TSP Miller-Tucker-Zemlin

* $x_{i,j} := \begin{cases} 1 & \text{path goes from city } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$ $\quad c(i,j)$: cost of the travel from city i to city j.

* u_i wher $1 \leq i \leq n$, u_i is a dummy variable

$$\min: \sum_{1 \leq i \neq j \leq n} c(i,j) x_{i,j}$$

$u_i \in \mathbb{Z}$
$2 \leq i \leq n$

* $u_i - u_j + n x_{i,j} \leq n-1 \quad \forall 2 \leq i \neq j \leq n$

* $\sum_{i=1, i \neq j}^{n} x_{i,j} = 1 \quad \forall 1 \leq j \leq n$

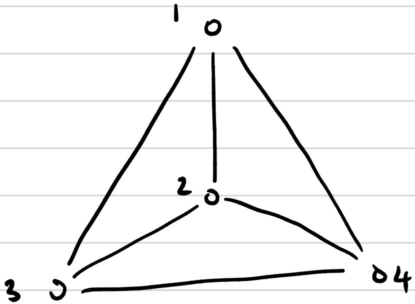
* $\sum_{j=1, i \neq j}^{n} x_{i,j} = 1 \quad \forall 1 \leq i \leq n$

* $0 \leq u_i \leq n-1 \quad \forall 2 \leq i \leq n$

$$\min: \sum_e (c_e x_e : e \in E) \quad s.t \quad x(\delta(v)) = 2 \ \forall v \in V$$

$$x(\delta(S)) \geq 2 \quad \forall \emptyset \neq S \neq V$$

$$0 \leq x_e \leq 1 \quad \forall e \in E$$



$$u_i - u_j + n x_{i,j} \leq n-1 \Leftrightarrow 1 + u_i - u_j + n x_{i,j} \leq n$$

$$\forall\ 2 \leq i \neq j \leq n$$

$$1 + u_i - u_j + n(x_{i,j} - 1) \leq 0$$

$$n(x_{i,j} - 1) \leq u_j - u_i - 1$$

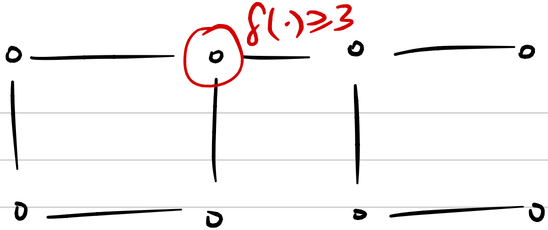
$$(u_2 - u_3) + (u_3 - u_4) + \cdots (u_{k-1} - u_k) + n(x_{2,3} + x_{3,4} + \cdots x_{k-1,k}) \leq (n-1) * k \Rightarrow (u_2 - u_k) + kn \leq (n-1) * k$$

$$(k-1) + kn \leq (n-1) * k$$

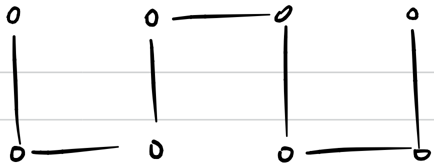
$$k(n-1) \leq (n-1) * k$$

$$u_i = \begin{cases} t & \text{if "i" is visted at step } i \\ \text{free} & \end{cases}$$

This is saying, $\forall$ path in the solution, with length k, must have $k-1$ edges in it.



Not correct



$\forall$ path in this solution, with length k, it has $k-1$ edges in it.

Consider a sub-cycle involving u_2
Name cycle C.
Consider all paths in C, any length of k:

$$(u_2 - u_k) + \sum_{e \in C \cap p} x_e \leq (n-1) * k \quad \text{true } \forall k$$

This case, $\forall 2 \leq i \leq k$, u_i is fixed, but there will be mutiple solutions.

if $\delta(C) \neq \phi$, then we are at the same vertex 2 times,
if $\delta(C) = \phi$, the graph unconnected.

Consider a permutation of $[n]$, name it p, and indexed by i.

$p_1 \to p_2 \to p_3 \cdots \to p_n \to p_1$ It's a valid tour of vertex.

$$x_{p_1,p_2} = x_{p_2,p_3} = x_{p_4,p_5} = \cdots x_{p_{n-1},p_n} = x_{p_n,p_1} = 1$$

The magical constraint is: $u_i - u_j + n x_{i,j} \leq n-1$

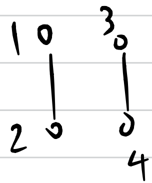
$$u_i - u_j \leq n - n x_{i,j} - 1$$
$$u_i - u_j \leq n(1 - x_{i,j}) - 1$$

Then $x_{i,j} = 1 \Rightarrow u_i - u_j \leq -1$

$$\Rightarrow u_j - u_i \geq 1$$

$$x_{i,j} = 0 \Rightarrow u_i - u_j \leq n-1$$

$\forall$ valid tour, the constraint is satisfied



$x_{1,2} = 1$

$x_{3,4} = 1$

$$u_2 - u_3 + u_3 - u_4 + 4\underbrace{(x_{2,3} + x_{3,4})}_{1} \leq 2(3)$$

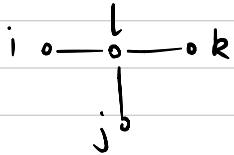
$$u_3 - u_4 + 4 \leq 3$$
$$u_3 - u_4 \leq -1$$

$$\Rightarrow u_2 - u_3 + u_3 - u_4 \leq 2$$

$$u_2 - u_4 \leq 2$$

(1) $\boxed{u_i - u_j + n x_{i,j} \le n-1 \quad \forall 2 \le i \le n}$

Consider any T-junction



then $x_{i,k}=1 \dot\vee x_{k,i}=1$

$x_{l,j}=1 \dot\vee x_{j,l}=1$

$x_{l,k}=1 \dot\vee x_{k,l}=1$

(2)

WLOG, Assume: $x_{i,l}=x_{l,j}=1$ (Rotate graph to get different cases)

$$\begin{matrix} u_i - u_l \le -1 \\ u_l - u_j \le -1 \end{matrix} \Rightarrow \begin{matrix} u_l - u_i = 1 \\ u_j - u_l = 1 \end{matrix}$$ because $\forall 2 \le i \le n : u_i \in \mathbb{Z}$ and the assumption (1), (2)

$$\begin{cases} u_l - u_k + n x_{l,k} \le n-1 \\ u_k - u_l + n x_{k,l} \le n-1 \end{cases}$$ with $x_{l,k}=1 \dot\vee x_{k,l}=1$

Assume $x_{l,k}=1$

$$u_l - u_k \le -1 \Rightarrow u_k - u_l \ge 1$$

$$u_l = 1 + u_i$$

$$u_j = 2 + u_i$$

$$u_k - (1+u_i) \ge 1$$

$$u_k - u_i \ge 2$$

LP is phrased as following:

$$x_{i,j} = \begin{cases} 1 & \text{path goes from } i \text{ to } j \text{ at some point} \\ 0 & \text{therwise} \end{cases}$$

$\#1 \wedge \#2 \Rightarrow |\delta(v)| = 2 \ \forall v$

$$\min \sum_{1 \le i < j \le n} c(i,j) x_{i,j} \quad \text{s.t}$$

Constraint #1 $\sum_{\substack{i=1 \\ i \neq j}}^{n} x_{i,j} = 1 \quad \forall 1 \le j \le n$

Constraint #2 $\sum_{j=1, j\neq i}^{n} x_{i,j} = 1 \quad \forall 1 \le i \le n$

Constraint #3 $u_i - u_j + n x_{i,j} \le n-1 \quad \forall 2 \le i \neq j \le n$

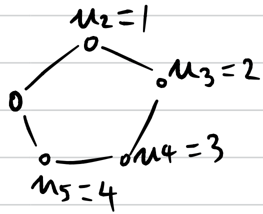
Constraint #4 $0 \le u_i \le n-1$

Dummy Variable

$u_i \ \forall 2 \le i \le n$

$0 \le u_i \le n-1 \ \forall 2 \le i \le n$

$u_i = t$
means vertex i has been
visited at step i of the algorithm



v_1 : the vertex that starts and
end the tour with.

proof (or at least I tried)

prove that a valid solution, a tour satisfies the constraint.

Consider the solution G, which is a cycle, and any path $P \subseteq G$ and P contains v_1.

Consider : $\forall e=(i,j) \in P : u_i - u_j + n x_{i,j} \leq n-1$

* Then suming up all of them, assuming p has k edges

$$\sum_{i \neq 1} (e=(i,j) \in P : u_i - u_j) + n\left(\sum_{e \in P} x_e\right) \leq k(n-1)$$

then it's equal to:

$$u_2 - u_k + n(k) \leq k(n-1)$$

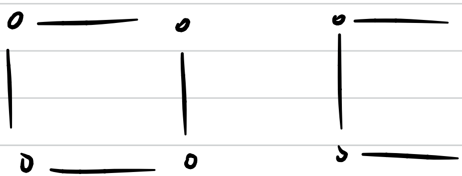
which is true for all k,

inductively, $u_i - u_{i+1} = -1$, hence the solution for dummy variable will be: $u_2 - u_k = (k-1)$

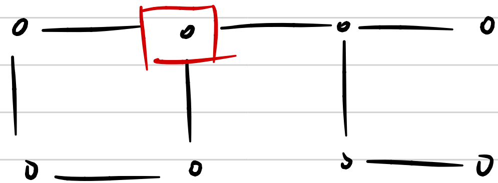
then $k(n-1) \leq k(n-1)$ $\square$

If a solution is not valid (Not a tour) then the constraint is not satisfied.

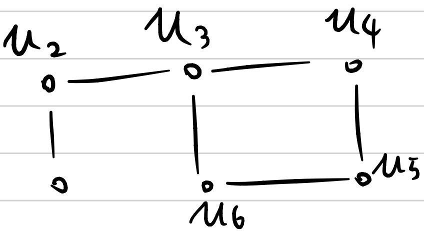
Assume constraint #1, #2 are satisfied, here is some non-trivial examples that are not a tour and satisfies the constraints:



Sub cycle



T junction



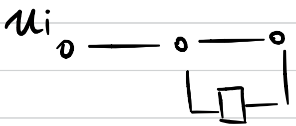
$u_3 - u_2 = 1$

but edges involve is 5

then $-1 + 6*5 \le (5)*5$

$29 \le 25$

Constraints #3 prevents Tea spoon configuration (Results will always be a path)



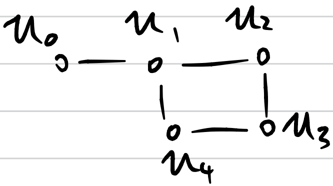
this config is prevented.

Say path length is k

$P_i, P_{i+1} \cdots P_{i+k}$

but one of the vertex repeated.

let's see a non-trivial example, which can be generalized easily.



$(u_0-u_1)+(u_1-u_2)+(u_2-u_3)+(u_3-u_4)+(u_4-u_1)$

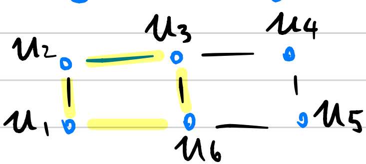
$\Rightarrow u_0-u_1=-1$

but then $k(x_{i,j})$ will be $5*5$

and $k(n-1)$ is $5*(5-1)=20$

$24 \leq 20$ is FALSE

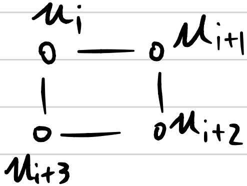
Double cycle configuration



$u_2-u_2=0$

oh No...
kinda hard.

if any sub cycle exists, say this one:



$u_i - u_i = 0$ cycle sum

$n\sum_{e \in P} x_e = 4*4 = 16$

$n(n-1) = 4*3 = 12$

$16 \le 12$ is false