Chapitre 13

TD

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Première partie
Exercice 5

$$(S) : AX = 0 \text{ où } X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} A = \mathrm{GL}_n(\mathbb{C}) &\iff \mathrm{rg}(A) = n \\ &\iff \mathrm{rg}(S) = n \\ &\iff S \text{ est de Cramer} \\ &\iff X = 0 \text{ est la seule solution de } (S) \end{aligned}$$

On suppose $X \neq 0$ tel que $AX = 0$ et on cherche une contradiction.

$$\begin{aligned} (S) &\iff \forall i \in [\![1, n]\!], \sum_{j=1}^{n} a_{ij} x_j = 0 \\ &\implies \forall i \in [\![1, n]\!] - a_{ii} x_i = \sum_{j \neq i} a_{ij} x_j \\ &\implies \forall i \in [\![1, n]\!], |a_{ii}| \, |x_i| = \left| \sum_{j \neq i} a_{ij} x_j \right| \\ &\implies \forall i \in [\![1, n]\!] \, |a_{ii}| \, |x_i| \leqslant \sum_{j \neq i} |a_{ij}| \, |x_j| \end{aligned}$$

Soit i_0 tel que $|x_{i_0}| = \max\limits_{i \leqslant j \leqslant n} (|x_j|)$ ($i_0 = \mathrm{argmax}\, 1 \leqslant j \leqslant n\, (|x_j|)$)
Comme $X \neq 0$, $x_{i_0} > 0$, donc

$$|a_{i_0, i_0}| \leqslant \sum_{j \neq i_0} |a_{i_0, j}| \frac{|x_j|}{|x_{i_0}|} \leqslant \sum_{j \neq i_0} |a_{i_0, j}|$$

une contradiction $\lightning$

Deuxième partie

Exercice 10

Soient a, b, c, d différents de -1.

$$(S) : \begin{cases} x = by + cz + dt \\ y = cz + dt + ax \\ z + dt + ax + by \\ t = ax + by + cz \end{cases} \iff \begin{cases} x = by + cz + dt \\ y - x = ax - by \\ z - y = by - cz \\ t - z = cz - dt \end{cases}$$

$$\iff \begin{cases} z = \dfrac{d+1}{c+1}t \\ y = \dfrac{c+1}{b+1}z = \dfrac{d+1}{b+1}t \\ x = \dfrac{b+1}{a+1}y = \dfrac{d+1}{a+1}t \\ \dfrac{d+1}{a+1}t = \dfrac{b}{b+1}(d+1)t + \dfrac{c}{c+1}(d+1)t + dt \end{cases}$$

$$\iff \begin{cases} x = \dfrac{d+1}{a+1}t \\ y = \dfrac{d+1}{b+1}t \\ z = \dfrac{d+1}{c+1}t \\ t\left(\dfrac{b}{b+1} + \dfrac{c}{c+1} + \dfrac{d}{d+1} - \dfrac{1}{a+1}\right) = 0 \end{cases}$$

(S) a des solutions non nulles ssi

$$\frac{b}{b+1} + \frac{c}{c+1} + \frac{d}{d+1} - \frac{1}{a+1} = 0$$

ssi

$$\frac{b}{b+1} + \frac{c}{c+1} + \frac{d}{d+1} - \frac{a+1-a}{a+1} = 0$$

ssi

$$\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} + \frac{d}{d+1} = 1$$

Troisième partie

Exercice 9

$$\left(\begin{array}{ccc|c} \boxed{1} & a & -a^2 & a^4 \\ 1 & b & -b^2 & b^4 \\ 1 & c & -c^2 & c^4 \end{array}\right) \underset{\substack{L_2 \leftarrow \frac{L_2 - L_1}{b-a} \\ L_3 \leftarrow \frac{L_3 - L_1}{c-a}}}{\sim} \left(\begin{array}{ccc|c} \boxed{1} & a & -a^2 & a^4 \\ 0 & \boxed{1} & -(a+b) & b^3 + ab^2 + a^2b + a^3 \\ 0 & 1 & -(a+c) & c^3 + ac^2 + a^2c + a^3 \end{array}\right)$$

$$\underset{L_3 \leftarrow \frac{L_3 - L_2}{b-c}}{\sim} \left(\begin{array}{ccc|c} \boxed{1} & a & -a^2 & a^4 \\ 0 & \boxed{1} & -(a+b) & b^3 + ab^2 + a^2b + a^3 \\ 0 & 0 & \boxed{1} & -c^2 - bc - b^2 - ac - ab - a^2 \end{array}\right)$$

$$\underset{\substack{L_2 \leftarrow L_2 + (a+b)L_3 \\ L_1 \leftarrow L_1 - aL_2 + a^2 L_3}}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \end{array}\right)$$