Draft: Modeling the value of PNK in terms of the number of disputes

This document attempts to model the tokenomics of the Kleros PNK token. Users who stake PNK tokens have a possibility of being randomly chosen (weighted according to how many tokens they stake) to be "jurors" in disputes that have designated the Kleros network as their arbitrator. See [2] for details.

The following analysis is an extension of the simple model described in [1], considering PNK as a "work" token in the sense of [5]. As such we employ a time-discounted utility model, similar to how one might model the value of taxicab medallions [4].

Note that a popular model for pricing cryptocurrencies is to estimate their velocity and apply Fischer's equation of exchange $MV = PQ$. This model is applicable when the cryptocurrency being evaluated is the underlying currency of some "economy" or marketplace. In a context of work tokens, and particularly in the context of PNK, which is not being used to purchase arbitration services, this model is less applicable. Nevertheless, the velocity of a token is still relevant to token price even in the context of work tokens, as if large amounts of tokens are locked up this decreases available supply. As a result, even though attempting to estimate "economy size" (PQ) are problematic, there have been nevertheless some attempts to analyze work tokens in this framework. See [3] for an evaluation of PNK from this perspective.

It should be remarked that these models, and the market data on which they are applied, are preliminary and one could expect to further develop this analysis in the future to obtain results that are more precise.

1 Model

In the following discussion, we consider a heuristic model for the value of a PNK token to users in terms of the number of available disputes. Kleros allows for the specialization of different kinds of disputes in a tree of subcourts, allowing potential jurors to stake in subcourts they are more qualified for and avoiding those they are less qualified for, see [2]. However, for the moment we will assume that all disputes are decided in a single (sub)court. In Section 2 we will relax that assumption.

Denote the following variables

- S_J - the total fees charged to arbitrate a case

- e - the cost to the user to arbitrate a dispute in time and effort
- d - the deposit (in PNK) that is lost by a juror that votes incoherently
- p - the probability that a juror that makes an honest effort to evaluate a case is nonetheless incoherent with the ultimate ruling
- M - the number of jurors in a given case
- K - the number of PNK that are drawn to rule on the cases in a given period
- r - the prevailing interest rate per arbitration period, namely the average interest rate that someone could obtain, for example in a bank, during the time it takes to arbitrate a dispute
- N - the number of PNK that are staked

Suppose a given juror has a single vote. Further, suppose all of the other $M-1$ participants make the honest effort at cost e (with appropriate parameter choices, this can be shown to be a Nash equilibrium). Then the number of coherent votes X from among these jurors is distributed as $Binom(M-1, p)$. So

$$
\begin{aligned}
E[\text{return per draw}] &= p \cdot E\left[\frac{S_j + d(M-X-1)}{X+1}\right] - (1-p)d - e \\
&= p \cdot (S_j + dM)E\left[\frac{1}{X+1}\right] - pd - (1-p)d - e \\
&= p(S_J + dM) \cdot \frac{1}{Mp}(1-(1-p)^M) - d - e,
\end{aligned}
$$

(where we have used the standard calculation of $E\left[\frac{1}{X+1}\right]$ when X is binomial)

$$
= \frac{S_J + dM}{M}(1-(1-p)^M) - d - e.
$$

As K tokens out of N are chosen with replacement, the number of times that a user is selected as a juror per period is also binomial, distributed as $Binom(K, 1/N)$. So

$$
E(\#\text{ of draws in period } i) = \frac{K}{N}.
$$

Then

$$
\text{Value of staking PNK to user} \geq \sum_{\text{period } i=1}^{\infty} E(\text{return in period } i)(1+r)^{-i}
$$

$$= \sum_{\text{period } i=1}^{\infty} E(\{\text{return per draw }\} \cdot \{\#\text{ of draws in period } i\})(1+r)^{-i}$$

$$= \sum_{\text{period } i=0}^{\infty} E(\text{return per draw }) \cdot E(\#\text{ of draws in period } i)(1+r)^{-i}$$

(where we have used the independence of the return in any given draw and the number of draws per period)

$$= \sum_{\text{period } i=1}^{\infty} \left(\frac{S_J + dM}{M}(1-(1-p)^M) - d - e \right) \frac{K}{N}(1+r)^{-i}$$

$$= \left(\frac{S_J + dM}{M}(1-(1-p)^M) - d - e \right) \frac{K}{Nr}$$

Remark 1. *Note that a more sophisticated model might attempt to take in demand and supply curves, allowing for an estimation of consumer and producer surplus from jurors and parties requiring dispute resolution, rather than taking fixed values of* e *and* f*. This would require substantially more data, but may be a subject of future work.*

2 Effects of multiple subcourts

In the previous section, we modeled the value to a potential user of a PNK-like token in the framework of having a single subcourt where all tokens are staked. In this section, we relax that framework.

Suppose we have now a total of N PNK that is staked across different subcourts. Suppose that N_i is staked in the ith subcourt. We take the $0th$ court to be the general court, i.e. the root of the subcourt tree, so $N_0 = N$.

For simplicity we suppose that these subcourts do not require particularly specialized skills. (This is of course not a reasonable assumption for more technical subcourts - much of the idea behind having a subcourt tree in the first place is to allow jurors to specialize in cases in which they are skilled. However, this framework may provide a reasonable heuristic for some less-skilled subcourts, where jurors may choose one subcourt over another for reasons of taste.) Then in the ith subcourt we have parameters S_{Ji}, e_i, d_i, p_i, M_i, K_i, $f_i = S_{Ji}/M_i$ that are the same for all participants. We assume that the prevailing interest remains constant.

We can consider the amount of value given to PNK by any given subcourt V_i as the value that PNK would have if this was the only subcourt available. Namely if $N_i = N$ and all other parameters were the same. Then we have seen

$$V_i = \left((f_i + d_i)(1-(1-p_i)^{M_i}) - d_i - e_i\right) \frac{K_i}{Nr}.$$

We denote

$$x_i = \left((f_i + d_i)(1-(1-p_i)^{M_i}) - d_i - e_i\right) K_i.$$

Theorem 1. *Suppose that the parameters e_i, p_i are the same for all participants in a given court (namely that there is no specialization of skills among the pool of potential jurors). Further, suppose that $V_j \geq 0$ for all subcourts. Then*

$$\textit{Value of staking PNK to user} \geq \sum_{j \text{ subcourts}} V_j.$$

Proof. Note that, as we are assuming that there is no differentiation in the skills of the jurors, it will always be optimal for a juror to stake at a leave in the tree of subcourts, namely a subcourt that has no child subcourts. (This is generally true unless the parameters for that subcourt are set such that it is not profitable to adopt an honest strategy when staked in that court, a case we have excluded with the assumption that $V_j \geq 0$ for all j.) Moreover, in equilibrium (where each of the jurors is thought of as employing a mixed strategy where they stake in each leaf with some probability), a given participant should be indifferent between the different leaves. Namely, staking in each of the leaves should produce an equal expected return.

Claim: Suppose that there is (sub)branch B of the tree of subcourts from r to l, where l is a leaf of the tree (but r is not necessarily a root for the whole tree). We will see

$$E\left[N_r \sum_{j \in B} \frac{x_j}{N_j}\right] \geq \sum_{j \text{ in subtree whose root is } r} x_j$$

by induction on the depth of r.

Suppose that r is a leaf, i.e. $r = l$. Then the claim holds trivially as it becomes

$$E\left[N_r \frac{x_r}{N_r}\right] = x_r.$$

Now take r the root of a (sub)branch B and suppose we know the claim for all deeper nodes. List all of the children of r: $k_1, k_2, \ldots$. Then through each child, take a (sub)branch to a leaf B_{k_1}, $B_{k_2}, \ldots$. (Note that one of these children is in B, so we can take the corresponding B_{k_i} to be the truncated B.)

Then

$$E\left[N_r \sum_{j \in B} \frac{x_j}{N_j}\right] = E\left[x_r + \left(\sum_{j=k_1,k_2,\ldots \text{ is a child of } r} N_j\right) \cdot \left(\sum_{j \in B, j \neq r} \frac{x_j}{N_j}\right)\right].$$

As we are in equilibrium, we saw staking in any of the leaves should be gives an equal return so canceling the contributions to this expected return from common ancestors, we have

$$E\left[\sum_{j \in B, j \neq r} \frac{x_j}{N_j}\right] = E\left[\sum_{j \in B_t} \frac{x_j}{N_j}\right]$$

for all $t = k_1, k_2, \ldots$.

Take t as one of of the children of r: $k_1, k_2, \ldots$ Note that for $B_t \neq B$,

$$E\left[\sum_{j \in B, j \neq r} \frac{x_j}{N_j} \mid N_t = y\right]$$

is monotonically increasing in y. (Note that, in equilibrium, the N_j are determined by multinomial draws where the the possible outcomes are the leaves of the subcourt tree. Then, conditional on a fixed value of $N_t = y$, the N_j that correspond to j not in the subtree B_t are still determined by such multinomial distributions with fewer possible outcomes, corresponding to however many leaves are in the subtree rooted at t, and a number of trials that decreases as y increases. Equivalently, as more trials are pre-assumed as being assigned to as the more PNK are staked in the t branch, the less is staked in the other branches increasing their average return). Similarly,

$$E\left[\sum_{j \in B_t} \frac{x_j}{N_j} \mid N_t = y\right]$$

is monotonically decreasing in y. Hence

$$E\left[N_t\left(\sum_{j \in B, j \neq r} \frac{x_j}{N_j} - \sum_{j \in B_t} \frac{x_j}{N_j}\right)\right] \geq E[N_t] \cdot E\left[\sum_{j \in B, j \neq r} \frac{x_j}{N_j} - \sum_{j \in B_t} \frac{x_j}{N_j}\right] = 0.$$

So

$$E\left[N_t \sum_{j \in B, j \neq r} \frac{x_j}{N_j}\right] \geq E\left[N_t \sum_{j \in B_t} \frac{x_j}{N_j}\right]$$

for all $t = k_1, k_2, \ldots$.

Then, by the induction hypothesis, as the k_1, $k_2, \ldots$ are deeper than r, we have

$$E\left[N_{k_t} \sum_{j \in B_t} \frac{x_j}{N_j}\right] \geq \sum_{j \text{ in subtree whose root is } k_t} x_j.$$

So

$$E\left[N_r \sum_{j \in B} \frac{x_j}{N_j}\right] \geq x_r + \sum_{t \text{ such that } k_t \text{ is a child of } r} \left(\sum_{j \text{ in subtree whose root is } k_t} x_j\right)$$

$$= \sum_{j \text{ in subtree whose root is } r} x_j$$

establishing the claim.

Then applying the claim to $r = 0$, i.e. the general court where the subtree whose root is r is the entire tree, we have that the expected value of staking on any leaf is at least

$$\frac{\sum_j x_j}{N \cdot r} = \sum_j V_j.$$

□

Remark 2. *Note the importance of the constance of the parameters e_i and p_i across users in Theorem 1, namely that the subcourts considered do not require particularly specialized skills. Indeed, if the Kleros court tree was exclusively composed of courts that are so highly specialized that it is impossible for new entrants to become competitive, then the value of PNK would be determined by the jurors in the subcourt for whom PNK has the most value, hence one would heuristically expect an estimated value of $\max_{j\ subcourts}\{V_j\}$. This would be particularly true if the "most valuable" subcourt was substantially more valuable to its jurors than other subcourts were to theirs. In practice, one would expect effects between this and the additivity observed in Theorem 1 depending on the difficulty of obtaining the skills required to be a competent juror in other courts.*

Remark 3. *In cases where one might want to take $M = 1$ to remain cost competitive with other dispute resolution methods that use a single reviewer, the design choice of having fees and lost deposits split between the jurors of a given round becomes problematic; whereas when $M > 1$ the probability that all jurors are wrong is relatively low so the same pool of money is split between them and coherences and incoherences largely average out, in the case when $M = 1$ "everyone" will be incoherent a $1 - p$ proportion of the time, resulting in a significant drag on the expected value for the jurors.*

A solution to this is to have deposits that are lost like this redistributed to jurors from other cases in a given subcourt over some period of time. Then as there is $(1 - p)(f + d)$ lost on average per case to split between a p proportion of the cases, the expected value of honest for a first round where $M = 1$ in this model is

$$E[honest] = p\left(f + \frac{(1-p)(f+d)}{p}\right) - (1-p)d - e = f - e$$

and

$$E[lazy] = t\left(f + \frac{(1-p)(f+d)}{p}\right) - (1-t)d.$$

(Compare to the discussion in [6].)

Then following the above reasoning, we arrive at

$$x_i = (f_i - e_i)K_i$$

and

$$V_i = (f_i - e_i)\frac{K_i}{Nr}$$

in this case, and Theorem 1 still applies.

References

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