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Norm convergence of Multiple Fourier Series

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Contents

Notations

- $\mathbb{R}$ is the set of real numbers.
- $$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$$
- $\mathrm{L}^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ is the set of functions such that
  $$\int |f(x)|^p dx < \infty$$
- $$\|f\|_p := (\int |f(x)|^p dx)^{1/p}$$
  where the integral is over $\mathbb{R}^n$ or $\mathbb{T}^n$ depending on whether the function f belongs to $\mathrm{L}^p(\mathbb{R}^n)$ or $\mathrm{L}^p(\mathbb{T}^n)$. The integral in this case as in everywhere else in the report is the Lebesgue integrals on Euclidean domains.
- $$f * g = \frac{1}{2\pi} \int f(t-\tau)g(\tau)d\tau$$
- We define an operator T from a function space X to a function space Y if for all $f \in X$, $Tf \in Y$.

  If (X, d_1) and (Y, d_2) are metric spaces, then we can define a norm on the operator T as
  $$\|T\|_{(X,Y)} = \sup_{f \in X} \frac{\|Tf\|_{d_2}}{\|f\|_{d_1}}.$$
  So, if we have
  $$\|Tf\|_{d_2} \leq C\|f\|_{d_1} \qquad \text{for all } f \in X$$
  then $\|T\|_{(X,Y)} \leq C$ and hence T is a bounded operator. Also, it is easy to show that an operator is bounded if and only is it is continuous. And so, we will often use the two terms interchangeably.

Chapter 1

Fourier Series

1.1 Introduction

Fourier, in 1822, inspired by his solution to the heat equation, claimed that any 2π periodic function can be written as a trigonometric series:

$$f = \sum_{n=0}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Using complex numbers we can rewrite this as

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\mathrm{e}^{int}$$

where

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{-int}\mathrm{d}t.$$

Due to a lack of a good definition of function during Fourier's time this question could not be answered very well.

Now although this does not hold for any general function we can define a Fourier series associated with any 2π periodic function f to be

$$\mathrm{S}[f] := \sum_{n=-\infty}^{\infty} \widehat{f}(n)\mathrm{e}^{int}.$$

The first question we could ask is whether $\mathrm{S}[f]$ equals f. To formalize this we define the partial sums

$$\mathrm{S}_N[f] := \sum_{n=-N}^{N} \widehat{f}(n)\mathrm{e}^{int}$$

and then we ask the question under what conditions on f does $\mathrm{S}_n[f]$ converges to f. Ideally we would like to know when $\mathrm{S}_n[f]$ converges to f point-wise but often this is not possible. So to better understand the Fourier series we will be satisfied by asking when the convergence occurs in some norm of the function space, more specifically for our project in the $L^p(\mathbb{T})$ norm, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. That is

$$\lim_{N\to\infty} \|\mathrm{S}_N[f] - f\|_p = 0.$$

Some elementary properties of Fourier coefficients:

Theorem 1.1.1. *Let f, $g \in \mathrm{L}^p(\mathbb{T})$,*

1. $\widehat{(f+g)}(n) = \widehat{f}(n) + \widehat{g}(n)$
2. *For any complex number α,*
$$\widehat{(\alpha f)} = \alpha \widehat{f}$$
3. *Let $\bar{f}$ be the conjugate of f, then*
$$\widehat{\bar{f}}(n) = \overline{\widehat{f}}(-n)$$
4. *Let $f_\tau(t) = f(t-\tau)$, $\tau \in \mathbb{T}$, then*
$$\widehat{f_\tau}(n) = \widehat{f}(n)e^{-in\tau}$$
5. $|\widehat{f}(n)| \leq \frac{1}{2\pi}\int |f(t)|dt = \|f\|_1$

Proof. 1.

$$\begin{aligned}
\widehat{(f+g)}(n) &= \frac{1}{2\pi}\int_0^{2\pi} \{f(t)+g(t)\}e^{-int}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t) + \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-int}dt \\
&= \widehat{f}(n) + \widehat{g}(n)
\end{aligned}$$

2.

$$\begin{aligned}
\widehat{(\alpha f)}(n) &= \frac{1}{2\pi}\int_0^{2\pi} \alpha f(t)e^{-int}dt \\
&= \frac{1}{2\pi}\alpha\int_0^{2\pi} f(t)e^{-int}dt \\
&= \alpha\widehat{f}(n)
\end{aligned}$$

3.

$$\begin{aligned}\widehat{\overline{f}}(n) &= \frac{1}{2\pi}\int_0^{2\pi} \overline{f}(t)e^{-int}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \overline{f}(t)\overline{e^{int}}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \overline{(f(t)e^{int})}dt \\ &= \overline{\widehat{f}}(-n)\end{aligned}$$

4.

$$\begin{aligned}\widehat{f_\tau}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f_\tau(t)e^{-int}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t-\tau)e^{-int}dt \\ &= e^{-in\tau}\frac{1}{2\pi}\int_0^{2\pi} f(t-\tau)e^{-in(t-\tau)}dt \\ &= \widehat{f}(n)e^{-in\tau}\end{aligned}$$

5.

$$\begin{aligned}|\widehat{f}(n)| &= |\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)e^{-int}|dt \\ &\leq \frac{1}{2\pi}\int |f(t)|dt = \|f\|_1\end{aligned}$$

□

Now we define the convolution operator on $\mathrm{L}^1(\mathbb{T})$. Let $f, g \in \mathrm{L}^1(\mathbb{T})$. Then the convolution of f and g is

$$f * g = \frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau$$

Theorem 1.1.2. *Let $f, g \in \mathrm{L}^1(\mathbb{T})$. Then $f * g \in \mathrm{L}^1(\mathbb{T})$ and*

$$\|f * g\|_1 \leq \|f\|_1\|g\|_1$$

$$\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)\forall n$$

Proof. Let $F(t,\tau) = f(t-\tau)g(\tau)$. Clearly F is measurable as f and g are measurable as functions of two variables. Now,

$$\frac{1}{2\pi}\int(\frac{1}{2\pi}\int|F(t,\tau)|dt)d\tau = \frac{1}{2\pi}\int|g(\tau)|\|f\|_1 = \|f\|_1\|g\|_1$$

By the Fubini's theorem, $f(t-\tau)g(\tau)$ is integrable as a function of τ for almost all t. Also,

$$\begin{aligned}\frac{1}{2\pi}\int|f*g(t)|dt &= \frac{1}{2\pi}\int|\frac{1}{2\pi}\int F(t,\tau)d\tau|dt \\ &\leq \frac{1}{4\pi^2}\int\int|F(t,\tau)|d\tau dt \\ &= \|f\|_1\|g\|_1\end{aligned}$$

Now,

$$\begin{aligned}\widehat{f*g}(n) &= \frac{1}{2\pi}\int f*g(t)e^{-int}dt \\ &= \frac{1}{2\pi}\int(\frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau)e^{-int}dt \\ &= \frac{1}{4\pi^2}\int\int f(t-\tau)e^{-in(t-\tau)}g(\tau)e^{-in\tau}dtd\tau \\ &= \frac{1}{2\pi}\int f(t)e^{-int}dt.\frac{1}{2\pi}\int g(\tau)e^{-in\tau}d\tau \\ &= \widehat{f}(n)\widehat{g}(n)\end{aligned}$$

□

Also, note that $f*g = g*f$.

1.2 Summability Kernel

We know that

$$\begin{aligned}S_N f(x) &= \sum_{j=-N}^{N}\widehat{f}(j)e^{ijx} \\ &= \sum_{j=-N}^{N}\frac{1}{2\pi}\int_0^{2\pi}f(t)\mathrm{e}^{-int}\mathrm{d}te^{ijx} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{j=-N}^{N}\mathrm{e}^{-in(x-t)}\right]f(t)\mathrm{d}t\end{aligned}$$

Now if we let

$$D_N(t) = \sum_{j=-N}^{N}\mathrm{e}^{-in(x-t)} = \frac{\sin\left[N+1/2\right]s}{\sin(s/2)}$$

then the the above equation can be written as

$$\begin{aligned} S_N f(x) &= \frac{1}{2\pi} \int_0^{2\pi} D_N(x-t) f(t) \mathrm{d}t \\ &= D_N * f(x). \end{aligned}$$

Here the family of functions D_N is termed as the Dirichlet kernel

A summability kernel is a family $\{k_N\}$ of continuous 2π periodic functions satisfying the following three conditions:

1. $$\frac{1}{2\pi} \int k_N(t) dt = 1$$

2. $$\frac{1}{2\pi} \int |k_N(t)| dt \leq C \qquad \text{for all } N$$

3. For all $0 < \delta < \pi$,
$$\lim_{n \to \infty} \int_{\delta}^{2\pi - \delta} |k_N(t)| dt = 0$$

Some examples of Summability kernels are :

1. Fejér kernels:
$$F_N(t) = \frac{1}{N+1} \frac{\sin \frac{(N+1)t}{2}}{\sin \frac{t}{2}}$$

2. Poisson kernels:
$$P_r(t) = \frac{1 - r^2}{1 - 2r \cos t + r^2} \qquad \text{for all } 0 \leq r < 1$$

   In this case we take the limit as $r \to 1^-$

It is easy to check that

Theorem 1.2.1. *Let $\{k_N\}$ be a family of summability kernels If f is a continuous function on $\mathbb{T}$, then*

$$\lim_{N \to \infty} f * k_N(x) = f(x)$$

Proof. Let $\epsilon > 0$. Since f is a continuous function on $\mathbb{T}$, f is uniformly continuous. So, we can choose a δ such that for all $|t| < \delta$, $|f(x+t) - f(x)| < \epsilon$ for all $x \in \mathbb{T}$. Also, let M be the bound of f.

$$
\begin{aligned}
|\frac{1}{2\pi}\int_0^{2\pi} k_N(t)f(x-t)\mathrm{d}t - f(x)| &= |\frac{1}{2\pi}\int_0^{2\pi} k_N(t)f(x-t)\mathrm{d}t - \frac{1}{2\pi}\int_0^{2\pi} f(x)k_N(t)\mathrm{d}t| \\
&= |\frac{1}{2\pi}\int_0^{2\pi} [f(x-t)-f(x)]k_N(t)\mathrm{d}t| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |[f(x-t)-f(x)]||k_N(t)|\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{\{t:|t|<\delta\}} |[f(x-t)-f(x)]||k_N(t)|\mathrm{d}t \\
&\quad + \frac{1}{2\pi}\int_{\{t:|t|\geq\delta\}} |[f(x-t)-f(x)]||k_N(t)|\mathrm{d}t
\end{aligned}
$$

For small enough δ and large enough N

$$
\begin{aligned}
&\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} \epsilon|k_N(t)|\mathrm{d}t \\
&\quad + \frac{1}{2\pi}\int_{\{t:|t|\geq\delta\}} 2M\epsilon\mathrm{d}t \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \epsilon|k_N(t)|\mathrm{d}t \\
&\quad + \frac{1}{2\pi}\int_{\{t:|t|\geq\delta\}} 2M\epsilon\mathrm{d}t \\
&\leq C\epsilon + 2M\epsilon
\end{aligned}
$$

Hence we have

$$\lim_{N\to\infty} f * k_N(x) = f(x)$$

$\square$

1.3 Convergence in $L^2(\mathbb{T})$

It is easy to check convergence of Fourier series of functions in $\mathrm{L}^2(\mathbb{T})$ in L^2 norm. This is because $\mathrm{L}^2(\mathbb{T})$ is a Hilbert space with its inner product defined as

$$\langle f, g\rangle = \frac{1}{2\pi}\int f(t)\overline{g(t)}dt$$

and the exponential functions form a complete orthogonal system.

For any Hilbert space we have the following:

Lemma 1.3.1. *Let $\{\phi_i\}_1^\infty$ be an orthogonal system of the Hilbert space and let $\{a_i\}_1^\infty$ be a sequence of complex numbers such that $\sum |a_i|^2 \leq \infty$. Then $\sum_{n=1}^\infty a_i\phi_i$ converges in the Hilbert space.*

Proof. To show that $\sum_{n=1}^{\infty} a_i\phi_i$ converges we show that the sequence of partial sums, $S_N = \sum_{j=1}^{N} a_j\phi_j$, is a Cauchy sequence.

For $N > M$, we have

$$\begin{aligned}
\|S_N - S_M\|^2 &= \|\sum_{M+1}^{N} a_j\phi_j\|^2 \\
&= \langle \sum_{M+1}^{N} a_j\phi_j, \sum_{M+1}^{N} a_j\phi_j\rangle \\
&= \sum_{M+1}^{N} a_j.\langle\phi_j, \sum_{M+1}^{N} a_j\phi_j\rangle \\
&= \sum_{M+1}^{N} a_j\bar{a_j} \\
&= \sum_{M+1}^{N} |a_j|^2 \to 0 \text{ as } M \to \infty
\end{aligned}$$

□

Lemma 1.3.2 (Bessel's Inequality)**.** *Let $\{\phi_j\}$ be a family of orthonormal functions. For any function f in the Hilbert space if we write $a_j = \langle f, \phi_j\rangle$, then*

$$\sum |a_j|^2 \leq \|f\|^2.$$

Proof.

$$\begin{aligned}
\|f - \sum_{-N}^{N} a_j\phi_j\|^2 &= \langle f - \sum_{-N}^{N} a_j\phi_j, f - \sum_{-N}^{N} a_j\phi_j\rangle \\
&= \|f\|^2 - \sum_{-N}^{N} \bar{a_j}\langle f, \phi_j\rangle - \sum_{-N}^{N} a_j\langle f, \phi_j\rangle + \sum_{-N}^{N} |a_j|^2 \quad = \|f\| - \sum_{-N}^{N} |a_j|^2
\end{aligned}$$

Now the left hand side of the above equation is always greater that 0. So we have

$$\|f\|^2 - \sum_{-N}^{N} |a_j|^2 \geq 0$$

Now taking the limit as N tends to infinity, we get

$$\|f\|^2 - \sum |a_j|^2 \geq 0$$

□

Theorem 1.3.3. *Let $f \in \mathrm{L}^2(\mathbb{T})$. Then*

1.
$$\sum |\widehat{f}(n)|^2 = \frac{1}{2\pi}\int |f(t)|^2 dt$$

2.
$$f = \lim_{N\to\infty}\sum_{-N}^{N}\widehat{f}(n)e^{int} \qquad \textit{in the } \mathrm{L}^2(\mathbb{T}) \textit{ norm.}$$

3. *For any square summable sequence $\{a_j\}_{j\in\mathbb{Z}}$ of complex numbers there exists a unique $f \in \mathrm{L}^2(\mathbb{T})$ such that $a_n = \widehat{f}(n)$.*

4. *Let $f, g \in \mathrm{L}^2(\mathbb{T})$. Then*
$$\frac{1}{2\pi}\int f(t)\overline{g(t)}dt = \sum_{-\infty}^{\infty}\widehat{f}(n)\overline{\widehat{g}(n)}.$$

Proof. Let $\phi_n = e^{int}$, then ϕ_n form a complete orthonormal basis of $\mathrm{L}^2(\mathbb{T})$. Let $a_n = \langle f, \phi_n\rangle$, then by Bessel's Inequality we have
$$\sum |a_j|^2 \leq \|f\|^2 < \infty$$
So by Lemma 4 we can say that $\sum\langle f, \phi_n\rangle\phi_n$ converges in $\mathrm{L}^2(\mathbb{T})$. If we denote $g = \sum\langle f, \phi_n\rangle\phi_n$, then $\langle g, \phi_n\rangle = \langle f, \phi_n\rangle$ or in other words, $g - f$ is orthogonal to $\{\phi_n\}$. Hence
$$f = \sum\langle f, \phi_n\rangle\phi_n = \lim_{N\to\infty}\sum_{-N}^{N}\widehat{f}(n)e^{int}$$
in $\mathrm{L}^2(\mathbb{T})$ norm. Now,
$$\begin{aligned}
\|f\|^2 &= \langle f, f\rangle \\
&= \langle \sum\langle f, \phi_n\rangle\phi_n, f\rangle \\
&= \sum\langle f, \phi_n\rangle\langle\phi_n, f\rangle \\
&= \sum a_n\overline{a_n} \\
&= \sum |a_n|^2
\end{aligned}$$
Hence
$$\sum |\widehat{f}(n)|^2 = \frac{1}{2\pi}\int |f(t)|^2 dt.$$

For Part 3 let $\{a_j\}_{j\in\mathbb{Z}}$ be any square summable sequence of complex numbers. Then by Lemma 4 we can say that $\sum a_n\phi_n$ converges in $\mathrm{L}^2(\mathbb{T})$. If we denote $f = \sum a_n\phi_n$, then $a_n = \widehat{f}(n)$. Suppose $g \in \mathrm{L}^2(\mathbb{T})$ such that $\langle g, \phi_n\rangle = a_n = \langle f, \phi_n\rangle$. So $\langle g - f, \phi_n\rangle$, or in other words, $g - f$ is orthogonal to $\{\phi_n\}$. Hence $g = f$ and so we have a unique representation.

For part 4 we have

$$\begin{aligned}\frac{1}{2\pi}\int f(t)\overline{g(t)}dt &= \langle f, g\rangle \\ &= \langle \sum_n \langle f, \phi_n\rangle \phi_n, g\rangle \\ &= \sum_n \langle f, \phi_n\rangle \langle \phi_n, g\rangle \\ &= \sum_n \widehat{f}(n)\overline{\widehat{g}(n)}.\end{aligned}$$

□

1.4 Partial Summation as an operator

Looking at the problem of convergence of $\mathrm{S}_n f$ to f in L^p norm from the functional analytic point of view gives us the following theorem:

Theorem 1.4.1. *$S_N f$ converges to f in L^p norm, $1 \leq p < \infty$, if and only if there exists a constant C_p, which is independent of N such that*

$$\|S_N f\|_p \leq C_p \|f\|_p$$

or in other words operator norm of S_N, represented by $\|S_N\|_{(L^p(\mathbb{T}^n), L^p(\mathbb{T}^n))}$, is uniformly bounded.

Proof. First assume that $\mathrm{S}_N f$ converges to f in L^p norm for all f in $\mathrm{L}^p(\mathbb{T})$. Then $\mathrm{S}_N f$ is bounded for all f in $\mathrm{L}^p(\mathbb{T})$. So, by Hahn Banach theorem S_N is a uniformly bounded operator.

For the converse assume that S_N is a uniformly bounded sequence of operators. Also, let $f \in \mathrm{L}^p(\mathbb{T})$ and a constant $\epsilon > 0$. Since the set of trigonometric polynomials is dense in $\mathrm{L}^p(\mathbb{T})$, there exists a trigonometric polynomial P such that

$$\|f - P\|_p \leq \frac{\epsilon}{C_p + 1}.$$

For any N greater that the degree of trigonometric polynomial(define) P, say N_0, $\mathrm{S}_N[P] = P$. So, for $N > N_0$

$$\begin{aligned}\|\mathrm{S}_N[f] - f\|_p &= \|\mathrm{S}_N[f] - \mathrm{S}_N[P] + P - f\|_p \\ &\leq \|\mathrm{S}_N[f] - \mathrm{S}_N[P]\|_p + \|P - f\|_p \\ &\leq \|\mathrm{S}_N[f - P]\|_p + \|P - f\|_p \\ &\leq C_p \frac{\epsilon}{C_p + 1} + \frac{\epsilon}{C_p + 1} \\ &\leq \epsilon\end{aligned}$$

□

So we have now reduced the problem of proving the norm convergence of the Fourier series to the problem of proving that the partial summation operator is uniformly bounded. We now proceed to reduce it to proving that a different operator, the Hilbert transform, is bounded.

Hilbert transform

We now introduce a piece of terminology that will be very useful later on, that of a multiplier operator.

Let $\Lambda = \{\lambda_j\}_{j=-\infty}^{\infty}$ be a sequence of numbers then the multiplier operator M_Λ is defined as :

$$M_\Lambda : f(x) \longmapsto \sum_{j=-\infty}^{\infty} \lambda_j \widehat{f}(j) \exp^{ijx}$$

Now if we let $\Lambda = \{-i\,\mathrm{sgn}(j)\}_{j=-\infty}^{\infty}$ the we get the Hilbert multiplier operator as:

$$\mathrm{H} : f(x) \longmapsto \sum_{i=-\infty}^{\infty} -i \ \mathrm{sgn} j \widehat{f}(j) \exp^{ijx} .$$

For the Hilbert multiplier we have the following theorem:

Theorem 1.4.2. *If operator H is bounded then* $\|S_N f - f\|_p \to 0$

Proof. Before we begin with the proof notice that

$$\widehat{f}(x - N) = \exp^{ijN} \widehat{f}(x).$$

We have $\mathrm{H}[f] = \sum_{j=-\infty}^{\infty} -i\,\mathrm{sgn}(j)\widehat{f}(j)$ So,

$$\mathrm{H}[\exp^{iNx} f] = \sum_{j=-\infty}^{\infty} -i\,\mathrm{sgn}(j - N)\widehat{f}(j)e^{ijx}$$

Similarly,

$$\mathrm{H}[e^{-iNx} f] = \sum_{j=-\infty}^{\infty} -i\,\mathrm{sgn}(j + N)\widehat{f}(j)e^{ijx}$$

This tells us that

$$\frac{1}{2} i.e.^{-iNt}\mathrm{H}[e^{iNx} f] - \frac{1}{2} e^{iNt}\mathrm{H}[e^{-iNx} f] = \mathrm{S}_{N-1} f(e^{it})$$

Now, $\|e^{int} f(t)\|_p = \|f\|_p$. So,

$$\begin{aligned} \|S_{N-1}\|_{(L^p,L^p)} &\leq \frac{1}{2}\|H\|_{(L^p,L^p)} + \frac{1}{2}\|H\|_{(L^p,L^p)} \\ &\leq \|H\|_{(L^p,L^p)} \end{aligned}$$

□

Let f be a real continuous functions on $\mathbb{T}$. Let $u = P_r * f$ where P_r is the Poisson kernel , so u is harmonic on the disc D and vanishes at 0. Let v be the harmonic conjugate of u on D such that $v(0) = 0$. So,

$$v_r(t) = Q_r * f(t)$$

where $Q_r(t) = \dfrac{r\sin(t)}{2\pi|1-re^{it}|^2}$

If we let $r \to 1^-$, then

$$\begin{aligned}
\lim_{r\to 1^-} Q_r(t) = &= \frac{r\sin(t)}{2\pi|1-re^{it}|^2} \\
&= \frac{1\sin(t)}{2\pi|1-e^{it}|^2} \\
&= \frac{1\sin(t)}{2\pi(1-2\cos t+1)} \\
&= \frac{1\sin(t)}{4\pi(1-\cos t)} \\
&= \frac{1}{4\pi}\cot(\frac{t}{2})
\end{aligned}$$

From now on let us call $\lim_{r\to 1^-} Q_r(t)$ as kernel k.

Now observe that

$$\widehat{Q_r}(n) = -i\,\mathrm{sgn}(n)e^{ir}$$
$$\widehat{k}(n) = -i\,\mathrm{sgn}(n)$$

Hence Q_r is the family of kernels associated with the Hilbert transform, that is

Let us look at $Q_r * f(t)$ under the limits $r \to 1^-$. This becomes

$$\lim_{r\to 1^-} Q_r * f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cot(\frac{\tau}{2})f(\tau - t)d\tau$$

The function inside the integral is unbounded at $t = 0$. We resolve this problem by defining the Cauchy principal value of the integral.

$$\text{p.v.}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\cot(\frac{\tau}{2})f(\tau-t)d\tau := \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\epsilon<|t|\le\pi}\cot(\frac{\tau}{2})f(\tau-t)d\tau$$

Hence we can also look at the Hilbert transform as

$$\mathrm{H} : f(x) \longmapsto \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\epsilon<|t|\le\pi} f(\tau-t)\cot(\frac{\tau}{2})d\tau.$$

Proposition 1.4.3. *The Hilbert transform is bounded on $L^p(\mathbb{T})$ when $p = 2k$ is a positive even integer.*

Proof. Let f be a real continuous functions on $\mathbb{T}$. Add a constant to f so that $\int f dx = 0$. Let $u = P_r * f$ where P_r is the Poisson kernel , so u is harmonic on the disc D and vanishes at 0. Let v be the harmonic conjugate of u on D such that $v(0) = 0$. Then v is the convolution kernel of the Hilbert transform and $u + iv$ is holomorphic and $h(0) = 0$.

Take a $0 < r < 1$. Then

$$\begin{aligned}
0 &= 2\pi h^p(0) \\
&= \int_0^{2\pi} h^{2k}(re^{i\theta}) d\theta \\
&= \int_0^{2\pi} [u(re^{i\theta}) + iv(re^{i\theta})]^{2k} d\theta \\
&= \int_0^{2\pi} u^{2k} d\theta + i\binom{2k}{1} \int_0^{2\pi} u^{2k-1} v d\theta + \dots \\
&\quad + i^{2k-1}\binom{2k}{2k-1} \int_0^{2\pi} uv^{2k-1} d\theta + i^{2k} \int_0^{2\pi} v^{2k} d\theta
\end{aligned}$$

This gives us

$$i^{2k} \int_0^{2\pi} v^{2k} d\theta = \int_0^{2\pi} u^{2k} d\theta + i\binom{2k}{1} \int_0^{2\pi} u^{2k-1} v d\theta + \dots + i^{2k-1}\binom{2k}{2k-1} \int_0^{2\pi} uv^{2k-1} d\theta$$

Taking modulus on both sides we get

$$\int_0^{2\pi} v^{2k} d\theta \leq \int_0^{2\pi} |u^{2k}| d\theta + \binom{2k}{1} \int_0^{2\pi} |u^{2k-1} v| d\theta + \dots + \binom{2k}{2k-1} \int_0^{2\pi} |uv^{2k-1}| d\theta$$

Now applying Hölder's inequality to each term with the exponents $2k/(2k-j)$ and $2k/j$ on the j-th term of the above summation and letting $S = \int (u^{2k} d\theta)^{1/2k}$ and $T = \int (V^{2k} d\theta)^{1/2k}$ we get

$$\binom{2k}{j} \int_0^{2\pi} |u^{2k-j} v^j| d\theta \leq \binom{2k}{j} S^{2k-j} T^j$$

And so

$$T^{2k} \leq S^{2k} + \binom{2k}{1} S^{2k-1} T + \dots + \binom{2k}{2k-1} S T^{2k-1}$$

Now dividing both sides of the inequality by S^{2k} and letting $U = T/U$ we get

$$U^{2k} \leq 1 + \binom{2k}{1} U + \dots + \binom{2k}{2k-1} U^{2k-1}$$

Now dividing both sides by U^{2k-1} we get

$$U \leq U^{-2k+1} + \binom{2k}{1} U^{-2k-2} + \dots + \binom{2k}{2k-2} U^{-1} + \binom{2k}{2k-1} 1$$

This implies that U is bounded because, if $U \geq 1$, then

$$\begin{aligned} U &\leq 1 + \binom{2k}{1} + \ldots + \binom{2k}{2k-2} + \binom{2k}{2k-1} 1 \\ &\leq 2^{2k} \end{aligned}$$

Hence

$$U \leq 2^{2k}$$
$$\|v\|_{2k} \leq 2^{2k} \|u\|_{2k}$$

Now as $v(re^{i\theta})$ is the Hilbert transform of $u(re^{i\theta})$ we are done.

□

Theorem 1.4.4. *The Hilbert transform is bounded on L^p, $1 < p < \infty$.*

Proof. Because of the earlier lemma the Hilbert transform is bounded in L^p for $p = 2, 4, 6, \ldots$. Now applying Riesz-Thorin interpolation we get that Hilbert transform is bounded for $2 < p < 4$, $4 < p < 8$, ... Hence Hilbert transform is bounded for $2 \leq p < \infty$. Now, for any $f \in L^p$ for $1 < p < 2$, let ϕ be any function in $L^{p'}$ where $p' = p/(p-1)$ with norm 1. Notice that $2 < p' < \infty$. So,

$$\begin{aligned} \int H[f].\phi &= \int [\int f(\psi) cot(\frac{\theta - \psi}{2}) d\psi] \psi(\theta) d\theta \\ &= \int [\int \psi(\theta) cot(\frac{\theta - \psi}{2}) d\theta] f(\psi) d\psi \qquad \text{By Fubini's Theorem} \\ &= - \int [\int \psi(\theta) cot(\frac{\psi - \theta}{2}) d\theta] f(\psi) d\psi \\ &= - \int H[\phi].f \end{aligned}$$

Now calculating the modulus of both sides we get

$$\begin{aligned} |\int H[f].\phi| &= |\int H[\phi].f| \\ &\leq \|H[\phi]\|_{p'} \|f\|_p \qquad \text{By Hölder's Inequality} \\ &\leq C \|\phi\|_{p'} \|f\|_p \\ &\leq C \|f\|_p \end{aligned}$$

Now since this is true for any such ϕ, Hilbert transform is bounded on L^p for $1 < p < 2$. □

1.5 Multiple Fourier series

In a similar fashion to what we have defined in the chapter so far, we can define the Fourier series for a two variable function $f(x, y) \in L^p(\mathbb{T}^2)$ to be

$$\mathrm{S}[f(x,y)] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{f}(m,n) \mathrm{e}^{i(mx+ny)}$$

where $\hat{f}(m,n) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(s,t)\mathrm{e}^{-i(ms+nt)}\mathrm{d}s\mathrm{d}t$.

And more generally for any $f \in L^p(\mathbb{T}^n)$

$$\mathrm{S}[f(\mathbf{x})] = \sum_{j\in\mathbb{Z}^n} \hat{f}(j)e^{ij.\mathbf{x}}$$

where $\hat{f}(j) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(s,t)\mathrm{e}^{-i\mathbf{x}.j}\mathrm{d}\mathbf{x}$.

In these cases it is no longer obvious as to how to the define partial sums. For example we could define the partial sum as :

$$\mathrm{S}_N^{\mathrm{sq}}[f(x,y)] = \sum_{m=-N}^{N}\sum_{n=-N}^{N} \hat{f}(m,n)\mathrm{e}^{i(mx+ny)}$$

Or we could define it as :

$$\mathrm{S}_N^{\mathrm{sp}}[f(x,y)] = \sum\sum_{m^2+n^2\leq N^2} \hat{f}(m,n)\mathrm{e}^{i(mx+ny)}$$

Here $\mathrm{S}_N^{\mathrm{sq}}$ is called the square summation of Fourier series. And $\mathrm{S}_N^{\mathrm{sp}}$ is the spherical summation of the Fourier series.

The very interesting result that we will investigate further is that the square summation converges whereas the spherical sum can diverge for certain functions. For this purpose we will restrict our attentions to the two variable case.

Chapter 2

Fourier Transform

2.1 Introduction

In a manner similar to the Fourier series we can define the Fourier transform for a function $f \in \mathrm{L}^1(\mathbb{R}^n)$ to be

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix.\xi} dx \tag{2.1}$$

where $x.\xi = x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n$. And we define the partial sums as

$$\mathrm{S}_R[f] = \int_{B_R} \widehat{f}(\xi) e^{-ix.\xi} d\xi$$

where $B_R = \{\, Rx : x \in B \,\}$,$B$ is an open convex neighbourhood of the origin.

Proposition 2.1.1. *If $f \in \mathrm{L}^1(\mathbb{R}^n)$, then*

$$\|\widehat{f}\|_\infty \leq \|f\|_1$$

Proof.

$$\begin{aligned}
|\widehat{f}(\xi)| &= |\int_{\mathbb{R}^n} f(x) e^{ix.\xi} dx| \\
&\leq \int_{\mathbb{R}^n} |f(x) e^{ix.\xi}| dx| \\
&= \|f\|_1
\end{aligned}$$

□

Proposition 2.1.2. *If $f \in \mathrm{L}^1(\mathbb{R}^n)$, f is differentiable and $\dfrac{\partial f}{\partial x_j} \in \mathrm{L}^1(\mathbb{R}^n)$, then*

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = -i\xi_j \widehat{f}(\xi)$$

Proof. We will first prove it for functions with compact support.

Assume that $f \in C_c^\infty(\mathbb{R})$ then

$$\begin{aligned}\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \int \frac{\partial f}{\partial x_j} e^{ix.\xi} dx \\ &= \int \dots \int [\int \frac{\partial f}{\partial x_j} e^{ix.\xi} dx_j] dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n\end{aligned}$$

By Gauss divergence theorem we have

$$\begin{aligned}&= -\int \dots \int [\int f \frac{\partial e^{ix.\xi}}{\partial x_j} dx_j] dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n \\ &= -i\xi_j \int \dots \int f e^{ix.\xi} dx \\ &= -i\xi_j \widehat{f}(\xi)\end{aligned}$$

Since there exist a sequence of C_c^∞ functions, $\{f_k\}$ converging to any $f \in \mathrm{L}^1(\mathbb{R}^n)$ such that $\{\frac{\partial f_k}{\partial x_j}\}$ converges to $\{\frac{\partial f}{\partial x_j}\}$, we are done. $\square$

Proposition 2.1.3. *If $f \in \mathrm{L}^1(\mathbb{R}^n)$, f is differentiable and $ix_j f \in \mathrm{L}^1(\mathbb{R}^n)$, then*

$$\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) = \widehat{(ix_j f)}(\xi)$$

Proof. By taking the derivative inside the integral we obtain the desired equation. $\square$

Proposition 2.1.4. *If $f, g \in L^1$, then*

$$\widehat{f * g} = \widehat{f}.\widehat{g}$$

The proof goes exactly as in the case of Fourier series. Also, we can define the partial sums as

$$\mathrm{S}_R[f] = \int_{B^R} \hat{f}(\xi) e^{-2\pi i x.\xi} d\xi$$

where $B^R = \{\, Rx : x \in B \,\}$,$B$ is an open convex neighbourhood of the origin. Again we can ask whether this partial summation converges to f. We will come back to this question latter.

We end this section by calculating the Fourier transform of $e^{-\pi|x|^2}$.

Lemma 2.1.5. *If $f(x) = e^{-\pi|x|^2}$ then $\widehat{f}(x) = (2\pi)^{n/2} e^{-\pi|x|^2}$*

Proof.

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{(2\pi)}n/2 \int_{\mathbb{R}^n} f(x) e^{ix.\xi} dx \\
&= \frac{1}{(2\pi)}n/2 \int_{\mathbb{R}^n} f(x) e^{x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n} dx \\
&= \frac{1}{(2\pi)}n/2 [\int_{\mathbb{R}^n} f(x) e^{x_1\xi_1} dx + \int_{\mathbb{R}^n} f(x) e^{x_2\xi_2} dx \ldots + \int_{\mathbb{R}^n} f(x) e^{x_n\xi_n} dx]
\end{aligned}$$

As a result of this, it is sufficient to prove the above proposition for the one variable case.

$$\frac{d\widehat{f}}{d\xi} = \int_{-\infty}^{\infty} ixe^{-x^2/2} e^{ix\xi} dx$$

Integrating by parts with $dv = xe^{-x^2/2}dx$ and $u = ie^{ix\xi}$ and taking the boundary term to be 0 due to the presence of $e^{-x^2/2}$ we get

$$\frac{d\widehat{f}}{d\xi} = -\xi \int_{-\infty}^{\infty} e^{-x^2/2} e^{ix\xi} dx = -\xi \widehat{f}(\xi)$$

Solving this partial differential equation with $\widehat{f}(0) = \int f(x)dx = \sqrt{2\pi}$ we get

$$\widehat{f}(\xi) = \sqrt{2\pi} e^{-\xi^2/2}.$$

And for $\mathbb{R}^n$ we get

$$\widehat{f} = (\sqrt{2\pi})^n e^{-|\xi|^2/2}$$

□

2.2 Schwartz Space and Tempered Distribution

We define Schwartz space, $\mathcal{S}(\mathbb{R}^n)$, to be the set of all infinitely differentiable functions, all of whose derivatives decrease rapidly at infinity. If we call

$$p_{\alpha,\beta}(f) = \sup_x |x^\alpha D^\beta f(x)|$$

for $\alpha, \beta \in \mathbb{N}^n$. Here we have used the multi-index notation. Using $p_{\alpha,\beta}(f)$ we can define,

$$\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : p_{\alpha,\beta}(f) < \infty \ \forall \alpha, \beta \in \mathbb{N}^n \}.$$

We can enumerate the $p_\alpha \beta$ as p_k. Then we can define a metric on $\mathcal{S}$ as

$$d(\phi, \psi) = \sum \frac{1}{2^k} . \frac{d_k(\phi - \psi)}{1 + d_k(\phi - \psi)}.$$

We can define the space of Tempered distribution $\mathcal{S}'$ as the dual space of $\mathcal{S}$, that is, the space of linear maps from $\mathcal{S}$ to $\mathbb{C}$ such that if $T \in \mathcal{S}'$ and $\lim_{k\to\infty} \phi_k = 0$ then

$$\lim_{k\to\infty} T(\phi_k) = 0.$$

Theorem 2.2.1. *The Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$ such that*

1. $$\int f\hat{g} = \int \hat{f}g;$$

2. $$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{-ix.\xi}$$

Proof. From Propositions – and – we have

$$\xi^\alpha D^\beta \hat{f}(\xi) = C(\widehat{D^\alpha x^\beta} f)(\xi)$$

$$|\xi^\alpha D^\beta \hat{f}(\xi)| = C\|D^\alpha x^\beta f\|_1$$

The L^1 norm can be bounded by a finite linear combinations of $p_{\alpha\beta}(f)$. Hence Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$.

By a simple application of Fubini's theorem we get

$$\begin{aligned}\int f\hat{g} &= \int f(\xi)\int g(x)e^{i\xi.x}dxd\xi\\ &= \int\int f(\xi)e^{i\xi.x}d\xi g(x)dx\\ &= \int \hat{f}g\end{aligned}$$

By a change of variable and application of Fubini's theorem we have

$$\begin{aligned}\int f(x)\hat{g}(\lambda x)dx &= \int \hat{f}(x)\lambda^{-n}g(x/\lambda)dx\\ \int f(x/\lambda)\hat{g}(x)dx &= \int \hat{f}(x)g(x/\lambda)dx\\ \text{Let } \lambda \to \infty&,\\ f(0)\int \hat{g}(x)dx &= g(0)\int \hat{f}(x)dx\end{aligned}$$

Now let $g(x) = e^{-|x|^2}$, then

$$f(0) = \int \hat{f}(\xi)d\xi$$

Now of we replace $f(x)$ by $h(x) = f(y-x)$, then

$$f(y) = \int \hat{h}(\xi)d\xi = \int \hat{f}(\xi)e^{-iy.\xi}$$

$\square$

We define the Fourier transform of $T \in \mathcal{S}'$ as $\widehat{T} \in \mathcal{S}'$ such that

$$\widehat{T}(f) = T(\hat{f}), \qquad f \in \mathcal{S}$$

Notice that if T is an integrable function then $\widehat{T}$ coincides with the definition of Fourier transform given earlier.

Theorem 2.2.2. *The Fourier transform is a bounded linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also bounded.*

Proof. If $T_n \to T$ in $\mathcal{S}'$, then for any $f \in \mathcal{S}$,

$$\widehat{T}_n(f) = T_n(\hat{f}) \to T(\hat{f}) = \widehat{T}(f).$$

Also, the inverse of Fourier transform is equivalent to applying the Fourier transform thrice as Fourier transform has a period 4

$$\widehat{(\hat{f})} = f(-x).$$

Hence the inverse is also continuous. $\square$

As in the case of Schwartz space we have for $T \in \mathcal{S}'$

$$T(x) = \int \widehat{T}(\xi)e^{-ix.\xi}d\xi.$$

If $f \in L^p$, $1 \le p \le \infty$, then f can be identified with a tempered distribution:

$$T_f(\phi) = \int f\phi < \infty$$

Suppose $\phi_k \to 0$ in $\mathcal{S}$ as $k \to \infty$. Then by Hölder's inequality

$$|T_f(\phi_k)| \le \|f\|_p \|\phi_k\|_{p'}.$$

Then $\|\phi_k\|_{p'}$ is dominated by the L^∞ norm of functions of the form $x^a\phi_k$, and so by a finite linear combination of $p_{\alpha\beta}(\phi_k)$. Therefore the left-hand side tends to 0 as $k \to \infty$. Thus T_f is a continuous operator. Moreover, when $1 \le p \le 2$, $\hat{f}$ is a function.

Theorem 2.2.3. *The Fourier transform is an isometry on L^2, that is, $\hat{f} \in L^2$ and $\|\hat{f}\|_2 = \|f\|_2$. Furthermore,*

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x)e^{ix.\xi}dx \quad \textit{and}$$

$$f(x) = \lim_{R\to\infty} \int_{|\xi|<R} f(x)e^{ix.\xi}d\xi$$

The identity $\|\hat{f}\|_2 = \|f\|_2$ is referred to as the Plancherel theorem.

Proof. Let $f, h \in \mathcal{S}$ and $g = \bar{\hat{h}}$ which implies that $\hat{g} = \bar{h}$. Now,

$$\int f\hat{g} = \int \hat{f}g$$

So,

$$\int f\bar{h} = \int \hat{f}\bar{\hat{h}}$$

If we let $h = f$ then we get

$$\|\hat{f}\|_2 = \|f\|_2 \text{ for } f \in \mathcal{S}.$$

Now since $\mathcal{S}$ is dense in L^2, the Fourier transform extends to all f in L^2.

Now,

$$\hat{f}(\xi)\chi_{|x|<R} = \int_{\mathbb{R}} f(x)\chi_{|x|<R}(x)e^{ix.\xi}dx$$

Taking the limit $R \to \infty$ we get

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x)e^{ix.\xi}dx$$

$\square$

2.3 Partial Summation as an Operator

In a manner analogous to Section 1.4 we have

Theorem 2.3.1. *$S_N f$ converges to f in $L^p(\mathbb{R}^n)$ norm, $1 \leq p < \infty$, if and only if there exists a constant C_p, which is independent of N such that*

$$\|S_N f\|_p \leq C_p\|f\|_p$$

or in other words operator norm of S_N, represented by $\|S_N\|_{(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))}$, is uniformly bounded.

Now let us first restrict our attentions to the single variable case. The multi variable case will be treated in the next chapter.

As with the Fourier series, an operator $\mathcal{M}_s$ is called a multiplier operator if

$$\widehat{\mathcal{M}_s f}(\xi) = s(\xi)\widehat{f}(\xi)$$

and we call the function $s(\xi)$, the Fourier multiplier. We also have

$$M_s : f(x) \longmapsto \int_{\mathbb{R}^n} s(\xi)\widehat{f}(\xi)\exp^{-ix.\xi} d\xi$$

So, Hf as a multiplier operator with a the multiplier function $-i\ \mathrm{sgn}(\xi)$ because

$$\widehat{\mathrm{H}f}(\xi) = -i\ \mathrm{sgn}(\xi)\widehat{f}(\xi).$$

Hence,

$$\mathrm{H}f = \int_{-\infty}^{\infty} -i\ \mathrm{sgn}(\xi)\widehat{f}(\xi)e^{-ix\xi}d\xi$$

As in the case of Hilbert transform on $\mathrm{L}^p(\mathbb{T})$,Hilbert transform can also be written as

$$Hf = \text{p.v.}\ \frac{1}{(2\pi)}\int_{\mathbb{R}} f(x-t)\cot(t/2)dt$$

Proposition 2.3.2. *In $\mathcal{S}'$,*

$$\lim_{t\to 0} Q_t = \frac{1}{\pi}\ p.v.\ \frac{1}{x}$$

Proof. For each $\epsilon > 0$, the function $\psi_\epsilon(x) = \frac{1}{x}\chi_{|x|>\epsilon}$ are bounded and define tempered distributions. Thus in $\mathcal{S}'$,

$$\lim_{\epsilon\to 0} = \text{p.v.}\ \frac{1}{x}.$$

So, it is enough to prove that in $\mathcal{S}'$

$$\lim_{t\to 0}(Q_t - \frac{1}{\pi}\psi_t) = 0.$$

Take $\phi \in \mathcal{S}$, then

$$\begin{aligned}
(\pi - \psi_t)(\phi) &= \int_{\mathbb{R}} \frac{x\phi(x)}{t^2+x^2}dx - \int_{|x|>t}\frac{\phi(x)}{x}dx \\
&= \int_{|x|<t}\frac{x\phi(x)}{t^2+x^2}dx - \int_{|x|>t}(\frac{x}{t^2+x^2} - \frac{1}{x})\phi(x)dx \\
&= \int_{|x|<1}\frac{x\phi(tx)}{1+x^2}dx - \int_{|x|>1}\frac{\phi(tx)}{x(1+x^2)}dx
\end{aligned}$$

Now taking the limit $t \to 0$ we have,

$$\lim_{t\to 0}(\pi - \psi_t)(\phi) = \int_{|x|<1}\frac{x\phi(0)}{1+x^2}dx - \int_{|x|>1}\frac{\phi(0)}{x(1+x^2)}dx$$

Both of these terms are integral of odd functions on symmetric domains. Hence,

$$\lim_{t\to 0}(\pi - \psi_t)(\phi) = 0$$

$\square$

Thus we can also write the Hilbert transform as

$$Hf = \text{p.v.}\, \frac{1}{(2\pi)} \int_{\mathbb{R}} \frac{f(x-t)}{x} dt$$

By repeating the calculations done for the case of Hilbert transform of 2π-periodic functions we can show that

Theorem 2.3.3. *The operator H is bounded if and only if $\|S_N f - f\|_p \to 0$, i.e., the operator S_N is bounded.*

We proceed further by proving that the Hilbert transform is bounded for the case of $\mathrm{L}^p(\mathbb{R})$, $1 < p \leq \infty$, thereby proving that the single variable partial summation converges.

In $\mathbb{R}^n$ we define the unit cube, open on the right, to be the set $[0,1)^n$ and we let Q_0 be the collection of cubes in $\mathbb{R}^n$ which are congruent to $[0,1)^n$ and whose vertices lie on the lattice $\mathbb{Z}^n$. If we dilate this family of cubes by a factor of 2^{-k} we get the collection Q_k, $k \in \mathbb{Z}$; that is Q_k is a family of cubes, open on the right, whose vertices are adjacent points of the lattice $(2^{-k}\mathbb{Z})^n$. The cubes in $\cup_k Q_k$ are called the dyadic cubes.

For this purpose we state the following theorem, called the Calderón-Zygmond decomposition theorem.

Theorem 2.3.4. *Given a function f which is integrable and non-negative and given a positive number λ, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that*

1. $f(x) \leq \lambda$ *for almost every* $x \notin \bigcup_j Q_j$;
2. $|\bigcup_j Q_j| \leq \frac{1}{\lambda}\|f\|_1$;
3. $\lambda < \frac{1}{|Q_j|}\int_{Q_j} f \leq 2^n \lambda$.

We will not be proving this theorem but we will use it prove that the Hilbert transform is a bounded linear operator on $L^p(\mathbb{R})$.

Theorem 2.3.5. *For $\mathcal{S}(\mathbb{R})$, the following are true:*

1. *H is weak* $(1,1)$*:*

$$|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

2. *H is strong* (p,p). $1 < p < \infty$

$$\|Hf\|_p \leq C_p \|f\|_p$$

Proof. Fix $\lambda > 0$ and f to be some non-negative function. From the Calderón Zygmund decomposition of f at height λ, we get a sequence of disjoint intervals $\{I_j\}$ such that

$$f(x) \leq \lambda \text{ for a.e. } x \notin \Omega = \bigcup_j I_j,$$

$$|\omega| \leq \frac{1}{\lambda}\|f\|_1,$$
$$\lambda < \frac{1}{|I_j|}\int_{I_j} f \leq 2\lambda.$$

Using this decomposition of $\mathbb{R}$ we define a decomposition of f as a sum of two functions, g and b, defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega \\ \frac{1}{|I_j|}\int_{I_j} f & \text{if } x \in \Omega \end{cases}$$

and

$$b(x) = \sum_j b_j(x) \text{ where } b_j(x) = (f(x) - \frac{1}{|I_j|}\int_{I_j} f)\chi_{I_j}(x).$$

So we have $g(x) \leq 2\lambda$ almost everywhere and b_j is supported on I_j and has zero integral. Since $Hf = Hg + Hb$,

$$\begin{aligned} &|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}| \\ &\qquad \leq |\{\, x \in \mathbb{R} : |\, Hg(x)| > \lambda/2 \,\}| + |\{\, x \in \mathbb{R} : |\, Hb(x)| > \lambda/2 \,\}| \end{aligned}$$

Now we estimate the two terms separately.

$$\begin{aligned} |\{\, x \in \mathbb{R} : |\, Hg(x)| > \lambda/2 \,\}| &\leq (\frac{2}{\lambda})^2 \int_{\mathbb{R}} |H(g)|^2 dx \\ &= \frac{4}{\lambda^2}\int g(x)^2 dx \\ &\leq \frac{8}{\lambda}\int g(x) dx \\ &= \frac{8}{\lambda}\int f(x) dx. \end{aligned}$$

be the interval with the same centre as I_j and twice the length, and let $\Omega* = \cup_j 2I_j$. Then $|\Omega*| \leq 2|\Omega|$ and

$$\begin{aligned} |\{\, x \in \mathbb{R} : |\, Hb(x)| > \lambda/2 \,\}| &\leq |\Omega*| + |\{\, x \notin \Omega* : |\, Hb(x)| > \lambda/2 \,\}| \\ &\leq \frac{1}{\lambda}\|f\|_1 + \frac{2}{\lambda}\int_{\mathbb{R}\setminus\Omega*} |Hb(x)| dx. \end{aligned}$$

Now $|Hb(x)| \leq \sum_j |Hb_j(x)|$ almost everywhere: this is immediate if the sum is finite and otherwise it follows from the fact that $\sum b_j$ and $\sum Hb_j$ converge to b in Hb in L^2. Hence, it is now sufficient to show that

$$\sum_j \int_{\mathbb{R}\setminus 2I_j} |Hb(x)| dx \leq C\|f\|_1.$$

Now

$$Hb_j(x) = \int_{I_j} \frac{b_j(y)}{x-y} dy.$$

Denote the centre of I_j by c_j, then

$$\begin{aligned}\int_{\mathbb{R}\backslash 2I_j} |Hb(x)|dx &= \int_{\mathbb{R}\backslash 2I_j} |\int_{I_j} \frac{b_j(y)}{x-y}dy|dx \\ &= \int_{\mathbb{R}\backslash 2I_j} |\int_{I_j} b_j(y)(\frac{1}{x-y} - \frac{1}{x-c_j})dy|dx \\ &\leq \int_{I_j} |b_j(y)| \int_{\mathbb{R}\backslash 2I_j} \frac{|y-c_j|}{|x-y||x-c_j|}dxdy\end{aligned}$$

Now as $|y - c_j| < |I_j|/2$ and $|x - y| > |x - c_j|/2$.

$$\int_{\mathbb{R}\backslash 2I_j} |Hb(x)|dx \leq \int_{I_j} |b_j(y)| \int_{\mathbb{R}\backslash 2I_j} \frac{|I_j|}{|x-c_j|^2}dxdy$$

We have

$$\int_{\mathbb{R}\backslash 2I_j} \frac{|I_j|}{|x-c_j|^2}dx = 2$$

so,

$$\int_{\mathbb{R}\backslash 2I_j} |Hb(x)|dx \leq 2\int_{I_j} |b_j(y)|dy \leq 4\|f\|_1.$$

Any real function f and be decomposed into positive and negative parts as $f = f^+ - f^-$. Thus we have proved the first part

$$|Hf(x)| > \lambda| \leq \frac{C}{\lambda}\|f\|_1.$$

We also know that

$$\|Hf\|_2 = \|f\|_2$$

So, by the Marcinkiewicz interpolation theorem we have, for $1 < p < 2$,

$$\|Hf\|_p \leq C_p\|f\|_p$$

Now by duality argument this is also true for $2 < p < \infty$. $\square$

Chapter 3

Summation of Fourier Series

3.1 Fourier Multipliers

A multiplier operator,$\mathcal{M}_m$ is termed as a Fourier multiplier for L^p if it is bounded on L^p.

For Fourier multiplier on L^2 we have the following result

Proposition 3.1.1. *Let μ be a measurable function on $\mathbb{R}^n$. Then μ is a Fourier multiplier for L^2 if and only if μ is essentially bounded or in other words $\mu \in L^\infty$.*

Proof. ($\Rightarrow$) Let μ be an essentially bounded function with bound M. If $f \in L^2$, then both $\widehat{f}$ and $\mu\widehat{f}$ lie in L^2. By Plancherel theorem we have

$$\|(2\pi)^{n/2}\mathcal{M}_\mu f\|_2 = \|\widehat{\mathcal{M}_\mu f}\|_2 = \|\mu\widehat{f}\|_2 \leq \|\mu\|_\infty \|\widehat{f}\|_2.$$

($\Leftarrow$) Let $\mu \notin L^\infty$. Suppose for $a > 0$ and $0 < R < \infty$

$$E_{a,R} = \{\, x : |\ \mu(x)| > a, |x| < R \,\}.$$

Then

$$m(E_{a,r}) < (2R)^n < \infty.$$

For any fixed a and large enough R we have $m(E_{a,R}) > 0$ as $\mu \notin L^\infty$. Now,

$$\|\mu\chi_{Ea,r}\|_2 \geq a\|\chi_{E_{a,R}}\|_2$$

which can be made as large as required. Hence,

$$\|(2\pi)^{n/2}\mathcal{M}_\mu f\|_2 = \|\mu\widehat{f}\|_2$$

is unbounded. $\square$

Lemma 3.1.2. *If m is a Fourier multiplier for some L^p with $1 < p < \infty$. Let $p' = p/(p-1)$ be the conjugate exponent for p. The m is a Fourier multiplier for $L^{p'}$.*

Proof. For $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\|\mathcal{M}_m f\|_{p'} &= \sup_{\phi \in L^p,\ \|\phi\|_p = 1} \left| \int (\mathcal{M}_m f).\tilde{\phi} dx \right| \\
&= \sup_{\phi \in L^p,\ \|\phi\|_p = 1} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(\xi) \widehat{f}(\xi) \exp^{ix.\xi} d\xi . \tilde{\phi} dx \right|
\end{aligned}$$

By Fubini's Theorem

$$\begin{aligned}
&= \sup_{\phi \in L^p,\ \|\phi\|_p = 1} \left| \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \tilde{\phi}(x) \exp^{ix.\xi} dx \right] m(\xi) \widehat{f}(\xi) d\xi . \right| \\
&= \sup_{\phi \in L^p,\ \|\phi\|_p = 1} \left| \int_{\mathbb{R}^n} m \widehat{\phi} \widehat{f} dx . \right| \\
&= \sup_{\phi \in L^p,\ \|\phi\|_p = 1} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(\xi) \widehat{\phi}(\xi) \exp^{ix.\xi} d\xi . \tilde{f} dx \right| \\
&\leq \sup_{\phi \in L^p,\ \|\phi\|_p = 1} \|f\|_{p'} \|\mathcal{M}_m \phi\|_p \\
&\leq C' \sup_{\phi \in L^p,\ \|\phi\|_p = 1} \|f\|_{p'} \|\phi\|_p \\
&= C' \|f\|_{p'}
\end{aligned}$$

And now since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^{p'}$ we obtain the desired result. $\square$

Lemma 3.1.3. *Let f be a continuous function on $\mathbb{R}^n$. If f is 2π-periodic in each variable then*

$$\lim_{\epsilon \to 0} \frac{1}{(2\pi)^{n/2}} \epsilon^{n/2} \int_{\mathbb{R}^n} f(x) e^{-\epsilon |x|^2/2} dx = \frac{1}{(2\pi)^N} \int_{Q_n} f(x) dx$$

where Q_n is defined to be the set $\{ (x_1, \ldots, x_n) : -\pi \leq x_j < \pi \}$.

Proof. Let $f = e^{ij.x}$ where $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$. Now

$$\text{LHS} = \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{n/2}} \epsilon^{n/2} \int_{\mathbb{R}^n} e^{ij.x} e^{-\epsilon |x|^2/2} dx$$

By Lebesgue differentiation theorem we get

$$\text{LHS} = 1.$$

Now clearly

$$\text{RHS} = \frac{1}{(2\pi)^N} \int_{Q_n} e^{ij.x} dx = 1$$

Now we know that linear combinations of the functions $e^{ij.x}$ are uniformly dense in the continuous, 2π-periodic function. So we have the desired equation. $\square$

Let us introduce a few notations to simplify the next lemma. First let

$$\eta(x) = \frac{1}{(2\pi)^{-n/2}} e^{-|x|^2/2}.$$

Notice that this function is positive everywhere and $\int \eta(x)dx = 1$. Next let

$$\eta_{\sqrt{\epsilon}}(x) = \frac{1}{(2\pi)^{-n/2}} \epsilon^{n/2} e^{-\epsilon|x|^2/2}.$$

Notice that we can get this by replacing x by $\sqrt{\epsilon}x$ in the earlier equation and so this too is positive and $\int \eta_{\sqrt{\epsilon}}(x)dx = 1$. The following lemma provides passage from $\mathbb{R}^n$ to $\mathbb{T}^n$ and vice versa.

Lemma 3.1.4. *Let P, Q be trigonometric polynomials. Let S be a L^p multiplier operator on $\mathbb{R}^n$, $1 < p < 2$, with multiplier function s. Let for each $j \in \mathbb{Z}^n$*

$$\lim_{\epsilon \to 0} \epsilon^{-N} \int_{|t| \leq \epsilon} |s(j-t) - s(j)| dt = 0.$$

Also assume that $f(x) = \sum a_j e^{ij.x}$ is a trigonometric polynomial with

$$\tilde{S}f(x) = \sum_{j \in \mathbb{Z}^n} s(-j) a_j e^{ij.x}.$$

Let $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Then

$$\lim_{\epsilon \to 0} (\frac{2\pi}{\alpha\beta\epsilon})^{n/2} \int_{\mathbb{R}^n} [S(P.\eta_{\sqrt{\epsilon\alpha}})(x)][\overline{Q(x).\eta_{\sqrt{\epsilon\beta}}(x)}]dx = \frac{1}{(2\pi)^n} \int_{Q_n} [(\tilde{S}P)(x)][\overline{Q(x)}]dx$$

Proof. Let $P(x) = e^{ij.x}$ and $Q(x) = e^{ik.x}$. By applying Plancherel's Theorem to the left hand side we obtain

$$\begin{aligned} &(\frac{2\pi}{\alpha\beta\epsilon})^{n/2} \int_{\mathbb{R}^n} [S(P.\eta_{\sqrt{\epsilon\alpha}})(x)][\overline{Q(x).\eta_{\sqrt{\epsilon\beta}}(x)}]dx \\ &\quad = (\frac{2\pi}{\alpha\beta\epsilon})^{n/2} \frac{1}{(2\pi)^{-n}} \int_{\mathbb{R}^n} s(\xi)\phi(\xi).\overline{\psi(\xi)}d\xi \end{aligned}$$

where

$$\phi(\xi) = \widehat{P.\eta_{\sqrt{\epsilon\alpha}}} = e^{-|\xi+j|^2/2\epsilon\alpha}$$

and

$$\psi(\xi) = \widehat{Q.\eta_{\sqrt{\epsilon\beta}}} = e^{-|\xi+k|^2/2\epsilon\beta}$$

Now we divide the problem into two case when $j = k$ and when $j \neq k$.

If $j \neq k$ We have $|j-k| \geq 1$. Since s is a Fourier multiplier s is bounded by some constant, say C. So,

$$
\begin{aligned}
&(\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\frac{1}{(2\pi)^{-n}}\int_{\mathbb{R}^n} s(\xi)\phi(\xi).\overline{\psi(\xi)}d\xi \\
&\qquad = (\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\frac{1}{(2\pi)^{-n}}\int_{\mathbb{R}^n} s(\xi)e^{-|\xi+j|^2/2\epsilon\alpha}e^{-|\xi+k|^2/2\epsilon\beta}d\xi \\
&\qquad \leq (\frac{2\pi}{\alpha\beta\epsilon})^{n/2}C\frac{1}{(2\pi)^{-n}}\int_{\mathbb{R}^n} e^{-|\xi+j|^2/2\epsilon\alpha}e^{-|\xi+k|^2/2\epsilon\beta}d\xi \\
&\qquad \leq C'\epsilon^{n/2}[\int_{|\xi+j|\geq 1/2} + \int_{|\xi+k|\geq 1/2}] \\
&\qquad = I + J \text{ say.}
\end{aligned}
$$

Now for the first term

$$
\begin{aligned}
\lim_{\epsilon\to 0} I &= \lim_{\epsilon\to 0} C'\epsilon^{n/2}\int_{|\xi+j|\geq 1/2} e^{-|\xi+j|^2/2\epsilon\alpha}e^{-|\xi+k|^2/2\epsilon\beta}d\xi \\
&\leq \lim_{\epsilon\to 0} C'\epsilon^{n/2}\int_{|\xi+j|\geq 1/2} e^{-1/8\epsilon\alpha}e^{-|\xi+k|^2/2\beta}d\xi
\end{aligned}
$$

Hence $I \to 0$ as $\epsilon \to 0$. Similarly for J we have the same result, hence

$$
\lim_{\epsilon\to 0}(\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\int_{\mathbb{R}^n}[S(P.\eta_{\sqrt{\epsilon\alpha}})(x)][\overline{Q(x).\eta_{\sqrt{\epsilon\beta}}(x)}]dx = 0
$$

for the given choice of P and Q. Now,

$$
\text{RHS} = \frac{1}{(2\pi)^{-n}}\int_{Q_n} s(-j)e^{ij.x}e^{-ik.x}dx = 0
$$

as $j \neq k$. Therefore the relation holds for this case.

If $j = k$ Since $\alpha + \beta = 1$

$$
\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\alpha\beta}.
$$

$$
\begin{aligned}
&\lim_{\epsilon\to 0}(\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\frac{1}{(2\pi)^{-n}}\int_{\mathbb{R}^n} s(\xi)e^{-|\xi+j|^2/2\epsilon\alpha}e^{-|\xi+k|^2/2\epsilon\beta}d\xi \\
&\qquad = \lim_{\epsilon\to 0}(\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\frac{1}{(2\pi)^{-n}}\int_{\mathbb{R}^n} s(\xi)e^{-|\xi+j|^2/2\epsilon\alpha\beta}d\xi \\
&\qquad = s(-j) \\
&\qquad = \text{RHS}
\end{aligned}
$$

Additionally the linearity of the relation implies that the relation is true for any trigonometric polynomials P and Q. $\square$

Now we have the following theorem implying that convergence of a multiplier operator in $L^p(\mathbb{T}^n)$ is enough to show the convergence of a corresponding multiplier operator in $L^p(\mathbb{R}^n)$.

Theorem 3.1.5. *Let $1 < p < \infty$ and S be an multiplier operator with the multiplier function s for $(R)^n$. If for each $j \in \mathbb{Z}^n$*

$$\lim_{\epsilon\to 0} \epsilon^{-N} \int_{|t|\leq\epsilon} |s(j-t) - s(j)|dt = 0$$

then there is a unique operator $\tilde{S}$ such that

$$\tilde{S}f := \sum_{j\in\mathbb{Z}^n} s(-j)\widehat{f}(j)e^{ij.x}$$

where $f \in L^p(\mathbb{T}^n)$, that is $\tilde{S}$ is a Fourier multiplier for $L^p(\mathbb{T}^n)$ with

$$\|\tilde{S}\|_{(L^p(\mathbb{T}^n),L^p(\mathbb{T}^n))} \leq \|S\|_{(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))}$$

Proof. Let $p' = p/(p-1)$ be the conjugate exponent. We first restrict ourselves to trigonometric polynomials. So, let P, Q be trigonometric polynomials then with $\alpha = 1/p$ and $\beta = 1/p'$.

$$\begin{aligned}
(\frac{2\pi}{\alpha\beta\epsilon})^{n/2} &\int_{\mathbb{R}^n} [S(P.\eta_{\sqrt{\epsilon\alpha}})(x)][\overline{Q(x).\eta_{\sqrt{\epsilon\beta}}(x)}]dx \\
&\leq (\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\|S\|_{\mathrm{L}^p(\mathbb{R}^n),\mathrm{L}^p(\mathbb{R}^n)}\|P.\eta_{\sqrt{\epsilon\alpha}}\|_p\|Q(x).\eta_{\sqrt{\epsilon\beta}}\|_{p'} \\
&= (\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\|S\|_{\mathrm{L}^p(\mathbb{R}^n),\mathrm{L}^p(\mathbb{R}^n)} \\
&\quad \times [\int_{\mathbb{R}^n} |P(x)|^p[\frac{1}{(2\pi)^{-n/2}}(\epsilon\alpha)^{n/2}]^p e^{-\epsilon\alpha p|x|^2/2}dx]^{1/p} \\
&\quad \times [\int_{\mathbb{R}^n} |Q(x)|^{p'}[\frac{1}{(2\pi)^{-n/2}}(\epsilon\beta)^{n/2}]^{p'} e^{-\epsilon\beta p'|x|^2/2}dx]^{1/p'} \\
&= (\frac{2\pi}{\alpha\beta\epsilon})^{n/2}\|S\|_{\mathrm{L}^p(\mathbb{R}^n),\mathrm{L}^p(\mathbb{R}^n)} \\
&\quad \times [\int_{\mathbb{R}^n} |P(x)|^p[\frac{1}{(2\pi)^{-n/2}}(\epsilon\alpha)^{n/2}]^p e^{-\epsilon|x|^2/2}dx]^{1/p} \\
&\quad \times [\int_{\mathbb{R}^n} |Q(x)|^{p'}[\frac{1}{(2\pi)^{-n/2}}(\epsilon\beta)^{n/2}]^{p'} e^{-\epsilon|x|^2/2}dx]^{1/p'} \\
&= \|S\|_{\mathrm{L}^p(\mathbb{R}^n),\mathrm{L}^p(\mathbb{R}^n)} \\
&\quad \times [\int_{\mathbb{R}^n} |P(x)|^p[\frac{1}{(2\pi)^{-n/2}}(\epsilon)^{n/2}]^p e^{-\epsilon|x|^2/2}dx]^{1/p} \\
&\quad \times [\int_{\mathbb{R}^n} |Q(x)|^{p'}[\frac{1}{(2\pi)^{-n/2}}(\epsilon)^{n/2}]^{p'} e^{-\epsilon|x|^2/2}dx]^{1/p'}
\end{aligned}$$

As $\epsilon \to 0^+$, the RHS tends to

$$\|S\|_{\mathrm{L}^p(\mathbb{R}^n),\mathrm{L}^p(\mathbb{R}^n)}[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|P(x)|^p dx]^{1/p}[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|Q(x)|^{p'}]^{1/p'}$$

and the LHS reduces to

$$\frac{1}{(2\pi)^n}\int_{Q_n}[(\tilde{S}P)(x)][\overline{Q(x)}]dx.$$

So we have

$$|\frac{1}{(2\pi)^n}\int_{Q_n}[(\tilde{S}P)(x)][\overline{Q(x)}]dx| \leq \|S\|_{\mathrm{L}^p(\mathbb{R}^n),\mathrm{L}^p(\mathbb{R}^n)}\|P\|_p\|Q\|_{p'}$$

Now since trigonometric polynomials are dense in L^p, $1 < p < \infty$, $\tilde{S}$ is a bounded operator. $\square$

Lemma 3.1.6. *For a fix p, $1 < p < \infty$, if $\mathcal{M}_Q$ is bounded on $\mathrm{L}^p(\mathbb{R}^n)$, then*

$$\mathcal{M}_{Q_R} : f \longmapsto \frac{1}{(2\pi)}n\int_{\mathbb{R}^n}\chi_{Q_E}(\xi)\widehat{f}(\xi)e^{-ix.\xi}d\xi$$

is bounded independent of R. Here Q_R is the R-fold dilate of the unit cube Q.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\mathcal{M}_{Q_R}f &= \frac{1}{(2\pi)}n\int\chi_{Q_r}(\xi)\widehat{f}(\xi)e^{-ix.\xi}d\xi \\
&= R^n\frac{1}{(2\pi)}n\int\chi_{Q_r}(R\xi)\widehat{f}(R\xi)e^{-ix.R\xi}d\xi \\
&= R^n\frac{1}{(2\pi)}n\int\chi_Q(\xi)\widehat{\alpha^R f}(\xi)e^{-ix.R\xi}d\xi \\
&= R^n[M_Q(\alpha^R f)](Rx).
\end{aligned}$$

Taking the norm we get

$$\begin{aligned}
\|\mathcal{M}_{Q_R}f\|_p &= R^n[\int|\mathcal{M}_Q[\alpha^R f](Rx)|^p dx]^{1/p} \\
&= R^n R^{-n/p}[\int|\mathcal{M}_Q[\alpha^R f](x)|^p dx]^{1/p} \\
&\leq CR^n R^{-n/p}\|\alpha^R f\|_p \\
&= CR^n R^{-n/p}[\int|R^{-n}f(R^{-1}x)|^p dx]^{1/p} \\
&= CR^n R^{-n/p}[\int R^n R^{-np}|f(x)|^p dx]^{1/p} \\
&= C[\int|f(x)|^p dx]^{1/p}.
\end{aligned}$$

Notice that the C here is independent of R. Hence we have shown that $\mathcal{M}_{Q_R}$ is bounded if $\mathcal{M}_Q$ is bounded.

(Although we have shown that this is true for functions in Schwartz space, we can say it for general functions because $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathrm{L}^p(\mathbb{R}^n)$) $\square$

Theorem 3.1.7. *Let P be a point of $\mathbb{R}^2$, $v \in \mathbb{R}^2$ be a unit vector and set $E_v = \{\, x \in \mathbb{R}^2 : (x-P).v \geq 0 \,\}$. Then the operator*

$$\mathcal{M}_{E_v} : f \mapsto \frac{1}{(2\pi)^2} \int_{E_v} \widehat{f}(\xi) e^{-ix.\xi} d\xi$$

is bounded on L^p, $1 < p < \infty$.

The proof of this theorem revolves around reducing this half-plane problem to that of a half line problem which is resolved using the Hilbert transform for $L^p(\mathbb{R})$.

Proof. Notice that in the 1-dimensional case, since the Hilbert transform H is bounded on $\mathrm{L}^p(\mathbb{R})$ for $1 < p < \infty$, then so is the operator $M = 1/2(I + iH)$ which has a Fourier multiplier $m = \chi_{[0,\infty)}$.

Without loss of generality we may assume that $P = 0$ and the vector v is the vector $(0, 1)$. And let us from now on denote E_v as merely E.

Now select a $1 < p < \infty$. We again make estimates for functions in the Schwartz space and then use the fact that $\mathcal{S}(\mathbb{R}^2)$ is dense in $\mathrm{L}^p(\mathbb{R}^2)$.

Let us define a slice $f_{x_1}(x_2)$ of the function $f(x_1, x_2)$ for a fixed x_1 as

$$f_{x_1}(x_2) := f(x_1, x_2).$$

By Fubini's Theorem,

$$\int_{\mathbb{R}} \|f_{x_1}\|_p^p dx_1 = \|f\|_p^p.$$

Now we have

$$\begin{aligned}
\mathcal{M}_E f(x_1, x_2) &= \frac{1}{(2\pi)^2} \int_0^\infty \int_{\mathbb{R}} \widehat{f}(\xi_1, \xi_2) e^{-ix_1\xi_1} e^{-ix_2\xi_2} d\xi_1 d\xi_2 \\
&= \frac{1}{(2\pi)^2} \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t_1, t_2) e^{-it_1\xi_1} e^{-it_2\xi_2} dt_1 dt_2 e^{-ix_1\xi_1} e^{-ix_2\xi_2} d\xi_1 d\xi_2 \\
&= \frac{1}{(2\pi)} \int_{\mathbb{R}} \int_{\mathbb{R}} [\frac{1}{(2\pi)} \int_0^\infty [\int_{\mathbb{R}} f(t_1, t_2) e^{-it_2\xi_2} dt_2] e^{-ix_2\xi_2} d\xi_2] e^{-it_1\xi_1} dt_1 e^{-ix_1\xi_1} d\xi_1 \\
&= \frac{1}{(2\pi)} \int_{\mathbb{R}} \int_{\mathbb{R}} M f_{t_1}(x_2) e^{it\xi_1} dt_1 e^{-i\xi_1 x_1} d\xi_1
\end{aligned}$$

Let us define a function in two variables,

$$F(x_2, t_1) \equiv M f_{t_1}(x_2)$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |F(x_2,t_1)|^p dx_2 dt_1 &= \int_{\mathbb{R}}\int_{\mathbb{R}} |Mf_{t_1}(x_2)|^p dx_2 dt_1 \\ &= \int_{\mathbb{R}} \|Mf_{t_1}\|_p^p dt_1 \\ &\leq C \int_{\mathbb{R}} \|f_{t_1}\|_p^p dt_1 \\ &= C\|f\|^p_{\mathrm{L}^p(\mathbb{R}^2)}\end{aligned}$$

Hence $F \in \mathrm{L}^p(\mathbb{R}^2)$. So we can say that the slice $F_{x_2}(t_1) \in \mathrm{L}^p(\mathbb{R})$. Also, by Fubini's theorem we have

$$\int_{\mathbb{R}} \|F_{x_2}\|_p^p dt_1 \leq C\|f\|^p_{\mathrm{L}^p(\mathbb{R}^2)}$$

So we have

$$\begin{aligned}\mathcal{M}_E f(x_1,x_2) &= \frac{1}{(2\pi)} \int_{\mathbb{R}}\int_{\mathbb{R}} F_{x_2}(t_1) e^{it\xi_1} dt_1 e^{-i\xi_1 x_1} d\xi_1 \\ &= \lim \epsilon \to 0 \int_{\mathbb{R}} \frac{1}{(2\pi)} \int_{\mathbb{R}} F_{x_2}(t_1) e^{it\xi_1} dt_1 e^{-i\xi_1 x_1} e^{-\epsilon|\xi_1|^2} d\xi_1\end{aligned}$$

□

If we define

$$T_i : f \longmapsto \int_{E_i} \widehat{f}(\xi) e^{-2\pi i x.\xi} d\xi \qquad \text{for } i = 1,2,3,4$$

where

$$\begin{aligned} E_1 &= \{\, (x,y) \in \mathbb{R}^2 : (-1,0).[(x,y)-(1,0)] \geq 0 \,\} \\ E_2 &= \{\, (x,y) \in \mathbb{R}^2 : (1,0).[(x,y)-(-1,0)] \geq 0 \,\} \\ E_3 &= \{\, (x,y) \in \mathbb{R}^2 : (0,-1).[(x,y)-(0,1)] \geq 0 \,\} \\ E_4 &= \{\, (x,y) \in \mathbb{R}^2 : (0,1).[(x,y)-(0,-1)] \geq 0 \,\} \end{aligned}$$

Then each of these operators is bounded by the previous theorem. Hence, the composition of these four multiplier operators, which is nothing but the square summation operator χ_Q, is also bounded. This implies that the square summation converges in L^p norm, for $1 < p < \infty$.

Chapter 4

Kakeya Set

In this chapter we take a small detour from our world of Fourier analysis into the world of geometry.

4.1 The Kakeya Problem

In the year 1917 the Japanese mathematician S. Kakeya asked the following question :

In the class of figures in which a segment of length 1 *can be turned around through* 360^o*, remaining always within the figure, which one has the smallest area.*

In the year 1920 A.S. Besicovitch independently answered the following question:

Given a function of two variables, Riemann-integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that the repeated simple integration along the two direction exists and gives the value of the integral over the domain?

This reduces to proving the existence of a set of Jordan plane measure 0 which is the union of segments of all directions each of length ≥ 1.

Besicovitch after discovering the Kakeya problem, managed to modify his solution to his own problem to solve the Kakeya problem.

Here we describe a modified version of Besicovitch's solution an a much later construction by Cunningham which solves the problem of finding a simply connected solution to Kakeya's problem.

4.2 Modified Besicovitch's Construction

Let DEF and GHI be a pair of elementary triangles translated by a distance ϵ, an arbitrary small number.

Then it is very easy to see that

$$|LMN| = |GHK| < \frac{\epsilon}{8}|GHI|.$$

The part consisting of lines GL, DL and the triangle LMN will be called the join.

If the lengths of all the triangles is appropriate then we can move a unit line segment from GI to DE without going outside either the triangles DEF and GHI or the join.

Now if we manage to find a construction of parallel translation of elementary triangles such that the area covered by them is small, then we would be done.

Perron in 1929, found a simple construction to do so. This has been outlined in the following diagrams.

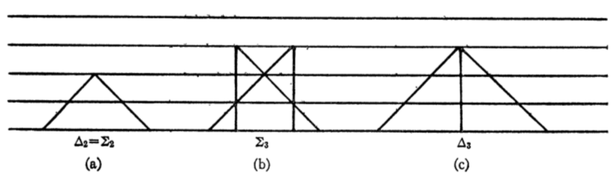

Figure 4.1:

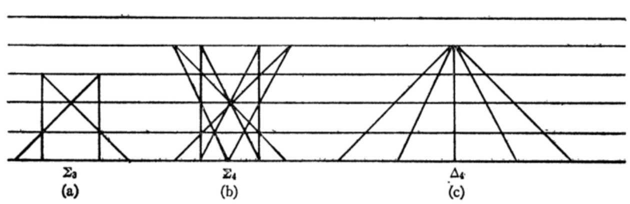

Figure 4.2:

Looking at these without the grid lines

A simple calculation tells us that the area of the figure can be made as small as we please. So, the area of the joins associated with these triangles is also very

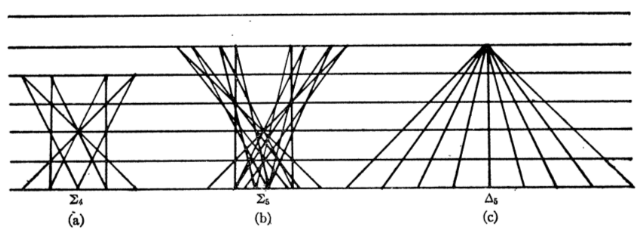

Figure 4.3:

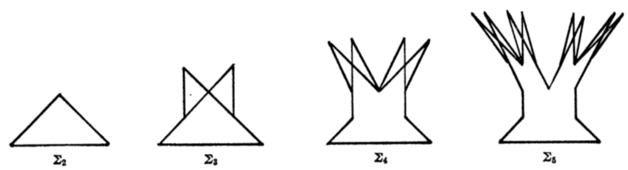

Figure 4.4:

small. Hence we have found a solution to the Kakeya problem with arbitrarily small area.

4.3 Cunningham's Construction

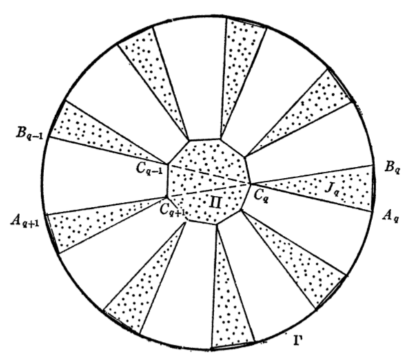

Figure 4.5:

Cunningham starts with the following figure in which we can rotate a unit line segment by starting from $B_{q-1}c_{q-1}$ translating it to C_qB_q, rotating to C_qA_q and then going to $A_{q+1}C_{q+1}$, rotating within the triangle and so on.

Now we select one of these triangles $C_{q-1}C_qC_{q+1}$ and $C_qB_qA_q$ and relabel them as follows.

Now using a sprouting process described later we convert it to the following

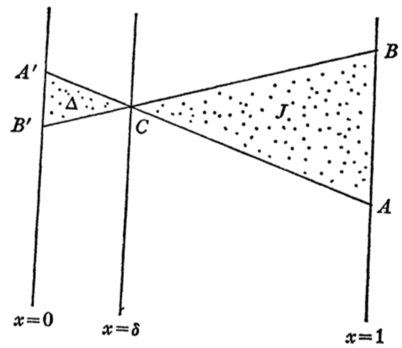

Figure 4.6:

figure. Notice that we can still rotate a unit needle all the way through from $B'B$ to $A'A$, by going through the regions J_1 to J_m. We call this whole construction $K^{(1)}$ with the earlier stage being $K^{(0)}$. Notice that the area of $K^{(1)}$ is ra where a is the area of the original triangle.

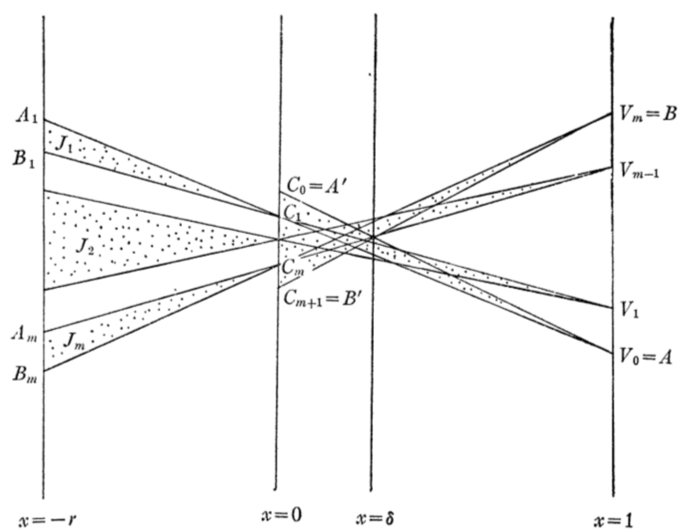

Figure 4.7:

As might be obvious from the diagram the major part of the area of the figure is in the extensions J_1 to J_m. So to reduce the area we again apply the same sprouting procedure to each of J_i's and obtain a refinement $K(2)$. If we repeat this procedure large enough number of times, we can make the area of the figure as small as we like, while still allowing ourselves to rotate the unit segment through the whole angle. We will skip the proof that such a figure will be simply connected.

Sprouting procedure

Let ABC be a triangle with height h. We create a sprouting of the triangle of height h' as follows.

We first extend the side AC and BC to heights h_1. And connect these points A' and B' to the mid-point D of AB. This is shown in the following diagram. Also, this is very similar to how the trees were constructed in the Besicovitch?s construction.

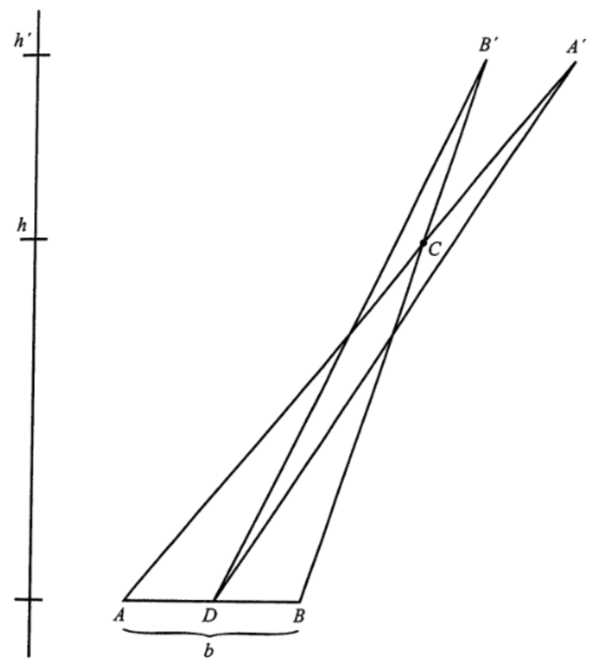

Figure 4.8: Details of the sprouting

Using simple geometric arguments we can show that area of the sprouting is

$$\frac{1}{2}bh + \frac{b(h_1 - h)^2}{2h_1 - h}$$

where $b = |AB|$.

We can repeat the process for each of the sprouts with a new height h_2 and so on. So we can start with a triangle with very small width and height (and hence a very small area). The increase in area with each successive sprouts is small because of the width factor. By appropriately increasing the heights we can obtain a sprout of an given height and as small a area as we like.

Chapter 5

Spherical summation of Multiple Fourier Series

5.1 Vector Valued Inequality

Many non-linear expressions in Fourier analysis can be viewed as linear quantities by viewing them in appropriate Banach spaces. For example, let T be a linear operator acting on L^p of some measure space (X, μ) and taking values in the set of measurable functions of another measure space (Y, ν). Hence the inequality

$$\|(\sum_j |T(f_j)|^2)^{1/2}\|_p \leq C_p \|(\sum_j |f_j|^2)^{1/2}\|_p$$

can looked as a linear inequality on the Banach space $L^p(X, \ell^2)$ where all sequences of measurable functions satisfy

$$\|\{f_j\}_j\|_{L^p(X,\ell^2)} = (\int_X (\sum_j |f_j|^2)^{p/2} d\mu)^{1/p}.$$

If we define a linear on such sequences by setting

$$\vec{T}(\{f_j\}_j) = \{T(f_j)\}_j.$$

Then the above inequality can be rewritten as

$$\|\vec{T}(\{f_j\}_j)\|_{L^p(X,\ell^2)} \leq \|\{f_j\}_j\|_{L^p(X,\ell^2)}$$

in which $\vec{T}$ can be thought of as a linear operator acting on the L^p space of ℓ^2-valued functions on X. This is the basic idea of vector-valued inequalities.

We will be primarily focused on the following vector-valued inequality:

Theorem 5.1.1 (A. Zygmund)**.** *If T is a bounded linear operator on $\mathrm{L}^p(\mathbb{R}^2)$, $1 < p < \infty$, then for a sequence of functions $\{g_j\}$, where $g_j \in \mathrm{L}^p(\mathbb{R}^2)$*

$$\|(\sum_j |T(f_j)|^2)^{1/2}\|_p \le C_p\|(\sum_j |f_j|^2)^{1/2}\|_p$$

This requires the following lemma which do not prove:

Lemma 5.1.2. *For any $0 < r < \infty$, take*

$$A_r = (\frac{\Gamma(\frac{r+1}{2})}{\pi^{\frac{r+1}{2}}}).$$

Then for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ we have

$$(\int_{\mathbb{R}^n} |\lambda_1 x_1 + \ldots + \lambda_n x_n|^r e^{-\pi|x|^2 dx})^{1/r} = A_r(\lambda_1^2 + \ldots + \lambda_n^2)^1/2$$

Let us return to the proof the Theorem

Proof. Using the measure $e^{-\pi|z|^2}dz$ and the fact that T is a bounded linear operator, we can write

$$\begin{aligned}
\|(\sum_{j=1}^n |T(f_j)|^2)^1/2\|^p_{L^p(\mathbb{R}^2)} &= A_p^{-p}\int_{\mathbb{R}^2}\int_{\mathbb{R}^n} |z_1T(f_1) + \ldots + z_nT(f_n)|^q e^{-\pi|z|^2}dzd\nu \\
&= A_p^{-p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^2} |T(z_1f_1 + \ldots + z_nf_n)|^p d\nu e^{-\pi|z|^2}dz \\
&\le A_p^{-p}C_p^p\int_{\mathbb{R}^n}\int_{\mathbb{R}^2} |z_1f_1 + \ldots + z_nf_n|^p d\nu e^{-\pi|z|^2}dz \\
&= A_p^{-p}C_p^pA_p^p\int_{\mathbb{R}^2}(\sum_{j=1}^n |f_j|^2)^{p/2}d\nu \\
&= C_p^p\|(\sum_j |f_j|^2)^{1/2}\|_p^p
\end{aligned}$$

Now taking the p-th root on both sides we get

$$\|(\sum_{j=1}^n |T(f_j)|^2)^{1/2}\|_p \le C_p\|(\sum_{j=1}^n |f_j|^2)^{1/2}\|_p$$

Now taking $n \to \infty$ we get the desired inequality. □

If we assume that the spherical summation converges then using the vector-valued inequality we get the following result about any sequence of functions in $\mathrm{L}^p(\mathbb{R}^2)$. Later we will find a sequence of functions which do not satisfy this inequality, thereby proving that the spherical summation does not converge.

Lemma 5.1.3 (Y. Meyer). *Let $v_1, v_2, \dots$ be a sequence of unit vectors in $\mathbb{R}^2$. Let $H_j := \{\, x \in \mathbb{R}^2 : x \cdot v_j \geq 0 \,\}$. We define operators $\mathcal{T}_1, \mathcal{T}_2, \dots$ by*

$$\widehat{(\mathcal{T}_j f)} = \chi_{H_j} \cdot \widehat{f}.$$

Assume that the ball operator $\mathcal{T} = \mathcal{M}_{\chi_B}$ defined by

$$\widehat{\mathcal{T} f} = \chi_B \cdot \widehat{f}$$

is bounded on L^p, $1 < p < \infty$. Then for any sequence of function $f_j \in \mathrm{L}^p(\mathbb{R}^2)$, we have

$$\|(\sum_j |\mathcal{T}_j f_j|^2)^1/2\|_p \leq \|(\sum_j |f_j|^2)^1/2\|_p$$

Proof. Let $D(j,r) = \{\, x \in \mathbb{R}^2 : |\, x - r v_j| \leq r \,\}$. We set $\mathcal{T}_{j,r} \equiv \mathcal{M}_{\chi_{D(j,r)}}$. If $f \in C_c^\infty(\mathbb{R}^2)$, then for every x,

$$\begin{aligned}
\lim_{r\to\infty} \mathcal{T}_{j,r} f(x) &= \lim_{r\to\infty} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{f}(\xi) \chi_{D(j,r)}(\xi) e^{-i\xi\cdot x} d\xi \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{f}(\xi) \chi_{H_j}(\xi) e^{-i\xi\cdot x} d\xi \\
&= \mathcal{T}_j f(x)
\end{aligned}$$

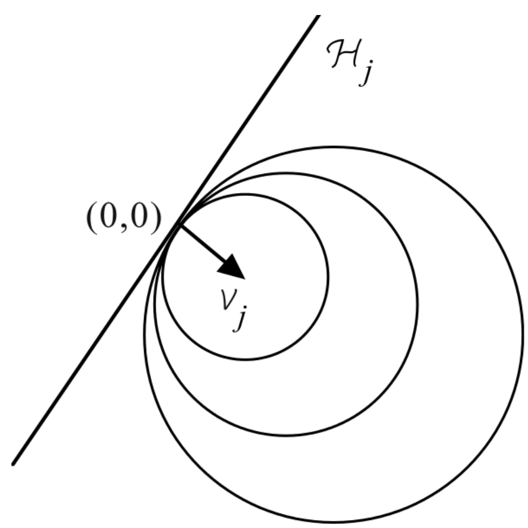

Figure 5.1: A sequence of disks converging to a half-plane.

Also, by application of Fatou's lemma we obtain

$$\|(\sum_{j=1}^{k} |\mathcal{T}_j f_j|^2)^{1/2}\|_p \leq \liminf_{r\to\infty} \|(\sum_{j=1}^{k} |\mathcal{T}_{j,r} f_j|^2)^{1/2}\|_p$$

Now, we also have

$$\|(\sum_j |\mathcal{T}_{j,r} f_j|^2)^{1/2}\|_p \le C.\|(\sum_j |f_j|^2)^{1/2}\|_p$$

with C independent of r. This we prove as follows

Now we might as well let $r = 1$.

$$\mathcal{T}_{j,1} f(x) = e^{-iv_j.x} \cdot \mathcal{T}(e^{iv_j.(.)} f)(x))$$

And so we get

$$\begin{aligned}\|(\sum_{j=1}^{k} |\mathcal{T}(e^{iv_j.(.)} f_j)|^2)^{1/2}\|_p &\le C.\|(\sum_{j=1}^{k} |e^{iv_j.(x)} f_j(x)|^2)^{1/2}\|_p \\ &= \|(\sum_{j=1}^{k} |f_j(x)|^2)^{1/2}\|_p\end{aligned}$$

Now as $C_c^\infty(\mathbb{R}^2)$ is dense in $L^p(\mathbb{R}2)$ we get the desired inequality. □

5.2 Fefferman's counterexample

Lemma 5.2.1. *Let $R \subseteq R^2$ be a rectangle, and let $\tilde{R}$ denote the union of the two copies of R that are adjacent to R as shown in the figure. Assume that the long side of $R \cup \tilde{R}$ (the side which contains parts of the boundaries of all three rectangles) runs parallel to the x_2-axis and the other side parallel to the x_2-axis. Assume further that R is centered at the origin. Let ℓ be the length of the side of R that points in the x_2-direction and w the length of the other side (in the x_1-direction). Let $\tilde{\tilde{R}} \equiv \{ x = (x_1, x_2 : x \in \tilde{R}, w/4 < |\, x_1| < w/2) \}$, v be a unit vector parallel to the longer side of $R \cup \tilde{R}$, $f = \chi_R$, and $H = \{ x \in \mathbb{R}^2 : x \cdot v \ge 0 \}$, and $S = \mathcal{M}_{\chi_H}$. Then*

$$|Sf(x)| \ge 1/20 \text{ for } x \in \tilde{\tilde{R}}.$$

Proof. Let us assume R to be a square of side length 2 with sides parallel to axes and centre 0 and that $v = (0, 1)$. So

$$\begin{aligned}\widehat{Sf}(\xi) &= \chi_{\xi_2 \ge 0}(\xi) \int_{-1}^{1} e^{i\xi_1 t_1} dt_1 \int_{-1}^{1} e^{i\xi_2 t_2} dt_2 \\ &= 4\chi_{\xi_2 \ge 0}(\xi) \frac{\sin \xi_1}{\xi_1} . \frac{\sin \xi_2}{\xi_2}\end{aligned}$$

Restricting our attention to $-1 < x_1 < 1$ and $|x_2| > 1$, we see that

$$Sf(x) = \frac{4}{(2\pi)^2} \int_0^\infty \int_{-\infty}^\infty \frac{\sin \xi_1}{\xi_1} . \frac{\sin \xi_2}{\xi_2} e^{-i\xi.x} d\xi_1 d\xi_2.$$

Calculating a lower bound for $Sf(x)$,

$$
\begin{aligned}
|Sf(x)| &\leq \mathrm{Re} Sf(x) \\
&= \pi^{-2} \int_0^\infty \int_{-\infty}^\infty \frac{\sin \xi_1}{\xi_1} . \frac{\sin \xi_2}{\xi_2} \cos(\xi \cdot x) d\xi_1 d\xi_2 \\
&= 2\pi^{-2} \int_0^\infty \int_0^\infty \frac{\sin \xi_1}{\xi_1} . \frac{\sin \xi_2}{\xi_2} \cos(\xi \cdot x) d\xi_1 d\xi_2 \\
&= 2\pi^{-2} \int_0^\infty \int_0^\infty \frac{\sin \xi_1}{\xi_1} . \frac{\sin \xi_2}{\xi_2} [\cos \xi_1 x_1 \cos \xi_2 x_2 - \sin \xi_1 x_1 \sin \xi_2 x_2] d\xi_1 d\xi_2 \\
&= 2\pi^{-2} \int_0^\infty \int_0^\infty \frac{\sin \xi_1(1+x_1) + \sin \xi_1(1-x_1)}{2\xi_1} \times \frac{\sin \xi_2(1+x_2) + \sin \xi_1(1-x_2)}{2\xi_2} d\xi_1 d\xi_2 \\
&+ 2\pi^{-2} \int_0^\infty \int_0^\infty \frac{\cos \xi_1(1+x_1) - \cos \xi_1(1-x_1)}{2\xi_1} \times \frac{\cos \xi_2(1+x_2) - \cos \xi_1(1-x_2)}{2\xi_2} d\xi_1 d\xi_2 \\
&\equiv A(x) + B(x)
\end{aligned}
$$

Now,

$$
A(x) = 2\pi^{-2} \int_0^\infty \frac{\sin \xi_1(1+x_1) + \sin \xi_1(1-x_1)}{2\xi_1} d\xi_1 \times \int_0^\infty \frac{\sin \xi_2(1+x_2) + \sin \xi_1(1-x_2)}{2\xi_2} d\xi_2
$$

The second integral is 0 as $|x_2| > 1$.

$$
\begin{aligned}
\int_0^\infty \frac{\cos \xi_1(1+x_1) - \cos \xi_1(1-x_1)}{2\xi_1} &= \lim_{\epsilon \to 0\ \eta \to \infty} [\int_{\epsilon(1+x_1)}^{\eta(1+x_1)} \frac{\cos \xi_1}{2\xi_1} d\xi_1 - \int_{\epsilon(1-x_1)}^{\eta(1-x_1)} \frac{\cos \xi_1}{2\xi_1} d\xi_1] \\
&= \lim_{\epsilon \to 0\ \eta \to \infty} [\int_{\eta(1-x_1)}^{\eta(1+x_1)} \frac{\cos \xi_1}{2\xi_1} d\xi_1 - \int_{\epsilon(1-x_1)}^{\epsilon(1+x_1)} \frac{\cos \xi_1}{2\xi_1} d\xi_1]
\end{aligned}
$$

Here the first integral goes to 0 as $\eta \to 0$ whereas the second integral can be calculated as

$$
\begin{aligned}
-\lim_{\epsilon \to 0} \int_{\epsilon(1-x_1)}^{\epsilon(1+x_1)} \frac{\cos \xi_1}{2\xi_1} d\xi_1 &\leq \lim_{\epsilon \to 0} \int_{\epsilon(1-x_1)}^{\epsilon(1+x_1)} \frac{1}{2\xi_1} d\xi_1 \\
&= \frac{1}{2} \log(\frac{1+x_1}{1-x_1}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{\cos \xi_2(1+x_2) + \cos \xi_1(1-x_2)}{2\xi_2} d\xi_1 d\xi_2 \\
&= \lim_{\epsilon \to 0\ \eta \to \infty} [\int_{\epsilon(1+x_2)}^{\eta(1+x_2)} \frac{\cos \xi_2}{2\xi_2} d\xi_2 - \int_{\epsilon(1-x_2)}^{\eta(1-x_2)} \frac{\cos \xi_2}{2\xi_2} d\xi_2] \\
&= \lim_{\epsilon \to 0\ \eta \to \infty} [\int_{\eta(1-x_2)}^{\eta(1+x_2)} \frac{\cos \xi_2}{2\xi_2} d\xi_2 - \int_{\epsilon(1-x_2)}^{\epsilon(1+x_2)} \frac{\cos \xi_2}{2\xi_1} d\xi_2]
\end{aligned}
$$

As in the previous case,

$$
\leq \frac{1}{2} \log(\frac{1+x_2}{1-x_2}).
$$

So, we have

$$|B(x) = c.|\log|\frac{1+x_1}{1-x_1}|||\log|\frac{1+x_2}{1-x_2}||.$$

Now since $1/2 < |x_1| < 1$ and $1 < |x_2| < 3$ we have

$$|Sf(x) \geq \mathrm{Re}Sf(x) = |B(x)| \geq \frac{\log 3 \log 2}{10} \geq 1/20$$

□

Let us denote the Lebesgue measure of a set $A \subseteq \mathbb{R}^2$ as $|A|$.

Lemma 5.2.2. *Let $\eta > 0$. Then there is a set $E \subseteq \mathbb{R}^2$ and a collection $\mathcal{R} = \{R_j\}$ of pairwise disjoint rectangles such that*

1. $|\tilde{\tilde{R}}_j \cap E| \geq |\tilde{R}_j|/80$;
2. $|E| \leq \eta \sum_j |R_j|$.

Proof. We use the notations used in the Cunningham's construction of the simply connected Kakeya Set. We constructed a set E with area not exceeding $2\epsilon^2$ by repeatedly sprouting an initial equilateral triangle of side ϵ. Now let $\epsilon = 1$.

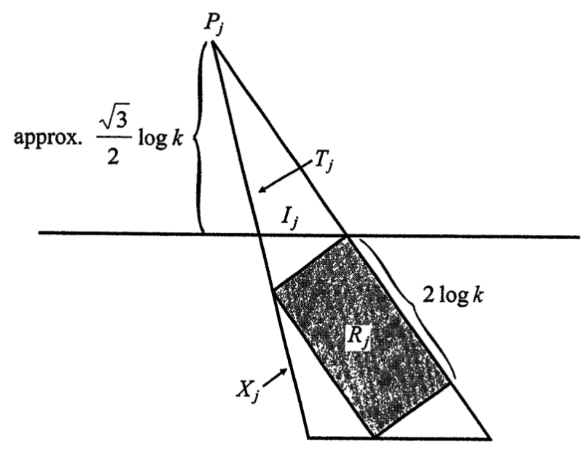

Figure 5.2: The rectangle R_j

Fix $k \in \{1, 2, \ldots\}$. Each dyadic subinterval $I_j \equiv I_j^k \equiv [j/2^k, (j+l)/2^k] \subseteq [0,1]$ is the base of precisely one sprouted triangle $T\alpha$, where $\alpha = (\alpha_0, ..., \alpha_k)$ has the property that $\alpha_0 = 0$ and each $\alpha_0, ..., \alpha_k$ is either 1 or 2. Call this triangle T_j). Let P_j be its upper vertex. Construct R_j as in Figure . Notice that R_j lies below the base of T_j and is uniquely determined by T_j and the fact that the side that is parallel to a side of T_j has length $2\log k$. Now for each j, $R_j \cap ? \subseteq \tilde{\tilde{R}}_j \cap T_j \equiv W_j$. Hence

$$\begin{aligned}
|\tilde{\tilde{R}}_j \cap E| &\geq |W_j| \\
&\geq \frac{1}{5}.\frac{\sqrt{3}}{2}.\log k.2^{-k} \\
&\geq \frac{1}{80}.2.(6\log k.2^{-k}) \\
&\geq \frac{|\tilde{R}_j|}{80}.
\end{aligned}$$

This completes the proof of the first part.

Now using induction we can see that each of the extended trapezoids are disjoint. This gives us the following estimate

$$\begin{aligned}
|\bigcup_{j=0}^{2^k-1} R_j| &= \sum_{j=0}^{2^k-1} |R_j| \\
&\geq \sum_{j=0}^{2^k-1} \frac{1}{2}(\log k)2^{-k} \\
&= \frac{1}{2}(\log k) \\
&\geq \frac{2}{\eta} \\
&\geq \frac{|E|}{\eta}
\end{aligned}$$

□

Theorem 5.2.3 (Fefferman's Theorem)**.** *Let $\mathcal{T}$ be the multiplier operator associated with the spherical summation. Then $\mathcal{T}$ is bounded on L^2, but if $1 < p < \infty$ and $p \neq 2$, then $\mathcal{T}$ does not map $\mathrm{L}^p(\mathbb{R}^n)$ into itself, i.e. $\mathcal{T}$ is not a bounded operator on $\mathrm{L}^p(\mathbb{R}^n)$.*

Proof. We prove this theorem by contradiction, so we take $\mathcal{T}$ to be a bounded operator. Assume $p > 2$. Let $E \subseteq \mathbb{R}^2$ and a collection $\mathcal{R} = \{R_j\}$ of pairwise disjoint rectangles which satisfy the conditions of the previous lemma. Also, suppose that $f_j = \chi_{R_j}$ and v_j be unit vector parallel to the long side of $(R_j \cup \tilde{R}_j)$

for each j. Let $H_j = \{ x \in \mathbb{R}^2 : x \cdot v_j \geq 0 \}$ and $\mathcal{T}_j = \mathcal{M}_{\chi_{H_j}}$. Then by lemma –

$$\begin{aligned}
\int_E (\sum_j |\mathcal{T}_j f_j(x)|^2) dx &= \sum_j \int_E (|\mathcal{T}_j f_j(x)|^2) dx \\
&\geq \frac{1}{400} \sum_j |E \cap \tilde{\tilde{R}}_j \\
&\geq \frac{1}{32000} \sum_j |\tilde{R}_j| \\
&= \frac{1}{16000} \sum_j |R_j|.
\end{aligned}$$

Applying Hölder's inequality to χ_E and $\sum_j |\mathcal{T}_j f_j(x)|^2$ with the exponents $p/(p-2)$ and $p/2$, we get

$$\int_E (\sum_j |\mathcal{T}_j f_j(x)|^2) dx = |E|^{(p-2)/p} \|(\sum_j |\mathcal{T}_j f_j(x)|^2)^{1/2}\|_p^2$$

By Meyer's Lemma we have

$$\begin{aligned}
&\leq C|E|^{(p-2)/p} \|(\sum_j |f_j(x)|^2)^{1/2}\|_p^2 \\
&= C|E|^{(p-2)/p} (\sum_j |R_j|)^{2/p} \\
&\geq C\eta^{(p-2)/p} \sum_j |R_j|
\end{aligned}$$

So we have

$$\sum_j |R_j| \leq C\eta^{(p-2)/p} \sum_j |R_j|$$

Since we can choose η to be arbitrarily small, we have a contradiction. Hence $\mathcal{T}$ cannot be a bounded on $\mathrm{L}^p(\mathbb{R}^n)$ for $2 < p < \infty$. By duality argument we can say that $\mathcal{T}$ cannot be a bounded on $\mathrm{L}^p(\mathbb{R}^n)$ for $1 < p < 2$. $\square$

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