An approach to review calibration

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Abstract

In this report, we develop a method to calibrate a set of reviews of people from different reviewers on different days. Given such data, we take into account the improvement of people as the days progress and the biases and different scales with which reviewers review.

1 Introduction and overview

Consider a competition amongst many people that lasts multiple days. Consider a set of reviewers who each review some subset of people on various different days. How can we determine an objective, absolute rating for each person? The issues to tackle are as follows. First, each reviewer has a different reviewing scale; what one reviewer means by 2 stars may be different from what another means by 2 stars. Second, the people may improve each day; if one reviewer reviews somebody on the first day and another on the fifth day, we should expect the person to have improved.

We would like to calibrate the reviewers so that everybody has the same absolute rating scale. We would also like to determine how much each person is improving over the course of the competition.

The remainder of the report is as follows. In Section 2, we will encode our calibration problem into the minimization of an objective function. In Section 3, we will discuss practical solutions. In particular, in Section 3.4 we provide an efficient, deterministic protocol for calibration. Sections 3.1 to 3.3 lead up to this result by providing evidence that the protocol in Section 3.4 does indeed give a good solution. Finally, we discuss possible extensions in Section 3.5.

2 Formulation as an optimization problem

Here we define the relevant terms. See Table 1 for a summary.

We have a set of people $\mathcal{P}$ and a set of reviewers $\mathcal{R}$. Each reviewer $r \in \mathcal{R}$ reviews a subset of people $\mathcal{P}_r \subseteq \mathcal{P}$. For convenience, also denote the set of reviewers who reviewed p by $\mathcal{R}_p$. Reviewer $r \in \mathcal{R}$ gives person $p \in \mathcal{P}_r$ a rating $y_p^r \in \mathbb{R}$ (usually the rating will be restricted to e.g. 1 through 5 stars, but for now we only assume that it is a real number). We are hoping to somehow calibrate the reviews; that is, reviewers r_1 and r_2 may have a different definition of what good and bad are, a different definition of the relative difference between 2 and 3 stars, etc. Hence, to each reviewer $r \in \mathcal{R}$, we associate a function σ_r that maps from reviewer r's scale to some absolute scale. In other words, given that y_p^r is reviewer r's actual rating of person p, $\sigma_r(y_p^r)$ will be the review on the "absolute scale" that we associate to person p from reviewer r. This way, each person p has a set of *calibrated* reviews $\left\{\sigma_r(y_p^r) \mid r \in \mathcal{R}_p\right\}$.

There is one minor caveat, though, which stems from the fact that reviewers may have reviewed the people at different times. We deal with this as follows. We let $\mathcal{D}_p^r$ be the set of days that reviewer r reviewed person p. Then for each reviewer $r \in \mathcal{R}$, person $p \in \mathcal{P}_r$, and day $d \in \mathcal{D}_p^r$, we have a review $y_{p,d}^r$. We then associate to each person $p \in \mathcal{P}$ an improvement function f_p. This function will take a day d and map it to an offset $f_p(d)$ associated to the absolute scale. Ultimately, the calibrated reviews will still be $\left\{\sigma_r(y_p^r) \mid r \in \mathcal{R}_p\right\}$, but the inclusion of f_p in our analysis will result in better choices of σ_r.

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$\mathcal{P}$	set of people reviewed
$\mathcal{R}$	set of reviewers
$\mathcal{P}_r$	set of people reviewed by reviewer $r \in \mathcal{R}$
$\mathcal{R}_p$	the set of reviewers who reviewed person $p \in \mathcal{P}$
$\mathcal{D}_p^r$	set of days that reviewer $r \in \mathcal{R}$ reviewed person $p \in \mathcal{P}_r$
$y_{p,d}^r$	reviewer r's ($r \in \mathcal{R}$) rating of person $p \in \mathcal{P}_r$ on day $d \in \mathcal{D}_p^r$
$y_{\min}$	minimum allowed raw rating
$y_{\max}$	maximum allowed raw rating
Δy	$\Delta y \coloneqq y_{\max} - y_{\min}$
σ_r	scaling function associated to reviewer $r \in \mathcal{R}$, $\sigma_r \colon [y_{\min}, y_{\max}] \to \mathbb{R}$
σ	$\sigma \coloneqq \{\sigma_r \mid r \in \mathcal{R}\}$
f_p	improvement function associated to person $p \in \mathcal{P}$, $f_p \colon \bigcup_r D_p^r \to \mathbb{R}$
f	$f \coloneqq \{f_p \mid p \in \mathcal{P}\}$
λ	multiplier

Table 1: A list of our definitions.

Our goal is to find the σ and f that *minimizes* the distinguishability of reviewers while maintaining the distinguishability of people. In the case of the former, we want that σ_r takes r's reviews and maps them to an absolute scale. Ignoring the improvement factor for a moment, for two different reviewers r_1, r_2, we want $\sigma_{r_1}(y_p^{r_1})$ to be very close to $\sigma_{r_2}(y_p^{r_2})$. In the case of the latter, we choose an absolute scale to be *roughly* between 0 and 1, and so we enforce that the average (over $r \in \mathcal{R}$) difference $\sigma_r(y_{\max}) - \sigma_r(y_{\min})$ is 1.

We thus arrive at the following objective function to minimize:

$$\begin{aligned} \mathcal{O}(\sigma, f) = \frac{1}{\mathcal{N}} \sum_{p \in \mathcal{P}} \sum_{r_1 \in \mathcal{R}_p} \sum_{d_1 \in \mathcal{D}_p^{r_1}} \sum_{r_2 \in \mathcal{R}_p} \sum_{d_2 \in \mathcal{D}_p^{r_2}} \Big(\sigma_{r_1}(y_{p,d_1}^{r_1}) + f_p(d_1) - \sigma_{r_2}(y_{p,d_2}^{r_2}) - f_p(d_2) \Big)^2 \\ + \frac{\lambda}{|\mathcal{R}|} \sum_{r \in \mathcal{R}} \left(\sigma_r(y_{\max}) - \sigma_r(y_{\min}) - 1 \right)^2, \end{aligned} \tag{1}$$

where λ is an arbitrary multiplier that we usually set to 1, and

$$\mathcal{N} = \left| \left\{ (p, r_1, d_1, r_2, d_2) \mid p \in \mathcal{P}, r_1 \in \mathcal{R}_p, d_1 \in \mathcal{D}_p^{r_1}, r_2 \in \mathcal{R}_p, d_2 \in \mathcal{D}_p^{r_2} \right\} \right|. \tag{2}$$

In summary, let σ^* and f^* be

$$(\sigma^*, f^*) = \underset{(\sigma, f)}{\operatorname{argmin}}\, \mathcal{O}(\sigma, f), \tag{3}$$

where $\mathcal{O}$ is from Eq. (1). Then, the set of calibrated reviews for person p is $\left\{ \sigma_r(y_{p,d}^r) \mid r \in \mathcal{R}_p, d \in \mathcal{D}_p^r \right\}$. Depending on what is desired – that is, whether or not we want to include improvement in our final averaging – we associate one of the two the calibrated, averaged rating to person $p \in \mathcal{P}$:

$$\frac{1}{\left| \left\{ (r, d) \mid r \in \mathcal{R}_p, d \in \mathcal{D}_p^r \right\} \right|} \sum_{r \in \mathcal{R}_p} \sum_{d \in \mathcal{D}_p^r} \sigma_r(y_{p,d}^r). \quad \text{or} \quad \frac{1}{\left| \left\{ (r, d) \mid r \in \mathcal{R}_p, d \in \mathcal{D}_p^r \right\} \right|} \sum_{r \in \mathcal{R}_p} \sum_{d \in \mathcal{D}_p^r} \sigma_r(y_{p,d}^r) + f_p(d). \tag{4}$$

3 Optimization in practice

In the previous section, we defined the objective function $\mathcal{O}(\sigma, f)$ that we are trying to minimize. The minimization is with respect to the set of all σ_r and f_p, which are themselves arbitrary functions. In practice, we restrict our attention to certain types of functions. In other words, we consider some set of functions Σ and another set of functions F, and we are looking for

$$(\sigma^*, f^*) = \underset{\substack{(\sigma,f) \text{ s.t.} \\ \sigma_r \in \Sigma, f_p \in F}}{\operatorname{argmin}} \mathcal{O}(\sigma, f). \tag{5}$$

For example, suppose we parameterize each σ_r by a set of parameters θ^r, and we parameterize each f_p by a set of parameters φ^p. Let $\theta = \{\theta^r \mid r \in \mathcal{R}\}$ and $\varphi = \{\varphi^p \mid p \in \mathcal{P}\}$. Then we are looking for $\operatorname{argmin}_{(\theta,\phi)} \mathcal{O}(\theta, \phi)$. We are therefore interested in $\nabla_{(\theta,\varphi)}\mathcal{O}(\theta, \varphi) = 0$.

Given the simplicity of $\mathcal{O}$, for certain classes of Σ and F, we can easily evaluate $\mathcal{O}$ and $\nabla\mathcal{O}$ on a computer. We can therefore perform a gradient-based minimization procedure to attempt to find σ^* and f^*. On the other hand, for certain classes of Σ and F, we can optimize analytically. This will be the subject of the remaining subsections of this section. In particular, in Sections 3.1 to 3.3, we will consider extremely simplified classes Σ and F in order to gain intuition as to what the optimization is doing. The nice intuition will serve as an argument that the objective function given in Eq. (1) is indeed encoding our problem adequately. In Section 3.4, we consider the general situation in which Σ and F contain all linear function. Even in this arbitrary case, we can efficiently exactly minimize the objective function and therefore contain a good numerical solution. Finally, in Section 3.5, we briefly comment on expanding Σ and F to include even nonlinear functions.

3.1 Case 0: when $f_p(d) = 0$ and $\sigma_r(x) = x + b_r$

We consider the case when

$$f_p(d) = 0 \tag{6}$$

$$\sigma_r(x) = x + b_r, \tag{7}$$

for parameters b_r. This corresponds to the case where we ignore any improvement of people, and we assume that reviewers review scales differ by a constant offset. In this case, the objective function in Eq. (1) $\mathcal{O}(\sigma, f)$ becomes $\mathcal{O}(b)$, where $b = \{b_r \mid r \in \mathcal{R}\}$, and we are therefore interested in $\frac{\partial\mathcal{O}}{\partial b_r}$. Plugging in the forms of f and σ, Eq. (1) becomes

$$\begin{aligned}\mathcal{O}(b) = \frac{1}{\mathcal{N}} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}_p} \sum_{d_1\in\mathcal{D}_p^{r_1}} \sum_{r_2\in\mathcal{R}_p} \sum_{d_2\in\mathcal{D}_p^{r2}} \left(y^{r_1}_{p,d_1} - y^{r_2}_{p,d_2} + b_{r_1} - b_{r_2}\right)^2 \\ + \frac{\lambda}{|\mathcal{R}|} \sum_{r\in\mathcal{R}} \left(y_{\max} - y_{\min} - 1\right)^2.\end{aligned} \tag{8}$$

We need that $\frac{\partial\mathcal{O}}{\partial b_r} = 0$, for each $r \in \mathcal{R}$. Let's compute the derivative:

$$\frac{\partial\mathcal{O}(b)}{\partial b_r} = \frac{2}{\mathcal{N}} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}_p} \sum_{d_1\in\mathcal{D}_p^{r_1}} \sum_{r_2\in\mathcal{R}_p} \sum_{d_2\in\mathcal{D}_p^{r_2}} \left(-y^{r_1}_{p,d_1} + y^{r_2}_{p,d_2} - b_{r_1} + b_{r_2}\right)\left(\delta_{r,r_2} - \delta_{r,r_1}\right) \tag{9}$$

$$= \frac{4}{\mathcal{N}} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}_p} \sum_{d_1\in\mathcal{D}_p^{r_1}} \sum_{r_2\in\mathcal{R}_p} \sum_{d_2\in\mathcal{D}_p^{r_2}} \delta_{r,r_2}\left(-y^{r_1}_{p,d_1} + y^{r}_{p,d_2} - b_{r_1} + b_r\right) \tag{10}$$

$$= \frac{4}{\mathcal{N}} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}_p} \sum_{d_1\in\mathcal{D}_p^{r_1}} \sum_{d_2\in\mathcal{D}_p^{r}} \delta_{p\in\mathcal{P}_r}\left(-y^{r_1}_{p,d_1} + y^{r}_{p,d_2} - b_{r_1} + b_r\right) \tag{11}$$

$$= \frac{4}{\mathcal{N}} \sum_{p\in\mathcal{P}_r} \sum_{r_1\in\mathcal{R}_p} \sum_{d_1\in\mathcal{D}_p^{r_1}} \sum_{d_2\in\mathcal{D}_p^{r}} \left(-y^{r_1}_{p,d_1} + y^{r}_{p,d_2} - b_{r_1} + b_r\right). \tag{12}$$

Setting all of these to 0 is easily recast into a linear matrix equation which can be solved. The matrix will be of dimension $|\mathcal{R}| \times |\mathcal{R}|$. Since the number of reviewers it typically very small (compared to the number of people), solving this linear matrix equation is simple numerically. Notice that if b_r^* is a solution, then $b_r^* + c$ is also a solution, for any constant c. Hence, the matrix will have nontrivial kernel.

To get some intuition for the solution, let's consider the case when each reviewer reviews every person exactly once. In other words, we consider the case when $\mathcal{P}_r = \mathcal{P}$ for each r and hence $\mathcal{R}_p = \mathcal{R}$ for each p, and when $\left|\mathcal{D}_p^r\right| = 1$ for each r and p. In this case, we find that (we suppress all d indices since they are trivial in this case)

$$\frac{\partial \mathcal{O}}{\partial b_r} = \frac{4}{|\mathcal{R}|\,|\mathcal{P}|} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}} \left(-y_p^{r_1} + y_p^r - b_{r_1} + b_r\right). \tag{13}$$

It follows that in this case, for each $r \in \mathcal{R}$, we want

$$\begin{aligned} b_r^* &= \frac{1}{|\mathcal{P}|\,|\mathcal{R}|} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}} \left(y_p^{r_1} - y_p^r\right) + c & (14)\\ &= (\text{average rating that the average reviewer gives}) - (\text{average rating that reviewer } r \text{ gives}) + c, & (15) \end{aligned}$$

where c is any arbitrary constant that we might as well set to 0[1]. So b_r^* translates a reviewer by a certain average amount so as to bring that reviewer's review scale closer to the mean. Thus, given a review y_p^r, we have found a calibrated review $y_p^r + b_r^*$. We therefore associate to person $p \in \mathcal{P}$ the average calibrated review

$$\frac{1}{|\mathcal{R}|} \sum_{r\in\mathcal{R}} (y_p^r + b_r^*) = \frac{1}{|\mathcal{R}|} \sum_{r\in\mathcal{R}} y_p^r, \tag{16}$$

Hence, we see that in this restricted case, – indeed, with all the restrictions we have enacted, this is the simplest possible case one can consider – we have reproduced the most naive form of calibration which is simply to average all the different ratings that a person gets from all the different reviewers.

3.2 Case 1: when $f_p(d) = 0$ and $\sigma_r(x) = a_r x$

In the previous subsection, we carefully worked through the case of $\sigma_r(x) = x + b_r$. We then restricted to the special case when $\mathcal{P}_r = \mathcal{P}$ and $\left|\mathcal{D}_p^r\right| = 1$. In Section 3.4, we will carefully work through the completely general linear case. Therefore, in this subsection, for brevity we will immediately assume the $\mathcal{P}_r = \mathcal{P}$ and $\left|\mathcal{D}_p^r\right| = 1$ conditions. We are just trying to get some intuition for what each parameter controls, and we return to the fully general case in Section 3.4. Recall we suppress all d indices since $\mathcal{D}_p^r$ is trivial.

With these assumptions, we assume the trivial improvement function $f_p(d) = 0$ and the reviewer scaling functions $\sigma_r(x) = a_r x$ for parameters a_r. Let $a = \{a_r \mid r \in \mathcal{R}\}$. Then, the objection function $\mathcal{O}(\sigma, f)$ from Eq. (1) becomes

$$\mathcal{O}(a) = \frac{1}{|\mathcal{P}|\,|\mathcal{R}|^2} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}} \sum_{r_2\in\mathcal{R}} \left(a_{r_1} y_p^{r_1} - a_{r_2} y_p^{r_2}\right)^2 + \frac{\lambda}{|\mathcal{R}|} \sum_{r\in\mathcal{R}} \left(a_r \Delta y - 1\right)^2, \tag{17}$$

where $\Delta y \coloneqq y_{\max} - y_{\min}$. We consider the derivative

$$\begin{aligned} \frac{\partial \mathcal{O}}{\partial a_r} &= \frac{2}{|\mathcal{P}|\,|\mathcal{R}|^2} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}} \sum_{r_2\in\mathcal{R}} \left(a_{r_1} y_p^{r_1} - a_{r_2} y_p^{r_2}\right) \left(\delta_{r,r_1} - \delta_{r,r_2}\right) y_p^r + \frac{2\lambda}{|\mathcal{R}|} \Delta y (a_r \Delta y - 1) & (18)\\ &= \frac{4}{|\mathcal{P}|\,|\mathcal{R}|^2} \sum_{p\in\mathcal{P}} y_p^r \sum_{r_2\in\mathcal{R}} \left(a_r y_p^r - a_{r_2} y_p^{r_2}\right) + \frac{2\lambda}{|\mathcal{R}|} \Delta y (a_r \Delta y - 1) & (19)\\ &= \frac{4}{|\mathcal{R}|} \left[a_r \left\langle (y_p^r)^2 \right\rangle_p - \left\langle a_{r_2} y_p^{r_2} y_p^r \right\rangle_{r_2,p} + \frac{\lambda}{2} a_r (\Delta y)^2 - \frac{\lambda}{2} \Delta y\right], & (20) \end{aligned}$$

where we defined the average quantities

$$\begin{aligned} \langle\cdot\rangle_p &\coloneqq \frac{1}{|\mathcal{P}|} \sum_{p\in\mathcal{P}} (\cdot) & (21)\\ \langle\cdot\rangle_{r_2,p} &\coloneqq \frac{1}{|\mathcal{R}|\,|\mathcal{P}|} \sum_{r_2\in\mathcal{R}} \sum_{p\in\mathcal{P}} (\cdot). & (22) \end{aligned}$$

[1] c being arbitrary is a result of the fact that we did not restrict the range of the absolute rating scale.

Let's investigate the solution in the extreme cases. We begin with the case where we set $\lambda = 0$. Here, we are ignoring the fact that we'd like people to be distinguishable and instead only focusing on the requirement that reviewers all agree on a score for each person. The solution is, for each $r \in \mathcal{R}$, $a_r = \frac{\left\langle a_{r_2} y_p^{r_2} y_p^r \right\rangle_{r_2,p}}{\left\langle (y_p^r)^2 \right\rangle_p}$. We see that if a_r^* is a solution, then so is ca_r^* for any constant c. It follows that $a_r^* = 0$ for all r is a solution; here, we therefore see no distinguishability between reviewers, which is what we want, but also no distinguishability between people because everyone is just rated 0! This solution can only occur when $\lambda = 0$. In general, though, nonzero a_r^* also satisfy. For example, again just for intuition, let's assume that there is only one person being reviewed. In this trivial case (we suppress p subscripts because there is only one person), $a_r^* = \frac{\left\langle a_{r_2}^* y^{r_2} \right\rangle_{r_2}}{y^r}$. In other words, the score that reviewer r gives is to the person $a_r^* y^r$ is just the average score that the person received from all the reviewers $\left\langle a_{r_2}^* y^{r_2} \right\rangle_{r_2}$.

On the other extreme, we consider $\lambda \to \infty$. In this case, the solution is $a_r^* = \frac{1}{\Delta y}$. This enforces that every reviewer's absolute scale has a range of exactly 1 (i.e. $a_r^* y_{\max} - a_r^* y_{\min} = a_r^* \Delta y = 1$).

For more general λ, we are somewhere between the two extremes, and therefore enforcing both conditions. Again, for intuition, let's go back to the trivial case when there is only one person, $|\mathcal{P}| = 1$. We consider $\lambda = 1$ to be roughly the sweet-spot. Then the score that reviewer r assigns to the person is

$$a_r^* y^r = \frac{\left\langle a_{r_2}^* y^{r_2} \right\rangle_{r_2} + \frac{\Delta y}{2y^r}}{1 + \frac{(\Delta y)^2}{2(y^r)^2}}. \tag{23}$$

Again for the sake of intuition, we make the assumption that the ratings of the person are relatively concentrated so that we can approximate $\langle g(y^r) \rangle_r$ by $g(\langle y^r \rangle)$. Then, we find that the averaged score for the person is

$$\langle a_r^* y^r \rangle_r = \frac{\left\langle a_{r_2}^* y^{r_2} \right\rangle_{r_2} + \frac{\Delta y}{2\langle y^r \rangle_r}}{1 + \frac{(\Delta y)^2}{2\langle y^r \rangle_r^2}} \qquad \Longrightarrow \langle a_r^* y^r \rangle_r = \frac{\langle y^r \rangle_r}{\Delta y}. \tag{24}$$

In other words, the average rating of the person on the absolute scale is precisely the average rating of the person divided by the scale factor Δy. The division brings us from the original scale to the absolute scale that has a range of 1.

3.3 Case 2: when $f_p(d) = \alpha_p(D - d)$ and $\sigma_r(x) = x$

As in the previous subsection, since we are working only on an intuitive level until Section 3.4, we restrict to when $\mathcal{P}_r = \mathcal{P}$. Also, for simplicity, we just assume that every reviewer reviews every person every day, so that $\mathcal{D}_p^r = \mathcal{D}$ for every r and p. We consider when

$$f_p(d) = \alpha_p(D - d) \tag{25}$$

$$\sigma_r(x) = x, \tag{26}$$

for some constant D and some set of parameters $\alpha = \{\alpha_p \mid p \in \mathcal{P}\}$. Here D is just some constant which intuitively should be the final day of the reviews (note that D makes no appearance in $\mathcal{O}$ and is therefore irrelevant, but we include it here purely for aesthetics). In this case, the objective function $\mathcal{O}(\sigma, f)$ from Eq. (1) becomes

$$\begin{aligned} \mathcal{O}(\alpha) = \frac{1}{\mathcal{N}} \sum_{p \in \mathcal{P}} \sum_{r_1, r_2 \in \mathcal{R}} \sum_{d_1, d_2 \in \mathcal{D}} \left(y_{p,d_1}^{r_1} - y_{p,d_2}^{r_2} + \alpha_p(d_2 - d_1) \right)^2 \\ + \frac{\lambda}{|\mathcal{R}|} \sum_{r \in \mathcal{R}} \left(y_{\max} - y_{\min} - 1 \right)^2. \end{aligned} \tag{27}$$

The derivative is

$$\frac{\partial \mathcal{O}}{\partial \alpha_p} = \frac{2}{\mathcal{N}} \sum_{r_1, r_2 \in \mathcal{R}} \sum_{d_1, d_2 \in \mathcal{D}} \left(y_{p,d_1}^{r_1} - y_{p,d_2}^{r_2} + \alpha_p(d_2 - d_1) \right) (d_2 - d_1) \tag{28}$$

$$= \frac{2}{\mathcal{N}} \sum_{r_1, r_2 \in \mathcal{R}} \sum_{d_1, d_2 \in \mathcal{D}} \left(d_2 y^{r_1}_{p,d_1} - d_2 y^{r_2}_{p,d_2} - d_1 y^{r_1}_{p,d_1} + d_1 y^{r_2}_{p,d_2} + (d_1 - d_2)^2 \alpha_p \right) \tag{29}$$

$$= \frac{4}{|\mathcal{P}|} \left(\langle d \rangle_d \left\langle y^r_{p,d} \right\rangle_{r,d} - \left\langle d y^r_{p,d} \right\rangle_{r,d} + \frac{1}{2} \left\langle (d_1 - d_2)^2 \right\rangle_{d_1, d_2} \alpha_p \right) \tag{30}$$

$$= \frac{4}{|\mathcal{P}|} \left(\langle d \rangle_d \left\langle y^r_{p,d} \right\rangle_{r,d} - \left\langle d y^r_{p,d} \right\rangle_{r,d} + \left(\left\langle d^2 \right\rangle_d - \langle d \rangle_d^2 \right) \alpha_p \right), \tag{31}$$

where again we defined averaged quantities $\langle \cdot \rangle_d \coloneqq \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} (\cdot)$, etc. Thus, we estimate that person $p \in \mathcal{P}$ has improved at a rate of α^*_p per day, where

$$\alpha^*_p = \frac{\left\langle d y^r_{p,d} \right\rangle_{r,d} - \langle d \rangle_d \left\langle y^r_{p,d} \right\rangle_{r,d}}{\langle d^2 \rangle_d - \langle d \rangle_d^2} \tag{32}$$

$$= \frac{\text{covariance of days with ratings}}{\text{variance of days; i.e. average square distance between days}} \tag{33}$$

$$= (\text{correlation between days and ratings}) \times \frac{\text{standard deviation of ratings}}{\text{standard deviation of days}}. \tag{34}$$

Indeed this is a very intuitive result. The rate at which p improved is equal to the correlation between days and ratings times the ratio of the standard deviation of the ratings to the standard deviation of the days. The aforementioned ratio gives the natural units with which to measure an improvement rate, and the correlation is then the improvement rate expressed in these units.

As a simple sanity check, we consider a case when days and ratings are uncorrelated. For example, for a fixed person p, consider when for each r, $y^r_{p,d} = y^r_{p,d'}$ for all d, d'; this is the case when all the reviewers rate the person the same every day. Then the numerator of α^*_p is zero, since the correlation is zero. Hence, we correctly identified no improvement.

3.4 Case 3: arbitrary linear case

The previous subsections singled out a specific part of the optimization procedure to give some intuition for what's going on. Everything was linear. In this section, we will allow everything to be arbitrary while maintaining the linearity condition. This is the most arbitrary case where we can still get something efficiently solvable on a computer. The reason is because when each function is linear, the partial derivatives of the objective function are linear in each parameter. Hence, we are left with a simple matrix equation that we can efficiently solve. In the next subsection, we will consider more arbitrary functions that are not linear; there, we will need to resort to gradient based minimization methods which will almost certainly not find the optimal result.

In addition to arbitrary linear being the most general case that we can still efficiently get exact answers, it is also the simplest case that has a few of the necessary properties that we require. Indeed, there are a few properties that we desire in order for our protocol to be independent of the rating scale used.

Firstly, suppose that σ^*_r and f^*_p minimizes the objective function for a set of ratings $y^r_{p,d}$. Then it follows that $\sigma^{*'}_r$ and $f^{*'}_p$ – where $\sigma^{*'}_r(x) = \sigma^*_r(x - c)$ and $f^{*'}_p(d) = f^*_p(d)$ – minimizes the objective function for the set of ratings $y^r_{p,d} + c$ for any constant c (where additionally $y$min and $y_{\max}$ also translate to $y_{\min} + c$ and $y_{\max} + c$). Our protocol should be independent of the constant c since it should be independent of the arbitrary scale with which the original ratings are performed. We therefore must choose a parameterization that gives us this freedom. In this case, suppose each σ_r is parameterized as $x \mapsto a_r x + b_r$. Then $\sigma^*_r(x) = a^*_r x + b^*_r$ and $\sigma^{*'}_r(x) = a^{*'}_r x + b^{*'}_r$, where $a^{*'}_r = a_r$ and $b^{*'}_r = b^*_r - c$.

Secondly, suppose that σ^*_r and f^*_p minimizes the objective function for a set of ratings $y^r_{p,d}$. Then it follows that $\sigma^{*'}_r$ and $f^{*'}_p$ – where $\sigma^{*'}_r(x) = \sigma^*_r(x/c)$ and $f^{*'}_p(d) = f^*_p(d)$ – minimizes the objective function for the set of ratings $c y^r_{p,d}$ for any constant c (where additionally $y$min and $y_{\max}$ also translate to $c y_{\min}$ and $c y_{\max}$). Our protocol should be independent of the constant c since it should be independent of the arbitrary scale with which the original ratings are performed. We therefore must choose a parameterization that gives us this freedom. In this case, suppose each σ_r is parameterized as $x \mapsto a_r x + b_r$. Then $\sigma^*_r(x) = a^*_r x + b^*_r$ and $\sigma^{*'}_r(x) = a^{*'}_r x + b^{*'}_r$, where $a^{*'}_r = a_r / c$ and $b^{*'}_r = b^*_r$.

Finally, similarly to above, we consider transforming each day as $d \mapsto cd + e$ for some constant c and e, and therefore transforming the final day as $D \mapsto cD + e$. Under this transformation, we therefore need that $\sigma_r^{*'}(x) = \sigma_r^*(x)$ and $f_p^{*'}(d) = f_p^*((d-e)/c)$. In our case when f_p is parameterized as $f_p(d) = \alpha_p(D-d)$, the transformation results in $\alpha_p^{*'} \mapsto \alpha_p^*/c$.

In this subsection, therefore, we consider the case when

$$f_p(d) = \alpha_p(D-d) \tag{35}$$

$$\sigma_r(x) = a_r x + b_r, \tag{36}$$

where D is some fixed constant, which intuitively should be the final day of the reviews (note that D makes no appearance in $\mathcal{O}$ and is therefore irrelevant, but we include it here purely for aesthetics). This form of f_p assumes that each person improves linearly at a rate of α_p per day over the days.

Let $a = \{a_r \mid r \in \mathcal{R}\}$, $b = \{b_r \mid r \in \mathcal{R}\}$, and $\alpha = \{\alpha_p \mid p \in \mathcal{P}\}$. Then the objective function $\mathcal{O}(\sigma, f)$ from Eq. (1) becomes $\mathcal{O}(a, b, \alpha)$ and we need that for each r and p,

$$\frac{\partial \mathcal{O}}{\partial a_r} = 0 \qquad \frac{\partial \mathcal{O}}{\partial b_r} = 0 \qquad \frac{\partial \mathcal{O}}{\partial \alpha_p} = 0. \tag{37}$$

Our task in this subsection is to write out the explicit forms of these derivative so that it can be easily recast as a linear matrix equation to be solved on a computer.

The objective function becomes

$$\begin{aligned}\mathcal{O}(a,b,\alpha) = \frac{1}{\mathcal{N}} \sum_{p\in\mathcal{P}} \sum_{r_1\in\mathcal{R}_p} \sum_{d_1\in\mathcal{D}_p^{r_1}} \sum_{r_2\in\mathcal{R}_p} \sum_{d_2\in\mathcal{D}_p^{r_2}} \Big(a_{r_1} y_{p,d_1}^{r_1} - a_{r_2} y_{p,d_2}^{r_2} + b_{r_1} - b_{r_2} + \alpha_p(d_2 - d_1)\Big)^2 \\ + \frac{\lambda}{|\mathcal{R}|} \sum_{r\in\mathcal{R}} \left(a_r \Delta y - 1\right)^2.\end{aligned} \tag{38}$$

Let $M = |\mathcal{R}| + |\mathcal{R}| + |\mathcal{P}|$. Let z be the $M \times 1$ matrix of parameters $z = \begin{pmatrix} a \\ b \\ \alpha \end{pmatrix}$. We now want to find the $M \times M$ matrix A and the $M \times 1$ matrix c such that the vanishing derivative conditions are written as $Az = c$. This can be done with a few nested for loops, and it is done in the code.

One minor caveat; recall that the matrix A will be singular. The reason is because if a set of b_r^* are solutions, then so will the set $b_r^* + \text{const}$ for any constant. This is due to the fact that we never fixed a range for the absolute scale. We can set the scale arbitrarily by adding a condition to one of the b_r. Indeed, suppose that $r_0 \in \mathcal{R}$ is some fixed reviewer. We replace the condition that $\frac{\partial \mathcal{O}}{\partial b_{r_0}} = 0$ with the condition that $b_{r_0} = 0$. In other words, take out the row of A corresponding to the index of b_{r_0}, and we replace it with 1 in the column corresponding to b_{r_0} and all 0's elsewhere. This will ensure that A is not singular, and therefore there is a unique solution for z.

This final step is optional, but aesthetically pleasing. We now have a bunch of calibrated data. Call the minimum rating in the calibrated data y_0 and the maximum y_1. Then for each rating y in the calibrated data, we transform it to $\frac{y-y_0}{y_1-y_0}$ to put everything exactly between 0 and 1. This corresponds to the transformations

$$a_r \to \frac{a_r}{y_1 - y_0} \tag{39}$$

$$b_r \to \frac{b_r - y_0}{y_1 - y_0} \tag{40}$$

$$\alpha_p \to \frac{\alpha_p}{y_1 - y_0}. \tag{41}$$

This gives us our final result: a bunch of calibrated reviews normalized between 0 and 1, and a bunch of parameters telling us something about the reviewers and the people. For example, the calibrated α_p tells us how much person p improves per day on the absolute scale.

Furthermore, since all the functions are linear and therefore invertible, we can use our result as a recommendation system. Suppose you and I review a bunch of people. One of the people I review is person p_0, but you did not review p_0. Given the calibrated absolute rating y of p_0, the a and b corresponding to you, and the α_{p_0} corresponding to p_0, we can determine that you would have rated person p_0 on day d with a rating on your scale of x, where $y = ax + b + \alpha_{p_0}(D - d)$.

3.5 Case 4: going beyond linear

As described in the previous subsection, if we generalize the allowed forms of f_p and σ_r, the minimization cannot be done optimally. Indeed, even if we only go one order beyond linear – quadratic – we should not expect to be able to efficiently find the optimal solution in general unless `P = NP` or something crazy like that happens[2]. However, this does not necessarily mean that we cannot still get useful approximate solutions.

In particular, one could consider generalizing to the case where each σ_r is a sigmoid function parameterized via translation and scale.

[2]See e.g. https://cs.stackexchange.com/q/127282.