Lecture 11: Fast Reinforcement Learning [1]

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CS234 Reinforcement Learning

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[1] With many slides from or derived from David Silver, Examples new

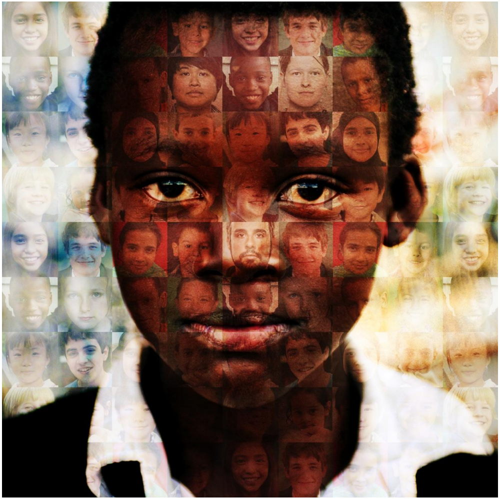
Class Structure

- Last time: Midterm
- **This time: Fast Learning**
- Next time: Fast Learning

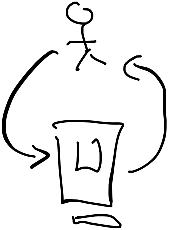
Up Till Now

- Discussed optimization, generalization, delayed consequences

Teach Computers to Help Us



education
healthcare
consumer
marketing



Computational Efficiency and Sample Efficiency

Q learning

Computational Efficiency	Sample Efficiency
driving car at 60mph	experience costly/hard to gather
simulators	- education - students
	- patients
	- customers

Algorithms Seen So Far

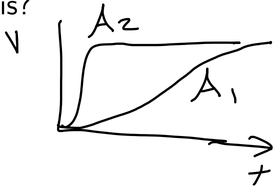
the algorithms we've seen so far are not sample efficient!

- How many steps did it take for DQN to learn a good policy for pong?

  5 millions

Evaluation Criteria

- How do we evaluate how "good" an algorithm is?
- If converges? $t \to \infty$
- If converges to optimal policy? $t \to \infty$
- How quickly reaches optimal policy?
- Mistakes make along the way?
- Will introduce different measures to evaluate RL algorithms



Settings, Frameworks & Approaches

- Over next couple lectures will consider 2 settings, multiple frameworks, and approaches
- Settings: Bandits (single decisions), MDPs tabular
- Frameworks: evaluation criteria for formally assessing the quality of a RL algorithm
- Approaches: Classes of algorithms for achieving particular evaluation criteria in a certain set
- Note: We will see that some approaches can achieve multiple frameworks in multiple settings

Today

regret is just a tool to analyze our algorithm. We don't actually use it in the algorithms themselves to affect which actions we take. The algos also don't compute regret.

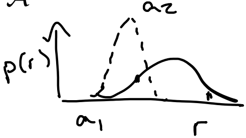
- Setting: Introduction to multi-armed bandits
- Framework: Regret
- Approach: Optimism under uncertainty
- Framework: Bayesian regret
- Approach: Probability matching / Thompson sampling

Multiarmed Bandits

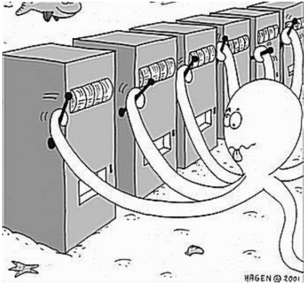
- Multi-armed bandit is a tuple of $(\mathcal{A}, \mathcal{R})$
- $\mathcal{A}$: known set of $\underline{m}$ actions (arms)

for each of the arms there is a
distribution of rewards you could get

- $\mathcal{R}^a(r) = \mathbb{P}[r \mid a]$ is an unknown probability distribution over rewards
- At each step t the agent selects an action $a_t \in \mathcal{A}$
- The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
- Goal: Maximize cumulative reward $\sum_{\tau=1}^{t} r_\tau$



there's no state or transition function.
For each timestep you take an action
(pull an arm) and observe a reward
(which is sampled from that arm's
reward distribution). Our goal is to
maximize cumulative reward



Regret

- **Action-value** is the mean reward for action a

the true expected reward is unknown to the agent. we learn this through observing samples

$$\underline{Q(a)} = \mathbb{E}[r \mid a]$$

- **Optimal value** V^*

unknown to the agent

$$V^* = Q(a^*) = \underbrace{\max_{a \in \mathcal{A}} Q(a)}$$

arm select

- **Regret** is the opportunity loss for one step

$$Q(a^*) - Q(a_t)$$

$$l_t = \mathbb{E}[V^* - Q(a_t)]$$

- **Total Regret** is the total opportunity loss

unknown

unknown

so this isn't something we can evaluate unless we're in a simulated domain

$$L_t = \mathbb{E}[\sum_{\tau=1}^{t} V^* - Q(a_\tau)]$$

$$r_t \sim P(r \mid a_t)$$

- Maximize cumulative reward $\iff$ minimize total regret

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = \underbrace{V^* - Q(a_i)}$ $\Delta_{a^*} = 0$
- Regret is a function of gaps and counts

$$
\begin{aligned}
L_t &= \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right] \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)](V^* - Q(a)) \\
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a
\end{aligned}
$$

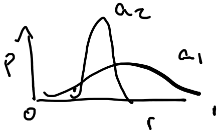
- A good algorithm ensures small counts for large gap, but gaps are not known

the algorithm will reduce the number of times you
pull arms with large gaps

- We consider algorithms that estimate $\hat{Q}_t(a) \approx Q(a)$
- Estimate the value of each action by Monte-Carlo evaluation

just average the rewards we've seen so far for an action

$$\hat{Q}_t(a) = \frac{1}{N_t(a)} \sum_{t=1}^{T} r_t \mathbb{1}(\underline{a_t = a})$$



- The **greedy** algorithm selects action with highest value

.2 a1
.5 a2

$$a_t^* = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$$

- Greedy can lock onto suboptimal action, forever

in this case, if we got unlucky and sampled 0.2 for a1 and 0.5 for a2, then we will forever sample a2 even though its true mean is lower than that for a1

ϵ-Greedy Algorithm

- The ϵ-**greedy** algorithm proceeds as follows:
  - With probability $1 - \epsilon$ select $a_t = \arg\max_{a \in \mathcal{A}} \hat{Q}_t(a)$
  - With probability ϵ select a random action
- Always will be making a sub-optimal decision ϵ fraction of the time
  $(1 - 1/m)$
- Already used this in prior homeworks

Toy Example: Ways to Treat Broken Toes[1]

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 possible options: (1) surgery (2) buddy taping the broken toe with another toe, (3) do nothing
- Outcome measure / reward is binary variable: whether the toe has healed (+1) or not healed (0) after 6 weeks, as assessed by x-ray

[1] Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes[1]

- Consider deciding how to best treat patients with broken toes
- Imagine have 3 common options: (1) surgery (2) surgical boot (3) buddy taping the broken toe with another toe
- Outcome measure is binary variable: whether the toe has healed (+1) or not (0) after 6 weeks, as assessed by x-ray
- Model as a multi-armed bandit with 3 arms, where each arm is a Bernoulli variable with an unknown parameter θ_i
- Check your understanding: what does a pull of an arm / taking an action correspond to? Why is it reasonable to model this as a multi-armed bandit instead of a Markov decision process?

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Toy Example: Ways to Treat Broken Toes[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$

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Toy Example: Ways to Treat Broken Toes, Greedy[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
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  - doing nothing: $Q(a^3) = \theta_3 = .1$
- Greedy
  1. Sample each arm once
     - Take action a^1 ($r \sim$Bernoulli(0.95)), get $+1$, $\hat{Q}(a^1) = 1$
     - Take action a^2 ($r \sim$Bernoulli(0.90)), get $+1$, $\hat{Q}(a^2) = 1$
     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. What is the probability of greedy selecting each arm next? Assume ties are split uniformly.

     $p(a_1) = p(a_2) = 1/2$

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Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret of Greedy

- True (unknown) Bernoulli reward parameters for each arm (action) are
  - surgery: $Q(a^1) = \theta_1 = .95$ ← optimal
  - buddy taping: $Q(a^2) = \theta_2 = .9$
  - doing nothing: $Q(a^3) = \theta_3 = .1$

$\underbrace{Q(a^*)}_{V^*} - Q(a_t)$

- Greedy

r	Action	Optimal Action	Regret	
1	a^1	a^1	0	init
1	a^2	a^1	$.95 - .9 = .05$	init
0	a^3	a^1	$.95 - .1 = .85$	init
	a^1	a^1	0	
	a^2	a^1	.05	

- Will greedy ever select a^3 again? no If yes, why? If not, is this a problem?

our current estimate for a3=0, and since we've already sampled 1 from a1 and a2, they will never go to 0. So we will never sample a3 again...

Toy Example: Ways to Treat Broken Toes, ϵ-**Greedy**[1]

- Imagine true (unknown) Bernoulli reward parameters for each arm (action) are
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     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Let $\epsilon = 0.1$
  3. What is the probability ϵ-greedy will pull each arm next? Assume ties are split uniformly.

$\frac{\epsilon}{3}$: a_1 or a_2 or a_3

$\frac{1-\epsilon}{2}$ a_1 or a_2

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- UCB1 (Auer, Cesa-Bianchi, Fischer 2002)

Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

- Will ϵ-greedy ever select a^3 again? If ϵ is fixed, how many times will each arm be selected? yes

Check Your Understanding

- **Count** $N_t(a)$ is expected number of selections for action a
- **Gap** Δ_a is the difference in value between action a and optimal action a^*, $\Delta_i = V^* - Q(a_i)$
- Regret is a function of gaps and counts

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L_t &= \mathbb{E}\left[\sum_{\tau=1}^{t} V^* - Q(a_\tau)\right] \\
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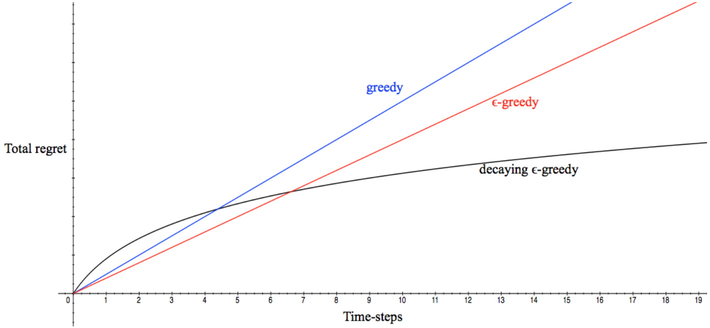
&= \sum_{a \in \mathcal{A}} \mathbb{E}[N_t(a)]\Delta_a
\end{aligned}
$$

- A good algorithm ensures small counts for large gap, but gaps are not known
- Check your understanding: Does fixed $\epsilon = 0.1$ greedy have large regret ?

"Good": Sublinear or below regret



- **Explore forever**: have linear total regret T
- **Explore never**: have linear total regret
- Is it possible to achieve sublinear regret?

Types of Regret bounds

most MDPs are
this

- **Problem independent**: Bound how regret grows as a function of T, the total number of time steps the algorithm operates for
- **Problem dependent**: Bound regret as a function of the number of times we pull each arm and the gap between the reward for the pulled arm a^*

most bandits are
this

a_1	a_2
$E[r]$	$E[r]$
1	.001
.53	.525

Lower Bound

- Use lower bound to determine how hard this problem is
- The performance of any algorithm is determined by similarity between optimal arm and other arms
- Hard problems have similar looking arms with different means
- This is described formally by the gap Δ_a and the similarity in distributions $D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})$ KL div
- Theorem (Lai and Robbins): Asymptotic total regret is at least logarithmic in number of steps

regret

$$\lim_{t\to\infty} L_t \geq \underbrace{\log t}_{} \sum_{a|\Delta_a>0} \frac{\Delta_a}{D_{KL}(\mathcal{R}^a \| \mathcal{R}^{a^*})}$$

- Promising in that lower bound is sublinear

Framework: Regret
Setting: Bandits
Approach: Optimism!

Kaelbling
1993

- Choose actions that that might have a high value
- Why?
- Two outcomes: a_1
  1) a_1 has high reward
  2) ~~a_1 has~~ $r(a_1)$ with low reward
  information

the a1 really is great, or a1 is not great, but we learn something about the world, i.e. that a1 does not have high reward

Upper Confidence Bounds

- Estimate an upper confidence $U_t(a)$ for each action value, such that $Q(a) \leq U_t(a)$ with high probability
- This depends on the number of times $N_t(a)$ action a has been selected
- Select action maximizing Upper Confidence Bound (UCB)

$$a_t = \arg\max_{a \in \mathcal{A}} [U_t(a)]$$

UCB init phase: pull each arm once, compute $U_t(a_t)$ $\forall a$

for $t = 1 \ldots$

$a_t = \arg\max_a U_t(a)$

$r \sim$ R reward distrib a_t

Update $U_t(a_t)$ & all other arms

Hoeffding's Inequality

if confidence bounds hold

$$U_t(a_t) = \hat{Q}(a_t) + \sqrt{\frac{1}{2n(a_t)} \log(t^2/\delta)}$$

$$> Q(a_t)$$

- Theorem (Hoeffding's Inequality): Let $X_1, \ldots, X_n$ be i.i.d. random variables in $[0, 1]$, and let $\bar{X}_n = \frac{1}{n}\sum_{\tau=1}^{n} X_\tau$ be the sample mean. Then

constant

$$\mathbb{P}\left[\underbrace{\mathbb{E}[X]}_{\text{true mean}} > \underbrace{\bar{X}_n}_{\text{empirical mean}} + u\right] \leq \exp(-2nu^2) = \delta/t^2$$

samples

$$\exp(-2nu^2) = \delta/t^2$$

$$u = \sqrt{\frac{1}{2n} \log(t^2/\delta)}$$

$$\bar{X}_n + u \geq E[X] \quad \text{w.prob} \geq 1 - \delta/t^2$$

$$U_t(a_t) = \hat{Q}(a_t) + \sqrt{1/2n(a_t) \cdot \log(t^2/\delta)}$$

High Probability Regret Bound for UCB Multi-armed Bandit

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High Probability Regret Bound for UCB Multi-armed Bandit

$$Regret(UCB, T) = \sum_{t=1}^{T} Q(a^*) - Q(a_t)$$

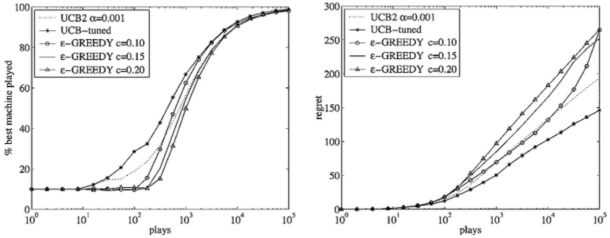
UCB Bandit Regret

- This leads to the UCB1 algorithm

$$a_t = \arg\max_{a \in \mathcal{A}} [\hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}]$$

- Theorem: The UCB algorithm achieves logarithmic asymptotic total regret

$$\lim_{t \to \infty} L_t \leq 8 \log t \sum_{a | \Delta_a > 0} \Delta_a$$



Toy Example: Ways to Treat Broken Toes, Thompson Sampling[1]

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- Optimism under uncertainty, UCB1 (Auer, Cesa-Bianchi, Fischer 2002)
  1. Sample each arm once

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     - Take action a^3 ($r \sim$Bernoulli(0.1)), get 0, $\hat{Q}(a^3) = 0$
  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

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  3. $t = 3$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

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  2. Set $t = 3$, Compute upper confidence bound on each action

$$UCB(a) = \hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}$$

  3. $t = t + 1$, Select action $a_t = \arg\max_a UCB(a)$,
  4. Observe reward 1
  5. Compute upper confidence bound on each action

Toy Example: Ways to Treat Broken Toes, Optimism, Assessing Regret

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Action	Optimal Action	Regret
a^1	a^1	
a^2	a^1	
a^3	a^1	
a^1	a^1	
a^2	a^1	

Check Your Understanding

- An alternative would be to always select the arm with the highest lower bound
- Why can this yield linear regret?
- Consider a two arm case for simplicity

Bayesian Bandits

- So far we have made no assumptions about the reward distribution $\mathcal{R}$
  - Except bounds on rewards
- **Bayesian bandits** exploit prior knowledge of rewards, $p[\mathcal{R}]$
- They compute posterior distribution of rewards $p[\mathcal{R} \mid h_t]$, where $h_t = (a_1, r_1, \ldots, a_{t-1}, r_{t-1})$
- Use posterior to guide exploration
  - Upper confidence bounds (Bayesian UCB)
  - Probability matching (Thompson Sampling)
- Better performance if prior knowledge is accurate

Regret and Bayesian Regret

- Frequentist regret assumes a true (unknown) set of parameters

$$Regret(\mathcal{A}, T; \theta) = \sum_{t=1}^{T} \mathbb{E}\left[Q(a^*) - Q(a_t) \leq \sum_{t=1}^{T} U_t(a_t) - Q(a_t) | \theta \right]$$

- Bayesian regret assumes there is a prior over parameters

$$BayesRegret(\mathcal{A}, T; \theta) =$$

$$\mathbb{E}_{\theta \sim p_\theta}\left[\sum_{t=1}^{T} \mathbb{E}\left[Q(a^*) - Q(a_t) \leq \sum_{t=1}^{T} U_t(a_t) - Q(a_t) | \theta \right] \right]$$

- *Note: Bayes regret and regret can be related using Markov inequality

Bayesian UCB Example: Independent Gaussians

- Assume reward distribution is Gaussian, $\mathcal{R}_a(r) = \mathcal{N}(r; \mu_a, \sigma_a^2)$
- Compute Gaussian posterior over μ_a and σ_a^2 (by Bayes law)

$$p[\mu_a, \sigma_a^2 \mid h_t] \propto p[\mu_a, \sigma_a^2] \prod_{t \mid a_t = a} \mathcal{N}(r_t; \mu_a, \sigma_a^2)$$

- Pick action that maximizes standard deviation of $Q(a)$

$$a_t = \arg\max_{a \in \mathcal{A}} \mu_a + c \frac{c\sigma_a}{\sqrt{N(a)}}$$

Probability Matching

- Assume have a parametric distribution over rewards for each arm
- **Probability matching** selects action a according to probability that a is the optimal action

$$\pi(a \mid h_t) = \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t]$$

- Probability matching is optimistic in the face of uncertainty
  - Uncertain actions have higher probability of being max
- Can be difficult to compute analytically from posterior

Thompson sampling implements probability matching

- Thompson sampling:

$$\begin{aligned}\pi(a \mid h_t) &= \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t] \\ &= \mathbb{E}_{\mathcal{R}|h_t}\left[\mathbb{1}(a = \arg\max_{a \in \mathcal{A}} Q(a))\right]\end{aligned}$$

Thompson sampling implements probability matching

- Thompson sampling:

$$\begin{aligned}\pi(a \mid h_t) &= \mathbb{P}[Q(a) > Q(a'), \forall a' \neq a \mid h_t] \\ &= \mathbb{E}_{\mathcal{R}|h_t}\left[\mathbb{1}(a = \arg\max_{a \in \mathcal{A}} Q(a))\right]\end{aligned}$$

- Use Bayes law to compute posterior distribution $p[\mathcal{R} \mid h_t]$
- **Sample** a reward distribution $\mathcal{R}$ from posterior
- Compute action-value function $Q(a) = \mathbb{E}[\mathcal{R}_a]$
- Select action maximizing value on sample, $a_t = \arg\max_{a \in \mathcal{A}} Q(a)$
- Update posterior

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose Beta(1,1) (Uniform)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1):

Toy Example: Ways to Treat Broken Toes, Thompson Sampling[2]

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) =$

[2]Note:This is a made up example. This is not the actual expected efficacies of the various treatment options for a broken toe

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Per arm, sample a Bernoulli θ given prior: 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0
  4. Update the posterior over the $Q(a_t) = Q(a^3)$ value for the arm pulled

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 3$
  3. Observe the patient outcome's outcome: 0
  4. Update the posterior over the $Q(a_t) = Q(a^1)$ value for the arm pulled
     - Beta(c_1, c_2) is the conjugate distribution for Bernoulli
     - If observe 1, $c_1 + 1$ else if observe 0 $c_2 + 1$
  5. New posterior over Q value for arm pulled is:
  6. New posterior $p(Q(a^3)) = p(\theta(a_3) = Beta(1, 2)$

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

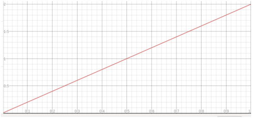
- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,1): 0.3 0.5 0.6
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 0
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(1,2)$

Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose θ_i ~Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3

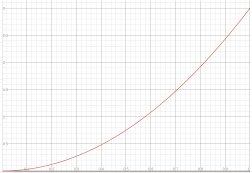
Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(1,1), Beta(1,1), Beta(1,2): 0.7, 0.5, 0.3
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(2, 1)$



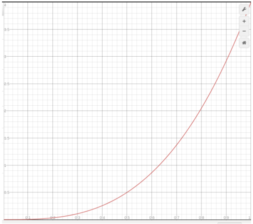
Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(2,1), Beta(1,1), Beta(1,2): 0.71, 0.65, 0.1
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(3,1)$



Toy Example: Ways to Treat Broken Toes, Thompson Sampling

- True (unknown) Bernoulli parameters for each arm/action
- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Thompson sampling:
- Place a prior over each arm's parameter. Here choose $\theta_i \sim$Beta(1,1)
  1. Sample a Bernoulli parameter given current prior over each arm Beta(2,1), Beta(1,1), Beta(1,2): 0.75, 0.45, 0.4
  2. Select $a_t = \arg\max_{a \in A} Q(a) = \arg\max_{ainA} \theta(a) = 1$
  3. Observe the patient outcome's outcome: 1
  4. New posterior $p(Q(a^1)) = p(\theta(a_1) = Beta(4, 1)$



Toy Example: Ways to Treat Broken Toes, Thompson Sampling vs Optimism

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- How does the sequence of arm pulls compare in this example so far?

Optimism	TS	Optimal	Regret Optimism	Regret TS
a^1	a^3			
a^2	a^1			
a^3	a^1			
a^1	a^1			
a^2	a^1			

Toy Example: Ways to Treat Broken Toes, Thompson Sampling vs Optimism

- Surgery: $\theta_1 = .95$ / Taping: $\theta_2 = .9$ / Nothing: $\theta_3 = .1$
- Incurred regret?

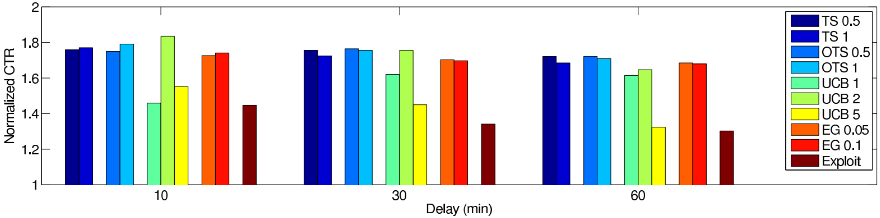
Optimism	TS	Optimal	Regret Optimism	Regret TS
a^1	a^3	a^1	0	0
a^2	a^1	a^1	0.05	
a^3	a^1	a^1	0.85	
a^1	a^1	a^1	0	
a^2	a^1	a^1	0.05	

Thompson sampling implements probability matching

- Thompson sampling(1929) achieves Lai and Robbins lower bound
- Bounds for optimism are tighter than for Thompson sampling
- But empirically Thompson sampling can be extremely effective

Thompson Sampling for News Article Recommendation (Chapelle and Li, 2010)

- Contextual bandit: input context which impacts reward of each arm, context sampled iid each step
- Arms = articles
- Reward = click (+1) on article ($Q(a)$=click through rate)



Bayesian Regret Bounds for Thompson Sampling

- Regret(UCB,T)

$$BayesRegret(TS, T) = E_{\theta \sim p_\theta}\left[\sum_{t=1}^{T} f^*(a^*) - f^*(a_t)\right]$$

- Posterior sampling has the same (ignoring constants) regret bounds as UCB

Optimistic Initialization

- Simple and practical idea: initialize Q(a) to high value
- Update action value by incremental Monte-Carlo evaluation
- Starting with $N(a) > 0$

$$\hat{Q}_t(a_t) = \hat{Q}_{t-1} + \frac{1}{N_t(a_t)}(r_t - \hat{Q}_{t-1})$$

- Encourages systematic exploration early on
- But can still lock onto suboptimal action
- Depends on how high initialize Q
- Check your understanding: What is the downside to initializing Q too high?

Greedy Bandit Algorithms and Optimistic Initialization

- **Greedy**: Linear total regret
- **Constant ϵ-greedy**: Linear total regret
- **Decaying ϵ-greedy**: Sublinear regret but schedule for decaying ϵ requires knowledge of gaps, which are unknown
- **Optimistic initialization**: Sublinear regret if initialize values sufficiently optimistically, else linear regret
- Check your understanding: why does fixed ϵ-greedy have linear regret? (Do a proof sketch)

Consider Montezuma's revenge

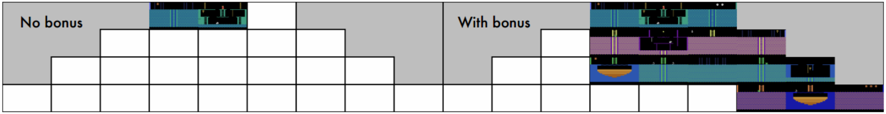


Figure 3: "Known world" of a DQN agent trained for 50 million frames with (**right**) and without (**left**) count-based exploration bonuses, in MONTEZUMA'S REVENGE.

- EB: move this to generalization and efficiency later on
- Bellemare et al. "Unifying Count-Based Exploration and Intrinsic Motivation"
- Enormously better than standard DQN with ϵ-greedy approach
- Uses principle of optimism under uncertainty which we will see today

Calculating UCB

- Pick a probability p that true value exceeds UCB
- Now solve for $U_t(a)$

$$\exp(-2N_t(a)U_t(a)^2) = p$$

$$U_t(a) = \sqrt{\frac{-\log p}{2N_t(a)}}$$

- Reduce p as we observe more rewards, e.q. $p = t^{-4}$
- Ensures we select optimal action as $t \to \infty$

$$U_t(a) = \sqrt{\frac{2\log t}{N_t(a)}}$$

UCB1

- This leads to the UCB1 algorithm

$$a_t = \arg\max_{a \in \mathcal{A}} [\hat{Q}(a) + \sqrt{\frac{2 \log t}{N_t(a)}}]$$