

# Assignment I: Reducing the variance of Monte Carlo Estimators.

Computational Statistics  
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## Background

Suppose we are interested in an integrable function  $\varphi(\cdot)$  and its expectation under a probability distribution with law  $\pi$ ,  $I = \int_{\mathbb{X}} \varphi(x)\pi(x)dx$ . Monte Carlo is a large class of sampling algorithms to approximate integrals. It depends on constructing an estimator

$$\hat{I}_n = \frac{1}{n} \sum_{i=1}^n \varphi(X_i), \quad (1)$$

where  $X_i \sim \pi$ . The estimator in (1) can be improved in a number of ways. In particular, the procedure known as *Rao-Blackwellisation* allows one to obtain estimators that have lower variance with the same bias, i.e., are more efficient.

## Questions

*Hint:* I suggest you employ  $\varphi(x) = x^2$ .

1. Show that, if  $\pi(x) = \int_{\mathbb{Y}} g(x | y)h(y) dy$ , then both

$$\begin{aligned} \hat{I}_n^{\text{MC}} &= \frac{1}{n} \sum_{i=1}^n \varphi(X_i), \quad X_i \sim \pi, \text{ and} \\ \hat{I}_n^{\text{RB}} &= \frac{1}{n} \sum_{i=1}^n E[\varphi(X) | Y_i], \quad Y_i \sim h, \end{aligned}$$

converge to  $E_{\pi}[\varphi(X)]$ .

2. For each of the following cases, generate random variables  $X_i$  and  $Y_i$  and compute the average and variance of the “vanilla” Monte Carlo and Rao-Blackwellised estimators:
  - (a)  $X | Y \sim \text{Poisson}(Y)$ ,  $Y \sim \text{Gamma}(a, b) \implies X \sim \text{Negative-Binomial}$ ;
  - (b)  $X | Y \sim \text{Normal}(0, Y)$ ,  $Y \sim \text{Gamma}(a, b) \implies X \sim \text{Generalised-t}$ ;
  - (c)  $X | Y \sim \text{Binomial}(n, Y)$ ,  $Y \sim \text{Beta}(a, b) \implies X \sim \text{Beta-Binomial}$ ;
3. Propose a Rejection Control (see Question 5 in exercise sheet 1) algorithm to sample from each of the targets in the previous question;

4. Compute (empirically) the  $n_{\text{eff}}$  for the results in the previous question. How does the algorithm perform in each case? Do you have to fiddle with the value of  $c > 0$  in order to obtain better performance? Discuss.
5. Compare the variance of the Rejection Control estimate with the variance of the “vanilla” Monte Carlo estimator and the Rao-Blackwellised estimator;
6. What other improvements (in terms of variance and bias) would you propose? Can you come up with an algorithm that dominates all algorithms studied here for the targets under consideration?