An Analysis of Griefs and Griefing Factors

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Abstract

We consider griefing factors, a measure of the cost-effectiveness of sabotage introduced by V. Buterin. We study mathematical properties of this notion, particularly related the effect of the presence of players willing to perform sabotage at varying griefing factors impacts the equilibria present in games. We subsequently analyze the griefing factors of several sabotage strategies present in blockchain-based applications.

1 Introduction

There is a rich economics literature studying strategies that rely on committing sabotage against other participants in games [15]. Indeed, in certain games, such as contests with a fixed prize that is awarded to one of a finite number of players, strategies based on attempting to reduce the chances of an opponent in order to improve one's own can be incentivized [23]. Typically these articles take a perspective of attempting to incorporate the incentives of participants to harm others directly into the payoff matrix of the games considered.

A related idea, introduced by Buterin in the context of analyzing blockchain systems [10], defines a "grief" as a strategy that causes harm to both the participant carrying it out and to other participants of a game. The word "grief" draws upon the slang of video game communities for styles of play that seek to harm others despite not being useful or indeed being harmful to the player carrying them out [14]. Then Buterin provides a measure of the effectiveness of griefs that he calls the griefing factor as the ratio of the harm that an attack does compared to its cost to the attacker [10].

For example, in the following game in which it is a (weakly) dominant strategy for each player to play A.

Alice \ Bob strategies	A	B
A	$(2,1)$	$(0,0)$
B	$(0,0)$	$(0,0)$

If Alice then expects Bob to play A, she can nonetheless play B. This costs her the the opportunity to earn \$1, but it costs Bob \$2. Hence, Alice carries out a grief with a griefing factor of 2.

Griefing factors provide a useful measure to the mechanism designer in that they allow one to identify when there are particularly effective sabotage strategies in a game without needing to know in advance what rewards an attacker may have for successful sabotage.

In this article, we will analyze mathematical properties of griefing factors. While this analysis is general, our examples will largely be drawn from the setting of blockchain. Indeed, this notion is perhaps particularly relevant in the context of blockchain systems due to their heavy reliance on mechanism design, in both consensus algorithms [27], [7] as well as in applications. Moreover,

- due to the ability of smart contracts to atomically link interactions between different blockchain applications, even permitting such novelties as flash loans [28], attackers will often have interests that are not well observed by the internal incentives of the game designed by any given developer.
- as the blockchain oracle problem [4] limits the information available to a given blockchain protocol, it can be difficult to identify parties as attackers and penalize them accordingly; this is the case in situations of "speaker-listener equivalence" where it is observed by Buterin [8] that there must always exist griefs with a griefing factor of at least one.
- due to the cultural of current communities of blockchain users [19], [24], one might expect there to exist "trolls" who are willing to suffer small financial losses to grief others for social or ideological reasons without necessarily requiring a financial incentive.

2 Related Work

A long literature [15], [23] considers games in which it is in the interests of players to cause harm to other participants. Further behavioural research [1] has shown that there are populations that do not require financial incentives to be incentived to harm other players in economic games. This phenomenon is related to psychological studies on "trolls" [25] who exert effort to antagonize others, particular in semi-anonymous Internet settings.

In defining griefing factors [8], [9], [10], Buterin has provided a measure for how effective a strategy is in doing harm to other participants relative to its cost, without requiring a priori knowledge of what incentives participants may have to seek to do this harm. Buterin has discussed this concept principally in the context of analyzing blockchain settings. He particularly the notes that griefs with griefing factors at least equal to one must exist in settings where there is a "speaker-listener equivalence" [8], and in [9], he considers the impacts of a specific grief on economic equilibria of supply and demand for validators in a simplified model of a proof-of-stake system.

In [14], griefing in a specific model of proof-of-work mining is related to a generalized notion of evolutionarily stable allocations explored in [29] in the context of games with a fixed reward. In that setting, it is observed that participants are incentivized to act "spitefully" to drive competitors out of the competition. Indeed, in [14] it is observed that the Nash equilibrium in the traditional sense in the mining game is not evolutionarily stable in this sense, and instead evolutionarily stable allocations have a griefing factor of one with respect to the Nash equilibrium.

3 Contribution of this Work

We expand on the mathematical framework allowing one to study notions of griefing and griefing factors. Particularly, we explore questions related to how the presence of players willing to grief affects the equilibria of a game. We provide results that facilitate such an analysis and note that this can be represented geometrically through diagrams that are inspired by the phase portraits used to consider differential equations and dynamical systems [31]. We provide several results which limit structure of these diagrams, with particular applications to situations where one can say that equilibria do not change or only change continuously under small perturbations in the griefing factors of participants.

Moreover, we examine examples of griefs in several prominent blockchain applications, particularly in relation to our abstract results.

4 Notation and Model of Actors

We consider games of the form (N, A, u), where

- N is a finite set of players of cardinality $N = n$
- For each $i \in N$, A_i is the set of actions that can be taken by i, and $A = A_1 \times \ldots \times A_n$ is the set of possible assignments of an action to each player,
- $u = (u_1, \ldots, u_n) : A \to \mathbb{R}^n$ is a set of utility functions.

We denote i's action by $x_i \in A_i$. Moreover, we denote the tuple of actions of the other users by $x_{-i} \in A_{-i}$, which consists of an action $x_j \in A_j$ for all $j \neq i$. Then one can compute the jth participant's utility given these actions as $u_j(x_i, x_{-i}) = u_j(x_1, \ldots, x_n)$ for any $j \in N$. In some cases, we will specify actions for all participants outside of some set $S \subseteq \{1, \ldots, n\}$. Then we will similarly denote by $x_{-S} \in A_{-S}$ a tuple of the actions of all participants not in S.

We take

$$\Delta_i = \left\{ s_i = (s_i(a))_{a \in A_i} \in \prod_{a \in A_i} [0,1] : \sum_{a \in A} s_i(a) = 1 \right\}$$

as naturally identified with the set of mixed strategies that participant $i \in N$ can take; particularly, for $s_i \in \Delta_i$ and $a \in A_i$, we denote by $s_i(a)$ the probability that i selects the alternative a while playing the mixed strategy s_i. Then $\Delta = \prod_{i=1}^{n} \Delta_i$ corresponds to the set of tuples of a mixed strategy for each participant.

We assume that the actors in the games we consider have perfect information on the payoff structure of the game. They are economically rational in that they seek to maximize utility functions that may be modified to include griefing factors as described below. Furthermore, we will suppose that they take actions simultaneously and independently unless otherwise noted.

Finally, we fix definitions of griefing factors that follow those of [14], both in terms of an attacker griefing another individual participant as well as griefing the broader set of other participants.

Definition 1. *Let $i, j \in N$, $x_i, x_i' \in A_i$, and $x_{-i} \in A_{-I}$ such that $u_i(x_i, x_{-i}) > u_i(x_i', x_{-i})$. Then we define the individual griefing factor of the action x_i' against j with respect to $x^* = (x_i, x_{-i})$ as:*

$$GF_{i,j}((x_i', x_{-i}), x^*) = \frac{u_j(x^*) - u_j(x_i', x_{-i})}{u_i(x^*) - u_i(x_i', x_{-i})}. \tag{1}$$

and the total griefing factor of the action x_i' with respect to x^* as:

$$GF_i((x_i', x_{-i}), x^*) = \frac{\sum_{j \neq i} (u_j(x^*) - u_j(x_i', x_{-i}))}{u_i(x^*) - u_i(x_i', x_{-i})}. \tag{2}$$

One extends these concepts to the case where $u_i(x_i, x_{-i}) = u_i(x_i', x_{-i})$, but $u_j(x^*) - u_j(x_i', x_{-i}) > 0$ for some j or $\sum_{j \neq i} (u_j(x^*) - u_j(x_i', x_{-i}))$ by taking the convention that the corresponding griefing factors are infinite.

Remark 1. *Note that Definition 1 allows for the possibility that $u_i(x_i, x_{-i}) > u_i(x_i', x_{-i})$, but $u_j(x^*) - u_j(x_i', x_{-i}) < 0$ which results in negative griefing factors. In most of the results of this work we will limit ourselves to considering positive griefing factors; however, the perspective of negative griefing factors may be relevant to analyze the behaviour of a participant i who is altruistic.*

5 Examples of griefs on cryptoeconomic systems

We further illustrate the notion of griefing factors by calculating the griefing factors of several griefs on the Ethereum 2.0 consensus protocol [11].

Example 1. *We consider the economic games of the Ethereum 2.0 Beacon Chain, as they are implemented after the Altair hard fork [12]. In this game, validators perform roles several roles. Validators are randomly selected to produce new blocks, and additionally once per epoch (approximately 6.4 minutes) each validator is tasked with attesting to the most recent block in the preceding slot (a 1/32 division of an epoch). Included in attestations are references to preceding "target" and "source" blocks, with the information on target and source blocks being used in the process to determine when a block finalizes, see [13]. The system is calibrated so that if all actors, including block producers and attestors, perform their roles perfectly that each participant will earn on average a base reward of B per epoch [12]. This base reward is a function of the number of active validators [11] and as of January 2022 is roughly 21000 Gwei [6]. Attestors are rewarded $B \cdot y_{task} \cdot \frac{x}{N}$ for providing an attestation that correctly identifies the previous block, target, and source as they are ultimately finalized where $y_{block} = \frac{14}{64}$, $y_{target} = \frac{26}{64}$, and $y_{source} = \frac{14}{64}$, N is the number of attestors assigned to this task, and x is the number of attestors who correctly perform the corresponding task. If an attestor fails on one of these points, she is penalized $B \cdot y_{task}$. Block producers that include an attestation are rewarded $\frac{B}{8} \cdot y_{task}$, and are not penalized for the non-inclusion of an attestation. In order for the attestation to be counted towards completion of a task, it must be "timely" [12], namely in the next slot for the task of indicating the preceding block, in the next 32 slots for the target, and the next 5 slots for the source.*

We analyze several griefs in this system. For all of these analyses we assume that the result of which blocks are finalized does not ultimately change and we take griefing factors with respect to default strategies of honest participation. We will denote by x the number of validators other than a given attestor i that complete relevant tasks. We provide an approximate value of each griefing factor after the symbol "≈", taking estimates $N = 280,000$ the current number of validators [6] and $x = 277200$ so that the percentage of validators completing each task is $\frac{x}{N} = .99$.

- *The attestor i griefs the block producer and the other attestors by publishing her attestation beyond the timeliness deadline for the header (noting that a subsequent block producer receives the reward for other components of this attestation)*

$$GF_i = \frac{\frac{B}{8} - \frac{B}{8}\left(\frac{26+14}{14+26+14}\right) + \left[\frac{14B}{64}\frac{x+1}{N} - \frac{14B}{64}\frac{x}{N}\right]x}{\frac{14B}{64}\frac{x+1}{N} + \frac{14B}{64}} \approx .572.$$

  This can also be considered as a grief against the block producer j individually, noting that the block producer is also an attestor

$$GF_{i,j} = \frac{\frac{B}{8} + \left[\frac{14B}{64}\frac{x+1}{N} - \frac{14B}{64}\frac{x}{N}\right]}{\frac{14B}{64}\frac{x+1}{N} + \frac{14B}{64}} \approx .287.$$

- *A block producer j can grief the community of attestors (other than himself) by not including an attestation from i, hence causing it to fall beyond the one slot period for timeliness of reporting the block header. This has a griefing factor of*

$$GF_j = \frac{\frac{14B}{64}\frac{x+1}{N} + \frac{14B}{64} - \frac{B}{8}\left(\frac{26+14}{14+26+14}\right) + \left[\frac{14B}{64}\frac{x+1}{N} - \frac{14B}{64}\frac{x}{N}\right](x-1)}{\frac{B}{8} + \left[\frac{14B}{64}\frac{x+1}{N} - \frac{14B}{64}\frac{x}{N}\right]} \approx 4.47.$$

As a grief against the censored attestor i individually this gives a griefing factor of

$$GF_{j,i} = \frac{\frac{14B}{64}\frac{x+1}{N} + \frac{14B}{64}}{\frac{B}{8} + \left[\frac{14B}{64}\frac{x+1}{N} - \frac{14B}{64}\frac{x}{N}\right]} \approx 3.48.$$

Note that all of these griefs are indistinguishable from the point of view of the consensus algorithm. In particular, the two individual griefs between the parties i and j are an example of a speaker-listener equivalence, as discussed in [8]; hence their griefing factors are reciprocals and particularly one of them must be at least one.

One can consider other griefs involving attestors not including correct, timely targets and sources. The griefing factors of these griefs are stated in the following table for the above, approximate values. Note that under the current [12] parameters for timeliness, failing to provide either a correct, timely target or source implies that the attestor also fails to provide a correct, timely previous block. Hence, the effects of failing to timely indicate the previous block are included in the griefing factors in the corresponding columns below. Also note that block producers can only delay an attestation by one block, so it is not typically possible for them to prevent its inclusion within the time limits for timely targets and sources.

Failed task / *Griefer, Victim*	*Prev. Block*	*Target*	*Source*	*All*
Attestor i vs block producer j, other attestors	.572	.572	.572	.572
Attestor i vs block producer j	.287	.101	.144	.074
Block producer j vs attestors	4.47	*NA*	*NA*	*NA*
Block producer j vs attestor i	3.48	*NA*	*NA*	*NA*

Another example of a grief on a cryptoeconomic system is considered in [14] in the setting of proof-of-work mining games. We will further consider this game in Example 3 using the tools we develop in Section 7. Note that it is also possible to have griefs that do harm that is not purely monetary. We briefly note cryptoeconomic systems in which griefers can expend resources to cause delays in the normal operation of the system.

Example 2 (Delay griefs)**.**

- *In Arbtrium [16], if there are contrasting states of the rollup submitted by validators, then there is a process designed to prove that at least one of them is incorrect [17]. As the final states differ, there is guaranteed to be some specific step in the off-chain computation on which the two parties disagree. Via an interactive game where parties progressively commit to points of disagreement in their series of computations such a step can be identified and verified on-chain. Attackers can place malicious stakes that require the system to use this process. The effectiveness of this attack is considered in [17].*

- *In the blockchain-based dispute resolution system of [18], small randomly selected panels are chosen from among token holders to provide subjective judgments on off-chain disputes. Then there is an appeal system by which parties can invoke larger panels that are generally more statistically representative of the community of crowd-sourced participants. In the event that a party invokes the appeal process frivolously, she sacrifices appeal fees to delay the finalization of the decision in the case.*

- *Many blockchain-based curated lists, such as [22], make use of a mechanism where individuals submit elements to the list having some fixed criteria for inclusion. Then submitters include a deposit, and their submission can be "challenged" by other participants who also provide a deposit within some fixed period. Then in the case a challenge, some blockchain oracle mechanism, for example a vote of holders of some token, decides whether the entry satisfy the criteria and either the submitter or the challenger loses their deposit accordingly. As the process of invoking the oracle is time-consuming, an attacker can sacrifice a deposit to frivolously challenge an entry, resulting in delaying its inclusion in the list.*

Finally, we consider the mining game considered in [14] and [2], allowing us to develop upon the results of [14] on griefs in proof-of-work mining.

Example 3 (Mining game)**.** *Suppose that there exist $c_1, \dots, c_n \in \mathbb{R}_{>0}$ that represent the per unit cost of producing a given unit of hashing power. Then one takes a simple model of miner rewards that distributes a fixed total mining reward proportional to hash power and deducts the cost of hash power. Namely, for any $q = (q_1, \dots, q_n) \in \mathbb{R}^n_{\geq 0}$, one takes $u_i : A = \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ given by:*

$$u_i(q) = \frac{q_i}{\sum_k q_k} - c_i q_i$$

for each $i \in N$.

Then a modified utility function as in Equation 4, that reflects the idea that a miner can grief other miners by spending additional energy to mine in excess of her natural equilibrium level, is given by:

$$u_i'(q) = \frac{q_i}{\sum_k q_k} - c_i q_i - \lambda_i^{-1} \sum_{j \neq i} \left(\frac{q_j}{\sum_k q_k} - c_j q_j \right).$$

Following a generalization of the proof of Theorem 1 in [2] that considers the varying values of $\lambda_1, \dots, \lambda_n$, we compute the equilibria of this game. Denote by

$$x_i(q) = \frac{q_i}{\sum_k q_i}$$

the percentage of mining power corresponding to i for a given profile q. Note that, for any i, the profiles given by $q_i = \epsilon > 0$ and $q_j = 0$ for all $j \neq i$ are not equilibria. Indeed, in these cases $u_i'(q) = u_i(q) = 1 - c_i q_i$, and i can unilaterally increase her rewards by playing $q_i = \epsilon/2$.

Then, note that $u_i'(q)$ is differentiable in q_i for each i such that $\sum_{k \neq i} q_i > 0$. Moreover, we compute

$$\frac{\partial}{\partial q_i} u_i'(q) = (1 + \lambda_i^{-1}) \left(\frac{1 - x_i(q)}{\sum_k q_k} \right) - c_i.$$

Then, as we note that $\frac{\partial^2}{\partial^2 q_i} u_i'(q) \leq 0$ for $q_i > 0$, $u_i'(q)$ is maximized either when

$$\frac{\partial}{\partial q_i} u_i'(q) = 0 \Leftrightarrow x_i(q) = 1 - \frac{c_i}{1 + \lambda_i^{-1}} \sum_k q_k$$

or $q_i = 0$. However, if $\sum_{k \neq i} q_i > 0$ then $q_i = 0 \Leftrightarrow x_i(q) = 0$. As the case where $q_j = 0$ for all $j \neq i$ is excluded, we then have that one has a Nash equilibrium if and only if

$$x_i(q) = \max \left\{ 1 - \frac{c_i}{1 + \lambda_i^{-1}} \sum_k q_k, 0 \right\}$$

for all $i \in N$.

As the percentages of hash power sum to one, in equilibrium we have

$$1 = \sum_i x_i(q) = \sum_i \max\left\{1 - \frac{c_i}{1+\lambda_i^{-1}} \sum_k q_k, 0\right\}$$

which implies by Lemma 1 of [2] that $\frac{1+\lambda_i^{-1}}{\sum_k q_k} = c^*$, *where* $c^* \in \mathbb{R}_{>0}$ *is a constant that is independent of* i *and the values* λ_i *(but can depend on the choices of* $c_1, \dots, c_n$*). Thus, in equilibrium,*

$$q_i = x_i(q) \sum_k q_k = \frac{1+\lambda_i^{-1}}{c^*}\left(\max\left\{1 - \frac{c_i}{c^*}, 0\right\}\right).$$

Thus, there is a single equilibrium that various continuous in $\lambda_i \in \mathbb{R}_{>0}$. *Note, in particular, that when* $\lambda_i = n-1$ *for all* i, *corresponding to the case of individual griefing factors of* 1 *by all participants against all other participants, one recovers the allocation of Theorem 6iii of [14].*

6 Equilibria in the Presence of Griefing

One can take the perspective that, in order for i to be indifferent between not griefing and voting x_i and griefing by voting for some other $x_i' \in A_i$ with $u_i(x_i', x_{-i}) \leq u_i(x_i, x_{-i})$, an adjusted utility function that considers the utility she obtains from the grief should give:

$$u'_{i,x_i}(x_i', x_{-i}) = u_i(x_i, x_{-i}).$$

However, rearranging Equation 2, one has

$$u_i(x^*) = u_i(x_i', x_{-i}) + GF_i((x_i', x_{-i}), x^*)^{-1} \cdot \sum_{j \neq i} \left(u_j(x^*) - u_j(x_i', x_{-i})\right).$$

Thus, if i is willing to grief at griefing factors at or greater than $\lambda_i \in (0, \infty)$, we are motivated to define the modified utility as:

$$u'_{i,x_i}(x_i', x_{-i}) = u_i(x_i', x_{-i}) + \lambda_i^{-1} \cdot \sum_{j \neq i} \left(u_j(x^*) - u_j(x_i', x_{-i})\right). \tag{3}$$

Definition 2. *Take* $x_i^* \in A_i$ *for each* $i \in N$. *Then we define the* $(\lambda_1, \dots, \lambda_n)$ *Nash-equilibria to be the equilibria that arise in the modified game where each participant* i *is willing to make griefs with a griefing factor of at least* λ_i, *namely the game induced by the utility functions* u'_{i,x_i^*} *given by Equation 3 for each* i.

Another related utility function we will consider is the following, where i receives a negative utility proportional to the utilities of other players:

$$u_i'(x_i, x_{-i}) = u_i(x_i, x_{-i}) - \lambda_i^{-1} \sum_{j \neq i} u_j(x_i, x_{-i}) \tag{4}$$

Then, we see:

Proposition 1. *The* $(\lambda_1, \dots, \lambda_n)$ *Nash-equilibria are independent of the choices of* x_i^* *and are hence well-defined. Moreover, these are the same Nash-equilibria that are induced by the utility functions* u_i' *of Equation 4*

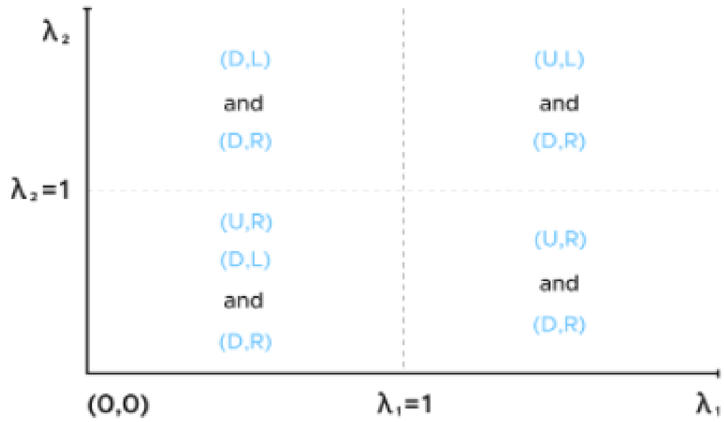

Figure 1: Diagram of equilibria in Example 4

Proof. We note that, for all $x_i \in A_i$, $x_{-i} \in A_{-i}$:

$$u'_{i,x_i^*}(x_i, x_{-i}) - u'_i(x_i, x_{-i}) = \lambda_i^{-1} \sum_{j \neq i} u_j(x_i^*, x_{-i}).$$

However, this quantity is independent of x_i. Hence, $\arg\max_{x_i \in \Delta_i} E[u'_{i,x_i^*}(x_i, x_{-i})] = \arg\max_{x_i \in \Delta_i} E[u'_i(x_i, x_{-i})]$ for all x_{-i}, where the expected value is taken over the potentially mixed strategies of other players; so $u'_{i,x_i^*}(x_i, x_{-i})$ and $u'_i(x_i, x_{-i})$ induce the same Nash equilibria (cf Lemma 2.1 of [21]). In particular, as $u'_i(x_i, x_{-i})$ is independent of x_i^*, then so are the $(\lambda_1, \dots, \lambda_n)$ Nash-equilibria. □

We denote the set of $(\lambda_1, \dots, \lambda_n)$ Nash-equilibria by $NE(\lambda_1, \dots, \lambda_n)$ By representing the $(\lambda_1, \dots, \lambda_n)$ Nash-equilibria for each choice of $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{>0}$ geometrically, we obtain a diagram showing how equilibria change as one varies the attackers' willingness to grief. This type of representation is similar to phase portraits used in analyzing differential equations and dynamical systems [31].

Example 4. *Consider the payoff table given below (left) and its transformation including utility from griefs under the perspective of Equation 4 (right):*

Alice \ Bob strategies	*A*	*B*
A	$(1,1)$	$(0,0)$
B	$(0,0)$	$(0,0)$

Alice \ Bob strategies	*A*	*B*
A	$(1-\lambda_1^{-1}, 1-\lambda_2^{-1})$	$(0,0)$
B	$(0,0)$	$(0,0)$

This yields the diagram of equilibra in Figure 1

Moreover, we can consider these phase spaces as subsets of $\mathbb{R}^n \times \Delta \subseteq \mathbb{R}^n \times [0,1]^{\sum_{i \in N}|A_i|}$ as:

$$\mathcal{L} = \{(\lambda_1, \dots, \lambda_n, s) \in \mathbb{R}^n \times \Delta : s \in NE(\lambda_1, \dots, \lambda_n)\}.$$

Note that while this framework allows one to consider equilibria in games with attackers that have sabotage incentives similar to work of [15], [23], it allows for a unified perspective where one can consider attackers with varying external incentives at once.

Example 5. *Using our calculations in Example 3, we see that in the mining game*

$$\mathcal{L} = \left\{ (\lambda_1, \ldots, \lambda_n, q_1, \ldots, q_n) \in \mathbb{R}^n_{>0} \times \Delta : q_i = \frac{1 + \lambda_i^{-1}}{c^*} \left(\max\left\{ 1 - \frac{c_i}{c^*}, 0 \right\} \right) \ \ \forall i \in N \right\}.$$

Namely, the diagram of equilibria consists of a single equilibrium at each $(\lambda_1, \ldots, \lambda_n)$ that continuously varies in the λ_i.

7 Structure of Diagrams of Equilibria

In this section we explore several results that limit the structure of the diagrams of equilibria introduced in Section 6. These results can give a mechanism designer insight into how a system may respond to griefs.

A first result considers situations in which the diagram of equilibria is constant with respect to the willingness of actors to grief.

Proposition 2. *Let $G =< S_i, u_i >_{i \in N}$ be a game where, for all $i \in N$ and for all $x_1, x_2 \in A$ and $v_{-i} \in A^{n-1}$,*

$$(u_i(x_1, v_{-i}) - u_i(x_2, v_{-i})) \cdot \sum_{j \neq i} (u_j(x_1, v_{-i}) - u_j(x_2, v_{-i})) \leq 0.$$

Then, the equilibria of this game are constant for all values of $\lambda_1, \ldots, \lambda_k > 0$.

Proof. Denote $a_i = u_i(x_1, v_{-i})$ and $b_i = u_i(x_2, v_{-i})$. Then, the modified utilities given by the perspective of Equation 4 are

$$a_i' = a_i - \lambda_i^{-1} \sum_{j \neq i} u_j(x_1, v_{-i})$$

and

$$b_i' = b_i - \lambda_i^{-1} \sum_{j \neq i} u_j(x_2, v_{-i}).$$

Then,

$$a_i' \geq b_i' \Leftrightarrow a_i - b_i \geq \lambda_i^{-1} \left(\sum_{j \neq i} (u_j(x_1, v_{-i}) - u_j(x_2, v_{-i})) \right).$$

Then, by the hypothesis, if $a_i' - b_i' \geq 0$, we must have $a_i - b_i \geq 0$ and $\sum_{j \neq i} (u_j(x_1, v_{-i}) - u_j(x_2, v_{-i})) \leq 0$. On the other hand, if $a_i - b_i \geq 0$, then by the hypothesis we have $\sum_{j \neq i} (u_j(x_1, v_{-i}) - u_j(x_2, v_{-i})) \leq 0$. Hence,

$$a_i - b_i \geq \lambda_i^{-1} \left(\sum_{j \neq i} (u_j(x_1, v_{-i}) - u_j(x_2, v_{-i})) \right) \Rightarrow a_i' - b_i' \geq 0.$$

Namely,

$$a_i' \geq b_i' \Leftrightarrow a_i \geq b_i.$$

As this holds for all strategies and all participants, these games then have the same equilibria.

□

Then, we have the immediate consequence:

Corollary 1. *Consider a constant sum game. Then, the equilibria of this game are constant for all values of $(\lambda_1, \ldots, \lambda_k)$.*

Proof. Let s be the constant value of the sum of payoffs. Hence,

$$\sum_j u_j(x_1, v_{-i}) = \sum_j u_j(x_2, v_{-i}) = s.$$

So,

$$u_i(x_1, v_{-i}) - u_i(x_2, v_{-i}) = -\sum_{j \neq i} \left(u_j(x_1, v_{-i}) - u_j(x_2, v_{-i})\right).$$

□

Remark 2. *Note that while games satisfying the conditions of these results may have equilibria that are constant with respect to actors' willingness to grief and hence provide ease of analysis to a mechanism designer, this constant equilibrium may nevertheless be an attack or otherwise undesirable. Indeed, in [9], it is argued that various actions in the validation games of Ethereum 2.0 should result in penalties for both the actor that performs them as well as others. While this leads to the opportunities for griefing at varying thresholds observed in Example 1, this is considered preferable in [9] to other design possibilities.*

Now we analyze the possibilities of the boundaries in these diagrams between regions with different equilibria. For all values $(x_1, \ldots, x_n) \in A$, we denote the regions where $(x_1, \ldots, x_n)$ is a (pure) Nash Equilibrium by:

$$\mathcal{R}_{(x_1,\ldots,x_n)} = \{(\lambda_1, \ldots, \lambda_n) : (x_1, \ldots, x_n) \in NE(\lambda_1, \ldots, \lambda_n)\}.$$

Proposition 3. *For all values $(x_1, \ldots, x_n) \in A$, boundaries of regions $\mathcal{R}_{(x_1,\ldots,x_n)}$ consist of a finite union of hyperplanes: $\cup_k H_k$, where each hyperplane H_k is of the form*

$$H_k = \{(\lambda_1, \ldots, \lambda_n) : \lambda_{i_k} = c_k\}$$

for some $i_k \in \{1, \ldots, n\}$ and $c_k \in \mathbb{R}$.

Proof. Note that $x^* \in A$ is a pure Nash equilibrium of the game defined by the utility functions u'_i given in Definition 3 if and only if

$$u'_i(x^*_{-i}, x_i) \leq u'_i(x^*) \tag{5}$$

for all $i \in N$ and $x_i \in A_i$. However, Inequality 5 can be rewritten as

$$u_i(x^*_{-i}, x_i) - \lambda_i^{-1} \sum_{j \neq i} u_j(x^*_{-i}, x_i) \leq u_i(x^*) - \lambda_i^{-1} \sum_{j \neq i} u_j(x^*).$$

If $u_i(x^*) = u_i(x^*_{-i}, x_i)$, then this inequality either holds or does not hold for all λ_i. In other cases, one solves Inequality 5 for λ_i, noting whether $u_i(x^*) - u_i(x^*_{-i}, x_i)$ is positive or negative, to see that $x^* \in A$ is a Nash equilibrium if and only if

$$\max_{x_i \in \{A_i : u_i(x^*) > u_i(x^*_{-i}, x_i)\}} \frac{\sum_{j \neq i} u_j(x^*) - u_j(x^*_{-i}, x_i)}{u_i(x^*) - u_i(x^*_{-i}, x_i)} \leq \lambda_i$$

$$\leq \min_{x_i \in \{A_i : u_i(x^*) < u_i(x^*_{-i}, x_i)\}} \frac{\sum_{j \neq i} u_j(x^*_{-i}, x_i) - u_j(x^*)}{u_i(x^*_{-i}, x_i) - u_i(x^*)}.$$

However, these boundary conditions are of the form of those in the statement.

□

However, we note in the following example that regions in which mixed equilibria have a given support do not necessarily have rectilinear boundaries.

Example 6. *We consider a three player coordination game, where $A = \{c, n\}$ consists of a cooperation strategy c and a non-cooperation strategy n. Then we take* $u_i(x_i, x_{-i}) = \left\{\begin{cases} 2, & \text{if } x_j = c \ \forall j = 1,2,3 \\ 1, & \text{if } x_i = n \\ 0, & \text{if } x_i = c \text{ but } \exists j \text{ such that } x_j = n \end{cases}\right\}$ *for each $i = 1, 2, 3$.*

Then, transformed under the perspective of Equation 4 gives a payoff table of:

Player 3 plays c

Player 1 \ Player 2 strategies	c	n
c	$(2 - 4\lambda_1^{-1}, 2 - 4\lambda_2^{-1}, 2 - 4\lambda_3^{-1})$	$(-\lambda_1^{-1}, 1, -\lambda_3^{-1})$
n	$(1, -\lambda_2^{-1}, -\lambda_3^{-1})$	$(1 - \lambda_1^{-1}, 1 - \lambda_2^{-1}, -2\lambda_3^{-1})$

Player 3 plays n

Player 1 \ Player 2 strategies	c	n
c	$(-\lambda_1^{-1}, -\lambda_2^{-1}, 1)$	$(-2\lambda_1^{-1}, 1 - \lambda_2^{-1}, 1 - \lambda_3^{-1})$
n	$(1 - \lambda_1^{-1}, -2\lambda_2^{-1}, 1 - \lambda_3^{-1})$	$(1 - 2\lambda_1^{-1}, 1 - 2\lambda_2^{-1}, 1 - 2\lambda_3^{-1})$

We notice that (n, n, n) is a pure Nash equilibrium for all λ_i. Furthermore, noting that $2 - 4\lambda_i^{-1} \geq 1 \Leftrightarrow \lambda_i \geq 4$, we have that (c, c, c) is a pure Nash equilibrium if and only if $\lambda_i \geq 4$ for all $i = 1, 2, 3$.

Then, one can compute all mixed equilibria of this game by exhaustively considering each possible choice of support. This gives:

a mixed equilibrium of $(\frac{1}{2-4\lambda_2^{-1}}c + \frac{1-4\lambda_2^{-1}}{2-4\lambda_2^{-1}}n, \frac{1}{2-4\lambda_1^{-1}}c + \frac{1-4\lambda_1^{-1}}{2-4\lambda_1^{-1}}n, c)$ *if*

$$\begin{cases} \lambda_1 \geq 4 \\ \lambda_2 \geq 4 \\ \lambda_3 \geq 4 \\ \left(\frac{1}{2-4\lambda_1^{-1}}\right) \cdot \left(\frac{1}{2-4\lambda_2^{-1}}\right)(2 - 4\lambda_3^{-1}) \geq 1 \end{cases}$$

a mixed equilibrium of $(\frac{1}{2-4\lambda_3^{-1}}c + \frac{1-4\lambda_3^{-1}}{2-4\lambda_3^{-1}}n, c, \frac{1}{2-4\lambda_1^{-1}}c + \frac{1-4\lambda_1^{-1}}{2-4\lambda_1^{-1}}n)$ *if*

$$\begin{cases} \lambda_1 \geq 4 \\ \lambda_2 \geq 4 \\ \lambda_3 \geq 4 \\ \left(\frac{1}{2-4\lambda_1^{-1}}\right) \cdot \left(\frac{1}{2-4\lambda_3^{-1}}\right)(2 - 4\lambda_2^{-1}) \geq 1 \end{cases}$$

a mixed equilibrium of $(c, \frac{1}{2-4\lambda_3^{-1}}c + \frac{1-4\lambda_3^{-1}}{2-4\lambda_3^{-1}}n, c, \frac{1}{2-4\lambda_2^{-1}}c + \frac{1-4\lambda_2^{-1}}{2-4\lambda_2^{-1}}n)$ *if*

$$\begin{cases} \lambda_1 \geq 4 \\ \lambda_2 \geq 4 \\ \lambda_3 \geq 4 \\ \left(\frac{1}{2-4\lambda_2^{-1}}\right) \cdot \left(\frac{1}{2-4\lambda_3^{-1}}\right)(2 - 4\lambda_1^{-1}) \geq 1 \end{cases}$$

and a mixed equilibrium of

$$\left(\sqrt{\frac{2-\lambda_1^{-1}}{(2-4\lambda_2^{-1})(2-4\lambda_3^{-1})}}c + \left(1 - \sqrt{\frac{2-\lambda_1^{-1}}{(2-4\lambda_2^{-1})(2-4\lambda_3^{-1})}}\right)n, \sqrt{\frac{2-\lambda_2^{-1}}{(2-4\lambda_1^{-1})(2-4\lambda_3^{-1})}}c + \left(1 - \sqrt{\frac{2-\lambda_2^{-1}}{(2-4\lambda_1^{-1})(2-4\lambda_3^{-1})}}\right)n, \sqrt{\frac{2-\lambda_3^{-1}}{(2-4\lambda_1^{-1})(2-4\lambda_2^{-1})}}c + \left(1 - \sqrt{\frac{2-\lambda_3^{-1}}{(2-4\lambda_1^{-1})(2-4\lambda_2^{-1})}}\right.$$

if

$$\begin{cases} \lambda_1 \geq 4 \\ \lambda_2 \geq 4 \\ \lambda_3 \geq 4 \\ \left(\frac{1}{2-4\lambda_1^{-1}}\right) \cdot \left(\frac{1}{2-4\lambda_2^{-1}}\right)(2-4\lambda_3^{-1}) \leq 1 \\ \left(\frac{1}{2-4\lambda_1^{-1}}\right) \cdot \left(\frac{1}{2-4\lambda_3^{-1}}\right)(2-4\lambda_2^{-1}) \leq 1 \\ \left(\frac{1}{2-4\lambda_2^{-1}}\right) \cdot \left(\frac{1}{2-4\lambda_3^{-1}}\right)(2-4\lambda_1^{-1}) \leq 1 \end{cases}$$

It is also relevant to consider situations where small changes in the griefing factors that participants are willing to accept lead to completely different equilibria. Such phenomena give us a sense of how stable, or not, the behaviour of a game is to the presence of griefing players. Situations such as that of Example 5, where $\mathcal{L}$ consists of a single equilibrium for all $(\lambda_1, \dots, \lambda_n)$ that varies continuously can be seen as being particularly stable in the presence of griefing players.

We will see that, generally, for any two $(\lambda_1, \dots, \lambda_n, s)$, $(\lambda'_1, \dots, \lambda'_n, s)$ in the diagram of equilibria, there exists an equilibrium that can be connected by a path along which some equilibrium varies continuously. To apply some of the tools we use in the following proof, we require compactness. Hence we consider compact approximations to this set. Namely, let $\epsilon > 0$ and $K > \epsilon$; we define:

$$\mathcal{L}_{\epsilon,K} = \{(\lambda_1, \dots, \lambda_n, s) \in [\epsilon, K]^n \times \Delta : s \in NE(\lambda_1, \dots, \lambda_n)\}.$$

Theorem 7.1. *Let $0 < \epsilon < K$. Suppose that A_i is finite for all $i \in N$. There exists a connected component of $\mathcal{L}_{\epsilon,K}$, which we denote $\mathcal{L}^c_{\epsilon,K}$, such that for all $\lambda_1, \dots, \lambda_n \in [\epsilon, K]$, there exists some $s \in \Delta$ with $(\lambda_1, \dots, \lambda_n, s) \in \mathcal{L}^c_{\epsilon,K}$. Moreover, for all $(\lambda_1, \dots, \lambda_n), (\lambda'_1, \dots, \lambda'_n) \in \mathbb{R}^n_{>0}$, there exist $s, s' \in \Delta$ such that there is a path between $(\lambda_1, \dots, \lambda_n, s), (\lambda'_1, \dots, \lambda'_n, s')$ in $\mathcal{L}$.*

Proof. We follow a similar argument to that of Proof II in [20]. We define $\beta_i : [\epsilon, K]^n \times \Delta \to \Delta_i$ by

$$\beta_i(\lambda_1, \dots, \lambda_n, s) = \arg\max_{s_i^* \in \Delta_i} \left\{ u_i(s_{-i}, s_i^*) - \lambda_i^{-1} \sum_{j \neq i} u_j(s_{-i}, s_i^*) - \|s_i - s_i^*\|^2 \right\}.$$

For fixed choices of $\lambda_1, \dots, \lambda_n$ and $s \in \Delta$, note that, if $s_i^*, s_i^{**} \in \Delta$ and $t \in [0,1]$,

$$u'_i(s_{-i}, ts_i^* + (1-t)s_i^{**}) = \sum_{a \in A} (ts_i^*(a) + (1-t)s_i^{**}(a)) \left(u_i(s_{-i}, a) - \lambda_i^{-1} \sum_{j \neq i} u_j(s_{-i}, a) \right) \tag{6}$$

$$= t \cdot u'_i(s_{-i}, s_i^*) + (1-t) \cdot u'_i(s_{-i}, s_i^{**}).$$

Then the $u'_i(s_{-i}, s_i^*) = u_i(s_{-i}, s_i^*) - \lambda_i^{-1} \sum_{j \neq i} u_j(s_{-i}, s_i^*)$ are linear (and particularly concave) in s_i^*. Thus, using standard arguments, see Proposition 3.a of [26], the argmax is unique and each β_i is well-defined.

Moreover, by the same reasoning involving decomposing $u'_i(s_{-i}, s_i^*)$ into a sum over A, $u'_i(s_{-i}, s_i^*)$ is continuous in s_i^* and in each position of s_{-i}. Moreover, $u'_i(s_{-i}, s_i^*)$ is continuous in $\lambda_i \in [\epsilon, K]$. So by Berge's Maximum theorem [5], using the compactness of $[\epsilon, K]^n \times \Delta$, each β_i is a continuous function.

Then we define $f : [\epsilon, K]^n \times \Delta \to \Delta$ by

$$f(\lambda_1, \dots, \lambda_n, s) = (\beta_1(\lambda_1, \dots, \lambda_n, s), \dots, \beta_n(\lambda_1, \dots, \lambda_n, s)).$$

Note that as N is finite and A_i is finite for all $i \in N$, Δ is contained in a finite dimensional Euclidean space. Hence, Theorem 2.1 of [30] implies that the set

$$C_f = \{(\lambda_1, \dots, \lambda_n, s) \in [\epsilon, K]^n \times \Delta : f(\lambda_1, \dots, \lambda_n, s) = s\}$$

has a connected component C_f^c whose projection to the first n coordinates is $[\epsilon, K]^n$.

However, if $(\lambda_1, \ldots, \lambda_n, s) \in \mathcal{L}_{\epsilon,K}$, then

$$s_i \in \arg\max_{s_i^* \in \Delta_i} \left\{ u_i(s_{-i}, s_i^*) - \lambda_i^{-1} \sum_{j \neq i} u_j(s_{-i}, s_i^*) \right\}$$

for all i. So as $-||s_i - s_i^*||^2 \leq 0$ for all s_i^*, $\beta_i(\lambda_1, \ldots, \lambda_n, s) = s_i$, namely $(\lambda_1, \ldots, \lambda_n, s) \in C_f$. Suppose we have $(\lambda_1, \ldots, \lambda_n, s) \in C_f$ that is not in $\mathcal{L}_{\epsilon,K}$. Following the argument of Proof II in [20], for some $i \in N$ and $s_i^* \in \Delta_i$ we must have $u_i'(s_{-i}, s_i^*) - u_i'(s) = h > 0$. Let $\delta = \frac{h}{2||s_i^* - s_i||^2} > 0$. Then, using a decomposition into pure strategies as in Equation 6,

$$\begin{aligned} u_i'(s_{-i}, \delta s_i^* + (1-\delta)s_i) - u_i'(s) &= \delta \cdot u_i'(s_{-i}, s_i^*) + (1-\delta) \cdot u_i'(s) - u_i'(s) \\ &= \delta \left(u_i'(s_{-i}, s_i^*) - u_i'(s) \right) = \delta \cdot h. \end{aligned}$$

Then,

$$\begin{aligned} u_i'(s_{-i}, \delta s_i^* + (1-\delta)s_i) - ||\delta s_i^* + (1-\delta)s_i - s_i||^2 &= u_i'(s_{-i}, \delta s_i^* + (1-\delta)s_i) - \delta^2 ||s_i^* - s_i||^2 \\ &= u_i'(s) + \delta \cdot h - \delta^2 ||s_i^* - s_i||^2. \end{aligned}$$

However, by our choice of δ, $\delta \cdot h - \delta^2 ||s_i^* - s_i||^2 > 0$, so

$$u_i'(s_{-i}, \delta s_i^* + (1-\delta)s_i) - ||\delta s_i^* + (1-\delta)s_i - s_i||^2 > u_i'(s) - ||s_i - s_i||^2,$$

which contradicts the maximality property of $(\lambda_1, \ldots, \lambda_n, s) \in C_f$. Thus, $\mathcal{L}_{\epsilon,K} = C_f$, and we can take $\mathcal{L}_{\epsilon,K}^c$ to be C_f^c.

Then as $\mathcal{L}_{\epsilon,K}$, being a set of Nash equilibria, is defined via a finite collections of inequalities (as N is finite), it is semi-algebraic. Thus, its connected components are in fact path-connected [3]. So, given any $(\lambda_1, \ldots, \lambda_n)$, $(\lambda_1', \ldots, \lambda_n') \in \mathbb{R}_{>0}^n$, we can take ϵ sufficiently small and K sufficiently large such that $(\lambda_1, \ldots, \lambda_n)$, $(\lambda_1', \ldots, \lambda_n') \in [\epsilon, K]^n$ and produce a path in the corresponding $\mathcal{L}_{\epsilon,K}^c$.

□

Theorem 7.1 combined with Proposition 3 imposes constraints on the geometry of such discontinuities in $\mathcal{L}$. One might hope to use these constraints to gain insight into structures of games that have relative stability under some changes in the griefing factors that participants are willing to accept.

8 Conclusions

We have considered a variety of griefs, particularly relating to blockchains and decentralized systems. Moreover, we have noted reasons that griefing might be particularly relevant in this setting. However, by analyzing diagrams of equilibria in the presence of griefing players, we have provided a geometric tool to gain global insight into how stable a given game is with respect to griefing behaviours.

References

[1] Klaus Abbink and Abdolkarim Sadrieh. The pleasure of being nasty. *Economics Letters*, 105(3):306–308, 2009.

[2] Nick Arnosti and S. Matthew Weinberg. Bitcoin: A natural oligopoly. In Avrim Blum, editor, *10th Innovations in Theoretical Computer Science, ITCS 2019*, Leibniz International Proceedings in Informatics, LIPIcs, Germany, January 2019. Schloss Dagstuhl- Leibniz-Zentrum fur Informatik GmbH, Dagstuhl Publishing.

[3] Saugata Basu, Richard Pollack, and Marie Françoise Roy. Computing roadmaps of semi-algebraic sets on a variety. *Journal of the AMS*, 13:55–82, 1999.

[4] Abdeljalil Beniiche. A study of blockchain oracles. ArXiv Preprint, 2020. arXiv:2004.07140.

[5] Claude Berge. *Topological Spaces*. Macmillan, 1963.

[6] Bitfly. Open source ethereum 2.0 beacon chain explorer. Consulted January 2022.

[7] Vitalik Buterin. Ethereum, a next-generation smart contract and decentralized application platform. `https://github.com/ethereum/wiki/wiki/White-Paper`, 2014.

[8] Vitalik Buterin. The triangle of harm. `https://vitalik.ca/general/2017/07/16/triangle_of_harm.html`, 2017.

[9] Vitalik Buterin. Discouragement attacks. Online, `https://github.com/ethereum/research/blob/master/papers/discouragement/discouragement.pdf`, 2018. Consulted, January 2022.

[10] Vitalik Buterin. A griefing factor analysis model. Ethresear.ch Forum, `https://ethresear.ch/t/a-griefing-factor-analysis-model/2338`, 2018.

[11] Vitalik Buterin. Serenity design rationale. Online, `https://notes.ethereum.org/@vbuterin/serenity_design_rationale`, 2022. Consulted, January 2022.

[12] Vitalik Buterin, David Goldberg, and Lion Dapplion. Ethereum 2.0 Altair beacon chain changes. Github, `https://github.com/ethereum/annotated-spec/blob/master/altair/beacon-chain.md`, 2022. Consulted, January 2022.

[13] Vitalik Buterin and Virgil Griffith. Casper the friendly finality gadget. *Computing Research Repository (CoRR)*, arxiv.org/abs/1710.09437v4, 2017.

[14] Yun Kuen Cheung, Stefanos Leonardos, Georgios Piliouras, and Shyam Sridhar. From griefing to stability in blockchain mining economies. ArXiv Preprint, 2021. arXiv:2106.12332.v1.

[15] Subhasish Chowdhury and Oliver Gürtler. Sabotage in contests: a survey. *Public Choice*, 164(1):135–155, July 2015.

[16] Offchain Labs Developers. Arbitrum rollup protocol. Online, `https://developer.offchainlabs.com/docs/rollup_protocol`, 2021. Consulted, December 2021.

[17] Offchain Labs Developers. Making progress and dispute resolution. Online, `https://developer.offchainlabs.com/docs/dispute_resolution`, 2021. Consulted, December 2021.

[18] Clément Lesaege Federico Ast and William George. Kleros short paper v1.0.7. `https://kleros.io/whitepaper_en.pdf`, 2019.

[19] Meg Gladieux. The stock market has always been a game. now Gen Z is learning to play. 34th Street, `https://www.34st.com/article/2021/02/reddit-robinhood-gamestop-amc-dogecoin-stockmarket-stocks-stonks-meme-wallstreetbets-investment-trading-genz-game-corruption`, 2021.

[20] P. Jean-Jacques Herings. Two simple proofs of the feasibility of the linear tracing procedure. *Economic Theory*, 15(2):485–490, 2000.

[21] Joseph Lee Heyman. *On the Computation of Strategically Equivalent Games*. PhD thesis, The Ohio State University, 2019.

[22] Stuart James. Proof of Humanity - an explainer. Kleros Blog, `https://blog.kleros.io/proof-of-humanity-an-explainer/`, 2021.

[23] Kai A. Konrad. Sabotage in rent-seeking contests. *The Journal of Law, Economics, and Organization*, 16(1):155–165, 04 2000.

[24] Taylor Locke. Younger investors are calling bitcoin "boomer coin" here's why they prefer dogecoin and other altcoins. CNBC, `https://www.cnbc.com/2021/05/20/some-millennials-call-bitcoin-boomer-coin-prefer-dogecoin-altcoins.html`, 2021.

[25] Evita March. Psychopathy, sadism, empathy, and the motivation to cause harm: New evidence confirms malevolent nature of the internet troll. *Personality and Individual Differences*, 141:133–137, 2019.

[26] Jean Jacques Moreau. Proximité et dualité dans un espace hilbertien. *Bulletin de la Société Mathématique de France*, 93:273–299, 1965.

[27] Satoshi Nakamoto. Bitcoin: A peer-to-peer electronic cash system. `https://bitcoin.org/bitcoin.pdf/`, 2008.

[28] Kaihua Qin, Liyi Zhou, Benjamin Livshits, and Arthur Gervais. Attacking the DeFi ecosystem with flash loans for fun and profit. *CoRR*, abs/2003.03810, 2020.

[29] Martha E. Schaffer. Evolutionarily stable strategies for a finite population and a variable contest size. *Journal of theoretical biology*, 132 4:469–78, 1988.

[30] Eilon Solan and Omri Solan. Browder's theorem with general parameter space. ArXiv Preprint, 2021. arXiv:2104.14612.v1.

[31] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*. Westview Press, 2000.