EE5027 Adaptive Signal Processing
Homework Assignment #1

Notice

- Due 9:00pm, October 13, 2020 (Tuesday) $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/3062`)
- ***All answers have to be fully justified***.
- No extensions, unless granted by the instructor one day before T_d.

Problems

1. (Complex Gaussian random vectors) Assume that the complex random vector $\mathbf{Z} = [Z_1, Z_2, Z_3]^T$ follows the complex multivariate circularly-symmetric Gaussian distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{R_z}$. The covariance matrix $\mathbf{R_z}$ has the following form

$$\mathbf{R_z} = \begin{bmatrix} 2 & 1+2j & 0.1 \\ \times & 3 & -1+j \\ \times & \times & 5 \end{bmatrix}. \tag{1}$$

   (a) (5 points) Complete the lower-triangular part of $\mathbf{R_z}$ such that $\mathbf{R_z}$ is a valid covariance matrix.

   (b) (5 points) Now let us consider another random vector $\mathbf{W} \triangleq \mathbf{AZ}$, where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \\ 2 & 0 & 3 \end{bmatrix}. \tag{2}$$

   Evaluate the mean vector of $\mathbf{W}$ and the correlation matrix of $\mathbf{W}$.

   (c) (5 points) Calculate $\mathbb{E}\left[\left(\sqrt{3}Z_1 - \frac{1}{\sqrt{5}}Z_2 + 2Z_3 + 2\right)^2\right]$.

2. (Stochastic models) Consider a first-order AR process with the following difference equation

$$s(n) = \alpha s(n-1) + v(n), \tag{3}$$

   where $|\alpha| < 1$ and $v(n)$ is a zero-mean, circularly-symmetric complex Gaussian, white random process with variance σ_v^2.

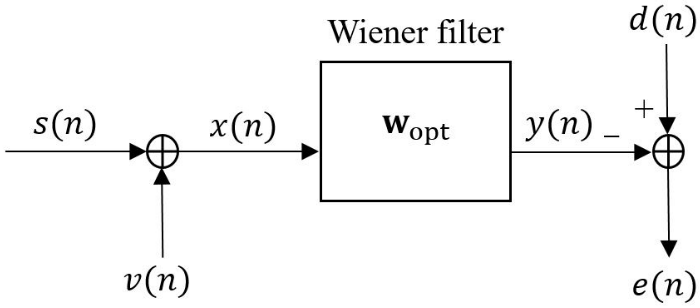

Figure 1: A block diagram for Problem 3.

(a) (5 points) Find the autocorrelation function of $v(n)$.

(b) (5 points) Find the autocorrelation function of $s(n)$.

(c) (5 points) Find the mean-value function of $s(n)$.

(d) (5 points) Find the power spectral density of $s(n)$.

3. (Wiener filters) We consider a block diagram associated Wiener filter in Figure 1, where the number of taps M is 2. The signals $s(n)$ and $v(n)$ are defined in (3). The desired signal $d(n) = v(n) + v(n-1)$. In the lecture, it was shown that the optimal weight vector of the Wiener filter is a solution to the Wiener-Hopf equation

$$\mathbf{R}\mathbf{w}_{\text{opt}} = \mathbf{p}, \tag{4}$$

where $\mathbf{R} = \mathbb{E}[\mathbf{x}(n)\mathbf{x}^H(n)]$ and $\mathbf{p} = \mathbb{E}[\mathbf{x}(n)d^*(n)]$.

(a) (4 points) Find $\mathbf{R}$ and $\mathbf{p}$.

(b) (3 points) Find $\sigma_d^2 = \mathbb{E}[d(n)d^*(n)]$.

(c) (3 points) Find the optimal weight vector $\mathbf{w}_{\text{opt}}$.

(d) (2 points) Find the impulse response $h(n)$ of the Wiener filter in Problem 3c.

(e) (5 points) Find the autocorrelation of $y(n)$.

(f) (5 points) Sketch the power spectral density of $y(n)$, assuming $\alpha = 0.1$. and $\sigma_v^2 = 1$.

(g) (3 points) Find $J_{\min}$.

4. (Matrix calculus) In the lecture, we focus on the derivatives with respect to a *vector*. In this problem, you will derive some properties for derivatives with respect to a *matrix*. Consider an N-by-N real-valued matrix $\mathbf{X}$ with the following layout

$$\mathbf{X} = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,N} \\ x_{2,1} & x_{2,2} & \dots & x_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N,1} & x_{N,2} & \dots & x_{N,N} \end{bmatrix} \tag{5}$$

Let $f(\mathbf{X})$ be a scalar-valued function of the matrix $\mathbf{X}$. Then we write

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} \triangleq \begin{bmatrix} \frac{\partial f(\mathbf{X})}{\partial x_{1,1}} & \frac{\partial f(\mathbf{X})}{\partial x_{1,2}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{1,N}} \\ \frac{\partial f(\mathbf{X})}{\partial x_{2,1}} & \frac{\partial f(\mathbf{X})}{\partial x_{2,2}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{2,N}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(\mathbf{X})}{\partial x_{N,1}} & \frac{\partial f(\mathbf{X})}{\partial x_{N,2}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{N,N}} \end{bmatrix}. \tag{6}$$

The eigenvalues of $\mathbf{X}$ are denoted by $\lambda_1(\mathbf{X}), \lambda_2(\mathbf{X}), \ldots, \lambda_N(\mathbf{X})$.

(a) (10 points) For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$, show that

$$\frac{\partial\left(\mathbf{a}^T\mathbf{X}\mathbf{b}\right)}{\partial \mathbf{X}} = \mathbf{a}\mathbf{b}^T. \tag{7}$$

(b) (5 points) Assume that $N = 2$. Find the closed-form expressions of

$$\frac{\partial\left(\lambda_1(\mathbf{X}) + \lambda_2(\mathbf{X})\right)}{\partial \mathbf{X}} \tag{8}$$

Hint: How to find the eigenvalues of any 2*-by-*2 *matrix?*

(c) (5 points) Assume that the matrix $\mathbf{X}$ is nonsingular. Find the closed-form expressions of

$$\frac{\partial\left(\prod_{k=1}^{N}\lambda_k(\mathbf{X})\right)}{\partial \mathbf{X}}. \tag{9}$$

Hint: Express you answer in terms of $\mathbf{X}$.

5. (The error-performance surface, 20 points) Consider a Wiener filter with a complex input process $x(n)$, a complex desired process $d(n)$, and the weight vector $\mathbf{w} = [w_0, w_1, w_2]^T$. Suppose the covariance matrix $\mathbf{R}$, the cross-correlation vector $\mathbf{p}$, and the variance of $d(n)$ are given by

$$\mathbf{R} = \begin{bmatrix} 1.1 & 0.5 & 0.1 \\ 0.5 & 1.1 & 0.5 \\ 0.1 & 0.5 & 1.1 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} 0.5 \\ -0.4 \\ -0.2 \end{bmatrix}, \qquad \sigma_d^2 = 1.0. \tag{10}$$

In this problem, we will generate the error-performance surface using MATLAB. The real part and the imaginary part of a complex number are denoted by $\mathrm{Re}\{z\}$ and $\mathrm{Im}\{z\}$, respectively.

(a) Write a MATLAB function that computes the mean square error (MSE) J given the covariance matrix $\mathbf{R}$, the weight vector $\mathbf{w}$, the cross-correlation vector $\mathbf{p}$, and σ_d^2 (`sd2`). The syntax is as follows:

```
J = ASP_Wiener_MSE(R, w, p, sd2);
```
(11)

This function returns error messages if any of the following occurs: 1) $\mathbf{R}$ is not positive semidefinite, 2) The dimensions of these input arguments are not suitable, or 3) `sd2` is negative or complex-valued.

(b) Using (11), compute the MSE J if $\mathbf{w}$ is the optimal weight vector.

(c) Use the MATLAB function `plot` to generate a 1-D plot of the MSE. The horizontal axis is $\mathrm{Re}\{w_0\}$, consisting of 101 uniform samples from -5 to 5. The vertical axis is the MSE J. We assume that $\mathrm{Im}\{w_0\} = 1$, $w_1 = -0.5 + j$, and $w_2 = -1$. Be sure to include 1) the labels for the x-axis and the y-axis, 2) the optimal point and the associated minimum J.

(d) Use the MATLAB function `surf` to generate a surface plot of the MSE. The parameter $w_1 = 1 + j$ and $w_2 = 0.5$. The real part of w_0, as the x-axis, takes 101 uniform samples from -5 to 5. The imaginary part of w_0, as the y-axis, takes 81 uniform samples from -4 to 4. The z-axis is the MSE J. Also include the axis labels and the color bar in your plot.

(e) Use the MATLAB function `contour` to generate a contour plot of the MSE. The imaginary parts of w_0 and w_2 are fixed to be 0, and $w_1 = -0.7683$. The x-axis is $\mathrm{Im}\{w_0\}$ while the y-axis is $\mathrm{Re}\{w_2\}$. We take 101 uniform samples from -2 to 2 along both axes. The contour levels contain $0.35, 0.6, 1, 2, 3, 4, 5$. Your contour plot also has to include 1) the optimal point and the associated minimum J, 2) labels on the contour lines, and 3) labels for the axes.

Note: Five MATLAB scripts with the following file names in a compressed file have to be submitted for this problem:

(a) `ASP_Wiener_MSE.m`

(b) `ASP_HW1_Problem_5b.m`

(c) `ASP_HW1_Problem_5c.m`

(d) `ASP_HW1_Problem_5d.m`

(e) `ASP_HW1_Problem_5e.m`

Make sure that the MATLAB codes in parts 5b to 5e generate the final results and plots, which have to be identical to those in your solution.

Last updated September 24, 2020.