Hi Michael,

You wrote me a confusing email, so I will try to respond here in the simplest and plainest fashion that I can. I will use language that is so plain and simple that even undergrads should be able to follow; so excuse me if it seems too simple. The goal is to avoid confusion and misunderstanding.

Linas Vepstas 27 April 2024

Spectral basics

Lets say we have some Hamiltonian operator H and we are able to solve it to find a discrete eigenvalue spectrum for it:

$$H\psi_n = \omega_n \psi_n$$

Written in this form, this H could be anything: a vibrating guitar string, a vibrating airplane wing, the Schroedinger eqn for the hydrogen atom, a radar cavity for some electronics, a harmonic operator on some quotient of Lie groups, a Dirac operator on some n-dimensional spacetime. Don't care, doesn't matter as long as the spectrum is discrete (and thus countable).

The eigenfunctions are assumed to be orthonormal. Pick your favorite notation, I don't care:

$$\delta_{mn} = \langle \psi_m | \psi_n \rangle = \int \overline{\psi_m} \psi_n = \langle m|n \rangle = \int \overline{\psi_m}(x)\, \psi_n(x)\, dx$$

Whatever, its all good.

In bra-ket notation, of course one writes

$$H\,|\psi_n\rangle = \omega_n\,|\psi_n\rangle$$

The vacuum expectation value of the Hamiltonian is

$$\langle H \rangle = \sum_n \langle \psi_n | H\,|\psi_n\rangle = \sum_n \omega_n$$

and note that this expression is "valid" even for guitar strings! This has nothing to do with quantum. It's generic for any operator.

When this sum is absolutely convergent, then such an operator H is called "trace class". https://en.wikipedia.org/wiki/Trace_class For the case where H is not an operator, but is instead a map between spaces, see https://en.wikipedia.org/wiki/Nuclear_operator

Returning to simple things:

The issue is that, for guitar strings, the eigenvalues are unbounded: $\omega_n = 2\pi n/L$ with L the length of the guitar string, and so $\langle H \rangle = (2\pi/L)\sum_{n=0}^{\infty} n$ and the sum is not convergent: the expectation value is infinity.

This is a generic problem: Hamiltonians have spectrums that are unbounded. For radar cavities, the radial part has ω_n similar to the guitar string, but sphere symmetry means sphere harmonics which means there are $\ell(\ell+1)$ of them for each radial excitation, and so the sum goes like $\langle H \rangle \sim \sum_{n=0}^{\infty} n^3$ which is even more divergent. For general D-dimensional space, it will go like $\langle H \rangle \sim \sum_{n=0}^{\infty} n^D$. This includes harmonic operators (aka "wave equations") on homogeneous spaces, which are quotients of Lie groups. https://en.wikipedia.org/wiki/Homogeneous_space

It gets interesting when H is a Dirac operator. In this case, the spectrum is both positive and negative. If the spectrum is perfectly symmetric, so that for each positive ω_n there is exactly one other $-\omega_n$, one can argue (hand-wave) that these should exactly cancel one-another to get exactly zero. The sum $\sum_n \omega_n$ is now "conditionally convergent". It's not absolutely convergent in the formal mathematical sense. The whole point of regulators is to deal with this.

BTW, for Dirac operators, if the positive spectrum almost pairs up with the negative spectrum, but not quite, then you can "split the difference", and the sum $\langle H \rangle = \sum_n \omega_n$ now measures the "spectral asymmetry". If the spectrum is perfectly symmetric, then $\langle H \rangle = \sum_n \omega_n = 0$ but otherwise its not. There are some famous theorems about this; the most important of these is the Atiyah–Singer theorem.

https://en.wikipedia.org/wiki/Spectral_asymmetry

Regulators

This is a quasi-repeat of my earlier email, but now in the Hamiltonian setting. Given some complex number s, write

$$\langle H \rangle_s = \sum_n \omega_n \left| \omega_n \right|^{-s}$$

For the guitar string example, ignore the assorted constants and take $\omega_n = n$ for positive n only. Then

$$\langle H \rangle_s = \sum_{n=1}^{\infty} n \left| n \right|^{-s} = \sum_{n=1}^{\infty} n^{-s+1}$$

and this sum is absolutely convergent for $\sigma > 2$ where $s = \sigma + i\tau$ are the real and imaginary parts of s. More interesting is that this is just some plain-old complex-analytic series, and can be analytically continued to the "entire" complex plane. It has a simple pole at $s = 0$ and no poles anywhere else. Simple pole means it behaves like $1/s$ near $s = 0$ and so now one can talk about the expression

$$\langle H \rangle_{\text{finite}} = \langle H \rangle_s - \frac{1}{s}$$

and make some hand-waving arguments about how $\langle H \rangle_{\text{finite}}$ is the vacuum expectation value for a vibrating guitar string, after removal of the divergences. In fact, for the guitar string, we have $\langle H \rangle_{\text{finite}} = 0$.

Now let me take the conventional big leap. If one studies quantum field theory, using the standard formalism used in the 1960's through 1990's, one discovers that the quantized bosonic field looks more-or-less like a guitar-string at every point in spacetime. More formally, it is a simple harmonic oscillator at every point in spacetime. Well, actually, it is a simple harmonic oscillator at every point in momentum space, and it was wrong to talk about position space, because in momentum space, the propagator blah-blah-blah. For the hand-waving part, it is sufficient to claim that there's an SHO at every point, and since the SHO has $\omega_n = n + 1/2$ you plug through the sum above to get $\langle H\rangle_s$ and you remove the the obvious simple pole to get $\langle H\rangle_{\text{finite}}$ and since its a non-interacting bosonic field, its $\langle H\rangle_{\text{finite}} = 0$ or I guess its $\langle H\rangle_{\text{finite}} = 1/2$; I'm too lazy to do the sum and the analytic continuation: this is just the Hurwicz zeta function, and this is just a plain-old-ordinary complex-analytic function and conventional 19th century math handles this just fine.

Of course, there's an infinite number of points in space, so you have to multiply the above by infinity. The "correct" solution for this is not to count points but to use measure theory. There's actually some formal mathematically-rigorous ways of doing this; one I've tripped across is "abstract Weiner space" which is pretty cool. https://en.wikipedia.org/wiki/Abstract_Wiener_space

There are other ways of making some amount of mathematically-rigorous forward progress within this conventional QFT framework. They don't go all the way. The other alternative is to throw up your hands and say "screw it" and use string theory, in which you can explicitly show that these kinds of sums are explicitly finite, using completely unrelated mechanisms. But then you get the string swampland. At any rate, we're totally off-topic.

Returning to the main topic: for an airplane wing, or for a vibrating drum (instead of a guitar string) one has $\omega_n = n + \mathscr{O}(1)$ but you have ℓ of these (for a "perfectly round wing" aka a "spherical cow"), and so now your right-most pole is at $s = 1$ and maybe more poles in other locations, and for a spherical radar cavity, the modes are $\ell(\ell+1)$ degenerate so you get a pole at $s = 2$ and maybe more poles in other places, and so on down the line: a pole at $s = D - 1$ for D dimensions. All this is more-or-less 19th century complex analysis, with no particular magic. Except for the spectrum, it's not physics; the sums are just classical sums.

As already noted, the exponential regulator is

$$\langle H\rangle_t = \sum_n \omega_n \exp -t\,|\omega_n|$$

and as you already know, this can be related to $\langle H\rangle_s$ with the Laplace transform & Mellin transform.

The gaussian regulator is

$$\langle H\rangle_{gt} = \sum_n \omega_n \exp -t^2\,|\omega_n|^2$$

and I'm not sure how to relate this to the other regulators, but again, the place to look is not physics books, cause this is not physics, but instead books on complex analysis. In my experience, books from the late 19th and early 20th century are better places to look for obscure stuff like this, because the modern texts have decided that the obscure stuff is boring and they skip it in favor of modern developments. And honestly, the obscure stuff really is mind-numbingly dull, unless you actually need to actually have some specific obscure formula, in which case it's fascinating. So I'm thinking, for example, the Hardy&Littlewood (or is it Hardy & Wright?) book from 1921, which is filled with mathematical gems on complex analysis that you'll never find in modern texts, and which will also drain from your mind as soon as you turn the page. Frack, I can't even remember the names of the authors of some of the things I've seen. In short, I've seen this stuff, but I haven't memorized it, so I can't tell you what it is. Sorry, I have to say "happy hunting".

Anyway, I hope this clarifies what it means for something "to be finite" or "to be convergent".