An approach to review calibration

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Abstract

In this report, we develop a method to calibrate a set of reviews of people from different reviewers on different days. Given such data, we take into account the improvement of people as the days progress and the biases and different scales with which reviewers review.

Contents

1 Introduction and overview

Consider a competition amongst many people that lasts multiple days. Consider a set of reviewers who each review some subset of people on various different days. How can we determine an objective, absolute rating for each person? The issues to tackle are as follows. First, each reviewer has a different reviewing scale; what one reviewer means by 2 stars may be different from what another means by 2 stars. Second, the people may improve each day; if one reviewer reviews somebody on the first day and another on the fifth day, we should expect the person to have improved.

We would like to calibrate the reviewers so that everybody has the same absolute rating scale. We would also like to determine how much each person is improving over the course of the competition.

The remainder of the report is as follows. In Section 2, we will encode our calibration problem into the minimization of an objective function. In Appendix A, we will gain some intuition for the solutions to the

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objective function by performing simple case studies. If one so wishes, they may desire to skip Appendix A as it mostly build intuition for Section 3. In Section 3, we provide an efficient, deterministic protocol for calibration. Appendices A.1 to A.3 lead up to this result by providing evidence that the protocol in Section 3 does indeed give a good solution. Finally, we discuss possible extensions in Section 4.

2 Formulation as an optimization problem

Here we define the relevant terms. See Table 1 for a summary.

$\mathcal{P}$	set of people reviewed
$\mathcal{R}$	set of reviewers
$\mathcal{P}_r$	set of people reviewed by reviewer $r \in \mathcal{R}$
$\mathcal{R}_p$	the set of reviewers who reviewed person $p \in \mathcal{P}$
$\mathcal{D}_p^r$	set of days that reviewer $r \in \mathcal{R}$ reviewed person $p \in \mathcal{P}_r$
$y_{p,d}^r$	reviewer r's ($r \in \mathcal{R}$) rating of person $p \in \mathcal{P}_r$ on day $d \in \mathcal{D}_p^r$
$y_{\min}$	minimum allowed raw rating
$y_{\max}$	maximum allowed raw rating
Δy	$\Delta y \coloneqq y_{\max} - y_{\min}$
σ_r	scaling function associated to reviewer $r \in \mathcal{R}$, $\sigma_r \colon [y_{\min}, y_{\max}] \to \mathbb{R}$
σ	$\sigma \coloneqq \{\sigma_r \mid r \in \mathcal{R}\}$
g_p	performance function associated to person $p \in \mathcal{P}$, $g_p \colon \bigcup_r D_p^r \to \mathbb{R}$
g	$g \coloneqq \{g_p \mid p \in \mathcal{P}\}$
λ	multiplier
$\widetilde{\sum}_{x\in X}$	$\frac{1}{\lvert X\rvert}\sum_{x\in X}$

Table 1: A list of our definitions.

We have a set of people $\mathcal{P}$ and a set of reviewers $\mathcal{R}$. Each reviewer $r \in \mathcal{R}$ reviews a subset of people $\mathcal{P}_r \subseteq \mathcal{P}$. For convenience, also denote the set of reviewers who reviewed p by $\mathcal{R}_p \subseteq \mathcal{R}$. Reviewer $r \in \mathcal{R}$ gives person $p \in \mathcal{P}_r$ a rating $y_p^r \in \mathbb{R}$ (usually the rating will be restricted to e.g. 1 through 5 stars, but for now we only assume that it is a real number). We are hoping to somehow calibrate the reviews; that is, reviewers r_1 and r_2 may have a different definition of what good and bad are, a different definition of the relative difference between 2 and 3 stars, etc. Hence, to each reviewer $r \in \mathcal{R}$, we associate a function σ_r that maps from reviewer r's scale to some absolute scale. In other words, given that y_p^r is reviewer r's actual rating of person p, $\sigma_r(y_p^r)$ will be the review on the "absolute scale" that we associate to person p from reviewer r. This way, each person p has a multiset of *calibrated* reviews $\{\sigma_r(y_p^r) \mid r \in \mathcal{R}_p\}$.

There is one minor caveat, though, which stems from the fact that reviewers may have reviewed the people at different times. We deal with this as follows. We let $\mathcal{D}_p^r$ be the set of days that reviewer r reviewed person p. Then for each reviewer $r \in \mathcal{R}$, person $p \in \mathcal{P}_r$, and day $d \in \mathcal{D}_p^r$, we have a review $y_{p,d}^r$. We then associate to each person $p \in \mathcal{P}$ a performance function g_p. This function will take a day d and map it to an averaged calibrated rating for person p on day d.

Our goal is to find the σ and g that minimizes the distinguishability of reviewers. We want that σ_r takes r's reviews and maps them to an absolute scale; thus, we want $\sigma_r(y_{p,d}^r)$ to be close to $g_p(d)$. We thus arrive at the following objective function to minimize:

$$\mathcal{O}(\sigma, g) = \widetilde{\sum_{p\in\mathcal{P}}}\ \widetilde{\sum_{r\in\mathcal{R}_p}}\ \widetilde{\sum_{d\in\mathcal{D}_p^r}} \left(\sigma_r(y_{p,d}^r) - g_p(d)\right)^2 + \lambda \widetilde{\sum_{r\in\mathcal{R}}} \left(\sigma_r(y_{\max}) - \sigma_r(y_{\min}) - 1\right)^2, \tag{1}$$

where λ is an arbitrary multiplier that we discuss below, and we define the symbol $\widetilde{\sum}$ as $\widetilde{\sum}_{x\in X} \equiv \frac{1}{\lvert X\rvert}\sum_{x\in X}$.

Remark 1. The second term in Eq. (1) – the term with the λ (and hence I will call it the λ term) – is needed to ensure that the trivial solution $\sigma_r = 0$ and $g_p = 0$ is not the minimum. However, the λ-term in Eq. (1) is not exactly what we want. Consider for example each σ_r being something resembling a step function, where $\sigma_r(y) \approx 0$ for almost all y, but then it quickly jumps to 1 near $y_{\max}$. The result would be that almost all ratings $y^r_{p,d}$ would be set to $\sigma_r(y^r_{p,d}) \approx 0$, and so each g_p would be set to approximately zero. Such σ_r and g_p would minimize the objective function. One can imagine changing the λ term to include integrals over different orders of derivatives, which could counteract this problem; indeed, the necessary choice of the λ term depends on the choice of parameterization of the σ functions, which we will return to in the next subsection. For our purposes, though, the current λ term is sufficient because we will restrict to certain parameterizations of the σ functions that do not allow something resembling the step function example above to occur. See e.g. the next subsection.

In summary, let σ^* and g^* be

$$(\sigma^*, g^*) = \underset{(\sigma,g)}{\operatorname{argmin}}\, \mathcal{O}(\sigma, g), \tag{2}$$

where $\mathcal{O}$ is from Eq. (1). Then, the multiset of calibrated reviews for person p is $\{\sigma^*_r(y^r_{p,d}) \mid r \in \mathcal{R}_p, d \in \mathcal{D}^r_p\}$. Furthermore, to each person, we have a performance function g^*_p where $g^*_p(d)$ represents roughly the average calibrated rating of person p on day d. Note that since the calibration scale is arbitrary, we typically choose to renormalize the σ and g functions such that $0 \le \sigma^*_r(y^r_{p,d}) \le 1$ for all r, p, and d, with equality on both sides for the min and max ratings.

We now describe this renormalization procedure more carefully. Denote the procedure as $\operatorname{Renorm}(\sigma, g)$. Suppose that $y_1 = \max\{\sigma_r(y^r_{p,d}) \mid r \in \mathcal{R}, p \in \mathcal{P}_r, d \in \mathcal{D}^r_p\}$, and y_0 the same but with max $\to$ min. Let σ' and g' be the output $(\sigma', g') = \operatorname{Renorm}(\sigma, g)$. Then, $\sigma'_r(x) = \frac{\sigma_r(x) - y_0}{y_1 - y_0}$ and $g'_p(x) = \frac{g_p(x) - y_0}{y_1 - y_0}$. It follows that all the calibrated ratings will be exactly between 0 and 1 and the improvement functions will denote improvement with respect to this calibrated scale. If one desires, one may wish to ignore outlying data when renormalizing. Specifically, one may wish to set y_1 to be the maximum calibrated review that is less than z standard deviations away from the mean of all the calibrated reviews, where z is some fixed positive number (e.g. $2 \le z \le 3$ is probably a good range). Similarly, one may set y_0 to be the minimum calibrated review that is less than z standard deviations from the mean. When renormalizing with these chosen y_0 and y_1, some of the resulting renormalized calibrated data may not be ≤ 1, in which case we just set these to be equal to 1. Similarly, some may not be ≥ 0, in which case we set them to 0. Ignoring outliers when renormalizing is thus a good way to make sure that a few exceptional reviews do not dominate the scale[1].

Finally, we now discuss the multiplier λ. When $\lambda = 0$, the objective function is trivially minimized by e.g. $\sigma_r(x) = 0$ and $g_p(d) = 0$. This clearly gives us no information. However, for any nonzero λ, this trivial solution disappears, and the renormalization described in the previous paragraph ensures again that the calibrated scale is precisely $[0, 1]$. Let $\operatorname{Minim}(\lambda) = \operatorname{argmin}_{(\sigma,g)} \mathcal{O}(\sigma, g)$. It follows that what we are really interested in is

$$(\sigma^*, g^*) = \lim_{\lambda \to 0^+} \operatorname{Renorm}(\operatorname{Minim}(\lambda)). \tag{3}$$

(Note that this is *not* the same as $\operatorname{Renorm}(\lim_{\lambda \to 0^+} \operatorname{Minim}(\lambda))$, which itself would just result in $\sigma^*_r = g^*_p = 0$). We describe a simple example in Appendix A.2.1 where we take this limit. Numerically, however, this ultimately just corresponds to choosing a very small positive value of λ, performing the minimization, and then renormalizing (although we will see some cases where the limit can be taken exactly; e.g. Section 3.1). Hence, throughout the remainder of this report we will, unless otherwise stated, assume that λ is small.

The reason we are interested in this limit as $\lambda \to 0^+$ is because the requirement that $\sigma_r(y_{\max}) - \sigma_r(y_{\min}) \approx 1$ is arbitrary considering that the calibrated absolute review scale is arbitrary. The requirement is really only necessary insofar as it ensures that we do not get the trivial solution $\sigma_r = g_p = 0$. Since any arbitrarily small but nonzero λ achieves this, we desire the limit to be taken.

[1]One may ask why such outliers emerge in the first place. There are two possibilities. The first is that there really is a person who is so much better or worse than everybody else. The second is that the reviewers unintentionally (or intentionally) on rare occasion do not review faithfully to their scale. I suspect that second is most often the reason.

2.1 Making necessary assumptions

In the previous section, we defined the objective function $\mathcal{O}(\sigma, g)$ that we are trying to minimize. The minimization is with respect to the set of all σ_r and g_p, which are themselves arbitrary functions. We mentioned in Remark 1 one reason why one needs to restrict to certain classes of functions to avoid getting a useless output. In particular, we consider some set of functions Σ and another set of functions G, and we are looking for

$$(\sigma^*, g^*) = \underset{\substack{(\sigma,g) \text{ s.t.} \\ \sigma_r \in \Sigma, g_p \in G}}{\operatorname{argmin}} \ \mathcal{O}(\sigma, g). \tag{4}$$

The reason for this is two-fold. Firstly, we need to make some assumptions on the reviewers and the people; if their σ and g functions could be arbitrary, then we can get completely arbitrary, nonsensical results. Restricting to a certain class Σ enforces an assumption that reviewers do not choose completely arbitrary numbers with which to choose for ratings. Secondly, in practice, we should only expect to be able to minimize the objective function with respect to certain especially nice classes Σ and G.

Suppose we parameterize each σ_r by a set of parameters θ^r, and we parameterize each g_p by a set of parameters φ^p. Let $\theta = \{\theta^r \mid r \in \mathcal{R}\}$ and $\varphi = \{\varphi^p \mid p \in \mathcal{P}\}$. Then we are looking for $\operatorname{argmin}_{(\theta,\phi)} \mathcal{O}(\theta, \phi)$. We are therefore interested in $\nabla_{(\theta,\varphi)}\mathcal{O}(\theta, \varphi) = 0$. As mentioned in Remark 1, what exactly we choose for the λ term in $\mathcal{O}$ depends heavily on what we choose for Σ. In this report, the λ term written in Eq. (1) will be sufficient for the choices we make for Σ.

Given the simplicity of $\mathcal{O}$, for certain classes of Σ and G, we can easily evaluate $\mathcal{O}$ and $\nabla\mathcal{O}$ on a computer. We can therefore perform a gradient-based minimization procedure to attempt to find σ^* and g^*. On the other hand, for certain classes of Σ and G, we can optimize analytically. This will be the subject of the remaining sections. In particular, in Appendices A.1 to A.3, we will consider extremely simplified classes Σ and G in order to gain intuition as to what the optimization is doing. The nice intuition will serve as an argument that the objective function given in Eq. (1) is indeed encoding our problem adequately. If one so wishes, they may desire to skip Appendix A as it is included mostly to build intuition for Section 3. In Section 3, we consider the general situation in which Σ and G contain all linear functions. Even in this arbitrary case, we can efficiently exactly minimize the objective function and therefore contain a good numerical solution. Finally, in Section 4, we briefly comment on expanding Σ and G to include more general functions.

2.2 An alternate characterization of the objective function

There is another nice interpretation of the minimization of $\mathcal{O}$ from Eq. (1) that we discuss now. Let $g_p(d) = f_p(d) + \beta_p$ where β_p is a parameter and f_p is a function. Intuitively, f_p singles out the improvement from $g_p(d)$ since $g_p(d_1) - g_p(d_2) = f_p(d_1) - f_p(d_2)$. Denote $\mathcal{O}(\sigma, g)$ with this parameterization by $\mathcal{O}(\sigma, f, \beta)$. Let $(\sigma^*, f^*, \beta^*) = \operatorname{argmin}_{(\sigma,f,\beta)} \mathcal{O}(\sigma, f, \beta)$. Then from the equation $\frac{\partial \mathcal{O}}{\partial \beta_p} = 0$, one easily finds that

$$\beta_p^* = \widetilde{\sum_{r\in\mathcal{R}_p}} \widetilde{\sum_{d\in\mathcal{D}_p^r}} \left(\sigma_r(y_{p,d}^r) - f_p(d)\right). \tag{5}$$

We are therefore interested in $(\sigma^*, f^*) = \operatorname{argmin}_{(\sigma,f)} \mathcal{O}(\sigma, f, \beta^*)$. Plugging this in and expanding the squares, one can easily verify that

$$\begin{aligned}\mathcal{O}(\sigma, f, \beta^*) = \frac{1}{2} \widetilde{\sum_{p\in\mathcal{P}}} \widetilde{\sum_{r_1\in\mathcal{R}_p}} \widetilde{\sum_{d_1\in\mathcal{D}_p^{r_1}}} \widetilde{\sum_{r_2\in\mathcal{R}_p}} \widetilde{\sum_{d_2\in\mathcal{D}_p^{r_2}}} \Big(\sigma_{r_1}(y_{p,d_1}^{r_1}) - f_p(d_1) - \sigma_{r_2}(y_{p,d_2}^{r_2}) + f_p(d_2)\Big)^2 \\ + \lambda \widetilde{\sum_{r\in\mathcal{R}}} \left(\sigma_r(y_{\max}) - \sigma_r(y_{\min}) - 1\right)^2.\end{aligned} \tag{6}$$

Minimizing $\mathcal{O}(\sigma, f, \beta^*)$ with respect to σ and f has an intuitive understanding. $\sigma_{r_1}(y_{p,d_1}^{r_1}) - f_p(d_1)$ is the calibrated rating that r_1 gives to p on day d_1 *projected* via the improvement function f_p to some fixed day (say day 0). Thus, by minimizing $\mathcal{O}(\sigma, f, \beta^*)$, we are saying that for any two reviewers r_1 and r_2, r_1's calibrated rating of p projected to day 0 should be similar equal to r_2's calibrated rating of p projected to day 0. This is precisely what we are attempting to do when calibrating; find σ_r functions so that different reviewers can be calibrated to the same scale.

Once more, by definition minimizing $\mathcal{O}(\sigma, f, \beta^*)$ is equivalent to minimizing $\mathcal{O}(\sigma, g)$. Numerically, however, it is (usually) better to minimize $\mathcal{O}(\sigma, g)$. The reason is because evaluating $\mathcal{O}(\sigma, f, \beta^*)$ (or its derivatives) requires us to iterate over the five nested sums in Eq. (6), whereas evaluated $\mathcal{O}(\sigma, g)$ (or its derivatives) requires us to iterate over only the three nested sums in Eq. (1).

2.3 Preprocessing the data

Suppose that reviewer r^* reviewed just one person, that one person being p^*, and further suppose that no other reviewers have reviewed p^*. Of course in this situation, there is no possible calibration procedure that can yield calibrated σ_{r^*} or g_{p^*} since there is nothing to calibrated with respect to. Hence, when calibrating the data, one should remove the review $y^{r^*}_{p^*,d}$ from the data and simply keep track of the fact that r^*'s reviews are not calibrated.

We generalize this preprocessing as follows. Given a set of data $\{y^r_{p,d} \mid r \in \mathcal{R}, p \in \mathcal{P}_r, d \in \mathcal{D}^r_p\}$, we can construct an undirected graph G with vertices $\mathcal{R}$ and with an edge between two reviewers if there is at least one person that they both review; that is, there is an edge between r_1 and r_2 if $\mathcal{P}_{r_1} \cap \mathcal{P}_{r_2} \neq \emptyset$. Each connected component of G can be calibrated separately, but disconnected components cannot be calibrated together. Suppose that $V_1, \ldots, V_n$ are the vertex sets for each connected component of G. Then, for each $i \in \{1, \ldots, n\}$, we calibrated the data $\{y^r_{p,d} \mid r \in V_i, p \in \mathcal{P}_r, d \in \mathcal{D}^r_p\}$. One must keep track of the fact that reviewers from different connected components are calibrated separately and it does not make sense to compare calibration results across connected components.

Note that this ignores the possibility of making assumptions on the data. For example, suppose there are two sets of reviewers $\mathcal{R}_1$ and $\mathcal{R}_2$ that review disconnected sets of people $\mathcal{P}_1$ and $\mathcal{P}_2$. Since they are disconnected, it does not make sense to compare calibrated scales between the two *unless* one assumes that the sample $\mathcal{P}_1$ statistically matches the sample $\mathcal{P}_2$. In other words, if we can assume that (1) $|\mathcal{P}_1|$ and $|\mathcal{P}_2|$ are large enough, and (2) $\mathcal{P}_1$ and $\mathcal{P}_2$ are drawn independently from the same distribution of people, then we can assume that the two resulting calibration scales for $\mathcal{R}_1$ and $\mathcal{R}_2$ are comparable. However, in many situations, these assumptions are too strong. Data is often sparse, and there could be unintended/intended reasons why a certain set of reviewers always reviews a better distribution of people than another set of reviewers.

2.3.1 Comparison to z-score

Indeed, the restrictiveness of these assumptions is the same reason why the z-score is often a bad method to calibrate reviews. For each reviewer r, suppose that $\sigma_r(x) = (x - \mu_r)/s_r$, where μ_r is the mean rating given by reviewer r and s_r is the standard deviation of reviewer r's ratings. Hence, $\sigma_r(x)$ is the z-score of x. Using the z-score as a calibration method — that is, comparing the z-scores from two different reviewers — only makes sense if we make the strong assumptions that (1) both reviewers have reviewed many people, and (2) both reviewers have reviewed roughly the same distribution of (the quality of) people. The first assumption is obviously invalid in many application. The second assumption may be invalid *by design*. For example, suppose an organizing committee trusts a reviewer r_1 more than r_2 and therefore gives r_1 more challenging people to review on average than they give r_2. It follows that the corresponding z-scores will be incomparable because of a systematic bias resulting in totally different calibration scales. Even if not by design, given a data set that is not sufficiently large, statistical fluctuations will naturally cause situations where certain reviewers receive on average lower quality people to review than other reviewers.

The method we presented above, summarized by Eq. (1), does not require such strong assumptions since reviewers are inherently being compared to each other. As long as the graph described above (with vertices $\mathcal{R}$ and edges between two reviewers in $\mathcal{R}$ if they review at least one person in common) is connected, the resulting calibrated data from all the reviewers can be reasonably compared.

3 Arbitrary linear case

In this section, we consider the case when g_p and σ_r are parameterized as

$$g_p(d) = \alpha_p d + \beta_p \tag{7}$$

$$\sigma_r(x) = a_r x + b_r, \tag{8}$$

where a_r, b_r, α_p, and β_p are parameters to learn. This form of g_p assumes that each person improves linearly at a rate of α_p per day over the days.

In Appendix A, we single out the specific parameters a, b, and α one at a time to give some intuition for what's going on, and recall that in Section 2.2 we showed that β is completely determined by a, b, and α. In this section, we discuss how to perform the general calibration with this arbitrary linear calibration of σ_r and g_p. This is a nice case where we can still get something efficiently solvable on a computer. The reason is because when each function is linear in the parameters, the partial derivatives of the objective function are linear in each parameter. Hence, we are left with a simple matrix equation that we can efficiently solve. In the next section, we will consider more arbitrary functions that may not have this property; there, we will need to resort to gradient based minimization methods which will almost certainly not find the optimal result.

In addition, the arbitrary linear case is also the simplest case that has a few of the necessary properties that we require. Indeed, there are a few properties that we desire in order for our protocol to be independent of the rating scale used.

Firstly, suppose that σ_r^* and g_p^* minimizes the objective function for a set of ratings $y_{p,d}^r$. Then it follows that $\sigma_r^{*'}$ and $g_p^{*'}$ – where $\sigma_r^{*'}(x) = \sigma_r^*(x-c)$ and $g_p^{*'}(d) = g_p^*(d)$ – minimizes the objective function for the set of ratings $y_{p,d}^r + c$ for any constant c (where additionally $y_{\min}$ and $y_{\max}$ also translate to $y_{\min}+c$ and $y_{\max}+c$). Our protocol should be independent of the constant c since it should be independent of the arbitrary scale with which the original ratings are performed. We therefore must choose a parameterization that gives us this freedom. In this case, suppose each σ_r is parameterized as $x \mapsto a_r x + b_r$. Then $\sigma_r^*(x) = a_r^* x + b_r^*$ and $\sigma_r^{*'}(x) = a_r^{*'} x + b_r^{*'}$, where $a_r^{*'} = a_r$ and $b_r^{*'} = b_r^* - c$.

Secondly, suppose that σ_r^* and g_p^* minimizes the objective function for a set of ratings $y_{p,d}^r$. Then it follows that $\sigma_r^{*'}$ and $g_p^{*'}$ – where $\sigma_r^{*'}(x) = \sigma_r^*(x/c)$ and $g_p^{*'}(d) = f_p^*(d)$ – minimizes the objective function for the set of ratings $cy_{p,d}^r$ for any constant c (where additionally $y_{\min}$ and $y_{\max}$ also translate to $cy_{\min}$ and $cy_{\max}$). Our protocol should be independent of the constant c since it should be independent of the arbitrary scale with which the original ratings are performed. We therefore must choose a parameterization that gives us this freedom. In this case, suppose each σ_r is parameterized as $x \mapsto a_r x + b_r$. Then $\sigma_r^*(x) = a_r^* x + b_r^*$ and $\sigma_r^{*'}(x) = a_r^{*'} x + b_r^{*'}$, where $a_r^{*'} = a_r/c$ and $b_r^{*'} = b_r^*$.

Finally, similarly to above, we consider transforming each day as $d \mapsto cd + e$ for some constant c and e. Under this transformation, we therefore need that $\sigma_r^{*'}(x) = \sigma_r^*(x)$ and $g_p^{*'}(d) = g_p^*((d-e)/c)$. In our case when g_p is parameterized as $g_p(d) = \alpha_p d + \beta_p$, the transformation results in $\alpha_p^{*'} \mapsto \alpha_p^*/c$ and $\beta_p^{*'} \mapsto \beta_p^* - \alpha_p^* e/c$.

Let $a = \{a_r \mid r \in \mathcal{R}\}$, $b = \{b_r \mid r \in \mathcal{R}\}$, $\alpha = \{\alpha_p \mid p \in \mathcal{P}\}$, and $\beta = \{\beta_p \mid p \in \mathcal{P}\}$. Then the objective function $\mathcal{O}(\sigma, g)$ from Eq. (1) becomes $\mathcal{O}(a, b, \alpha, \beta)$ and we need that for each r and p,

$$\frac{\partial \mathcal{O}}{\partial a_r} = 0, \qquad \frac{\partial \mathcal{O}}{\partial b_r} = 0, \qquad \frac{\partial \mathcal{O}}{\partial \alpha_p} = 0, \qquad \frac{\partial \mathcal{O}}{\partial \beta_p} = 0. \tag{9}$$

Since $\mathcal{O}$ is quadratic, these derivatives are linear in the parameters so that this equation describes a system of linear equations.

Let $M = 2|\mathcal{R}| + 2|\mathcal{P}|$. Let z be the $M \times 1$ matrix of parameters $z = \begin{pmatrix} a & b & \alpha & \beta \end{pmatrix}^T$. We now want to find the $M \times M$ matrix A and the $M \times 1$ matrix c such that the vanishing derivative conditions are written as $Az = c$. This can be done by simply iterating over the data $y_{p,d}^r$ once.

One minor caveat; the matrix A will be singular. The reason is because if a set of b_r^* are solutions, then so will the set $b_r^* + \text{const}$ for any constant. This is due to the fact that we never fixed a range for the absolute scale. We can set the scale arbitrarily by adding a condition to one of the b_r. Indeed, suppose that $r_0 \in \mathcal{R}$ is some fixed reviewer. We replace the condition that $\frac{\partial \mathcal{O}}{\partial b_{r_0}} = 0$ with the condition that $b_{r_0} = 0$. In other words, take out the row of A corresponding to the index of b_{r_0}, and we replace it with 1 in the column corresponding to b_{r_0} and all 0's elsewhere. This will ensure that A is not singular, and therefore there is a unique solution for z. Ultimately, we will renormalize (see below), so this has no effect on the final result.

Another reason that A may be singular is if a certain person p only has ratings given to them on a single day. In this case, their improvement rate α_p could be chosen arbitrarily. We choose by convention to set α_p to zero in this case, which corresponds to setting every column in the row corresponding to α_p to zero except for the column corresponding to α_p which we set to 1.

Finally, A may be (close to) singular if a certain reviewer r has only given a single value rating; that is, if $|\{y^r_{p,d} \mid p \in \mathcal{P}_r, d \in \mathcal{D}^r_p\}| = 1$. In this case, there is no way to determine both a_r and b_r (although, for finite λ, A may only be close to singular and one may be able to find both; however, these will not be at all representative of the true parameters). Hence, in this case, we choose by convention to set $a_r = 0$. This is done by setting every column in the row corresponding to a_r to zero except for the column corresponding to a_r which is set to 1, and column corresponding to a_r in c is then set to zero.

Recall that we want to take λ to be very small, and after the minimization procedure we want to renormalize. Here we discuss the renormalization. We now have a bunch of calibrated data. Call the minimum rating in the calibrated data y_0 and the maximum y_1. Then for each rating y in the calibrated data, we transform it to $\frac{y-y_0}{y_1-y_0}$ to put everything exactly between 0 and 1. This corresponds to the transformations

$$a_r \to \frac{a_r}{y_1 - y_0}, \qquad b_r \to \frac{b_r - y_0}{y_1 - y_0}, \qquad \alpha_p \to \frac{\alpha_p}{y_1 - y_0}, \qquad \beta_p \to \frac{\beta_p - y_0}{y_1 - y_0}. \tag{10}$$

This gives us our final result: a bunch of calibrated reviews normalized between 0 and 1, and a bunch of parameters telling us something about the reviewers and the people. For example, the calibrated α_p tells us how much person p improves per day on the absolute scale.

Furthermore, since all the functions are linear and therefore invertible, we can use our result as a recommendation system. Suppose that I, say reviewer r_0, did not review the person p_0. I can estimate that if I *had* reviewed p_0 on day d, then I *would* have given a rating $\sigma^{-1}_{r_0}(g_{p_0}(d))$.

We implement this arbitrary linear parameterization in the Python package `rcal`[2].

3.1 Performing the limit exactly

One can check that $A = A' + \lambda B$ and $c = (\lambda\Delta y)c'$, where c' is a vector independent of λ and Δy, A' is a matrix independent of λ and Δy, and B is a matrix independent of λ. The equation $Az = c$ becomes $(\mathbb{I} + \lambda B)z = (\lambda\Delta y)A'^{-1}c'$. We consider the Neumann series expansion of $(\mathbb{I} + \lambda B)$, which gives $z = (\lambda\Delta y)A'^{-1}c' + \mathcal{O}(\lambda^2)$. It follows that

$$\lim_{\lambda\to 0^+} \mathrm{Renorm}(A^{-1}c) = \mathrm{Renorm}(A'^{-1}c'). \tag{11}$$

On the right-hand side, as desired all dependence on λ and Δy has disappeared.

We have thus performed the limit analytically whenever A' is invertible. However, it can be the case that A is invertible for arbitrarily small but nonzero λ while A' is not invertible. In this case, the above analytic solution fails. Suppose however that A' has rank $M-1$. We can solve this case analytically as well. Recall that Cramer's rule says that $z_i = \det(A_i)/\det(A)$, where A_i is the matrix A with its i^{th} column replaced with c. Notice though that $\mathrm{Renorm}((\det(A_1)/\det(A), \ldots, \det(A_M)/\det(A))) = \mathrm{Renorm}((\det(A_1), \ldots, \det(A_M)))$. It then easily follows that $\lim_{\lambda\to 0} \mathrm{Renorm}((\det(A_1), \ldots, \det(A_M))) = \mathrm{Renorm}((\det(A'_1), \ldots, \det(A'_M)))$, where A'_i is the matrix A' with its i^{th} column replaced with c'. We therefore find that, regardless of whether or not A' is invertible,

$$\lim_{\lambda\to 0^+} \mathrm{Renorm}(A^{-1}c) = \mathrm{Renorm}((\det(A'_1), \ldots, \det(A'_M)))\,. \tag{12}$$

We have thus found a general analytic solution and as desired the dependence on λ and Δy has dropped out. Interestingly, the dependence on B has also dropped out.

We note that numerically, simply setting λ to be small and performing a linear system solve (e.g. with Gaussian elimination) of $Az = c$ may be more efficient and numerically stable than computing each $\det(A'_i)$. *Possible future work; which one is better? Is there any way we can get around computing M determinants?*

[2]See https://github.com/jtiosue/rcal.

In the end, whenever the data is sufficiently dense, A' should be invertible and therefore one can simply solve $A'z = c'$. However, in the instances when the data is sparse enough that A' is (close to) singular but is rank $M-1$, one can compute $z_i = \det(A'_i)$ for each $1 \leq i \leq M$. We show a simple example of this technique in Appendix A.2.1.

We have still not discussed the case when the rank of A' is less than $M-1$. First, we multiply our equation by A^T on both sides to get the new linear system

$$\left(A'^T A' + \lambda(A'^T B + B^T A') + \lambda^2 B^T B\right) z = A'^T c + \lambda B^T c. \tag{13}$$

Let P be the projector onto the nullspace of $\tilde{A} := A'^T A'$. As $\lambda \to 0$, the inverse of $\tilde{A}$ restricted to the image of P will blow up, while the rest will remain finite. Thus, we need to solve the equation when restricting to the image of P, giving us the equation $PB^T BPz = PB^T c'$. In order for A to be invertible at finite λ, we must have that the nullspace of B is orthogonal to the nullspace of A'. Thus, our equation has the unique solution

$$\lim_{\lambda \to 0^+} \operatorname{Renorm}(A^{-1}c) = \operatorname{Renorm}\left((PB^T BP)^+ B^T c'\right), \tag{14}$$

where $(\cdot)^+$ denote the Moore-Penrose inverse. Notice that all dependence on λ and Δy has vanished, as desired.

We will now study this problem more formally. Such a problem (without Renorm) goes by the name singular matrix perturbation theory, and is a well-studied problem [1–4]. We will make use of Ref. [1] to rigorously prove Eq. (14).

Proof. First, notice that we are trying to find $\lim_{\lambda\to 0} \frac{(A^T A)^{-1} A^T c}{\|(A^T A)^{-1} A^T c\|_\infty}$. Suppose that $(A^T A)^{-1} = \sum_{n=-m}^{\infty} X_{m+n}\lambda^n$. If $m = 0$, then $\operatorname{rank}(A') = M$, and so this is not the case we are interested in. Instead, we are concerned with the case that $m > 0$, so that the Laurent series for the inverse has a pole at $\lambda = 0$. If $m > 0$ and $X_0 A'^T = 0$, then it easily follows that

$$\lim_{\lambda \to 0} \frac{(A^T A)^{-1} A^T c}{\|(A^T A)^{-1} A^T c\|_\infty} = \frac{X_0 B^T c'}{\|X_0 B^T c'\|_\infty}. \tag{15}$$

We must therefore find X_0, and in particular we will prove that $X_0 = (PB^T BP)^+$, where $(\cdot)^+$ denote the Moore-Penrose pseudoinverse. Notice that with this X_0, the condition $X_0 A'^T = 0$ is satisfied.

We will show below that $m = -2$ for our equation. Define $X = \sum_{n=0}^{\infty} X_n \lambda^n$, so that $\tilde{A}^{-1} = \frac{1}{\lambda^m} X$. The equations $A^T A (A^T A)^{-1} = (A^T A)^{-1} A^T A = \mathbb{I}$ yields [1]

$$\begin{aligned}
\tilde{A} X_0 &= 0 = X_0 \tilde{A} \\
\tilde{A} X_1 + (A'^T B + B^T A') X_0 &= 0 = X_1 \tilde{A} + X_0 (A'^T B + B^T A') \\
\tilde{A} X_2 + (A'^T B + B^T A') X_1 + B^T B X_0 &= \mathbb{I} = X_2 \tilde{A} + X_1 (A'^T B + B^T A') + X_0 B^T B \\
\tilde{A} X_3 + (A'^T B + B^T A') X_2 + B^T B X_1 &= 0 = X_3 \tilde{A} + X_2 (A'^T B + B^T A') + X_1 B^T B \\
&\vdots
\end{aligned} \tag{16}$$

Given this, the proof is completed due to the following simple argument. From the first equation and because $\tilde{A}$ is symmetric, clearly X_0 must equal PX_0P. Similarly, by multiplying the second equations on the right or left by P, X_1 must equal $(1-P)X_1(1-P)$. Using these and multiplying the third equations on the left and right by P yields $PB^T BPX_0P = PX_0PB^T BP = P$. This is the defining equation for $X_0 = (PB^T BP)^+$ as desired.

It therefore only remains to show that $m = 2$. From [1, Cor. 2.2], it is sufficient to show that

$$\begin{aligned}
&\operatorname{rank}(C_2) - \operatorname{rank}(C_1) = M, \quad \text{where} \\
&C_1 = \begin{pmatrix} \tilde{A} & 0 \\ A'^T B + B^T A' & \tilde{A} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \tilde{A} & 0 & 0 \\ A'^T B + B^T A' & \tilde{A} & 0 \\ B^T B & A'^T B + B^T A' & \tilde{A} \end{pmatrix}.
\end{aligned} \tag{17}$$

Using that we are assuming that the nullspace of A' and the nullspace of B are orthogonal so that A is invertible at finite values of λ, one can check, as we do below, that $\dim \mathrm{Null}(C_1) = \dim \mathrm{Null}(C_2) = \dim \mathrm{Null}(A') + \dim(\mathrm{Im}(\tilde{A}) \cap \mathrm{Im}(A'^T BP))$, hence via the rank-nullity theorem proving Eq. (17). Note that the fact that $\dim \mathrm{Null}(C_1) > \dim \mathrm{Null}(A')$ shows that $m > 1$ [1].

We now show that $\dim \mathrm{Null}(C_1) = \dim \mathrm{Null}(A') + \dim(\mathrm{Im}(\tilde{A}) \cap \mathrm{Im}(A'^T BP))$. We parameterize the nullspace by $C_1(a\ b)^T = 0$. This immediately yields the constraints that $a \in \mathrm{Null}(A)$ and $A^T Ba + \tilde{A}b = 0$. The first $\dim \mathrm{Null}(A')$ simply comes from setting $a = 0$ and thus requiring $b \in \mathrm{Null}(A')$. We can then consider the case that $b \notin \mathrm{Null}(A')$. In this case, each b gives us an element $\tilde{b}$ of $\mathrm{Im}(A')$. We therefore need an $a \in \mathrm{Null}(A')$ such that $A^T Ba + \tilde{b} = 0$. We can therefore parameterize all the solutions by looking at the intersection of $\mathrm{Im}(A^T BP)$ with $\mathrm{Im}(\tilde{A})$.

We now show that $\dim \mathrm{Null}(C_2) = \dim \mathrm{Null}(A') + \dim(\mathrm{Im}(\tilde{A}) \cap \mathrm{Im}(A'^T BP))$. We parameterize the nullspace by $C_2(a\ b\ c)^T = 0$. This yields the same constraints as above, plus the constraint $B^T Ba + A'^T Bb + B^T A'b + \tilde{A}c = 0$. As before, the $\dim \mathrm{Null}(A')$ comes from setting $a = b = 0$ and $c \in \mathrm{Null}(A')$. For the remaining term, we consider $c \notin \mathrm{Null}(A')$. By multiplying the third constraint on the left with P, we find $PB^T Ba + PB^T A'b = 0$, so that we are interested in the intersection of $\mathrm{Im}(BP)$ with $\mathrm{Im}(A')$. This constraint is already taken care of by the $\mathrm{Im}(A^T BP) \cap \mathrm{Im}(\tilde{A})$. Thus, we have shown that $\dim \mathrm{Null}(C_2) \leq \dim \mathrm{Null}(A') + \dim(\mathrm{Im}(\tilde{A}) \cap \mathrm{Im}(A'^T BP))$. It follows that $\mathrm{rank}(C_2) - \mathrm{rank}(C_1) \geq M$. Note that $\mathrm{rank}(C_2) = \mathrm{rank}(C_2^T) \leq \mathrm{rank}(C_1^T) + M = \mathrm{rank}(C_1) + M$ simply because $C_2^T = \begin{pmatrix} C_1^T & D_1 \\ 0 & D_2 \end{pmatrix}$ for matrices D_1, D_2. Thus, $\mathrm{rank}(C_2) - \mathrm{rank}(C_1) = M$. □

4 General functions

Provided that σ_r and g_p are linear in the parameters with which they are parameterized, the minimization can be performed efficiently optimally via a linear solver. For example, one could consider the parameterization $\sigma_r(x) = a_r x + b_r + c_r x^2$ and $g_p(d) = \alpha_p d + \beta_p + \gamma_p d^2$. However, one needs to be careful when adding more and more parameters to fit; one may quickly run into the issue of overfitting.

There are, however, natural parameterizations that one may expect to model this situation well. For example, σ_r could be a sigmoid function parameterized by translation and scale. Such a function is naturally bounded, has a section of near-linear growth, and has flat tails. Similarly, g_p could be parameterized as $\alpha_p d + \beta_p - \gamma_p/d$. Such a parameterization with positive α_p and γ_p models the tendency for improvement to be more rapid initially (i.e. when d is small) and to slow down as the days progress.

In the general situation in which the functions are not linear in the parameters, one should not expect to be able to optimally perform the minimization, and instead one must defer to gradient-based minimization techniques. Indeed, this is the case when σ_r is parameterized as a sigmoid function. One should therefore not expect to be able to perform the minimization optimally; nonetheless, approximate solutions may still be useful.

Finally, as mentioned in Remark 1, the λ term in $\mathcal{O}$ may need to be changed when we start parameterizing σ_r and g_p in more general ways.

Possible future work.

A Case study

This section will serve as a means to build intuition for the output of the minimization of the objective function $\mathcal{O}$ from Eq. (1). Recall from Section 2.2 that we can equivalently consider Eq. (6) and therefore ignore the β_p parameters. This is what we will do throughout this appendix. For convienience, we will write $\mathcal{O}(\sigma, f)$ to mean $\mathcal{O}(\sigma, f, \beta^*)$ from Eq. (6).

A.1 Case 0: when $f_p(d) = 0$ and $\sigma_r(x) = x + b_r$

We consider the case when

$$f_p(d) = 0 \tag{A1}$$

$$\sigma_r(x) = x + b_r, \tag{A2}$$

for parameters b_r. This corresponds to the case where we ignore any improvement of people, and we assume that reviewers review scales differ by a constant offset. In this case, the objective function in Eq. (6) $\mathcal{O}(\sigma, f)$ becomes $\mathcal{O}(b)$, where $b = \{b_r \mid r \in \mathcal{R}\}$, and we are therefore interested in $\frac{\partial \mathcal{O}}{\partial b_r}$. Plugging in the forms of f and σ, Eq. (6) becomes

$$\begin{aligned}\mathcal{O}(b) = \frac{1}{2} \widetilde{\sum_{p\in\mathcal{P}}} \widetilde{\sum_{r_1\in\mathcal{R}_p}} \widetilde{\sum_{d_1\in\mathcal{D}_p^{r_1}}} \widetilde{\sum_{r_2\in\mathcal{R}_p}} \widetilde{\sum_{d_2\in\mathcal{D}_p^{r_2}}} \left(y_{p,d_1}^{r_1} - y_{p,d_2}^{r_2} + b_{r_1} - b_{r_2}\right)^2 \\ + \lambda \widetilde{\sum_{r\in\mathcal{R}}} \left(y_{\max} - y_{\min} - 1\right)^2 .\end{aligned} \tag{A3}$$

We need that $\frac{\partial \mathcal{O}}{\partial b_r} = 0$, for each $r \in \mathcal{R}$. Let's compute the derivative:

$$\frac{\partial \mathcal{O}(b)}{\partial b_r} \propto \widetilde{\sum_{p\in\mathcal{P}_r}} \widetilde{\sum_{r_1\in\mathcal{R}_p}} \widetilde{\sum_{d_1\in\mathcal{D}_p^{r_1}}} \widetilde{\sum_{d_2\in\mathcal{D}_p^{r}}} \left(-y_{p,d_1}^{r_1} + y_{p,d_2}^{r} - b_{r_1} + b_r\right). \tag{A4}$$

Setting all of these to 0 is easily recast into a linear matrix equation which can be solved. The matrix will be of dimension $|\mathcal{R}| \times |\mathcal{R}|$. Notice that if b_r^* is a solution, then $b_r^* + c$ is also a solution, for any constant c. Hence, the matrix will have nontrivial kernel which we can account for as described in Section 3.

To get some intuition for the solution, let's consider the case when each reviewer reviews every person exactly once. In other words, we consider the case when $\mathcal{P}_r = \mathcal{P}$ for each r and hence $\mathcal{R}_p = \mathcal{R}$ for each p, and when $\left|\mathcal{D}_p^r\right| = 1$ for each r and p. In this case, we find that (we suppress all d indices since they are trivial in this case)

$$\frac{\partial \mathcal{O}}{\partial b_r} \propto \widetilde{\sum_{p\in\mathcal{P}}} \widetilde{\sum_{r_1\in\mathcal{R}}} \left(-y_p^{r_1} + y_p^{r} - b_{r_1} + b_r\right). \tag{A5}$$

It follows that in this case, for each $r \in \mathcal{R}$, we want

$$b_r^* = \widetilde{\sum_{p\in\mathcal{P}}} \widetilde{\sum_{r_1\in\mathcal{R}}} \left(y_p^{r_1} - y_p^{r}\right) + c \tag{A6}$$

$$= (\text{average rating that the average reviewer gives}) - (\text{average rating that reviewer } r \text{ gives}) + c, \tag{A7}$$

where c is any arbitrary constant that we might as well set to 0[3]. So b_r^* translates a reviewer by a certain average amount so as to bring that reviewer's review scale closer to the mean. Thus, given a review y_p^r, we have found a calibrated review $y_p^r + b_r^*$. We therefore associate to person $p \in \mathcal{P}$ the average calibrated review

$$\widetilde{\sum_{r\in\mathcal{R}}} (y_p^r + b_r^*) = \widetilde{\sum_{r\in\mathcal{R}}} y_p^r, \tag{A8}$$

Hence, we see that in this restricted case, – indeed, with all the restrictions we have enacted, this is the simplest possible case one can consider – we have reproduced the most naive form of calibration which is simply to average (recall that $\widetilde{\sum}_{r\in\mathcal{R}}$ denotes the average $\frac{1}{|\mathcal{R}|}\sum_{r\in\mathcal{R}}$) all the different ratings that a person gets from all the different reviewers.

A.2 Case 1: when $f_p(d) = 0$ and $\sigma_r(x) = a_r x$

In the previous subsection, we carefully worked through the case of $\sigma_r(x) = x + b_r$. We then restricted to the special case when $\mathcal{P}_r = \mathcal{P}$ and $\left|\mathcal{D}_p^r\right| = 1$. In Section 3, we carefully worked through the completely general linear case. Therefore, in this subsection, for brevity we will immediately assume the $\mathcal{P}_r = \mathcal{P}$ and $\left|\mathcal{D}_p^r\right| = 1$ conditions. We are just trying to get some intuition for what each parameter controls, and we return to the fully general case in Section 3. Recall we suppress all d indices since $\mathcal{D}_p^r$ is trivial.

[3] c being arbitrary is a result of the fact that we did not restrict the range of the absolute rating scale.

With these assumptions, we assume the trivial improvement function $f_p(d) = 0$ and the reviewer scaling functions $\sigma_r(x) = a_r x$ for parameters a_r. Let $a = \{a_r \mid r \in \mathcal{R}\}$. Then, the objection function $\mathcal{O}(\sigma, f)$ from Eq. (6) becomes

$$\mathcal{O}(a) = \frac{1}{2} \widetilde{\sum_{p\in\mathcal{P}}} \widetilde{\sum_{r_1\in\mathcal{R}}} \widetilde{\sum_{r_2\in\mathcal{R}}} \left(a_{r_1} y_p^{r_1} - a_{r_2} y_p^{r_2}\right)^2 + \lambda \widetilde{\sum_{r\in\mathcal{R}}} \left(a_r \Delta y - 1\right)^2, \tag{A9}$$

where $\Delta y \coloneqq y_{\max} - y_{\min}$. We consider the derivative

$$\frac{\partial \mathcal{O}}{\partial a_r} \propto a_r \left\langle (y_p^r)^2 \right\rangle_p - \left\langle a_{r_2} y_p^{r_2} y_p^r \right\rangle_{r_2,p} + \lambda a_r (\Delta y)^2 - \lambda \Delta y, \tag{A10}$$

where we defined the average quantities

$$\langle \cdot \rangle_p \coloneqq \widetilde{\sum_{p\in\mathcal{P}}} (\cdot), \qquad \langle \cdot \rangle_{r_2,p} \coloneqq \widetilde{\sum_{r_2\in\mathcal{R}}} \widetilde{\sum_{p\in\mathcal{P}}} (\cdot). \tag{A11}$$

A.2.1 Numerical example

We investigate a simple situation to get some intuition. Suppose that there are only two people p_1 and p_2. Further for simplicity, suppose that there are only two reviewers, r_1 and r_2. In this case, the system of equations to solve is, from Eq. (A10),

$$\lambda \Delta y = \frac{1}{2} a_{r_1} \left((y_{p_1}^{r_1})^2 + (y_{p_2}^{r_1})^2\right) - \frac{1}{4}\left(a_{r_1} y_{p_1}^{r_1} y_{p_1}^{r_1} + a_{r_1} y_{p_2}^{r_1} y_{p_2}^{r_1} + a_{r_2} y_{p_1}^{r_2} y_{p_1}^{r_1} + a_{r_2} y_{p_2}^{r_2} y_{p_2}^{r_1}\right) + \lambda a_{r_1} (\Delta y)^2 \tag{A12}$$

$$\lambda \Delta y = \frac{1}{2} a_{r_2} \left((y_{p_1}^{r_2})^2 + (y_{p_2}^{r_2})^2\right) - \frac{1}{4}\left(a_{r_1} y_{p_1}^{r_1} y_{p_1}^{r_2} + a_{r_1} y_{p_2}^{r_1} y_{p_2}^{r_2} + a_{r_2} y_{p_1}^{r_2} y_{p_1}^{r_2} + a_{r_2} y_{p_2}^{r_2} y_{p_2}^{r_2}\right) + \lambda a_{r_2} (\Delta y)^2. \tag{A13}$$

Suppose that $y_{\min} = 0$, $y_{\max} = 1$, $y_{p_1}^{r_1} = 0.5, y_{p_2}^{r_1} = 0.75, y_{p_1}^{r_2} = 0.4, y_{p_2}^{r_2} = 0.5$. One can check that this system is uniquely solved by

$$a_{r_1} = \frac{\lambda(\lambda + 0.2462)}{\lambda^2 + 0.30562\lambda + 0.000156} \tag{A14}$$

$$a_{r_2} = \frac{\lambda(\lambda + 0.3468)}{\lambda^2 + 0.30562\lambda + 0.000156}. \tag{A15}$$

Notice that as $\lambda \to 0$, both a_{r_1} and $a_{r_2} \to 0$. But recall that we are interested in

$$\left\{a_{r_1}^*, a_{r_2}^*\right\} = \lim_{\lambda\to 0^+} \text{Renorm}(\text{Minim}(\lambda)). \tag{A16}$$

In this case, suppose that we renormalize by dividing each a_{r_i} by the max score, $\max\{a_{r_1} y_{p_1}^{r_1}, a_{r_1} y_{p_2}^{r_1}, a_{r_2} y_{p_1}^{r_2}, a_{r_2} y_{p_2}^{r_2}\}$. One can check that this maximum is always $a_{r_1} y_{p_2}^{r_1}$. Thus

$$a_{r_1}^* = \frac{a_{r_1}}{a_{r_1} y_{p_2}^{r_1}} = 1/y_{p_2}^{r_1} = 4/3, \tag{A17}$$

$$a_{r_2}^* = \lim_{\lambda\to 0^+} \frac{a_{r_2}}{a_{r_1} y_{p_2}^{r_1}} \tag{A18}$$

$$= \lim_{\lambda\to 0^+} \frac{4}{3} \times \frac{\lambda^2 + 0.30562\lambda + 0.000156}{\lambda(\lambda + 0.2462)} \times \frac{\lambda(\lambda + 0.3468)}{\lambda^2 + 0.30562\lambda + 0.000156} \tag{A19}$$

$$= \lim_{\lambda\to 0^+} \frac{1.33333(\lambda + 0.3468)}{\lambda + 0.2462} \tag{A20}$$

$$= 1.87817. \tag{A21}$$

By taking the limit after renormalizing, we avoid getting the trivial zero solution. The set of calibrated, renormalized ratings is therefore $\left\{a_{r_1}^* y_{p_1}^{r_1}, a_{r_1}^* y_{p_2}^{r_1}, a_{r_2}^* y_{p_1}^{r_2}, a_{r_2}^* y_{p_2}^{r_2}\right\}$, which is $\{0.666667, 1., 0.751269, 0.939086\}$.

More generally, we can perform this limit exactly using the techniques from Section 3.1. From above, the equation we are trying to solve is $A\begin{pmatrix}a_{r_1}\\ a_{r_2}\end{pmatrix}=c$, where $A=A'+\lambda B$, $c=(\lambda\Delta y)c'$, $B=(\Delta y)^2\mathbb{I}$, $c'=\begin{pmatrix}1\\1\end{pmatrix}$, and

$$A'=\begin{pmatrix}\frac{1}{2}\left((y_{p_1}^{r_1})^2+(y_{p_2}^{r_1})^2\right)-\frac{1}{4}\left((y_{p_1}^{r_1})^2+(y_{p_2}^{r_1})^2\right) & -\frac{1}{4}(y_{p_1}^{r_1}y_{p_1}^{r_2}+y_{p_2}^{r_1}y_{p_2}^{r_2}) \\ -\frac{1}{4}(y_{p_1}^{r_1}y_{p_1}^{r_2}+y_{p_2}^{r_1}y_{p_2}^{r_2}) & \frac{1}{2}\left((y_{p_1}^{r_2})^2+(y_{p_2}^{r_2})^2\right)-\frac{1}{4}\left((y_{p_1}^{r_2})^2+(y_{p_2}^{r_2})^2\right)\end{pmatrix}. \tag{A22}$$

It follows from Section 3.1 that as $\lambda\to 0^+$, $\begin{pmatrix}a_{r_1} & a_{r_2}\end{pmatrix}$ is proportional to

$$\left(\det\begin{pmatrix}1 & -\frac{1}{4}(y_{p_1}^{r_1}y_{p_1}^{r_2}+y_{p_2}^{r_1}y_{p_2}^{r_2}) \\ 1 & \frac{1}{2}\left((y_{p_1}^{r_2})^2+(y_{p_2}^{r_2})^2\right)-\frac{1}{4}\left((y_{p_1}^{r_2})^2+(y_{p_2}^{r_2})^2\right)\end{pmatrix} \quad \det\begin{pmatrix}\frac{1}{2}\left((y_{p_1}^{r_1})^2+(y_{p_2}^{r_1})^2\right)-\frac{1}{4}\left((y_{p_1}^{r_1})^2+(y_{p_2}^{r_1})^2\right) & 1 \\ -\frac{1}{4}(y_{p_1}^{r_1}y_{p_1}^{r_2}+y_{p_2}^{r_1}y_{p_2}^{r_2}) & 1\end{pmatrix}\right), \tag{A23}$$

where the proportionality constant is set by renormalization. Plugging in the numbers from above, we find that $(a_{r_1},a_{r_2})\propto(0.24625,0.346875)$. We renormalize by dividing by $\max\{a_{r_1}y_{p_1}^{r_1},a_{r_1}y_{p_2}^{r_1},a_{r_2}y_{p_1}^{r_2},a_{r_2}y_{p_2}^{r_2}\}$, to get

$$(a_{r_1}^*,a_{r_2}^*)=(0.24625,0.346875)/\max\{0.123125,0.184688,0.13875,0.173438\}=(1.33333,1.87817), \tag{A24}$$

which matches the result above.

A.3 Case 2: when $f_p(d)=\alpha_p d$ and $\sigma_r(x)=x$

As in the previous subsection, since we are working only on an intuitive level (see the general case in Section 3), we restrict to when $\mathcal{P}_r=\mathcal{P}$. Also, for simplicity, we just assume that every reviewer reviews every person every day, so that $\mathcal{D}_p^r=\mathcal{D}$ for every r and p. We consider when

$$f_p(d)=\alpha_p d \tag{A25}$$
$$\sigma_r(x)=x, \tag{A26}$$

for some set of parameters $\alpha=\{\alpha_p\mid p\in\mathcal{P}\}$ which characterizes the improvement rate of each person. In this case, the objective function $\mathcal{O}(\sigma,f)$ from Eq. (6) becomes

$$\begin{aligned}\mathcal{O}(\alpha)=\frac{1}{2}\widetilde{\sum_{p\in\mathcal{P}}}\widetilde{\sum_{r_1,r_2\in\mathcal{R}}}\widetilde{\sum_{d_1,d_2\in\mathcal{D}}}&\left(y_{p,d_1}^{r_1}-y_{p,d_2}^{r_2}+\alpha_p(d_2-d_1)\right)^2\\ &+\lambda\widetilde{\sum_{r\in\mathcal{R}}}\left(y_{\max}-y_{\min}-1\right)^2.\end{aligned} \tag{A27}$$

The derivative is

$$\frac{\partial\mathcal{O}}{\partial\alpha_p}\propto\widetilde{\sum_{r_1,r_2\in\mathcal{R}}}\widetilde{\sum_{d_1,d_2\in\mathcal{D}}}\left(d_2y_{p,d_1}^{r_1}-d_2y_{p,d_2}^{r_2}-d_1y_{p,d_1}^{r_1}+d_1y_{p,d_2}^{r_2}+(d_1-d_2)^2\alpha_p\right) \tag{A28}$$
$$\propto\langle d\rangle_d\left\langle y_{p,d}^r\right\rangle_{r,d}-\left\langle dy_{p,d}^r\right\rangle_{r,d}+\left(\left\langle d^2\right\rangle_d-\langle d\rangle_d^2\right)\alpha_p, \tag{A29}$$

where again we defined averaged quantities $\langle\cdot\rangle_d\coloneqq\widetilde{\sum}_{d\in\mathcal{D}}(\cdot)$, etc. Thus, by setting this to zero, we estimate that person $p\in\mathcal{P}$ has improved at a rate of α_p^* per day, where

$$\alpha_p^*=\frac{\left\langle dy_{p,d}^r\right\rangle_{r,d}-\langle d\rangle_d\left\langle y_{p,d}^r\right\rangle_{r,d}}{\langle d^2\rangle_d-\langle d\rangle_d^2} \tag{A30}$$
$$=\frac{\text{covariance of days with ratings}}{\text{variance of days; i.e. average square distance between days}} \tag{A31}$$
$$=(\text{correlation between days and ratings})\times\frac{\text{standard deviation of ratings}}{\text{standard deviation of days}}. \tag{A32}$$

Indeed this is a very intuitive result. The rate at which p improved is equal to the correlation between days and ratings times the ratio of the standard deviation of the ratings to the standard deviation of the days. The aforementioned ratio gives the natural units with which to measure an improvement rate, and the correlation is then the improvement rate expressed in these units.

As a simple sanity check, we consider a case when days and ratings are uncorrelated. For example, for a fixed person p, consider when for each r, $y^r_{p,d} = y^r_{p,d'}$ for all d, d'; this is the case when all the reviewers rate the person the same every day. Then the numerator of α^*_p is zero, since the correlation is zero. Hence, we correctly identified no improvement.

B Alternate method for calibration

Here we describe an alternate method for calibration that is different from the method described in the main text. We discuss pros and cons.

Consider that person p's true rating on day d in $g_p(d)$. Consider functions χ_r which will behave like σ_r^{-1}; namely, $\chi_r(g_p(d))$ should be close to $y^r_{p,d}$. When reviewer r sees a person with true quality x, the reviewer rates them $\chi_r(x)$. It follows that we want to minimize

$$\mathcal{O}_{\text{alt}}(\chi, g) = \widetilde{\sum_{p\in\mathcal{P}}}\ \widetilde{\sum_{r\in\mathcal{R}_p}}\ \widetilde{\sum_{d\in\mathcal{D}^r_p}} \left(y^r_{p,d} - \chi_r(g_p(d))\right)^2 . \tag{B1}$$

Suppose that we parameterize $\chi_r(x) = x + c_r$ and $g_p(d) = \alpha_p d + \beta_p$. Then the derivatives of $\mathcal{O}_{\text{alt}}(c, \alpha, \beta)$ are all linear in the parameters so that we easily find (c^*, α^*, β^*) via a linear systems solve, where $(c^*, \alpha^*, \beta^*) = \operatorname{argmin}_{(c,\alpha,\beta)} \mathcal{O}_{\text{alt}}(c, \alpha, \beta)$.

The downside of this method is that we cannot parameterize χ_r to be arbitrarily linear and still use a linear system solve. If $\chi_r(x) = e_r x + c_r$, then $\chi_r(g_p(d))$ has a term $e_r\alpha_p d$ and a term $e_r\beta_p$, both of which are quadratic in the parameters. Thus, the derivatives of $\mathcal{O}_{\text{alt}}$ will have quadratic terms and so the system cannot be solved via a matrix equation $Az = b$. We therefore argue that the calibration method developed in the main text and Appendix A is superior to this alternative method. The objective function in this alternate method is simpler, but this comes at the cost of losing an important parameter[4].

Interestingly, we see that this alternate calibration method (combined with a renormalization analogous to the main text where we place all calibrated ratings within set bounds) matches the calibration method from the main text *when setting λ to be very large*. This is easy to prove by letting $\lambda \to \infty$ which fixes a_r. The intuition for this result is as follows. Unsurprisingly, both $\mathcal{O}$ and $\mathcal{O}_{\text{alt}}$ are encoding roughly equivalent conditions (indeed results similar to those in Appendices A.1 and A.3 also hold for the alternate method). However, in the alternate method, we don't have a e_r to play with and instead fix each e_r to be 1. This scale factor, e_r, is analogous to the scale factor a_r in the main text. Recall that when $\lambda \to \infty$, each a_r is restricted to be exactly the same as all the other, and their values are then fixed by the bounds set in the renormalization. Thus, when λ is large, we are essentially removing the scale factors a_r as parameters, which is analogous to the fact that each e_r is fixed in the alternate method.

References

[1] P. Howlett and K. Avrachenkov, "Laurent series for the inversion of perturbed linear operators on Hilbert space", in *Optimization and related topics*, edited by A. Rubinov and B. Glover (Springer US, Boston, MA, 2001), pp. 325–342.

[2] P. Howlett, A. Albrecht, and C. Pearce, "Laurent series for inversion of linearly perturbed bounded linear operators on Banach space", Journal of Mathematical Analysis and Applications **366**, 112–123 (2010).

[4] Indeed, if we keep the e_r parameter, we are tasked with minimizing a multivariate quartic polynomial, which is in general `NP-Hard` [5]. If however, one is only interested in the relative *ranking* of people in $\mathcal{P}$ as calibrated by $\mathcal{O}_{\text{alt}}$ as opposed to actual calibrated ratings assigned to each person, then I believe it is possible to efficiently find the ranking. The reason is because one can convert the ranking problem to a convex problem via Theorem 3.6 in Ref. [6]. However, we emphasize that even though one can efficiently determine the ranking, one cannot in general efficiently determine the actual calibration parameters.

[3] A. S. Deif, "The generalized inverse of a perturbed singular matrix", Zeitschrift für angewandte Mathematik und Physik ZAMP **34**, 291–300 (1983).

[4] P. Gonzalez-Rodriguez, M. Moscoso, and M. Kindelan, "Laurent expansion of the inverse of perturbed, singular matrices", Journal of Computational Physics **299**, 307–319 (2015).

[5] A. A. Ahmadi and J. Zhang, "Complexity aspects of local minima and related notions", Advances in Mathematics **397**, 108119 (2022).

[6] S. Tan, J. Wu, X. Bei, and H. Xu, "Least square calibration for peer reviews", Advances in Neural Information Processing Systems **34**, 27069–27080 (2021), arXiv:2110.12607 [cs.LG].