearity, that is true on $\mathbf{C}$ is also true on $\mathbf{R}$. The complex case is then a *special case* of the real one. Sometimes, this specialization is not purely formal. For example, theorem 12.7 of [3] asserts that, in a Hilbert space H equipped with the inner product $\langle \cdot \,|\, \cdot \rangle$, every nonzero linear continuous operator T "breaks orthogonality", in the sense that there always exists x = x(T) in H that satisfies $\langle Tx|x \rangle \neq 0$. The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exists. To see that, consider the $90°$ rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitely mentioned*, the exension to the real case will always be obvious. So, taking $\mathbf{C}$ as the scalar field shall mean "*Instead of letting the scalar field undefined, we choose $\mathbf{C}$ for the sake of expessivity. But considering $\mathbf{R}$ instead of $\mathbf{C}$ would actually make no difference here*".

3. **Finite dimensional spaces**. Let Y be a finite dimensional space. If $\dim Y = 0$, *i.e.* Y is a group of order 1, then $\{\emptyset, Y\}$ is the only possible topology for Y. For instance, in a quotient space X/N, the zero is N and $\{N\}$ is zero-dimensional in X/N.

Assume henceforth that $\dim Y > 0$, *i.e.* Y has a base $F_n = \{f_i : i = 1, \dots, n\}$ for some positive n. The cartesian power $\mathbf{C}^n = \prod_{j=1}^n \mathbf{C}$ is the vector space of all lists $(z_1, \dots, z_n)$, where $z_j \in \mathbf{C}$ (identify $\mathbf{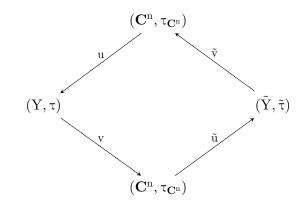C}^1$ with $\mathbf{C}$). The subset $E_n = \{e_j : j = 1, \dots, n\}$ is the *standard base* of $\mathbf{C}^n$, *i.e.* $e_j = 1_{\{j\}}$. So, $\dim \mathbf{C}^n = n$. Let $u : \mathbf{C}^n \to Y$ be the only linear mapping that verifies all $u(e_j) = f_j$. Since u is encoded as the identity matrix, both u and $v = u^{-1}$ exist as isomorphisms. Additionally, $\mathbf{C}^n$ can be equipped with various norms, *e.g.* the *p-norms* $\| \ \|_p$ (where $\| (z_1, \dots, z_n) \|_p^p = |z_1|^p + \cdots + |z_n|^p \, ; p \geq 1$) or $\| \ \|_\infty$ (where $\| (z_1, \dots, z_n) \|_\infty = \max |z_j|$). Note that Y inherits any norm $\| \ \|$ of $\mathbf{C}^n$, with $\| u(z_1, \cdots, z_n) \| = \| (z_1, \cdots, z_n) \|$; which turns u into a isometry of $\mathbf{C}^n$ onto Y. Let $\tau_{\| \ \|}$ denote the topology of a norm $\| \ \|$. We now go back to the proof of 1.21 of [3] and so equip Y with a its own norm $\| \ \|_2$; which turns u into a isometric isomorphim of $\mathbf{C}^n$ onto Y. Y can now be seen as a topological vector space, in at least one fashion; namely, the space $(Y, \tau_{\| \ \|_2})$. Let $\tau = \tau_Y$ stand for any arbitrary topology of Y. Hence the following commutative diagram

(1)

It is now clear that the *identity mapping* $u \circ v$ is an homeomorphism of Y onto Y, which implies that $\tau = \tau_{\| \ \|_2}$. In other words, there is only one topology τ for Y, as a topological vector space. This topology is normable, since $\tau = \tau_{\| \ \|_2}$. Let $\| \ \|_Y$ stand for any norm of Y. The special case $Y = \mathbf{C}^n, F_n = E_n, u = i$ is of considerable interest. TOTO. Now take $\tilde{Y}$ of dimension n then similarly define (obvious notations) $\tilde{u}$, $\tilde{v}$ and $\tilde{\tau}$.

(2)

The homeomorphism between Y and $\tilde{Y}$ leads to *the equivalence of norms at fixed dimension* n, as follows $A \| y \|_Y \leq \| \tilde{u} \circ v(y) \|_{\tilde{Y}} \leq B \| y \|_Y$ $(y \in Y)$ for some positive

A, B. Equip Y and $\tilde{Y}$ with the inherited norm $\| \ \|$. Y and $\tilde{Y}$ are homeomorphically isomorphic ($\equiv$) to $\mathbf{C}^n, \| \ \|$.

$$
\begin{array}{ccc}
 & (\mathbf{C}^n, \| \ \|) & \\
{\scriptstyle\equiv}\swarrow & & \nwarrow{\scriptstyle\equiv} \\
(Y, \| \ \|) & & (\tilde{Y}, \| \ \|) \\
{\scriptstyle\equiv}\searrow & & \nearrow{\scriptstyle\equiv} \\
 & (\mathbf{C}^n, \| \ \|) &
\end{array}
$$

(3)

From now the default norm will be $\| \ \|_\infty$.

Chapter 1

Topological Vector Spaces

1.1 Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X. *Prove the following statements from the axioms as given as in section 1.4.*

   (a) If x, y $\in$ X *there is a unique* z $\in$ X *such that* $x + z = y$.

   (b) $0 \cdot x = 0 = \alpha \cdot 0$ $(\alpha \in \mathbf{C}, x \in X)$.

   (c) $2A \subset A + A$.

   (d) A *is convex if and only if* $(s + t)A = sA + tA$ *for all positive scalars* s *and* t.

   (e) Every union (and intersection) of balanced sets is balanced.

   (f) Every intersection of convex sets is convex.

   (g) If Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

   (h) If A *and* B *are convex, so is* $A + B$.

   (i) If A *and* B *are balanced, so is* $A + B$.

   (j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

Proof.   1. Such property only depends on the group structure of X: Each x in X has an opposite -x. Let x' be any opposite of x, so that $x - x = 0 = x + x'$. Thus, $-x + x - x = -x + x + x'$, which is equivalent to $-x = x'$. So is established the uniqueness of -x. It is now clear that $x + z = y$ **iff** $z = -x + y$, which asserts both the existence and the uniqueness of z.

    2. Remark that

$$(1.1) \qquad\qquad 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$$
$$(1.2) \qquad\qquad\quad = (0 + 0) \cdot x = 0 + 0 \cdot x$$

then conclude from (a) that $0 \cdot x = 0$. So,

$$(1.3) \qquad\quad 0 = 0 \cdot x = (1 - 1) \cdot x = x + (\text{-}1) \cdot x \Rightarrow \text{-}1 \cdot x = \text{-}x.$$

Finally,

$$(1.4) \qquad \alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (x + (\text{-}1 \cdot x)) = \alpha \cdot x + \alpha \cdot (\text{-}1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0,$$

which proves (b).

3. Remark that

$$(1.5) \qquad\qquad\qquad 2x = (1 + 1)x = x + x$$

for every x in X, and so conclude that

$$(1.6) \qquad 2A = \{2x : x \in A\} = \{x + x : x \in A\} \subset \{x + y : (x, y) \in A^2\} = A + A$$

for all subsets A of X; which proves (c).

4. If A is convex, then

$$(1.7) \qquad\qquad\qquad A \subset \frac{s}{s+t}A + \frac{t}{s+t}A \subset A;$$

which is

$$(1.8) \qquad\qquad\qquad sA + tA = (s + t)A.$$

Conversely, the special case $s + t = 1$ is

$$(1.9) \qquad\qquad\qquad sA + (1 - s)A = A.$$

The latter extends to $s = 0$, since

$$(1.10) \qquad\qquad\qquad 0A + A \overset{(b)}{=} \{0\} + A = A.$$

The extension to $s = 1$ is analogously established (or simply use the fact that $+$ is commutative!). So ends the proof.

5. Let A range over B a collection of balanced subsets, so that

$$(1.11) \qquad\qquad\qquad \alpha \bigcap B \subset \alpha A \subset A \subset \bigcup B$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap B \subset A$ establishes the first part. Now remark that

$$(1.12) \qquad\qquad\qquad \alpha A \subset \bigcup B$$

implies

$$(1.13) \qquad\qquad\qquad \alpha \bigcup B \subset \bigcup B;$$

which achieves the proof.

6. Let A range over C a collection of convex subsets, so that

$$(1.14) \qquad (s + t) \bigcap C \subset s \bigcap C + t \bigcap C \subset sA + tA \overset{(d)}{=} (s + t)A$$

for all positives scalars s, t. Thus,

$$(1.15) \qquad (s + t) \bigcap C \subset s \bigcap C + t \bigcap C \subset (s + t) \bigcap C.$$

We now conclude from (d) that the intersection of C is convex. So ends the proof.

7. Pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i $(i = 1, 2)$ lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_1's is then in $C_2 \subset \bigcup \Gamma$, hence (g).

8. Simply remark that

$$(1.16) \qquad s(A + B) + t(A + B) = sA + tA + sB + tB = (s + t)(A + B)$$

for all positive scalars s and t, then conclude from (d) that $A + B$ is convex.

9. Given any α from the closed unit disc,

$$(1.17) \qquad \alpha(A + B) = \alpha A + \alpha B \subset A + B.$$

There is no more to prove.

10. Our proof will be based on the following lemma,

> *If* $S \subset X$*, then any assertion*
> *(i)* S *is a vector subspace of* X*;*
> *(ii)* S *is convex balanced such that* $S + S = S$*;*
> *(iii)* S *is convex balanced such that* $\lambda S = S$ $(\lambda > 0)$
> *implies the other ones.*

To prove the lemma, let S range over the subsets of X. First, assume that (i) holds: Clearly, every S is convex balanced. Moreover, $S + S \subset S$. Conversely, $S = S + \{0\} \subset S + S$; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$(1.18) \qquad nS = (n - 1)S + S = S + S = S \quad (n = 1, 2, 3, \dots)$$

with the help of (b) and (d). The special case $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$ $(\lambda > 0)$ yields

$$(1.19) \qquad nS \overset{(1.18)}{\subset} S \subset n\lambda S \subset n^2 S,$$

since S is balanced and that $1 < n\lambda < n^2$. Dividing the latter inclusions by n shows that

$$(1.20) \qquad S \subset \lambda S \subset nS \overset{(1.18)}{\subset} S,$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$(1.21) \qquad \alpha S + \beta S = |\alpha|S + |\beta|S \overset{(d)}{=} (|\alpha| + |\beta|)S \overset{(iii)}{=} S \quad (|\alpha| + |\beta| > 0);$$

where the equality at the left holds as S is balanced. Moreover (under the sole assumption that S is balanced), this extends to $|\alpha| + |\beta| = 0$, as follows,

$$(1.22) \qquad \alpha S + \beta S = 0S + 0S \overset{(b)}{=} \{0\} \overset{(b)}{=} 0S \subset S.$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let V be a collection of vector spaces of X, of intersection I and union U. First, remark that every member of V is convex balanced: So is I (combine (e) with (f)). Next, let Y range over V, so that

$$(1.23) \qquad\qquad I + I \subset Y + Y \subset Y;$$

which yields

$$(1.24) \qquad\qquad I + I \subset I.$$

Conversely,

$$(1.25) \qquad\qquad I = I + \{0\} \subset I + I.$$

It now follows from the lemma's (ii) $\Rightarrow$ (i) that I is a vector subspace of X. Now temporarily assume that S is totally ordered by set inclusion: Combining (e) with (g) establishes that U is convex balanced. To show that U is more specifically a vector subspace, we first remark that such total order implies that either $Z \subset Y$ or $Y \subset Z$, as Z ranges over V. A straightforward consequence is that

$$(1.26) \qquad\qquad Y \subset Y + Z \subset Y \cup Z.$$

Another one is that $Y \cup Z$ ranges over V as well. Combined with the latter inclusions, this leads to

$$(1.27) \qquad\qquad U \subset U + U \subset U.$$

It then follows from the lemma's (ii) $\Rightarrow$ (i) that U is a vector subspace of X. Finally, let A, B run through the vector subspaces of X: Combining (h) with (i) proves that $A + B$ is convex balanced as well. Furthermore,

$$(1.28) \qquad A + B \overset{(i)\Rightarrow(ii)}{=} (A + A) + (B + B) = (A + B) + (A + B),$$

where the equality at the right holds as X is an abelian group. We now conclude from (ii) that any $A + B$ is a vector subspace of X. So ends the proof.

$\square$

1.2   Exercise 2. Convex hull

The convex hull of a set A *in a vector space* X *is the set of all convex combinations of members of* A, *that is the set of all sums* $t_1 x_1 + \cdots + t_n x_n$ *in which* $x_i \in A$, $t_i \geq 0$, $\sum t_i = 1$; n *is arbitrary. Prove that the convex hull of a set* A *is convex and that is the intersection of all convex sets that contain* A.

Proof. The convex hull of a set S will be denoted by co(S). Remark that $S \subset co(S)$ (to see that, take $t_1 = 1$ for each x_1 in S) and that $co(A) \subset co(B)$ where $A \subset B$ (obvious). Our proof will directly derive from the following lemma,

> *Let* S *be a subset of a vector space* X: *Its convex hull* co(S) *is convex and the following statements*
>
> *(i)* S *is convex;*
> *(ii)* $s_1 S + \cdots + s_n S = (s_1 + \cdots + s_n)S$ *for all positive scalar variables* $s_1, \ldots, s_n$;
> *(iii)* $t_1 S + \cdots + t_n S = S$ *for all positive scalar variables* $s_1, \ldots, s_n$ *such that* $s_1 + \cdots + s_n = 1$;
> *(iv)* co(S) = S
>
> *are equivalent.*

More specifically, our proof of the second part will only depend on (i) $\Rightarrow$ (iv).

From now on, we skip the trivial case $S = \emptyset$ then only consider nonempty sets. To prove the first part, let *a*, *b* run through the convex combination(s) of S, so that $a = t_1 x_1 + \cdots + t_n x_n$ and $b = t_{n+1} x_{n+1} + \cdots + t_{n+p} x_{n+p}$ for some (t_i, x_i). Every sum $sa + (1-s)b$ $(0 \leq s \leq 1)$ is then a convex combination of $x_1, \ldots, x_{n+p}$, since

$$(1.29) \qquad sa + (1-s)b = \sum_{i=1}^{n} s t_i x_i + \sum_{i=n+1}^{n+p} (1-s) t_i x_i$$

and

$$(1.30) \qquad \sum_{i=1}^{n} s t_i + \sum_{i=n+1}^{n+p} (1-s) t_i = s \sum_{i=1}^{n} t_i + (1-s) \sum_{i=n+1}^{n+p} t_i = 1.$$

In terms of sets S, this reads

$$(1.31) \qquad s \, co(S) + (1-s) \, co(S) \subset co(S);$$

which was our fist goal. We now aim at the equivalence (i) $\Rightarrow \cdots \Rightarrow$ (iv) $\Rightarrow$ (i): An easy proof by induction makes the implication (i) $\Rightarrow$ (ii) directly come from (d) of the above exercise 1, chapter 1. (iii) is a special case of (ii), and the implication (iii) $\Rightarrow$ (iv) derives from the definition of the convex hull. We now close the chain with (iv) $\Rightarrow$ (i), by remarking that S is convex whether $S = co(S)$. The lemma being proved, let us establish the second part. To do so, start from $F \ni co(A)$ then possibly enrich F the following way:

$$(1.32) \qquad B \in F \Rightarrow B \text{ is convex and contains A.}$$

Note that our definition of F is weaker than the primary assumption "[F only encompasses] *all convex sets that contain A*", which is the special case

$$(1.33) \qquad B \in F \Leftrightarrow B \text{ is convex and contains A.}$$

In any case, the key ingredient is that $co(A) \in F$ implies

$$(1.34) \qquad\qquad co(A) \supset \bigcap_{B \in F} B.$$

Conversely, the next formula

$$(1.35) \qquad\qquad co(A) \subset co(B) \stackrel{(i)\Rightarrow(iv)}{=\!=} B \quad (B \in F)$$

is valid and implies

$$(1.36) \qquad\qquad co(A) \subset \bigcap_{B \in F} B.$$

So ends the proof $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

1.3 Exercise 3. Other basic results

1.4   Exercise 4. A nonempty set whose interior is not

1.5 Exercise 5. A first restatement of boundedness

1.6   Exercise 6. A second restatement of boundedness

1.7 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$, topologized by the family of seminorms

$$p_x(f) = |f(x)| \qquad (0 \le x \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \to \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \to \infty$ then $\{\gamma_n f_n\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0,1]$.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.

Proof. The family of the seminorms p_x is separating: By 1.37 of [3], the collection $\mathscr{B}$ of all finite intersections of the sets

$$(1.37) \qquad V(x,k) \triangleq \{p_x < 2^{-k}\} \qquad (x \in [0,1], k = 1,2,3,\dots)$$

is therefore a local base for a topology τ on X. So,

$$(1.38) \quad \sum_{n=1}^{\infty} [f_n \notin \cap_{i=1}^{m} U_i] \le \sum_{n=1}^{\infty} \sum_{i=1}^{m} [f_n \notin U_i] = \sum_{i=1}^{m} \sum_{n=1}^{\infty} [f_n \notin U_i] \qquad (f_n \in X, U_i \in \tau).$$

Now assume that $\{f_n\}$ τ-converges to some f, *i.e.*

$$(1.39) \qquad \sum_{n=1}^{\infty} [f_n \notin f + W] < \infty \qquad (W \in \mathscr{B}).$$

The special case $W = V(x,k)$ means that, given k, $|f_n(x) - f(x)| < 2^{-k}$ for almost all n, *i.e.* $\{f_n(x)\}$ converges to f(x). Conversely, assume that $\{f_n\}$ does not τ-converges in X, *i.e.*

$$(1.40) \qquad \forall f \in X, \exists W \in \mathscr{B} : \sum_{n=1}^{\infty} [f_n \notin f + W] = \infty.$$

W is now the (nonempty) intersection of finitely many $V(x,k)$, say $V(x_1, k_1), \dots, V(x_m, k_m)$. Thus,

$$(1.41) \qquad \sum_{i=1}^{m} \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)] \stackrel{(1.38)}{\ge} \sum_{n=1}^{\infty} [f_n \notin f + W] \stackrel{(1.40)}{=} \infty.$$

We can now conclude that, for some index *i*,

$$(1.42) \qquad \sum_{n=1}^{\infty} [f_n \notin f + V(x_i, k_i)] = \infty.$$

In other word, $\{f_n(x_i)\}$ fails to converge to f(x_i). We have so proved that τ-convergence is a rewording of pointwise convergence. We now establish the second part.

To do so, we split x into two variables: r if x is rational, α otherwise. The proof is based on the following well-known result: Each α has a *unique* binary expansion. More precisely,

there exists a bijection $b : [0,1] \setminus \mathbf{Q} \to \{\beta \in \{0,1\}^{\mathbf{N}_+} : \beta$ is not eventually periodic$\}$ where $b(\alpha) = (\beta_1, \beta_2, \dots)$ is the only bit stream such that

$$(1.43) \qquad\qquad \alpha = \sum_{k=1}^{\infty} \beta_k \cdot 2^{-k}.$$

Remark that $b(\alpha)_1 + \cdots + b(\alpha)_n \xrightarrow[n\to\infty]{} \infty$, since $b(\alpha)$ has infinite support, then fix

$$(1.44) \qquad\qquad f_n(\alpha) \triangleq \frac{1}{b(\alpha)_1 + \cdots + b(\alpha)_n} \xrightarrow[n\to\infty]{} 0.$$

The actual values $f_n(r)$ are of no interest, as long as every sequence $\{f_n(r) : n = 1, 2, 3, \dots\}$ converges to 0. For example, put $f_n(r) = r/n$, or just $f_n(r) = 0$. We also take $\gamma_n \longrightarrow \infty$, *i.e.* given any counting number p, γ_n is greater than p for almost all n. Next, we choose n_p among those *almost all n* that are large enough to satisfy

$$(1.45) \qquad\qquad n_p - n_{p-1} > p$$

(start with $n_0 = 0$). So, every list $n_p, n_{p'}, n_{p''}, \dots$ that satisfies $n_{p'} - n_p = n_{p''} - n_{p'} = \dots$ is finite (otherwise, $n_{p'} - n_p \geq n_{p+1} - n_p > p \to \infty$ would hold; see (1.45)). In other words, *the distribution of $n_1, n_2, \dots$ displays no periodic pattern.* As a consequence, the *characteristic function* $\chi : k \mapsto [\, k \in \{n_1, n_2, \dots\}\,]$ is not eventually periodic. Combined with (1.43), this establishes that

$$(1.46) \qquad\qquad \alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k}$$

is irrational. Conversely, still with (1.43),

$$(1.47) \qquad\qquad b(\alpha_\gamma)_k = \chi_k.$$

Now remark that

$$(1.48) \qquad\qquad \chi_1 + \cdots + \chi_{n_1} \qquad\qquad\qquad = 1$$
$$(1.49) \qquad\qquad \chi_1 + \cdots + \chi_{n_1} + \cdots + \chi_{n_2} \qquad\quad = 2$$
$$\vdots$$
$$(1.50) \qquad\qquad \chi_1 + \cdots + \chi_{n_1} + \cdots + \chi_{n_2} + \cdots + \chi_{n_p} = p.$$

Combined with (1.44), this yields

$$(1.51) \qquad\qquad \gamma_{n_p} f_{n_p}(\alpha_\gamma) = \frac{\gamma_{n_p}}{p} > 1.$$

There so exists a subsequence $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. Since $\{\gamma_n\}$ was arbitrary, this proves (b). $\square$

1.9 Exercise 9. Quotient map

Suppose

1. X *and* Y *are topological vector spaces,*

2. $\Lambda : X \to Y$ *is linear.*

3. N *is a closed subspace of* X,

4. $\pi : X \to X/N$ *is the quotient map, and*

5. $\Lambda x = 0$ *for every* $x \in N$.

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$, *that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$. *Prove that f is linear and that* Λ *is continuous if and only if f is continuous. Also,* Λ *is open if and only if f is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$(1.52) \qquad f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y.$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$(1.53) \qquad \Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'.$$

It straightforwardly derives from (1.52) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$(1.54) \qquad f(\pi x) = 0 \overset{(1.52)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$(1.55) \qquad \Lambda(U) = f(\pi(U)) \subset V.$$

Since π is open (1.41 (a) of [3]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$(1.56) \qquad W = \pi(V)$$

for some neighborhood V of 0_X. So,

$$(1.57) \qquad f(W) = f(\pi(V)) = \Lambda(V).$$

As a consequence, if Λ is open, then f(W) is a neighborhood of 0_Y. So ends the proof. $\square$

1.10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim Y < \infty$, $\Lambda : X \to Y$ *is linear, and* $\Lambda(X) = Y$.

1. *Prove that* Λ *is an open mapping.*

2. *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. Discard the trivial case $\Lambda = 0$ and so assume that $\dim Y = n$ for some positive n. Let e range over a base of B of Y then pick W an arbitrary neighborhood of the origin: There so exists V a balanced neighborhood of the origin such that

$$(1.58) \qquad \sum_e V \subset W,$$

since addition is continuous. Moreover, for each e, there exists x_e in X such that $\Lambda(x_e) = e$, simply because Λ is onto. So,

$$(1.59) \qquad y = \sum_e y_e \cdot \Lambda x_e \qquad (y \in Y).$$

As a finite set, $\{x_e : e \in B\}$ is bounded: There so exists a positive scalar s such that

$$(1.60) \qquad \forall e \in B, x_e \in s \cdot V.$$

Combining this with (1.59) shows that

$$(1.61) \qquad y \in \sum_e y_e \cdot s \cdot \Lambda(V).$$

We now come back to (1.58) and so conclude that

$$(1.62) \qquad y \in \sum_e \Lambda(V) \subset \Lambda(W)$$

whether $|y_e| < 1/s$; which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$(1.63) \qquad \dim X/N = n.$$

f is then an homeomorphism of $X/N \equiv \mathbf{C}^n$ onto Y; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = f \circ \pi$. $\qquad \square$

1.12 Exercise 12. Topology stays, completeness leaves

1.14   Exercise 14. $\mathscr{D}_K$ equipped with other seminorms

Put $K = [0, 1]$ *and define* $\mathscr{D}_K$ *as in Section 1.46. Show that the following three families of seminorms (where* $n = 0, 1, 2, \dots$ *) define the same topology on* $\mathscr{D}_K$. *If* $D = d/dx$:

   1. $\|D^n f\|_\infty = \sup\{|D^n f(x)| : \infty < x < \infty\}$

   2. $\|D^n f\|_1 = \int_0^1 |D^n f(x)| \, dx$

   3. $\|D^n f\|_2 = \left\{ \int_0^1 |D^n f(x)|^2 dx \right\}^{1/2}$.

Proof. First, remark that

(1.64) $$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty < \infty$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of $D^n f$ lies in K; which yields

(1.65) $$|D^n f(x)| = \left| \int_0^x D^{n+1} f \right| \leq \int_0^x \left| D^{n+1} f \right| \leq \|D^{n+1} f\|_1.$$

So,

(1.66) $$\|D^n f\|_\infty \leq \|D^{n+1} f\|_1.$$

We now combine (1.64) with (1.66) and so obtain

(1.67) $$\|D^n f\|_1 \leq \|D^n f\|_2 \leq \|D^n f\|_\infty \leq \|D^{n+1} f\|_1 \leq \cdots \quad (n = 0, 1, 2, \dots).$$

Put

(1.68) $$V_n^{(i)} \triangleq \{ f \in \mathscr{D}_K : \| f \|_i < 2^{-n} \} \quad (i = 1, 2, \infty)$$

(1.69) $$\mathscr{B}^{(i)} \triangleq \{ V_n^{(i)} : n = 0, 1, 2, \dots \},$$

so that (1.67) is mirrored in terms of neighborhood inclusions, as follows,

(1.70) $$V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \cdots .$$

Since $V_n^{(i)} \supset V_{n+1}^{(i)}$, $\mathscr{B}^{(i)}$ is a local base of a topology τ_i. But the chain (1.70) forces

(1.71) $$\tau_1 = \tau_2 = \tau_\infty.$$

To see that, choose a set S that is τ_1-open at f, *i.e.* $V_n^{(1)} \subset S - f$ for some n. Next, concatenate this with $V_n^{(2)} \subset V_n^{(1)}$ (see (1.70)) and so obtain $V_n^{(2)} \subset S - f$; which implies that S is τ_2-open at f. Similarly, we deduce, still from (1.70), that

(1.72) $$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}.$$

So ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

1.16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ *does not depend on the particular choice of* $\{K_n\}$, *as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $C^\infty(\Omega)$ *(Section 1.46).*

Comment   This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms p_n, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1   *Let* X *be a topological space with a countable local base* $\{V_n : n = 1, 2, 3, \dots\}$. *If* $\tilde{V}_n = V_1 \cap \cdots \cap V_n$, *then every subsequence* $\{\tilde{V}_{\varrho(n)}\}$ *is a decreasing (i.e.* $\tilde{V}_{\varrho(n)} \supset \tilde{V}_{\varrho(n+1)}$) *local base of* X.

Proof. The decreasing property is trivial. Now remark that $V_n \supset \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X. Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supset \tilde{V}_{\rho(n)}$. $\square$

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$**)**   *Under the same notations of Lemma 1, if* $\{V_n\}$ *is a decreasing local base, then so is* $\{V_{\varrho(n)}\}$.

Corollary 2   *If* $\{Q_n\}$ *is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence* $\{Q_{\varrho(n)}\}$ *also satisfies theses conditions. Furthermore, if* τ_Q *is the* $C(\Omega)$*'s (respectively* $C^\infty(\Omega)$*'s) topology of the seminorms* p_n, *as defined in section 1.44 (respectively 1.46), then the seminorms* $p_{\varrho(n)}$ *define the same topology* τ_Q.

Proof. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X. Moreover,

$$(1.73) \qquad Q_{\rho(n)} \subset \mathring{Q}_{\rho(n)+1} \subset Q_{\rho(n)+1} \subset Q_{\rho(n+1)}.$$

Thus,

$$(1.74) \qquad Q_{\rho(n)} \subset \mathring{Q}_{\rho(n+1)}.$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(n)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(n)}\}$ is a local base. So, $\tau_{Q_\rho} \subset \tau_Q$. Conversely, the above corollary asserts that $\{V_{\rho(n)}\}$ is a local base of τ_Q, which yields $\tau_Q \subset \tau_{Q_\rho}$. $\square$

Lemma 2   *If a sequence of compact sets* $\{Q_n\}$ *satisfies the conditions specified in section 1.44, then every compact set* K *lies in allmost all* Q_n°, *i.e. there exists m such that*

$$(1.75) \qquad K \subset \mathring{Q}_m \subset \mathring{Q}_{m+1} \subset \mathring{Q}_{m+2} \subset \cdots.$$

Proof. The following definition

$$(1.76) \qquad\qquad C_n \triangleq K \setminus \overset{\circ}{Q}_n$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset Q_m^\circ$, for some m. Finally,

$$(1.77) \qquad K \subset \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots .$$

$\square$

We are now in a fair position to establish the following:

Theorem   *The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.*

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subset L_{m_i}^\circ \subset L_{m_i+1}^\circ \subset \cdots$ for some m_i. In particular, the special case $\varkappa_i = m_i + i$ is

$$(1.78) \qquad\qquad K_i \subset \overset{\circ}{L}_{\varkappa_i}.$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$(1.79) \qquad\qquad L_j \subset \overset{\circ}{K}_{\lambda_j}$$

Combine (1.78) with (1.79) and so obtain

$$(1.80) \qquad K_1 \subset \overset{\circ}{L}_{\varkappa_1} \subset L_{\varkappa_1} \subset \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subset K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots ,$$

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$(1.81) \qquad\qquad \tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L.$$

So ends the proof $\square$

---

[1] See (b) of 2.5 of [2].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [2].

1.17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$, *for every multi-index* α.

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$(1.82) \qquad V_N = \left\{ \varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \leq N, x \in K_N\} < 1/N \right\}.$$

Now pick g in $V_{N+|\alpha|}$, so that

$$(1.83) \qquad \max\{|D^\gamma g(x)| : |\gamma| \leq N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}.$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$(1.84) \qquad \max\{|D^\beta D^\alpha g(x)| : |\beta| \leq N, x \in K_N\} < \frac{1}{N}.$$

We have just proved that

$$(1.85) \qquad g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad i.e. \quad D^\alpha\left(V_{N+|\alpha|}\right) \subset V_N,$$

which establishes the continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^\alpha|_{\mathscr{D}_K}: \mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$(1.86) \qquad D^\alpha|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) = D^\alpha\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right)$$
$$(1.87) \qquad\qquad\qquad \subset D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha\left(\mathscr{D}_K\right)$$
$$(1.88) \qquad\qquad\qquad \subset V_N \cap \mathscr{D}_K \quad (\text{see } (1.85))$$

So ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Chapter 2

Completeness

2.3   Exercise 3. An equicontinous sequence of measures

Put $K = [-1, 1]$; define $\mathscr{D}_K$ as in section 1.46 (with $\mathbf{R}$ in place of $\mathbf{R}^n$). Supose $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\Lambda \varphi = \lim\limits_{n \to \infty} \int_{-1}^{1} f_n(t) \varphi(t) dt$ exists for every $\varphi \in \mathscr{D}_K$. Show that Λ is a continuous linear functional on $\mathscr{D}_K$. Show that there is a positive integer p and a number $M < \infty$ such that

$$\left| \int_{-1}^{1} f_n(t) \varphi(t) dt \right| \leq M \| D^p \|_{\infty}$$

for all n. For example, if $f_n(t) = n^3 t$ on $[-1/n, 1/n]$ and 0 elsewhere, show that this can be done with $p = 1$. Construct an example where it can be done with $p = 2$ but not with $p = 1$.

We will also consider the case $p = 0$. Since all supports of $\varphi, \varphi', \varphi'', \ldots$, are in K, we make a specialization of the mean value theorem:

Lemma   If $\varphi \in \mathscr{D}_{[a,b]}$, then

$$(2.1) \qquad \| D^\alpha \varphi \|_\infty \leq \| D^p \varphi \|_\infty \left(\frac{\lambda}{2} \right)^{p - \alpha} \qquad (\alpha = 0, 1, \ldots, p)$$

at every order $p = 0, 1, 2, \ldots$; where λ is the length $| b - a |$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 $(a < x_1 < x_0)$, such that

$$(2.2) \qquad \varphi(x_0) = \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a).$$

Since every $D^p \varphi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$(2.3) \qquad \varphi(x_0) = D^0 \varphi(x_0)$$
$$(2.4) \qquad \qquad = D^1 \varphi(x_1)(x_0 - a)$$
$$\qquad \qquad = \cdots$$
$$(2.5) \qquad \qquad = D^p \varphi(x_p)(x_0 - a) \cdots (x_{p-1} - a),$$

20

for all p. More compactly,

$$(2.6) \qquad D^\alpha \varphi(x_0) = D^p \varphi(x_p) \prod_{k=\alpha}^{p-1} (x_k - a);$$

which yields,

$$(2.7) \qquad |D^\alpha \varphi(x)| \le \| D^p \varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c])$$

The case $x_0 \ge c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$(2.8) \qquad |D^\alpha \varphi(x)| \le \| D^p \varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$(2.9) \qquad \| D^\alpha \varphi \|_\infty \le \| D^p \varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha}.$$

$\square$

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$(2.10) \qquad \langle u | \varphi \rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})).$$

The following inequalities

$$(2.11) \qquad |\langle u | \varphi \rangle| \le \int_{\mathbf{R}} |u\varphi| \le \| u \|_{L^1} \quad (\| \varphi \|_\infty \le 1)$$

imply that every linear functional

$$(2.12) \qquad \langle u | : C_0(\mathbf{R}) \to \mathbf{C}$$
$$\varphi \mapsto \langle u | \varphi \rangle$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u |$, since u is determined (a.e) by the integrals $\langle u | \varphi \rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u |$, of norm

$$(2.13) \qquad \sup\{| \langle u | \varphi \rangle | : \| \varphi \|_\infty = 1\} = \| u \|_{L^1}.$$

Note that, in the latter equality, $\le \| u \|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$(2.14) \qquad g_n : \mathbf{R} \to \mathbf{R}$$
$$x \mapsto \begin{cases} n^3 x & \left(x \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right) \\ 0 & \left(x \notin \left[-\frac{1}{n}, \frac{1}{n}\right]\right). \end{cases}$$

---

[1] see 1.32, 4.1 of [3]
[2] See 7.26 of [1].

First, remark that $g_n(x) \longrightarrow 0$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$(2.15) \qquad\qquad\qquad \langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi,$$

where φ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\varphi_1^+, \ldots, \varphi_j^+, \ldots$, from $C_K^\infty(\mathbf{R})$. More specifically,

  (i) $\varphi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

  (ii) $\varphi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

  (iii) $0 \leq \varphi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those φ_j^+. Let $\varphi_1^-, \ldots, \varphi_j^-, \ldots$, mirror the φ_j^+, in the sense that $\varphi_j^-(x) = \varphi_j^+(-x)$, so that

  (iv) $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$ is odd, as g_n is;

  (v) every φ_j is in $C_K^\infty(\mathbf{R})$;

  (vi) The sequence $\{\varphi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\|\varphi_j\|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$(2.16) \qquad \langle g_n|\varphi_j\rangle = 2\int_0^1 g_n(t)\varphi_j^+(t)dt \underset{j\to\infty}{\longrightarrow} 2\int_0^1 g_n(t)dt = \|g_n\|_{L^1} = n.$$

Finally,

$$(2.17) \qquad \|g_n\|_{L^1} \overset{(2.16)}{\leq} \sup\{|\langle g_n|\varphi\rangle| : \|\varphi\|_\infty = 1\} \overset{(2.13)}{\leq} \|g_n\|_{L^1};$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$(2.18) \qquad\qquad\qquad |\langle g_n|\varphi\rangle| \leq C_n \quad (\|\varphi\|_\infty = 1);$$

see (2.11). Furthermore, $\|g_n\|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\varphi_{\rho(n)}\rangle\}$ such that $\langle g_n|\varphi_{\rho(n)}\rangle$ is greater than, say, $n - 0.01$, as $\|\varphi_{\rho(n)}\|_\infty = 1$. Consequently, there is no bound M such that

$$(2.19) \qquad\qquad |\langle g_n|\varphi\rangle| \leq M \quad (\|\varphi\|_\infty = 1; n = 1, 2, 3, \ldots).$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinuous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$(2.20) \qquad\qquad\qquad \langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi \quad (\varphi \in C_0(\mathbf{R}))$$

---

  [3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that φ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$(2.21) \qquad \Lambda\varphi = \lim_{n\to\infty} \int_K f_n\varphi \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

for some linear form Λ. Since φ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$(2.22) \qquad \Lambda\varphi = \lim_{n\to\infty} \langle f_n|\varphi\rangle \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

Let $K_1, K_2, \ldots$, be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha\varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\varphi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\varphi \in \mathscr{D}_K : \|\varphi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\varphi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|f_n\|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [3].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi$ for all φ, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$(2.23) \qquad |\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i,$$
$$(2.24) \qquad |\Lambda\varphi - \langle f_n|\varphi\rangle| < 1/2 \quad \text{if } \varphi \in V_j.$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$(2.25) \qquad |\langle f_n|\varphi\rangle| \leq |\Lambda\varphi - \langle f_n|\varphi\rangle| + |\Lambda\varphi| < 1 \quad \text{if } \varphi \in V_p.$$

Now remark that every $\psi = \psi[\mu, \varphi]$, where

$$(2.26) \qquad \psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\varphi\|_p)\varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

(2.27) $\qquad\qquad\qquad\qquad |\langle f_n | \psi \rangle| < 1$

(2.28) $\qquad\qquad\qquad\qquad |\langle f_n | \varphi \rangle| < 2^p \| \varphi \|_p \cdot \mu$

(2.29) $\qquad\qquad\qquad\qquad |\langle f_n | \varphi \rangle| \leq 2^p \| \varphi \|_p$

(2.30) $\qquad\qquad\qquad\qquad |\langle f_n | \varphi \rangle| \leq 2^p \{ \| D^0 \varphi \|_\infty + \cdots + \| D^p \varphi \|_\infty \}.$

Finally, with the help of (2.1),

(2.31) $\qquad\qquad\qquad\qquad |\langle f_n | \varphi \rangle| \leq 2^p (p+1) \| D^p \varphi \|_\infty.$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3 x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\varphi = \lim\limits_{n\to\infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n | \varphi \rangle| \leq M \| D^p \varphi \|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let φ keep running on $\mathscr{D}_K$ and so define

(2.32) $\qquad\qquad\qquad B_n(\varphi) \triangleq \max\{ |\varphi(x)| : x \in [-1/n, 1/n] \},$

(2.33) $\qquad\qquad\qquad \Delta_n(\varphi) \triangleq \max\{ |\varphi(x) - \varphi(0)| : x \in [-1/n, 1/n] \}.$

The mean value asserts that

(2.34) $\qquad\qquad |\varphi(1/n) - \varphi(\text{-}1/n)| \leq B_n(\varphi') |1/n - (\text{-}1/n)| = \dfrac{2}{n} B_n(\varphi').$

Independently, an integration by parts shows that

(2.35) $\qquad\qquad \langle f_n | \varphi \rangle = \left[\dfrac{n^3 t^2}{2} \varphi(t) \right]_{-1/n}^{1/n} - \dfrac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt$

(2.36) $\qquad\qquad\qquad = \dfrac{n}{2} \left(\varphi(1/n) - \varphi(\text{-}1/n) \right) - \dfrac{n^3}{2} \int_{-1/n}^{1/n} t^2 \varphi'(t) dt.$

Combine (2.34) with (2.36) and so obtain

(2.37) $\qquad\qquad |\langle f_n | \varphi \rangle| \leq \dfrac{n}{2} |\varphi(1/n) - \varphi(\text{-}1/n)| + \dfrac{n^3}{2} \int_{-1/n}^{1/n} t^2 |\varphi'(t)| dt$

(2.38) $\qquad\qquad\qquad \leq B_n(\varphi') + \dfrac{n^3}{2} B_n(\varphi') \int_{-1/n}^{1/n} t^2 dt$

(2.39) $\qquad\qquad\qquad \leq \dfrac{4}{3} B_n(\varphi')$

(2.40) $\qquad\qquad\qquad \leq \dfrac{4}{3} \| \varphi' \|_\infty.$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\varphi')$ tends to $|\varphi'(0)|$, we may expect that f_n tends to $\frac{4}{3}\varphi'(0)$. This is actually true: A straightforward computation shows that

$$(2.41) \qquad \langle f_n | \varphi \rangle - \frac{4}{3}\varphi'(0) \overset{(2.36)}{=} \frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n}(\varphi' - \varphi'(0))t^2 dt.$$

So,

$$(2.42) \qquad \left| \langle f_n | \varphi \rangle - \frac{4}{3}\varphi'(0) \right| \leq \left| \frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0) \right| + \frac{1}{3}\Delta_n(\varphi') \underset{n\to\infty}{\longrightarrow} 0.$$

We have just proved that

$$(2.43) \qquad \langle f_n | \varphi \rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\varphi'(0) \quad (\varphi \in \mathscr{D}_K).$$

In other words,

$$(2.44) \qquad \langle f_n | \underset{n\to\infty}{\longrightarrow} \text{-}\frac{4}{3}\delta',$$

where δ is the *Dirac measure* and $\delta', \delta'', \dots$, its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $-\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$(2.45) \qquad |\langle f_n | \varphi \rangle| \leq \frac{4}{3}\|\varphi'\|_\infty \quad (n = 1, 2, 3, \dots)$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$(2.46) \qquad \{\langle f_n | \varphi_{\rho(n)} \rangle : \|\varphi_{\rho(n)}\|_\infty = 1; n = 1, 2, 3, \dots\}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$(2.47) \qquad |\langle f_n | \varphi \rangle| \leq M\|\varphi\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})).$$

The previous bound of $\langle f_n |$ - see (2.40), is therefore the best possible one, *i.e.* p = 1 is the smallest possible p and, given p = 1, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*[4] f_n', as follows

$$(2.48) \qquad f_n' : \mathscr{D}_K \to \mathbf{C}$$
$$\varphi \mapsto \text{-}\langle f_n | \varphi' \rangle.$$

It has been proved that every $\langle f_n |$ is continuous. So is

$$(2.49) \qquad D : \mathscr{D}_K \to \mathscr{D}_K$$
$$\varphi \mapsto \varphi';$$

---

[4] See 6.1 of [3] for a further discussion.

see Exercise 1.17. f_n' is therefore continuous. Now apply (2.43) with φ' and so obtain

$$- \langle f_n | \varphi' \rangle \xrightarrow[n \to \infty]{} \frac{4}{3} \varphi''(0) \quad (\varphi \in \mathscr{D}_K),$$

i.e.

(2.50)
$$f_n' \xrightarrow[n \to \infty]{} \frac{4}{3} \delta''.$$

It follows from (2.40) that,

(2.51)
$$|\langle f_n | \varphi' \rangle| \le \frac{4}{3} \| \varphi'' \|_\infty \quad (n = 1, 2, 3, \dots).$$

It is therefore possible to uniformly bound f_n' with respect to a norm $\| D^p \cdot \|_\infty$, namely $\| D^2 \cdot \|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

(2.52)
$$|\langle f_n | \varphi' \rangle| \le M \| \varphi' \|_\infty \quad (n = 1, 2, 3, \dots).$$

Define

(2.53)
$$\Phi_j(x) = \int_{-1}^{x} \varphi_j.$$

The oddness of φ_j forces Φ_j to vanish outside $[-1, 1]$: φ_j is therefore in $\mathscr{D}_K$. So, under our assumption,

(2.54)
$$|\langle f_n | \Phi_j' \rangle| \le M \| \Phi_j' \|_\infty \quad (n = 1, 2, 3, \dots);$$

which is

(2.55)
$$|\langle f_n | \varphi_j \rangle| \le M \quad (n = 1, 2, 3, \dots).$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n | \varphi_{\rho(n)} \rangle\}$) and so conclude that there is no constant M such that

(2.56)
$$|\langle | f_n \varphi' \rangle| \le M \| \varphi' \|_\infty \quad (n = 1, 2, 3, \dots).$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

(2.57)
$$|\langle f_n | \varphi' \rangle| \le M \| \varphi \|_\infty.$$

The mean value theorem (see (2.1)) asserts that

(2.58)
$$|\langle f_n | \varphi' \rangle| \le M \| \varphi \|_\infty \le M \| \varphi' \|_\infty;$$

which is, again, a desired contradiction. So ends the proof.

$\square$

2.6   Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ of a function $f \in L^2(T)$ (T is the unit circle) by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

for all $n \in \mathbf{Z}$ (the integers). Put

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f \text{ exists}\}$ is a dense subspace of $L^2(T)$ of the first category.

Proof. Let $f(\theta)$ stand for $f(e^{i\theta})$, so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

(2.59) $$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

We believe it is customary to write

(2.60) $$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n).$$

Moreover, a well known (and easy to prove) result is

(2.61) $$(e_n, e_{n'}) = [n = n'], \text{ i.e. } \{e_n : n \in \mathbf{Z}\} \text{ is an orthormal subset of } L^2(T).$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

(2.62) $$\| \Lambda_n \|^2 \overset{(2.60)}{=} \| e_{-n} + \cdots + e_n \|^2 \overset{(2.61)}{=} \| e_{-n} \|^2 + \cdots + \| e_n \|^2 \overset{(2.61)}{=} 2n + 1.$$

We now assume, to reach a contradiction, that

(2.63) $$B \triangleq \{f \in L^2(T) : \sup\{| \Lambda_n f | : n = 1, 2, 3, \ldots\} < \infty\}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

(2.64) $$\| \Lambda_n \| \overset{(2.62)}{=} \sqrt{2n + 1} \underset{n\to\infty}{\longrightarrow} \infty.$$

We have just established that B is actually of the first category; and so is its subset $L = \{f \in L^2(T) : \lim_{n \longrightarrow \infty} \Lambda_n f \text{ exists}\}$. We now prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be $\text{span}\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

(2.65) $$P \subset L \subset L^2(T).$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.65), that

(2.66) $$L^2(T) = \overline{P} = \overline{L}.$$

So ends the proof   □

2.9   Exercise 9. Boundedness without closedness

Suppose X, Y, Z *are Banach spaces and*

$$B : X \times Y \to Z$$

is bilinear and continuous. Prove that there exists $M < \infty$ *such that*

$$\| B(x, y) \| \le M \| x \| \| x \| \quad (x \in X, y \in Y).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Since B is continous at the origin, there exists a positive r such that

$$(2.67) \qquad\qquad \| x \| + \| y \| < r \Rightarrow \| B(x, y) \| < 1.$$

Given nonzero x, y, let s range over $]0, r[$, so that the folllowing bound

$$(2.68) \qquad \| B(x, y) \| = \frac{4 \| x \| \| y \|}{s^2} \left\| B \left(\frac{s}{2 \| x \|} x, \frac{s}{2 \| y \|} y \right) \right\| \overset{(2.67)}{<} \frac{4 \| x \| \| y \|}{s^2}$$

is effective. It is now obvious that

$$(2.69) \qquad B(x, y) \le \frac{4}{s^2} \| x \| \| y \| \underset{s \to r}{\longrightarrow} \frac{4}{r^2} \| x \| \| y \| \quad ((x, y) \in X \times Y);$$

which achieves the proof.

As a concrete example, choose $X = Y = Z = C_c(\mathbf{R})$, topologized by the supremum norm. $C_c(\mathbf{R})$ is not complete (see 5.4.4 of [4]), nevertheless the bilinear product

$$\begin{aligned} B : \quad C_c(\mathbf{R})^2 &\to C_c(\mathbf{R}) \\ (f, g) &\mapsto f \cdot g \end{aligned}$$

is bounded (since $\| f \cdot g \|_\infty \le \| f \|_\infty \cdot \| g \|_\infty$), and continuous. To show this, pick a positive scalar ε smaller than 1 then put

$$(2.70) \qquad\qquad r \triangleq \frac{\varepsilon}{1 + \| f \|_\infty + \| g \|_\infty} < 1$$

for all (f, g). Obviously, there exists a segment S that contains both supports of f and g: Now take T a segment that is strictly greater than S ($S \subsetneq T$) then consider a continuous mapping ϕ that equals 1 on S , 0 outside T, and that keeps in $[0, 1]$ otherwise (see 1.46 of [3] for a construction of ϕ). The Stone-Weierstrass theorem (see 7.26 of [1]) asserts the existence of polynomials p, q such that

$$(2.71) \qquad\qquad \| f - p \|_\infty + \| g - q \|_\infty < r.$$

on S. It is now clear that $u = \phi p \in C_c(\mathbf{R})$ with $v = \phi q C_c(\mathbf{R})$ yield

$$(2.72) \qquad\qquad \| f - u \|_\infty + \| g - v \|_\infty < r.$$

Next, remark that $\| u \|_\infty \le r + \| f \|_\infty$ and so obtain (bear in mind that $r < 1$)

$$(2.73) \qquad\qquad \| fg - uv \|_\infty = \| (f - u) \cdot g + u \cdot (g - v) \|_\infty$$
$$(2.74) \qquad\qquad\qquad\qquad \le \| f - u \|_\infty \cdot \| g \|_\infty + \| u \|_\infty \cdot \| g - v \|_\infty$$
$$(2.75) \qquad\qquad\qquad\qquad < r \cdot \| g \|_\infty + (r + \| f \|_\infty) \cdot r$$
$$(2.76) \qquad\qquad\qquad\qquad < r \cdot (r + \| f \|_\infty + \| g \|_\infty)$$
$$(2.77) \qquad\qquad\qquad\qquad < \varepsilon.$$

Since ε was arbitrary, it is now established that B continuous. $\square$

2.10   Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let (X_1, X_2, Z) be topological spaces and B a bilinear mapping

$$(2.78) \qquad\qquad\qquad B : X_1 \times X_2 \to Z$$

From now on, $x = (x_1, x_2)$ denotes an arbitrary element of $X_1 \times X_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $X_1 \times X_2$, *i.e.* given an arbitrary balanced open subset W of Z, there exists in X_i $(i = 1, 2)$ a balanced open subset U_i such that

$$(2.79) \qquad\qquad\qquad B(U_1 \times U_2) \subset W.$$

Let $\nu_i(x)$ denote any scalar that is greater than $\mu_i(x_i) = \inf\{r > 0 : x_i \in r \cdot U_i\}$. So,

$$(2.80) \qquad\qquad B(x_1, x_2) = \nu_1(x)\nu_2(x) \cdot B\left(\nu_1(x)^{-1}x_1, \nu_2(x)^{-1}x_2\right)$$
$$(2.81) \qquad\qquad\qquad\quad \in \nu_1(x)\nu_2(x) \cdot B(U_1 \times U_2)$$
$$(2.82) \qquad\qquad\qquad\quad \subset \nu_1(x)\nu_2(x) \cdot W.$$

Now pick $p = (p_1, p_2)$ in $X_1 \times X_2$: It directly follows from (2.82) that

$$(2.83) \quad B(p_1, p_2) - B(x_1, x_2) = B(p_1, p_2 - x_2) + B(p_1 - x_1, x_2 - p_2) + B(p_1 - x_1, p_2)$$
$$(2.84) \qquad\qquad \in \nu_1(p)\nu_2(p-x) \cdot W + \nu_1(p-x)\nu_2(x-p) \cdot W + \nu_1(p-x)\nu_2(p) \cdot W.$$

Let us henceforth assume that

$$(2.85) \qquad\qquad p_i - x_i \in [\mu_1(p) + \mu_2(p) + 2]^{-1} \cdot U_i;$$

which yields

$$(2.86) \qquad\qquad \mu_i(p_i - x_i) \leq [\mu_1(p) + \mu_2(p) + 2]^{-1}.$$

Finally, combine the special case

$$(2.87) \qquad\qquad \nu_i(p - x) = [\mu_1(p) + \mu_2(p) + 1]^{-1},$$
$$(2.88) \qquad\qquad \nu_i(p) = \mu_1(p) + \mu_2(p) + 1$$

with (2.84) and so obtain

$$(2.89) \qquad\qquad\qquad B(p_1, p_2) - B(x_1, x_2) \in W + W + W.$$

W being arbitrary, we have so established the continuousness of B at (p_1, p_2). Since (p_1, p_2) is also arbitrary, the proof is complete. $\square$

2.12   Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\| f \| = \int_0^1 |f(t)| \ dt.$$

Put $B(f, g) = \int_0^1 f(t)g(t)dt$, *and show that* B *is a bilinear continuous functional on* $X \times X$ *which is separately but not continuous.*

Proof. Let f denote the first variable, g the second one. Remark that

$$(2.90) \qquad\qquad | B(f, g) | < \| f \| \cdot \max_{[0,1]} | g | ;$$

which is sufficient (1.18 of [3]) to assert that any $f \mapsto B(f, g)$ is continuous. The continuity of all $g \mapsto B(f, g)$ follows (Put $C(g, f) = B(f, g)$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$(2.91) \qquad\qquad | B(f, g) | < M\| f \|\| g \|.$$

Put

$$(2.92) \qquad\qquad f_n(x) \triangleq 2\sqrt{n} \cdot x^n \in \mathbf{R}[x] \qquad (n = 1, 2, 3, \dots),$$

so that

$$(2.93) \qquad\qquad \| f_n \| = \frac{2\sqrt{n}}{n + 1} \xrightarrow[n\to\infty]{} 0.$$

On the other hand,

$$(2.94) \qquad\qquad B(f_n, f_n) = \frac{4n}{2n + 1} > 1.$$

Finally, we combine (2.94) and (2.91) with (2.93) and so obtain

$$(2.95) \qquad\qquad 1 < B(f_n, f_n) < M\| f_n \|^2 \xrightarrow[n\to\infty]{} 0.$$

Our continuousness assumption is then contradicted. So ends the proof.   □

2.15 Exercise 15. Baire cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* Y = X. *Hint:* Y *must intersect* x + Y *for every* x ∈ X.

Proof. Assume Y is a subgroup of X. Under our assumptions, there exists a sequence $\{E_n : n = 1, 2, 3, \dots\}$ of X such that

(i) $(\overline{E}_n)^\circ = \emptyset$;

(ii) $X \setminus Y = \bigcup_{n=1}^{\infty} E_n$.

By (i), the complement V_n of $\overline{E}_n$ is a dense open set. Since X is an F-space, it follows from the Baire's theorem that the intersection S of the V_n's is dense in X: So is $x + S$ ($x \in X$). To see that, remark that

$$(2.96) \qquad\qquad X = x + \overline{S} \subset \overline{x + S}$$

follows from 1.3 (b) of [3]. Since S and $x + S$ are both dense open subsets of X, the Baire's theorem asserts that

$$(2.97) \qquad\qquad \overline{(x + S) \cap S} = X.$$

Thus,

$$(2.98) \qquad\qquad (x + S) \cap S \neq \emptyset.$$

Moreover, it follows from (ii) that $X \setminus Y \subset \bigcup_n \overline{E}_n$, *i.e.* $Y \supset S$. Combined with (2.98), this shows that $x + Y$ cuts Y. Therefore, our arbitrary x is an element of the subgroup Y. We have thus established that $X \subset Y$, which achieves the proof. □

2.16   Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $f : X \to K$ *is a closed subset of* $X \times K$*. Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

Proof. Choose a sequence $\{x_n : n = 1, 2, 3, \dots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ of $\{(x_n, f(x_n))\}$ that converges to some (a, b) of $X \times K$. G is closed; therefore, $\{(x_{\rho(n)}, f(x_{\rho(n)}))\}$ converges in G. So, $b = f(a)$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$(2.99) \qquad\qquad f : [0, \infty) \to [0, \infty)$$

$$x \mapsto \begin{cases} 1/x & (x > 0) \\ 0 & (x = 0). \end{cases}$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$(2.100) \qquad\qquad G = \{(x, 1/x) : x > 0\} \cup \{(0, 0)\}.$$

Next, let $\{(x_n, 1/x_n)\}$ be a sequence in $G_+ = \{(x, 1/x) : x > 0\}$ that converges to (a, b). To be more specific: $a = 0$ contradicts the boundedness of $\{(x_n, 1/x_n)\}$: a is necessarily positive and $b = 1/a$, since $x \mapsto 1/x$ is continuous on R_+. This establishes that $(a, b) \in G_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. $\qquad\square$

Chapter 3

Convexity

3.3 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subset X$ an internal point of A if $A - x_0$ is an absorbing set.

(a) *Suppose A and B are disjoint convex sets in X, and A has an internal point. Prove that there is a nonconstant linear functional Λ such that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) *Show (with $X = \mathbf{R}^2$, for example) that it may not possible to have $\Lambda(A)$ and $\Lambda(B)$ disjoint, under the hypotheses of (a).*

Proof. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [3] to establish the existence of a linear functional $\Lambda : X \to \mathbf{R}$ such that

$$(3.1) \qquad\qquad \Lambda \leq p$$

and

$$(3.2) \qquad\qquad \Lambda(b_0 - x_0) = 1.$$

Then

$$(3.3) \quad \Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B).$$

Hence

$$(3.4) \qquad\qquad \Lambda a \leq \Lambda b.$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There so exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

$$(3.5) \qquad\qquad \Lambda a_i = \Lambda b_i = y_i.$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$(3.6) \qquad\qquad 2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad .$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

$$(3.7) \qquad\qquad\qquad\qquad \Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1 \quad;$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$(3.8) \qquad\qquad\qquad S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\},$$

$$(3.9) \qquad\qquad\qquad S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\},$$

$$(3.10) \qquad\qquad\qquad A \triangleq S_1 \cup S_2,$$

$$(3.11) \qquad\qquad\qquad B \triangleq X \setminus A.$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$(3.12) \qquad t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1 - t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1 - t) \cdot x_2 \\ t \cdot y_1 + (1 - t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subset A.$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.* A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a linear functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$(3.13) \qquad\qquad\qquad \Lambda(L) = \{0\} \subset \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad.$$

So ends the proof.   □

---

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [2].

3.11 Exercise 11. Meagerness of the polar

Let X be an infinite-dimensional Fréchet space. Prove that X, with its weak*-topology, is of the first category in itself.*

This is actually a consequence of the below lemma, which we prove first. The proof that X* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual X* separates points on X, then the polar*

(3.14) $$K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \ on \ A\}$$

of any absorbing subset A is a weak-closed set that has empty interior.*

Proof. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous; see 3.14 of [3]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X* (see 3.11 of [3]). Put

(3.15) $$W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \qquad (r_x > 0)$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X*. Pick one of those W and remark that the following subspace

(3.16) $$M \triangleq \operatorname{span}(F)$$

is finite dimensional. Assume, to reach a contradiction, that $A \subset M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subset M$: Now pick *a* in $A \setminus M$ and so conclude that

(3.17) $$b \triangleq \frac{a}{t_a} \in A$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see 1.21 (b) of [3] for a proof): By the Hahn-Banach theorem 3.5 of [3], there exists Λ_a in X* such that

(3.18) $$\Lambda_a b > 2$$

and

(3.19) $$\Lambda_a(M) = \{0\}.$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

(3.20) $$|\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1.$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

(3.21) $$\Lambda + W \not\subset K_A.$$

Since W and Λ are both arbitrary, this achieves the proof. $\square$

We now give a proof of the original statement.

Corollary.   *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$(3.22) \qquad B_n \triangleq \{x \in X : d(0,x) < 1/n\} \qquad (n = 1,2,3,\dots)$$

form a local base of X. If Λ is in X^*, then

$$(3.23) \qquad |\Lambda| \leq m \text{ on } B_n$$

for some $(n,m) \in \{1,2,3,\dots\}^2$; see 1.18 of [3]. Hence, X^* is the countable union of all

$$(3.24) \qquad m \cdot K_n \qquad (m,n = 1,2,3,\dots),$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [3]. As a consequence, the above lemma implies

$$(3.25) \qquad \left(\overline{K_n}\right)^\circ = (K_n)^\circ = \emptyset.$$

Since the multiplication by m is an homeomorphism (see 1.7 of [3]), this is equivalent to

$$(3.26) \qquad \left(\overline{m \cdot K_n}\right)^\circ = m \cdot (K_n)^\circ = \emptyset.$$

So ends the proof. $\square$

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

4.1 Exercise 1. Basic results

Let φ be the embedding of X into X^{**} decribed in Section 4.5. Let τ be the weak topology of X, and let σ be the weak*- topology of X^{**}- the one induced by X^*.

(a) *Prove that φ is an homeomorphism of (X, τ) onto a dense subspace of (X^{**}, σ).*

(b) *If B is the closed unit ball of X, prove that $\varphi(B)$ is σ-dense in the closed unit ball of X^{**}. (Use the Hahn-Banach separation theorem.)*

(c) *Use (a), (b), and the Banach-Alaoglu theorem to prove that X is reflexive if and only if B is weakly compact.*

(d) *Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.*

(e) *If X is reflexive and Y is a closed subspace of X, prove that X/Y is reflexive.*

(f) *Prove that X is reflexive if and only X^* if reflexive.*
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\varphi(X)$ of X^{**}.

Proof. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$(4.1) \qquad\qquad\qquad \varphi^{-1} : \varphi(X) \to X$$

$$(4.2) \qquad\qquad\qquad \varphi(x) \mapsto x$$

is an homeomorphism (with respect to τ and σ). We first consider

$$(4.3) \qquad V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}|\psi x^*\rangle| < r\} \qquad\qquad (x^* \in X^*, r > 0);$$

$$(4.4) \qquad U \triangleq \{x \in X : |\langle x|x^*\rangle| < r\} \qquad\qquad (x^* \in X^*, r > 0).$$

and remark that the so defined V's (respectively U's) shape a local subbase $\mathscr{S}_\sigma$ (respectively $\mathscr{S}_\tau$) of σ (respectively τ). We now observe that

$$(4.5) \qquad U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathscr{S}_\sigma,\ U \in \mathscr{S}_\tau) \quad ,$$

since φ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathscr{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that φ^{-1} and its inverse φ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$(4.6) \qquad\qquad \overline{\varphi(X)}_\sigma \subset X^{**} \quad .$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\varphi(X)$. By [3.5], there so exists y^* in X^* such that

$$(4.7) \qquad\qquad \langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ;$$
$$(4.8) \qquad\qquad \langle z^{**}, \psi y^* \rangle = 1$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$(4.9) \qquad\qquad \varphi(B) \subset B^{**} \quad ,$$

since φ is isometric. Hence

$$(4.10) \qquad\qquad \overline{\varphi(B)}_\sigma \subset \overline{(B^{**})}_\sigma = B^{**} \quad .$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\varphi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$(4.11) \qquad\qquad |\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_\sigma \quad ;$$
$$(4.12) \qquad\qquad \langle z^{**}, \psi y^* \rangle > 1 \quad .$$

It follows from (4.11) that

$$(4.13) \qquad\qquad |\psi y^*| \leq 1 \text{ on } \varphi(B), \ i.e. |y^*| \leq 1 \text{ on } B \quad .$$

We have so proved that

$$(4.14) \qquad\qquad y^* \in B^* \quad .$$

Since z^{**} lies in B^{**}, it is now clear that

$$(4.15) \qquad\qquad |\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ;$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

$$(4.16) \qquad\qquad B \text{ is weakly compact if and only if } \varphi(B) \text{ is weak*-compact.}$$

If B is weakly compact, then $\varphi(B)$ is weak*-closed. So,

$$(4.17) \qquad\qquad \varphi(B) = \overline{\varphi(B)}_\sigma \overset{(b)}{=} B^{**} \quad .$$

φ is therefore onto, *i.e.* X is reflexive.

Conversely, keep φ as onto: one easily checks that $\varphi(B) = B^{**}$. The image $\varphi(B)$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $B \cap Y$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let ≡ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(4.18) \qquad\qquad (X/Y)^* \equiv Y^{\perp} \quad,$$

next,

$$(4.19) \qquad\qquad (X/Y)^{**} \equiv (Y^{\perp})^* \equiv X^{**}/(Y^{\perp})^{\perp} \equiv X/Y \quad.$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

> *Given a reflexive Banach space* Z, *the weak*-topology of* Z^* *is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\varphi(X)$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\varphi(X)$ is reflexive, by (d). It now follows from (c) that $B^{**} \cap \varphi(X)$ is weakly compact, *i.e.* weak*-compact (to see this, apply (L) with $Z = X^*$).

By (a), B is therefore weakly compact, *i.e.* X is reflexive; see (c). So ends the proof. □

4.13   Exercise 13. Operator compactness in a Hilbert space

4.15 Exercise 15. Hilbert-Schmidt operators

Chapter 6

Distributions

6.1 Exercise 1. Test functions are almost polynomial

6.6 Exercise 6. Around the supports of some distributions

6.9 Exercise 9. Convergence in $\mathscr{D}(\Omega)$ vs. convergence in $\mathscr{D}'(\Omega)$

(a) *Prove that a set* $E \subset \mathscr{D}(\Omega)$ *is bounded if and only if*

$$\sup\{|\Lambda\varphi| : \varphi \in E\} < \infty$$

for every $\Lambda \in \mathscr{D}(\Omega)$.

(b) *Suppose* $\{\varphi_j\}$ *is a sequence in* $\mathscr{D}(\Omega)$ *such that* $\{\Lambda\varphi_j\}$ *is a bounded sequence of numbers, for every* $\Lambda \in \mathscr{D}'(\Omega)$. *Prove that some subsequence of* $\{\varphi_j\}$ *converges, in the topology of* $\mathscr{D}(\Omega)$.

(c) *Suppose* $\{\Lambda_j\}$ *is a sequence in* $\mathscr{D}'(\Omega)$ *such that* $\{\Lambda_j \varphi\}$ *is bounded, for every* $\varphi \in \mathscr{D}(\Omega)$. *Prove that some subsequence of* $\{\Lambda_j\}$ *converges in* $\mathscr{D}'(\Omega)$ *and that the convergence is uniform on every bounded subset of* $\mathscr{D}(\Omega)$. *Hint: By the Banach-Steinhaus theorem, the restrictions of the* Λ_j *to* $\mathscr{D}_K$ *are equicontinuous. Apply Ascoli's theorem.*

PROOF. Since $\mathscr{D}(\Omega)$ locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $E = \{\varphi_j : j \in \mathbf{N}\}$ is bounded: so is $\overline{E}$. By (c) of [6.5], there exists some $\mathscr{D}_K$ that contains $\overline{E}$. Since $\mathscr{D}_K$ has the Heine-Borel property (see [1.46]), $\overline{E}$ is τ_K-compact. Apply [A4] with the metrizable space $\mathscr{D}_K$ (see [1.46]) to conclude that $\overline{E}$ has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds.

Bibliography

[1] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 1976.

[2] Walter Rudin. *Real and Complex Analysis*. McGraw-Hill, 1986.

[3] Walter Rudin. *Functional Analysis*. McGraw-Hill, 1991.

[4] Laurent Schwartz. *Analyse*, volume III (in French). Hermann, 1997.