Solutions to some exercises from Walter Rudin's *Functional Analysis*

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Contents

Notations and Conventions

0.1 Logic

1. **Halmos' iff. iff** is a short for "if and only if".

2. **Definitions (of values) with $\triangleq$.** Given variables a and b, $\mathrm{a} \triangleq \mathrm{b}$ means that a is defined as equal to b.

3. **$\equiv$.** $\mathrm{a} \equiv \mathrm{b}$ means that there exists a "natural" bijection $\rightarrow$ that maps a to b; which let us identify a with b. In a metric space context, $\mathrm{a} \equiv \mathrm{b}$ means that $\rightarrow$ is isometric.

4. **Definitions (formulæ).** Definitions use the **iff** format. In other words, every definition has a "only if".

5. **Iverson notation.** Given a boolean expression Φ, $[\,\Phi\,]$ returns the truth value of Φ, encoded as follows,

$$[\,\Phi\,] \triangleq \begin{cases} 0 & \text{if } \Phi \text{ is false;} \\ 1 & \text{if } \Phi \text{ is true.} \end{cases}$$

   For example, $[\,1 > 0\,] = 1$ but $\left[\,\sqrt{2} \in \mathbf{Q}\,\right] = 0$.

0.2 Topological vector spaces

1. **Product space**

2. **Scalar field.** The usual (complete) scalar field is $\mathbf{C}$. A property, *e.g.* linearity, that is true on $\mathbf{C}$ is also true on $\mathbf{R}$. The complex case is then a *special case* of the real one. Sometimes, this specialization is not purely formal. For example, theorem 12.7 of [3] asserts that, in a Hilbert space H equipped with the inner product $\langle \cdot \,|\, \cdot \rangle$, every nonzero linear continuous operator T "breaks orthogonality", in the sense that there always exists $\mathrm{x} = \mathrm{x}(\mathrm{T})$ in H that satisfies $\langle \mathrm{Tx} | \mathrm{x} \rangle \neq 0$. The proof of this theorem strongly depends on the complex field. Actually, a real counterpart does not exists. To see that, consider the 90° rotations of the euclidian plane. Nevertheless, *unless the contrary is explicitely mentioned*, the exension to the real case will always be obvious. So, taking $\mathbf{C}$ as the scalar field shall mean "*Instead of letting the scalar field undefined, we choose* $\mathbf{C}$ *for the sake of expessivity. But considering* $\mathbf{R}$ *instead of* $\mathbf{C}$ *would actually make no difference here*".

3. **Finite dimensional spaces**. Let Y be a finite dimensional space. If $\dim \mathrm{Y} = 0$, *i.e.* Y is a group of order 1, then $\{\emptyset, \mathrm{Y}\}$ is the only possible topology for Y. For instance, in a quotient space X/N, the zero is N and $\{\mathrm{N}\}$ is zero-dimensional in X/N.

Assume henceforth that $\dim \mathrm{Y} > 0$, *i.e.* Y has a base $\mathrm{F_n} = \{\mathrm{f_i} : \mathrm{i} = 1, \ldots, \mathrm{n}\}$ for some positive n. The cartesian power $\mathbf{C}^{\mathrm{n}} = \prod_{\mathrm{j}=1}^{\mathrm{n}} \mathbf{C}$ is the vector space of all lists $(\mathrm{z}_1, \ldots, \mathrm{z}_\mathrm{n})$, where $\mathrm{z_j} \in \mathbf{C}$ (identify $\mathbf{C}^1$ with $\mathbf{C}$). The subset $\mathrm{E_n} = \{\mathrm{e_j} : \mathrm{j} = 1, \ldots, \mathrm{n}\}$ is the *standard base* of $\mathbf{C}^{\mathrm{n}}$, *i.e.* $\mathrm{e_j} = 1_{\{\mathrm{j}\}}$. So, $\dim \mathbf{C}^{\mathrm{n}} = \mathrm{n}$. Let $\mathrm{u} : \mathbf{C}^{\mathrm{n}} \to \mathrm{Y}$ be the only linear mapping that verifies all $\mathrm{u}(\mathrm{e_j}) = \mathrm{f_j}$. Since u is encoded as the identity matrix, both u and $\mathrm{v} = \mathrm{u}^{-1}$ exist as isomorphisms. Additionally, $\mathbf{C}^{\mathrm{n}}$ can be equipped with various norms, *e.g.* the *p-norms* $\|\ \|_\mathrm{p}$ (where $\|(\mathrm{z}_1, \ldots, \mathrm{z}_\mathrm{n})\|_\mathrm{p}^\mathrm{p} = |\mathrm{z}_1|^\mathrm{p} + \cdots + |\mathrm{z}_\mathrm{n}|^\mathrm{p} ; \mathrm{p} \geq 1$) or $\|\ \|_\infty$ (where $\|(\mathrm{z}_1, \ldots, \mathrm{z}_\mathrm{n})\|_\infty = \max |\mathrm{z_j}|$). Note that Y inherits any norm $\|\ \|$ of $\mathbf{C}^{\mathrm{n}}$, with $\|\mathrm{u}(\mathrm{z}_1, \cdots, \mathrm{z}_\mathrm{n})\| = \|(\mathrm{z}_1, \cdots, \mathrm{z}_\mathrm{n})\|$; which turns u into a isometry of $\mathbf{C}^{\mathrm{n}}$ onto Y. Let $\tau_{\|\ \|}$ denote the topology of a norm $\|\ \|$. We now go back to the proof of 1.21 of [3] and so equip Y with a its own norm $\|\ \|_2$; which turns u into a isometric isomorphim of $\mathbf{C}^{\mathrm{n}}$ onto Y. Y can now be seen as a topological vector space, in at least one fashion; namely, the space $(\mathrm{Y}, \tau_{\|\ \|_2})$. Let $\tau = \tau_\mathrm{Y}$ stand for any arbitrary topology of Y. Hence the following commutative diagram

$$\begin{tikzcd}
 & (\mathbf{C}^{\mathrm{n}}, \tau_{\|\ \|_2}) \arrow[dl, "\mathrm{u}"'] & \\
(\mathrm{Y}, \tau_{\|\ \|_2}) \arrow[dr, "\mathrm{v}"'] & & (\mathrm{Y}, \tau) \arrow[ul, "\mathrm{v}"'] \\
 & (\mathbf{C}^{\mathrm{n}}, \tau_{\|\ \|_2}) \arrow[ur, "\mathrm{u}"'] &
\end{tikzcd} \tag{1}$$

It is now clear that the *identity mapping* $\mathrm{u} \circ \mathrm{v}$ is an homeomorphism of Y onto Y, which implies that $\tau = \tau_{\|\ \|_2}$. In other words, there is only one topology τ for Y, as a topological vector space. This topology is normable, since $\tau = \tau_{\|\ \|_2}$. Let $\|\ \|_\mathrm{Y}$ stand for any norm of Y. The special case $\mathrm{Y} = \mathbf{C}^{\mathrm{n}}, \mathrm{F_n} = \mathrm{E_n}, \mathrm{u} = \mathrm{i}$ is of considerable interest. TOTO. Now take $\tilde{\mathrm{Y}}$ of dimension n then similarly define (obvious notations) $\tilde{\mathrm{u}}$, $\tilde{\mathrm{v}}$ and $\tilde{\tau}$.

$$\begin{tikzcd}
 & (\mathbf{C}^{\mathrm{n}}, \tau_{\mathbf{C}^{\mathrm{n}}}) \arrow[dl, "\mathrm{u}"'] & \\
(\mathrm{Y}, \tau) \arrow[dr, "\mathrm{v}"'] & & (\tilde{\mathrm{Y}}, \tilde{\tau}) \arrow[ul, "\tilde{\mathrm{v}}"'] \\
 & (\mathbf{C}^{\mathrm{n}}, \tau_{\mathbf{C}^{\mathrm{n}}}) \arrow[ur, "\tilde{\mathrm{u}}"'] &
\end{tikzcd} \tag{2}$$

The homeomorphism between Y and $\tilde{\mathrm{Y}}$ leads to *the equivalence of norms at fixed dimension* n, as follows $\mathrm{A}\|\mathrm{y}\|_\mathrm{Y} \leq \|\tilde{\mathrm{u}} \circ \mathrm{v}(\mathrm{y})\|_{\tilde{\mathrm{Y}}} \leq \mathrm{B}\|\mathrm{y}\|_\mathrm{Y}$ $(\mathrm{y} \in \mathrm{Y})$ for some positive

A, B. Equip Y and $\tilde{\mathrm{Y}}$ with the inherited norm $\| \ \|$. Y and $\tilde{\mathrm{Y}}$ are homeomorphically isomorphic ($\equiv$) to $\mathbf{C}^{\mathrm{n}}, \| \ \|$.

$$\begin{tikzcd}
& (\mathbf{C}^{\mathrm{n}}, \| \ \|) \arrow[dl, "\equiv"'] & \\
(\mathrm{Y}, \| \ \|) \arrow[dr, "\equiv"] & & (\tilde{\mathrm{Y}}, \| \ \|) \arrow[ul, "\equiv"'] \\
& (\mathbf{C}^{\mathrm{n}}, \| \ \|) \arrow[ur, "\equiv"'] &
\end{tikzcd}$$

(3)

From now the default norm will be $\| \ \|_\infty$.

Chapter 1

Topological Vector Spaces

1.1 Exercise 1. Basic results

Suppose X *is a vector space. All sets mentioned below are understood to be subsets of* X*. Prove the following statements from the axioms as given as in section 1.4.*

(a) If $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ *there is a unique* $\mathrm{z} \in \mathrm{X}$ *such that* $\mathrm{x} + \mathrm{z} = \mathrm{y}$.

(b) $0 \cdot \mathrm{x} = 0 = \alpha \cdot 0 \quad (\alpha \in \mathbf{C}, \mathrm{x} \in \mathrm{X})$.

(c) $2\mathrm{A} \subset \mathrm{A} + \mathrm{A}$.

(d) A *is convex if and only if* $(\mathrm{s} + \mathrm{t})\mathrm{A} = \mathrm{sA} + \mathrm{tA}$ *for all positive scalars* s *and* t.

(e) Every union (and intersection) of balanced sets is balanced.

(f) Every intersection of convex sets is convex.

(g) If Γ *is a collection of convex sets that is totally ordered by set inclusion, then the union of all members of* Γ *is convex.*

(h) If A *and* B *are convex, so is* $\mathrm{A} + \mathrm{B}$.

(i) If A *and* B *are balanced, so is* $\mathrm{A} + \mathrm{B}$.

(j) Show that parts (f), (g) and (h) hold with subspaces in place of convex sets.

Proof. 1. Such property only depends on the group structure of X: Each x in X has an opposite -x. Let x' be any opposite of x, so that $\mathrm{x} - \mathrm{x} = 0 = \mathrm{x} + \mathrm{x}'$. Thus, $\text{-x} + \mathrm{x} - \mathrm{x} = \text{-x} + \mathrm{x} + \mathrm{x}'$, which is equivalent to $\text{-x} = \mathrm{x}'$. So is established the uniqueness of -x. It is now clear that $\mathrm{x} + \mathrm{z} = \mathrm{y}$ **iff** $\mathrm{z} = \text{-x} + \mathrm{y}$, which asserts both the existence and the uniqueness of z.

2. Remark that

$$0 \cdot \mathrm{x} = (0 + 0) \cdot \mathrm{x} = 0 \cdot \mathrm{x} + 0 \cdot \mathrm{x} \tag{1.1}$$
$$= (0 + 0) \cdot \mathrm{x} = 0 + 0 \cdot \mathrm{x} \tag{1.2}$$

then conclude from (a) that $0 \cdot \mathrm{x} = 0$. So,

$$0 = 0 \cdot \mathrm{x} = (1 - 1) \cdot \mathrm{x} = \mathrm{x} + (\text{-}1) \cdot \mathrm{x} \Rightarrow \text{-}1 \cdot \mathrm{x} = \text{-x}. \tag{1.3}$$

Finally,

$$\alpha \cdot 0 \overset{(1.3)}{=} \alpha \cdot (x + (\text{-}1 \cdot x)) = \alpha \cdot x + \alpha \cdot (\text{-}1) \cdot x = (\alpha - \alpha) \cdot x = 0 \cdot x = 0, \tag{1.4}$$

which proves (b).

3. Remark that

$$2x = (1 + 1)x = x + x \tag{1.5}$$

for every x in X, and so conclude that

$$2A = \{2x : x \in A\} = \{x + x : x \in A\} \subset \{x + y : (x, y) \in A^2\} = A + A \tag{1.6}$$

for all subsets A of X; which proves (c).

4. If A is convex, then

$$A \subset \frac{s}{s+t}A + \frac{t}{s+t}A \subset A; \tag{1.7}$$

which is

$$sA + tA = (s + t)A. \tag{1.8}$$

Conversely, the special case $s + t = 1$ is

$$sA + (1 - s)A = A. \tag{1.9}$$

The latter extends to $s = 0$, since

$$0A + A \overset{(b)}{=} \{0\} + A = A. \tag{1.10}$$

The extension to $s = 1$ is analogously established (or simply use the fact that + is commutative!). So ends the proof.

5. Let A range over B a collection of balanced subsets, so that

$$\alpha \bigcap B \subset \alpha A \subset A \subset \bigcup B \tag{1.11}$$

for all scalars α of magnitude ≤ 1. The inclusion $\alpha \bigcap B \subset A$ establishes the first part. Now remark that

$$\alpha A \subset \bigcup B \tag{1.12}$$

implies

$$\alpha \bigcup B \subset \bigcup B; \tag{1.13}$$

which achieves the proof.

6. Let A range over C a collection of convex subsets, so that

$$(s + t) \bigcap C \subset s \bigcap C + t \bigcap C \subset sA + tA \overset{(d)}{=} (s + t)A \tag{1.14}$$

for all positives scalars s, t. Thus,

$$(s + t) \bigcap C \subset s \bigcap C + t \bigcap C \subset (s + t) \bigcap C. \tag{1.15}$$

We now conclude from (d) that the intersection of C is convex. So ends the proof.

7. Pick x_1, x_2 in $\bigcup \Gamma$, so that each x_i $(i = 1, 2)$ lies in some $C_i \in \Gamma$. Since Γ is totally ordered by set inclusion, we henceforth assume without loss of generality that C_1 is a subset of C_2. So, x_1, x_2 are now elements of the convex set C_2. Every convex combination of our x_1's is then in $C_2 \subset \bigcup \Gamma$, hence (g).

8. Simply remark that

$$s(A + B) + t(A + B) = sA + tA + sB + tB = (s + t)(A + B) \tag{1.16}$$

for all positive scalars *s* and *t*, then conclude from (d) that $A + B$ is convex.

9. Given any α from the closed unit disc,

$$\alpha(A + B) = \alpha A + \alpha B \subset A + B. \tag{1.17}$$

There is no more to prove.

10. Our proof will be based on the following lemma,

If $S \subset X$*, then any assertion*

(i) S *is a vector subspace of* X*;*

(ii) S *is convex balanced such that* $S + S = S$*;*

(iii) S *is convex balanced such that* $\lambda S = S \quad (\lambda > 0)$

implies the other ones.

To prove the lemma, let *S* range over the subsets of X. First, assume that (i) holds: Clearly, every S is convex balanced. Moreover, $S + S \subset S$. Conversely, $S = S + \{0\} \subset S + S$; which establishes (ii). Next, assume (only) (ii): A proof by induction shows that

$$nS = (n - 1)S + S = S + S = S \quad (n = 1, 2, 3, \dots) \tag{1.18}$$

with the help of (b) and (d). The special case $n = \lceil 1/\lambda \rceil + \lceil \lambda \rceil$ $(\lambda > 0)$ yields

$$nS \overset{(1.18)}{\subset} S \subset n\lambda S \subset n^2 S, \tag{1.19}$$

since S is balanced and that $1 < n\lambda < n^2$. Dividing the latter inclusions by n shows that

$$S \subset \lambda S \subset nS \overset{(1.18)}{\subset} S, \tag{1.20}$$

which is (iii). Finally, dropping (ii) in favor of (iii) leads to

$$\alpha S + \beta S = |\alpha| S + |\beta| S \overset{(d)}{=} (|\alpha| + |\beta|) S \overset{(iii)}{=} S \quad (|\alpha| + |\beta| > 0); \tag{1.21}$$

where the equality at the left holds as S is balanced. Moreover (under the sole assumption that S is balanced), this extends to $|\alpha| + |\beta| = 0$, as follows,

$$\alpha S + \beta S = 0S + 0S \overset{(b)}{=} \{0\} \overset{(b)}{=} 0S \subset S. \tag{1.22}$$

Hence (i), which achieves the lemma's proof. We will now offer a straightforward proof of (j).

Let V be a collection of vector spaces of X, of intersection I and union U. First, remark that every member of V is convex balanced: So is I (combine (e) with (f)). Next, let Y range over V, so that

$$\mathrm{I} + \mathrm{I} \subset \mathrm{Y} + \mathrm{Y} \subset \mathrm{Y}; \tag{1.23}$$

which yields

$$\mathrm{I} + \mathrm{I} \subset \mathrm{I}. \tag{1.24}$$

Conversely,

$$\mathrm{I} = \mathrm{I} + \{0\} \subset \mathrm{I} + \mathrm{I}. \tag{1.25}$$

It now follows from the lemma's (ii) $\Rightarrow$ (i) that I is a vector subspace of X. Now temporarily assume that S is totally ordered by set inclusion: Combining (e) with (g) establishes that U is convex balanced. To show that U is more specifically a vector subspace, we first remark that such total order implies that either $\mathrm{Z} \subset \mathrm{Y}$ or $\mathrm{Y} \subset \mathrm{Z}$, as Z ranges over V. A straightforward consequence is that

$$\mathrm{Y} \subset \mathrm{Y} + \mathrm{Z} \subset \mathrm{Y} \cup \mathrm{Z}. \tag{1.26}$$

Another one is that $\mathrm{Y} \cup \mathrm{Z}$ ranges over V as well. Combined with the latter inclusions, this leads to

$$\mathrm{U} \subset \mathrm{U} + \mathrm{U} \subset \mathrm{U}. \tag{1.27}$$

It then follows from the lemma's (ii) $\Rightarrow$ (i) that U is a vector subspace of X. Finally, let A, B run through the vector subspaces of X: Combining (h) with (i) proves that $\mathrm{A} + \mathrm{B}$ is convex balanced as well. Furthermore,

$$\mathrm{A} + \mathrm{B} \overset{\text{(ii)}}{=} (\mathrm{A} + \mathrm{A}) + (\mathrm{B} + \mathrm{B}) = (\mathrm{A} + \mathrm{B}) + (\mathrm{A} + \mathrm{B}), \tag{1.28}$$

where the equality at the right holds as X is an abelian group. We now conclude from (ii) that any $\mathrm{A} + \mathrm{B}$ is a vector subspace of X. So ends the proof.

□

1.2 Exercise 2. Convex hull

The convex hull of a set A *in a vector space* X *is the set of all convex combinations of members of* A*, that is the set of all sums* $\mathrm{t_1 x_1} + \cdots + \mathrm{t_n x_n}$ *in which* $\mathrm{x_i} \in \mathrm{A}$, $\mathrm{t_i} \geq 0$*. Prove that the convex hull of a set* A *is convex and that is the intersection of all convex sets that contain* A*.*

Proof. The convex hull of a set S will be denoted by co(S). Remark that $\mathrm{S} \subset \mathrm{co(S)}$. (to see that, take $\mathrm{t_1} = 1$ for each $\mathrm{x_1}$ in S).

Our proof will directly derive from the following lemma,

Let S *be a subset of a vector space* X*: The following assertions*

(i) S *is convex;*

(ii) $\mathrm{s_1 S} + \cdots + \mathrm{s_n S} = \mathrm{S}$ *for all positive scalar variables* $\mathrm{s_1}, \ldots, \mathrm{s_n}$*;*

(iii) $\mathrm{co(S) = S}$

are then equivalent.

An easy proof by induction makes the implication (i) $\Rightarrow$ (ii) directly come from (d) of the above exercise 1, chapter 1. The implication (ii) $\Rightarrow$ (iii) is an immediate consequence of the definition of the convex hull. We now prove that any co(S), *e.g.* co(A), is convex. To do so, skip the trivial case $\mathrm{S} = \emptyset$ and let a, b run through the convex combination(s) of S, so that $\mathrm{a = t_1x_1 + \cdots + t_nx_n}$ and $\mathrm{b = t_{n+1}x_{n+1} + \cdots + t_{n+p}x_{n+p}}$ for some tuple $\mathrm{((t_1, x_1), \ldots, (t_{n+p}, x_{n+p}))}$. Every sum $\mathrm{sa+(1-s)b}$ $(0 \leq \mathrm{s} \leq 1)$ is then a convex combination of $x_1, x_2, \ldots$, since

$$\mathrm{sa + (1-s)b = \sum_{i=1}^{n} st_ix_i + \sum_{i=n+1}^{n+p} (1-s)t_ix_i} \tag{1.29}$$

and

$$\mathrm{\sum_{i=1}^{n} st_i + \sum_{i=n+1}^{n+p} (1-s)t_i = s\sum_{i=1}^{n} t_i + (1-s)\sum_{i=n+1}^{n+p} t_i = 1.} \tag{1.30}$$

As a consequence, S is convex whether S = co(S); which is (i) $\Rightarrow$ (ii). We now prove the second part: Start from $\mathrm{F \ni co(A)}$ then possibly enrich F with subset(s) B such that

$$\mathrm{B \in F \Rightarrow B\ is\ convex\ and\ contains\ A} \tag{1.31}$$

(bear in mind that co(A) is convex and contains A, see above). Note that our definition of F is weaker than the primary assumption "[F only encompasses] *all convex sets that contain A*", which is the special case

$$\mathrm{B \in F \Leftrightarrow B\ is\ convex\ and\ contains\ A.} \tag{1.32}$$

In any case, the key ingredient is that $\mathrm{co(A) \in F}$ forces $\mathrm{co(A) \supset \bigcap F}$. Conversely, the following formula

$$\mathrm{co(A) \subset co(B) \overset{(i)\Rightarrow(iii)}{=} B \quad (B \in F)} \tag{1.33}$$

is valid and implies

$$\mathrm{co(A) \subset \bigcap_{B \in F} B.} \tag{1.34}$$

So ends the proof □

1.3 Exercise 3. Other basic results

1.4 Exercise 4. A nonempty set whose interior is not

1.5 Exercise 5. A first restatement of boundedness

1.6 Exercise 6. A second restatement of boundedness

1.7 Exercise 7. Metrizability & number theory

Let be X *the vector space of all complex functions on the unit interval* $[0, 1]$*, topologized by the family of seminorms*

$$p_x(f) = |f(x)| \qquad (0 \leq \mathrm{x} \leq 1).$$

This topology is called the topology of pointwise convergence. Justify this terminology. Show that there is a sequence $\{f_n\}$ in X such that (a) $\{f_n\}$ converges to 0 as $n \to \infty$, but (b) if $\{\gamma_n\}$ is any sequence of scalars such that $\gamma_n \to \infty$ then $\{\gamma_n f_n\}$ does not converge to 0. (Use the fact that the collection of all complex sequences converging to 0 has the same cardinality as $[0,1]$.) This shows that metrizability cannot be omited in (b) of Theorem 1.28.

Proof. The family of the seminorms p_x is separating: By 1.37 of [3], the collection $\mathscr{B}$ of all finite intersections of the sets

$$\mathrm{V}(x,k) \triangleq \{\mathrm{p}_x < 2^{-k}\} \qquad (x \in [0,1], k = 1,2,3,\dots) \tag{1.35}$$

is therefore a local base for a topology τ on X. So,

$$\sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \cap_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{U_i} \,] \le \sum_{\mathrm{n}=1}^{\infty} \sum_{\mathrm{i}=1}^{\mathrm{m}} [\, \mathrm{f_n} \notin \mathrm{U_i} \,] = \sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{U_i} \,] \qquad (\mathrm{f_n} \in \mathrm{X}, \mathrm{U_i} \in \tau). \tag{1.36}$$

Now assume that $\{\mathrm{f_n}\}$ τ-converges to some f, *i.e.*

$$\sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{W} \,] < \infty \qquad (\mathrm{W} \in \mathscr{B}). \tag{1.37}$$

The special case $\mathrm{W} = \mathrm{V(x,k)}$ means that, given k, $|\,\mathrm{f_n(x)} - \mathrm{f(x)}\,| < 2^{-\mathrm{k}}$ for almost all n, *i.e.* $\{\mathrm{f_n(x)}\}$ converges to f(x). Conversely, assume that $\{\mathrm{f_n}\}$ does not τ-converges in X, *i.e.*

$$\forall \mathrm{f} \in \mathrm{X}, \exists \mathrm{W} \in \mathscr{B} : \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{W} \,] = \infty. \tag{1.38}$$

W is now the (nonempty) intersection of finitely many $\mathrm{V}(x,k)$, say $\mathrm{V(x_1,k_1)}, \dots, \mathrm{V(x_m,k_m)}$. Thus,

$$\sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{V(x_i,k_i)} \,] \overset{(1.36)}{\ge} \sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{W} \,] \overset{(1.38)}{=} \infty. \tag{1.39}$$

We can now conclude that, for some index i,

$$\sum_{\mathrm{n}=1}^{\infty} [\, \mathrm{f_n} \notin \mathrm{f} + \mathrm{V(x_i,k_i)} \,] = \infty. \tag{1.40}$$

In other word, $\{\mathrm{f_n(x_i)}\}$ fails to converge to $\mathrm{f(x_i)}$. We have so proved that τ-convergence is a rewording of pointwise convergence. We now establish the second part.

To do so, we split x into two variables: r if x is rational, α otherwise. The proof is based on the following well-known result: Each α has a *unique* binary expansion. More precisely, there exists a bijection $\mathrm{b} : [0,1] \setminus \mathbf{Q} \to \{\beta \in \{0,1\}^{\mathbf{N}_+} : \beta \text{ is not eventually periodic}\}$ where $\mathrm{b}(\alpha) = (\beta_1, \beta_2, \dots)$ is the only bit stream such that

$$\alpha = \sum_{\mathrm{k}=1}^{\infty} \beta_{\mathrm{k}} \cdot 2^{-\mathrm{k}}. \tag{1.41}$$

Remark that $b(\alpha)_1 + \cdots + b(\alpha)_n \underset{n\to\infty}{\longrightarrow} \infty$, since $b(\alpha)$ has infinite support, then fix

$$f_n(\alpha) \triangleq \frac{1}{b(\alpha)_1 + \cdots + b(\alpha)_n} \underset{n\to\infty}{\longrightarrow} 0. \tag{1.42}$$

The actual values $f_n(r)$ are of no interest, as long as every sequence $\{f_n(r) : n = 1, 2, 3, \dots\}$ converges to 0. For example, put $f_n(r) = r/n$, or just $f_n(r) = 0$. We also take $\gamma_n \longrightarrow \infty$, *i.e.* given any counting number *p*, γ_n is greater than *p* for almost all *n*. Next, we choose n_p among those *almost all n* that are large enough to satisfy

$$n_p - n_{p-1} > p \tag{1.43}$$

(start with $n_0 = 0$). So, every list $n_p, n_{p'}, n_{p''}, \dots$ that satisfies $n_{p'} - n_p = n_{p''} - n_{p'} = \dots$ is finite (otherwise, $n_{p'} - n_p \geq n_{p+1} - n_p > p \to \infty$ would hold; see (1.43)). In other words, *the distribution of $n_1, n_2, \dots$ displays no periodic pattern.* As a consequence, the *characteristic function* $\chi : k \mapsto [\, k \in \{n_1, n_2, \dots\}\,]$ is not eventually periodic. Combined with (1.41), this establishes that

$$\alpha_\gamma \triangleq \sum_{k=1}^{\infty} \chi_k 2^{-k} \tag{1.44}$$

is irrational. Conversely, still with (1.41),

$$b(\alpha_\gamma)_k = \chi_k. \tag{1.45}$$

Now remark that

$$\chi_1 + \cdots + \chi_{n_1} = 1 \tag{1.46}$$

$$\chi_1 + \cdots + \chi_{n_1} + \cdots + \chi_{n_2} = 2 \tag{1.47}$$

$$\vdots$$

$$\chi_1 + \cdots + \chi_{n_1} + \cdots + \chi_{n_2} + \cdots + \chi_{n_p} = p. \tag{1.48}$$

Combined with (1.42), this yields

$$\gamma_{n_p} f_{n_p}(\alpha_\gamma) = \frac{\gamma_{n_p}}{p} > 1. \tag{1.49}$$

There so exists a subsequence $\{\gamma_{n_p}\}$ such that $\{\gamma_{n_p} f_{\gamma_{n_p}}\}$ fails to converge pointwise to 0. Since $\{\gamma_n\}$ was arbitrary, this proves (b). □

1.9 Exercise 9. Quotient map

Suppose

1. X *and* Y *are topological vector spaces,*
2. $\Lambda : X \to Y$ *is linear.*
3. N *is a closed subspace of* X*,*
4. $\pi : X \to X/N$ *is the quotient map, and*
5. $\Lambda x = 0$ *for every* $x \in N$.

Prove that there is a unique $\mathrm{f} : \mathrm{X}/\mathrm{N} \to \mathrm{Y}$ *which satisfies* $\Lambda = f \circ \pi$*, that is,* $\Lambda x = f(\pi(x))$ *for all* $x \in X$*. Prove that* f *is linear and that* Λ *is continuous if and only if* f *is continuous. Also,* Λ *is open if and only if* f *is open.*

Proof. Bear in mind that π continously maps X onto the topological (Hausdorff) space X/N, since N is closed (see 1.41 of [3]). Moreover, the equation $\Lambda = \mathrm{f} \circ \pi$ has necessarily a unique solution, which is the binary relation

$$\mathrm{f} \triangleq \{(\pi \mathrm{x}, \Lambda \mathrm{x}) : \mathrm{x} \in \mathrm{X}\} \subset \mathrm{X}/\mathrm{N} \times \mathrm{Y}. \tag{1.50}$$

To ensure that f is actually a mapping, simply remark that the linearity of Λ implies

$$\Lambda \mathrm{x} \neq \Lambda \mathrm{x}' \Rightarrow \pi \mathrm{x}' \neq \pi \mathrm{x}'. \tag{1.51}$$

It straightforwardly derives from (1.50) that f inherits linearity from π and Λ.

Remark. The special case $\mathrm{N} = \{\Lambda = 0\}$, *i.e.* $\Lambda \mathrm{x} = 0$ **iff** $\mathrm{x} \in \mathrm{N}$ (*cf.* (e)), is the first isomorphism theorem in the topological spaces context. To see this, remark that this strenghtening of (e) yields

$$\mathrm{f}(\pi \mathrm{x}) = 0 \overset{(1.50)}{\Rightarrow} \Lambda \mathrm{x} = 0 \Rightarrow \mathrm{x} \in \mathrm{N} \Rightarrow \pi \mathrm{x} = \mathrm{N} \tag{1.52}$$

and so conclude that f is also one-to-one.

Now assume f to be continuous. Then so is $\Lambda = \mathrm{f} \circ \pi$, by 1.41 (a) of [3]. Conversely, if Λ is continuous, then for each neighborhood V of 0_{Y} there exists a neighborhood U of 0_{X} such that

$$\Lambda(\mathrm{U}) = \mathrm{f}(\pi(\mathrm{U})) \subset \mathrm{V}. \tag{1.53}$$

Since π is open (1.41 (a) of [3]), $\pi(\mathrm{U})$ is a neighborhood of $\mathrm{N} = 0_{\mathrm{X}/\mathrm{N}}$: This is sufficient to establish that the linear mapping f is continuous. If f is open, so is $\Lambda = \mathrm{f} \circ \pi$, by 1.41 (a) of [3]. To prove the converse, remark that every neighborhood W of $0_{\mathrm{X}/\mathrm{N}}$ satisfies

$$\mathrm{W} = \pi(\mathrm{V}) \tag{1.54}$$

for some neighborhood V of 0_{X}. So,

$$\mathrm{f}(\mathrm{W}) = \mathrm{f}(\pi(\mathrm{V})) = \Lambda(\mathrm{V}). \tag{1.55}$$

As a consequence, if Λ is open, then f(W) is a neighborhood of 0_{Y}. So ends the proof. □

1.10 Exercise 10. An open mapping theorem

Suppose that X and Y are topological vector spaces, $\dim \mathrm{Y} < \infty$, $\Lambda : \mathrm{X} \to \mathrm{Y}$ *is linear, and* $\Lambda(\mathrm{X}) = \mathrm{Y}$*.*

1. *Prove that* Λ *is an open mapping.*

2. *Assume, in addition, that the null space of* Λ *is closed, and prove that* Λ *is continuous.*

Proof. Discard the trivial case $\Lambda = 0$ and so assume that $\dim \mathrm{Y} = \mathrm{n}$ for some positive n. Let e range over a base of B of Y then pick W an arbitrary neighborhood of the origin: There so exists V a balanced neighborhood of the origin such that

$$\sum_{\mathrm{e}} \mathrm{V} \subset \mathrm{W}, \tag{1.56}$$

since addition is continuous. Moreover, for each e, there exists $\mathrm{x_e}$ in X such that $\Lambda(\mathrm{x_e}) = \mathrm{e}$, simply because Λ is onto. So,

$$\mathrm{y} = \sum_{\mathrm{e}} \mathrm{y_e} \cdot \Lambda \mathrm{x_e} \qquad (\mathrm{y} \in \mathrm{Y}). \tag{1.57}$$

As a finite set, $\{\mathrm{x_e} : \mathrm{e} \in \mathrm{B}\}$ is bounded: There so exists a positive scalar s such that

$$\forall \mathrm{e} \in \mathrm{B}, \mathrm{x_e} \in \mathrm{s} \cdot \mathrm{V}. \tag{1.58}$$

Combining this with (1.57) shows that

$$\mathrm{y} \in \sum_{\mathrm{e}} \mathrm{y_e} \cdot \mathrm{s} \cdot \Lambda(\mathrm{V}). \tag{1.59}$$

We now come back to (1.56) and so conclude that

$$\mathrm{y} \in \sum_{\mathrm{e}} \Lambda(\mathrm{V}) \subset \Lambda(\mathrm{W}) \tag{1.60}$$

whether $|\,\mathrm{y_e}\,| < 1/\mathrm{s}$; which proves (a).

To prove (b), assume that the null space $\{\Lambda = 0\}$ is closed and let f, π be as in Exercise 1.9, $\{\Lambda = 0\}$ playing the role of N. Since Λ is onto, the first isomorphism theorem (see Exercise 1.9) asserts that f is an isomorphism of X/N onto Y. Consequently,

$$\dim \mathrm{X/N} = \mathrm{n}. \tag{1.61}$$

f is then an homeomorphism of $\mathrm{X/N} \equiv \mathbf{C}^{\mathrm{n}}$ onto Y; see 1.21 of [3]. We have thus established that f is continuous: So is $\Lambda = \mathrm{f} \circ \pi$. □

1.12 Exercise 12. Topology stays, completeness leaves

1.14 Exercise 14. $\mathscr{D}_{\mathrm{K}}$ equipped with other seminorms

Put $\mathrm{K} = [0, 1]$ *and define* $\mathscr{D}_{\mathrm{K}}$ *as in Section 1.46. Show that the following three families of seminorms (where* $\mathrm{n} = 0, 1, 2, \ldots$ *) define the same topology on* $\mathscr{D}_{\mathrm{K}}$*. If* $\mathrm{D} = \mathrm{d/dx}$*:*

1. $\|\mathrm{D^n f}\|_\infty = \sup\{|\mathrm{D^n f(x)}| : \infty < \mathrm{x} < \infty\}$
2. $\|\mathrm{D^n f}\|_1 = \int_0^1 |\mathrm{D^n f}(x)|\, \mathrm{d}x$
3. $\|\mathrm{D^n f}\|_2 = \left\{\int_0^1 |\mathrm{D^n f}(x)|^2 \mathrm{d}x\right\}^{1/2}.$

Proof. First, remark that

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty < \infty \tag{1.62}$$

holds, since K has length 1 (the inequality on the left is a Cauchy-Schwarz one). Next, that the support of $D^n f$ lies in K; which yields

$$|\, D^n f(x)\,| = \left| \int_0^x D^{n+1} f \right| \le \int_0^x |\, D^{n+1} f\,| \le \|D^{n+1} f\|_1. \tag{1.63}$$

So,

$$\|D^n f\|_\infty \le \|D^{n+1} f\|_1. \tag{1.64}$$

We now combine (1.62) with (1.64) and so obtain

$$\|D^n f\|_1 \le \|D^n f\|_2 \le \|D^n f\|_\infty \le \|D^{n+1} f\|_1 \le \cdots \quad (n = 0, 1, 2, \dots). \tag{1.65}$$

Put

$$V_n^{(i)} \triangleq \{f \in \mathscr{D}_K : \|\, f\,\|_i < 2^{-n}\} \quad (i = 1, 2, \infty) \tag{1.66}$$

$$\mathscr{B}^{(i)} \triangleq \{V_n^{(i)} : n = 0, 1, 2, \dots\}, \tag{1.67}$$

so that (1.65) is mirrored in terms of neighborhood inclusions, as follows,

$$V_n^{(1)} \supset V_n^{(2)} \supset V_n^{(\infty)} \supset V_{n+1}^{(1)} \supset \cdots. \tag{1.68}$$

Since $V_n^{(i)} \supset V_{n+1}^{(i)}$, $\mathscr{B}^{(i)}$ is a local base of a topology τ_i. But the chain (1.68) forces

$$\tau_1 = \tau_2 = \tau_\infty. \tag{1.69}$$

To see that, choose a set S that is τ_1-open at f, *i.e.* $V_n^{(1)} \subset S - f$ for some n. Next, concatenate this with $V_n^{(2)} \subset V_n^{(1)}$ (see (1.68)) and so obtain $V_n^{(2)} \subset S - f$; which implies that S is τ_2-open at f. Similarly, we deduce, still from (1.68), that

$$\tau_2\text{-open} \Rightarrow \tau_\infty\text{-open} \Rightarrow \tau_1\text{-open}. \tag{1.70}$$

So ends the proof. □

1.16 Exercise 16. Uniqueness of topology for test functions

Prove that the topology of $C(\Omega)$ *does not depend on the particular choice of* $\{K_n\}$, *as long as this sequence satisfies the conditions specified in section 1.44. Do the same for* $C^\infty(\Omega)$ *(Section 1.46).*

Comment This is an invariance property: The function test topology only depends on the existence of the supremum-seminorms p_n, then, eventually, only on the ambient space itself. This should be regarded as a very part of the textbook [3] The proof consists in combining trivial consequences of the local base definition with a well-known result (*e.g.* [2.6] in [2]) about intersection of nonempty compact sets.

Lemma 1 *Let* X *be a topological space with a countable local base* $\{V_n : n = 1, 2, 3, \dots\}$*. If* $\tilde{V}_n = V_1 \cap \cdots \cap V_n$*, then every subsequence* $\{\tilde{V}_{\varrho(n)}\}$ *is a decreasing (i.e.* $\tilde{V}_{\varrho(n)} \supset \tilde{V}_{\varrho(n+1)}$*) local base of X.*

Proof. The decreasing property is trivial. Now remark that $V_n \supset \tilde{V}_n$: This shows that $\{\tilde{V}_n\}$ is a local base of X. Then so is $\{\tilde{V}_{\rho(n)}\}$, since $\tilde{V}_n \supset \tilde{V}_{\rho(n)}$. □

The following special case $V_n = \tilde{V}_n$ is one of the key ingredients:

Corollary 1 (special case $V_n = \tilde{V}_n$**)** *Under the same notations of Lemma 1, if* $\{V_n\}$ *is a decreasing local base, then so is* $\{V_{\varrho(n)}\}$*.*

Corollary 2 *If* $\{Q_n\}$ *is a sequence of compact sets that satisfies the conditions specified in section 1.44, then every subsequence* $\{Q_{\varrho(n)}\}$ *also satisfies theses conditions. Furthermore, if* τ_Q *is the* $C(\Omega)$*'s (respectively* $C^\infty(\Omega)$*'s) topology of the seminorms* p_n*, as defined in section 1.44 (respectively 1.46), then the seminorms* $p_{\varrho(n)}$ *define the same topology* τ_Q*.*

Proof. Let X be $C(\Omega)$ topologized by the seminorms p_n (the case $X = C^\infty(\Omega)$ is proved the same way). If $V_n = \{p_n < 1/n\}$, then $\{V_n\}$ is a decreasing local base of X. Moreover,

$$Q_{\rho(n)} \subset \overset{\circ}{Q}_{\rho(n)+1} \subset Q_{\rho(n)+1} \subset Q_{\rho(n+1)}. \tag{1.71}$$

Thus,

$$Q_{\rho(n)} \subset \overset{\circ}{Q}_{\rho(n+1)}. \tag{1.72}$$

In other words, $Q_{\rho(n)}$ satisfies the conditions specified in section 1.44. $\{p_{\rho(n)}\}$ then defines a topology τ_{Q_ρ} for which $\{V_{\rho(n)}\}$ is a local base. So, $\tau_{Q_\rho} \subset \tau_Q$. Conversely, the above corollary asserts that $\{V_{\rho(n)}\}$ is a local base of τ_Q, which yields $\tau_Q \subset \tau_{Q_\rho}$. □

Lemma 2 *If a sequence of compact sets* $\{Q_n\}$ *satisfies the conditions specified in section 1.44, then every compact set* K *lies in allmost all* Q_n°*, i.e. there exists* m *such that*

$$K \subset \overset{\circ}{Q}_m \subset \overset{\circ}{Q}_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots. \tag{1.73}$$

Proof. The following definition

$$C_n \triangleq K \setminus \overset{\circ}{Q}_n \tag{1.74}$$

shapes $\{C_n\}$ as a decreasing sequence of compact[1] sets. We now suppose (to reach a contradiction) that no C_n is empty and so conclude[2] that the C_n's intersection contains a point that is not in any Q_n°. On the other hand, the conditions specified in [1.44] force the Q_n°'s collection to be an open cover. This contradiction reveals that $C_m = \emptyset$, *i.e.* $K \subset Q_m^\circ$, for some m. Finally,

$$K \subset \overset{\circ}{Q}_m \subset Q_m \subset \overset{\circ}{Q}_{m+1} \subset Q_{m+1} \subset \overset{\circ}{Q}_{m+2} \subset \cdots. \tag{1.75}$$

□

We are now in a fair position to establish the following:

[1] See (b) of 2.5 of [2].

[2] In every Hausdorff space, the intersection of a decreasing sequence of nomempty compact sets is nonempty. This is a corollary of 2.6 of [2].

Theorem *The topology of $C(\Omega)$ does not depend on the particular choice of $\{K_n\}$, as long as this sequence satisfies the conditions specified in section 1.44. Neither does the topology of $C^\infty(\Omega)$, as long as this sequence satisfies the conditions specified in section 1.44.*

Proof. With the second corollary's notations, $\tau_K = \tau_{K_\lambda}$, for every subsequence $\{K_{\lambda(n)}\}$. Similarly, let $\{L_n\}$ be another sequence of compact subsets of Ω that satisfies the condition specified in [1.44], so that $\tau_L = \tau_{L_\varkappa}$ for every subsequence $\{L_{\varkappa(n)}\}$. Now apply the above Lemma 2 with K_i $(i = 1, 2, 3, \dots)$ and so conclude that $K_i \subset L^\circ_{m_i} \subset L^\circ_{m_i+1} \subset \cdots$ for some m_i. In particular, the special case $\varkappa_i = m_i + i$ is

$$K_i \subset \overset{\circ}{L}_{\varkappa_i}. \tag{1.76}$$

Let us reiterate the above proof with K_n and L_n in exchanged roles then similarly find a subsequence $\{\lambda_j : j = 1, 2, 3, \dots\}$ such that

$$L_j \subset \overset{\circ}{K}_{\lambda_j} \tag{1.77}$$

Combine (1.76) with (1.77) and so obtain

$$K_1 \subset \overset{\circ}{L}_{\varkappa_1} \subset L_{\varkappa_1} \subset \overset{\circ}{K}_{\lambda_{\varkappa_1}} \subset K_{\lambda_{\varkappa_1}} \subset \overset{\circ}{L}_{\varkappa_{\lambda_{\varkappa_1}}} \subset \cdots, \tag{1.78}$$

which means that the sequence $Q = (K_1, L_{\varkappa_1}, K_{\lambda_{\varkappa_1}}, \dots)$ satisfies the conditions specified in section 1.44. It now follows from the corollary 2 that

$$\tau_K = \tau_{K_\lambda} = \tau_Q = \tau_{L_\varkappa} = \tau_L. \tag{1.79}$$

So ends the proof □

1.17 Exercise 17. Derivation in some non normed space

In the setting of Section 1.46, prove that $f \mapsto D^\alpha f$ *is a continuous mapping of* $C^\infty(\Omega)$ *into* $C^\infty(\Omega)$ *and also of* $\mathscr{D}_K$ *into* $\mathscr{D}_K$*, for every multi-index* α.

Proof. In both cases, D^α is a linear mapping. It is then sufficient to establish continuousness at the origin. We begin with the $C^\infty(\Omega)$ case.

Let U be an aribtray neighborhood of the origin. There so exists N such that U contains

$$V_N = \left\{\varphi \in C^\infty(\Omega) : \max\{|D^\beta\varphi(x)| : |\beta| \le N, x \in K_N\} < 1/N\right\}. \tag{1.80}$$

Now pick g in $V_{N+|\alpha|}$, so that

$$\max\{|D^\gamma g(x)| : |\gamma| \le N + |\alpha|, x \in K_N\} < \frac{1}{N + |\alpha|}. \tag{1.81}$$

(the fact that $K_N \subset K_{N+|\alpha|}$ was tacitely used). The special case $\gamma = \beta + \alpha$ yields

$$\max\{|D^\beta D^\alpha g(x)| : |\beta| \le N, x \in K_N\} < \frac{1}{N}. \tag{1.82}$$

We have just proved that

$$g \in V_{N+|\alpha|} \Rightarrow D^{\alpha} g \in V_N, \quad \textit{i.e.} \quad D^{\alpha}\left(V_{N+|\alpha|}\right) \subset V_N, \tag{1.83}$$

which establishes the continuity of $D^{\alpha} : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$.

To prove the continuousness of the restriction $D^{\alpha}|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathscr{D}_K$, we first remark that the collection of the $V_N \cap \mathscr{D}_K$ is a local base of the subspace topology of $\mathscr{D}_K$. $V_{N+|\alpha|} \cap \mathscr{D}_K$ is then a neighborhood of 0 in this topology. Furthermore,

$$D^{\alpha}|_{\mathscr{D}_K}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) = D^{\alpha}\left(V_{N+|\alpha|} \cap \mathscr{D}_K\right) \tag{1.84}$$

$$\subset D^{\alpha}\left(V_{N+|\alpha|}\right) \cap D^{\alpha}\left(\mathscr{D}_K\right) \tag{1.85}$$

$$\subset V_N \cap \mathscr{D}_K \quad \text{(see (1.83))} \tag{1.86}$$

So ends the proof. □

Chapter 2

Completeness

2.1 Exercise 3. An equicontinous sequence of measures

Put $K = [-1, 1]$*; define* $\mathscr{D}_K$ *as in section 1.46 (with* $\mathbf{R}$ *in place of* $\mathbf{R}^n$*). Supose* $\{f_n\}$ *is a sequence of Lebesgue integrable functions such that* $\Lambda\varphi = \lim_{n\to\infty} \int_{-1}^{1} f_n(t)\varphi(t)dt$ *exists for every* $\varphi \in \mathscr{D}_K$*. Show that* Λ *is a continuous linear functional on* $\mathscr{D}_K$*. Show that there is a positive integer* p *and a number* $M < \infty$ *such that*

$$\left| \int_{-1}^{1} f_n(t)\varphi(t)dt \right| \leq M \| D^p \|_\infty$$

for all n*. For example, if* $f_n(t) = n^3 t$ *on* $[-1/n, 1/n]$ *and* 0 *elsewhere, show that this can be done with* $p = 1$*. Construct an example where it can be done with* $p = 2$ *but not with* $p = 1$*.*

We will also consider the case $p = 0$. Since all supports of $\varphi, \varphi', \varphi'', \ldots,$ are in K, we make a specialization of the mean value theorem:

Lemma If $\varphi \in \mathscr{D}_{[a,b]}$, then

$$\| D^\alpha \varphi \|_\infty \leq \| D^p \varphi \|_\infty \left(\frac{\lambda}{2}\right)^{p-\alpha} \qquad (\alpha = 0, 1, \ldots, p) \tag{2.1}$$

at every order $p = 0, 1, 2, \ldots$; where λ is the length $|\, b - a \,|$.

Proof. Let x_0 be in (a, b) . We first consider the case $x_0 \leq c = (a + b)/2$: The mean value theorem asserts that there exists x_1 $(a < x_1 < x_0)$, such that

$$\varphi(x_0) = \varphi(x_0) - \varphi(a) = D\varphi(x_1)(x_0 - a). \tag{2.2}$$

Since every $D^p\varphi$ lies in $\mathscr{D}_{[a,b]}$, a straightforward proof by induction shows that there exists a partition $a < \cdots < x_p < \cdots < x_0$ such that

$$\varphi(x_0) = D^0\varphi(x_0) \tag{2.3}$$
$$= D^1\varphi(x_1)(x_0 - a) \tag{2.4}$$
$$= \cdots$$
$$= D^p\varphi(x_p)(x_0 - a)\cdots(x_{p-1} - a), \tag{2.5}$$

for all p. More compactly,

$$D^{\alpha}\varphi(x_0) = D^{p}\varphi(x_p)\prod_{k=\alpha}^{p-1}(x_k - a); \tag{2.6}$$

which yields,

$$|D^{\alpha}\varphi(x)| \leq \| D^{p}\varphi \|_{\infty} \left(\frac{\lambda}{2}\right)^{p-\alpha} \quad (x \in [a, c]) \tag{2.7}$$

The case $x_0 \geq c$ outputs a "reversed" result, with $b > \cdots > x_p > \cdots > x_0$ and $x_k - b$ playing the role of $x_k - a$: So,

$$|D^{\alpha}\varphi(x)| \leq \| D^{p}\varphi \|_{\infty} \left(\frac{\lambda}{2}\right)^{p-\alpha} \tag{2.8}$$

Finally, we combine (2.7) with (2.8) and so obtain

$$\| D^{\alpha}\varphi \|_{\infty} \leq \| D^{p}\varphi \|_{\infty} \left(\frac{\lambda}{2}\right)^{p-\alpha}. \tag{2.9}$$

□

Proof. We first consider $C_0(\mathbf{R})$ topologized by the supremum norm. Given a Lebesgue integrable function u, we put

$$\langle u|\varphi\rangle \triangleq \int_{\mathbf{R}} u\varphi \quad (\varphi \in C_0(\mathbf{R})). \tag{2.10}$$

The following inequalities

$$|\langle u|\varphi\rangle| \leq \int_{\mathbf{R}} |u\varphi| \leq \| u \|_{L^1} \quad (\| \varphi \|_{\infty} \leq 1) \tag{2.11}$$

imply that every linear functional

$$\begin{aligned} \langle u| : C_0(\mathbf{R}) &\to \mathbf{C} \\ \varphi &\mapsto \langle u|\varphi\rangle \end{aligned} \tag{2.12}$$

is bounded on the open unit ball. It is therefore continuous; see 1.18 of [3]. Conversely, u can be identified with $\langle u|$, since u is determined (a.e) by the integrals $\langle u|\varphi\rangle$. In the Banach spaces terminology, u is then (identified with) a linear *bounded*[1] operator $\langle u|$, of norm

$$\sup\{|\langle u|\varphi\rangle| : \| \varphi \|_{\infty} = 1\} = \| u \|_{L^1}. \tag{2.13}$$

Note that, in the latter equality, $\leq \| u \|_{L^1}$ comes from (2.11), as the converse comes from the Stone-Weierstrass theorem[2]. We now consider the special cases $u = g_n$, where g_n is

$$\begin{aligned} g_n : \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \begin{cases} n^3 x & (x \in [-\frac{1}{n}, \frac{1}{n}]) \\ 0 & (x \notin [-\frac{1}{n}, \frac{1}{n}]) . \end{cases} \end{aligned} \tag{2.14}$$

[1] see 1.32, 4.1 of [3]

[2] See 7.26 of [1].

First, remark that $g_n(x) \longrightarrow 0$, as the sequence $\{g_n\}$ fails to converge in $C_0(\mathbf{R})$ (since $g_n(1/n) = n^2 \geq 1$), and also in L^1 (since $\int_{\mathbf{R}} |g_n| = n^2 \longrightarrow \infty$) . Nevertheless, we will show that the $\langle g_n|$ converge pointwise[3] on $\mathscr{D}_K$ *i.e.* there exists a τ_K-continuous linear form Λ such that

$$\langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi, \tag{2.15}$$

where φ ranges over $\mathscr{D}_K$. We now prove (2.13) in the special cases $u = g_n$. To do so, we fetch $\varphi_1^+, \ldots, \varphi_j^+, \ldots,$ from $C_K^\infty(\mathbf{R})$. More specifically,

(i) $\varphi_j^+ = 1$ on $[e^{-j}, 1 - e^{-j}]$;

(ii) $\varphi_j^+ = 0$ on $\mathbf{R} \setminus [-1, 1]$;

(iii) $0 \leq \varphi_j^+ \leq 1$ on $\mathbf{R}$;

see [1.46] of [3] for a possible construction of those φ_j^+. Let $\varphi_1^-, \ldots, \varphi_j^-, \ldots,$ mirror the φ_j^+, in the sense that $\varphi_j^-(x) = \varphi_j^+(-x)$, so that

(iv) $\varphi_j \triangleq \varphi_j^+ - \varphi_j^-$ is odd, as g_n is;

(v) every φ_j is in $C_K^\infty(\mathbf{R})$;

(vi) The sequence $\{\varphi_j\}$ converges (pointwise) to $1_{[0,1]} - 1_{[-1,0]}$, and $\| \varphi_j \|_\infty = 1$.

Thus, with the help of the Lebesgue's convergence theorem,

$$\langle g_n|\varphi_j\rangle = 2\int_0^1 g_n(t)\varphi_j^+(t)dt \underset{j\to\infty}{\longrightarrow} 2\int_0^1 g_n(t)dt = \| g_n \|_{L^1} = n. \tag{2.16}$$

Finally,

$$\| g_n \|_{L^1} \overset{(2.16)}{\leq} \sup\{| \langle g_n|\varphi\rangle | : \| \varphi \|_\infty = 1\} \overset{(2.13)}{\leq} \| g_n \|_{L^1}; \tag{2.17}$$

which is the desired result. So, in terms of boundedness constants: Given n, there exists $C_n < \infty$ such that

$$|\langle g_n|\varphi\rangle| \leq C_n \quad (\| \varphi \|_\infty = 1); \tag{2.18}$$

see (2.11). Furthermore, $\| g_n \|_{L^1}$ is actually the best, *i.e.* lowest, possible C_n; see (2.17). But, on the other hand, (2.16) shows that there exists a subsequence $\{\langle g_n|\varphi_{\rho(n)}\rangle\}$ such that $\langle g_n|\varphi_{\rho(n)}\rangle$ is greater than, say, $n - 0.01$, as $\| \varphi_{\rho(n)} \|_\infty = 1$. Consequently, there is no bound M such that

$$|\langle g_n|\varphi\rangle| \leq M \quad (\| \varphi \|_\infty = 1; n = 1, 2, 3, \ldots). \tag{2.19}$$

In other words, the g_n have no *uniform bound* in L^1, *i.e.* the collection of all continous linear mappings $\langle g_n|$ is not equicontinous (see discussion in 2.6 of [3]). As a consequence, the $\langle g_n|$ do not converge pointwise (or "vaguely", in Radon measure context): A vague (*i.e.* pointwise) convergence would be (by definition)

$$\langle g_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi \quad (\varphi \in C_0(\mathbf{R})) \tag{2.20}$$

[3] See 3.14 of [3] for a definition of the related topology.

for some $\Lambda \in C_0(\mathbf{R})^*$, which would make (2.19) hold; see 2.6, 2.8 of [3]. This by no means says that the $\langle g_n|$ do not converge pointwise, in a relevant space, to some Λ (see (2.15).

From now on, unless the contrary is explicitely stated, we asume that φ only denotes an element of $C_K^\infty(\mathbf{R})$. Let f_n be a Lebesgue integrable function such that

$$\Lambda\varphi = \lim_{n\to\infty} \int_K f_n\varphi \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.21}$$

for some linear form Λ. Since φ vanishes outside K, we can suppose without loss of generality that the support of f_n lies in K. So, (2.21) can be restated as follows,

$$\Lambda\varphi = \lim_{n\to\infty} \langle f_n|\varphi\rangle \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.22}$$

Let $K_1, K_2, \dots,$ be compact sets that satisfy the conditions specified in 1.44 of [3]. $\mathscr{D}_K$ is $C_K^\infty(\mathbf{R})$ topologized by the related seminorms $p_1, p_2, \dots$; see 1.46, 6.2 of [3] and Exercise 1.16. We know that $K \subset K_m$ for some index m (see Lemma 2 of Exercise 1.16): From now on, we only consider the indices $N \geq m$, so that

(a) $p_N(\varphi) = \|\varphi\|_N \triangleq \max\{|D^\alpha\varphi(x)| : \alpha \leq N, x \in \mathbf{R}\}$, for $\varphi \in \mathscr{D}_K$;

(b) The collection of the sets $V_N = \{\varphi \in \mathscr{D}_K : \|\varphi\|_N < 2^{-N}\}$ is a (decreasing) local base of τ_K, the subspace topology of $\mathscr{D}_K$; see 6.2 of [3] for a more complete discussion.

Let us specialize (2.11) with $u = f_n$ and $\varphi \in V_m$ then conclude that $\langle f_n|$ is bounded by $\|f_n\|_{L^1}$ on V_m: Every linear functional $\langle f_n|$ is therefore τ_K-continuous; see 1.18 of [3].

To sum it up:

(i) $\mathscr{D}_K$, equipped the topology τ_K, is a Fréchet space (see section 1.46 of [3]);

(ii) Every linear functional $\langle f_n|$ is continuous with respect to this topology;

(iii) $\langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \Lambda\varphi$ for all φ, *i.e.* $\Lambda - \langle f_n| \underset{n\to\infty}{\longrightarrow} 0$.

With the help of [2.6] and [2.8] of [3], we conclude that Λ is continuous and that the sequence $\{\langle f_n|\}$ is equicontinuous. So is the sequence $\{\Lambda - \langle f_n|\}$, since addition is continuous. There so exists i, j such that, for all n,

$$|\Lambda\varphi| < 1/2 \quad \text{if } \varphi \in V_i, \tag{2.23}$$

$$|\Lambda\varphi - \langle f_n|\varphi\rangle| < 1/2 \quad \text{if } \varphi \in V_j. \tag{2.24}$$

Choose $p = \max\{i, j\}$, so that $V_p = V_i \cap V_j$: The latter inequalities imply that

$$|\langle f_n|\varphi\rangle| \leq |\Lambda\varphi - \langle f_n|\varphi\rangle| + |\Lambda\varphi| < 1 \quad \text{if } \varphi \in V_p. \tag{2.25}$$

Now remark that every $\psi = \psi[\mu, \varphi]$, where

$$\psi[\mu, \varphi] \triangleq \begin{cases} (1/\mu \cdot 2^p\|\varphi\|_p)\varphi & (\varphi \neq 0, \mu > 1) \\ 0 & (\varphi = 0, \mu > 1), \end{cases} \tag{2.26}$$

keeps in V_p. Finally, it is clear that each below statement implies the following one.

$$|\langle f_n|\psi\rangle| < 1 \tag{2.27}$$
$$|\langle f_n|\varphi\rangle| < 2^p\|\,\varphi\,\|_p\cdot\mu \tag{2.28}$$
$$|\,\langle f_n|\varphi\rangle\,| \le 2^p\|\,\varphi\,\|_p \tag{2.29}$$
$$|\,\langle f_n|\varphi\rangle\,| \le 2^p\{\|\,D^0\varphi\,\|_\infty + \cdots + \|\,D^p\varphi\,\|_\infty\}. \tag{2.30}$$

Finally, with the help of (2.1),

$$|\,\langle f_n|\varphi\rangle\,| \le 2^p(p+1)\|\,D^p\varphi\,\|_\infty. \tag{2.31}$$

The first part is so proved, with *some* p and $M = 2^p(p+1)$.

We now come back to the special case $f_n = g_n$ (see the first part). From now on, $f_n(x) = n^3x$ on $[\text{-}1/n, 1/n]$, 0 elsewhere. Actually, we will prove that

(a) $\Lambda\varphi = \lim\limits_{n\to\infty}\int_{-1}^{1} f_n(t)\varphi(t)dt$ exists for every $\varphi \in \mathscr{D}_K$;

(b) A *uniform* bound $|\langle f_n|\varphi\rangle| \le M\|\,D^p\varphi\,\|_\infty$ $(n = 1, 2, 3, \dots)$ exists for all those f_n, with $p = 1$ as the smallest possible p.

Bear in mind that $K \subset K_m$ and shift the K_N's indices, so that K_{m+1} becomes K_1, K_{m+2} becomes K_2, and so on. The resulting topology τ_K remains unchanged (see Exercise 1.16). We let φ keep running on $\mathscr{D}_K$ and so define

$$B_n(\varphi) \triangleq \max\{|\,\varphi(x)\,| : x \in [\text{-}1/n, 1/n]\}, \tag{2.32}$$
$$\Delta_n(\varphi) \triangleq \max\{|\,\varphi(x) - \varphi(0)\,| : x \in [\text{-}1/n, 1/n]\}. \tag{2.33}$$

The mean value asserts that

$$|\,\varphi(1/n) - \varphi(\text{-}1/n)\,| \le B_n(\varphi')\,|\,1/n - (\text{-}1/n)\,| = \frac{2}{n}B_n(\varphi'). \tag{2.34}$$

Independently, an integration by parts shows that

$$\langle f_n|\varphi\rangle = \left[\frac{n^3t^2}{2}\varphi(t)\right]_{-1/n}^{1/n} - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\varphi'(t)dt \tag{2.35}$$
$$= \frac{n}{2}\left(\varphi(1/n) - \varphi(\text{-}1/n)\right) - \frac{n^3}{2}\int_{-1/n}^{1/n} t^2\varphi'(t)dt. \tag{2.36}$$

Combine (2.34) with (2.36) and so obtain

$$|\,\langle f_n|\varphi\rangle\,| \le \frac{n}{2}\,|\,\varphi(1/n) - \varphi(\text{-}1/n)\,| + \frac{n^3}{2}\int_{-1/n}^{1/n} t^2|\varphi'(t)|dt \tag{2.37}$$
$$\le B_n(\varphi') + \frac{n^3}{2}B_n(\varphi')\int_{-1/n}^{1/n} t^2dt \tag{2.38}$$
$$\le \frac{4}{3}B_n(\varphi') \tag{2.39}$$
$$\le \frac{4}{3}\|\,\varphi'\,\|_\infty. \tag{2.40}$$

Futhermore, (2.39) gives a hint about the convergence of f_n: Since $B_n(\varphi')$ tends to $|\varphi'(0)|$, we may expect that f_n tends to $\frac{4}{3}\varphi'(0)$. This is actually true: A straightforward computation shows that

$$\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0) \overset{(2.36)}{=} \frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0) - \frac{n^3}{2}\int_{-1/n}^{1/n}(\varphi' - \varphi'(0))t^2 dt. \tag{2.41}$$

So,

$$\left|\langle f_n|\varphi\rangle - \frac{4}{3}\varphi'(0)\right| \le \left|\frac{\varphi(1/n) - \varphi(\text{-}1/n)}{1/n - (\text{-}1/n)} - \varphi'(0)\right| + \frac{1}{3}\Delta_n(\varphi') \underset{n\to\infty}{\longrightarrow} 0. \tag{2.42}$$

We have just proved that

$$\langle f_n|\varphi\rangle \underset{n\to\infty}{\longrightarrow} \frac{4}{3}\varphi'(0) \quad (\varphi \in \mathscr{D}_K). \tag{2.43}$$

In other words,

$$\langle f_n| \underset{n\to\infty}{\longrightarrow} \text{-}\frac{4}{3}\delta', \tag{2.44}$$

where δ is the *Dirac measure* and $\delta', \delta'', \ldots$, its *derivatives*; see 6.1 and 6.9 of [3].

It follows from the previous part that $\text{-}\frac{4}{3}\delta'$ is τ_K-continuous, and from (2.40) that

$$|\langle f_n|\varphi\rangle| \le \frac{4}{3}\|\,\varphi'\,\|_\infty \quad (n = 1, 2, 3, \ldots) \tag{2.45}$$

(which is a constructive version of (2.31)). Furthermore, we have already spotlighted a sequence

$$\{\langle f_n|\varphi_{\rho(n)}\rangle : \|\,\varphi_{\rho(n)}\,\|_\infty = 1; n = 1, 2, 3, \ldots\} \tag{2.46}$$

that is not bounded. We then restate (2.19) in a more precise fashion: There is no constant M such that

$$|\langle f_n|\varphi\rangle| \le M\|\,\varphi\,\|_\infty \quad (\varphi \in C_K^\infty(\mathbf{R})). \tag{2.47}$$

The previous bound of $\langle f_n|$ - see (2.40), is therefore the best possible one, *i.e.* $p = 1$ is the smallest possible p and, given $p = 1$, $M = \frac{4}{3}$ is the smallest possible M (to see that, compare (2.39) with (2.43)); which is (b).

In order to construct the second requested example, we give f_n a *derivative*[4] ${f_n}'$, as follows

$$\begin{aligned} {f_n}' : \mathscr{D}_K &\to \mathbf{C} \\ \varphi &\mapsto \text{-}\left\langle f_n\middle|\varphi'\right\rangle. \end{aligned} \tag{2.48}$$

It has been proved that every $\langle f_n|$ is continuous. So is

$$\begin{aligned} D : \mathscr{D}_K &\to \mathscr{D}_K \\ \varphi &\mapsto \varphi'; \end{aligned} \tag{2.49}$$

[4] See 6.1 of [3] for a further discussion.

see Exercise 1.17. $\mathrm{f_n}'$ is therefore continuous. Now apply (2.43) with φ' and so obtain

$$-\langle \mathrm{f_n}|\varphi'\rangle \underset{\mathrm{n}\to\infty}{\longrightarrow} \frac{4}{3}\varphi''(0) \quad (\varphi \in \mathscr{D}_\mathrm{K}),$$

i.e.

$$\mathrm{f_n}' \underset{\mathrm{n}\to\infty}{\longrightarrow} \frac{4}{3}\delta''. \tag{2.50}$$

It follows from (2.40) that,

$$|\langle \mathrm{f_n}|\varphi'\rangle| \le \frac{4}{3}\| \varphi'' \|_\infty \quad (\mathrm{n} = 1, 2, 3, \dots). \tag{2.51}$$

It is therefore possible to uniformly bound $\mathrm{f_n}'$ with respect to a norm $\| \mathrm{D^p} \cdot \|_\infty$, namely $\| \mathrm{D}^2 \cdot \|_\infty$. Then arises a question: Is 2 the smallest p? The answer is: Yes. To show this, we first assume, to reach a contradiction, that there exists a positive constant M such that

$$|\langle \mathrm{f_n}|\varphi'\rangle| \le \mathrm{M}\| \varphi' \|_\infty \quad (\mathrm{n} = 1, 2, 3, \dots). \tag{2.52}$$

Define

$$\Phi_\mathrm{j}(\mathrm{x}) = \int_{-1}^{\mathrm{x}} \varphi_\mathrm{j}. \tag{2.53}$$

The oddness of φ_j forces Φ_j to vanish outside [-1, 1]: φ_j is therefore in $\mathscr{D}_\mathrm{K}$. So, under our assumption,

$$|\langle \mathrm{f_n}|\Phi_\mathrm{j}'\rangle| \le \mathrm{M}\| \Phi_\mathrm{j}' \|_\infty \quad (\mathrm{n} = 1, 2, 3, \dots); \tag{2.54}$$

which is

$$|\langle \mathrm{f_n}|\varphi_\mathrm{j}\rangle| \le \mathrm{M} \quad (\mathrm{n} = 1, 2, 3, \dots). \tag{2.55}$$

We have thus reached a contradiction (again with the sequence $\{\langle \mathrm{f_n}|\varphi_{\rho(\mathrm{n})}\rangle\}$) and so conclude that there is no constant M such that

$$|\langle|\mathrm{f_n}\varphi'\rangle| \le \mathrm{M}\| \varphi' \|_\infty \quad (\mathrm{n} = 1, 2, 3, \dots). \tag{2.56}$$

Finally, assume, to reach a contradicton, that there exists a constant M such that

$$|\langle \mathrm{f_n}|\varphi'\rangle| \le \mathrm{M}\| \varphi \|_\infty. \tag{2.57}$$

The mean value theorem (see (2.1)) asserts that

$$|\langle \mathrm{f_n}|\varphi'\rangle| \le \mathrm{M}\| \varphi \|_\infty \le \mathrm{M}\| \varphi' \|_\infty; \tag{2.58}$$

which is, again, a desired contradiction. So ends the proof.

□

2.2 Exercise 6. Fourier series may diverge at 0

Define the Fourier coefficient $\hat{f}(n)$ *of a function* $f \in L^2(T)$ *(*T *is the unit circle) by*

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

for all $n \in \mathbf{Z}$ *(the integers). Put*

$$\Lambda_n f = \sum_{k=-n}^{n} \hat{f}(k).$$

Prove that $\{f \in L^2(T) : \lim_{n\infty} \Lambda_n f$ *exists*$\}$ *is a dense subspace of* $L^2(T)$ *of the first category.*

Proof. Let $f(\theta)$ stand for $f(e^{i\theta})$, so that $L^2(T)$ is identified with a closed subset of $L^2([-\pi, \pi])$, hence the inner product

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta. \tag{2.59}$$

We believe it is customary to write

$$\Lambda_n(f) = (f, e_{-n}) + \cdots + (f, e_n). \tag{2.60}$$

Moreover, a well known (and easy to prove) result is

$$(e_n, e_{n'}) = [n = n'], \textit{i.e.}\ \{e_n : n \in \mathbf{Z}\} \text{ is an orthormal subset of } L^2(T). \tag{2.61}$$

For the sake of brevity, we assume the isometric ($\equiv$) identification $L^2 \equiv (L^2)^*$. So,

$$\| \Lambda_n \|^2 \overset{(2.60)}{=} \| e_{-n} + \cdots + e_n \|^2 \overset{(2.61)}{=} \| e_{-n} \|^2 + \cdots + \| e_n \|^2 \overset{(2.61)}{=} 2n + 1. \tag{2.62}$$

We now assume, to reach a contradiction, that

$$B \triangleq \{f \in L^2(T) : \sup\{| \Lambda_n f | : n = 1, 2, 3, \ldots\} < \infty\} \tag{2.63}$$

is of the second category. So, the Banach-Steinhaus theorem 2.5 of [3] asserts that the sequence $\{\Lambda_n\}$ is norm-bounded; which is a desired contradiction, since

$$\| \Lambda_n \| \overset{(2.62)}{=} \sqrt{2n + 1} \underset{n\to\infty}{\longrightarrow} \infty. \tag{2.64}$$

We have just established that B is actually of the first category; and so is its subset L=$\{f \in L^2(T) : \lim_{n\longrightarrow\infty} \Lambda_n f \text{ exists}\}$. We now prove that L is nevertheless dense in $L^2(T)$. To do so, we let P be span$\{e_k : k \in Z\}$, the collection of the trignometric polynomials $p(\theta) = \sum \lambda_k e^{ik\theta}$: Combining (2.60) with (2.61) shows that $\Lambda_n(p) = \sum \lambda_k$ for almost all n. Thus,

$$P \subset L \subset L^2(T). \tag{2.65}$$

We know from the Fejér theorem (the Lebesgue variant) that P is dense in $L^2(T)$. We then conclude, with the help of (2.65), that

$$L^2(T) = \overline{P} = \overline{L}. \tag{2.66}$$

So ends the proof ∎

2.3 Exercise 9. Boundedness without closedness

Suppose $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ *are Banach spaces and*

$$\mathrm{B} : \mathrm{X} \times \mathrm{Y} \to \mathrm{Z}$$

is bilinear and continuous. Prove that there exists $\mathrm{M} < \infty$ *such that*

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y} \| \leq \mathrm{M} \| \mathrm{x} \| \| \mathrm{x} \| \quad (\mathrm{x} \in \mathrm{X}, \mathrm{y} \in \mathrm{Y}).$$

Is completeness needed here?

Proof. The answer is: No. To prove this, we only assume that X, Y, Z are normed spaces. Let (x, y) range over $\mathrm{X} \times \mathrm{Y}$: Since B is continous at the origin, there exists a positive r such that

$$\| \mathrm{x} \| + \| \mathrm{x} \| < \mathrm{r} \Rightarrow \| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| < 1. \tag{2.67}$$

Now consider all scalars s, t such that $2 \| \mathrm{x} \| < \mathrm{rs}$ and $2 \| \mathrm{y} \| < \mathrm{rt}$: The following bound

$$\| \mathrm{B}(\mathrm{x}, \mathrm{y}) \| = \mathrm{st} \| \mathrm{B}\left(\mathrm{x}/\mathrm{s}, \mathrm{y}/\mathrm{t}\right) \| \overset{2.67}{<} \mathrm{st} \tag{2.68}$$

is effective, since $\mathrm{r} > \| \mathrm{x} \| / \mathrm{s} + \| \mathrm{y} \| / \mathrm{t}$. Finally, remark that s (t) have infimum bounf $2 \| \mathrm{x} \| / \mathrm{r}$ $(2 \| \mathrm{y} \| / \mathrm{s})$ then so obtain

$$\mathrm{B}(\mathrm{x}, \mathrm{y}) \leq \frac{4}{\mathrm{r}^2} \| \mathrm{x} \| \| \mathrm{y} \|; \tag{2.69}$$

which achieves the proof.

As a concrete example, choose $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = \mathrm{C_c}(\mathbf{R})$, topologized by the supremum norm. $\mathrm{C_c}(\mathbf{R})$ is not complete (see 5.4.4 of [4]), nevertheless the bilinear product

$$\begin{array}{rlcl} \mathrm{B}: & \mathrm{C_c}(\mathbf{R})^2 & \to & \mathrm{C_c}(\mathbf{R}) \\ & (\mathrm{f}, \mathrm{g}) & \mapsto & \mathrm{f} \cdot \mathrm{g} \end{array}$$

is bounded (since $\| \mathrm{f} \cdot \mathrm{g} \|_\infty \leq \| \mathrm{f} \|_\infty \cdot \| \mathrm{g} \|_\infty$), and continuous. To show this, pick a positive scalar ε smaller than 1 then put

$$\mathrm{r} \triangleq \frac{\varepsilon}{1 + \| \mathrm{f} \|_\infty + \| \mathrm{g} \|_\infty} \tag{2.70}$$

(bear in mind that $\mathrm{r} < 1$). The Stone-Weierstrass theorem (see 7.26 of [1]) asserts the existence of (u, v) in $\mathrm{C_c}(\mathbf{R})^2$ such that

$$\| \mathrm{f} - \mathrm{u} \|_\infty + \| \mathrm{g} - \mathrm{v} \|_\infty < \mathrm{r}. \tag{2.71}$$

Next, remark that $\| \mathrm{u} \|_\infty \leq \mathrm{r} + \| \mathrm{f} \|_\infty$ and so obtain

$$\begin{aligned}
\| \mathrm{fg} - \mathrm{uv} \|_\infty &= \| (\mathrm{f} - \mathrm{u}) \cdot \mathrm{g} + \mathrm{u} \cdot (\mathrm{g} - \mathrm{v}) \|_\infty && (2.72)\\
&\leq \| \mathrm{f} - \mathrm{u} \|_\infty \cdot \| \mathrm{g} \|_\infty + \| \mathrm{u} \|_\infty \cdot \| \mathrm{g} - \mathrm{v} \|_\infty && (2.73)\\
&< \mathrm{r} \cdot \| \mathrm{g} \|_\infty + (\mathrm{r} + \| \mathrm{f} \|_\infty) \cdot \mathrm{r} && (2.74)\\
&< \mathrm{r} \cdot (\mathrm{r} + \| \mathrm{f} \|_\infty + \| \mathrm{g} \|_\infty) && (2.75)\\
&< \varepsilon. && (2.76)
\end{aligned}$$

Since ε was arbitrary, it is now established that B continuous. □

2.4 Exercise 10. Continuousness of bilinear mappings

Prove that a bilinear mapping is continuous if it is continuous at the origin $(0,0)$.

Proof. Let $(\mathrm{X}_1, \mathrm{X}_2, \mathrm{Z})$ be topological spaces and B a bilinear mapping

$$\mathrm{B} : \mathrm{X}_1 \times \mathrm{X}_2 \to \mathrm{Z} \tag{2.77}$$

From now on, $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)$ denotes an arbitrary element of $\mathrm{X}_1 \times \mathrm{X}_2$. We henceforth assume that B is continuous at the origin $(0,0)$ of $\mathrm{X}_1 \times \mathrm{X}_2$, *i.e.* given an arbitrary balanced open subset W of Z, there exists in X_i $(\mathrm{i} = 1, 2)$ a balanced open subset U_i such that

$$\mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) \subset \mathrm{W}. \tag{2.78}$$

Let $\nu_\mathrm{i}(\mathrm{x})$ denote any scalar that is greater than $\mu_\mathrm{i}(\mathrm{x}_\mathrm{i}) = \inf\{\mathrm{r} > 0 : \mathrm{x}_\mathrm{i} \in \mathrm{r} \cdot \mathrm{U}_\mathrm{i}\}$. So,

$$\begin{aligned}
\mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) &= \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{B}\left(\nu_1(\mathrm{x})^{-1}\mathrm{x}_1, \nu_2(\mathrm{x})^{-1}\mathrm{x}_2\right) && (2.79)\\
&\in \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{B}(\mathrm{U}_1 \times \mathrm{U}_2) && (2.80)\\
&\subset \nu_1(\mathrm{x})\nu_2(\mathrm{x}) \cdot \mathrm{W}. && (2.81)
\end{aligned}$$

Now pick $\mathrm{p} = (\mathrm{p}_1, \mathrm{p}_2)$ in $\mathrm{X}_1 \times \mathrm{X}_2$: It directly follows from (2.81) that

$$\begin{aligned}
\mathrm{B}(\mathrm{p}_1, \mathrm{p}_2) - \mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) &= \mathrm{B}(\mathrm{p}_1, \mathrm{p}_2 - \mathrm{x}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{x}_1, \mathrm{x}_2 - \mathrm{p}_2) + \mathrm{B}(\mathrm{p}_1 - \mathrm{x}_1, \mathrm{p}_2) && (2.82)\\
&\in \nu_1(\mathrm{p})\nu_2(\mathrm{p} - \mathrm{x}) \cdot \mathrm{W} + \nu_1(\mathrm{p} - \mathrm{x})\nu_2(\mathrm{x} - \mathrm{p}) \cdot \mathrm{W} + \nu_1(\mathrm{p} - \mathrm{x})\nu_2(\mathrm{p}) \cdot \mathrm{W}. && (2.83)
\end{aligned}$$

Let us henceforth assume that

$$\mathrm{p}_\mathrm{i} - \mathrm{x}_\mathrm{i} \in [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 2]^{-1} \cdot \mathrm{U}_\mathrm{i}; \tag{2.84}$$

which yields

$$\mu_\mathrm{i}(\mathrm{p}_\mathrm{i} - \mathrm{x}_\mathrm{i}) \leq [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 2]^{-1}. \tag{2.85}$$

Finally, combine the special case

$$\begin{aligned}
\nu_\mathrm{i}(\mathrm{p} - \mathrm{x}) &= [\mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 1]^{-1}, && (2.86)\\
\nu_\mathrm{i}(\mathrm{p}) &= \ \mu_1(\mathrm{p}) + \mu_2(\mathrm{p}) + 1 && (2.87)
\end{aligned}$$

with (2.83) and so obtain

$$\mathrm{B}(\mathrm{p}_1, \mathrm{p}_2) - \mathrm{B}(\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{W} + \mathrm{W} + \mathrm{W}. \tag{2.88}$$

W being arbitrary, we have so established the continuousness of B at $(\mathrm{p}_1, \mathrm{p}_2)$. Since $(\mathrm{p}_1, \mathrm{p}_2)$ is also arbitrary, the proof is complete. □

2.5 Exercise 12. A bilinear mapping that is not continuous

Let X be the normed space of all real polynomials in one variable, with

$$\| \mathrm{f} \| = \int_0^1 |\mathrm{f(t)}| \ \mathrm{dt}.$$

Put $\mathrm{B(f,g)} = \int_0^1 \mathrm{f(t)g(t)dt}$, *and show that* B *is a bilinear continuous functional on* $\mathrm{X} \times \mathrm{X}$ *which is separately but not continuous.*

Proof. Let f denote the first variable, g the second one. Remark that

$$|\,\mathrm{B(f,g)}\,| < \|\,\mathrm{f}\,\| \cdot \max_{[0,1]} |\,\mathrm{g}\,|\,; \tag{2.89}$$

which is sufficient (1.18 of [3]) to assert that any $\mathrm{f} \mapsto \mathrm{B(f,g)}$ is continuous. The continuity of all $\mathrm{g} \mapsto \mathrm{B(f,g)}$ follows (Put $\mathrm{C(g,f)} = \mathrm{B(f,g)}$ and proceed as above). Suppose, to reach a contradiction, that B is continuous. There so exists a positive M such that,

$$|\,\mathrm{B(f,g)}\,| < \mathrm{M}\|\,\mathrm{f}\,\|\|\,\mathrm{g}\,\|. \tag{2.90}$$

Put

$$\mathrm{f_n(x)} \triangleq 2\sqrt{\mathrm{n}} \cdot \mathrm{x^n} \in \mathbf{R}[\mathrm{x}] \qquad (\mathrm{n} = 1, 2, 3, \dots), \tag{2.91}$$

so that

$$\|\,\mathrm{f_n}\,\| = \frac{2\sqrt{\mathrm{n}}}{\mathrm{n}+1} \underset{\mathrm{n}\to\infty}{\longrightarrow} 0. \tag{2.92}$$

On the other hand,

$$\mathrm{B(f_n, f_n)} = \frac{4\mathrm{n}}{2\mathrm{n}+1} > 1. \tag{2.93}$$

Finally, we combine (2.93) and (2.90) with (2.92) and so obtain

$$1 < \mathrm{B(f_n, f_n)} < \mathrm{M}\|\,\mathrm{f_n}\,\|^2 \underset{\mathrm{n}\to\infty}{\longrightarrow} 0. \tag{2.94}$$

Our continuousness assumption is then contradicted. So ends the proof. □

2.6 Exercise 15. Baire cut

Suppose X *is an F-space and* Y *is a subspace of* X *whose complement is of the first category. Prove that* $\mathrm{Y}=\mathrm{X}$. Hint: Y *must intersect* $\mathrm{x}+\mathrm{Y}$ *for every* $\mathrm{x}\in\mathrm{X}$.

Proof. Assume Y is a subgroup of X. Under our assumptions, there exists a sequence $\{\mathrm{E_n} : \mathrm{n}=1,2,3,\dots\}$ of X such that

(i) $(\overline{\mathrm{E}}_\mathrm{n})^\circ=\emptyset$;

(ii) $\mathrm{X}\setminus\mathrm{Y}=\bigcup_{\mathrm{n}=1}^{\infty}\mathrm{E_n}$.

By (i), the complement $\mathrm{V_n}$ of $\overline{\mathrm{E}}_\mathrm{n}$ is a dense open set. Since X is an F-space, it follows from the Baire's theorem that the intersection S of the $\mathrm{V_n}$'s is dense in X: So is $\mathrm{x}+\mathrm{S}$ $(\mathrm{x}\in\mathrm{X})$. To see that, remark that

$$\mathrm{X}=\mathrm{x}+\overline{\mathrm{S}}\subset\overline{\mathrm{x}+\mathrm{S}} \tag{2.95}$$

follows from 1.3 (b) of [3]. Since S and $\mathrm{x}+\mathrm{S}$ are both dense open subsets of X, the Baire's theorem asserts that

$$\overline{(\mathrm{x}+\mathrm{S})\cap\mathrm{S}}=\mathrm{X}. \tag{2.96}$$

Thus,

$$(\mathrm{x}+\mathrm{S})\cap\mathrm{S}\neq\emptyset. \tag{2.97}$$

Moreover, it follows from (ii) that $\mathrm{X}\setminus\mathrm{Y}\subset\bigcup_\mathrm{n}\overline{\mathrm{E}}_\mathrm{n}$, *i.e.* $\mathrm{Y}\supset\mathrm{S}$. Combined with (2.97), this shows that $\mathrm{x}+\mathrm{Y}$ cuts Y. Therefore, our arbitrary x is an element of the subgroup Y. We have thus established that $\mathrm{X}\subset\mathrm{Y}$, which achieves the proof. □

2.7 Exercise 16. An elementary closed graph theorem

Suppose that X *and* K *are metric spaces, that* K *is compact, and that the graph of* $\mathrm{f} : \mathrm{X} \to \mathrm{K}$ *is a closed subset of* $\mathrm{X} \times \mathrm{K}$. *Prove that* f *is continuous (This is an analogue of Theorem 2.15 but much easier.) Show that compactness of* K *cannot be omitted from the hypothese, even when* X *is compact.*

Proof. Choose a sequence $\{\mathrm{x_n} : \mathrm{n} = 1, 2, 3, \ldots\}$ whose limit is an arbitrary a. By compactness of K, the graph G of f contains a subsequence $\{(\mathrm{x}_{\rho(\mathrm{n})}, \mathrm{f}(\mathrm{x}_{\rho(\mathrm{n})}))\}$ of $\{(\mathrm{x_n}, \mathrm{f}(\mathrm{x_n}))\}$ that converges to some (a, b) of $\mathrm{X} \times \mathrm{K}$. G is closed; therefore, $\{(\mathrm{x}_{\rho(\mathrm{n})}, \mathrm{f}(\mathrm{x}_{\rho(\mathrm{n})}))\}$ converges in G. So, $\mathrm{b} = \mathrm{f}(\mathrm{a})$; which establishes that f is sequentially continuous. Since X is metrizable, f is also continuous; see [A6] of [3]. So ends the proof.

To show that compactness cannot be omitted from the hypotheses, we showcase the following counterexample,

$$
\begin{aligned}
\mathrm{f} : [0, \infty) &\to [0, \infty) \\
\mathrm{x} &\mapsto \begin{cases} 1/\mathrm{x} & (\mathrm{x} > 0) \\ 0 & (\mathrm{x} = 0). \end{cases}
\end{aligned}
\tag{2.98}
$$

Clearly, f has a discontinuity at 0. Nevertheless the graph G of f is closed. To see that, first remark that

$$
\mathrm{G} = \{(\mathrm{x}, 1/\mathrm{x}) : \mathrm{x} > 0\} \cup \{(0, 0)\}. \tag{2.99}
$$

Next, let $\{(\mathrm{x_n}, 1/\mathrm{x_n})\}$ be a sequence in $\mathrm{G}_+ = \{(\mathrm{x}, 1/\mathrm{x}) : \mathrm{x} > 0\}$ that converges to (a, b). To be more specific: $\mathrm{a} = 0$ contradicts the boundedness of $\{(\mathrm{x_n}, 1/\mathrm{x_n})\}$: a is necessarily positive and $\mathrm{b} = 1/\mathrm{a}$, since $\mathrm{x} \mapsto 1/\mathrm{x}$ is continuous on R_+. This establishes that $(\mathrm{a}, \mathrm{b}) \in \mathrm{G}_+$, hence the closedness G_+. Finally, we conclude that G is closed, as a finite union of closed sets. □

Chapter 3

Convexity

3.1 Exercise 3.

Suppose X is a real vector space (without topology). Call a point $x_0 \in A \subset X$ *an internal point of* A *if* $A - x_0$ *is an absorbing set.*

(a) *Suppose* A *and* B *are disjoint convex sets in* X*, and* A *has an internal point. Prove that there is a nonconstant linear functional* Λ *such that* $\Lambda(A) \cap \Lambda(B)$ *contains at most one point. (The proof is similar to that of Theorem 3.4)*

(b) *Show (with* $X = \mathbf{R}^2$*, for example) that it may not possible to have* $\Lambda(A)$ *and* $\Lambda(B)$ *disjoint, under the hypotheses of (a).*

Proof. Take A and B as in (a); the trivial case $B = \emptyset$ is discarded. Since $A - x_0$ is absorbing, so is its convex superset $C = A - B - x_0 + b_0$ ($b_0 \in B$). Note that C contains the origin. Let p be the Minkowski functional of C. Since A and B are disjoint, $b_0 - x_0$ is not in C, hence $p(b_0 - x_0) \geq 1$. We now proceed as in the proof of the Hahn-Banach theorem 3.4 of [3] to establish the existence of a linear functional $\Lambda : X \to \mathbf{R}$ such that

$$\Lambda \leq p \tag{3.1}$$

and

$$\Lambda(b_0 - x_0) = 1. \tag{3.2}$$

Then

$$\Lambda a - \Lambda b + 1 = \Lambda(a - b + b_0 - x_0) \leq p(a - b + b_0 - x_0) \leq 1 \quad (a \in A, b \in B). \tag{3.3}$$

Hence

$$\Lambda a \leq \Lambda b. \tag{3.4}$$

We now prove that $\Lambda(A) \cap \Lambda(B)$ contains at most one point. Suppose, to reach a contradiction, that this intersection contains y_1 and y_2. There so exists (a_i, b_i) in $A \times B$ ($i = 1, 2$) such that

$$\Lambda a_i = \Lambda b_i = y_i. \tag{3.5}$$

Assume without loss of generality that $y_1 < y_2$. Then,

$$2 \cdot y_1 = \Lambda b_1 + \Lambda b_1 < \Lambda(a_1 + a_2) = (y_1 + y_2) \quad . \tag{3.6}$$

Remark that $a_3 = \frac{1}{2}(a_1 + a_2)$ lies in the convex set A. This implies

$$\Lambda b_1 \overset{(3.6)}{<} \Lambda a_3 \overset{(3.4)}{\leq} \Lambda b_1 \quad ; \tag{3.7}$$

which is a desired contradiction. (a) is so proved and we now deal with (b).

From now on, the space X is $\mathbf{R}^2$. Fetch

$$S_1 \triangleq \{(x, y) \in \mathbf{R}^2 : x \leq 0, y \geq 0\}, \tag{3.8}$$

$$S_2 \triangleq \{(x, y) \in \mathbf{R}^2 : x > 0, y > 0\}, \tag{3.9}$$

$$A \triangleq S_1 \cup S_2, \tag{3.10}$$

$$B \triangleq X \setminus A. \tag{3.11}$$

Pick (x_i, y_i) in S_i. Let t range over the unit interval, and so obtain

$$t \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (1-t) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} t \cdot x_1 + (1-t) \cdot x_2 \\ t \cdot y_1 + (1-t) \cdot y_2 \end{pmatrix} \in \mathbf{R} \times \mathbf{R}_+ \subset A. \tag{3.12}$$

Thus, every segment that has an extremity in S_1 and the other one in S_2 lies in A. Moreover, each S_i is convex. We can now conclude that A is so. The convexity of B is proved in the same manner. Furthermore, A hosts a non degenerate triangle, *i.e.* A° is nonempty[1]: A contains an internal point.

Let L be a vector line of $\mathbf{R}^2$. In other words, L is the null space of a linear functional $\Lambda : \mathbf{R}^2 \to \mathbf{R}$ (to see this, take some nonzero u in $L^\perp$ and set $\Lambda x = (x, u)$ for all x in $\mathbf{R}^2$). One easily checks that both A and B cut L. Hence

$$\Lambda(L) = \{0\} \subset \Lambda(A) \cap \Lambda(B) \neq \emptyset \quad . \tag{3.13}$$

So ends the proof. □

3.2 Exercise 11. Meagerness of the polar

Let X be an infinite-dimensional Fréchet space. Prove that X^*, with its weak-topology, is of the first category in itself.*

This is actually a consequence of the below lemma, which we prove first. The proof that X^* is of the first category in itself comes right after, as a corollary.

Lemma. *f X is an infinite dimensional topological vector space whose dual X^* separates points on X, then the polar*

$$K_A \triangleq \{\Lambda \in X^* : |\Lambda| \leq 1 \ on\ A\} \tag{3.14}$$

of any absorbing subset A is a weak-closed set that has empty interior.*

[1]For a immediate proof of this, remark that a triangle boundary is compact/closed and apply [1.10] or 2.5 of [2].

Proof. Let x range over X. The linear form $\Lambda \mapsto \Lambda x$ is weak*-continuous; see 3.14 of [3]. Therefore, $P_x = \{\Lambda \in X^* : |\Lambda x| \leq 1\}$ is weak*- closed: As the intersection of $\{P_a : a \in A\}$, K_A is also a weak*-closed set. We now prove the second half of the statement.

From now on, X is assumed to be endowed with its weak topology: X is then locally convex, but its dual space is still X^* (see 3.11 of [3]). Put

$$W_{F,x} \triangleq \bigcap_{x \in F} \{\Lambda \in X^* : |\Lambda x| < r_x\} \qquad (r_x > 0) \tag{3.15}$$

where F runs through the nonempty finite subsets of X. Clearly, the collection of all such W is a local base of X^*. Pick one of those W and remark that the following subspace

$$M \triangleq \mathrm{span}(F) \tag{3.16}$$

is finite dimensional. Assume, to reach a contradiction, that $A \subset M$. So, every x lies in $t_x M = M$ for some $t_x > 0$, since A is absorbing. As a consequence, X is the finite dimensional space M, which is a desired contradiction. We have just established that $A \not\subset M$: Now pick a in $A \setminus M$ and so conclude that

$$b \triangleq \frac{a}{t_a} \in A \tag{3.17}$$

Remark that $b \notin M$ (otherwise, $a = t_a b \in t_a M = M$ would hold) and that M, as a finite dimensional space, is closed (see 1.21 (b) of [3] for a proof): By the Hahn-Banach theorem 3.5 of [3], there exists Λ_a in X^* such that

$$\Lambda_a b > 2 \tag{3.18}$$

and

$$\Lambda_a(M) = \{0\}. \tag{3.19}$$

The latter equality implies that Λ_a vanishes on F; hence Λ_a is an element of W. On the other hand, given an arbitrary $\Lambda \in K_A$, the following inequalities

$$|\Lambda_a b + \Lambda b| \geq 2 - |\Lambda b| > 1. \tag{3.20}$$

show that $\Lambda + \Lambda_a$ is not in K_A. We have thus proved that

$$\Lambda + W \not\subset K_A. \tag{3.21}$$

Since W and Λ are both arbitrary, this achieves the proof. □

We now give a proof of the original statement.

Corollary. *If X is an infinite-dimensional Fréchet space, then X^* is meager in itself.*

Proof. From now on, X^* is only endowed with its weak*-topology. Let d be an invariant distance that is compatible with the topology of X, so that the following sets

$$B_n \triangleq \{x \in X : d(0, x) < 1/n\} \qquad (n = 1, 2, 3, \dots) \tag{3.22}$$

form a local base of X. If Λ is in X^*, then

$$|\Lambda| \leq m \text{ on } B_n \tag{3.23}$$

for some $(\mathrm{n},\mathrm{m}) \in \{1,2,3,\dots\}^2$; see 1.18 of [3]. Hence, X^* is the countable union of all

$$\mathrm{m}\cdot\mathrm{K_n} \qquad (\mathrm{m},\mathrm{n}=1,2,3,\dots), \tag{3.24}$$

where $\mathrm{K_n}$ is the polar of $\mathrm{B_n}$. Clearly, showing that every $\mathrm{m}\cdot\mathrm{K_n}$ is nowhere dense is now sufficient. To do so, we use the fact that X^* separates points; see 3.4 of [3]. As a consequence, the above lemma implies

$$\left(\overline{\mathrm{K_n}}\right)^\circ = (\mathrm{K_n})^\circ = \emptyset. \tag{3.25}$$

Since the multiplication by m is an homeomorphism (see 1.7 of [3]), this is equivalent to

$$\left(\,\overline{\mathrm{m}\cdot\mathrm{K_n}}\,\right)^\circ = \mathrm{m}\cdot(\mathrm{K_n})^\circ = \emptyset. \tag{3.26}$$

So ends the proof. □

Chapter 4

Banach Spaces

Throughout this set of exercises, X and Y denote Banach spaces, unless the contrary is explicitly stated.

4.1 Exercise 1. Basic results

Let φ be the embedding of X *into* X^{**} *decribed in Section 4.5. Let τ be the weak topology of* X*, and let σ be the weak***- topology of* X^{**}*- the one induced by* X^**.*

(a) *Prove that φ is an homeomorphism of* (X, τ) *onto a dense subspace of* (X^{**}, σ)*.*

(b) *If* B *is the closed unit ball of* X*, prove that* $\varphi(B)$ *is σ-dense in the closed unit ball of* X^{**}*. (Use the Hahn-Banach separation theorem.)*

(c) *Use (a), (b), and the Banach-Alaoglu theorem to prove that* X *is reflexive if and only if* B *is weakly compact.*

(d) *Deduce from (c) that every norm-closed subspace of a reflexive space is reflexive.*

(e) *If* X *is reflexive and* Y *is a closed subspace of* X*, prove that* X/Y *is reflexive.*

(f) *Prove that* X *is reflexive if and only* X^* *if reflexive.*
Suggestion: One half follows from (c); for the other half, apply (d) to the subspace $\varphi(X)$ *of* X^{**}*.*

Proof. Let ψ be the isometric embedding of X^* into X^{***}. The dual space of (X^{**}, σ) is then $\psi(X^*)$.

It is sufficient to prove that

$$\varphi^{-1} : \varphi(X) \to X \tag{4.1}$$

$$\varphi(x) \mapsto x \tag{4.2}$$

is an homeomorphism (with respect to τ and σ). We first consider

$$V \triangleq \{x^{**} \in X^{**} : |\langle x^{**}|\psi x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0); \tag{4.3}$$

$$U \triangleq \{x \in X : |\langle x|x^*\rangle| < r\} \qquad (x^* \in X^*, r > 0). \tag{4.4}$$

and remark that the so defined V 's (respectively U 's) shape a local subbase $\mathscr{S}_\sigma$ (respectively $\mathscr{S}_\tau$) of σ (respectively τ). We now observe that

$$U = \varphi^{-1}(V \cap \varphi(X)) = \varphi^{-1}(V) \cap X \quad (V \in \mathscr{S}_\sigma,\ U \in \mathscr{S}_\tau) \quad , \tag{4.5}$$

since φ^{-1} is one-to-one. This remains true whether we enrich each subbase $\mathscr{S}$ with all finite intersections of its own elements, for the same reason. It then follows from the very definition of a local base of a weak / weak*-topology that φ^{-1} and its inverse φ are continuous.

The second part of (a) is a special case of [3.5] and is so proved. First, it is evident that

$$\overline{\varphi(X)}_\sigma \subset X^{**} \quad . \tag{4.6}$$

and we now assume- to reach a contradiction- that (X^{**}, σ) contains a point z^{**} outside the σ-closure of $\varphi(X)$. By [3.5], there so exists y^* in X^* such that

$$\langle \varphi x, \psi y^* \rangle = \langle y^*, \varphi x \rangle = \langle x, y^* \rangle = 0 \quad (x \in X) \quad ; \tag{4.7}$$

$$\langle z^{**}, \psi y^* \rangle = 1 \tag{4.8}$$

(4.7) forces y^* to be a the zero of X^*. The functional ψy^* is then the zero of X^{***}: (4.8) is contradicted. Statement (a) is so proved; we next deal with (b).

The unit ball B^{**} of X^{**} is weak*-closed, by (c) of [4.3]. On the other hand,

$$\varphi(B) \subset B^{**} \quad , \tag{4.9}$$

since φ is isometric. Hence

$$\overline{\varphi(B)}_\sigma \subset \overline{(B^{**})}_\sigma = B^{**} \quad . \tag{4.10}$$

Now suppose, to reach a contradiction, that $B^{**} \setminus \overline{\varphi(B)}_\sigma$ contains a vector z^{**}. By [3.7], there exists y^* in X^* such that

$$|\psi y^*| \leq 1 \quad \text{on } \overline{\varphi(B)}_\sigma \quad ; \tag{4.11}$$

$$\langle z^{**}, \psi y^* \rangle > 1 \quad . \tag{4.12}$$

It follows from (4.11) that

$$|\psi y^*| \leq 1 \text{ on } \varphi(B), \ \textit{i.e.} |y^*| \leq 1 \text{ on } B \quad . \tag{4.13}$$

We have so proved that

$$y^* \in B^* \quad . \tag{4.14}$$

Since z^{**} lies in B^{**}, it is now clear that

$$|\langle z^{**}, \psi y^* \rangle| \leq 1 \quad ; \tag{4.15}$$

what it contradicts (4.12), and thus proves (b). We now aim at (c).

It follows from (a) that

(4.16) B is weakly compact if and only if $\varphi(B)$ is weak*-compact.

If B is weakly compact, then $\varphi(B)$ is weak*-closed. So,

$$\varphi(B) = \overline{\varphi(B)}_\sigma \overset{(b)}{=} B^{**} \quad . \tag{4.17}$$

φ is therefore onto, *i.e.* X is reflexive.
Conversely, keep φ as onto: one easily checks that $\varphi(\mathrm{B}) = \mathrm{B}^{**}$. The image $\varphi(\mathrm{B})$ is then weak*-compact by (c) of [4.3]. The conclusion now follows from (4.16).

Next, let X be a reflexive space X, whose closed unit ball is B. Let Y be a norm-closed subspace of X: Y is then weakly closed (*cf.* [3.12]). On the other hand, it follows from (c) that B is weakly compact. We now conclude that the closed unit ball $\mathrm{B} \cap \mathrm{Y}$ of Y is weakly compact. We again use (c) to conclude that Y is reflexive. (d) is therefore established. Now proceed to (e).

Let $\equiv$ stand for "isometrically isomorphic" and apply twice [4.9] to obtain, first

$$(\mathrm{X}/\mathrm{Y})^* \equiv \mathrm{Y}^{\perp} \quad , \tag{4.18}$$

next,

$$(\mathrm{X}/\mathrm{Y})^{**} \equiv (\mathrm{Y}^{\perp})^* \equiv \mathrm{X}^{**}/(\mathrm{Y}^{\perp})^{\perp} \equiv \mathrm{X}/\mathrm{Y} \quad . \tag{4.19}$$

Combining (4.18) with (4.19) makes (e) to hold.

It remains to prove (f). To do so, we state the following trivial lemma (L)

Given a reflexive Banach space Z*, the weak***-topology of* Z^* *is its weak one.*

Assume first that X is reflexive. Since B^* is weak* compact, by (c) of [4.3], (L) implies that B^* is also weakly compact. Then (c) turns X^* into a reflexive space.

Conversely, let X^* be reflexive. What we have just proved that makes X^{**} reflexive. On the other hand, $\varphi(\mathrm{X})$ is a norm-closed subspace of X^{**}; *cf.* [4.5]. Hence $\varphi(\mathrm{X})$ is reflexive, by (d). It now follows from (c) that $\mathrm{B}^{**} \cap \varphi(\mathrm{X})$ is weakly compact, *i.e.* weak*-compact (to see this, apply (L) with $\mathrm{Z} = \mathrm{X}^*$).

By (a), B is therefore weakly compact, *i.e.* X is reflexive; see (c). So ends the proof. □

4.13 Exercise 13. Operator compactness in a Hilbert space

4.15 Exercise 15. Hilbert-Schmidt operators

Chapter 5

Distributions

5.1 Exercise 1. Test functions are almost polynomial

5.6 Exercise 6. Around the supports of some distributions

5.9 Exercise 9. Convergence in $\mathscr{D}(\Omega)$ vs. convergence in $\mathscr{D}'(\Omega)$

(a) *Prove that a set* $\mathrm{E} \subset \mathscr{D}(\Omega)$ *is bounded if and only if*

$$\sup\{|\Lambda\varphi| : \varphi \in \mathrm{E}\} < \infty$$

for every $\Lambda \in \mathscr{D}(\Omega)$.

(b) *Suppose* $\{\varphi_j\}$ *is a sequence in* $\mathscr{D}(\Omega)$ *such that* $\{\Lambda\varphi_j\}$ *is a bounded sequence of numbers, for every* $\Lambda \in \mathscr{D}'(\Omega)$. *Prove that some subsequence of* $\{\varphi_j\}$ *converges, in the topology of* $\mathscr{D}(\Omega)$.

(c) *Suppose* $\{\Lambda_j\}$ *is a sequence in* $\mathscr{D}'(\Omega)$ *such that* $\{\Lambda_j\,\varphi\}$ *is bounded, for every* $\varphi \in \mathscr{D}(\Omega)$. *Prove that some subsequence of* $\{\Lambda_j\}$ *converges in* $\mathscr{D}'(\Omega)$ *and that the convergence is uniform on every bounded subset of* $\mathscr{D}(\Omega)$. *Hint: By the Banach-Steinhaus theorem, the restrictions of the* Λ_j *to* $\mathscr{D}_K$ *are equicontinuous. Apply Ascoli's theorem.*

PROOF. Since $\mathscr{D}(\Omega)$ locally convex space (see (b) of [6.4]), [3.18] states that E is bounded if and only if it is weakly bounded. That is (a).

To prove (b), we first use (a) to conclude that $\mathrm{E} = \{\varphi_j : j \in \mathbf{N}\}$ is bounded: so is $\overline{\mathrm{E}}$. By (c) of [6.5], there exists some $\mathscr{D}_K$ that contains $\overline{\mathrm{E}}$. Since $\mathscr{D}_K$ has the Heine-Borel property (see [1.46]), $\overline{\mathrm{E}}$ is τ_K-compact. Apply [A4] with the metrizable space $\mathscr{D}_K$ (see [1.46]) to conclude that $\overline{\mathrm{E}}$ has a τ_K limit point. It then follows from (b) of [6.5] that (b) holds.

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