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On Sidon Sets

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Contents

Notation

- $\mathbb{R}$ is the set of real numbers.
- $$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$$
- $\mathrm{L}^p(\mathbb{T})$ for $1 \leq p < \infty$ is the set of functions such that
$$\int_{\mathbb{T}} |f(x)|^p dx < \infty$$
- $$\|f\|_p := (\int |f(x)|^p dx)^{1/p}$$
where the integral is over $\mathbb{T}$. The integral in this case as in everywhere else in the report is the Lebesgue integrals on Euclidean domains.
- $$f * g = \frac{1}{2\pi} \int f(t-\tau)g(\tau)d\tau$$
- We define an operator T from a function space X to a function space Y if for all $f \in X$, $Tf \in Y$.

  If (X, d_1) and (Y, d_2) are metric spaces, then we can define a norm on the operator T as
$$\|T\|_{(X,Y)} = \sup_{f \in X} \frac{\|Tf\|_{d_2}}{\|f\|_{d_1}}.$$

  So, if we have
$$\|Tf\|_{d_2} \leq C\|f\|_{d_1} \qquad \text{for all } f \in X$$
then $\|T\|_{(X,Y)} \leq C$ and hence T is a bounded operator. Also, it is easy to show that an operator is bounded if and only is it is continuous. And so, we will often use the two terms interchangeably.
- For any 2π-periodic function f, we can define its Fourier series as
$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)\mathrm{e}^{int}.$$

where

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \mathrm{e}^{-int} \mathrm{d}t.$$

- The Fourier series of a 2π-periodic function f is termed as absolutely convergent if

$$\sum |\hat{f}(n)| \leq \infty$$

Chapter 1

Introduction

Definition 1.1 (Lacunary set). *A set $\{n_k\}$ of positive integers which for some λ satisfy*

$$\frac{n_{k+1}}{n_k} > \lambda > 1$$

is often termed as a lacunary set or Hadamard set. The above condition is often termed as the Hadamard's lacunarity condition.

In 1892, Hadamard proved that the Taylor series

$$f(z) = \sum_{1}^{\infty} a_n z^{\lambda_n} \text{ with } \limsup_{n\to\infty} |a_n|^{1/\lambda_n} = 1$$

has $|z|$ as a natural boundary, if $\{\lambda_n\}$ is a Hadamard set. A natural boundary is when a function is continuous inside the set but is cannot be *extended* to a continuous function on the boundary. To see that f is continuous on all points $|z| < 1$, we compare it with a geometric series. to make the proof slightly neater we first assume all the coefficients as 1. Then

$$f(z) \leq 1 + z + z^2 + z^3 + \dots$$

And $\sum z^n$ converges for $|z| < 1$, so $f(z)$ exists and is clearly analytic for all such z's. Now analytic functions are holomorphic, so $f(z)$ is continuous in $|z| < 1$. Now to see it in the general case, notice that $\limsup_{n\to\infty} |a_n|^{1/\lambda_n} = 1$, so for large enough n, $a_n < 1 + \epsilon$. So these terms can be bounded by $(1+\epsilon)$ times the geometric series and the same proof will work.

Enforcing this lacunarity on Fourier series gives us the Lacunary Fourier series. This is a function f such that $\hat{f}(n) \neq 0$ only if $n = \lambda_k$. A general introduction to lacunary series can be found in Kahane[3]. Most importantly For these functions we have the following result.

Theorem 1.2. *1. If $f \in \mathrm{L}^1(\mathbb{T})$ and*

$$f(x) \sim \sum_{k=1}^{\infty} a_k\, e^{i n_k x}$$

where $\{n_k\}$ *is a lacunary set, then* $f \in \mathrm{L}^p(\mathbb{T})$ *for all* $p < \infty$.

2. *If additionally,* f *is bounded, then* $\sum |a_k| < \infty$.

If we generalize the set $\{n_k\}$ such that f defined in the above manner forces f to be in $\mathrm{L}^p(\mathbb{T})$ for some p's, but not necessarily all p's, then the set of integers are termed as $\Lambda(p)$ sets.

If we consider the set of integers for which the second result holds then we have the Sidon sets.

The aim of this project was to study properties of these sets, as these, much like the lacunary series provide a rich set of counter examples in the topic of Fourier analysis. Some of which we will also find in this report. We will mainly be following the outlines provided in Rudin [6]

Chapter 2

Sidon Sets

Definition 2.1. *For a given set of integers E, a trignometric polynomial $f = \sum a_n\, e^{inx}$, is called an E-polynomial if $a_n \neq 0$ only if $n \in E$.*

Definition 2.2. *For a given set of integers E, a given function f is called an E-function if $\hat{f}(n) \neq 0$ only when $n \in E$.*

Definition 2.3 (Sidon Set)**.** *A set of integers E is called a Sidon set if there exists a constant B such that for all E-polynomials f,*

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| \leq B\|f\|_{\infty}.$$

We can say a lot about a function merely by knowing where its fourier transform is supported as we will see in the following theorem for characterizing a Sidon set via analytic methods.

2.1 Analytic Properties

Theorem 2.4. *The following statements are equivalent:*

1. *E is a Sidon set.*
2. *Every bounded E-function has an absolutely convergent Fourier series*
3. *Every continuous E-function has an absolutely convergent. Fourier series.*
4. *For every bounded function b on E, there is a measure $\mu \in \mathcal{M}(\mathbb{T})$ such that*
$$\hat{\mu}(n) = b(n) \quad \text{for all } n \in E$$
5. *If $b(n) \to 0$ as $|n| \to \infty$, there exists a $f \in \mathrm{L}^1(\mathbb{T})$ such that*
$$\hat{f}(n) = b(n) \quad \text{for all } n \in E$$

Proof. We prove the theorem in following steps:

$1 \implies 2$ We assume that E is a Sidon set. So there exists a constant B such that for all E-polynomials f we have

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| \leq B\|f\|_\infty.$$

For any bounded E function g we have the Fejér means

$$\sigma_N(g,x) = \sum_{-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{g}(n)\, \mathrm{e}^{inx}\,.$$

From basic theory of Fourier series we have $\sigma_N(g) \to g$. Also $\|\sigma_N(g)\|_\infty \leq \|g\|_\infty$. Additionally since σ_N's are trigonometric polynomials $\sum_{-\infty}^{\infty} |\hat{\sigma}_N(g,n)| \leq B\|\sigma_N(g)\|_\infty$. This implies

$$\sum_{-N}^{N} \left(1 - \frac{|n|}{N}\right) |\hat{g}(n)| \leq B\|g\|_\infty$$

Taking the limit $n \to \infty$ we get

$$\sum_{-\infty}^{\infty} |\hat{g}(n)| \leq B\|g\|_\infty$$

$2 \implies 3$ This is obvious as every continuous E-function is also bounded.

$3 \implies 1$ Let the space of all continuous E-function be called C_E. Also, consider the space of functions on the set E (a sequence) with the norm

$$\|\phi\| = \sum_{n \in E} |\phi(n)| < \infty.$$

These two space are clearly isomorphic via the map $f \leftrightarrow \hat{f}$. Then we have two equivalent norms on the space C_E. Hence we have

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| \leq B\|f\|_\infty.$$

for all E-functions f. Thus E is a Sidon set

$1 \implies 4$ Let E be a Sidon set and $b(n)$ be a sequence such that $|b(n)| \leq 1$. Consider the mapping

$$f \mapsto \sum_{n \in E} \hat{f}(n) b(n).$$

Clearly this defines a bounded linear functional on the space C_E with the bound B. By Hahn Banach theorem, this functional can be extended to a functional on the entire space $C(\mathbb{T})$. Now Riesz representation theorem tells us that there is a measure $\mu \in \mathcal{M}(\mathbb{T})$ such that $\|\mu\| \leq B$ and for all E-functions f,

$$\sum_{n\in E} \hat{f}(n)b(n) = \frac{1}{(2\pi)} \int_{-\pi}^{\pi} f(-x)d\mu(x).$$

Now putting $f(x) = \mathrm{e}^{inx}$ where $n \in E$, we get that

$$\hat{\mu}(n) = b(n) \quad \forall n \in E.$$

$4 \implies 5$ If we have a sequence $b(n) \to 0$ as $|n| \to \infty$ then for $n \geq 0$ there exists a positive convex sequence $\{c(n)\}$ such that $c(n) \to 0$ and the ratio $b(n)/c(n)$ is bounded. Extend the sequence $c(n)$ to the negative numbers by letting $c(-n) = c(n)$. So by our assumption there exists a measure $\mu \in \mathcal{M}(\mathbb{T})$ with $\hat{\mu}(n) = b(n)/c(n)$ for all $n \in E$. Also there exists a $f \in \mathrm{L}^1(\mathbb{T})$ such that $\hat{f}(n) = c(n)$.(Katznelson [5] pg 23). So we have

$$\begin{aligned} b(n) &= \hat{f}(n)\hat{\mu}(n) \\ &= (f * \mu)\hat{\ }(n). \end{aligned}$$

Since $f \in \mathrm{L}^1(\mathbb{T})$ and μ is of bounded variation, $f * \mu \in \mathrm{L}^1(\mathbb{T})$ and we have our desired candidate.

$5 \implies 3$ Consider a continuous E-function f. Let g be any integrable function. So $f * g \in \mathrm{L}^1(\mathbb{T})$ and its Fourier transform is Cesàro summable. Hence we have the series $\sum \hat{f}(n)\hat{g}(n)$ as Cesàro summable. Now take any sequence $\{c_n\}$ converging to zero. We have $\sum |\hat{f}(n)c_n|$ as Cesàro summable because we can always find a function g such that $\{\hat{g}(n)\}$ restricted to E is an arbitrary sequence converging to zero. Since the above series consists of positive terms, $\sum |\hat{f}(n)c_n|$ converges for all sequences c_n. Thus by bounded convergence theorem, $\sum |\hat{f}(n)|$ converges and hence we are done.

$\square$

In the part 4 of the above theorem if we can construct a measure which is exactly equals any given bounded sequence on a set then the set is Sidon. In the next theorem we show that even if we can "approximate" a bounded sequence on a set then the set has to be Sidon

Theorem 2.5. *Let E be a set of integers and $\{b(n)\}$ be a sequence such that $|b(n)| = 1$ for all $n \in E$. Suppose there exists a measure μ with*

$$|\hat{\mu}(n) - b(n)| < 1 - \delta$$

for all $n \in E$, then E is a Sidon set.

Proof. Let f be a continuous E-function. Define a sequence $b(n)$ in the following manner

$$b(n) = \begin{cases} \frac{|\hat{f}(n)|}{\hat{f}(n)} & \text{if } \hat{f}(n) \neq 0 \\ 1 & \text{if } \hat{f}(n) = 0 \end{cases}$$

By our assumption there exists a measure μ with

$$|\hat{\mu}(n) - b(n)| < 1 - \delta.$$

Multiplying by $|\hat{f}(n)|$ on both sides we get

$$|\hat{\mu}(n)\hat{f}(n) - b(n)\hat{f}(n)| < (1-\delta)|\hat{f}(n)|.$$

So clearly,

$$\mathrm{Re}[\hat{\mu}(n)\hat{f}(n)] \geq \delta|\hat{f}(n)|.$$

We have already seen that $\sum \hat{\mu}(n)\hat{f}(n)$ is Cesàro summable. So $\mathrm{Re}\sum \hat{\mu}(n)\hat{f}(n)$ is also Cesàro summable. This is a summation of positive terms and hence convergent. Thus we have

$$\sum |\hat{f}(n)| \leq \frac{1}{\delta}\mathrm{Re}[\sum \hat{\mu}(n)\hat{f}(n)] < \infty.$$

Now by part 3 of the above theorem we are done. $\square$

2.2 Arithmetic properties

Consider a set of integers E whose elements are enumerated as $n_1, n_2, \ldots$. Take an integer n and represent it in terms of s elements of E as

$$n = n_{k_1} + n_{k_2} + \ldots + n_{k_s}$$

where $k_1 < k_2 < \ldots < k_s$. Denote by $R_s(E, n)$, the number of such possible representations. Then we have the following way to decide whether the given set is Sidon via its $R_s(E, n)$ function.

Theorem 2.6. *Let E be a set such that*

1. *If $n \in E$ then $-n \notin E$.*
2. *If there exists a finite constant B and a decomposition of E into a union of disjoint sets $E_1, E_2, \ldots, E_t$ such that for all $n \in E$ and for $n = 0$ we have*

$$R_s(E, n) \leq B^s \quad 1 \leq j \leq t \text{ and } s = 1, 2, \ldots$$

Then the set E is a Sidon set.

Proof. Notice that $B >\geq 1$ as for $s=1$ and $n \in E$ $R_s(E,n) = 1$. Now we choose $\beta = (3tB^2)^{-1} < 1/2$ and let $b(n)$ be a sequence of integers with $|b(n)| = \beta$ for $n \in E$. Now fix a j $((1 \leq j \leq t))$ and let $n_1, n_2, n_3, \ldots$ be an enumeration of elements in E_j. Now consider the Riesz product

$$P_N(x) = \prod_{k=1}^{N} (1 + b(n_k)\, \mathrm{e}^{in_k x} + \overline{b(n_k)}\, \mathrm{e}^{-in_k x})$$

Notice that $P_N(x) \geq 0$ for all x as $\beta < 1/2$. Now by expanding the above product we get

$$P_N(x) = 1 + \sum_{k=1}^{N} b(n_k)\, \mathrm{e}^{in_k x} + \sum_{k=1}^{N} \overline{b(n_k)}\, \mathrm{e}^{-in_k x} + \sum_{-\infty}^{\infty} c_n^{(N)}\, \mathrm{e}^{inx}$$

where $c_n^{(N)} = \sum_{n=2}^{k} \sum |b(n_{k_1}) \ldots b(n_{k_s})|$ and the second summation is over all $n_{k_1}, \ldots, n_{k_s}$. We try to get an upper bound.

$$\begin{aligned}
|c_n^{(N)}| &= |\sum_{s=2}^{N} \sum (b(n_{k_1}) \ldots b(n_{k_s})| \\
&\leq \sum_{s=2}^{N} \sum |b(n_{k_1}) \ldots b(n_{k_s})| \\
&\leq \sum_{s=2}^{\infty} (B\beta)^s \\
&= \frac{b^2\beta^2}{1 - B\beta} \\
&\leq \frac{1}{6t^2B^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\|P_N\|_1 &= \int_0^{2\pi} |1 + \sum_{k=1}^{N} b(n_k)\, \mathrm{e}^{in_k x} + \sum_{k=1}^{N} \overline{b(n_k)}\, \mathrm{e}^{-in_k x} + \sum_{-\infty}^{\infty} c_n^{(N)}\, \mathrm{e}^{inx} |dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} 1 + \sum_{k=1}^{N} b(n_k)\, \mathrm{e}^{in_k x} + \sum_{k=1}^{N} \overline{b(n_k)}\, \mathrm{e}^{-in_k x} + \sum_{-\infty}^{\infty} c_n^{(N)}\, \mathrm{e}^{inx}\, dx \\
&= 1 + c_0^{(N)} \\
&\leq 1 + \frac{1}{6t^2B^2}
\end{aligned}$$

Since $\{\|P_N\|_1\}$ is bounded as $N \to \infty$, so there is a subsequence of $\{P_N\}$ converges weakly to a measure μ_j such that

$$\begin{aligned}
|\hat{\mu}_j(n_k) - b(n_k) &\leq \frac{1}{6t^2B^2} \quad (n_k \in E_j) \text{ and} \\
|\hat{\mu}_j(n)| &\leq \frac{1}{6t^2B^2} \quad (n \in E, n \notin E_j).
\end{aligned}$$

We can similarly get measures for other values of j. Take $\mu = \mu_1 + \ldots + \mu_t$. So we get

$$|\hat{\mu}(n) - b(n)| = \frac{t}{6t^2B^2} < \frac{\beta}{2}$$

Dividing by β on both sides we get

$$|\frac{\hat{\mu}(n)}{\beta} - \frac{b(n)}{\beta}| < \frac{1}{2}$$

Hence by the previous "approximation" theorem, E is a Sidon set. $\square$

Using this theorem we have an alternative method to show that a Hadamard set is a Sidon set. This is done by first noticing that, for Hadamard sets with $n_{k+1}/n_k > 3$, $R_s(E, n) \leq 1$ for all n. And then the fact that any Hadamard set can be written as union of sets with $n_{k+1}/n_k > 3$. This allows us to use the above theorem to show that a Hadamard set is Sidon.

It is very instructive to compare this theorem with the part 2 of theorem 3.9.

Chapter 3

$\Lambda(p)$ Sets

3.1 Definition and Basic properties

We begin with the following definition.

Definition 3.1. *E is of* **type(r, s)** *if there is a finite constant B such that for every E-polynomial f,*

$$\|f\|_s \leq B\|f\|_r$$

These set have the following two useful properties.

Proposition 3.2. *A set of integers E is of type(r, s) if and only if $L_E^r = L_E^s$*

Proof. If E is of *type*(r, s) then obviously $L_E^r = L_E^s$.

For the other side we have L_E^r and L_E^s are closed subspaces of L^r and L^s respectively. Hence $\|.\|_r$ and $\|.\|_s$ are two norms on the same Banach space and hence are equivalent. Thus we are done. □

Proposition 3.3. *Let $0 < r < s < t < \infty$. Then a set of integers E is of type(r, t) if an only if E is of type(s, t).*

Proof. It is clear that *type*$(r, t) \subset$ *type*(s, t). For the other direction suppose E is of *type*(s, t). Then by Hölder's inequality we have

$$[\frac{1}{2\pi}\int |f|^s dx]^{t-r} \leq [\frac{1}{2\pi}\int |f|^r dx]^{t-s}[\frac{1}{2\pi}\int |f|^t dx]^{s-r}$$

which can be rewritten as

$$\|f\|_s^{s(t-r)} \leq \|f\|_r^{r(t-s)}\|f\|_t^{t(s-r)}.$$

By our assumption we have $\|f\|_t \leq B_{s,t}\|f\|_s$. Using this we get

$$\|f\|_s^{s(t-r)} \leq B_{s,t}^{r(t-s)}\|f\|_r^{r(t-s)}\|f\|_s^{t(s-r)}$$

Thus we have

$$\|f\|_s^{r(t-s)} \leq B_{s,t}^{r(t-s)} \|f\|_r^{r(t-s)}$$

By denoting $B_{s,t}^{\frac{r(t-s)}{t(s-r)}}$ by $B_{r,s}$ (as both the left and right side of the inequality is independent of t), we get

$$\|f\|_s \leq B_{r,s}^{r(t-s)} \|f\|_r^{r(t-s)}$$

Therefore we have

$$\|f\|_t \leq B_{s,t}\|f\|_s \leq B_{s,t}B_{r,s}\|f\|_r$$

and so E is of $type(r, t)$. □

Now we define the $\Lambda(p)$ set.

Definition 3.4. *Suppose* $0 < s < \infty$*. A set* E *is of type* $\Lambda(s)$ *is* E *is of* $type(r, s)$ *for some* $r < s$*.*

Thus if E is a $\Lambda(s)$ set then for all $0 < t < s$ and for any E-polynomials f, there is a constant B_t (dependent on t) so that

$$\|f\|_t \leq B_t\|f\|_s.$$

For $\Lambda(p)$ sets we have the following results.

3.2 Analytic Properties

We have the following analytic ways of characterizing a Sidon set.

Theorem 3.5. *Let* $1/p + 1/q = 1$ *where* $1 \leq p, q \leq \infty$*. Then the following statements are equivalent*

1. E *is of type* $\Lambda(p)$*.*
2. *If* μ *is an* E*-measure, then there exists a function* $f \in \mathrm{L}^p(\mathbb{T})$ *such that* $d\mu(x) = f(x)dx$*.*
3. *Every* E*-function in* $\mathrm{L}^1(\mathbb{T})$ *is also in* $\mathrm{L}^p(\mathbb{T})$*.*
4. *For every function* $g \in \mathrm{L}^q(\mathbb{T})$ *there is a bounded function* h *such that* $\hat{h}(n) = \hat{g}(n)$ *for all* $n \in E$*.*
5. *For every function* $g \in \mathrm{L}^q(\mathbb{T})$ *there is a continuous function* h *such that* $\hat{h}(n) = \hat{g}(n)$ *for all* $n \in E$*.*

Proof. We prove the theorem in the following steps.

1 $\implies$ 2 Consider the Cesàro Means of the Fourier -Stieljes series of μ

$$c_n = \frac{1}{n}\sum_{j=1}^{n} \hat{\mu}(j)\, \mathrm{e}^{ixj}\, .$$

Then c_n's are bounded in $\mathrm{L}^1(\mathbb{T})$. Now since E is a $\Lambda(p)$ set, c_n are also bounded in $\mathrm{L}^p(\mathbb{T})$. So by uniform boundedness principle μ is also bounded in $\mathrm{L}^p(\mathbb{T})$. Thus there exists a function $f \in \mathrm{L}^p(\mathbb{T})$ such that $d\mu(x) = f(x)dx$.

2 $\implies$ 3 If $f \in \mathrm{L}^1(\mathbb{T})$ is a E-function, then define a measure such that $d\mu(x) = f(x)dx$. That measure is an E measure. Thus by our assumption we have a function $g \in \mathrm{L}^p(\mathbb{T})$ such that $d\mu(x) = g(x)dx$. But that implies that $f(x) = g(x)$ almost everywhere and thus $f \in \mathrm{L}^p(\mathbb{T})$.

3 $\implies$ 1 Our assumption implies that L^1_E and L^p_E are the same Banach space with two different norms. Therefore these two norms are equivalent. Thus E is of type $\Lambda(p)$.

1 $\implies$ 4 Since E is a $\Lambda(p)$ set we can choose a constant B such that $\|f\|_p \leq B\|f\|_1$ for all E-polynomials f. Now let $g \in \mathrm{L}^q(\mathbb{T})$ where $1/p + 1/q = 1$, and define an linear functional T_g which take $f \in L^p_E(\mathbb{T})$ to $1/2\pi \int_{-\pi}^{\pi} f(-x)g(x)$. Then clearly T_g is a bounded functional on $L^p_E(\mathbb{T})$. Looking at the same operator on $L^p_E(\mathbb{T})$(this is well defined is obvious from part 3), we can see that it is bounded as

$$|Tf| \leq \|g\|_q \|f\|_p \leq B\|g\|_q\|f\|_1$$

Now using the Hahn Banach theorem we can extend the above operator to a bounded linear functional on the space $\mathrm{L}^1(\mathbb{T})$. Let us call the extension as T. The by Riesz representation theorem there is a bounded function h such that

$$Tf = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x)h(x)dx$$

Now substituting $f = \mathrm{e}^{inx}$, for $n \in E$, we get that

$$Tf = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-inx}\, h(x)dx$$

But since f is an E-polynomial, by our original definition we have

$$Tf = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-inx}\, g(x)dx.$$

Thus we have

$$\hat{g}(n) = \hat{f}(n) \text{ for all } n \in E$$

$4 \implies 5$ Let $g \in \mathrm{L}^q(\mathbb{T})$ by factorization of g (as described in the following lemma), we have a $f \in \mathrm{L}^1(\mathbb{T})$ and $g_1 \in \mathrm{L}^q(\mathbb{T})$ such that $g = f * g_1$. By our assumption there exists a bounded function h_1 such that $\hat{h_1}(n) = \hat{g_1}(n)$ for all $n \in E$. Let $h = f * h_1$, then h is a continuous function with $\hat{h}(n) = \hat{g}(n)$ for all $n \in E$.

$5 \implies 1$ Consider the space C_0 and L_0^q which are subspaces of $C(\mathbb{T})$ and $\mathrm{L}^q(\mathbb{T})$ respectively with $\hat{f}(n) = 0$ for all $n \in E$ and f in the respective spaces. Then by our assumption the two spaces C/C_0 and L^q/L_0^q are identical and so their quotient space norms are equivalent. Therefore there exists a constant B such that for every $g \in \mathrm{L}^q(\mathbb{T})$, there is a continuous function h such that

$$\hat{h}(n) = \hat{g}(n) \text{ and } \|h\|_\infty \leq B\|g\|_q$$

If f is an E-polynomial then

$$|\frac{1}{2\pi}\int f(-x)g(x)| = |\frac{1}{2\pi}\int f(-x)h(x)| \leq \|f\|_1\|h\|_\infty \leq B\|f\|_1\|h\|_q$$

Now by uniform boundedness principle we have

$$\|f\|_p \leq B\|f\|_1$$

□

We now provide the proof of the factorization lemma referred to earlier.

Lemma 3.6. *Let $1 < q < \infty$ and $h \in \mathrm{L}^q(\mathbb{T})$. Then there are function $f \in \mathrm{L}^1(\mathbb{T})$ and $g \in \mathrm{L}^q(\mathbb{T})$so that $h = f * g$*

This is often termed as factorization of the space $\mathrm{L}^q(\mathbb{T})$ into $\mathrm{L}^1(\mathbb{T})$ and $\mathrm{L}^q(\mathbb{T})$.

Proof. Let $s_k(x) = \sum_{-k}^{k} \hat{h}(n)\,\mathrm{e}^{inx}$ and $\epsilon_k = \|h - s_k\|_q$. Choose a sequence $\{a_k\}$ with $a_0 > a_1 > \ldots > 0$, $\sum_{k=0}^{\infty} a_k = \infty$ and $\sum_{k=1}^{\infty} a_k\epsilon_{k-1} < \infty$. So the series $\sum_{k=0}^{\infty} a_k(h - s_{k-1})$, with $s_{-1} = 0$, converges in norm. So let the function it converges to, be g. Then $g \in \mathrm{L}^q(\mathbb{T})$ and

$$\hat{g}(n) = \sum_{k=0}^{\infty} a_k[\hat{h}(n) - \hat{s_{k-1}}(n)] = \hat{h}(n)(a_0 + a_1 + \ldots + a_{|n|})$$

for $n = 0, \pm1, \pm2, \ldots$.Let $c_n = 1/(a_0 + a_1 + \ldots + a_{|n|})$. Then $\{c_n\}$ is an even sequence which tends to 0. Also, it is convex for positive n. Hence there is a function $f \in \mathrm{L}^1(\mathbb{T})$ such that $c_n = \hat{f}(n)$. So

$$\hat{g}(n) = \frac{\hat{h}(n)}{\hat{f}(n)}$$

which gives us

$$\hat{h}(n) = \hat{g}(n)\hat{f}(n)$$

Which gives us $h = f * g$ and so we are done. □

3.3 Arithmetic Properties

The arithmetic properties of a $\Lambda(p)$ set can be determined in terms of the following two function $\alpha_E(n)$ and $r_s(n, E)$ which we define in the following paragraphs.

Let $\alpha_E(n, a, b)$ be the number of elements of E in the arithmetic sequence $\{a+b, a+2b, \ldots, a+nb\}$. We define $\alpha_E(n) = \sup_{a,b} \alpha_E(n, a, b)$. Thus $\alpha_E(n)$ is the maximum number of elements of E in any arithmetic sequence of length n. For a $\Lambda(p)$ set we have a bound on $\alpha_E(n)$ as described in the following theorem.

Let E be a set of integers, then we denote by $r_s(E, n)$ the number of representations of n of the form

$$n = n_{i_1} + \ldots + n_{i_s}$$

where $n_{i_1}, \ldots, n_{i_s} \in E$ and are distinct. Thus we have that

$$(\sum_{k=1}^{\infty} z^{n_k})^s = \sum_{n=1}^{\infty} r_s(E, n) z^n$$

Theorem 3.7. *Let E be a $\Lambda(q)$ set for some $q > 2$ and given a constant B such that for every E-polynomial f, $\|f\|_q \leq B\|f\|_2$. Then for $N = 1, 2, \ldots$*

$$\alpha_E(N) \leq 4B^2 N^{2/q}$$

Proof. Let K_N be Fejér kernal that is

$$K_N(x) = \sum_{-N}^{N} (1 - \frac{|n|}{N}) \, \mathrm{e}^{inx} .$$

Then $\|K_N\|_1 = 1$ and $\|K_N\|_2 \leq N$.

Take p such that $1/p + 1/q = 1$. From Hölder's inequality we have

$$[\frac{1}{2\pi} \int |f|^s dx]^{t-r} \leq [\frac{1}{2\pi} \int |f|^r dx]^{t-s} [\frac{1}{2\pi} \int |f|^t dx]^{s-r}$$

Putting $f = K_N$, $s = p$, $t = 2$ and $r = 1$ we have

$$[\frac{1}{2\pi} \int |K_N|^p dx] \leq [\frac{1}{2\pi} \int |K_N| dx]^{2-p} [\frac{1}{2\pi} \int |K_N|^2 dx]^{p-1}$$

which gives us

$$\|K_N\|_p^p \leq \|K_N\|_{2-p} \|K_N\|_{2(p-1)}$$

So,

$$\|K_N\|_p \leq N^{1/q}.$$

Let $A = \{a+b, \ldots, a+Nb\}$. Take m to be $N/2$ if N is even and $(N+1)/2$ if N is odd. Also let $Q(x) = \mathrm{e}^{imx} K_N(x)$. Then

$$\hat{Q}(x) = \hat{K_N}(n-m) = 1 - \frac{|n-m}{N} \geq \frac{1}{2}$$

Let $A \cap E = \{m_1, \dots m_\alpha\}$ and let $f(x) = \sum_{k=1}^{\alpha} \mathrm{e}^{-in_k x}$. Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)Q(x) = \sum \hat{Q}(n_k) \geq \frac{\alpha}{2}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)Q(x) &\leq \|f\|_q \|Q\|_p \\ &\leq B\|f\|_2\|Q\|_p \\ &\leq B\|f\|_2\|K_N\|_p \\ &\leq B\sqrt{\alpha}N^{2/q}.\end{aligned}$$

Thus we have

$$\begin{aligned}\alpha/2 &\leq B\sqrt{\alpha}N^{1/q} \\ \sqrt{\alpha} &\leq 2BN^{1/q} \\ \alpha \leq 4B^2N^{2/q}\end{aligned}$$

□

Corollary 3.7.1. *If E is of type $\Lambda(q)$ for every $q < \infty$, then*

$$\lim_{N\to\infty} \frac{\alpha_E(N)}{N^\epsilon} = 0$$

Corollary 3.7.2. *Let $q > 2$. Then a $\Lambda(q)$ set cannot contain an arbitrarily long arithmetic sequence.*

This is because $\alpha_E(N) < N$ for sufficiently large N.

Now we show that the union of two $\Lambda(p)$ sets is again $\Lambda(p)$. This will also be used in the next theorem.

Theorem 3.8. *Suppose E_1 and E_2 be two sets of type $\Lambda(p)$ and let $E = E_1 \cup E_2$*

1. *If $p > 2$, then E is of type $\Lambda(p)$.*
2. *If E_1 is a set of non-negative integers and E_2 is a set of negative integers then E is of type $\Lambda(p)$ for $p > 1$.*

Proof. We can without loss of generality assume that E_1 and E_2 are dijoint.

1. If $p > 2$, the there are constants B_1 and B_2 such that

   $$\|f_i\|_p \leq B_i\|f_i\|_2 \text{ for } i = 1, 2$$

   for every E_i-polynomial f_i. Then we have

   $$\begin{aligned}\|f\|_p &\leq \|f_1\|_p + \|f_2\|_p \\ &\leq B_1\|f_1\|_2 + B_2\|f_2\|_2 \\ &\leq B_1\|f\|_2 + B_2\|f\|_2 \\ &= (B_1 + B_2)\|f\|_2\end{aligned}$$

2. Since E_1 and E_2 are of type $\Lambda(p)$, the there are constants B_1 and B_2 and an s $(1 < s < p)$ such that

$$\|f_i\|_p \leq B_i \|f_i\|_2 \text{ for } i = 1, 2$$

for every E_i-polynomial f_i. Now by boundedness of Hilbert transform we can find a constant A_s so that if $f = f_1 + f_2$, where f_i's are E_i-polynomials, then

$$\|f_i\|_s \leq A_s \|f\|_s$$

for $i = 1, 2$. Thus we have

$$\begin{aligned}
\|f\|_p &\leq \|f_1\|_p + \|f_2\|_p \\
&\leq B_1 \|f_1\|_s + B_2 \|f_2\|_s \\
&\leq B_1 A_s \|f\|_s + B_2 A_s \|f\|_s \\
&= (B_1 + B_2) A_s \|f\|_s
\end{aligned}$$

□

Theorem 3.9. *Let E be a set of non-negative integers and let s be an integer $s > 1$.*

1. *If E is of typr $\Lambda(2s)$ then the averages*

$$\frac{1}{N} \sum r_s^2(E, n)$$

are bounded as $N \to \infty$

2. *If E is a union of sets $E_1, \ldots, E_t$ such that $r_s(E_i, n)$ is a bounded function of n for each j, then E is of type $\Lambda(2s)$.*

Proof. Suppose $E = \{n_k\}$, where $0 \leq n_1 < n_2 < \ldots$.

1. Let $f(x) = \mathrm{e}^{in_1 x} + \ldots + \mathrm{e}^{in_k x}$. Then

$$(f(x))^s = r(0) + r(1)\,\mathrm{e}^{ix} + \ldots + r(n_k)\,\mathrm{e}^i\, n_k x + \ldots$$

where $r(n) = r_s(E, n)$ and for $n > n_k$ the coefficients may be different from $r_s(E, n)$. Now since E is of type $\Lambda(2s)$, there is a constant B such that for every E-polynomial f

$$\|f\|_{2s} \leq B \|f\|_2.$$

Now by Bessel's inequality on f^s we get

$$\begin{aligned}
\sum_{i=1}^{n_k} r^2(i) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^s(x)|^2 dx &\leq B^{2s} \|f\|_2^{2s} \\
&= B^{2s} (\int |f(x)|^2)^s \\
&\leq B^{2s} k^s
\end{aligned}$$

Now we already know that $\alpha_E(n_k) \leq 4B_2^2 n_k^{1/s}$. So $k \leq 4B_2^2 n_k^{1/s}$ and so we have

$$n_k \geq B_1 k^s$$

Thus

$$\frac{1}{B_1 k^s} \sum_{i=1}^{B_1 k^s} r^2(i) \leq \frac{1}{B_1 k^s} \sum_{i=1}^{n_k} r^2(i) \leq \frac{b^{2s} k^s}{B_1 k^s} = C$$

Now for any integer N we can find a k such that $B_1(k-1)^s \leq N \leq B_1 k^s$, so

$$\frac{1}{B_1(k-1)^s} \sum_{i=1}^{B_1(k-1)^s} r^2(i) \leq \frac{1}{N} \sum_{i=1}^{N} r^2(i) \leq \frac{1}{B_1 k^s} \sum_{i=1}^{B_1 k^s} r^2(i)$$

So by sandwich theorem we have the desired result.

2. Using the previous theorem it is enough to show that the given statement holds for $t = 1$.

   Let $f(x) = \sum a(k)\, \mathrm{e}^{i n_k x}$ be an E-polynomial. So

$$f^s(x) = \sum_{-\infty}^{\infty} b_m\, \mathrm{e}^{imx}$$

   where $b_m = \sum a(i_1) \ldots a(i_s)$ and summation is over all representations of m of the form $m = n_{i_1} + \ldots + n_{i_s}$. Let $i_1(m), \ldots, i_s(m)$ be the representation such that $|a(i_1(m)) \ldots a(i_s(m))|$ is maximum and let $r_s(E, n) \leq R$. Then

$$|b_m|^2 \leq R^2 |a(i_1(m)) \ldots a(i_s(m))|.$$

   So we have

$$\|f\|_2^2 = \frac{1}{2\pi} \int |f|^2 dx = \sum |b_m|^2 \leq R^2 (\sum |a(k)|^2)^s = R^2 \|f\|_2^{2s}$$

□

Notice the similarity of the second part with the earlier result for Sidon sets. In that case we had assumed that $R_s(E, n)$ was bounded whereas in this case we have $r_s(E, n)$ is bounded. The union result for $\Lambda(p)$ sets also suggests a similar result for Sidon sets. This was shown by Drury in 1970-71.

3.4 Relationship between $\Lambda(p)$ and Sidon sets

A Sidon set is $\Lambda(p)$ for all p. In fact we can say more:

Theorem 3.10. *For a Sidon set E, let B be a constant such that $\sum |\hat{f}(n)| \leq B\|f\|_\infty$, then for every E-polynomial*

$$\|f\|_q \leq B\sqrt{q}\|f\|_2 \text{ for } 2 < q < \infty \text{ and}$$

$$\|f\|_2 \leq 2B\|f\|_1$$

To prove this theorem we require the following lemma

Lemma 3.11. *Let $g(t) = \sum a_k \phi_k(t)$, where ϕ_k are the Rademacher functions. If $g \in \mathrm{L}^2(\mathbb{T})$ then for integers m*

$$\int_0^1 |g(t)|^{2m} dt \leq m^m (\sum |a_k|^2)^m$$

This has been shown via explicit calculations.

Proof. Let $s_n = \sum_{k=1}^n a_k \phi_k$. Then

$$\int_0^1 s_n^{2m}(t) dt = \sum A_{\alpha_1, \alpha_2, \ldots \alpha_j} a_{m_1}^{\alpha_1} \ldots a_{m_j}^{\alpha_j} \int_0^1 \phi_{m_1}^{\alpha_1} \ldots \phi_{m_j}^{\alpha_j}$$

where $A_{\alpha_1, \alpha_2, \ldots \alpha_j} = \frac{(\alpha_1 + \ldots + \alpha_j)!}{\alpha_1! \alpha_2! \ldots \alpha_j!}$ and the summation is over all positive integers $\alpha_1, \ldots, \alpha_j$ such that $\alpha_1 + \ldots + \alpha_j = 2m$. Notice that if any one of them is odd then the integral is 0. So we can restrict our attentions to terms with even α_i's. Also in these cases the integral gives the value of 1. Also we have

$$\begin{aligned}
\frac{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}}{A_{\beta_1, \beta_2, \ldots, \beta_j}} &= \frac{(2m)(2m-1)\ldots(m+1)}{\prod (2\beta_i)(2\beta_i - 1)(\beta_i + 1)} \\
&\leq \frac{(2m)\ldots(m+1)}{2^m} \\
&\leq m^m
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 s_n^{2m} dt &= \sum A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j} a_{m_1}^{2\beta_1} \ldots a_{m_j}^{2\beta_j} \\
&= \sum \frac{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}}{A_{\beta_1, \beta_2, \ldots, \beta_j}} A_{\beta_1, \beta_2, \ldots, \beta_j} a_{m_1}^{2\beta_1} \ldots a_{m_j}^{2\beta_j} \\
&\leq \sup \frac{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}}{A_{\beta_1, \beta_2, \ldots, \beta_j}} \sum A_{\beta_1, \beta_2, \ldots, \beta_j} a_{m_1}^{2\beta_1} \ldots a_{m_j}^{2\beta_j} \\
&\leq \sup \frac{A_{2\beta_1, 2\beta_2, \ldots, 2\beta_j}}{A_{\beta_1, \beta_2, \ldots, \beta_j}} (\sum_{i=0}^n a_i^2)^m \\
&\leq m^m (\sum a_i^2)^m
\end{aligned}$$

Taking limit $n \to \infty$ we get the desired result. □

More information about Rademacher function can be found in Kac [2]. We now begin with the proof the theorem

Proof. Let $g(t) = \sum a_k \phi_k(t)$ where $\phi(t)$ are Rademacher functions. Then

$$\int_0^1 |g(t)|^{2m} dt \leq m^m (\sum |a_k|^2)^m.$$

By Hölder's inequality we have

$$
\begin{aligned}
(\int g(t)^2 dt)^3 &\leq (\int g(t) dt)^2 (\int g(t)^4 dt)^1 \\
(\sum |a_k|^2)^3 &\leq (\int g(t) dt)^2 [2^2 (\sum |a_k|^2)^2] \\
(\sum |a_k|^2)^{1/2} &\leq 2 \int_0^1 g(t) dt
\end{aligned}
$$

These are two inequalities we use later in the proof.

Let E be a Sidon set and let $f(x) = \sum c_k \, \mathrm{e}^{i n_k x}$ be an E-polynomial. Now we define a two variable polynomial $g(t, x)$ which is Rademacher polynomial in one variable and an E-polynomial in the other variable that is

$$g(t, x) = \sum c_k \phi_k(t) \, \mathrm{e}^{i n_k x} .$$

For a fixed t we can find a measure $\mu \in \mathcal{M}(\mathbb{T})$ such that $\|\mu_t\| \leq B$ for a constant B and $\hat{\mu}(n_k) = \phi_k(t)$. Hence $f = g_t * \mu_t$ where $g_t = g(t, x)$. So we have,

$$\|f\|_q \leq \|g_t\|_q \|\mu_t\| \leq B \|g_t\|_q$$

But also notice that $g_t = f * \mu_t$, so

$$\|g_t\|_1 \leq B \|f\|_1 .$$

These statements are true for all $0 \leq t \leq 1$. So

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^{2m} dx \leq B^{2m} \frac{1}{2\pi} \int_{-\pi}^{\pi} |g_t(x)|^{2m} dx$$

Now integrating both with sides with respect to t and using Fubini's theorem we get,

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} |f(x)|^{2m} dx dt &\leq B^{2m} \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} |g(t, x)|^{2m} dx dt \\
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^{2m} dx &\leq B^{2m} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^1 |g(t, x)|^{2m} dt dx \\
&\leq B^{2m} m^m (\sum |c_k|^2)^m
\end{aligned}
$$

So we have

$$\|f\|_{2m} \leq B \sqrt{m} \|f\|_2$$

for all integral values of m. For a general q, there is an integer m so that $2m - 2 \leq q \leq 2m$. Then

$$\|f\|_q \leq \|f\|_{2m} \leq B \sqrt{m} \|f\|_2 \leq B \sqrt{q} \|f\|_2 .$$

Similarly, using the second inequality we obtain

$$\|f\|_2 \leq 2B \|f\|_1$$

□

3.5 Applications

We begin by extending the following classical theorem of Fourier series.

Theorem 3.12 (F. and M. Riesz)**.** *If $\mu \in \mathcal{M}(\mathbb{T})$ and $\hat{\mu}(n) = 0$ for all $n > 0$ then μ is absolutely continuous.*

This can be found in Katznelson [5] pg 101.

Theorem 3.13. *If E is of type $\Lambda(1)$ with positive elements and μ is a measure whose Fourier Stieltjes series is of the form*

$$\sum_{-\infty}^{0} c_n\, e^{inx} + \sum_{n \in E} c_n\, e^{inx},$$

then μ is absolutely continuous.

Proof. Let $f(z) = \sum_{n \in E} c_n z^n$ for $|z| < 1$. The f belongs to the space H_p for every $p < 1$. So f can be approximated by E-polynomials in H_p.

Now since $E \in \Lambda(p)$, these E-polynomials are bounded in $\mathrm{L}^1(\mathbb{T})$ so $\sum_{n \in E} c_n\, \mathrm{e}^{inx}$ is Fourier Stieltjes. Its difference with the orignal series is already known to be Fourier Stieltjes, so the orignal series is a Fourier series. Thus μ is absolutely continuous. □

Now we define permutations of $\mathrm{L}^p(\mathbb{T})$ functions. A permutation τ is said to carry $\mathrm{L}^p(\mathbb{T})$ to $\mathrm{L}^p(\mathbb{T})$ if the series $\sum a(\tau(n))\, \mathrm{e}^{inx}$ is a fourier series of an $\mathrm{L}^p(\mathbb{T})$ function whenever $\sum a(n)\, \mathrm{e}^{inx}$ is a Fourier series.

For a given p the set of all permutations from $\mathrm{L}^p(\mathbb{T})$ to $\mathrm{L}^p(\mathbb{T})$ forms a group. We call the group $G(p)$. The set $G(p)$ has the following properties.

Proposition 3.14.

1. *If $1/p + 1/q = 1$, then $G(p) = G(q)$.*
2. *If $2 \leq p_2 < p_1$, then $G(p_2) \supset G(p_1)$.*

Proof.

1. If we let T_τ be the operator defined by the permutation *tau* on $\mathrm{L}^p(\mathbb{T})$. Let T^* be the adjoint operator of T. Then T^* is a bounded linear operator from $\mathrm{L}^q(\mathbb{T})$ to $\mathrm{L}^q(\mathbb{T})$ $T^* = T_{\tau^{-1}}$. So $\tau \mapsto \tau^{-1}$ is an isomorphism from $G(p)$ to $G(q)$. Thus $G(p) = G(q)$.
2. Let $\tau \in G(p_1)$. Then T_τ is a bounded linear operator from $\mathrm{L}^{p_1}(\mathbb{T})$ to $\mathrm{L}^{p_1}(\mathbb{T})$ and also from $\mathrm{L}^{q_1}(\mathbb{T})$ to $\mathrm{L}^{q_1}(\mathbb{T})$ where q_1 is such that $1/p_1 + 1/q_1 = 1$. Applying Riesz-Thorin interpolation on T_τ we get that t_τ is a bounded operator from $\mathrm{L}^{p_2}(\mathbb{T})$ to $\mathrm{L}^{p_2}(\mathbb{T})$. So $\tau \in G(p_2)$.

□

Theorem 3.15. *If $2 \leq p_1 < p_2 < \infty$ and if $\Lambda(p_2)$ is properly contained in $\Lambda(p_1)$, then $G(p_2)$ is a proper subgroup of $G(p_1)$.*

Proof. Let E be of the type $\Lambda(p_1)$ but not $\Lambda(p_2)$. So there exists a function $g \in \mathrm{L}^{q_2}(\mathbb{T})$ such that

$$\sum_{n \in E} |\hat{g}(n)|^2 = \infty$$

We decompose the set E into two infinite set E_1 and E_2 so that E_2 is of type $\Lambda(p_2)$. So, E_1 is not of type $\Lambda(p_2)$. Then we can define a permutation τ which fixes the set $\mathbb{N} \setminus E$ and takes the set E_1 to E_2 and vice versa. Then

$$\sum_{n \in E_2} |\hat{g}(\tau(n))|^2 = \infty.$$

Hence the series $\sum \hat{g}(\tau(n))\, \mathrm{e}^{inx}$ is not in $\mathrm{L}^{q_2}(\mathbb{T})$. So, $\tau \notin G(q_2)$.

But if $f \in \mathrm{L}^{q_2}(\mathbb{T})$, then $\sum_{n \in E} |\hat{g}(\tau(n))|^2 \leq \infty$. So $\tau \in G(q_1)$.

Now since $G(q_1) = G(p_1)$ and $G(q_2) = G(p_2)$, we have $G(p_2) \subsetneq G(p_1)$. □

Lemma 3.16. *Let $s > 1$ be an integer. Then there is a set E consisting of non-negative integers such that*

1. $r_s(E, n) \leq s!$ *for all* n.
2. $\limsup \alpha_E(N)/N^{1/s} > 0$.

The above two conditions imply that E is of type $\Lambda(2s)$ but not of type $\Lambda(2s + \epsilon)$ for $\epsilon > 0$. Combining this with the previous theorem we get that $G(2s + \epsilon) \subsetneq G(2s)$.

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