Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Notations and Conventions **ii**
1.1 Logic . . . ii

1 Topological Vector Spaces **1**
1.1 Exercise 7. Metrizability & number theory . . . 2
1.2 Exercise 9. Quotient map . . . 4
1.3 Exercise 10. An open mapping theorem . . . 5
1.4 Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms . . . 6
1.5 Exercise 16. Uniqueness of topology for test functions . . . 7
1.6 Exercise 17. Derivation in some non normed space . . . 9

2 Completeness **10**
2.1 Exercise 3. An equicontinous sequence of measures . . . 10
2.2 Exercise 6. Fourier series may diverge at 0 . . . 17
2.3 Exercise 9. Boundedness without closedness . . . 18
2.4 Exercise 10. Continuousness of bilinear mappings . . . 19
2.5 Exercise 12. A bilinear mapping that is not continuous . . . 20
2.6 Exercise 15. Baire cut . . . 21
2.7 Exercise 16. An elementary closed graph theorem . . . 22

3 Convexity **23**
3.1 Exercise 3. . . . 23
3.2 Exercise 11. Meagerness of the polar . . . 25

4 Banach Spaces **27**
4.1 Exercise 1. Basic results . . . 27

Bibliography **30**

Notations and Conventions

1.1 Logic

1. **Halmos' iff**: **iff** is a short for "if and only if".

2. **Definitions (of values) with $\triangleq$**: Given a variables a and b, a $\triangleq$ b means that a is defined as equal to b.

3. **Definitions (formulæ)**: Definitions come from **iff** . In other words, both parts (the "if ..." part and the "only if ..." part) are explicitely stated.

4. **Iverson notation**: Given a boolean expression Φ,

$$[\,\Phi\,] \triangleq \begin{cases} 0 & \text{if } \Phi \text{ is false;} \\ 1 & \text{if } \Phi \text{ is true.} \end{cases} \tag{1.1}$$

   For example, $[\,1 > 0\,] = 1$ but $\left[\,\sqrt{2} \in \mathbf{Q}\,\right] = 0$

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \qquad (0 \leq x \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{\mathrm{f_n}\}$ *in X such that (a)* $\{\mathrm{f_n}\}$ *converges to* 0 *as* $\mathrm{n} \to \infty$*, but (b) if* $\{\gamma_\mathrm{n}\}$ *is any sequence of scalars such that* $\gamma_\mathrm{n} \to \infty$ *then* $\{\gamma_\mathrm{n}\mathrm{f_n}\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

Proof. The family of the seminorms p_x is separating: By 1.37 of [3], the collection $\mathscr{B}$ of all finite intersections of the sets

$$\mathrm{V}(x,k) \triangleq \{\mathrm{p}_x < 2^{-k}\} \qquad (x \in [0,1], k \in \mathbf{N}) \tag{1.1}$$

is therefore a local base for a topology τ on X. So,

$$\sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \cap_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{U_i} \,] \leq \sum_{\mathrm{n}=1}^{\infty} \sum_{\mathrm{i}=1}^{\mathrm{m}} [\, \mathrm{f_n} \notin \mathrm{U_i} \,] = \sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{U_i} \,] \qquad (\mathrm{f_n} \in \mathrm{X}, \mathrm{U_i} \in \tau). \tag{1.2}$$

Now assume that $\{\mathrm{f_n}\}$ τ-converges to some f, *i.e.*

$$\sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{W} \,] < \infty \qquad (\mathrm{W} \in \mathscr{B}). \tag{1.3}$$

The special case $\mathrm{W} = \mathrm{V}(x,k)$ means that $|\, \mathrm{f_n}(x) - \mathrm{f}(x) \,| < 2^{-k}$ for almost all n, *i.e.* $\{\mathrm{f_n}(x)\}$ converges to $\mathrm{f}(x)$. Conversely, assume that $\{\mathrm{f_n}\}$ does not τ-converges in X, *i.e.*

$$\forall \mathrm{f} \in \mathrm{X}, \exists \mathrm{W} \in \mathscr{B} : \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{W} \,] = \infty. \tag{1.4}$$

W is now the intersection of finitely many $\mathrm{V}(x,k)$, say $\mathrm{V}(\mathrm{x_1},\mathrm{k_1}), \ldots, \mathrm{V}(\mathrm{x_m},\mathrm{k_m})$. Thus,

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{V}(\mathrm{x_i},\mathrm{k_i}) \,] \overset{(1.2)}{\geq} \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{W} \,] \overset{(1.4)}{=} \infty. \tag{1.5}$$

We can now conclude that, for some index i,

$$\sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{V}(\mathrm{x_i},\mathrm{k_i}) \,] = \infty. \tag{1.6}$$

In other word, $\{\mathrm{f_n}(\mathrm{x_i})\}$ fails to converge to $\mathrm{f}(\mathrm{x_i})$. We have so proved that τ-convergence is a rewording of pointwise convergence. We now establish the second part.

To do so, we split x into two variables: r if x is rational, a otherwise. The proof is based on the following well-known result: Each a has a *unique* binary expansion. More precisely, there exists a bijection $\mathrm{b} : [0,1] \setminus \mathbf{Q} \to \{\beta \in \{0,1\}^{\mathbf{N}_+} : \beta \text{ is not eventually periodic}\}$ where $\mathrm{b}(a) = (\beta_1, \beta_2, \dots)$ is the only bit stream such that

$$a = \sum_{\mathrm{k}=1}^{\infty} \beta_{\mathrm{k}} \cdot 2^{-\mathrm{k}}. \tag{1.7}$$

Remark that $\mathrm{b}(a)_1 + \dots + \mathrm{b}(a)_{\mathrm{n}} \longrightarrow \infty$, since $\mathrm{b}(a)$ has infinite support, then fix

$$\mathrm{f}_{\mathrm{n}}(a) \triangleq \frac{1}{\mathrm{b}(a)_1 + \dots + \mathrm{b}(a)_{\mathrm{n}}} \underset{\mathrm{n}\to\infty}{\longrightarrow} 0. \tag{1.8}$$

The actual values $\mathrm{f}_{\mathrm{n}}(\mathrm{r})$ are of no interest, as long as every sequence $\{\mathrm{f}_{\mathrm{n}}(\mathrm{r}) : \mathrm{n} = 1, 2, 3, \dots\}$ converges to 0. For example, put $\mathrm{f}_{\mathrm{n}}(\mathrm{r}) = \mathrm{r}/\mathrm{n}$, or just $\mathrm{f}_{\mathrm{n}}(\mathrm{r}) = 0$. We also take $\gamma_{\mathrm{n}} \longrightarrow \infty$, *i.e.* given any counting number p, γ_{n} is greater than p for almost all n. Next, we choose n_{p} among those *almost all* n that are large enough to satisfy

$$\mathrm{n}_{\mathrm{p}} - \mathrm{n}_{\mathrm{p}-1} > \mathrm{p} \tag{1.9}$$

(start with $\mathrm{n}_0 = 0$). So, every list $\mathrm{n}_{\mathrm{p}}, \mathrm{n}_{\mathrm{p}'}, \mathrm{n}_{\mathrm{p}''}, \dots$ that satisfies $\mathrm{n}_{\mathrm{p}'} - \mathrm{n}_{\mathrm{p}} = \mathrm{n}_{\mathrm{p}''} - \mathrm{n}_{\mathrm{p}'} = \dots$ is finite (otherwise, $\mathrm{n}_{\mathrm{p}'} - \mathrm{n}_{\mathrm{p}} \geq \mathrm{n}_{\mathrm{p}+1} - \mathrm{n}_{\mathrm{p}} > \mathrm{p} \to \infty$ would hold from; see (1.9)). In other words, *the distribution of $n_1, n_2, \dots$ displays no periodic pattern.* As a consequence, the *characteristic function* $\chi : \mathrm{k} \mapsto [\,\mathrm{k} \in \{\mathrm{n}_1, \mathrm{n}_2, \dots\}\,]$ is not eventually periodic. Combined with (1.7), this establishes that

$$a_\gamma \triangleq \sum_{\mathrm{k}=1}^{\infty} \chi_{\mathrm{k}} 2^{-k} \tag{1.10}$$

is irrational. Conversely, still with (1.7),

$$\mathrm{b}(a_\gamma)_{\mathrm{k}} = \chi_{\mathrm{k}}. \tag{1.11}$$

Now remark that

$$\chi_1 + \dots + \chi_{\mathrm{n}_1} = 1 \tag{1.12}$$

$$\chi_1 + \dots + \chi_{\mathrm{n}_1} + \dots + \chi_{\mathrm{n}_2} = 2 \tag{1.13}$$

$$\vdots$$

$$\chi_1 + \dots + \chi_{\mathrm{n}_1} + \dots + \chi_{\mathrm{n}_2} + \dots + \chi_{\mathrm{n}_{\mathrm{p}}} = \mathrm{p}. \tag{1.14}$$

Combined with (1.8), this yields

$$\gamma_{\mathrm{n}_{\mathrm{p}}} \mathrm{f}_{\mathrm{n}_{\mathrm{p}}}(a_\gamma) = \frac{\gamma_{\mathrm{n}_{\mathrm{p}}}}{\mathrm{p}} > 1. \tag{1.15}$$

There so exists a subsequence $\{\gamma_{\mathrm{n}_{\mathrm{p}}}\}$ such that $\{\gamma_{\mathrm{n}_{\mathrm{p}}} \mathrm{f}_{\gamma_{\mathrm{n}_{\mathrm{p}}}}\}$ fails to converge pointwise to 0. In other words, (b) holds, which is in violent contrast with 1.28 of [3]: X is therefore not metrizable. So ends the proof. □

1.2 Exercise 9. Quotient map

Suppose

1. X *and* Y *are topological vector spaces,*
2. $\Lambda : X \to Y$ *is linear.*
3. N *is a closed subspace of* X*,*
4. $\pi : X \to X/N$ *is the quotient map, and*
5. $\Lambda x = 0$ *for every* $x \in N$*.*

Prove that there is a unique $f : X/N \to Y$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = f \circ \pi$ has necessarily a unique solution, which is the binary relation

$$f \triangleq \{(\pi x, \Lambda x) : x \in X\} \subset X/N \times Y. \tag{1.16}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda x \neq \Lambda x' \Rightarrow \pi x' \neq \pi x'. \tag{1.17}$$

It straightforwardly derives from (1.16) that f inherits linearity from π and Λ.

Remark. The special case $N = \{\Lambda = 0\}$, *i.e.* $\Lambda x = 0$ **iff** $x \in N$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$f(\pi x) = 0 \overset{(1.16)}{\Rightarrow} \Lambda x = 0 \Rightarrow x \in N \Rightarrow \pi x = N \tag{1.18}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_Y there exists a neighborhood U of 0_X such that

$$\Lambda(U) = f(\pi(U)) \subset V. \tag{1.19}$$

Since π is open (1.41 (a) of [3]), $\pi(U)$ is a neighborhood of $N = 0_{X/N}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = f \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{X/N}$ satisfies

$$W = \pi(V) \tag{1.20}$$

for some neighborhood V of 0_X. So,

$$f(W) = f(\pi(V)) = \Lambda(V). \tag{1.21}$$

As a consequence, if Λ is open, then $f(W)$ is a neighborhood of 0_Y. So ends the proof. □

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$*,* $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$*.*

1. *Prove that* Λ *is an open mapping.*
2. *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. Discard the trivial case $\Lambda = 0$ then assume that $\dim \mathrm{Y} = \mathrm{n}$ for some positive n. Let e range over a base of B of Y. Pick W an arbitrary neighborhood of the origin: There so exists V a balanced neighborhood of the origin such that

$$\underbrace{\mathrm{V} + \cdots + \mathrm{V}}_{\text{Put V exactly n time(s)}} \quad \subset \mathrm{W}, \tag{1.22}$$

since addition is continuous. Moreover, for each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, simply because Λ is onto. So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \cdot \Lambda \mathrm{x_e}, \tag{1.23}$$

given any element $\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \cdot \mathrm{e}$ of Y. As a finite set, $\{\mathrm{x_e} : \mathrm{e} \in \mathrm{B}\}$ is bounded: In particular, there exists a positive scalar s such that

$$\forall \mathrm{e} \in \mathrm{B}, \mathrm{x_e} \in \mathrm{s} \cdot \mathrm{V}. \tag{1.24}$$

Combining this with (1.23) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \mathrm{y_e} \cdot \mathrm{s} \cdot \Lambda(\mathrm{V}). \tag{1.25}$$

We now come back to (1.22) and so conclude that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \subset \Lambda(\mathrm{W}) \tag{1.26}$$

whether $|\,\mathrm{y_e}\,| < 1/\mathrm{s}$; which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$\dim \mathrm{X/N} = \mathrm{n}. \tag{1.27}$$

f is then an homeomorphism of $\mathrm{X/N} \equiv \mathbf{C}^{\mathrm{n}}$ onto Y; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$. ∎

1.4 Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms

Put $\mathrm{K} = [0,1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0,1,2,\dots$*) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d}/\mathrm{dx}$*:*

1. $\|\mathrm{D^n f}\|_\infty = \sup\{|\mathrm{D^n f(x)}| : \infty < \mathrm{x} < \infty\}$
2. $\|\mathrm{D^n f}\|_1 = \int_0^1 |\mathrm{D^n f}(x)|\,\mathrm{d}x$
3. $\|\mathrm{D^n f}\|_2 = \left\{\int_0^1 |\mathrm{D^n f}(x)|^2 \mathrm{d}x\right\}^{1/2}$.

Proof. First, remark that

$$\|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty < \infty \tag{1.28}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of $\mathrm{D^n f}$ lies in K; which yields

$$|\,\mathrm{D^n f(x)}\,| = \left|\int_0^{\mathrm{x}} \mathrm{D^{n+1} f}\right| \le \int_0^{\mathrm{x}} \left|\,\mathrm{D^{n+1} f}\,\right| \le \|\mathrm{D^{n+1} f}\|_1. \tag{1.29}$$

So,

$$\|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1. \tag{1.30}$$

We now combine (1.28) with (1.30) and so obtain

$$\|\mathrm{D^n f}\|_1 \le \|\mathrm{D^n f}\|_2 \le \|\mathrm{D^n f}\|_\infty \le \|\mathrm{D^{n+1} f}\|_1 \le \cdots \quad (\mathrm{n} = 0,1,2,\dots). \tag{1.31}$$

Put

$$\mathrm{V_n^{(i)}} \triangleq \{\mathrm{f} \in \mathscr{D}_{\mathrm{K}} : \|\,\mathrm{f}\,\|_{\mathrm{i}} < 2^{\text{-}\mathrm{n}}\} \quad (\mathrm{i} = 1,2,\infty) \tag{1.32}$$

$$\mathscr{B}^{(\mathrm{i})} \triangleq \{\mathrm{V_n^{(i)}} : \mathrm{n} = 0,1,2,\dots\}, \tag{1.33}$$

so that (1.31) is mirrored in terms of neighborhood inclusions, as follows,

$$\mathrm{V_n^{(1)}} \supset \mathrm{V_n^{(2)}} \supset \mathrm{V_n^{(\infty)}} \supset \mathrm{V_{n+1}^{(1)}} \supset \cdots. \tag{1.34}$$

Since $\mathrm{V_n^{(i)}} \supset \mathrm{V_{n+1}^{(i)}}$, $\mathscr{B}^{(\mathrm{i})}$ is a local base of a topology τ_{i}. But the chain (1.34) forces

$$\tau_1 = \tau_2 = \tau_\infty. \tag{1.35}$$

To see that, choose a set S that is τ_1-open at f, *i.e.* $\mathrm{V_n^{(1)}} \subset \mathrm{S} - \mathrm{f}$ for some n. Next, concatenate this with $\mathrm{V_n^{(2)}} \subset \mathrm{V_n^{(1)}}$ (see (1.34)) and so obtain $\mathrm{V_n^{(2)}} \subset \mathrm{S} - \mathrm{f}$; which implies that S is τ_2-open at f. Similarly, we deduce, still from (1.34), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.36}$$

So ends the proof. □

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $\mathrm{p_n}$, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1 Let X be a topological space with a countable local base $\{\mathrm{V_n} : \mathrm{n} = 1, 2, 3, \dots\}$. If $\tilde{\mathrm{V}}_\mathrm{n} = \mathrm{V}_1 \cap \dots \cap \mathrm{V_n}$, then every subsequence $\{\tilde{\mathrm{V}}_{\varrho(\mathrm{n})}\}$ is a decreasing (*i.e.* $\tilde{\mathrm{V}}_{\varrho(\mathrm{n})} \supset \tilde{\mathrm{V}}_{\varrho(\mathrm{n}+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_n} \supset \tilde{\mathrm{V}}_\mathrm{n}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{n}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\varrho(\mathrm{n})}\}$, since $\tilde{\mathrm{V}}_\mathrm{n} \supset \tilde{\mathrm{V}}_{\varrho(\mathrm{n})}$. □

The following special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$ is one of the key ingredients:

Corollary 1 (special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$**)** Under the same notations of Lemma 1, if $\{\mathrm{V_n}\}$ is a decreasing local base, then so is $\{\mathrm{V}_{\varrho(\mathrm{n})}\}$.

Corollary 2 If $\{\mathrm{Q_n}\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{\mathrm{Q}_{\varrho(\mathrm{n})}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $\mathrm{C}(\Omega)$'s (respectively $\mathrm{C}^\infty(\Omega)$'s) topology of the seminorms $\mathrm{p_n}$, as defined in section 1.44 (respectively 1.46), then the seminorms $\mathrm{p}_{\varrho(\mathrm{n})}$ define the same topology τ_Q.

Proof. Let X be $\mathrm{C}(\Omega)$ topologized by the seminorms $\mathrm{p_n}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_n} = \{\mathrm{p_n} < 1/\mathrm{n}\}$, then $\{\mathrm{V_n}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\varrho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\varrho(\mathrm{n})+1} \subset \mathrm{Q}_{\varrho(\mathrm{n})+1} \subset \mathrm{Q}_{\varrho(\mathrm{n}+1)}. \tag{1.37}$$

Thus,

$$\mathrm{Q}_{\varrho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\varrho(\mathrm{n}+1)}. \tag{1.38}$$

In other words, $\mathrm{Q}_{\varrho(\mathrm{n})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\varrho(\mathrm{n})}\}$ then defines a topology $\tau_{\mathrm{Q}_\varrho}$ for which $\{\mathrm{V}_{\varrho(\mathrm{n})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\varrho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V}_{\varrho(\mathrm{n})}\}$ is a local base of τ_Q, which yields $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\varrho}$. □

Lemma 2 If a sequence of compact sets $\{\mathrm{Q_n}\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in allmost all $\mathrm{Q}^\circ_\mathrm{n}$, *i.e.* there exists m such that

$$\mathrm{K} \subset \mathring{\mathrm{Q}}_\mathrm{m} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+1} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+2} \subset \cdots. \tag{1.39}$$

Proof. The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \quad (n = 1, 2, 3, \dots) \tag{1.40}$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset Q_m^\circ$, for some m. Finally,

$$K \subset \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots . \tag{1.41}$$

□

We are now in a fair position to establish the following:

Theorem The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subset L_{m_i}^\circ \subset L_{m_i+1}^\circ \subset \cdots$ for some m_i. In particular, the special case $\varkappa_i = m_i + i$ is

$$K_i \subset \overset{\circ}{L}_{\varkappa_i}. \tag{1.42}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \tag{1.43}$$

Combine (1.42) with (1.43) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\varkappa_1} \subset L_{\varkappa_1} \subset \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subset K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots , \tag{1.44}$$

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L. \tag{1.45}$$

So ends the proof □

[1] See (b) of 2.5 of [2].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [2].

1.6 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^a f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$*, for every multi-index* a.

Proof. In both cases, D^a is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{\varphi \in C^\infty(\Omega) : \max\{|D^\beta \varphi(x)| : |\beta| \le N, x \in K_N\} < 1/N\right\}. \tag{1.46}$$

Now pick g in $V_{N+|a|}$, so that

$$\max\{|D^\gamma g(x)| : |\gamma| \le N + |a|, x \in K_N\} < \frac{1}{N + |a|}. \tag{1.47}$$

(the fact that $K_N \subset K_{N+|a|}$ was tacitely used). The special case $\gamma = \beta + a$ yields

$$\max\{|D^\beta D^a g(x)| : |\beta| \le N, x \in K_N\} < \frac{1}{N}. \tag{1.48}$$

We have just proved that

$$g \in V_{N+|a|} \Rightarrow D^a g \in V_N, \quad \textit{i.e.} \quad D^a\left(V_{N+|a|}\right) \subset V_N, \tag{1.49}$$

which establishes the continuity of $D^a : C^\infty(\Omega) \to C^\infty(\Omega)$.

To prove the continuousness of the restriction $D^a|_{\mathscr{D}_K}$: $\mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|a|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$
\begin{aligned}
D^a|_{\mathscr{D}_K}\left(V_{N+|a|} \cap \mathscr{D}_K\right) &= D^a\left(V_{N+|a|} \cap \mathscr{D}_K\right) && (1.50)\\
&\subset D^a\left(V_{N+|a|}\right) \cap D^a\left(\mathscr{D}_K\right) && (1.51)\\
&\subset V_N \cap \mathscr{D}_K \quad \text{(see (1.49))} && (1.52)
\end{aligned}
$$

So ends the proof. □

Chapter 2

Completeness

2.1 Exercise 3. An equicontinous sequence of measures

Put $\mathrm{K} = [-1,1]$*; define* $\mathscr{D}_{\mathrm{K}}$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^{\mathrm{n}}$*). Supose* $\{\mathrm{f_n}\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\varphi = \lim\limits_{\mathrm{n}\to\infty} \int_{-1}^{1} \mathrm{f_n}(\mathrm{t})\varphi(\mathrm{t})\mathrm{dt}$ *exists for every* $\varphi \in \mathscr{D}_{\mathrm{K}}$*. Show that* Λ *is a continuous linear functional on* $\mathscr{D}_{\mathrm{K}}$*. Show that there is a positive integer* p *and a number* $\mathrm{M} < \infty$ *such that*

$$\left| \int_{-1}^{1} \mathrm{f_n}(\mathrm{t})\varphi(\mathrm{t})\mathrm{dt} \right| \le \mathrm{M} \| \mathrm{D}^{\mathrm{p}} \|_\infty$$

for all n*. For example, if* $\mathrm{f_n}(\mathrm{t}) = \mathrm{n}^3\mathrm{t}$ *on* [-1/n, 1/n] *and* 0 *elsewhere, show that this can be done with* $\mathrm{p} = 1$*. Construct an example where it can be done with* $\mathrm{p} = 2$ *but not with* $\mathrm{p} = 1$*.*

We will also consider the case $\mathrm{p} = 0$. Since all supports of $\varphi, \varphi', \varphi'', \ldots,$ are in K, we make a specialization of the mean value theorem:

Lemma If $\varphi \in \mathscr{D}_{[\mathrm{a,b}]}$, then

$$\| \mathrm{D}^a \varphi \|_\infty \le \| \mathrm{D}^{\mathrm{p}} \varphi \|_\infty \left(\frac{\lambda}{2} \right)^{\mathrm{p}-a} \qquad (a = 0, 1, \ldots, \mathrm{p}) \tag{2.1}$$

at every order $\mathrm{p} = 0, 1, 2, \ldots$; where λ is the length $|\,\mathrm{b} - \mathrm{a}\,|$.

Proof. Let $\mathrm{x_0}$ be in (a, b) . We first consider the case $\mathrm{x_0} \le \mathrm{c} = (\mathrm{a}+\mathrm{b})/2$: The mean value theorem asserts that there exists $\mathrm{x_1}$ ($\mathrm{a} < \mathrm{x_1} < \mathrm{x_0}$), such that

$$\varphi(\mathrm{x_0}) = \varphi(\mathrm{x_0}) - \varphi(\mathrm{a}) = \mathrm{D}\varphi(\mathrm{x_1})(\mathrm{x_0} - \mathrm{a}). \tag{2.2}$$

Since every $\mathrm{D}^{\mathrm{p}}\varphi$ lies in $\mathscr{D}_{[\mathrm{a,b}]}$, a straightforward proof by induction shows that there exists a partition $\mathrm{a} < \cdots < \mathrm{x_p} < \cdots < \mathrm{x_0}$ such that

$$\begin{aligned}
\varphi(\mathrm{x_0}) &= \mathrm{D}^0\varphi(\mathrm{x_0}) && (2.3)\\
&= \mathrm{D}^1\varphi(\mathrm{x_1})(\mathrm{x_0} - \mathrm{a}) && (2.4)\\
&= \cdots \\
&= \mathrm{D}^{\mathrm{p}}\varphi(\mathrm{x_p})(\mathrm{x_0} - \mathrm{a})\cdots(\mathrm{x_{p-1}} - \mathrm{a}), && (2.5)
\end{aligned}$$

for all p. More compactly,

$$D^a\varphi(x_0) = D^p\varphi(x_p)\prod_{k=a}^{p-1}(x_k - a); \tag{2.6}$$

which yields,

$$|D^a\varphi(x)| \le \| D^p\varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-a} \qquad (x \in [a, c]) \tag{2.7}$$

The case $x_0 \ge c$ outputs a “reversed” result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^a\varphi(x)| \le \| D^p\varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-a} \tag{2.8}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\| D^a\varphi \|_\infty \le \| D^p\varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-a}. \tag{2.9}$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\varphi\rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})). \tag{2.10}$$

The following inequalities

$$|\langle u|\varphi\rangle| \le \int_{\mathbf{R}} |u\varphi| \le \| u \|_{L^1} \quad (\| \varphi \|_\infty \le 1) \tag{2.11}$$

imply that every linear functional

$$\begin{aligned} \langle u| : C_0(\mathbf{R}) &\to \mathbf{C} \\ \varphi &\mapsto \langle u|\varphi\rangle \end{aligned} \tag{2.12}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u|$, since u is determined (a.e) by the integrals $\langle u|\varphi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\sup\{|\, \langle u|\varphi\rangle \,| : \| \varphi \|_\infty = 1\} = \| u \|_{L^1}. \tag{2.13}$$

Note that, in the latter equality, $\le \| u \|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$\begin{aligned} g_n : \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & \left(x \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right) \\ 0 & \left(x \notin \left[-\frac{1}{n}, \frac{1}{n}\right]\right). \end{cases} \end{aligned} \tag{2.14}$$

[1] see 1.32, 4.1 of [3]

[2] See 7.26 of [1].

First, remark that $g_n(x) \underset{n\to\infty}{\longrightarrow} 0$ ($x \in \mathbf{R}$), as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$\langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi, \tag{2.15}$$

where φ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\varphi_1^+, \dots, \varphi_j^+, \dots,$ from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\varphi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

(ii) $\varphi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

(iii) $0 \leq \varphi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those φ_j^+. Let $\varphi_1^-, \dots, \varphi_j^-, \dots,$ mirror the φ_j^+, in the sense that $\varphi_j^-(x) = \varphi_j^+(-x)$, so that

(iv) $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$ is odd, as g_n is;

(v) every φ_j is in $C_K^\infty(\mathbf{R})$;

(vi) The sequence $\{\varphi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\| \varphi_j \|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n|\varphi_j\rangle = 2\int_0^1 g_n(t)\varphi_j^+(t)dt \underset{j\to\infty}{\longrightarrow} 2\int_0^1 g_n(t)dt = \| g_n \|_{L^1} = n. \tag{2.16}$$

Finally,

$$\| g_n \|_{L^1} \overset{(2.16)}{\leq} \sup\{| \langle g_n|\varphi\rangle | : \| \varphi \|_\infty = 1\} \overset{(2.13)}{\leq} \| g_n \|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$|\langle g_n|\varphi\rangle| \leq C_n \quad (\| \varphi \|_\infty = 1); \tag{2.18}$$

see (2.11). Furthermore, $\| g_n \|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\varphi_{\varrho(n)}\rangle\}$ such that $\langle g_n|\varphi_{\varrho(n)}\rangle$ is greater than, say, $n - 0.01$, as $\| \varphi_{\varrho(n)} \|_\infty = 1$. Consequently, there is no bound M such that

$$|\langle g_n|\varphi\rangle| \leq M \quad (\| \varphi \|_\infty = 1; n = 1, 2, 3, \dots). \tag{2.19}$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi \quad (\varphi \in C_0(\mathbf{R})) \tag{2.20}$$

[3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that φ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$\Lambda\varphi = \lim_{n\to\infty} \int_K f_n \varphi \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.21}$$

for some linear form Λ. Since φ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$\Lambda\varphi = \lim_{n\to\infty} \langle f_n|\varphi\rangle \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.22}$$

Let $K_1, K_2, \ldots,$ be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \ldots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^a\varphi(x)| : a \leq N, x \in \mathbf{R}\}$, for $\varphi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\varphi \in \mathscr{D}_K : \|\varphi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\varphi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|f_n\|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [3].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi$ for all φ, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$|\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i, \tag{2.23}$$

$$|\Lambda\varphi - \langle f_n|\varphi\rangle| < 1/2 \quad \text{if } \varphi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\varphi\rangle| \leq |\,\Lambda\varphi - \langle f_n|\varphi\rangle\,| + |\,\Lambda\varphi\,| < 1 \qquad \text{if } \varphi \in V_p. \tag{2.25}$$

Now remark that every $\psi = \psi[\mu, \varphi]$, where

$$\psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p \|\varphi\|_p)\varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases} \tag{2.26}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\psi\rangle| < 1 \tag{2.27}$$

$$|\langle f_n|\varphi\rangle| < 2^p\|\,\varphi\,\|_p \cdot \mu \tag{2.28}$$

$$|\,\langle f_n|\varphi\rangle\,| \le 2^p\|\,\varphi\,\|_p \tag{2.29}$$

$$|\,\langle f_n|\varphi\rangle\,| \le 2^p\{\|\,D^0\varphi\,\|_\infty + \cdots + \|\,D^p\varphi\,\|_\infty\}. \tag{2.30}$$

Finally, with the help of (2.1),

$$|\,\langle f_n|\varphi\rangle\,| \le 2^p(p+1)\|\,D^p\varphi\,\|_\infty. \tag{2.31}$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3x$ on $[-1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\varphi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\varphi\rangle| \le M\|\,D^p\varphi\,\|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let φ keep running on $\mathscr{D}_K$ and so define

$$B_n(\varphi) \triangleq \max\{|\,\varphi(x)\,| : x \in [-1/n, 1/n]\}, \tag{2.32}$$

$$\Delta_n(\varphi) \triangleq \max\{|\,\varphi(x) - \varphi(0)\,| : x \in [-1/n, 1/n]\}. \tag{2.33}$$

The mean value asserts that

$$|\,\varphi(1/n) - \varphi(-1/n)\,| \le B_n(\varphi')\,|\,1/n - (-1/n)\,| = \frac{2}{n}B_n(\varphi'). \tag{2.34}$$

Independently, an integration by parts shows that

$$\langle f_n|\varphi\rangle = \left[\frac{n^3t^2}{2}\varphi(t)\right]_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\varphi'(t)dt \tag{2.35}$$

$$= \frac{n}{2}\left(\varphi(1/n) - \varphi(-1/n)\right) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\varphi'(t)dt. \tag{2.36}$$

Combine (2.34) with (2.36) and so obtain

$$|\,\langle f_n|\varphi\rangle\,| \le \frac{n}{2}\,|\,\varphi(1/n) - \varphi(-1/n)\,| + \frac{n^3}{2}\int_{-1/n}^{1/n} t^2|\varphi'(t)|dt \tag{2.37}$$

$$\le B_n(\varphi') + \frac{n^3}{2}B_n(\varphi')\int_{-1/n}^{1/n} t^2dt \tag{2.38}$$

$$\le \frac{4}{3}B_n(\varphi') \tag{2.39}$$

$$\le \frac{4}{3}\|\,\varphi'\,\|_\infty. \tag{2.40}$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\varphi')$ tends to $|\,\varphi'(0)\,|$, we may expect that f_n tends to $\frac{4}{3}\varphi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0) \overset{(2.36)}{=} \frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n}(\varphi' - \varphi'(0))t^2 dt. \tag{2.41}$$

So,

$$\left|\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0)\right| \le \left|\frac{\varphi(1/n) - \varphi(-1/n)}{1/n - (-1/n)} - \varphi'(0)\right| + \frac{1}{3}\Delta_n(\varphi') \underset{n\to\infty}{\longrightarrow} 0. \tag{2.42}$$

We have just proved that

$$\langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\varphi'(0) \quad (\varphi \in \mathscr{D}_K). \tag{2.43}$$

In other words,

$$\langle f_n| \underset{n\to\infty}{\longrightarrow} -\frac{4}{3}\delta', \tag{2.44}$$

where δ is the *Dirac measure* and $\delta', \delta'', \ldots,$ its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $-\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$|\langle f_n|\varphi\rangle| \le \frac{4}{3}\|\,\varphi'\,\|_\infty \quad (n = 1, 2, 3, \ldots) \tag{2.45}$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n|\varphi_{\varrho(n)}\rangle : \|\,\varphi_{\varrho(n)}\,\|_\infty = 1; n = 1, 2, 3, \ldots\} \tag{2.46}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle f_n|\varphi\rangle| \le M\|\,\varphi\,\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.47}$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*[4] ${f_n}'$, as follows

$$\begin{aligned} {f_n}' : \mathscr{D}_K &\to \mathbf{C} \\ \varphi &\mapsto -\big\langle f_n\big|\varphi'\big\rangle\,. \end{aligned} \tag{2.48}$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$\begin{aligned} D : \mathscr{D}_K &\to \mathscr{D}_K \\ \varphi &\mapsto \varphi'; \end{aligned} \tag{2.49}$$

[4] See 6.1 of [3] for a further discussion.

see Exercise 1.17. ${f_n}'$ is therefore continuous. Now apply (2.43) with φ' and so obtain

$$-\langle f_n | \varphi' \rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\varphi''(0) \quad (\varphi \in \mathscr{D}_K),$$

i.e.

$$f_n{}' \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\delta''. \tag{2.50}$$

It follows from (2.40) that,

$$|\langle f_n | \varphi' \rangle| \le \frac{4}{3} \| \varphi'' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.51}$$

It is therefore possible to uniformly bound ${f_n}'$ with respect to a norm $\| D^p \cdot \|_\infty$, namely $\| D^2 \cdot \|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle f_n | \varphi' \rangle| \le M \| \varphi' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.52}$$

Define

$$\Phi_j(x) = \int_{-1}^{x} \varphi_j. \tag{2.53}$$

The oddness of φ_j forces Φ_j to vanish outside [-1, 1]: φ_j is therefore in $\mathscr{D}_K$. So, under our assumption,

$$|\langle f_n | \Phi_j' \rangle| \le M \| \Phi_j' \|_\infty \quad (n = 1, 2, 3, \dots); \tag{2.54}$$

which is

$$|\langle f_n | \varphi_j \rangle| \le M \quad (n = 1, 2, 3, \dots). \tag{2.55}$$

We have thus reached a contradiction (again with the sequence $\{\langle f_n | \varphi_{\varrho(n)} \rangle\}$) and so conclude that there is no constant M such that

$$|\langle | f_n \varphi' \rangle| \le M \| \varphi' \|_\infty \quad (n = 1, 2, 3, \dots). \tag{2.56}$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$|\langle f_n | \varphi' \rangle| \le M \| \varphi \|_\infty. \tag{2.57}$$

The mean value theorem (see (2.1)) asserts that

$$|\langle f_n | \varphi' \rangle| \le M \| \varphi \|_\infty \le M \| \varphi' \|_\infty; \tag{2.58}$$

which is, again, a desired contradiction. So ends the proof.

□

2.2 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{\mathrm{f}}(\mathrm{n})$ *of a function* $\mathrm{f} \in \mathrm{L}^2(\mathrm{T})$ *(*T *is the unit circle) by*

$$\hat{\mathrm{f}}(\mathrm{n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(\mathrm{e}^{\mathrm{i}\vartheta})\mathrm{e}^{-\mathrm{in}\vartheta}\mathrm{d}\vartheta$$

for all $\mathrm{n} \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_{\mathrm{n}}\mathrm{f} = \sum_{\mathrm{k}=-\mathrm{n}}^{\mathrm{n}} \hat{\mathrm{f}}(\mathrm{k}).$$

Prove that $\{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \lim_{\mathrm{n}\infty} \Lambda_{\mathrm{n}}\mathrm{f}$ *exists*$\}$ *is a dense subspace of* $\mathrm{L}^2(\mathrm{T})$ *of the first category.*

Proof. Let $\mathrm{f}(\vartheta)$ stand for $\mathrm{f}(\mathrm{e}^{\mathrm{i}\vartheta})$, so that $\mathrm{L}^2(\mathrm{T})$ is identified with a closed subset of $\mathrm{L}^2([-\pi, \pi])$, hence the inner product

$$\hat{\mathrm{f}}(\mathrm{n}) = (\mathrm{f}, \mathrm{e}_{\mathrm{n}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(\vartheta)\mathrm{e}^{-\mathrm{in}\vartheta}\mathrm{d}\vartheta. \tag{2.59}$$

We believe it is customary to write

$$\Lambda_{\mathrm{n}}(\mathrm{f}) = (\mathrm{f}, \mathrm{e}_{-\mathrm{n}}) + \cdots + (\mathrm{f}, \mathrm{e}_{\mathrm{n}}). \tag{2.60}$$

Moreover, a well known (and easy to prove) result is

$$(\mathrm{e}_{\mathrm{n}}, \mathrm{e}_{\mathrm{n}'}) = [\mathrm{n} = \mathrm{n}'], \textit{i.e.}\ \{\mathrm{e}_{\mathrm{n}} : \mathrm{n} \in \mathbf{Z}\} \text{ is an orthormal subset of } \mathrm{L}^2(\mathrm{T}). \tag{2.61}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $\mathrm{L}^2 \equiv (\mathrm{L}^2)^*$. So,

$$\| \Lambda_{\mathrm{n}} \|^2 \overset{(2.60)}{=} \| \mathrm{e}_{-\mathrm{n}} + \cdots + \mathrm{e}_{\mathrm{n}} \|^2 \overset{(2.61)}{=} \| \mathrm{e}_{-\mathrm{n}} \|^2 + \cdots + \| \mathrm{e}_{\mathrm{n}} \|^2 \overset{(2.61)}{=} 2\mathrm{n} + 1. \tag{2.62}$$

We now assume, to reach a contradiction, that

$$\mathrm{B} \triangleq \{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \sup\{| \Lambda_{\mathrm{n}} \mathrm{f} | : \mathrm{n} = 1, 2, 3, \ldots\} < \infty\} \tag{2.63}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_{\mathrm{n}}\}$ is norm-bounded; which is a desired contradiction, since

$$\| \Lambda_{\mathrm{n}} \| \overset{(2.62)}{=} \sqrt{2\mathrm{n}+1} \underset{\mathrm{n}\to\infty}{\longrightarrow} \infty. \tag{2.64}$$

We have just established that B is actually of the first category; and so is its subset $\mathrm{L}=\{\mathrm{f} \in \mathrm{L}^2(\mathrm{T}) : \lim_{\mathrm{n}\longrightarrow\infty} \Lambda_{\mathrm{n}}\mathrm{f} \text{ exists}\}$. We now prove that L is nevertheless dense in $\mathrm{L}^2(\mathrm{T})$. To do so, we let P be span$\{\mathrm{e}_{\mathrm{k}} : \mathrm{k} \in \mathrm{Z}\}$, the collection of the trignometric polynomials $\mathrm{p}(\vartheta) = \sum \lambda_{\mathrm{k}}\mathrm{e}^{\mathrm{ik}\vartheta}$: Combining (2.60) with (2.61) shows that $\Lambda_{\mathrm{n}}(\mathrm{p}) = \sum \lambda_{\mathrm{k}}$ for almost all n. Thus,

$$\mathrm{P} \subset \mathrm{L} \subset \mathrm{L}^2(\mathrm{T}). \tag{2.65}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $\mathrm{L}^2(\mathrm{T})$. We then conclude, with the help of (2.65), that

$$\mathrm{L}^2(\mathrm{T}) = \overline{\mathrm{P}} = \overline{\mathrm{L}}. \tag{2.66}$$

So ends the proof □

2.3 Exercise 9. Boundedness without closedness

Suppose $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ *are Banach spaces and*

$$\mathrm{B} : \mathrm{X} \times \mathrm{Y} \to \mathrm{Z}$$

is bilinear and continuous. Prove that there exists $\mathrm{M} < \infty$ *such that*

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y} \| \leq \mathrm{M} \| \mathrm{x} \| \| \mathrm{y} \| \quad (\mathrm{x} \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Let (x, y) range over $\mathrm{X} \times \mathrm{Y}$. B is continous at the origin; thus, there exists a positive r such that

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| < 1 \quad (\max\{\| \mathrm{x} \|, \| \mathrm{y} \|\} < \mathrm{r}). \tag{2.67}$$

Given (x, y), we choose two scalars α, β such that $\mathrm{r}\alpha > \| \mathrm{x} \|$ and $\mathrm{r}\beta > \| \mathrm{y} \|$. Thus,

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| = \alpha\beta \| \mathrm{B} \left(\alpha^{-1}\mathrm{x}, \beta^{-1}\mathrm{y}\right) \| \tag{2.68}$$
$$< \alpha\beta. \tag{2.69}$$

We now conclude that

$$\mathrm{B}(\mathrm{x}, \mathrm{y}) \leq \mathrm{r}^{-2} \| \mathrm{x} \| \| \mathrm{y} \|. \tag{2.70}$$

So ends the proof.

As a concrete example, choose $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = \mathrm{C_c}(\mathbf{R})$, topologized by the supremum norm. $\mathrm{C_c}(\mathbf{R})$ is not complete[5], nevertheless the bilinear product

$$\mathrm{B}(\mathrm{f}, \mathrm{g}) = \mathrm{f} \times \mathrm{g} \quad ((\mathrm{f}, \mathrm{g}) \in \mathrm{C_c}(\mathbf{R})^2) \tag{2.71}$$

is bounded, (since $\| \mathrm{B}(\mathrm{f}, \mathrm{g}) \|_\infty = \| \mathrm{f} \|_\infty \| \mathrm{g} \|_\infty$) and continuous. To see that, pick (u, v) in $\mathrm{C_c}(\mathbf{R})^2$: Given any positive scalar ε, there exists another positive scalar r such that $\mathrm{r}(\mathrm{r} + \| \mathrm{u} \| + \| \mathrm{v} \|) < \varepsilon$. So, under the following assumption

$$\max\{\| \mathrm{f} - \mathrm{u} \|_\infty, \| \mathrm{g} - \mathrm{v} \|_\infty\} < \mathrm{r}, \tag{2.72}$$

we reach

$$\| \mathrm{fg} - \mathrm{uv} \|_\infty \leq \| \mathrm{f} - \mathrm{u} \|_\infty \cdot \| \mathrm{g} \|_\infty + \| \mathrm{u} \|_\infty \cdot \| \mathrm{g} - \mathrm{v} \|_\infty \tag{2.73}$$
$$< \mathrm{r}(\mathrm{r} + \| \mathrm{v} \|) + \| \mathrm{u} \| \mathrm{r} \tag{2.74}$$
$$< \mathrm{r}(\mathrm{r} + \| \mathrm{u} \| + \| \mathrm{v} \|) \tag{2.75}$$
$$< \varepsilon; \tag{2.76}$$

which establishes the continuousness of B. □

[5] See 5.4.4 [4]

2.4 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let $(\mathrm{X}_1, \mathrm{X}_2, \mathrm{Z})$ be topological spaces and B a bilinear mapping

$$\mathrm{B} : \mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{Z} \tag{2.77}$$

From now on, $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)$ denotes an arbitrary element of $\mathrm{X}_1 \times \mathrm{X}_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $\mathrm{X}_1 \times \mathrm{X}_2$, *i.e.* given an arbitrary balanced open subset W of Z, there exists in X_i $(\mathrm{i} = 1, 2)$ a balanced open subset U_i such that

$$\mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) \subset \mathrm{W}. \tag{2.78}$$

Let $\nu_\mathrm{i}(\mathrm{x})$ denote any scalar that is greater than $\mu_\mathrm{i}(\mathrm{x}_\mathrm{i}) = \inf\{\mathrm{r} > 0 : \mathrm{x}_\mathrm{i} \in \mathrm{r} \cdot \mathrm{U}_\mathrm{i}\}$. So,

$$\mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) = \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{B}\left(\nu_1(\mathrm{x})^{-1}\mathrm{x}_1, \nu_2(\mathrm{x})^{-1}\mathrm{x}_2\right) \tag{2.79}$$
$$\in \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) \tag{2.80}$$
$$\subset \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{W}. \tag{2.81}$$

Now pick $\mathrm{p} = (\mathrm{p}_1, \mathrm{p}_2)$ in $\mathrm{X}_1 \times \mathrm{X}_2$: It directly follows from (2.81) that

$$\mathrm{B}(\mathrm{p}_1, \mathrm{p}_2) - \mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) = \mathrm{B}(\mathrm{p}_1, \mathrm{p}_2 - \mathrm{x}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{x}_1, \mathrm{x}_2 - \mathrm{p}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{x}_1, \mathrm{p}_2) \tag{2.82}$$
$$\in \nu_1(\mathrm{p})\nu_2(\mathrm{p} - \mathrm{x}) \cdot \mathrm{W} + \nu_1(\mathrm{p} - \mathrm{x})\nu_2(\mathrm{x} - \mathrm{p}) \cdot \mathrm{W} + \nu_1(\mathrm{p} - \mathrm{x})\nu_2(\mathrm{p}) \cdot \mathrm{W}. \tag{2.83}$$

Let us henceforth assume that

$$\mathrm{p}_\mathrm{i} - \mathrm{x}_\mathrm{i} \in [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 2]^{-1} \cdot \mathrm{U}_\mathrm{i}; \tag{2.84}$$

which yields

$$\mu_\mathrm{i}(\mathrm{p}_\mathrm{i} - \mathrm{x}_\mathrm{i}) \leq [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 2]^{-1}. \tag{2.85}$$

Finally, combine the special case

$$\nu_\mathrm{i}(\mathrm{p} - \mathrm{x}) = [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 1]^{-1}, \tag{2.86}$$
$$\nu_\mathrm{i}(\mathrm{p}) = \ \mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 1 \tag{2.87}$$

with (2.83) and so obtain

$$\mathrm{B}(\mathrm{p}_1, \mathrm{p}_2) - \mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{W} + \mathrm{W} + \mathrm{W}. \tag{2.88}$$

W being arbitrary, we have so established the continuousness of B at $(\mathrm{p}_1, \mathrm{p}_2)$. Since $(\mathrm{p}_1, \mathrm{p}_2)$ is also arbitrary, the proof is complete. □

2.5 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\| \mathrm{f} \| = \int_0^1 |\mathrm{f(t)}| \ \mathrm{dt}.$$

Put $\mathrm{B(f,g)} = \int_0^1 \mathrm{f(t)g(t)dt}$*, and show that* B *is a bilinear continuous functional on* $\mathrm{X} \times \mathrm{X}$ *which is separately but not continuous.*

Proof. Let f denote the first variable, g the second one. Remark that

$$|\,\mathrm{B(f,g)}\,| < \|\,\mathrm{f}\,\| \cdot \max_{[0,1]} |\,\mathrm{g}\,|\,; \tag{2.89}$$

which is sufficient (1.18 of [3]) to assert that any $\mathrm{f} \mapsto \mathrm{B(f,g)}$ is continuous. The continuity of all $\mathrm{g} \mapsto \mathrm{B(f,g)}$ follows (Put $\mathrm{C(g,f)} = \mathrm{B(f,g)}$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$|\,\mathrm{B(f,g)}\,| \leq \mathrm{M}\|\,\mathrm{f}\,\|\|\,\mathrm{g}\,\|. \tag{2.90}$$

Put

$$\mathrm{f_n(X)} \triangleq 2\sqrt{\mathrm{n}} \cdot \mathrm{X^n} \in \mathbf{R}[\mathrm{X}] \quad (\mathrm{n} = 1, 2, 3, \dots), \tag{2.91}$$

so that

$$\|\,\mathrm{f_n}\,\| = \frac{2\sqrt{\mathrm{n}}}{\mathrm{n}+1} \xrightarrow[\mathrm{n}\to\infty]{} 0. \tag{2.92}$$

On the other hand,

$$\mathrm{B(f_n, f_n)} = \frac{4\mathrm{n}}{2\mathrm{n}+1} > 1. \tag{2.93}$$

Finally, we combine (2.92) and (2.93) with (2.90) and so obtain

$$1 < \mathrm{B(f_n, f_n)} \leq \mathrm{M}\|\,\mathrm{f_n}\,\|^2 \xrightarrow[\mathrm{n}\to\infty]{} 0. \tag{2.94}$$

Our continuousness assumption is then contradicted. So ends the proof. □

2.6 Exercise 15. Baire cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* $\mathrm{Y} = \mathrm{X}$*.* Hint: Y *must intersect* $\mathrm{x} + \mathrm{Y}$ *for every* $\mathrm{x} \in \mathrm{X}$*.*

Proof. Assume Y is a subgroup of X. Under our assumptions, there exists a sequence $\{\mathrm{E_n} : \mathrm{n} = 1, 2, 3, \dots\}$ of X such that

(i) $(\overline{\mathrm{E}}_\mathrm{n})^\circ = \emptyset$;

(ii) $\mathrm{X} \setminus \mathrm{Y} = \bigcup_{\mathrm{n}=1}^{\infty} \mathrm{E_n}$.

By (i), the complement $\mathrm{V_n}$ of $\overline{\mathrm{E}}_\mathrm{n}$ is a dense open set. Since X is an F-space, it follows from the Baire's theorem that the intersection S of the $\mathrm{V_n}$'s is dense in X: So is $\mathrm{x} + \mathrm{S}$ $(\mathrm{x} \in \mathrm{X})$. To see that, remark that

$$\mathrm{X} = \mathrm{x} + \overline{\mathrm{S}} \subset \overline{\mathrm{x} + \mathrm{S}} \tag{2.95}$$

follows from 1.3 (b) of [3]. Since S and $\mathrm{x} + \mathrm{S}$ are both dense open subsets of X, the Baire's theorem asserts that

$$\overline{(\mathrm{x} + \mathrm{S}) \cap \mathrm{S}} = \mathrm{X}. \tag{2.96}$$

Thus,

$$(\mathrm{x} + \mathrm{S}) \cap \mathrm{S} \neq \emptyset. \tag{2.97}$$

Moreover, it follows from (ii) that $\mathrm{X} \setminus \mathrm{Y} \subset \bigcup_\mathrm{n} \overline{\mathrm{E}}_\mathrm{n}$, *i.e.* $\mathrm{Y} \supset \mathrm{S}$. Combined with (2.97), this shows that $\mathrm{x} + \mathrm{Y}$ cuts Y. Therefore, our arbitrary x is an element of the subgroup Y. We have thus established that $\mathrm{X} \subset \mathrm{Y}$, which achieves the proof. $\square$

2.7 Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $\mathrm{f} : \mathrm{X} \to \mathrm{K}$ *is a closed subset of* $\mathrm{X} \times \mathrm{K}$*. Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

Proof. Choose a sequence $\{\mathrm{x_n} : \mathrm{n} = 1, 2, 3, \dots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(\mathrm{x}_{\varrho(\mathrm{n})}, \mathrm{f}(\mathrm{x}_{\varrho(\mathrm{n})}))\}$ of $\{(\mathrm{x_n}, \mathrm{f}(\mathrm{x_n}))\}$ that converges to some (a, b) of $\mathrm{X} \times \mathrm{K}$. G is closed; therefore, $\{(\mathrm{x}_{\varrho(\mathrm{n})}, \mathrm{f}(\mathrm{x}_{\varrho(\mathrm{n})}))\}$ converges in G. So, $\mathrm{b} = \mathrm{f}(\mathrm{a})$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$
\begin{aligned}
\mathrm{f} : [0, \infty) &\to [0, \infty) \\
\mathrm{x} &\mapsto \begin{cases} 1/\mathrm{x} & (\mathrm{x} > 0) \\ 0 & (\mathrm{x} = 0). \end{cases}
\end{aligned}
\tag{2.98}
$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$
\mathrm{G} = \{(\mathrm{x}, 1/\mathrm{x}) : \mathrm{x} > 0\} \cup \{(0, 0)\}. \tag{2.99}
$$

Next, let $\{(\mathrm{x_n}, 1/\mathrm{x_n})\}$ be a sequence in $\mathrm{G}_+ = \{(\mathrm{x}, 1/\mathrm{x}) : \mathrm{x} > 0\}$ that converges to (a, b). To be more specific: $\mathrm{a} = 0$ contradicts the boundedness of $\{(\mathrm{x_n}, 1/\mathrm{x_n})\}$: a is necessarily positive and $\mathrm{b} = 1/\mathrm{a}$, since $\mathrm{x} \mapsto 1/\mathrm{x}$ is continuous on R_+. This establishes that $(\mathrm{a}, \mathrm{b}) \in \mathrm{G}_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. □

Chapter 3

Convexity

3.1 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subset X$ *an internal point of* A *if* $A - x_0$ *is an absorbing set.*

(a) Suppose A *and* B *are disjoint convex sets in* X*, and* A *has an internal point. Prove that there is a nonconstant linear functional* Λ *such that* $\Lambda(A) \cap \Lambda(B)$ *contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) Show (with $X = \mathbf{R}^2$*, for example) that it may not possible to have* $\Lambda(A)$ *and* $\Lambda(B)$ *disjoint, under the hypotheses of (a).*

Proof. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [3] to establish the existence of a linear functional $\Lambda : X \to \mathbf{R}$ such that

$$\Lambda \leq p \tag{3.1}$$

and

$$\Lambda(b_0 - x_0) = 1. \tag{3.2}$$

Then

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B). \tag{3.3}$$

Hence

$$\Lambda a \leq \Lambda b. \tag{3.4}$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There so exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

$$\Lambda a_i = \Lambda b_i = y_i. \tag{3.5}$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad . \tag{3.6}$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

$$\Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1 \quad ; \tag{3.7}$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\}, \tag{3.8}$$

$$S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}, \tag{3.9}$$

$$A \triangleq S_1 \cup S_2, \tag{3.10}$$

$$B \triangleq X \setminus A. \tag{3.11}$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subset A. \tag{3.12}$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.* A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a linear functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$\Lambda(L) = \{0\} \subset \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad . \tag{3.13}$$

So ends the proof. $\square$

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [2].

3.2 Exercise 11. Meagerness of the polar

Let X *be an infinite-dimensional Fréchet space. Prove that* X^**, with its weak*-*topology, is of the first category in itself.*

This is actually a consequence of the below lemma, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. If X is an infinite dimensional topological vector space whose dual X^* separates points on X, then the polar

$$\mathrm{K_A} \triangleq \{\Lambda \in \mathrm{X}^* : |\,\Lambda\,| \leq 1 \text{ on } \mathrm{A}\} \tag{3.14}$$

of any absorbing subset A is a weak*-closed set that has empty interior.

Proof. Let x range over X. The linear form $\Lambda \mapsto \Lambda\mathrm{x}$ is weak*-continuous; see 3.14 of [3]. Therefore, $\mathrm{P_x} = \{\Lambda \in \mathrm{X}^* : |\,\Lambda\mathrm{x}\,| \leq 1\}$ is weak*- closed: As the intersection of $\{\mathrm{P_a} : \mathrm{a} \in \mathrm{A}\}$, $\mathrm{K_A}$ is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see 3.11 of [3]). Put

$$\mathrm{W} \triangleq \bigcap_{\mathrm{x}\in\mathrm{F}} \{\Lambda \in \mathrm{X}^* : |\,\Lambda\mathrm{x}\,| < \mathrm{r_x}\}, \tag{3.15}$$

where $\mathrm{r_x}$ runs on $\mathbf{R}_+$, as F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$\mathrm{M} \triangleq \mathrm{span}(\mathrm{F}) \tag{3.16}$$

is finite dimensional. Assume, to reach a contradiction, that $\mathrm{A} \subset \mathrm{M}$. So, every x lies in $\mathrm{t_x M} = \mathrm{M}$ for some $\mathrm{t_x} > 0$, since A is absorbing. As a consequence, $\mathrm{X} = \mathrm{M}$ is finite dimensional, which is a desired contradiction. We have just established that $\mathrm{A} \not\subset \mathrm{M}$: Now pick a in $\mathrm{A} \setminus \mathrm{M}$ and so conclude that

$$\mathrm{b} \triangleq \frac{\mathrm{a}}{\mathrm{t_a}} \in \mathrm{A} \tag{3.17}$$

Remark that $\mathrm{b} \notin \mathrm{M}$ (otherwise, $\mathrm{a} = \mathrm{t_a b} \in \mathrm{t_a M} = \mathrm{M}$ would hold) and that M is closed (see 1.21 (b) of [3]): By the Hahn-Banach theorem 3.5 of [3], there exists Λ_a in X^* such that

$$\Lambda_\mathrm{a}\mathrm{b} > 2 \tag{3.18}$$

and

$$\Lambda_\mathrm{a}(\mathrm{M}) = \{0\}. \tag{3.19}$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in \mathrm{K_A}$, the following inequalities

$$|\,\Lambda_\mathrm{a}\mathrm{b} + \Lambda\mathrm{b}\,| \geq 2 - |\,\Lambda\mathrm{b}\,| > 1. \tag{3.20}$$

show that $\Lambda + \Lambda_\mathrm{a}$ is not in $\mathrm{K_A}$. We have thus proved that

$$\Lambda + \mathrm{W} \not\subset \mathrm{K_A}. \tag{3.21}$$

Since W and Λ are both arbitrary, this achieves the proof. □

We now give a proof of the original statement.

Corollary. If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$B_n \triangleq \{x \in X : d(0,x) < 1/n\} \quad (n = 1,2,3,\dots) \tag{3.22}$$

form a local base of X. If Λ is in X^*, then

$$|\,\Lambda\,| \le m \text{ on } B_n \tag{3.23}$$

for some $(n,m) \in \{1,2,3,\dots\}^2$; see 1.18 of [3]. Hence, X^* is the countable union of all

$$m \cdot K_n \quad (m,n = 1,2,3,\dots), \tag{3.24}$$

where K_n is the polar of B_n. Clearly, showing that every $m \cdot K_n$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [3]. As a consequence, the above lemma implies

$$\left(\overline{K_n}\right)^\circ = (K_n)^\circ = \emptyset. \tag{3.25}$$

Since the multiplication by m is an homeomorphism (see 1.7 of [3]), this is equivalent to

$$\left(\overline{m \cdot K_n}\right)^\circ = m \cdot (K_n)^\circ = \emptyset. \tag{3.26}$$

So ends the proof. □

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

4.1 Exercise 1. Basic results

Let φ be the embedding of X *into* X^{**} *decribed in Section 4.5. Let τ be the weak topology of* X*, and let σ be the weak*- topology of* X^{**}*- the one induced by* X^**.*

(a) *Prove that φ is an homeomorphism of* (X, τ) *onto a dense subspace of* (X^{**}, σ)*.*

(b) *If* B *is the closed unit ball of* X*, prove that* $\varphi(B)$ *is σ-dense in the closed unit ball of* X^{**}*. (Use the Hahn-Banach separation theorem.)*

(c) *Use (a), (b), and the Banach-Alaoglu theorem to prove that* X *is reflexive if and only if* B *is weakly compact.*

(d) *Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.*

(e) *If* X *is reflexive and* Y *is a closed subspace of* X*, prove that* X/Y *is reflexive.*

(f) *Prove that* X *is reflexive if and only* X^* *if reflexive.*
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\varphi(X)$ *of* X^{**}*.*

Proof. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$\varphi^{-1} : \varphi(X) \to X \tag{4.1}$$

$$\varphi(x) \mapsto x \tag{4.2}$$

is an homeomorphism (with respect to τ and σ). We first consider

$$V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}|\psi x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0); \tag{4.3}$$

$$U \triangleq \{x \in X : |\langle x|x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0). \tag{4.4}$$

and remark that the so defined V 's (respectively U 's) shape a local subbase $\mathscr{S}_\sigma$ (respectively $\mathscr{S}_\tau$) of σ (respectively τ). We now observe that

$$U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathscr{S}_\sigma,\ U \in \mathscr{S}_\tau) \quad , \tag{4.5}$$

since φ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathscr{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that φ^{-1} and its inverse φ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$\overline{\varphi(X)}_\sigma \subset X^{**} \quad . \tag{4.6}$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\varphi(X)$. By [3.5], there so exists y^* in X^* such that

$$\langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ; \tag{4.7}$$

$$\langle z^{**}, \psi y^* \rangle = 1 \tag{4.8}$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$\varphi(B) \subset B^{**} \quad , \tag{4.9}$$

since φ is isometric. Hence

$$\overline{\varphi(B)}_\sigma \subset \overline{(B^{**})}_\sigma = B^{**} \quad . \tag{4.10}$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\varphi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$|\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_\sigma \quad ; \tag{4.11}$$

$$\langle z^{**}, \psi y^* \rangle > 1 \quad . \tag{4.12}$$

It follows from (4.11) that

$$|\, \psi y^* \,| \leq 1 \text{ on } \varphi(B)\,, \; i.e. |\, y^* \,| \leq 1 \text{ on } B \quad . \tag{4.13}$$

We have so proved that

$$y^* \in B^* \quad . \tag{4.14}$$

Since z^{**} lies in B^{**}, it is now clear that

$$|\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ; \tag{4.15}$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

$$\text{B is weakly compact if and only if } \varphi(B) \text{ is weak*-compact.} \tag{4.16}$$

If B is weakly compact, then $\varphi(B)$ is weak*-closed. So,

$$\varphi(B) = \overline{\varphi(B)}_\sigma \overset{(b)}{=} B^{**} \quad . \tag{4.17}$$

φ is therefore onto, *i.e.* X is reflexive.
Conversely, keep φ as onto: one easily checks that $\varphi(\mathrm{B}) = \mathrm{B}^{**}$. The image $\varphi(\mathrm{B})$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $\mathrm{B} \cap \mathrm{Y}$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(\mathrm{X}/\mathrm{Y})^* \equiv \mathrm{Y}^{\perp} \quad , \tag{4.18}$$

next,

$$(\mathrm{X}/\mathrm{Y})^{**} \equiv (\mathrm{Y}^{\perp})^* \equiv \mathrm{X}^{**}/(\mathrm{Y}^{\perp})^{\perp} \equiv \mathrm{X}/\mathrm{Y} \quad . \tag{4.19}$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z*, the weak***-topology of* Z^* *is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\varphi(\mathrm{X})$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\varphi(\mathrm{X})$ is reflexive, by (d). It now follows from (c) that $\mathrm{B}^{**} \cap \varphi(\mathrm{X})$ is weakly compact, *i.e.* weak*-compact (to see this, apply (L) with $\mathrm{Z} = \mathrm{X}^*$).

By (a), B is therefore weakly compact, *i.e.* X is reflexive; see (c). So ends the proof. $\square$

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