tags: **ASP**

ASP HW2

Problem 1

By the Levinson-Durbin Algorithm:
// Initialization
$a_0 = 1,\ P_0 = r(0) = 0.1482$
$\Delta_0 = r^*(1) = 0.05$
// Iteration 1
$\kappa_1 = -\frac{\Delta_0}{P_0} = -\frac{0.05}{0.1482} = -0.33738$
$a_1 = \begin{bmatrix} a_0 \\ 0 \end{bmatrix} + \kappa_1 \begin{bmatrix} 0 \\ a_0^{B*} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0.33738 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.33738 \end{bmatrix}$
$P_1 = P_0 \times (1 - |\kappa_1|^2) = 0.1482 \times (1 - |-0.33738|^2) = 0.13133$
$\Delta_1 = r_2^{BT} a_1 = \begin{bmatrix} r(-2) \\ r(-1) \end{bmatrix}^T a_1 = \begin{bmatrix} 0.017 \\ 0.05 \end{bmatrix}^T \begin{bmatrix} 1 \\ -0.33738 \end{bmatrix} = 0.000131$
// Iteration 2
$\kappa_2 = -\frac{\Delta_1}{P_1} = -\frac{0.000131}{0.13133} = -0.000997487$
$a_2 = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + \kappa_2 \begin{bmatrix} 0 \\ a_1^{B*} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.33738 \\ 0 \end{bmatrix} - 0.000997487 \begin{bmatrix} 0 \\ -0.33738 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.33704346783 \\ -0.000997487 \end{bmatrix}$
$P_2 = P_1 \times (1 - |\kappa_2|^2) = 0.13133 \times (1 - |-0.000997487|^2) = 0.13132986932$
$\Delta_2 = r_3^{BT} a_2 = \begin{bmatrix} r(-3) \\ r(-2) \\ r(-1) \end{bmatrix}^T a_2 = \begin{bmatrix} -0.0323 \\ 0.017 \\ 0.05 \end{bmatrix}^T \begin{bmatrix} 1 \\ -0.33704346783 \\ -0.000997487 \end{bmatrix} = -0.0380796133$
// Iteration 3
$\kappa_3 = -\frac{\Delta_2}{P_2} = -\frac{-0.0380796133}{0.13132986932} = 0.28995394191$
$a_3 = \begin{bmatrix} a_2 \\ 0 \end{bmatrix} + \kappa_3 \begin{bmatrix} 0 \\ a_2^{B*} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.33704346783 \\ -0.000997487 \\ 0 \end{bmatrix} + 0.28995394191 \begin{bmatrix} 0 \\ -0.000997487 \\ -0.33704346783 \\ 1 \end{bmatrix}$
$= \begin{bmatrix} 1 \\ -0.33733269311 \\ -0.09872456909 \\ 0.28995394191 \end{bmatrix}$
$P_3 = P_2 \times (1 - |\kappa_3|^2) = 0.13132986932 \times (1 - |0.28995394191|^2) = 0.12028853533$

Therefore,

a.

$\kappa_1 = -0.33738,\ \kappa_2 = -0.000997487,\ \kappa_3 = 0.28995394191$

b.

$\Delta_0 = 0.05,\ \Delta_1 = 0.000131,\ \Delta_2 = -0.0380796133$

c.

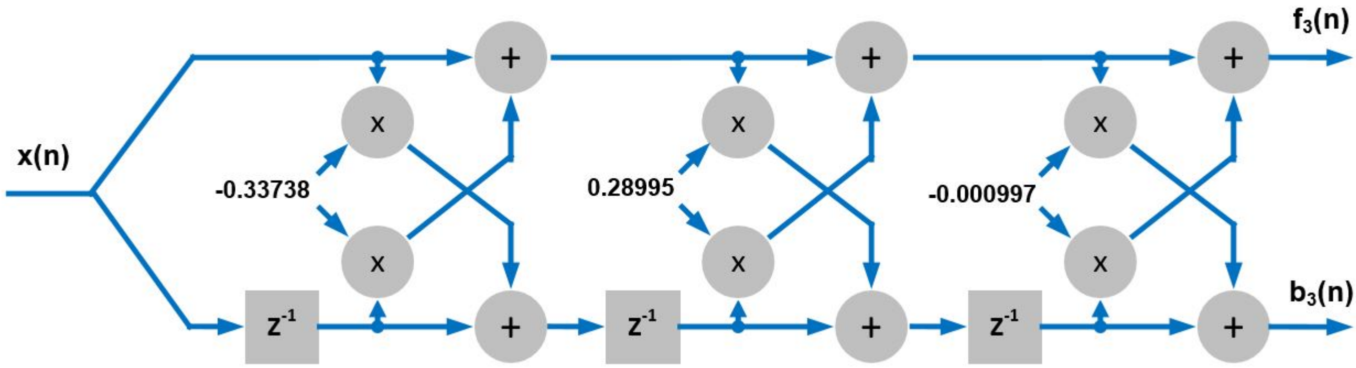
$$a_0 = 1,\ a_1 = \begin{bmatrix} 1 \\ -0.33738 \end{bmatrix},\ a_2 = \begin{bmatrix} 1 \\ -0.33704346783 \\ -0.000997487 \end{bmatrix},\ a_3 = \begin{bmatrix} 1 \\ -0.33733269311 \\ -0.09872456909 \\ 0.28995394191 \end{bmatrix}$$

d.

$P_0 = 0.1482,\ P_1 = 0.13133,\ P_2 = 0.13132986932,\ P_3 = 0.12028853533$

e.

The signal-flow graph for the lattice model when M is 3 is as follows:



Problem 2

a.

From HW1, we have

$S_x(e^{j2\pi f}) = \frac{1}{1-\alpha e^{-j2\pi f}} \times \frac{1}{1-\alpha^* e^{j2\pi f}}$

Since,

$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ for $z \in \mathbf{C}$ and $|z| < 1$

The prediction lower bound of x is

$\exp(\int_{-\frac{1}{2}}^{\frac{1}{2}} \log S_x(e^{j2\pi f})df)$
$= \exp(-\int_{-\frac{1}{2}}^{\frac{1}{2}} \log(1-\alpha e^{-j2\pi f})df - \int_{-\frac{1}{2}}^{\frac{1}{2}} \log(1-\alpha^* e^{j2\pi f})df)$
$= \exp(\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(\alpha e^{-j2\pi f})^n}{n} df + \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(\alpha^* e^{j2\pi f})^n}{n} df)$
$= \exp(\sum_{n=1}^{\infty} \frac{\alpha^n \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi fn} df}{n} + \sum_{n=1}^{\infty} \frac{\alpha^{*n} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi fn} df}{n})$
$= \exp(\sum_{n=1}^{\infty} \frac{\alpha^n (e^{-j\pi n} - e^{j\pi n})}{-2\pi j n^2} + \sum_{n=1}^{\infty} \frac{\alpha^{*n} (e^{j\pi n} - e^{-j\pi n})}{2\pi j n^2})$
$= \exp(0) = 1$

b.

Since,
$\int_{-\frac{1}{2}}^{\frac{1}{2}} S_x(e^{j2\pi f})df$
$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k=-\infty}^{\infty} r_x(k) e^{-2\pi jfk} df$
$= \sum_{k=-\infty}^{\infty} r_x(k) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi jfk} df$
$= r_x(0) + \sum_{k=-\infty}^{-1} \frac{r_x(k)(e^{-\pi jk} - e^{\pi jk})}{-2\pi jk} + \sum_{k=1}^{\infty} \frac{r_x(k)(e^{-\pi jk} - e^{\pi jk})}{-2\pi jk}$
$= r_x(0) = \frac{1}{1-|\alpha|^2}$

We have:
$\gamma_x^2 = \frac{1}{\int_{-\frac{1}{2}}^{\frac{1}{2}} S_x(e^{j2\pi f})df} = 1 - |\alpha|^2$

Problem 3

a.

$e(0) = d(0) - y(0) = d(0) - \hat{w}^*(0)x(0) = 1 - 0 \times 1 = 1$

b.

$\hat{w}(1) = \hat{w}(0) + \mu x(0) e^*(0) = 0 + \mu \times 1 \times 1 = \mu$

c.

$e(1) = d(1) - y(1) = d(1) - \hat{w}^*(1)x(1) = e^{j2\pi f_2} - \mu e^{j2\pi f_1}$
$\hat{w}(2) = \hat{w}(1) + \mu x(1) e^*(1) = \mu + \mu \times e^{j2\pi f_1} \times (e^{-j2\pi f_2} - \mu e^{-j2\pi f_1})$
$= \mu + \mu e^{j2\pi(f_1 - f_2)} - \mu^2$

d.

Since
$e(n-1) = d(n-1) - y(n-1) = d(n-1) - \hat{w}^*(n-1)x(n-1)$
$= e^{j2\pi f_2 (n-1)} - \hat{w}^*(n-1) e^{j2\pi f_1 (n-1)}$

$\hat{w}(n) = \hat{w}(n-1) + \mu x(n-1)e^*(n-1)$
$= \hat{w}(n-1) + \mu e^{j2\pi f_1(n-1)}(e^{-j2\pi f_2(n-1)} - \hat{w}(n-1)e^{-j2\pi f_1(n-1)})$
$= (1-\mu)\hat{w}(n-1) + \mu e^{-j2\pi(f_1-f_2)(n-1)}$
We have:
$\hat{w}(n)$
$= \mu[(1-\mu)^{n-1} + (1-\mu)^{n-2}e^{-j2\pi(f_1-f_2)} + \ldots + (1-\mu)e^{-j2\pi(f_1-f_2)(n-2)} + e^{-j2\pi(f_1-f_2)(n-1)}]$
$= \mu\sum_{k=0}^{n-1}(1-\mu)^k e^{-j2\pi(f_1-f_2)(n-1-k)}$

Problem 4

a.
The least perturbation property of ϵ-LMS is:

$\hat{\mathbf{w}}_{NLMS}(n+1) = \arg\min_{\mathbf{w}} |\mathbf{w} - \hat{\mathbf{w}}_{NLMS}(n)|^2$
subject to
$(d(n) - \mathbf{w}^H\mathbf{x}(n)) - (1 - \frac{\tilde{\mu}|\mathbf{x}(n)|^2}{\epsilon + |\mathbf{x}(n)^2|})(d(n) - \mathbf{w}_{NLMS}^H(n)\mathbf{x}(n)) = 0$

b.
The Lagrangian function for the optimization problem in a. is:
$L(\mathbf{w}, \mathbf{w}^*, \lambda)$
$= |\mathbf{w} - \hat{\mathbf{w}}_{NLMS}(n)|^2 + \Re\{\lambda^*(r(n) - \alpha(n)e(n))\}$
$= \mathbf{w}^H(\mathbf{w} - \hat{\mathbf{w}}_{NLMS}(n)) + \frac{\lambda^*}{2}(r(n) - \alpha(n)e(n))$
$- \hat{\mathbf{w}}_{NLMS}(n)^H(\mathbf{w} - \hat{\mathbf{w}}_{NLMS}(n)) + \frac{\lambda}{2}(r^*(n) - \alpha^*(n)e^*(n))$

Taking the Wirtinger derivative and set it to zero:
$\frac{\partial L}{\partial \mathbf{w}^*} = \mathbf{w} - \hat{\mathbf{w}}_{NLMS}(n) - \frac{\lambda^*}{2}\mathbf{x}(n) = 0$

Substiture back to the constraint:
$(d(n) - (\hat{\mathbf{w}}_{NLMS}(n) + \frac{\lambda^*}{2}\mathbf{x}(n))^H\mathbf{x}(n)) - (1 - \frac{\tilde{\mu}|x(n)|^2}{\epsilon + |\mathbf{x}(n)^2|})(d(n) - \mathbf{w}_{NLMS}^H(n)\mathbf{x}(n)) = 0$
$\Rightarrow -\frac{\lambda^*}{2}\mathbf{x}(n)^H\mathbf{x}(n) + (\frac{\tilde{\mu}|\mathbf{x}(n)|^2}{\epsilon + |\mathbf{x}(n)^2|})(d(n) - \mathbf{w}_{NLMS}^H(n)\mathbf{x}(n)) = 0$
$\Rightarrow -\frac{\lambda^*}{2}|\mathbf{x}(n)|^2 + \frac{\tilde{\mu}|\mathbf{x}(n)|^2}{\epsilon + |\mathbf{x}(n)^2|}e(n) = 0$
$\Rightarrow \lambda = \frac{2\tilde{\mu}}{\epsilon + |\mathbf{x}(n)^2|}e(n)$

Therefore,
$\mathbf{w} = \hat{\mathbf{w}}_{NLMS}(n) + \frac{\tilde{\mu}}{\epsilon + |\mathbf{x}(n)^2|}e^*(n)\mathbf{x}(n)$

Problem 5

a.

First, we calculate the impulse response.

$H(z) = \frac{X(z)}{V(z)} = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})} = \frac{9}{5}\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{4}{5}\frac{1}{1+\frac{1}{3}z^{-1}}$

$\Rightarrow h(n) = [\frac{9}{5}(\frac{1}{2})^n - \frac{4}{5}(-\frac{1}{3})^n]u(n)$

Then the cross correlation of x and y is:

$\mathbb{E}[v(n+k)x^*(n)] = \mathbb{E}[v(n+k)\sum_{t=-\infty}^{\infty}\{h^*(t)v^*(n-t)\}]$

$= \mathbb{E}[v(n+k)h^*(-k)v^*(n+k)\}] = h^*(-k)$

From all the equation above, we have:

$r_x(k) = \mathbb{E}[x(n+k)x^*(n)] = \mathbb{E}[\sum_{t=-\infty}^{\infty} h(t)v(n+k-t)x^*(n)]$

$= \sum_{t=-\infty}^{\infty} h(t)\mathbb{E}[v(n+k-t)x^*(n)]$

$= \sum_{t=-\infty}^{\infty} h(t)h^*(-k+t)$

$= \sum_{t=-\infty}^{\infty}[\frac{9}{5}(\frac{1}{2})^t - \frac{4}{5}(-\frac{1}{3})^t]u(t)[\frac{9}{5}(\frac{1}{2})^{t-k} - \frac{4}{5}(-\frac{1}{3})^{t-k}]u(-k+t)$

When $k \leq 0$, the equation can further be reduce to:

$\sum_{t=-\infty}^{\infty}[\frac{9}{5}(\frac{1}{2})^t - \frac{4}{5}(-\frac{1}{3})^t][\frac{9}{5}(\frac{1}{2})^{t-k} - \frac{4}{5}(-\frac{1}{3})^{t-k}]u(t)$

$= \sum_{t=0}^{\infty}[\frac{81}{25}(\frac{1}{2})^{2t-k} - \frac{36}{25}(-\frac{1}{3})^{-k}(-\frac{1}{6})^t - \frac{36}{25}(\frac{1}{2})^{-k}(-\frac{1}{6})^t + \frac{16}{25}(-\frac{1}{3})^{2t-k}]$

$= \frac{108}{25}2^k - \frac{216}{175}(-3)^k - \frac{216}{175}2^k + \frac{18}{25}(-3)^k$

$= \frac{108}{35}2^k - \frac{18}{35}(-3)^k$

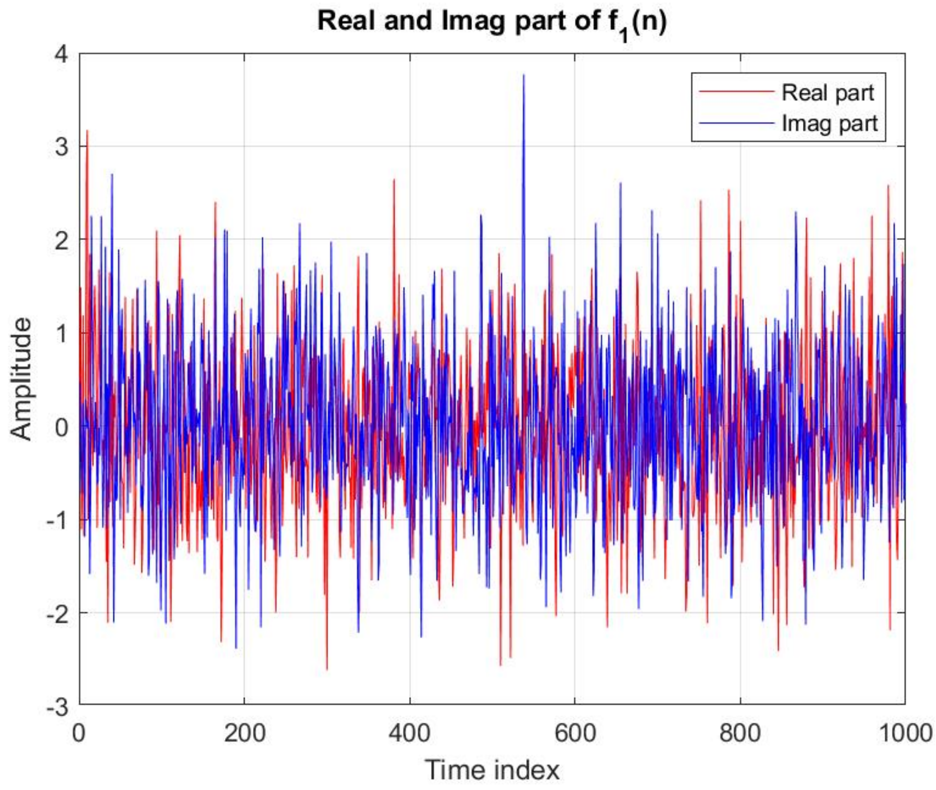
By the symmetric property of autocorrelation function, for $k \geq 0$:

$r_x(k) = r_x^*(-k) = \frac{108}{35}2^{-k} - \frac{18}{35}(-3)^{-k}$

So, $r_x(k) = \frac{108}{35}2^{-|k|} - \frac{18}{35}(-3)^{-|k|}$

b.

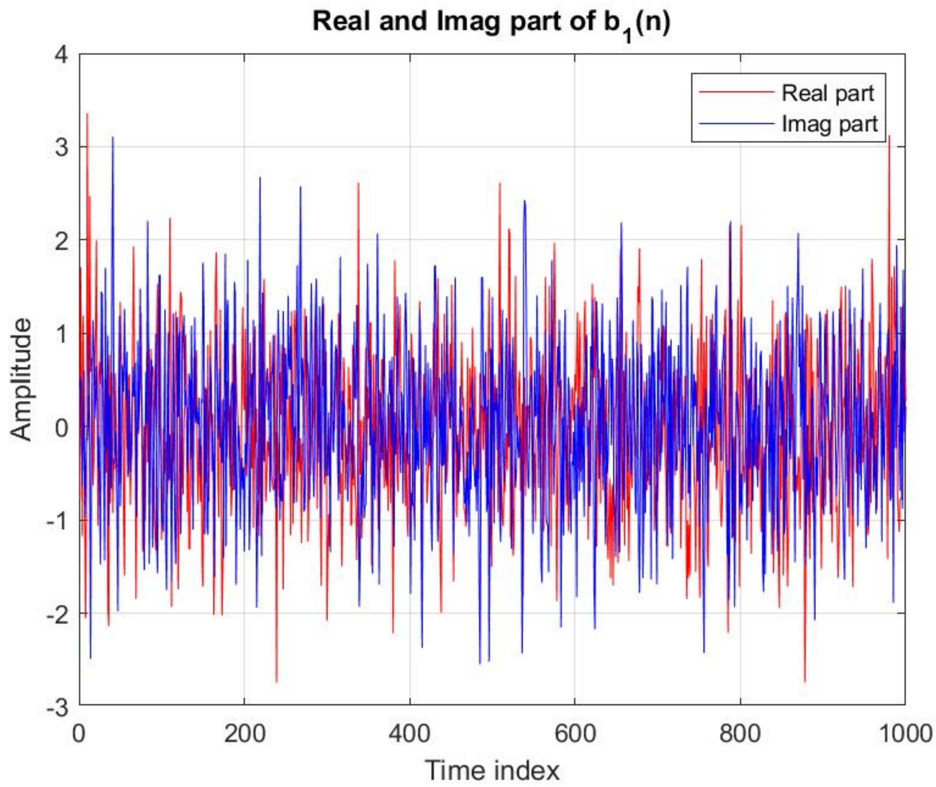
The magnitude of real and imaginary part of $f_1(n)$ is shown in the following figure:



And the average power of $f_1(n)$ is 1.4420.

c.

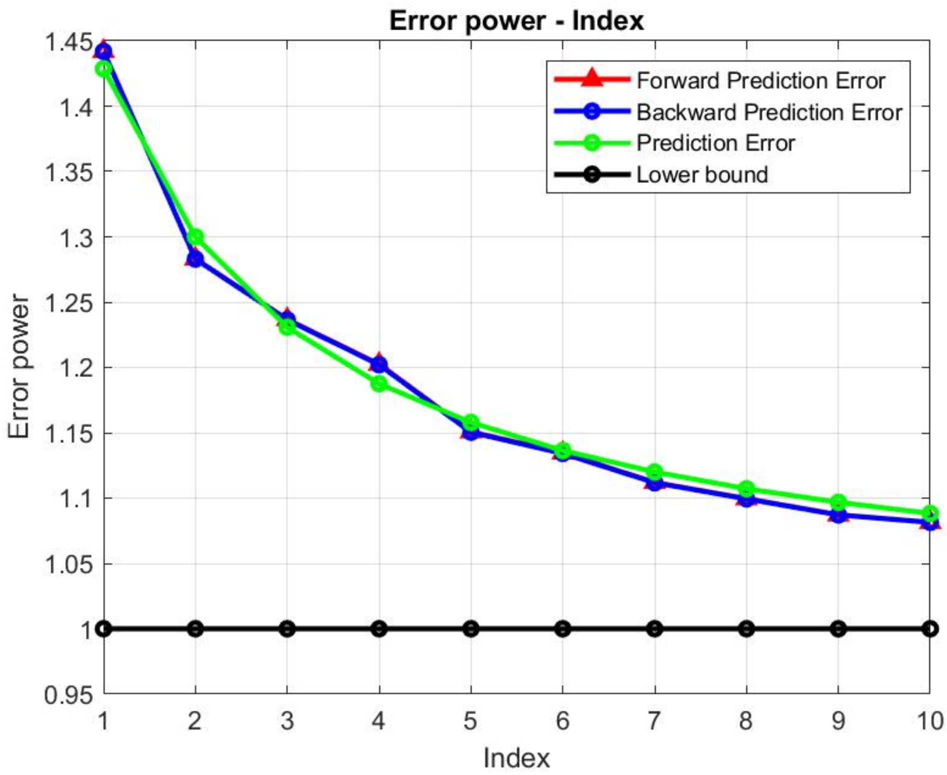
The magnitude of real and imaginary part of $b_1(n)$ is shown in the following figure:



And the average power of $b_1(n)$ is 1.4419.

d.

The prediction error curves is showns in the figure below.



Since the number of samples is 1000 which is really large, the power of forward prediction error and backward prediction error will be so close. We can see this phenomanon in the above figure.

Also, since the prediction error is the expectation of the forward and backward error bounds, there might be fluctuations on these errors.