Motivating Results

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1 Result 1 (static)

The following result shows that the fixed point residual converges linearly fast for contractive operators, but sub-linearly (with the square root of the iteration number) for averaged operators.

Proposition 1.1. *Consider an operator $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ and the Banach-Picard iteration*

$$\boldsymbol{x}^{\ell+1} = \mathcal{T}\boldsymbol{x}^{\ell}, \quad \ell \in \mathbb{N}.$$

Then, the fixed point residual $\{\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^{\ell}\|\}$ is bounded as:

- $\mathcal{T}$ ζ-contractive*:* $\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^{\ell}\| \leq (1+\zeta)\zeta^{\ell} \|\boldsymbol{x}^0 - \boldsymbol{x}^*\|$*;*
- $\mathcal{T}$ α-averaged*:* $\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^{\ell}\| \leq \frac{1}{\sqrt{\ell+1}}\sqrt{\frac{\alpha}{1-\alpha}} \|\boldsymbol{x}^0 - \boldsymbol{x}^*\|$.

Proof. $\mathcal{T}$ *contractive*: by [1, Theorem 1.50] we know that

$$\left\|\boldsymbol{x}^{\ell} - \boldsymbol{x}^*\right\| \leq \zeta^{\ell} \left\|\boldsymbol{x}^0 - \boldsymbol{x}^*\right\|;$$

using the triangle inequality we then have

$$\begin{aligned}
\left\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^{\ell}\right\| &\leq \left\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^*\right\| + \left\|\boldsymbol{x}^{\ell} - \boldsymbol{x}^*\right\| \\
&\leq \zeta \left\|\boldsymbol{x}^{\ell} - \boldsymbol{x}^*\right\| + \left\|\boldsymbol{x}^{\ell} - \boldsymbol{x}^*\right\| \\
&\leq (1+\zeta) \left\|\boldsymbol{x}^{\ell} - \boldsymbol{x}^*\right\| \\
&\leq (1+\zeta)\zeta^{\ell} \left\|\boldsymbol{x}^0 - \boldsymbol{x}^*\right\|.
\end{aligned}$$

$\mathcal{T}$ *averaged*: by the averagedness of $\mathcal{T}$ we can write, for some $\boldsymbol{x}^* \in \operatorname{fix}(\mathcal{T})$:

$$\left\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^*\right\|^2 \leq \left\|\boldsymbol{x}^{\ell} - \boldsymbol{x}^*\right\|^2 - \frac{1-\alpha}{\alpha} \left\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^{\ell}\right\|^2;$$

rearranging, and summing over time we get

$$\sum_{h=0}^{\ell} \left\|\boldsymbol{x}^{h+1} - \boldsymbol{x}^{h}\right\|^2 \leq \frac{\alpha}{1-\alpha} \left\|\boldsymbol{x}^0 - \boldsymbol{x}^*\right\|^2$$

where we used the telescoping sum and removed the negative term $-\left\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^{\ell}\right\|^2$.

Since $\{\left\|\boldsymbol{x}^{h+1} - \boldsymbol{x}^{h}\right\|^2\}$ is a monotonically decreasing sequence, we know that $(\ell+1)\left\|\boldsymbol{x}^{\ell+1} - \boldsymbol{x}^{h}\right\| \leq \sum_{h=0}^{\ell} \left\|\boldsymbol{x}^{h+1} - \boldsymbol{x}^{h}\right\|^2$; using this fact, rerranging and taking the square root yields the thesis. □

2 Result 2 (dynamic)

Consider the time-varying Banach-Picard

$$\boldsymbol{x}_{k+1} = \mathcal{T}_{k+1}\boldsymbol{x}_k, \quad k \in \mathbb{N},$$

and for simplicity assume that $\boldsymbol{x}_0 \in \operatorname{fix}(\mathcal{T}_0)$.

Proposition 2.1. *The cumulative fixed point residual is bounded as:*

- $\mathcal{T}_k$ ζ-contractive*:*

$$\frac{1}{k+1}\sum_{h=0}^{k}\|\boldsymbol{x}_{h+1} - \boldsymbol{x}_h\|^2 \leq \left(\sigma\frac{1+\zeta}{1-\zeta}\right)$$

where $\sigma \geq \max_k \left\|\boldsymbol{x}_k^* - \boldsymbol{x}_{k-1}^*\right\|$*;*

- $\mathcal{T}_k$ α-averaged *and defined on a compact domain:*

$$\frac{1}{k+1}\sum_{h=0}^{k}\|\boldsymbol{x}_{h+1} - \boldsymbol{x}_h\|^2 \leq \frac{\alpha}{1-\alpha}(\tau^2 + 2\delta\tau)$$

where τ *is the maximum distance between consecutive fixed point sets* $\operatorname{fix}(\mathcal{T}_k)$ *and* $\operatorname{fix}(\mathcal{T}_{k-1})$*, and* δ *is the diameter of* $\mathcal{T}_k$*'s domains.*

Proof. $\mathcal{T}_k$ *contractive*: using triangle inequality and contractiveness we can write

$$\begin{aligned}
\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| &\leq \left\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_{k+1}^*\right\| + \left\|\boldsymbol{x}_k - \boldsymbol{x}_{k+1}^*\right\| \\
&\leq (1+\zeta)\left\|\boldsymbol{x}_k - \boldsymbol{x}_{k+1}^*\right\| \\
&\leq (1+\zeta)(\|\boldsymbol{x}_k - \boldsymbol{x}_k^*\| + \sigma) \\
&\leq (1+\zeta)\left(\zeta^k\|\boldsymbol{x}_0 - \boldsymbol{x}_0^*\| + \frac{\sigma}{1-\zeta}\right) = \frac{1+\zeta}{1-\zeta}\sigma.
\end{aligned}$$

Taking the square and averaging over time we get:

$$\frac{1}{k+1}\sum_{h=0}^{k}\|\boldsymbol{x}_{h+1} - \boldsymbol{x}_h\|^2 \leq \left(\frac{1+\zeta}{1-\zeta}\sigma\right)^2.$$

$\mathcal{T}_k$ *averaged*: using the averagedness we have

$$\left\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_{k+1}^*\right\|^2 \leq \left\|\boldsymbol{x}_k - \boldsymbol{x}_{k+1}^*\right\|^2 - \frac{1-\alpha}{\alpha}\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\|^2;$$

rearranging and using triangle and Cauchy-Schwarz inequalities we have

$$\begin{aligned}
\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\|^2 &\leq \frac{\alpha}{1-\alpha}\left(\left\|\boldsymbol{x}_k - \boldsymbol{x}_{k+1}^*\right\|^2 - \left\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_{k+1}^*\right\|^2\right) \\
&\leq \frac{\alpha}{1-\alpha}\left(\|\boldsymbol{x}_k - \boldsymbol{x}_k^*\|^2 - \left\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_{k+1}^*\right\|^2 + \left\|\boldsymbol{x}_k^* - \boldsymbol{x}_{k+1}^*\right\|^2 + 2\left\|\boldsymbol{x}_k - \boldsymbol{x}_{k+1}^*\right\|\left\|\boldsymbol{x}_k^* - \boldsymbol{x}_{k+1}^*\right\|\right) \\
&\leq \frac{\alpha}{1-\alpha}\left(\|\boldsymbol{x}_k - \boldsymbol{x}_k^*\|^2 - \left\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_{k+1}^*\right\|^2 + \tau^2 + 2\delta\tau\right)
\end{aligned}$$

where the last inequality follows by $\left\|\boldsymbol{x}_k^* - \boldsymbol{x}_{k-1}^*\right\| \leq \tau$ and δ being the diameter of the compact domain.

Averaging over time, and noticing that the error terms cancel out we have

$$\frac{1}{k+1}\sum_{h=0}^{k}\|\boldsymbol{x}_{h+1} - \boldsymbol{x}_h\|^2 \leq \frac{\alpha}{1-\alpha}\left(\frac{1}{k+1}\|\boldsymbol{x}_0 - \boldsymbol{x}_0^*\|^2 + \tau^2 + 2\delta\tau\right) = \frac{\alpha}{1-\alpha}(\tau^2 + 2\delta\tau).$$

□

References

[1] H. H. Bauschke and P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*. CMS books in mathematics, Cham: Springer, 2 ed., 2017.