tags: **ASP**

ASP HW1

Problem 1

$\mathbf{Z}$ is circularly symmetric Gaussian $\rightarrow\ \mathbb{E}[\mathbf{Z}\mathbf{Z}^T]=0$
Zero mean $\rightarrow\ \mathbb{E}[\mathbf{Z}]=0$
Covariance $\mathbf{R_Z}=\mathbb{E}[\mathbf{Z}\mathbf{Z}^H]=\begin{bmatrix}2 & 1+2j & 0.1\\ \times & 3 & -1+j\\ \times & \times & 5\end{bmatrix}$

a.
Since covarianve matrix is Hermitian,

$$\mathbf{R_Z}=\mathbb{E}[\mathbf{Z}\mathbf{Z}^H]=\begin{bmatrix}2 & 1+2j & 0.1\\ 1-2j & 3 & -1+j\\ 0.1 & -1-j & 5\end{bmatrix}$$

b.
$\mu_\mathbf{W}=\mathbb{E}[\mathbf{A}\mathbf{Z}]=\mathbf{A}\mathbb{E}[\mathbf{Z}]=\mathbf{0}$
$(\because\ \mu_\mathbf{Z}=0)$

$$\mathbf{R_W}=\mathbb{E}[\mathbf{W}\mathbf{W}^H]=\mathbb{E}[\mathbf{A}\mathbf{Z}\mathbf{Z}^H\mathbf{A}^H]=\mathbf{A}\mathbb{E}[\mathbf{Z}\mathbf{Z}^H]\mathbf{A}^H$$
$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 2 & 0 & 2\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}2 & 1+2j & 0.1\\ 1-2j & 3 & -1+j\\ 0.1 & -1-j & 5\end{bmatrix}\begin{bmatrix}1 & 0 & 2 & 2\\ 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 3\end{bmatrix}$$
$$=\begin{bmatrix}2 & 1.1+2j & 4.2 & 4.3\\ 1.1-2j & 6 & 10.2-2j & 14.2-j\\ 4.2 & 10.2+j & 28.8 & 39\\ 4.3 & 14.2+j & 39 & 54.2\end{bmatrix}$$

c.
Let $\mathbf{u}=[\sqrt{3}\ \frac{-1}{\sqrt{5}}\ 2]^T$, then
$\mathbb{E}[.]=\mathbb{E}[(\mathbf{u}^T\mathbf{Z}+2)^2]=\mathbb{E}[\mathbf{u}^T\mathbf{Z}\mathbf{u}^T\mathbf{Z}+4\mathbf{u}^T\mathbf{Z}+4]=\mathbb{E}[\mathbf{u}^T\mathbf{Z}\mathbf{Z}^T\mathbf{u}+4\mathbf{u}^T\mathbf{Z}+4]$
$=\mathbf{u}^T\mathbb{E}[\mathbf{Z}\mathbf{Z}^T]\mathbf{u}+4\mathbf{u}^T\mathbb{E}[\mathbf{Z}]+4=4$
$(\because\ \mu_\mathbf{Z}=0,\ \mathbb{E}[\mathbf{Z}\mathbf{Z}^T]=0)$

Problem 2

a.
By the definition:
$\mathbb{E}[v(n)v^*(n-k)] = \sigma_v^2\delta(k)$

b.
First, we calculate the impulse response function.

$S_v(z) = \sum_{k=-\infty}^{\infty} r_v(k)z^{-k} = \sum_{k=-\infty}^{\infty} \sigma_v^2\delta(k)z^{-k} = \sigma_v^2$
Z-transform the difference equation:
$(1-\alpha z^{-1})S_s(z) = S_v(z)$
$\Rightarrow H(z) = \frac{S_s(z)}{S_v(z)} = \frac{\sigma_v^2}{1-\alpha z^{-1}}$
$\Rightarrow h(n) = \alpha^n u(n)$
where $u(n)$ is the unit step function.

Then, we prove a lemma [1]:
When $k > 0$,
$\mathbb{E}[v(n)s^*(n-k)] = \mathbb{E}[v(n)\sum_t h^*(t)v^*(n-t-k)]$
$= \sum_t h^*(t)\mathbb{E}[v(n)v^*(n-t-k)] = h^*(-k)\sigma_v^2 = 0$

By the above lemma we then calculate the autocorrelation function:
$r_s(0) = \mathbb{E}[s(n)s^*(n)] = |\alpha|^2 r_s(0) + r_v(0) + 2\mathbb{E}[\Re\{v(n)\alpha s^*(n-1)\}]$
$= |\alpha|^2 r_s(0) + r_v(0)$ (by lemma [1])
$\Rightarrow r_s(0) = \frac{\sigma_v^2}{1-|\alpha|^2}$ ([2])

Multiply $s^*(n-k)$ and take expectation on both side of the difference equation, we get:
$\mathbb{E}[s(n)s^*(n-k)] = \alpha\mathbb{E}[s(n-1)s^*(n-k)] + \mathbb{E}[v(n)s^*(n-k)]$
$\Rightarrow r_s(k) = \alpha r_s(k-1)$ (by lemma [1]), when $k > 0$ ([3])

By [2] and [3], we have:
$r_s(k) = \alpha^k r_s(0) = \alpha^k \frac{\sigma_v^2}{1-|\alpha|^2},\ k > 0$

Finally, by the Hermitian symmetric property of autocorrelation function, we have:
$$r_s(k) = \begin{cases} (\alpha^*)^{-k}\frac{\sigma_v^2}{1-|\alpha|^2}, k < 0 \\ \alpha^k \frac{\sigma_v^2}{1-|\alpha|^2}, k \geq 0 \end{cases}$$

c.
By substituting the difference equation into the expectation formula we get:
$\mu_s = \mathbb{E}[s(n)] = \mathbb{E}[\alpha s(n-1) + v(n)] = \alpha\mathbb{E}[s(n-1)] + \mathbb{E}[v(n)] = \alpha\mu_s + 0 = \alpha\mu_s$
$\Rightarrow \mu_s = 0$

d.

$S_s(z) = H(z)H^*(1/z^*)S_v(z) = \frac{1}{1-\alpha z^{-1}}\frac{1}{1-\alpha^* z}\sigma_v^2$

$\Rightarrow S_s(e^{j2\pi ft}) = \frac{\sigma_v^2}{1-2\Re\{\alpha e^{-j2\pi ft}\}+|\alpha|^2}$

Problem 3

a.

Since,

$r_x(k) = \mathbb{E}[x(n)x^*(n-k)] = \mathbb{E}[(s(n)+v(n))(s^*(n-k)+v^*(n-k))]$

$= r_s(k) + r_v(k) + h(k)\sigma_v^2 + h^*(-k)\sigma_v^2 = (r_s(k) + \delta(k) + h(k) + h^*(-k))\sigma_v^2$

The correlation matrix is:

$\mathbf{R} = \mathbb{E}[\mathbf{x}(n)\mathbf{x}(n)^H] = \begin{bmatrix} \mathbb{E}[x(n)x(n)] & \mathbb{E}[x(n)x^*(n-1)] \\ \mathbb{E}[x(n-1)x^*(n)] & \mathbb{E}[x(n-1)x^*(n-1)] \end{bmatrix} = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x^*(1) & r_x(0) \end{bmatrix}$

$= \sigma_v^2 \begin{bmatrix} 3+\frac{1}{1-|\alpha|^2} & \alpha+\frac{\alpha}{1-|\alpha|^2} \\ \alpha^*+\frac{\alpha^*}{1-|\alpha|^2} & 3+\frac{1}{1-|\alpha|^2} \end{bmatrix} = \frac{\sigma_v^2}{1-|\alpha|^2}\begin{bmatrix} 4-3|\alpha|^2 & 2\alpha-\alpha|\alpha|^2 \\ 2\alpha^*-\alpha^*|\alpha|^2 & 4-3|\alpha|^2 \end{bmatrix}$

Also, since

$\mathbb{E}[x(n-k)d^*(n)] = \mathbb{E}[(s(n-k)+v(n-k))(v^*(n)+v^*(n-1))]$

$= (h(-k) + h(-k+1) + \delta(k) + \delta(k-1))\sigma_v^2$

The cross correlation vector is:

$\mathbf{p} = \mathbb{E}[\mathbf{x}(n)d^*(n)] = \mathbb{E}\begin{bmatrix} x(n)d^*(n) \\ x(n-1)d^*(n) \end{bmatrix} = \sigma_v^2\begin{bmatrix} \alpha+2 \\ 2 \end{bmatrix}$

b.

$\mathbb{E}[d(n)d^*(n)] = \mathbb{E}[(v(n)+v(n-1))(v^*(n)+v^*(n-1))] = 2\sigma_v^2$

c.

By a, b and the Wiener-Hopf equation we have:

$\frac{\sigma_v^2}{1-|\alpha|^2}\begin{bmatrix} 4-3|\alpha|^2 & 2\alpha-\alpha|\alpha|^2 \\ 2\alpha^*-\alpha^*|\alpha|^2 & 4-3|\alpha|^2 \end{bmatrix} = \sigma_v^2\begin{bmatrix} \alpha+2 \\ 2 \end{bmatrix}$

Let matrix $\mathbf{B} = \begin{bmatrix} 4-3|\alpha|^2 & 2\alpha-\alpha|\alpha|^2 \\ 2\alpha^*-\alpha^*|\alpha|^2 & 4-3|\alpha|^2 \end{bmatrix}$

Since $\det(\mathbf{B}) = 16 - 28|\alpha|^2 + 13|\alpha|^4 - |\alpha|^6$ is always postivie for all $|\alpha| < 1$, the above linear system always has a solution:

$$\mathbf{w}_{opt} = (1-|\alpha|^2)\mathbf{B}^{-1}\mathbf{p} = \frac{1-|\alpha|^2}{\det(\mathbf{B})}\begin{bmatrix} 4-3|\alpha|^2 & -2\alpha+\alpha|\alpha|^2 \\ -2\alpha^*+\alpha^*|\alpha|^2 & 4-3|\alpha|^2 \end{bmatrix}\begin{bmatrix} \alpha+2 \\ 2 \end{bmatrix}$$

$$= \frac{1-|\alpha|^2}{16-28|\alpha|^2+13|\alpha|^4-|\alpha|^6}\begin{bmatrix} 8-\alpha|\alpha|^2-6|\alpha|^2 \\ 8-4\alpha^*-8|\alpha|^2+2\alpha^*|\alpha|^2+|\alpha|^4 \end{bmatrix}$$

$$= \frac{1}{|\alpha|^4-12|\alpha|^2+16}\begin{bmatrix} 8-\alpha|\alpha|^2-6|\alpha|^2 \\ 8-4\alpha^*-8|\alpha|^2+2\alpha^*|\alpha|^2+|\alpha|^4 \end{bmatrix}$$

d.

$$h_{wiener}(n) = \mathbf{w}_{opt}^H \begin{bmatrix} \delta(n) \\ \delta(n-1) \end{bmatrix}$$

$$= \frac{8-\alpha^*|\alpha|^2-6|\alpha|^2}{|\alpha|^4-12|\alpha|^2+16}\delta(n) + \frac{8-4\alpha-8|\alpha|^2+2\alpha|\alpha|^2+|\alpha|^4}{|\alpha|^4-12|\alpha|^2+16}\delta(n-1)$$

e.

$$r_y(k) = \mathbb{E}[y(n)y^*(n-k)] = \mathbb{E}[[w_0^*x(n)+w_1^*x(n-1)][w_0^*x(n-k)+w_1^*x(n-k-1)]^*]$$

$$= |w_0|^2 r_x(k) + |w_1|^2 r_x(k) + w_0 w_1^* r_x(k+1) + w_1 w_0^* r_x(k-1)$$

$$= (|w_0|^2+|w_1|^2) r_x(k) + w_0^* w_1 r_x(k+1) + w_1^* w_0 r_x(k-1)$$

Where

$$w_0 = \frac{8-\alpha|\alpha|^2-6|\alpha|^2}{|\alpha|^4-12|\alpha|^2+16}$$

$$w_1 = \frac{8-4\alpha^*-8|\alpha|^2+2\alpha^*|\alpha|^2+|\alpha|^4}{|\alpha|^4-12|\alpha|^2+16}$$

$$r_x(k) = (r_s(k) + \delta(k) + h(k) + h^*(-k))\sigma_v^2$$

$$r_s(k) = \begin{cases} (\alpha^*)^{-k}\frac{\sigma_v^2}{1-|\alpha|^2}, k < 0 \\ \alpha^k \frac{\sigma_v^2}{1-|\alpha|^2}, k \geq 0 \end{cases}$$

$$h(n) = \alpha^n u(n)$$

$u(n)$ is the unit step function

f.

Since

$$w_0|_{\alpha=0.1,\,\sigma_v^2=1} = 7.939/15.8801$$

$$w_1|_{\alpha=0.1,\,\sigma_v^2=1} = 7.5221/15.8801$$

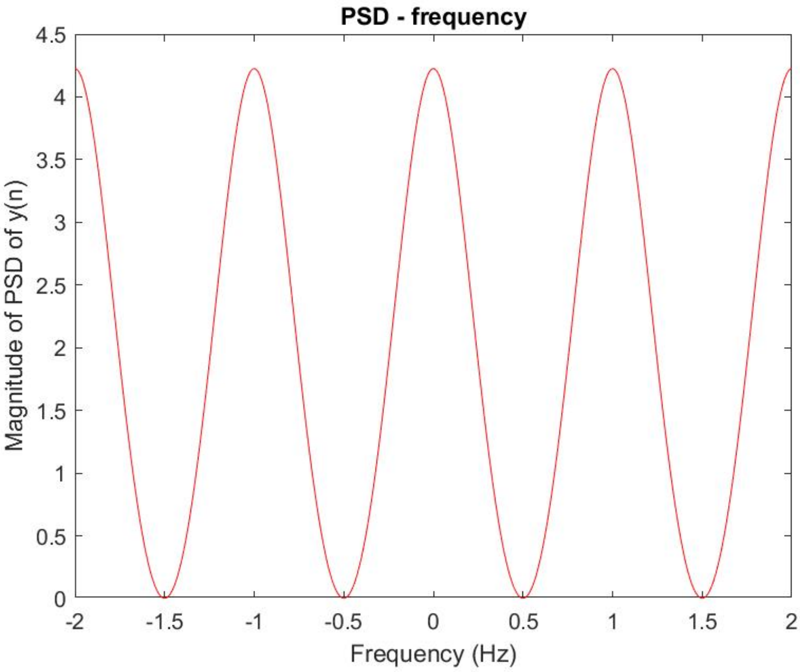
$$h(n)|_{\alpha=0.1,\,\sigma_v^2=1} = 0.1^n u(n)$$

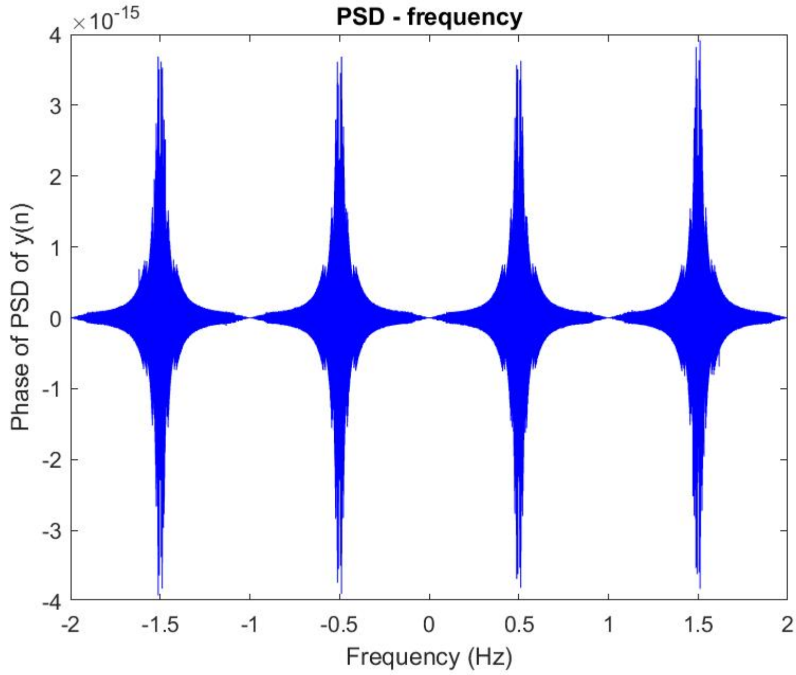
$$r_s(k)|_{\alpha=0.1,\,\sigma_v^2=1} = 0.1^{|k|}/0.99$$

$$r_x(k)|_{\alpha=0.1,\,\sigma_v^2=1} = 0.1^{|k|}/0.99 + \delta(k) + 0.1^k u(k) + 0.1^{-k}u(-k)$$

We have

$S_x(z)|_{\alpha=0.1,\,\sigma_v^2=1} = \sum_k r_x(k)z^{-k}|_{\alpha=0.1,\,\sigma_v^2=1} = \sum_k\{r_x(k)|_{\alpha=0.1,\,\sigma_v^2=1}\}z^{-k}$
$= \sum_k\{0.1^{|k|}/0.99 + \delta(k) + 0.1^k u(k) + 0.1^{-k}u(-k)\}z^{-k}$
$= \sum_{k=1\sim\infty}\{0.1^k/0.99 + 0.1^k\}z^{-k}$
$+ \sum_{k=-\infty\sim-1}\{0.1^{-k}/0.99 + 0.1^{-k}\}z^{-k}$
$+ 3.97/0.99$
$= \sum_{k=1\sim\infty}\{1.99(0.1z^{-1})^k/0.99\}$
$+ \sum_{k=1\sim\infty}\{1.99(0.1z)^k/0.99\}$
$+ 3.97/0.99$
$= (1.99/0.99)0.1z^{-1}/(1-0.1z^{-1})$
$+ (1.99/0.99)0.1z/(1-0.1z) + 3.97/0.99$
$= \frac{0.199}{0.99(z-0.1)} + \frac{0.199z}{0.99(1-0.1z)} + \frac{3.97}{0.99}$
Hence,
$S_y(z) = Z\{(|w_0|^2 + |w_1|^2)r_x(k) + w_0w_1^*r_x(k+1) + w_1w_0^*r_x(k-1)\}$
$= (|w_0|^2 + |w_1|^2)S_x(k) + w_0w_1^*S_x(k)z + w_1w_0^*S_x(k)z^{-1}$
$= \frac{7.939^2+7.5221^2+7.939(7.5221)z+7.5221(7.939)z^{-1}}{15.8801^2}S_x(z)$
The following figure shows the plot of PSD to frequency under this condition (take 1e6 samples from -2 to 2):





g.

$$J_{min} = \sigma_d^2 - \mathbf{p}^H \mathbf{w}_{opt}$$

$$= 2\sigma_v^2 - \begin{bmatrix} \alpha + 2 \\ 2 \end{bmatrix}^H \frac{\sigma_v^2}{|\alpha|^4 - 12|\alpha|^2 + 16} \begin{bmatrix} 8 - \alpha|\alpha|^2 - 6|\alpha|^2 \\ 8 - 4\alpha^* - 8|\alpha|^2 + 2\alpha^*|\alpha|^2 + |\alpha|^4 \end{bmatrix}$$

$$= \frac{|\alpha|^2(4 + |\alpha|^2 + 2\alpha^* + 2\alpha)}{|\alpha|^4 - 12|\alpha|^2 + 16}\sigma_v^2$$

Problem 4

a.

$$(\frac{\partial(\mathbf{a}^T\mathbf{X}\mathbf{b})}{\partial \mathbf{X}})_{i,j} = \frac{\partial(\mathbf{a}^T\mathbf{X}\mathbf{b})}{\partial x_{i,j}} = a_i b_j$$

$$\Rightarrow \frac{\partial(\mathbf{a}^T\mathbf{X}\mathbf{b})}{\partial \mathbf{X}} = \mathbf{a}\mathbf{b}^T$$

b.

For $\mathbf{X} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}$, from the charateristic polynomial we get:

$(x_{1,1} - \lambda)(x_{2,2} - \lambda) - x_{1,2}x_{2,1} = 0$

$\Rightarrow \lambda^2 - \mathrm{Tr}(\mathbf{X})\lambda + \det(\mathbf{X}) = 0$

$\Rightarrow \lambda_1(\mathbf{X}) + \lambda_2(\mathbf{X}) = \mathrm{Tr}(\mathbf{X})$

Hence,

$\frac{\partial(\lambda_1(\mathbf{X})+\lambda_2(\mathbf{X}))}{\partial \mathbf{X}} = I_2$

where I_k denotes the identity matrix

c.

By the charateristic polynomial, we can easily verify that $\prod \lambda_i(\mathbf{X}) = \det(\mathbf{X})$

Then,

$\frac{\partial(\prod \lambda_i(\mathbf{X}))}{\partial \mathbf{X}} = \frac{\det(\mathbf{X})}{\partial \mathbf{X}} = \mathrm{adj}(X)^T$

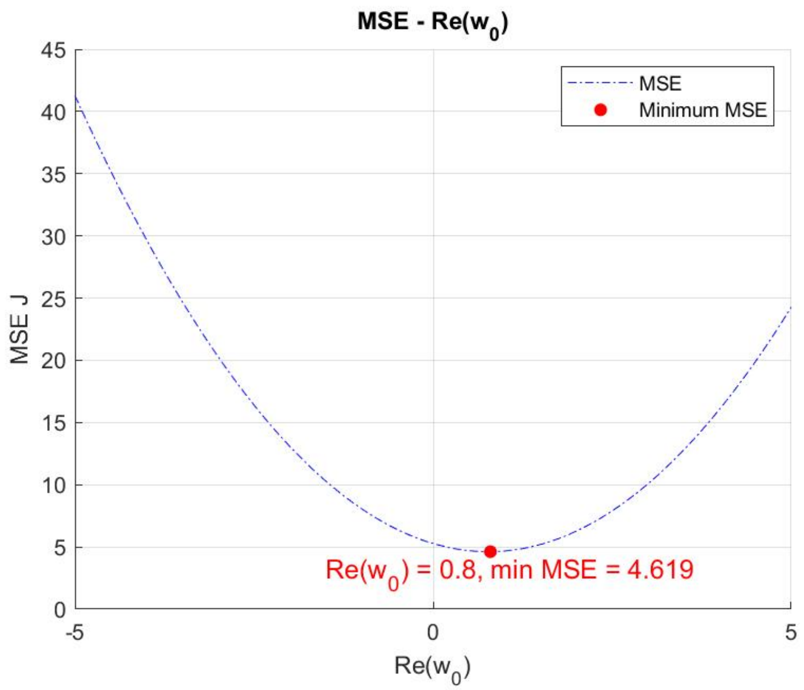
where $\mathrm{adj}(\mathbf{X})$ denotes the adjugate matrix.

Problem 5

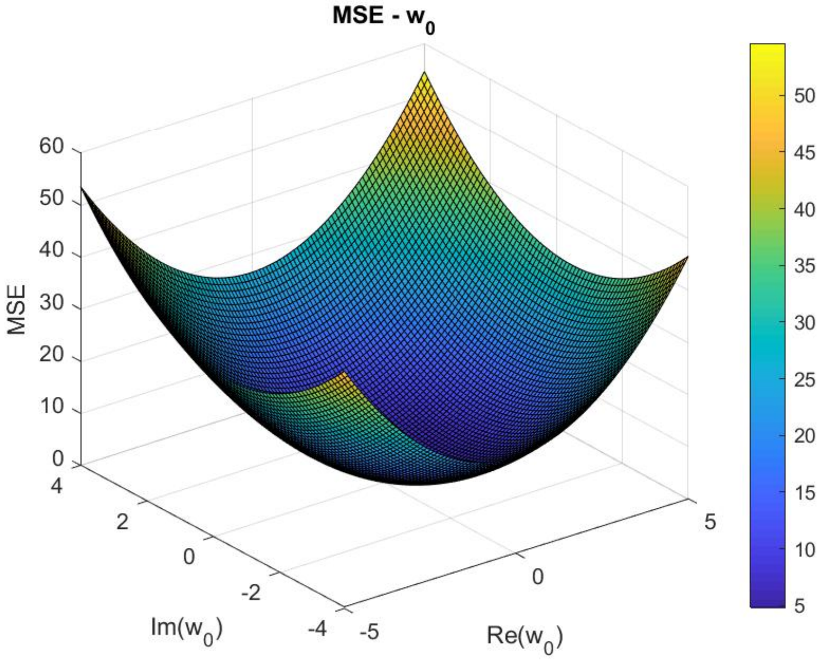
b.

The MSE of the optimal weight vector is: 0.31415

c.

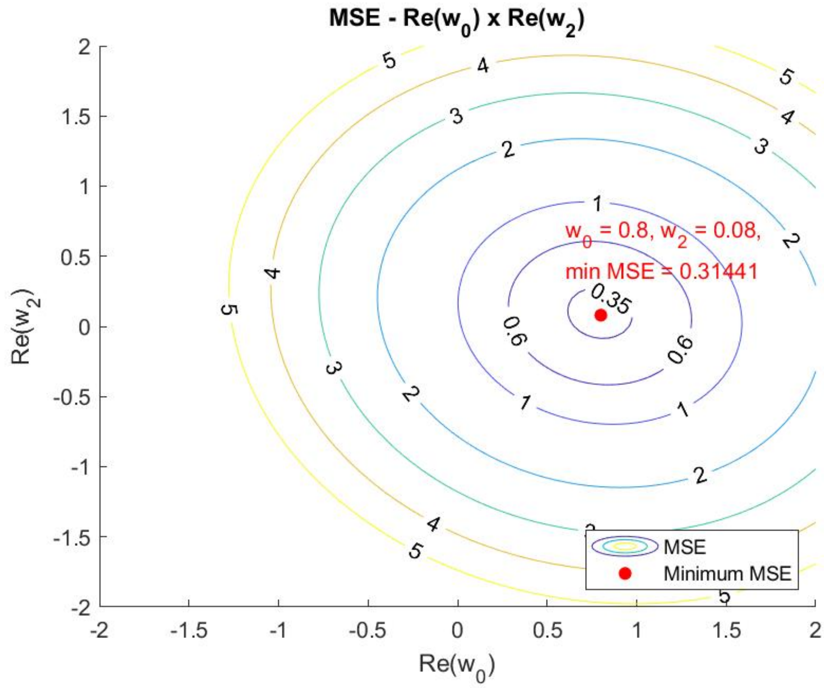


d.



MSE - w_0
MSE
Im(w_0)
Re(w_0)

e.



MSE - Re(w_0) x Re(w_2)
w_0 = 0.8, w_2 = 0.08,
min MSE = 0.31441
MSE
Minimum MSE
Re(w_0)
Re(w_2)