Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Chapter 1

Topological Vector Spaces

1.1 Exercise 7. Metrizability & number theory

Let be X the vector space of all complex functions on the unit interval $[0,1]$*, topologized by the family of seminorms*

$$\mathrm{p_x}(\mathrm{f}) = |\mathrm{f}(\mathrm{x})| \quad (0 \le \mathrm{x} \le 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{\mathrm{f_n}\}$ *in X such that (a)* $\{\mathrm{f_n}\}$ *converges to* 0 *as* $\mathrm{n} \to \infty$*, but (b) if* $\{\gamma_\mathrm{n}\}$ *is any sequence of scalars such that* $\gamma_\mathrm{n} \to \infty$ *then* $\{\gamma_\mathrm{n}\mathrm{f_n}\}$ *does not converge to* 0*. (Use the fact that the collection of all complex sequences converging to* 0 *has the same cardinality as* $[0,1]$*.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.*

Proof. Our justification consists in proving that τ-convergence and pointwise convergence are the same one. To do so, remark first that the family of the seminorms $\mathrm{p_x}$ is separating. By [1.37], the collection $\mathscr{B}$ of all finite intersections of the sets

$$\mathrm{V}^{((\mathrm{x},\mathrm{k})} \triangleq \{\mathrm{p_x} < 2^{\text{-}\mathrm{k}}\} \quad (\mathrm{x} \in [0,1], \mathrm{k} \in \mathbf{N}) \tag{1.1}$$

is then a local base for a topology τ on X. Given $\{\mathrm{f_n} : \mathrm{n} = 1, 2, 3, \dots\}$, we set

$$\texttt{off}(\mathrm{U}) \triangleq \sum_{\mathrm{n}=1}^{\infty} [\mathrm{f_n} \notin \mathrm{U}] \quad (\mathrm{U} \in \tau), \tag{1.2}$$

with the convention $\texttt{off}(\mathrm{U}) = \infty$ whether the sum has no finite support. So,

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \texttt{off}(\mathrm{U}^{(\mathrm{i})}) = \sum_{\mathrm{n}=1}^{\infty}\sum_{\mathrm{i}=1}^{\mathrm{m}} [\mathrm{f_n} \notin \mathrm{U}^{(\mathrm{i})}] \ge \texttt{off}(\mathrm{U}^{(1)} \cap \cdots \cap \mathrm{U}^{(\mathrm{m})}) \tag{1.3}$$

We first assume that $\{\mathrm{f_n}\}$ τ-converges to some f in X, *i.e.*

$$\texttt{off}(\mathrm{f} + \mathrm{V}) < \infty \quad (\mathrm{V} \in \mathscr{B}). \tag{1.4}$$

The special cases $\mathrm{V} = \mathrm{V}^{(\mathrm{x},\mathrm{k})}$ mean the pointwise convergence of $\{\mathrm{f_n}\}$. Conversely, assume that $\{\mathrm{f_n}\}$ does not τ-converges to any g in X, *i.e.*

$$\forall \mathrm{g} \in \mathrm{X}, \exists \mathrm{V}^{(\mathrm{g})} \in \mathscr{B} : \texttt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{g})}) = \infty. \tag{1.5}$$

Given g, $\mathrm{V}^{(\mathrm{g})}$ is then an intersection $\mathrm{V}^{(\mathrm{x}^{(1)},\mathrm{k}^{(1)})} \cap \cdots \cap \mathrm{V}^{(\mathrm{x}^{(\mathrm{m})},\mathrm{k}^{(\mathrm{m})})}$. Thus

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \texttt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{x}^{(\mathrm{i})},\mathrm{k}^{(\mathrm{i})})}) \overset{(1.3)}{\ge} \texttt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{g})}) \overset{(1.5)}{=} \infty. \tag{1.6}$$

One of the sum $\texttt{off}(\mathrm{g} + \mathrm{V}^{(\mathrm{x}^{(\mathrm{i})},\mathrm{k}^{(\mathrm{i})})})$ must then be ∞. This implies that convergence of $\mathrm{f_n}$ to g fails at point $\mathrm{x_i}$. g being arbitrary, we so conclude that $\mathrm{f_n}$ does not converge pointwise. We have just proved that τ-convergence is a rewording of pointwise convergence. We now aim to prove the second part. From now on, k, n and p run on $\mathbf{N}_+$. Let $\texttt{dyadic}(\mathrm{x})$ be the usual dyadic expansion of a real number x, so that $\texttt{dyadic}(\mathrm{x})$ is an aperiodic binary sequence **iff** x is irrational. Define

$$\mathrm{f_n}(\mathrm{x}) \triangleq \begin{cases} 2^{-\sum_{\mathrm{k}=1}^{\mathrm{n}} \texttt{dyadic}(\mathrm{x})_{-\mathrm{k}}} & (\mathrm{x} \in [0,1] \setminus \mathbf{Q}) \\ 0 & (\mathrm{x} \in [0,1] \cap \mathbf{Q}) \end{cases} \tag{1.7}$$

so that $f_n(x) \underset{n\to\infty}{\longrightarrow} 0$, and take scalars γ_n such that $\underset{n\to\infty}{\longrightarrow} \infty$, *i.e.* at fixed p, γ_n is greater than 2^p for almost all n. Next, choose $n^{(p)}$ among those *almost all* n that are large enough to satisfy

$$n^{(p-1)} - n^{(p-2)} < n^{(p)} - n^{(p-1)} \tag{1.8}$$

(start with $n^{(-1)} = n^{(0)} = 0$) and so obtain

$$2^p < \gamma_{n^{(p)}} : 0 < n^{(p)} - n^{(p-1)} \underset{p\to\infty}{\longrightarrow} \infty. \tag{1.9}$$

The indicator χ of $\{n^{(1)}, n^{(2)}, \dots\}$ is then aperiodic, *i.e.*

$$x^{(\gamma)} \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \tag{1.10}$$

is irrational. Consequently,

$$\texttt{dyadic}(x^{(\gamma)})_{-k} = \chi_k. \tag{1.11}$$

We now easily see that

$$\chi_1 + \cdots + \chi_{n^{(p)}} = p, \tag{1.12}$$

which, combined with (1.7), yields

$$f_{n^{(p)}}(x^{(\gamma)}) = 2^{-p}. \tag{1.13}$$

Finally,

$$\gamma_{n^{(p)}} f_{n^{(p)}}(x^{(\gamma)}) > 1. \tag{1.14}$$

We have so established that the subsequence $\{\gamma_{n^{(p)}} f_{n^{(p)}}\}$ does not tend pointwise to 0, hence neither does the whole sequence $\{\gamma_n f_n\}$. In other words, (b) holds, which is in violent contrast with [1.28]: X is then not metrizable. So ends the proof. □

1.2 Exercise 9. Quotient map

Suppose

(*a*) X *and* Y *are topological vector spaces,*

(*b*) $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear.*

(*c*) N *is a closed subspace of* X*,*

(*d*) $\pi : \mathrm{X} \to \mathrm{X}/\mathrm{N}$ *is the quotient map, and*

(*e*) $\Lambda\mathrm{x} = 0$ *for every* $\mathrm{x} \in \mathrm{N}$.

Prove that there is a unique $\mathrm{f} : \mathrm{X}/\mathrm{N} \to \mathrm{Y}$ *which satisfies* $\Lambda = \mathrm{f} \circ \pi$*, that is,* $\Lambda\mathrm{x} = \mathrm{f}(\pi(\mathrm{x}))$ *for all* $\mathrm{x} \in \mathrm{X}$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [2]). Moreover, the equation $\Lambda = \mathrm{f} \circ \pi$ has necessarily a unique solution, which is the binary relation

$$\mathrm{f} \triangleq \{(\pi\mathrm{x}, \Lambda\mathrm{x}) : \mathrm{x} \in \mathrm{X}\} \subset \mathrm{X}/\mathrm{N} \times \mathrm{Y}. \tag{1.15}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda\mathrm{x} \neq \Lambda\mathrm{x}' \Rightarrow \pi\mathrm{x}' \neq \pi\mathrm{x}'. \tag{1.16}$$

It straightforwardly derives from (1.15) that f inherits linearity from π and Λ.

Remark. The special case $\mathrm{N} = \{\Lambda = 0\}$, *i.e.* $\Lambda\mathrm{x} = 0$ **iff** $\mathrm{x} \in \mathrm{N}$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$\mathrm{f}(\pi\mathrm{x}) = 0 \overset{(1.15)}{\Rightarrow} \Lambda\mathrm{x} = 0 \Rightarrow \mathrm{x} \in \mathrm{N} \Rightarrow \pi\mathrm{x} = \mathrm{N} \tag{1.17}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = \mathrm{f} \circ \pi$, by (a) of [1.41]. Conversely, if Λ is continuous, then for each neighborhood V of 0_{Y} there exists a neighborhood U of 0_{X} such that

$$\Lambda(\mathrm{U}) = \mathrm{f}\,(\pi(\mathrm{U})) \subset \mathrm{V}. \tag{1.18}$$

Since π is open (see (a) of [1.41]), $\pi(\mathrm{U})$ is a neighborhood of $\mathrm{N} = 0_{\mathrm{X}/\mathrm{N}}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = \mathrm{f} \circ \pi$, by (a) of [1.41]. To prove the converse, remark that every neighborhood W of $0_{\mathrm{X}/\mathrm{N}}$ satisfies

$$\mathrm{W} = \pi(\mathrm{V}) \tag{1.19}$$

for some neighborhood V of 0_{X}. So,

$$\mathrm{f}(\mathrm{W}) = \mathrm{f}\,(\pi(\mathrm{V})) = \Lambda(\mathrm{V}). \tag{1.20}$$

As a consequence, if Λ is open, then f(W) is a neighborhood of 0_{Y}. So ends the proof. □

1.3 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$, $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$.

(*a*) *Prove that* Λ *is an open mapping.*

(*b*) *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. We discard the trivial case $\dim \mathrm{Y} = 0$ then henceforth assume that $\dim \mathrm{Y}$ has positive dimension n.

Let e range over a base of Y: For each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, since Λ is onto. So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \Lambda \mathrm{x_e} \quad (\mathrm{y} \in \mathrm{Y}). \tag{1.21}$$

The sequence $\{\mathrm{x_e}\}$ is finite hence bounded: Given V a balanced neighborhood of the origin, there exists a positive scalar s such that

$$\mathrm{x_e} \in \mathrm{sV} \tag{1.22}$$

for all $\mathrm{x_e}$. Combining this with (1.21) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \quad (\mathrm{y} \in \mathrm{Y} : |\mathrm{y_e}| < \mathrm{s}^{-1}), \tag{1.23}$$

which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, with $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$\dim \mathrm{X/N} = \mathrm{n}. \tag{1.24}$$

f is then an homeomorphism of $\mathrm{X/N} \equiv \mathbf{C}^{\mathrm{n}}$ onto Y; see 1.21 of [2]. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$. □

1.4 Exercise 14. $\mathscr{D}_K$ equipped with other seminorms

Put $K = [0,1]$ *and define* $\mathscr{D}_K$ *as in Section 1.46. Show that the following three families of seminorms (where* $n = 0, 1, 2, \dots$*) define the same topology on* $\mathscr{D}_K$*. If* $D = d/dx$*:*

(*a*) $\|D^n f\|_\infty = \sup\{|D^n f(x)| : \infty < x < \infty\}$

(*b*) $\|D^n f\|_1 = \int_0^1 |D^n f(x)|\, dx$

(*c*) $\|D^n f\|_2 = \left\{\int_0^1 |D^n f(x)|^2 dx\right\}^{1/2}$.

Proof. First, remark that

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty < \infty \tag{1.25}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, start from

$$D^n f(x) = \int_{-\infty}^{x} D^{n+1} f \tag{1.26}$$

(which is true, since f has a bounded support) to obtain

$$|D^n f(x)| \le \int_{-\infty}^{x} \left|D^{n+1} f\right| \le \|D^{n+1} f\|_1 \tag{1.27}$$

hence

$$\|D^n f\|_\infty \le \|D^{n+1} f\|_1. \tag{1.28}$$

Combining (1.25) with (1.28) yields

$$\|D^0 f\|_1 \le \cdots \le \|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty \le \|D^{n+1} f\|_1 \le \cdots. \tag{1.29}$$

We now define

$$V_n^{(i)} \triangleq \{f \in \mathscr{D}_K : \|f\|_i < 1/n\} \quad (i = 1, 2, \infty) \tag{1.30}$$

$$\mathscr{B}^{(i)} \triangleq \{V_n^{(i)} : n = 1, 2, 3, \dots\} \tag{1.31}$$

so that (1.29) is mirrored in terms of neighborhood inclusions, as follows,

$$V_1^{(1)} \supset \cdots \supset V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \cdots. \tag{1.32}$$

Since $V_n^{(i)} \supset V_{n+1}^{(i)}$, $\mathscr{B}^{(i)}$ is the local base of a topology τ_i. But the chain (1.32) forces the τ_i's to be equals. To see that, choose a set S that is τ_1-open at, say a, *i.e.* $V_n^{(1)} \subset S - a$ for some n. Next, concatenate this with $V_n^{(2)} \subset V_n^{(1)}$ (see (1.32)) and so obtain $V_n^{(2)} \subset S - a$, which implies that S is τ_2-open at a. Similarly, we deduce, still from (1.32), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.33}$$

So ends the proof. □

1.5 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $\mathrm{C}(\Omega)$ *does not depend on the particular choice of* $\{\mathrm{K_n}\}$*, as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $\mathrm{C}^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms $\mathrm{p_n}$, then, eventually, only on the ambient space itself. This should then be regarded as a very part of the textbook [2] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [1]) about intersection of nonempty compact sets.

Lemma 1 Let X be a topological space with a countable local base $\{\mathrm{V_n} : \mathrm{n} = 1, 2, 3, \dots\}$. If $\tilde{\mathrm{V}}_\mathrm{n} = \mathrm{V}_1 \cap \cdots \cap \mathrm{V_n}$, then every subsequence $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$ is a decreasing (*i.e.* $\tilde{\mathrm{V}}_{\rho(\mathrm{n})} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n}+1)}$) local base of X.

Proof. The decreasing property is trivial. Now remark that $\mathrm{V_n} \supset \tilde{\mathrm{V}}_\mathrm{n}$: This shows that $\{\tilde{\mathrm{V}}_\mathrm{n}\}$ is a local base of X. Then so is $\{\tilde{\mathrm{V}}_{\rho(\mathrm{n})}\}$, since $\tilde{\mathrm{V}}_\mathrm{n} \supset \tilde{\mathrm{V}}_{\rho(\mathrm{n})}$. □

The following special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$ is one of the key ingredients:

Corollary 1 (special case $\mathrm{V_n} = \tilde{\mathrm{V}}_\mathrm{n}$) Under the same notations of Lemma 1, if $\{\mathrm{V_n}\}$ is a decreasing local base, then so is $\{\mathrm{V}_{\rho(\mathrm{n})}\}$.

Corollary 2 If $\{\mathrm{Q_n}\}$ is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence $\{\mathrm{Q}_{\rho(\mathrm{n})}\}$ also satisfies theses conditions. Furthermore, if τ_Q is the $\mathrm{C}(\Omega)$'s (respectively $\mathrm{C}^\infty(\Omega)$'s) topology of the seminorms $\mathrm{p_n}$, as defined in section 1.44 (respectively 1.46), then the seminorms $\mathrm{p}_{\rho(\mathrm{n})}$ define the same topology τ_Q.

Proof. Let X be $\mathrm{C}(\Omega)$ topologized with the seminorms $\mathrm{p_n}$ (the case $\mathrm{X} = \mathrm{C}^\infty(\Omega)$ is proved the same way). If $\mathrm{V_n} = \{\mathrm{p_n} < 1/\mathrm{n}\}$, then $\{\mathrm{V_n}\}$ is a decreasing local base of X. Moreover,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n})+1} \subset \mathrm{Q}_{\rho(\mathrm{n}+1)}. \tag{1.34}$$

Thus,

$$\mathrm{Q}_{\rho(\mathrm{n})} \subset \mathring{\mathrm{Q}}_{\rho(\mathrm{n}+1)}. \tag{1.35}$$

In other words, $\mathrm{Q}_{\rho(\mathrm{n})}$ satisfies the conditions specified in section 1.44. $\{\mathrm{p}_{\rho(\mathrm{n})}\}$ then defines a topology τ_{Q_ρ} for which $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base. So, $\tau_{\mathrm{Q}_\rho} \subset \tau_\mathrm{Q}$. Conversely, the above corollary asserts that $\{\mathrm{V}_{\rho(\mathrm{n})}\}$ is a local base of τ_Q, which yields $\tau_\mathrm{Q} \subset \tau_{\mathrm{Q}_\rho}$. □

Lemma 2 If a sequence of compact sets $\{\mathrm{Q_n}\}$ satisfies the conditions specified in section 1.44, then every compact set K lies in allmost all $\mathrm{Q}^\circ_\mathrm{n}$, *i.e.* there exists m such that

$$\mathrm{K} \subset \mathring{\mathrm{Q}}_\mathrm{m} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+1} \subset \mathring{\mathrm{Q}}_{\mathrm{m}+2} \subset \cdots . \tag{1.36}$$

Proof. The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \quad (n = 1, 2, 3, \dots) \tag{1.37}$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset Q_m^\circ$, for some m. Finally,

$$K \subset \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots. \tag{1.38}$$

□

We are now in a fair position to establish the following:

Theorem The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subset L_{m_i}^\circ \subset L_{m_i+1}^\circ \subset \cdots$ for some m_i. In particular, the special case $\varkappa_i = m_i + i$ is

$$K_i \subset \overset{\circ}{L}_{\varkappa_i}. \tag{1.39}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \tag{1.40}$$

Combine (1.39) with (1.40) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\varkappa_1} \subset L_{\varkappa_1} \subset \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subset K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots, \tag{1.41}$$

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L. \tag{1.42}$$

So ends the proof □

[1] See (b) of 2.5 of [1].

[2] The intersection of a decreasing sequence of nomempty Hausdorff compact sets is nonempty. This is a corollary of 2.6 of [1].

1.6 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$*, for every multi-index* α*.*

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{\phi \in C^\infty(\Omega) : \max\{|D^\beta \phi(x)| : |\beta| \le N, x \in K_N\} < 1/N\right\}. \tag{1.43}$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^\gamma g(x)| : |\gamma| \le N + |\alpha|, x \in K_N\} < \frac{1}{N}. \tag{1.44}$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^\beta D^\alpha g(x)| : |\beta| \le N, x \in K_N\} < \frac{1}{N}. \tag{1.45}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^\alpha g \in V_N, \quad \textit{i.e.} \quad D^\alpha\left(V_{N+|\alpha|}\right) \subset V_N. \tag{1.46}$$

The continuity of $D^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$ is so established.

To prove the continuousness of the restriction $D^\alpha|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathscr{D}_K$, we first remark the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+\alpha} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$\begin{aligned} D^\alpha|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) &= D^\alpha\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) && (1.47)\\ &\subset D^\alpha\left(V_{N+|\alpha|}\right) \cap D^\alpha(\mathscr{D}_K) && (1.48)\\ &\subset V_N \cap \mathscr{D}_K \quad \text{(see 1.46)} && (1.49) \end{aligned}$$

So ends the proof. □

Chapter 2

Completeness

2.1 Exercise 3. An equicontinous sequence of measures

Put $K = [-1, 1]$*; define* $\mathscr{D}_K$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^n$*). Supose* $\{f_n\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\phi = \lim_{n\infty} \int_{-1}^{1} f_n\phi$ *exists for every* $\phi \in \mathscr{D}_K$*. Show that* Λ *is a continuous linear functional on* $\mathscr{D}_K$*. Show that there is a positive integer* p *and a number* $M < \infty$ *such that*

$$\left| \int_{-1}^{1} f_n(t)\phi(t)dt \right| \leq M\|D^p\phi\|_\infty$$

for all n*. For example, if* $f_n(t) = n^3 t$ *on* $[-1/n, 1/n]$ *and* 0 *elsewhere, show that this can be done with* $p = 1$*. Construct an example where it can be done with* $p = 2$ *but not with* $p = 1$*.*

We are going to extend the scope of our investigations to the case $p = 0$. The following version of the mean value theorem will be of a great deal of help.

Lemma If $\phi \in \mathscr{D}_{[a,b]}$, then

$$\|D^\alpha\phi\|_\infty \leq \|D^p\phi\|_\infty\, |b-a|^{p-\alpha} \quad (\alpha = 0, 1, \ldots, p) \tag{2.1}$$

at every order $p = 1, 2, 3, \ldots$.

Proof. Choose $x = c_0$ in (a, b) and remark that $\phi(x) = \phi(x) - \phi(a)$. The mean value theorem then asserts that there exists c_1 $(c_0 > c_1)$, such that $\phi(x) = D\phi(c_1)(c_0 - a)$. Since $D\phi, D^2\phi, \ldots,$ are in $\mathscr{D}_{[a,b]}$, the same reasoning aplies to them. It then exists $c_0 > c_1 \cdots > c_p > a$ such that

$$\begin{aligned} \phi(x) &= D^1\phi(c_1)(c_0 - a) \qquad (2.2)\\ &= \cdots \\ &= D^p\phi(c_p)(c_0 - a)(c_1 - a)\cdots(c_{p-1} - a). \qquad (2.3) \end{aligned}$$

More concisely,

$$D^\alpha\phi(x) = D^p\phi(c_p)(c_\alpha - a)\cdots(c_{p-1} - a). \tag{2.4}$$

Finally,

$$\|D^\alpha\phi\|_\infty \leq \|D^p\phi\|_\infty |b-a|^{p-\alpha} \tag{2.5}$$

□

Proof. Let $K_1, K_2, \dots,$ be compact sets that satisfy the conditions specified in 1.44 of [2]. $\mathscr{D}_K$ has a topology τ_K, and the collection of the below defined V_N

$$p_N(\phi) \triangleq \max\{|D^\alpha \phi(x)| : \alpha \leq N, x \in K_N\} \tag{2.6}$$
$$V_N \triangleq \{p_N < 2^{-N}\} \tag{2.7}$$

is a (decreasing) local base of τ_K. Moreover, $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

$$V_N = \{p_N < 2^{-N}\} \tag{2.8}$$
$$= \{\max\{\|D^\alpha \phi\|_\infty : \alpha \leq N\} < 2^{-N}\}. \tag{2.9}$$

As a first consequence,

$$|\langle f_n|\phi\rangle| \leq \int_{\mathbf{R}} |f_n(t)\phi(t)dt| < |f_n|_{L^1} \tag{2.10}$$

holds whether ϕ keeps in V_m. Every functional $\langle f_n|$ is then bounded on a neighborhood of 0, *i.e.* continuous, see 1.18 of [2]. To sum it up:

(*a*) $\mathscr{D}_K$ is a Fréchet space;

(*b*) Every linear functional $\langle f_n|$ is continuous;

(*c*) $\langle f_n|\phi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\phi$ for all ϕ, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [2], we conclude that Λ is continuous and that the $\langle f_n|$'s collection is equicontinuous. So is the $\Lambda - \langle f_n|$'s collection, since addition is continuous. There so exist i, j such that

$$|\langle f_n|\phi\rangle| < 1/2 \quad (\text{if } \phi \in V_i) \tag{2.11}$$
$$|\langle \phi|\Lambda\rangle - \langle f_n|\phi\rangle| < 1/2 \quad (\text{if } \phi \in V_j). \tag{2.12}$$

Choose $p = \max(i, j)$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\phi\rangle| \leq |\langle \phi|\Lambda\rangle - \langle f_n|\phi\rangle| + |\Lambda\phi| < 1 \qquad (\text{if } \phi \in V_p). \tag{2.13}$$

A relevant criterion for $\phi \in V_p$ is

$$2^{m+p+1}\|D^p\phi\|_\infty < 1, \tag{2.14}$$

which is established by putting forth the next following inequalities:

$$\|\phi\|_p \leq \|\phi\|_\infty + \|D\phi\|_\infty + \cdots + \|D^p\phi\|_\infty \tag{2.15}$$
$$\leq (2^p + \cdots + 2^0)\|D^p\phi\|_\infty \quad (\text{see } 2.1) \tag{2.16}$$
$$< 2^{p+1}\|D^p\phi\|_\infty \tag{2.17}$$
$$< 2^{-m} \quad (\text{see } 2.14). \tag{2.18}$$
$$\tag{2.19}$$

As a consequence, all

$$\phi^{(\mu)} \triangleq \begin{cases} (\mu \cdot 2^{m+p+1}\|D^p\phi\|_\infty)^{-1}\phi & (\phi \neq 0, \mu > 1) \\ 0 & (\phi = 0, \mu > 1) \end{cases} \tag{2.20}$$

keep in V_m. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\phi^{(\mu)}\rangle| < 1 \tag{2.21}$$
$$|\langle f_n|\phi\rangle| < 2^{m+p+1}\|D^m\phi\|_\infty \cdot \mu \tag{2.22}$$
$$|\langle f_n|\phi\rangle| \leq 2^{m+p+1}\,\|D^m\phi\|_\infty \tag{2.23}$$

The first part is so proved.

Now set

$$f_n(x) = n^3 \cdot [x \in [\text{-}1/n, 1/n]]. \tag{2.24}$$

We will prove that the $\langle f_n|$'s collection is not uniformly bounded with respect to the norm $\|\cdot\|_\infty$. To do so, we first assume (to reach a contradiction) that there exists a positive scalar C such that

$$|\langle f_n|\phi\rangle| < C\|\phi\|_\infty \quad (n = 1, 2, 3, \ldots). \tag{2.25}$$

Let $\phi^{[1]}, \ldots, \phi^{[j]}, \ldots$, be such that

$$0 \leq \phi^{[j]}(x) \leq \cdots \leq \phi^{[j]}(x) \underset{j\to\infty}{\longrightarrow} 1_{[0,1]}(x) \quad (x \in \mathbf{R}), \tag{2.26}$$

see [1.46] of [2] for a possible construction of those $\phi^{[j]}$. Under our boundedess assumption, all $\langle f_n|\phi^{[j]}\rangle$ are then smaller than C. But the Lebesgue's dominated convergence theorem yields

$$\langle f_n|\phi^{[j]}\rangle \underset{j\to\infty}{\longrightarrow} n^3 \int_0^{1/n} t\,dt = \frac{n}{2} \underset{n\to\infty}{\longrightarrow} \infty. \tag{2.27}$$

There so exists some $\langle f_n|\phi^{[j]}\rangle$ that are greater than C: We have thus reached a contradiction and so conclude that the linear functional $\langle f_n|$ are not uniformly bounded with respect to the norm $\|\cdot\|_\infty$.

We now prove that they are uniformly bounded with respect to the norm $\|D\cdot\|_\infty$. To do so, we define $M^{(n)}$ and $\Delta^{(n)}$ as, respectively, the maximum of $|D\phi|$ on $[\text{-}1/n, 1/n]$ and the maximum of $|D\phi - D\phi(0)|$ on $[\text{-}1/n, 1/n]$. The mean value asserts that

$$|\phi(1/n) - \phi(\text{-}1/n)| \leq M^{(n)}\,|1/n - (\text{-}1/n)| = \frac{2}{n}M^{(n)} \tag{2.28}$$

Combine the latter result with an integration by parts and so obtain

$$|\langle f_n|\phi\rangle| = \left[\frac{n^3t^2}{2}\phi(t)\right]_{\text{-}1/n}^{1/n} - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n} t^2 D\phi(t)dt \tag{2.29}$$
$$\leq M^{(n)} + \frac{n^3}{2}M^{(n)}\int_{\text{-}1/n}^{1/n} t^2 dt \tag{2.30}$$
$$\leq \frac{4}{3}M^{(n)} \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\,|D\phi(0)| \tag{2.31}$$

A straightforward computation shows that

$$\langle f_n|\phi\rangle - \frac{4}{3}D\phi(0) = \frac{\phi(1/n) - \phi(\text{-}1/n)}{1/n - (\text{-}1/n)} - D\phi(0) - \frac{n^3}{2}\int_{\text{-}1/n}^{1/n}(D\phi - D\phi(0)t^2 dt. \tag{2.32}$$

So,

$$\left|\langle f_n|\phi\rangle - \frac{4}{3}D\phi(0)\right| \leq \left|\frac{\phi(1/n)-\phi(\text{-}1/n)}{1/n-(\text{-}1/n)} - D\phi(0)\right| + \frac{1}{3}\Delta^{(n)} \underset{n\to\infty}{\longrightarrow} 0 \tag{2.33}$$

We have just proved that

$$\Lambda\phi = \frac{4}{3}D\phi(0) \quad (\phi \in \mathscr{D}_K) \quad . \tag{2.34}$$

In other words,

$$\Lambda = \text{-}\frac{4}{3}\delta' \quad , \tag{2.35}$$

where δ is the *Dirac measure*; see [6.9] of of [2]. Now set

$$T_n \triangleq \langle f_n| \circ D \in \mathscr{L}(\mathscr{D}_K, \mathbf{C}) \quad (n = 1, 2, 3, \dots) \quad . \tag{2.36}$$

It has been proved that every $\langle f_n|$ is continuous. So is D; see Exercise 1.17. Then

$$T_n \in \mathscr{D}_K^* \quad . \tag{2.37}$$

Consequently,

$$T_n\phi \underset{n\infty}{\longrightarrow} \frac{4}{3}D^2\phi(0) \quad (\phi \in \mathscr{D}_K)\,,\ \textit{i.e.} \tag{2.38}$$

$$T_n \underset{n\infty}{\longrightarrow} \frac{4}{3}\delta'' \quad ;\ \text{see [6.1]}\,. \tag{2.39}$$

It follows from (2.34) that, for some positive constant C,

$$|T_n\phi\,| \leq C\,\|D^2\phi\,\|_\infty \quad (n = 1, 2, 3, \dots) \quad . \tag{2.40}$$

It is therefore possible to uniformly bound T_n with respect to a norm $\|D^p\|_\infty$, namely $\|D^2\|$. Then arises a question: is 2 the smallest p? The answer is: yes. To show this, we fetch

$$\begin{aligned} \psi_j : \mathbf{R} &\to \mathbf{R} \qquad\qquad (j = 1, 2, 3, \dots) \\ x &\mapsto \int_{\text{-}\infty}^{x} (\phi_j - \check{\phi}_j)(t) \quad , \end{aligned} \tag{2.41}$$

as $\check{\phi}(t) \triangleq \phi(\text{-}t)$. Next, assume that

$$\exists B \in \mathbf{R}_+ :\ |D^2\psi_j(0)| \leq B\,\|D\psi_j\,\|_\infty \quad . \tag{2.42}$$

Then, similarly to (**??**),

$$B \underset{j\infty}{\sim} B\,\|\phi_j - \check{\phi}_j\|_\infty\,|D(\phi_j - \check{\phi}_j)(0)| \underset{j\infty}{\longrightarrow} \infty \tag{2.43}$$

Our assumption (2.42) is therefore contradicted: it is not possible to uniformly bound T_n with respect to the norm $\|D\,\|_\infty$ (not with $\|\,\|_\infty$ either, by the mean value theorem). □

Bibliography

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