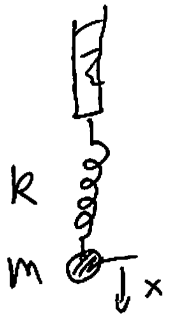
SPRING MOTION — CONTINUATION

4-VII-2021
STEGER

1



$$\boxed{m\ddot{X} = -kX}$$

$\dot{x} = \frac{dx}{dt} = v$ $\dot{v} = \frac{dv}{dt} = a$

$a = \ddot{x}$

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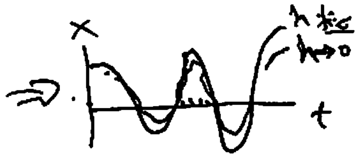
1) $t_0 = 0$ $x(0) \equiv x_0 = 1 \text{ m}$ $m = 1 \text{ kg}$

$v(0) \equiv \dot{x}_0 = 0$ $k = 20 \frac{\text{kg}}{s^2}$

$t_i = ih$ $x_i = x_{i-1} + v_i h$ $i = 1, 2, \dots$

$t_i = t_{i-1} + h$ $v_i = v_{i-1} + a_i h$

$a_i = -\frac{k}{m} x_i$ $\Rightarrow$



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2) ~~More accurate integration~~

$\boxed{\text{Conservation of energy}}$

$\Delta t = h$

$$\dot{x}(t) \equiv \frac{dx}{dt} \simeq \frac{\Delta x}{\Delta t}$$

$$\dot{x}\,(m\ddot{x}) = (-kx)\,\dot{x}$$

$$m(\ddot{x}\dot{x}) = -k(\dot{x}x) \quad (1)$$

~~e.g. $x = 2$, $\Delta x = 0.1$, $\Delta t = 0.1$, $x + \Delta x = 2.1$, $\Delta(x^2) = (x+\Delta x)^2 - x^2$~~

$$\left(\frac{dx}{dt}\right)x = \dot{x}\,x \simeq \frac{\Delta x}{\Delta t}\,x$$

	t	$t+\Delta t$
x	2	2.1
x^2	4	$(2.1)^2$

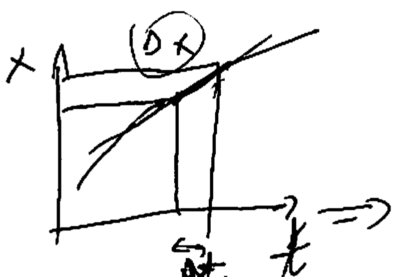
$\Delta x = 2.1 - 2 = 0.1$

$\Delta(x^2) = \underbrace{(2.1)^2}_{=(x+\Delta x)^2} - \underbrace{4}_{=x^2} = 0.41$

$(x+\Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$ $\Delta x \to 0$

$= (x+\Delta x)^2 - x^2 = \underbrace{2x\Delta x}_{=0.4} + \underbrace{\cancel{(\Delta x)^2}}_{=0.01}$

$\simeq 2x\Delta x$



$\Delta(x^2) \simeq 2x\Delta x$

$$\frac{\Delta(x^2)}{\Delta t} \simeq 2x\frac{\Delta x}{\Delta t} \underset{\Delta t \to 0}{\Longrightarrow} \frac{d(x^2)}{dt} = 2x\overbrace{\frac{dx}{dt}}^{=\dot{x}} \quad (2)$$

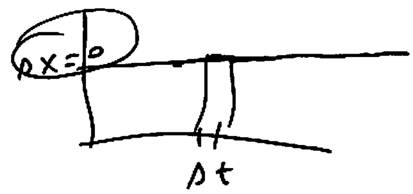
(1) & (2) $\longrightarrow$ $$m\frac{1}{2}\frac{d(\dot{x})^2}{dt} = -k\,\frac{1}{2}\,\frac{d(x^2)}{dt}$$

$$\dot{x}\,x = \frac{1}{2}\frac{dx^2}{dt}$$

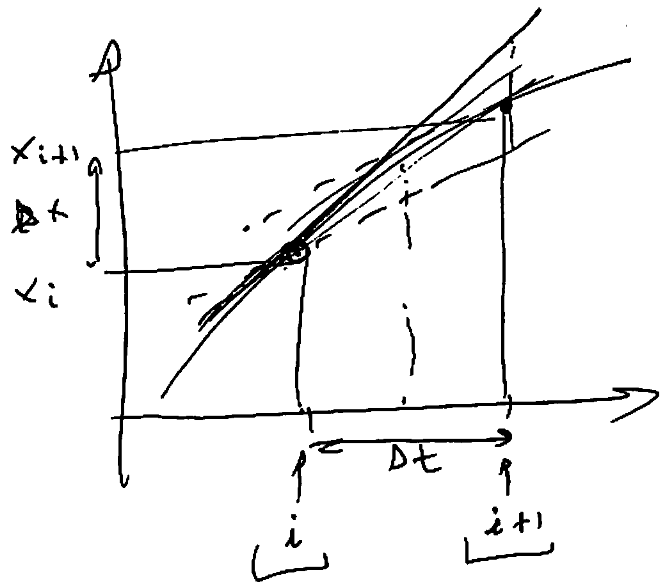
$$\ddot{x}\,\dot{x} = \frac{1}{2}\frac{d\dot{x}^2}{dt}$$

$$\dot{x} \equiv v \equiv \frac{dx}{dt}$$

$$\frac{d}{dt}\Big[\underbrace{\frac{1}{2}m\dot{x}^2 + \frac{k}{2}x^2}_{\Large E}\Big] = 0$$



$$E = \underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic energy}} + \underbrace{\frac{kx^2}{2}}_{\text{potential energy}}$$



$h = \Delta t$

$t_i \longrightarrow \underline{x_i\ ,\ \dot{x}_i}$

(4) $x_{i+1} = x_i + v\,h$

(3) $\dot{x}_{i+1} = \dot{x}_i + a\,h$

$$v = \frac{\dot{x}_i + \dot{x}_{i+1}}{2}$$

$$a = \frac{\ddot{x}_i + \ddot{x}_{i+1}}{2}$$

$$\ddot{x}_i = -\frac{k}{m}x_i$$

$$\ddot{x}_{i+1} = -\frac{k}{m}x_{i+1}$$

$$a = -\frac{k}{m}\,\frac{x_i + x_{i+1}}{2}$$

$$(3) \rightarrow \quad \dot{x}_{i+1} = \dot{x}_i + \overbrace{\left(\frac{-k}{m}\right) \frac{(x_i + x_{i+1})}{2}}^{a} h$$

$$(4) \rightarrow \quad x_{i+1} = x_i + \frac{\dot{x}_i + \dot{x}_{i+1}}{2} h$$

$$\dot{x}_{i+1} + \frac{kh}{2m} x_{i+1} = \dot{x}_i - \frac{kh}{2m} x_i \qquad (5)$$

$$\underbrace{-\frac{h}{2}\dot{x}_{i+1} + x_{i+1}}_{\text{at time } t_{i+1} = t_i + h} = \underbrace{x_i + \frac{h}{2}\dot{x}_i}_{\text{at point } i \text{ at time } t_i} \qquad (6)$$

$\frac{h}{2}(5) + (6)$

$$\hookrightarrow \left(1 + \frac{kh^2}{4m}\right) x_{i+1} = x_i \left(1 - \frac{kh^2}{4m}\right) + h\dot{x}_i$$

$$\hookrightarrow \boxed{x_{i+1} = x_i \left(\frac{1 - \frac{kh^2}{4m}}{1 + \frac{kh^2}{4m}}\right) + \frac{h \dot{x}_i}{\left(1 + \frac{kh^2}{4m}\right)}}$$

$(5) + \left(\frac{-kh}{2m}\right)(6)$

$$\hookrightarrow \left(1 + \frac{kh^2}{4m}\right) \dot{x}_{i+1} = -\frac{kh}{m} x_i + \left(1 - \frac{kh^2}{4m}\right) \dot{x}_i$$

$$\hookrightarrow \boxed{\dot{x}_{i+1} = \frac{-(kh/m)}{\left(1 + \frac{kh^2}{4m}\right)} x_i + \left(\frac{1 - \frac{kh^2}{4m}}{1 + \frac{kh^2}{4m}}\right) \dot{x}_i}$$

$t_i \rightarrow x_i, \dot{x}_i = v_i \quad \Rightarrow \quad t_{i+1} = t_i + h$

x_{i+1}

$\dot{x}_{i+1} = v_{i+1}$

TASKS

1) start progress report

$$E_i = \frac{1}{2} m \dot{x}_i^2 + \frac{k}{2} x_i^2$$

2)

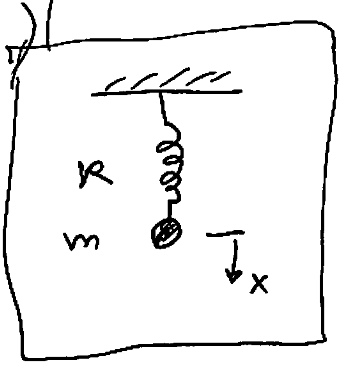
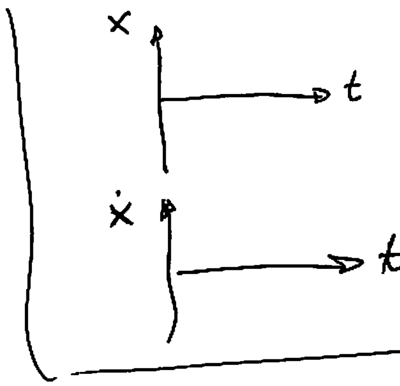


Fig. 1



E

method 1

method 2

3)

- limits
- derivatives

$$\lim_{x \to 0} f(x) =$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\left(\frac{dx}{dt} \equiv \dot{x}\right)$

$$(f(x)\, g(x))' = f'(x)\, g(x) + f(x)\, g'(x)$$

$$(f(x) + g(x))' = f'(x) + g(x)$$

a = constant

$$(a)' = 0$$

$$(x)' = 1$$

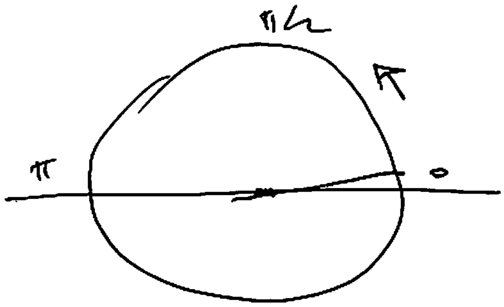
$$(x^n)' = n\, x^{n-1}$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

- $\sin(a+b) = \sin a \cos b + \sin b \cos a$

  $\cos(a+b) = \cos a \cos b - \sin a \sin b$

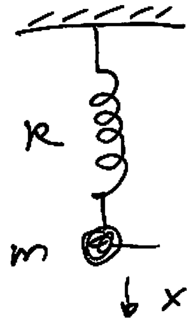


radians

$\sin x \simeq x$
$x \to 0$

$\cos x \simeq 1$

4)



$$m \ddot{x} = -kx - f\dot{x}$$