Problem Set - 1

Short Course on Quantum Computing

Problem 1: Polarizers, Photons and single qubits

The following questions are based on a single qubit representing the polarization state of a photon. The basis states are represented by $|0\rangle$ for vertically polarized light and $|1\rangle$ for horizontally polarized light respectively.

a) What is the angle made by the following polarization states with respect to the vertical axis?

   i. $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

   ii. $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

b) The polarization state of a photon is given by $\frac{4}{5}|0\rangle + x|1\rangle$, what is the allowed value(s) of x?

c) Consider photons of an unknown polarization passing through a polarizer **A**, oriented along the vertical direction. The photons that are transmitted through **A** subsequently pass through a polarizer **B** oriented along the horizontal direction.

   i. How many photons will be transmitted through polarizer **B**?

   ii. If a third polarizer **C** is inserted between **A** and **B**. What is the angle the orientation of C should make with the vertical axis such that the probability of photons being transmitted through polarizer **B**.

   iii. In the previous problem, maximum possible transmission probability for the three polarizer system?

Problem 2: Unitary transformations and orthonormal bases

The following questions are based on the properties of single qubit unitary transformations.

a) Given a single qubit unitary transformation U, such that

$$U|0\rangle = |a\rangle$$
$$U|1\rangle = |b\rangle$$

   Show that $\{|a\rangle, |b\rangle\}$ is an orthonormal basis.

b) If a qubit in the state $|\psi\rangle$ is measured in the $\{|a\rangle, |b\rangle\}$ basis, what are the possible measurement outcomes and what are the corresponding probabilities?

c) If a qubit in the state $U^\dagger|\psi\rangle$ is measured in the standard basis, what are the outcomes and the corresponding probabilities?

d) Are the answers for parts b and c related? If so, how are they related?

e) If a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, is measured in the Hadamard basis. What are the possible measurement outcomes and their corresponding probabilities?

Problem 3: The Bell basis

Consider the following quantum circuit:

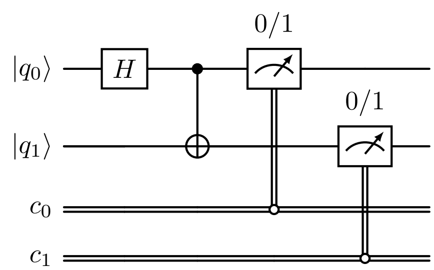

Figure 1: Circuit diagram for Problem 3

Both $|q_0\rangle$ and $|q_1\rangle$ can either be $|0\rangle$ or $|1\rangle$. The measurement is done in the computational basis and the measurement outcome is stored in the classical bits c_0 and c_1

a) Find the output state (before measurement) of the above quantum circuit for the input combinations:

   i. $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |0\rangle$
   ii. $|q_0\rangle = |1\rangle$ and $|q_1\rangle = |0\rangle$
   iii. $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |1\rangle$
   iv. $|q_0\rangle = |1\rangle$ and $|q_1\rangle = |1\rangle$

b) What are the possible measurement outcomes (c_0c_1 values) when the initial states are $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |1\rangle$?

c) Show that the four vectors obtained from part a) form a two-qubit basis.

d) If two-qubits in the state $|\Psi\rangle = |+\rangle|-\rangle$ is measured with the two qubit basis defined here, What are the possible measurement outcomes and their corresponding probabilities?

e) If two-qubits in the state $|\Phi\rangle = |0\rangle|0\rangle$ is measured with respect to the new basis. Answer the following questions with explanations.

   i. Is the initial state entangled?
   ii. Is the state of the system after measurement entangled?

Problem 4: Multi-qubit entanglement

In the following multi-qubit states, identify which qubits are entangled to one another:

a) $\frac{1}{\sqrt{2}}|010\rangle + \frac{1}{\sqrt{2}}|111\rangle$

b) $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$

c) $\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$

d) $\frac{1}{2}|0000\rangle + \frac{1}{2}|0011\rangle + \frac{1}{2}|1100\rangle + \frac{1}{2}|1111\rangle$

e) $\frac{1}{2}|0000\rangle + \frac{1}{2}|0101\rangle + \frac{1}{2}|1010\rangle + \frac{1}{2}|1111\rangle$