Coordinates and Transformations

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Robotics and Intelligent Systems
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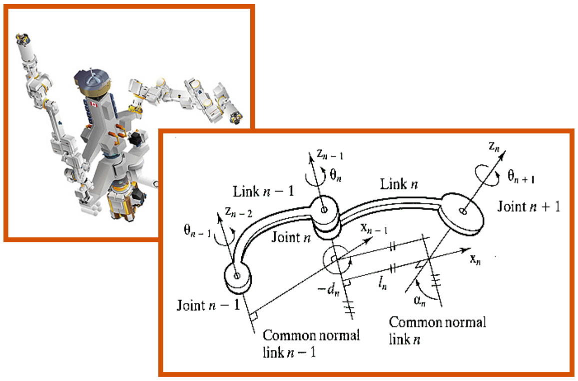
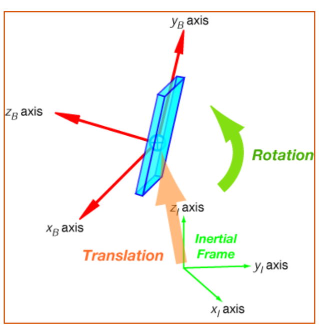
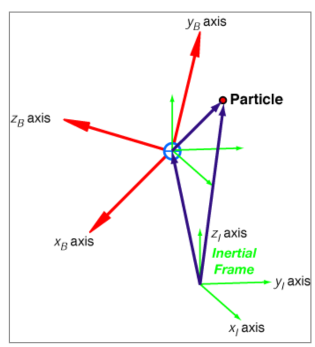


- **Homogeneous Coordinates**
- **Denavit-Hartenberg Transformation**
- **Forward and Inverse Transformations**

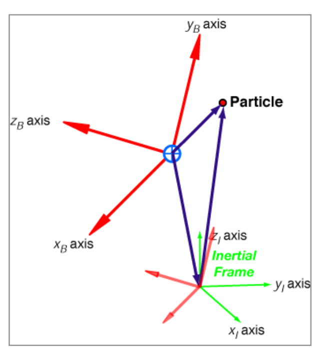


Measurement of Position in Alternative Frames



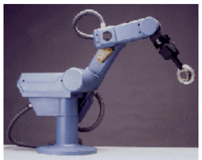
Inertial-axis view

$$\mathbf{r}_{particle_I} = H_B^I \Delta\mathbf{r}_B + \mathbf{r}_{body\ origin_I}$$



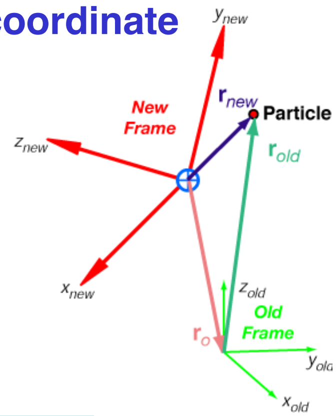
Body-axis view

$$\mathbf{r}_{particle_B} = H_I^B \Delta\mathbf{r}_I + \mathbf{r}_{inertial\ origin_B}$$



Rotation + Translation
("Forward Kinematics")

- **Expression of a vector in a new coordinate frame**
  - Displaced from old frame
  - Rotated w.r.t. old frame



$$\mathbf{r}_{new} = H_{old}^{new}\mathbf{r}_{old} + \mathbf{r}_{old_{new}}$$

Rotation matrix

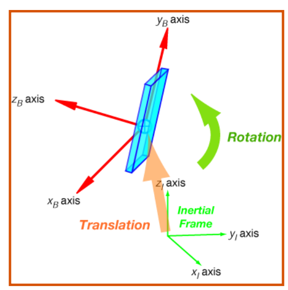
$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- **Augmented vector**
  - Concatenate a "1" to r

$$\mathbf{s} = \begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \equiv$$

Homogeneous coordinate

Homogeneous Transformation Matrix



$$\mathbf{s}_{new} = \left[\begin{array}{c|c} \begin{pmatrix} \text{Rotation} \\ \text{Matrix} \end{pmatrix}_{old}^{new} & \begin{pmatrix} \text{Location} \\ \text{of Old} \\ \text{Origin} \end{pmatrix}_{new} \\ \hline \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & 1 \end{array}\right] \mathbf{s}_{old} = \mathbf{A}_{old}^{new}\,\mathbf{s}_{old}$$

$$(4\times1)_{new} = \begin{bmatrix} (3\times3) & (3\times1) \\ (1\times3) & (1\times1) \end{bmatrix} (4\times1)_{old} = [(4\times4)]\,(4\times1)_{old}$$

Homogeneous Transformation

- Rotation <u>and</u> translation can be expressed in terms of homogeneous coordinates
  - Single matrix-vector product produces rotation and transformation

$$\mathbf{s}_{new} = \begin{bmatrix} H_{old}^{new} & \mathbf{r}_{old_{new}} \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & 1 \end{bmatrix} \mathbf{s}_{old} = \mathbf{A}\ \mathbf{s}_{old}$$

- or

$$\begin{bmatrix} x \\ y \\ x \\ 1 \end{bmatrix}_{new} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & x_o \\ h_{21} & h_{22} & h_{23} & y_o \\ h_{31} & h_{32} & h_{33} & z_o \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{old}$$

Equivalent Scalar Equations for Homogeneous Transformation

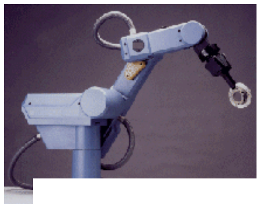
$$\mathbf{s}_{new} = \mathbf{A}_{old}^{new}\ \mathbf{s}_{old}$$

Matrix-Vector Multiplication

$$\begin{bmatrix} x \\ y \\ x \\ 1 \end{bmatrix}_{new} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & x_o \\ h_{21} & h_{22} & h_{23} & y_o \\ h_{31} & h_{32} & h_{33} & z_o \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{old}$$

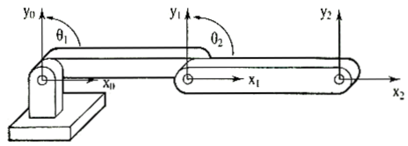
Individual Operations

$$\begin{aligned} x_{new} &= h_{11}x_{old} + h_{12}y_{old} + h_{13}z_{old} + x_o \\ y_{new} &= h_{21}x_{old} + h_{22}y_{old} + h_{23}z_{old} + y_o \\ z_{new} &= h_{31}x_{old} + h_{32}y_{old} + h_{33}z_{old} + z_o \\ &---- \\ 1 &= 1 \end{aligned}$$

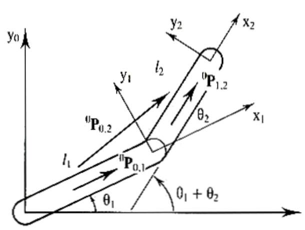


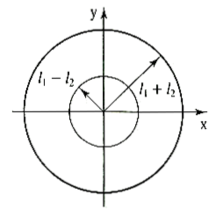
Two-Link/Three-Joint Manipulator

Parallel Rotation Axes



Manipulator in zero position





Workspace

Assignment of coordinate frames

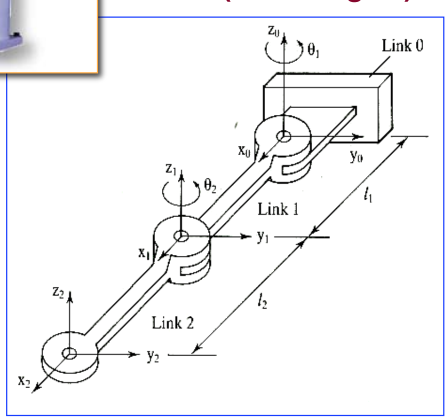
Parameters and Variables for 2-link manipulator

- Link lengths (fixed)
- Joint angles (variable)

McKerrow, 1991

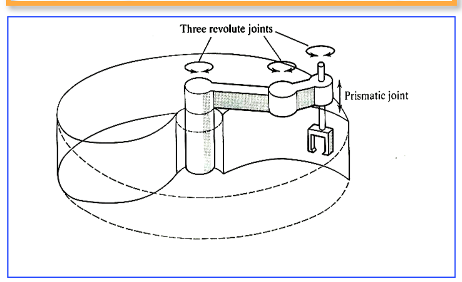
Four-Joint (SCARA*) Manipulator

Arm with Three Revolute Link Variables (Joint Angles)



Operation

http://www.youtube.com/watch?v=3-sbtCCyJXo

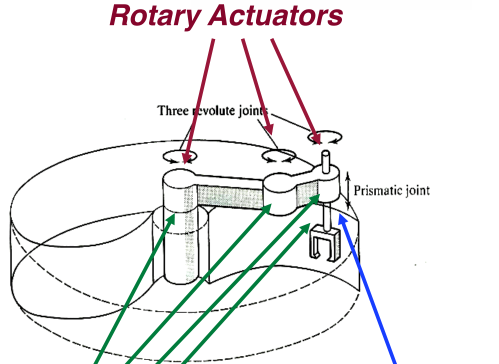


McKerrow, 1991

**Selective Compliant Articulated Robot Arm*

Link Variables Must Be Actuated and Observed for Control

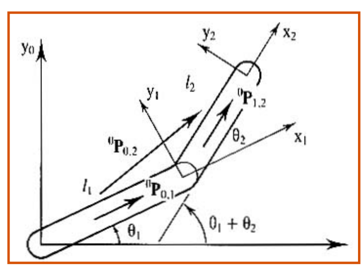
- Frames of Reference for Actuation and Control
  - World coordinates
  - Actuator coordinates
  - Joint coordinates
  - Tool coordinates



Series of Homogeneous Transformations

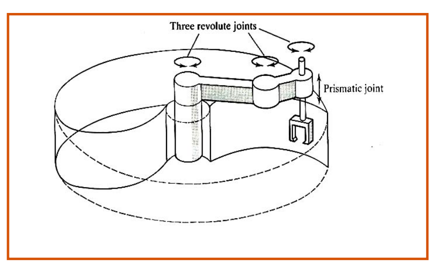
Two serial transformations can be combined in a single transformation

$$\mathbf{s}_2 = \mathbf{A}_1^2 \mathbf{A}_0^1 \, \mathbf{s}_0 = \mathbf{A}_0^2 \, \mathbf{s}_0$$



Four transformations for SCARA robot

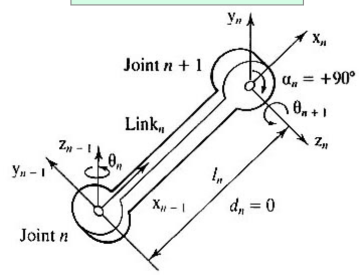
$$\mathbf{s}_4 = \mathbf{A}_3^4 \mathbf{A}_2^3 \mathbf{A}_1^2 \mathbf{A}_0^1 \, \mathbf{s}_0 = \mathbf{A}_0^4 \, \mathbf{s}_0$$



Transformation Between Robotic Joints

- Four transformations occur in going from one joint to the next:
  - Rotation about α
  - Translation along d
  - Translation along l
  - Rotation about θ

Type 2 Link

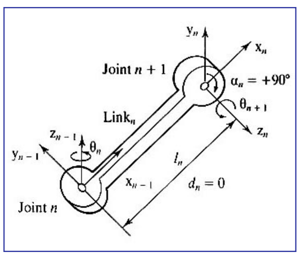


$$\mathbf{s}_{n+1} = \mathbf{A}_3^{n+1}\mathbf{A}_2^{3}\mathbf{A}_1^{2}\mathbf{A}_n^{1}\ \mathbf{s}_n = \mathbf{A}_n^{n+1}\ \mathbf{s}_n$$
$$= \mathbf{A}_\theta\mathbf{A}_d\mathbf{A}_l\mathbf{A}_\alpha\ \mathbf{s}_n = \mathbf{A}_n^{n+1}\ \mathbf{s}_n$$

- ... axes for each transformation (along or around) must be specified

4th 3rd 2nd 1st

$$\mathbf{s}_{n+1} = \mathbf{A}\left(z_{n-1},\theta_n\right)\mathbf{A}\left(z_{n-1},d_n\right)\mathbf{A}\left(x_{n-1},l_n\right)\mathbf{A}\left(x_{n-1},\alpha_n\right)\mathbf{s}_n = \mathbf{A}_n^{n+1}\ \mathbf{s}_n$$



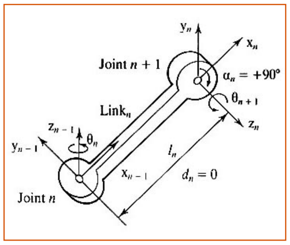
Denavit-Hartenberg Representation of Joint-Link-Joint Transformation

- Like Euler angle rotation, transformational effects of the 4 link parameters are defined in a specific application sequence (right to left): $\{\theta, d, l, \alpha\}$

- <u>4 link parameters</u>
  - Angle between 2 links, θ (revolute)
  - Distance (offset) between links, d (prismatic)
  - Length of the link between rotational axes, l, along the common normal (prismatic)
  - Twist angle between axes, α (revolute)

$$\mathbf{A}_n = \mathbf{A}\left(z_{n-1},\theta_n\right)\mathbf{A}\left(z_{n-1},d_n\right)\mathbf{A}\left(x_{n-1},l_n\right)\mathbf{A}\left(x_{n-1},\alpha_n\right)$$
$$= \mathrm{Rot}(z_{n-1},\theta_n)\ \mathrm{Trans}(z_{n-1},d_n)\ \mathrm{Trans}(x_{n-1},l_n)\ \mathrm{Rot}(x_{n-1},\alpha_n)$$
$$\equiv {}^{n}\mathbf{T}_{n+1} \quad \text{in some references (e.g., McKerrow, 1991)}$$

Denavit-Hartenberg Demo
http://www.youtube.com/watch?v=10mUtjfGmzw



Four Transformations from One Joint to the Next

(Single Link)

Rotation of θ_n about the z_{n-1} axis

$$\mathrm{Rot}(z_{n-1},\theta_n)=\begin{bmatrix}\cos\theta_n & -\sin\theta_n & 0 & 0\\ \sin\theta_n & \cos\theta_n & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Translation of d_n along the z_{n-1} axis

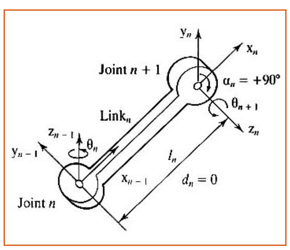
$$\mathrm{Trans}(z_{n-1},d_n)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d_n\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Translation of l_n along the x_{n-1} axis

$$\mathrm{Trans}(x_{n-1},l_n)=\begin{bmatrix}1 & 0 & 0 & l_n\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Rotation of α_n about the x_{n-1} axis

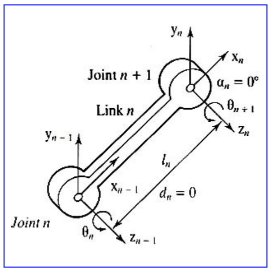
$$\mathrm{Rot}(x_{n-1},\alpha_n)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha_n & -\sin\alpha_n & 0\\ 0 & \sin\alpha_n & \cos\alpha_n & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Denavit-Hartenberg Representation of Joint-Link-Joint Transformation

$$\mathbf{A}_n=\begin{bmatrix}\cos\theta_n & -\sin\theta_n & 0 & 0\\ \sin\theta_n & \cos\theta_n & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & d_n\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & l_n\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha_n & -\sin\alpha_n & 0\\ 0 & \sin\alpha_n & \cos\alpha_n & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{A}_n=\begin{bmatrix}\cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & l_n\cos\theta_n\\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & l_n\sin\theta_n\\ 0 & \sin\alpha_n & \cos\alpha_n & d_n\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Example: Denavit-Hartenberg Representation of Joint-Link-Joint Transformation for Type 1 Link

Joint Variable = θ_n

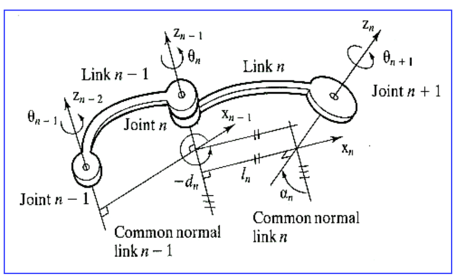
$$\mathbf{A}_n = \begin{bmatrix} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & l_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & l_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\theta = \text{variable}$
$d = 0\text{ m}$
$l = 0.25\text{ m}$
$\alpha = 90\text{ deg}$

$$\mathbf{A}_n = \begin{bmatrix} \cos\theta_n & 0 & \sin\theta_n & 0.25\cos\theta_n \\ \sin\theta_n & 0 & -\cos\theta_n & 0.25\sin\theta_n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\theta \triangleq 30\text{ deg}$
$d = 0\text{ m}$
$l = 0.25\text{ m}$
$\alpha = 90\text{ deg}$

$$\mathbf{A}_n = \begin{bmatrix} 0.866 & 0 & 0.5 & 0.217 \\ 0.5 & 0 & -0.866 & 0.125 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward and Inverse Transformations

Forward transformation through links requires pre-multiplication of matrices

$$\mathbf{s}_1 = \mathbf{A}_0^1\mathbf{s}_0 \quad ; \quad s_2 = \mathbf{A}_1^2\mathbf{s}_1 = \mathbf{A}_1^2\mathbf{A}_0^1\mathbf{s}_0 = \mathbf{A}_0^2\mathbf{s}_0$$

Reverse transformation uses the matrix inverse

$$\mathbf{s}_0 = \left(\mathbf{A}_0^2\right)^{-1}\mathbf{s}_2 = \mathbf{A}_2^0\mathbf{s}_2 = \mathbf{A}_1^0\mathbf{A}_2^1\mathbf{s}_2$$

...but Homogeneous Transformation Matrix is not Orthonormal

$$\boxed{\left(\mathbf{A}_0^2\right)^{-1} \neq \left(\mathbf{A}_0^2\right)^T}$$

Useful Identity for Matrix Inverse

Given: a square matrix, A, and its inverse, B

$$\mathbf{A} = \left[\begin{array}{c|c} \underset{m\times m}{\mathbf{A}_1} & \underset{m\times n}{\mathbf{A}_2} \\ \hline \underset{n\times m}{\mathbf{A}_3} & \underset{n\times n}{\mathbf{A}_4} \end{array}\right] \quad ; \quad \mathbf{B} \triangleq \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}$$

Then

$$\mathbf{A}\,\mathbf{B} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}_{m+n}$$

$$= \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} = \begin{bmatrix} \left(\mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_3\right) & \left(\mathbf{A}_1\mathbf{B}_2 + \mathbf{A}_2\mathbf{B}_4\right) \\ \left(\mathbf{A}_3\mathbf{B}_1 + \mathbf{A}_4\mathbf{B}_3\right) & \left(\mathbf{A}_3\mathbf{B}_2 + \mathbf{A}_4\mathbf{B}_4\right) \end{bmatrix}$$

Equating like parts, and solving for $\mathbf{B}_i$

$$\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix} = \left[\begin{array}{c|c} \left(\mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3\right)^{-1} & -\mathbf{A}_1^{-1}\mathbf{A}_2\left(\mathbf{A}_4 - \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2\right)^{-1} \\ \hline -\mathbf{A}_4^{-1}\mathbf{A}_3\left(\mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3\right)^{-1} & \left(\mathbf{A}_4 - \mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2\right)^{-1} \end{array}\right]$$

Inverse Homogeneous Transformation

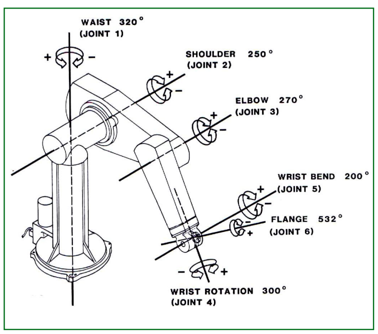
Forward transformation

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{old}^{new} & \mathbf{r}_o \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & 1 \end{bmatrix}$$

Inverse

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{new}^{old} & -\mathbf{H}_{new}^{old}\mathbf{r}_o \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} h_{11} & h_{21} & h_{31} & -\left(h_{11}x_o + h_{21}y_o + h_{31}z_o\right) \\ h_{12} & h_{22} & h_{32} & -\left(h_{12}x_o + h_{22}y_o + h_{32}z_o\right) \\ h_{13} & h_{23} & h_{33} & -\left(h_{13}x_o + h_{23}y_o + h_{33}z_o\right) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$



Manipulator Maneuvering Spaces

- **Joint space**: Vector of joint variables, e.g.,

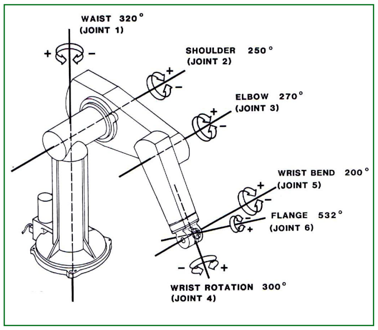
$$\mathbf{r}_J = \begin{bmatrix} \theta_{waist} & \theta_{shoulder} & \theta_{elbow} & \theta_{wrist-bend} & \theta_{flange} & \theta_{wrist-twist} \end{bmatrix}^T$$

- **End-effecter space**: Vector of end-effecter positions, e.g.,

$$\mathbf{r}_E = \begin{bmatrix} x_{tool} & y_{tool} & z_{tool} & \psi_{tool} & \theta_{tool} & \phi_{tool} \end{bmatrix}^T$$

- **Task space**: Vector of task-dependent positions, e.g., locating a symmetric grinding tool above a horizontal surface:

$$\mathbf{r}_T = \begin{bmatrix} x_{tool} & y_{tool} & z_{tool} & \psi_{tool} & \theta_{tool} \end{bmatrix}^T$$



Forward and Inverse Transformations of a Robotic Assembly

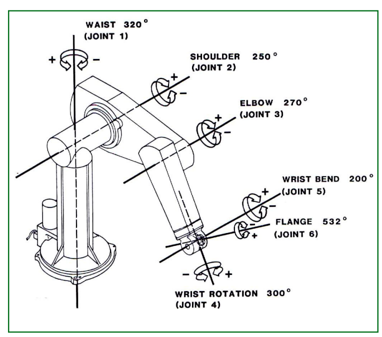
- **Forward Transformation**
  - Transforms homogeneous coordinates from tool frame to reference frame coordinates

$$s_{base} = \mathbf{A}_{waist}\mathbf{A}_{shoulder}\mathbf{A}_{elbow}\mathbf{A}_{wrist-bend}\mathbf{A}_{flange}\mathbf{A}_{wrist-twist}\mathbf{s}_{tool} = \mathbf{A}_{tool}^{base}\mathbf{s}_{tool}$$

- **Inverse Transformation**
  - Transform homogeneous coordinate from reference frame to tool frame coordinates

$$s_{tool} = \mathbf{A}_{wrist-twist}^{-1}\mathbf{A}_{flange}^{-1}\mathbf{A}_{wrist-bend}^{-1}\mathbf{A}_{elbow}^{-1}\mathbf{A}_{shoulder}^{-1}\mathbf{A}_{waist}^{-1}\mathbf{s}_{base} = \mathbf{A}_{base}^{tool}\mathbf{s}_{base}$$

Forward and Inverse Kinematics Between Joints and Tool Position and Orientation



Forward Kinematic Problem: Compute the position of the tool in the reference frame that corresponds to a given joint vector (i.e., vector of link variables)

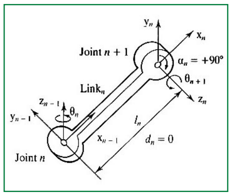
$$s_{base} = \mathbf{A}_{waist}\mathbf{A}_{shoulder}\mathbf{A}_{elbow}\mathbf{A}_{wrist-bend}\mathbf{A}_{flange}\mathbf{A}_{wrist-twist}\mathbf{s}_{tool} = \mathbf{A}_{tool}^{base}\mathbf{s}_{tool}$$

$$\text{To Be Determined} \Leftarrow \text{Given}$$

Inverse Kinematic Problem: Find the vector of link variables that corresponds to a desired task-dependent position

$$\mathbf{A}_{waist}\mathbf{A}_{shoulder}\mathbf{A}_{elbow}\mathbf{A}_{wrist-bend}\mathbf{A}_{flange}\mathbf{A}_{wrist-twist}\mathbf{s}_{tool} = \mathbf{A}_{tool}^{base}\mathbf{s}_0 = s_{base}$$

$$\text{To Be Determined} \quad \Leftarrow \quad \text{Given}$$



Forward and Inverse Kinematics Single-Link Example

Forward Kinematic Problem: Specify task-dependent position that corresponds to a given joint variable (= θ_n)

$$\mathbf{s}_{n-1} = \mathbf{A}(z_{n-1},\theta_n)\mathbf{A}(z_{n-1},d_n)\mathbf{A}(x_{n-1},l_n)\mathbf{A}(x_{n-1},\alpha_n)\,\mathbf{s}_n$$
$$= \begin{bmatrix} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & l_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & l_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{s}_n$$

Red: Known
Blue: Unknown

Inverse Kinematic Problem: Find the joint variable that corresponds to a desired task-dependent position

$$\mathbf{s}_{n-1} = \mathbf{A}(z_{n-1},\theta_n)\mathbf{A}(z_{n-1},d_n)\mathbf{A}(x_{n-1},l_n)\mathbf{A}(x_{n-1},\alpha_n)\,\mathbf{s}_n$$
$$= \begin{bmatrix} \cos\theta_n & -\sin\theta_n\cos\alpha_n & \sin\theta_n\sin\alpha_n & l_n\cos\theta_n \\ \sin\theta_n & \cos\theta_n\cos\alpha_n & -\cos\theta_n\sin\alpha_n & l_n\sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{s}_n$$

Forward and Inverse Kinematics Single-Link Example

Inverse Problem: Find the **joint variable, θ,** that corresponds to a desired task-dependent position

$$\mathbf{s}_{n-1} = \mathbf{A}(z_{n-1},\theta_n)\mathbf{A}(z_{n-1},0)\mathbf{A}(x_{n-1},l_n)\mathbf{A}(x_{n-1},90°)\,\mathbf{s}_n$$
$$= \begin{bmatrix} \cos\theta_n & 0 & \sin\theta_n & l_n\cos\theta_n \\ \sin\theta_n & 0 & -\cos\theta_n & l_n\sin\theta_n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{s}_n$$

Red: Known
Blue: Unknown

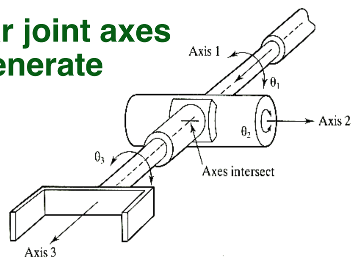
$$x_{n-1} = x_n\cos\theta_n + z_n\sin\theta_n + l_n\cos\theta_n$$
$$y_{n-1} = x_n\sin\theta_n - z_n\cos\theta_n + l_n\sin\theta_n$$

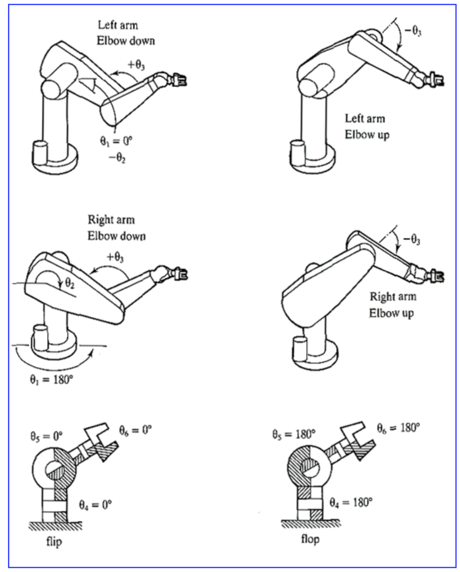
Solve by elimination and inverse trig functions

Manipulator Redundancy and Degeneracy

- More than one link configuration may provide a given end point if $\dim(x_J) \geq \dim(x_E) \geq \dim(x_T)$
- **Redundancy**: Finite number of joint vectors provide the same task-dependent vector
- **Degeneracy**: Infinite number of joint vectors provide the same task-dependent vector

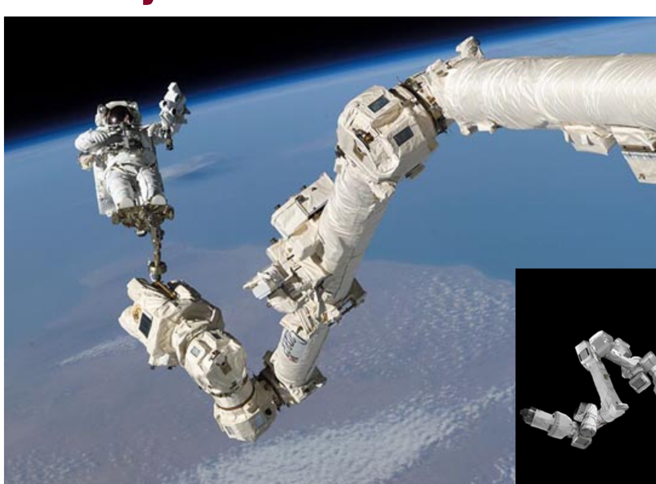
- Co-linear joint axes are degenerate

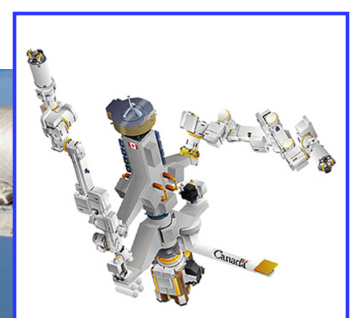


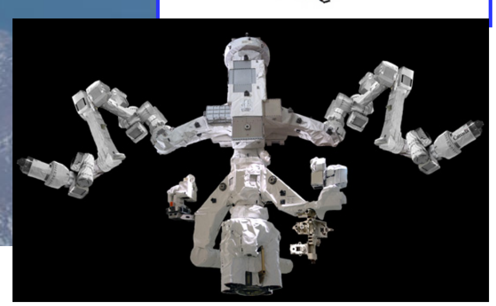


Space Robot Arms are Highly Redundant

- Why?







Link variable	θ	α	l	d
1	θ_1	θ_1	0	l_1	0
2	θ_2	θ_2	0	l_2	0

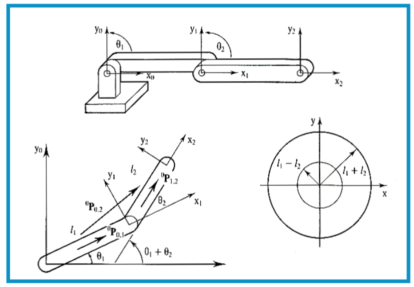
Transformations for a Two-Link Manipulator

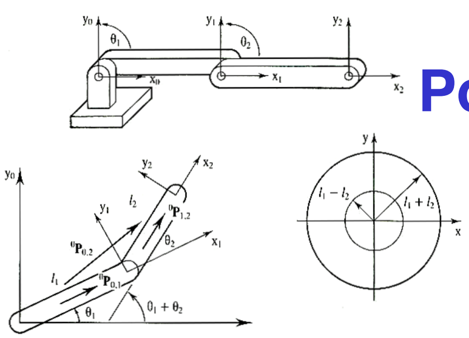
- **Example: Type 1 Two-Link Manipulator (e.g., Puma geometry without waist and wrist)**

$$\mathbf{H}_0^1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \mathbf{r}_0 = \begin{bmatrix} -l_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{H}_0^1 & \mathbf{r}_0 \\ (\,0 \quad 0 \quad 0\,) & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & -l_1 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{H}_1^2 & \mathbf{r}_1 \\ (\,0 \quad 0 \quad 0\,) & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & -l_2 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Position of Distal Joint Relative to the Base

(2-link manipulator)

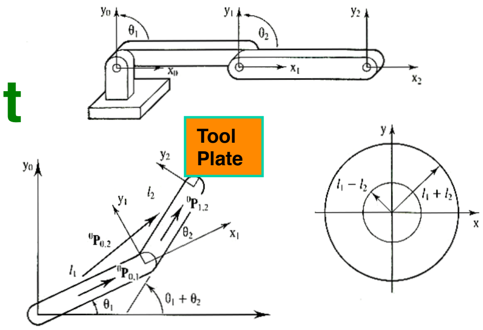
$$\theta_B = \theta_1 + \theta_2$$

$$\mathbf{s}_{base} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{base} = \mathbf{A}_1\mathbf{A}_2\mathbf{s}_{distal} = \begin{bmatrix} \cos\theta_B & -\sin\theta_B & 0 & l_1\cos\theta_1 + l_2\cos\theta_B \\ \sin\theta_B & \cos\theta_B & 0 & l_1\sin\theta_1 + l_2\sin\theta_B \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{distal}$$

$$= \begin{bmatrix} l_1\cos\theta_1 + l_2\cos\theta_B \\ l_1\sin\theta_1 + l_2\sin\theta_B \\ 0 \\ 1 \end{bmatrix}$$

Position of Distal Joint Relative to the Base

(2-link manipulator)



- **Suppose a tool plate is fixed to the distal joint at $(x\ y\ z)_{distal}^T$; then**

$$
\mathbf{s}_{base} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{base} = \mathbf{A}_1\mathbf{A}_2\mathbf{s}_{distal} = \begin{bmatrix} \cos\theta_B & -\sin\theta_B & 0 & l_1\cos\theta_1 + l_2\cos\theta_B \\ \sin\theta_B & \cos\theta_B & 0 & l_1\sin\theta_1 + l_2\sin\theta_B \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}_{distal}
$$

$$
= \begin{bmatrix} x\cos\theta_B - y\sin\theta_B + l_1\cos\theta_1 + l_2\cos\theta_B \\ x\sin\theta_B + y\cos\theta_B + l_1\sin\theta_1 + l_2\sin\theta_B \\ z \\ 1 \end{bmatrix}
$$

- Alternatively, straightforward trigonometry could be used in this example

Next Time: Mobile Robots, Personal Assistance, Toys, and Games

Supplemental Material

American Android Multi-Arm UGV

(David Handelman, *89)

http://www.youtube.com/watch?v=pOi6OdcPKfk

http://www.youtube.com/watch?v=tVZFJ7yivxI

http://www.youtube.com/watch?v=qdM48cAg0U4

Joint Variables for Different Link Types