

IBL-Infinity Model of String Topology from Perturbative Chern-Simons Theory

Dissertation

zur Erlangung des akademischen Grades

Dr. rer. nat.

eingereicht an der

Mathematisch-Naturwissenschaftlich-Technischen Fakultät

der Universität Augsburg

von

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Augsburg, Oktober 2019



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Tag der mündlichen Prüfung: Freitag, 20. Dezember, 2019

Abstract

Pavel Hájek: *IBL-Infinity Model of String Topology from Perturbative Chern-Simons Theory*

Following Cieliebak, Fukaya, Latschev and Volkov, we construct an IBL-infinity chain model for equivariant string topology on cyclic Hochschild cochains of de Rham cohomology. We study its properties and perform explicit computations.

Keywords: chain model for equivariant string topology, IBL-infinity algebra, Maurer-Cartan element, Hodge propagator, perturbative Chern-Simons theory, Poincaré duality models, BV-formalism, cyclic homology

The thesis was typeset in \LaTeX using KOMA-Script, the bibliography was created using \BibLaTeX and the figures were drawn in \TikZ . The font is Linux Libertine, 11 pt.
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Pavel Hájek: *IBL-Infinity Model of String Topology from Perturbative Chern-Simons
Theory*

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Acknowledgements

First and foremost, I thank my parents, *Dana Hájková* and *Libor Hájek*, for supporting me throughout my studies, both emotionally and materially, for discussing various decisions I had to make, for listening when I needed to talk and for wishing me all the best in whatever I wanted to do. Díky moc mami a tati!

I thank my grandparents, *Anděla Hájková*, *Anna Uhlíková* and *Jaroslav Uhlík*, for their love and support, although they would rather see me being happy and having a family than pursuing a doctorate. Díky moc babi a dědo!

Next, I thank my supervisor, Prof. Dr. Kai Cieliebak, for giving me time and support to get through the graduate studies, to develop academically and to finish the thesis. I enjoyed discussions with him very much and learned from his insights and methods. I appreciate and admire his skill to explain everything anytime starting from elementary facts, his passion for mathematics and physics, scientific productivity and lecturing skills. Danke schön, Kai!

I thank Dr. Evgeny Volkov for discussing his work on Chern-Simons theory, string topology, cyclic homology and Chen's iterated integrals, for providing me his preprints in progress and for supportive chats at the university. Thank you, Evgeny, good luck fighting with lions in Siberia!

I thank Prof. Robert Bryant for proposing an alternative notation for the Hodge propagator for spheres in an online discussion. I thank Prof. John Rognes for answering my question about spectral sequences in an online discussion. I thank Dr. Najib Idrissi for answering my question about formality in an online discussion. I also thank the online discussion — Mathoverflow and Math Stackexchange — for its existence.

I thank Dr. Alexandru Doicu for checking a tedious sign computation and for friendly talks. I thank Dr. Andreas Hermann for discussing the standard Hodge propagator, for sending me his notes on the Green kernel for spheres and for his interest in my work. I thank Prof. Dr. Hông Vân Lê for discussing her work on formality and Poincaré DGA's and for her interest in my work. I thank Ján Pulmann and Lada Peksová for discussing BV-formalism and properads. I thank Ondřej Hulík for a Skype discussion about physics and for friendly talks. I thank Jiří Zeman for pointing out that absolute convergence is

equivalent to invariance on resummations and for friendly talks. I thank Dr. Kyler Siegel for a discussion about A-infinity in Stony Brook. I thank Dr. Raymond Puzio for his interest in my work, for thinking about an action for our propagator and for friendly talks. I thank Prof. Alberto Cattaneo and Dr. Pavel Mnev for discussing Feynman integrals and an action for our propagator in Stony Brook and for sending me a list of relevant literature later. I thank Prof. Dr. Braňo Jurčo for discussing BV-formalism and for offering me a position in Prague.

I thank Prof. Dr. Christian Bär, Prof. Dr. Kaoru Ono and Prof. Dr. Janko Latschev for inviting me to their institutes to discuss and give a talk, for their hospitality and for valuable feedback and insights. I am grateful to Prof. Dr. Janko Latschev for giving me a position in Hamburg and for waiting until I finish the thesis.

Although everything mentioned in the last paragraphs occurred in the final fifth year of my Ph.D., it inspired me and motivated me to complete the work.

I thank members of the department for mathematics at the University of Augsburg for creating and maintaining a rich and inspiring academic environment. I thank Prof. Dr. Urs Frauenfelder for various discussions and for always coming up with a highly interesting topic for the symplectic seminar, which, at the beginning, I often preferred to my own work. I thank Prof. Dr. Edward Belbruno, a visiting professor from Princeton, for his series of talks about chaotic dynamics in celestial mechanics, for the explanation of the weak stability boundary, for organizing exciting events related to space travel and for stopping by our office to chat about various topics.

I thank my colleagues at the university; those who I liked for being friendly and encouraging, and those who I did not for putting up with me and for not fueling conflicts. Specifically, I thank Alexei Kudryashov and Thorsten Hertl, who hanged out with me in the final two years when I was spending the most of my time at the university, and who had to listen to my laments about lost years, starting to work on concrete problems too late, terrible topic which is not geometric at all, internet overuse and screwed up personal life. I can not forget to highlight their top-level expertise in model categories and algebraic topology, respectively.

Unrelated to academia, I thank my tennis trainer, Thorsten Moser, alias Mr. T, for his university tennis courses in a cheerful atmosphere, him being dressed in one of his crazy costumes and constantly making fun of my tennis skills. There were times when the regular tennis training was the only thing I was looking forward to and the only order I could stick to. I thank my sparring partners and friends, Lukas Moosbrugger, Thomas Gumpinger and Tobias Watzka, for playing tennis and drinking beer with me. I thank Surf Club Augsburg, Rügen Piraten and Windsurfing F4 for the great time I had

windsurfing on Mandichosee, Ostsee and Bol, respectively, during my holiday. Memories of planing along the coast when a storm was coming helped me to overcome times when I felt really down. I thank Cuban Salsa Power in Augsburg, where I learned to dance Salsa properly, and where I spent almost every other evening in the beginnings of my studies. I thank my dance partner, Beatrix Fertl, for visiting me in Augsburg and for going out with me. I thank Ushi Haller for her interest in my well-being and for baking cookies and cakes for me. Finally, I want to mention an old friend, Milan Vaňkát, and thank him for the occasional obscene phone calls we had and for his few visits when we made fun of every possible social norm. He also left me the two African snails (die Schnecken) who accompanied me from the beginning to the end; I never felt too slow with them by my side.



Contents

1. Introduction	1
1.1. String topology and Chen's iterated integrals	1
1.2. IBL-infinity chain model and Chern-Simons theory	5
1.3. Relevant work of others	10
1.4. Summary of my results	11
 Part I. IBL-infinity chain model	 14
2. Summary and results	15
3. Algebraic structures	23
3.1. Gradings, degree shifts and completions	24
3.2. Basics of IBL-infinity-algebras	36
3.3. Dual cyclic bar complex and cyclic cohomology	48
3.4. Canonical dIBL-structure on cyclic cochains	59
4. Chern-Simons Maurer-Cartan element and string topology	73
4.1. Canonical dIBL-structures on cyclic cochains of de Rham cohomology	74
4.2. Hodge propagator	76
4.3. Chern-Simons Maurer-Cartan element	87
4.4. Results about vanishing of Chern-Simons Maurer-Cartan element	99
4.5. Conjectured relation to string topology	103
5. Explicit computations	108
5.1. Construction of a Hodge propagator for spheres	109
5.2. Computation of Chern-Simons Maurer-Cartan element for spheres	120
5.3. Twisted IBL-infinity-structure for spheres	137
5.4. Homology of twisted IBL-infinity structure for complex projective space	144

Part II. Follow-up topics	147
6. IBL-infinity formality and Poincaré duality models	148
6.1. Orientation, Poincaré duality, Hodge decomposition	149
6.2. Poincaré DGA's and Poincaré duality models	165
6.3. Consequences for IBL-infinity theory	172
7. Standard Hodge propagator	176
7.1. Schwartz form and smooth extension to blow-up	177
7.2. Uniqueness of Hodge propagator	180
7.3. Approximation using heat form	184
7.4. Standard Hodge propagator for Euclidean space	186
7.5. Standard Hodge propagator for 1- and 2-sphere	192
8. BV-formalism for IBL-infinity theory on cyclic cochains	198
8.1. Summary of IBL-infinity theory for cyclic DGA	198
8.2. BV-formulation for dIBL-algebra on cyclic cochains	200
8.3. Homotopy transfer and effective action	204
Part III. Appendices	208
A. Evaluation of labeled ribbon graphs	209
A.1. Finite dimensional case	209
A.2. De Rham case	218
B. Reduced cyclic homology of A-infinity-algebras	223
B.1. Computational prerequisites	224
B.2. Homological algebra of bicomplexes	235
B.3. Final argument and remarks	252
C. Towards an invariant definition of canonical IBL-operations	255
C.1. Bracket and grafting of trees	256
C.2. Cobracket and odd symplectic geometry	264
D. Filtered IBL-infinity-algebras as filtered MV-algebras	272
D.1. Filtered bialgebras and exponential in convolution product	273
D.2. Filtered IBL-infinity-algebras in filtered MV-formalism	285
D.3. BV-complexes for IBL-infinity-algebras	309

Contents

D.4. Composition of polynomials in convolution product	311
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List of Figures

1.1. Bracket and cobracket in equivariant string topology.	2
1.2. Inserting fields on strings.	6
1.3. Operations \mathfrak{q}_{210} and \mathfrak{q}_{120}	7
1.4. Adding \mathfrak{d} and \wedge via S	8
1.5. Kontsevich-Soibelman evaluation of a decorated tree.	10
1.6. A 1-loop diagram and its contribution to the twisted cobracket.	11
3.1. IBL_∞ -relations, Maurer-Cartan equation and twisting graphically.	42
4.1. A tree and a circular graph.	97
4.2. The Y -graph and an O_6 -graph.	97
4.3. Trivalent vertices of types A , B and C	99
5.1. Retraction of the configuration space $C_2(\mathbb{S}^n)$ to \mathbb{S}^n	111
5.2. A curve approaching the diagonal in the configuration space $C_2(\mathbb{S}^n)$	116
5.3. The Y -graph for \mathbb{S}^n	120
5.4. Vanishing graphs P_1 and P_2 for \mathbb{S}^2	126
5.5. Graphs K_1 , K_2 and the tadpole graph from Chern-Simons theory.	128
5.6. Swap of adjacent vertices on a circular graph.	132
5.7. A fully labeled circular ribbon graph for \mathbb{S}^1	133
5.8. The mirror isomorphism for the O_k -graph.	134
A.1. Ribbon graphs for \mathfrak{q}_{210} and \mathfrak{q}_{120}	216
B.1. Illustration of weight normalization in the Loday's cyclic bicomplex.	245
B.2. Illustration of weight normalization in the Connes' bicomplex.	246
B.3. Isomorphisms of various versions of cyclic (co)homologies of an A_∞ -algebra.	254
C.1. The Gerstenhaber bracket as grafting of trees.	257
D.1. Natural filtrations of a bigraded vector space.	277
D.2. Bound for the filtration degree in the iterated convolution product.	283

List of Figures

D.3. Bubbling in IBL_∞ -relations.	304
D.4. The “heart” of the iterated bialgebra compatibility condition.	312

1. Introduction

1.1. String topology and Chen's iterated integrals

String topology of a manifold M is the study of the *free loop space*

$$LM = \{\gamma : \mathbb{S}^1 \rightarrow M \text{ continuous}\}$$

with the compact-open topology and natural structures on it. Each loop γ is parametrized, with base-point 1, and there is a natural \mathbb{S}^1 -action changing the base-point. Therefore, we distinguish two homology theories:

$$\begin{aligned} H(LM) &\quad \dots \quad \text{the \textit{singular homology} and} \\ H^{\mathbb{S}^1}(LM) &\quad \dots \quad \text{the \textit{equivariant homology} — “the singular homology of the} \\ &\quad \quad \quad \text{space of parametrized loops with the base-point forgotten.”} \end{aligned}$$

In this thesis, we consider $H^{\mathbb{S}^1}(LM)$ with coefficients in \mathbb{R} only.

If $M = \Sigma$ is an oriented surface, we consider immersed loops with transverse double points and define a bracket and cobracket by Figure 1.1. In words:

\mathfrak{m}_2 : Imagine putting one base-point b_1 on the first loop γ_1 and another base-point b_2 on the second loop γ_2 in all possible positions. Whenever $\gamma_1(b_1) = \gamma_2(b_2) = p$, construct a new loop $\gamma = \gamma_1 *_p \gamma_2$ by running first along γ_1 with double speed starting and ending at b_1 and continuing along γ_2 starting and ending at b_2 . Forget the base-points and multiply γ with the sign of the intersection $\varepsilon(p; \gamma_1, \gamma_2)$. Should more intersections occur, take the sum.

\mathfrak{c}_2 : Imagine putting the basepoints b_1 and b_2 on γ in all possible positions such that $b_1 \neq b_2$. Whenever $\gamma(b_1) = \gamma(b_2) = p$, split γ into γ_1 and γ_2 as follows. The first loop γ_1 is the portion of γ from b_1 to b_2 and the second loop γ_2 is the portion from b_2 to b_1 ran along with the correspondingly scaled speed. Forget the base-points, form the tensor product $\gamma_1 \otimes \gamma_2$ and multiply it with $\varepsilon(p; \gamma)$. Should more self-intersections occur, take the sum of the tensors.

1. Introduction

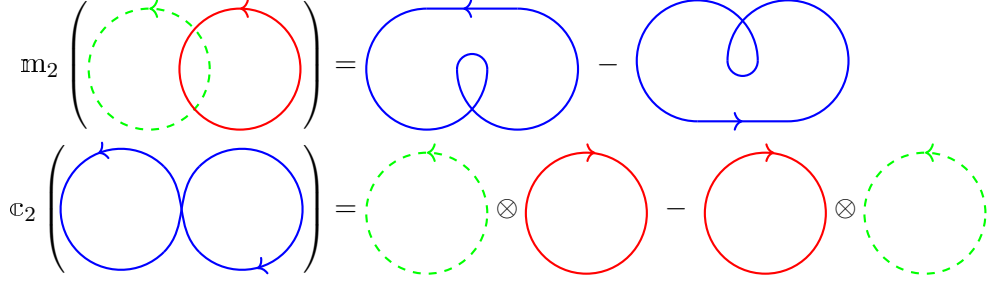


Figure 1.1.: Bracket and cobracket in equivariant string topology. Imagine some genus so that the operations are homotopically non-trivial.

The operations m_2 and c_2 are known as the *Goldman bracket* and the *Turaev cobracket* and were defined and studied in [Gol86] and [Tur91], respectively.

In [CS99], it was demonstrated that the construction of m_2 extends to families of loops and to an arbitrary dimension n of an oriented manifold M ; it produces a Lie bracket on the equivariant homology. The construction of c_2 generalizes too, which is explained for instance in [CL07], and gives a Lie cobracket. The picture is always 1.1, just the intersection points come from transverse intersections of smooth parameter spaces of points on loops in M ; thus, both m_2 and c_2 have degrees $2 - n$. Another definition of the coproduct is in [Bas11], where loops in M are viewed as open strings in $M \times M$ with endpoints at the diagonal. In order to make these geometric constructions rigorous, the most straightforward way is to use a *geometric homology theory* of M based on smooth chains, so that the transverse intersection of two smooth chains is again a smooth chain. Such theory for smooth manifolds was constructed in [Lip14]. A version for general topological spaces is proposed in [Cie13], and some details regarding triangulations are addressed in [Háj14]. Note that m_2 and c_2 are only “transversally defined” on the chain level and for the definition on homology it is important that we can homotop to a generic situation within the homology class.

The main theorem of [CS04] asserts that m_2 and c_2 induce the structure of an *involutive bi-Lie algebra* (abbreviated IBL) of degree $2 - n$ on the equivariant homology $H^{\mathbb{S}^1}(LM, M)$ relative to constant loops $M \hookrightarrow LM$. Modding out constant loops is necessary for c_2 to be well defined because of the phenomenon of “*vanishing of small loops*” illustrated in [CL07]: Let $\sigma \in C_1(LM)$ be a 1-chain supported at $[0, 1]$ which for $t = 0$ agrees with the loop in the argument of c_2 in 1.1, for $t \in (0, 1)$ the left knot L contracts to the mid-point p (thus the name) and for $t = 1$ only the right knot R remains. It is then easy

1. Introduction

to see that

$$0 \neq \partial \mathfrak{c}_2(\sigma) - \mathfrak{c}_2(\partial \sigma) = p \otimes R - L \otimes p \in C(M) \otimes C(LM) + C(LM) \otimes C(M).$$

In fact, there might be another problem with the restriction of \mathfrak{m}_2 to $H^{\mathbb{S}^1}(LM, M)$ as constant loops might not form an ideal (this is easy to see for $M = \mathbb{T}^2$). Our current understanding of this issue is that there should be an IBL-structure either on $H^{\mathbb{S}^1}(LM)$, if the Euler characteristics $\chi(M)$ vanishes, or on the homology $H^{\mathbb{S}^1}(LM, \text{pt})$ relative to one point, if $\chi(M) \neq 0$. However, the extension of \mathfrak{c}_2 requires choices (of a nowhere vanishing vector field, see [Bas11]), and it is not clear to which extent the IBL-structure depends on it. Some of these problems are (coincidentally) addressed in the work in progress [CVona], where Poincaré duality on the Rabinowitz-Floer homology of the unit cotangent bundle is introduced and related to the (non-equivariant) string topology. We will use $\overline{H}^{\mathbb{S}^1}(LM)$ as an avatar for either $H^{\mathbb{S}^1}(LM)$ or $H^{\mathbb{S}^1}(LM, \text{pt})$.

It is expected that the IBL-structure on $\overline{H}^{\mathbb{S}^1}(LM)$ is induced from a much richer (and in some context natural) algebraic structure on the chain level whose homotopy type is an invariant of M . In fact, there is the notion of a *strong homotopy involutive bi-Lie algebra* (abbreviated IBL_∞) which was developed in [CFL15]. It consists of operations (\mathfrak{q}_{klg}) for $k, l \geq 1$ and $g \geq 0$, where \mathfrak{q}_{110} is a boundary operator, \mathfrak{q}_{210} a bracket and \mathfrak{q}_{120} a cobracket, which satisfy the IBL-relations up to a coherent system of higher homotopies (\mathfrak{q}_{klg}) . Consider the string space

$$L_{\mathbb{S}^1}M := (E\mathbb{S}^1 \times LM)/\mathbb{S}^1$$

(the homotopically correct version of the quotient LM/\mathbb{S}^1), and let $(C(L_{\mathbb{S}^1}M), \partial)$ be the singular chain complex of $L_{\mathbb{S}^1}M$ where \mathfrak{m}_2 and \mathfrak{c}_2 are partially defined on transverse smooth chains. An IBL_∞ -chain model for equivariant string topology is an IBL_∞ -algebra $(\mathcal{M}(L_{\mathbb{S}^1}M), (\mathfrak{q}_{klg}))$ (\mathcal{M} stands for “model”) together with a weak homotopy equivalence ($:=$ zig-zag of quasi-isomorphisms) of $(C(L_{\mathbb{S}^1}M), \partial)$ and $(\mathcal{M}(L_{\mathbb{S}^1}M), \mathfrak{q}_{110})$ which induces an isomorphism of IBL-algebras

$$(\overline{H}^{\mathbb{S}^1}(LM), \mathfrak{m}_2, \mathfrak{c}_2) \simeq (H(\mathcal{M}(L_{\mathbb{S}^1}M), \mathfrak{q}_{110}), \mathfrak{q}_{210}, \mathfrak{q}_{120}).$$

Note that there can be various non-homotopically equivalent IBL_∞ -chain models. On the other hand, the properad IBL_∞ is a quasi-free resolution of the properad IBL and as such has convenient homotopy theoretical properties, e.g., homotopy inverses of quasi-isomorphisms exist. These properties imply that any two weakly homotopy equivalent IBL_∞ -algebras are (strongly) homotopy equivalent.

1. Introduction

In this thesis, we use a version of *perturbative Chern-Simons theory* to construct an IBL_∞ -chain model for the equivariant string topology starting with an oriented compact Riemannian manifold M . Chains are cyclic Hochschild cochains of the de Rham cohomology H_{dR} of M , and the homotopy type of the model is supposed to be an invariant of M (perhaps topological). As the construction involves a version of Feynman integrals, the proof of well-definedness relies on Stokes' theorem on compactifications of configuration spaces, which is currently being developped in [CVona] and [CVonb]. The concrete form of this chain model was sketched in [CFL15]. It is expected that the evaluation at boundaries of pseudo-holomorphic curves in the compactification induces an IBL_∞ -quasi-isomorphism of the SFT of the unit cotangent bundle of M and the IBL_∞ -chain model of string topology, see [CL07].

We now describe the underlying chain complex and the quasi-isomorphism to string topology in more details. Let $B^{\text{cyc}}\Omega$, where Ω is the space of de Rham forms on M , be the graded vector space generated by cyclic words

$$\omega_1 \cdots \omega_k = (-1)^{|\omega_k|(|\omega_1|+\cdots+|\omega_k|)} \omega_k \omega_1 \cdots \omega_{k-1}$$

with homogenous $\omega_1, \dots, \omega_k \in \Omega$ for $k \geq 1$. The grading satisfies $|\omega_i| = \deg(\omega_i) - 1$, where $\deg(\omega_i)$ denotes the form-degree of ω_i . This $B^{\text{cyc}}\Omega$ is called the *(reduced) cyclic bar complex of Ω* (reduced because we omit $k = 0$). On $B^{\text{cyc}}\Omega$, we consider the *cyclic Hochschild differential*

$$\begin{aligned} b(\omega_1 \cdots \omega_k) = & \sum_{i=1}^k (-1)^{|\omega_1|+\cdots+|\omega_{i-1}|} \omega_1 \cdots d\omega_i \cdots \omega_k \\ & + \sum_{i=1}^{k-1} (-1)^{|\omega_1|+\cdots+|\omega_{i-1}|+|\omega_i|+1} \omega_1 \cdots \omega_i \wedge \omega_{i+1} \cdots \omega_k \\ & + (-1)^{|\omega_k|(|\omega_1|+\cdots+|\omega_{k-1}|)+|\omega_k|+1} \omega_k \wedge \omega_1 \cdots \omega_{k-1}. \end{aligned}$$

It descends from the Hochschild differential on the bar construction $B\Omega$, which is standardly defined as the sum of the unique extensions of degree shifts of d and \wedge to coderivatives of $B\Omega$. The *Chen's iterated integrals map* is the map

$$\begin{aligned} I : B^{\text{cyc}}\Omega &\longrightarrow C^*(L_{\mathbb{S}^1}M) \\ \omega_1 \cdots \omega_k &\longmapsto \left(\sigma \mapsto \varepsilon(\omega) \int_{K_\sigma \times \Delta^k} \omega_1(\tilde{\sigma}(x, t_1)) \cdots \omega_k(\tilde{\sigma}(x, t_k)) \right), \end{aligned}$$

1. Introduction

where $\varepsilon(\omega)$ is the sign

$$\varepsilon(\omega) = (-1)^{(k-1)(|\omega_1|+1)+(k-2)(|\omega_2|+1)+\dots+|\omega_{k-1}|+1},$$

the map $\tilde{\sigma} : K_\sigma \times \mathbb{S}^1 \rightarrow M$, where K_σ is a smooth chain in M (a manifold with corners), is the projection of a lift $K_\sigma \xrightarrow{\sigma} L_{\mathbb{S}^1} M \dashrightarrow E\mathbb{S}^1 \times LM$ to the second factor, and we identify $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The map I is a chain map (of degree 0 with respect to the cohomological grading of $B^{\text{cyc}}\Omega$ by $|\omega_1 \dots \omega_k| = |\omega_1| + \dots + |\omega_k|$), and if $\pi_1(M) = \{1\}$, it induces an isomorphism

$$H(B^{\text{cyc}}\Omega, b)/\text{span}\{[1^{2k-1}] \mid k \in \mathbb{N}\} \simeq H^*(L_{\mathbb{S}^1} M)/\mathbb{R}[[u]], \quad (1.1)$$

where $\text{span}\{[1^{2k-1}] \mid k \in \mathbb{N}\} = H(B^{\text{cyc}}\mathbb{R}, b)$ (we prove later that this agrees with the homology of $\text{coker}(B^{\text{cyc}}\mathbb{R} \hookrightarrow B^{\text{cyc}}\Omega)$) and the polynomial ring $\mathbb{R}[u]$ with $|u| = 2$ comes from the module structure on $H^*(L_{\mathbb{S}^1} M)$ induced from $L_{\mathbb{S}^1} M = (E\mathbb{S}^1 \times LM)/\mathbb{S}^1 \rightarrow E\mathbb{S}^1/\mathbb{S}^1 = \mathbb{CP}^\infty$. The proof of this isomorphism is the goal of [CVonb], which is again under development; it is based on [GJP91] after identifying $(B^{\text{cyc}}\Omega, b)$ with the correct totalization of the Connes' cyclic bicomplex of Ω up to weak equivalence.

1.2. IBL-infinity chain model and Chern-Simons theory

We consider the *dual cyclic bar complex*

$$\hat{B}_{\text{cyc}}^* \Omega = \bigoplus_{d \in \mathbb{Z}} \prod_{k=1}^{\infty} (B_k^{\text{cyc}} \Omega)^{d*},$$

where $(B_k^{\text{cyc}} \Omega)^{d*}$ denotes the linear dual to the degree d component of the weight k component $B_k^{\text{cyc}} \Omega$ of $B^{\text{cyc}}\Omega$, equipped with the dual Hochschild differential b^* . By taking \prod , we allow bubbling of degree 1 forms and constants 1; namely, if $\omega \in \Omega^1$, then $|\omega| = 0$, and $|1| = -1$. Notice that if we take H_{dR} instead of Ω , then the completion is relevant only if $H_{\text{dR}}^1 \neq 0$. The dual of the Chen's iterated integral map provides a chain map

$$I^* : (C(L_{\mathbb{S}^1} M), \partial) \rightarrow (\hat{B}_{\text{cyc}}^* \Omega, b^*)$$

(in fact, $\hat{B}_{\text{cyc}}^* \Omega$ is the graded dual of $B^{\text{cyc}}\Omega$).

We introduce the following *physical interpretation*. Beware, the discussion is imprecise! We think of

- elements of Ω as *fields*,

1. Introduction

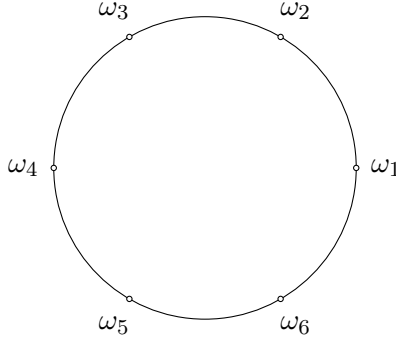


Figure 1.2.: Inserting fields on strings.

- elements of $B^{\text{cyc}}\Omega$ as *field strings* (not to confuse with string-fields) and
- elements of the space

$$\mathcal{F}(B^{\text{cyc}}\Omega[1]) := \hat{S}(B_{\text{cyc}}^*\Omega[1]) \hat{\otimes} \mathbb{R}((\hbar)),$$

where $\mathbb{R}((\hbar))$ is the ring of Laurent series in Planck's constant \hbar , \hat{S} denotes the completed symmetric algebra and $\hat{\otimes}$ the completed tensor product, as *observables on field strings*.

If $\sigma : \mathbb{S}^1 \rightarrow M$ is a string, the observable $I^*(\sigma)$ “localizes” on field strings which approximate σ ; for instance, it holds $I^*(\omega) = \int_{\sigma} \omega$, and so $I^*(\omega)$ “localizes” at fundamental forms of $\sigma(\mathbb{S}^1)$. We imagine that we decorate σ with a field string $\omega_1 \dots \omega_k$ as in Figure 1.2 and get a number $I(\sigma)(\omega_1 \dots \omega_k)$. The isomorphism (1.1) guarantees that the observable $I(\sigma)$ determines σ up to a boundary. We will be dealing with *two “dynamical” theories*: one is the theory of fields $\omega \in \Omega$ and one of field strings $\omega_1 \dots \omega_k \in B^{\text{cyc}}\Omega$.

The field theory is at hand. We know that $(\Omega, d, \wedge, \langle \cdot, \cdot \rangle)$, where $\langle \omega_1, \omega_2 \rangle = \int_M \omega_1 \wedge \omega_2$ for $\omega_1, \omega_2 \in \Omega$, is a *symmetric dg-Frobenius algebra*. It is well-known that finite-dimensional Frobenius algebras V are equivalent to 2-dimensional topological quantum field theories (TQFT). Finite-dimensionality is a sufficient (and necessary) condition so that the trivial cylinder corresponding to the identity satisfies $\mathbb{1} = \sum \langle \cdot, e^i \rangle e_i$. For now, we will ignore this issue and imagine $V = \Omega$, although we will soon transfer to H_{dR} , where everything works just fine. We will represent interactions of fields via Feynman graphs drawn on surfaces — the trivial cylinder for fields, i.e., the free propagation, will be a line and the pair of pants, i.e., the interaction via the intersection \wedge , will be a point with 3 segments emanating from it (we do not distinguish input and outputs by cyclic symmetry).

1. Introduction

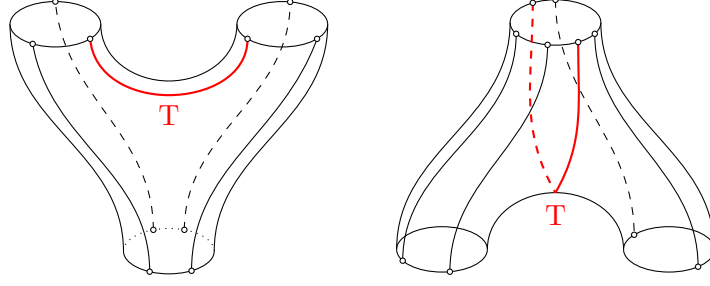


Figure 1.3.: Operations \mathfrak{q}_{210} and \mathfrak{q}_{120} . Fields propagate from the bottom to the top and an additional dualization, equivalently contraction with the identity propagator, is required when two outputs are connected; hence the red color.

Let us now consider field strings. Figure 1.3 defines the operations

$$\mathfrak{q}_{210} : \hat{E}_2 B_{\text{cyc}}^* V \rightarrow \hat{E}_1 B_{\text{cyc}}^* V \quad \text{and} \quad \mathfrak{q}_{120} : \hat{E}_1 B_{\text{cyc}}^* V \rightarrow \hat{E}_2 B_{\text{cyc}}^* V,$$

where $E(B_{\text{cyc}}^* V)$ is the exterior algebra canonically isomorphic to $S(B_{\text{cyc}}^* V[1])$. We read the diagram from the top to the bottom but imagine fields ω being fed into ψ 's from the bottom to the top. We think of these digrams as of *string interaction diagrams* for strings freely moving in a topological space M , connecting and disconnecting. This suggests that \mathfrak{q}_{210} and \mathfrak{q}_{120} are related to \mathfrak{m}_2 and \mathfrak{c}_2 , respectively. Formulas for \mathfrak{q}_{210} and \mathfrak{q}_{120} for a finite dimensional symmetric dg-Frobenius algebra V were written down in [CFL15] and it was proven that they indeed constitute an IBL-algebra on $B_{\text{cyc}}^* V$ (note that this is not a TQFT!). As for the formulas, one simply decorates the world-lines of fields with the identity propagator $T = \sum_i e^i \otimes e_i$ and evaluates. As a mathematical remark, we will show that \mathfrak{q}_{210} is obtained from the Gerstenhaber bracket on Hochschild cochains via cyclization by $\langle \cdot, \cdot \rangle$ and that \mathfrak{q}_{120} is a factorization of an extension of the canonical Schwarz's BV-operator on $\mathcal{F}(V[1])$ (odd degree shift of a finite-dimensional symmetric dg-Frobenius algebra is an odd symplectic vector space) to cyclic invariants with respect to the cyclic shuffle product.

Since $B_{\text{cyc}}^* V[1]$ is not naturally an odd symplectic vector space, there is no Schwarz's BV-operator on $\mathcal{F}(B_{\text{cyc}}^* V[1])$. However, the following canonical operator

$$\Delta_s := \hat{\mathfrak{q}}_{120} + \hbar \hat{\mathfrak{q}}_{210} : \mathcal{F}(B_{\text{cyc}}^* V[1]) \rightarrow \mathcal{F}(B_{\text{cyc}}^* V[1]),$$

where $\hat{\cdot}$ denote canonical extension to (co)derivatives, is a BV-operator with respect to the function multiplication; we call it the *string BV-operator*. Recall that in physics, a BV-operator Δ on $\mathcal{F}(U)$, where U is the space of fields, is related to the measure in the

1. Introduction

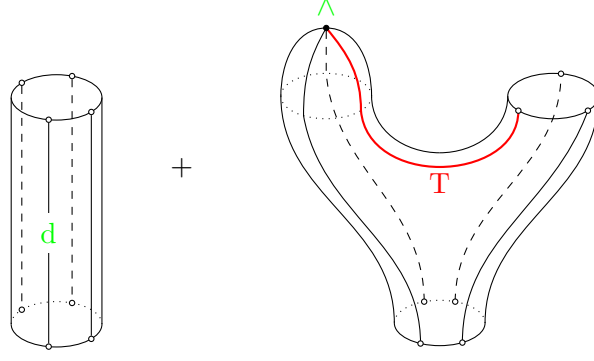


Figure 1.4.: Adding d and \wedge via S .

path integral $\int_U \mu$. An action $S \in \mathcal{F}(U)$ satisfying the *quantum master equation* (QME)

$$\Delta S + \frac{1}{2}\{S, S\} = 0$$

defines a new measure $e^{-S}\mu$, and the corresponding twisted BV-operator (or rather BV_∞ -operator) satisfies $\Delta^S = e^{-S} \Delta e^S$. For field strings, we define the following *actions* $S_{\text{free}}, S_{\text{int}} \in \mathcal{F}(B^{\text{cyc}}V[1])$, which remind us of the *Chern-Simons functional*:

$$S_{\text{free}}(\omega_1 \omega_2) := \pm \hbar^{-1} \int_M \omega_1 \wedge d\omega_2 \quad \text{and} \quad S_{\text{int}}(\omega_1 \omega_2 \omega_3) := \pm \hbar^{-1} \int_M \omega_1 \wedge \omega_2 \wedge \omega_3.$$

Note that S_{free} and S_{int} are linear functions on $B^{\text{cyc}}V[1]$ which vanish everywhere but field strings of lengths 2 and 3, respectively. It turns out that S_{free} and $S := S_{\text{free}} + S_{\text{int}}$ satisfy the QME for Δ_s . The twisted BV-operators look like

$$\Delta_s^{S_{\text{free}}} = \hat{\mathfrak{q}}_{110} + \Delta_s \quad \text{and} \quad \Delta_s^S = \hat{\mathfrak{q}}_{110} + \widehat{\mathfrak{q}_{210}(S_{\text{int}}, \cdot)} + \Delta_s.$$

Figure 1.4 depicts the new terms \mathfrak{q}_{110} and $\mathfrak{q}_{210}(S_{\text{int}}, \cdot)$ in Δ_s^S . The corresponding dIBL-algebra reads

$$(B_{\text{cyc}}^*V, \mathfrak{q}_{110}^{\mathfrak{m}} := \mathfrak{q}_{110} + \mathfrak{q}_{210}(S_{\text{int}}, \cdot), \mathfrak{q}_{210}, \mathfrak{q}_{120}),$$

where we recognize the Hochschild differential $\mathfrak{q}_{110}^{\mathfrak{m}}$. If this was well-defined for $V = \Omega$, then it would give the IBL_∞ -chain model of string topology.

As in quantum field theories, we are going to “formally” *integrate out redundant degrees of freedom in the path integral* of our ill-defined theory and obtain a well-defined theory on $B_{\text{cyc}}^*H_{\text{dR}}$, which is “formally” homotopy equivalent to the original one. We pick a Riemannian metric on M and consider the Hodge decomposition $\Omega = \mathcal{H} \oplus \text{im } d \oplus \text{im } d^*$,

1. Introduction

where $\mathcal{H} \simeq H_{\text{dR}}$ are the harmonic forms $d\omega = d^*\omega = 0$. For Chern-Simons fields $\omega \in \Omega$, one thinks of $d\omega = 0$ as of the Euler-Lagrange equation and of $d^*\omega = 0$ as of the Lorentz gauge. The “inverse” of d as a map $\text{im } d \rightarrow \text{im } d^*$ is the standard Hodge homotopy \mathcal{P}_{std} , i.e., the unique coexact solution of $d\mathcal{P} + \mathcal{P}d = \mathbb{1} - \pi_{\mathcal{H}}$, where $\pi_{\mathcal{H}} : \Omega \rightarrow \mathcal{H}$ is the orthogonal projection. The Schwartz kernel of \mathcal{P}_{std} is the *standard Hodge propagator* P_{std} . A formula for the *effective action* $W \in \mathcal{F}(B^{\text{cyc}}H_{\text{dR}}[1])$ was given in [CFL15]; in their terminology, it would be called the *(formal) pushforward Maurer-Cartan element*. We have

$$W = \hbar^{-1} \sum_{l \geq 1, g \geq 0} \mathbf{n}_{lg} \hbar^g,$$

where $\mathbf{n}_{lg} \in \hat{E}_l B_{\text{cyc}}^* H_{\text{dR}}$ is computed from $(l + g - 1)$ -loop Feynman diagrams with interaction vertices \wedge and propagator P_{std} . One can think of trying to push the field strings to their representatives consisting of harmonic forms. The twisted string BV-operator on $\mathcal{F}(B^{\text{cyc}}H_{\text{dR}}[1])$ reads

$$\Delta_s^W = \hat{\mathbf{q}}_{110}^n + \hbar \mathbf{q}_{210} + \sum_{l \geq 2, g \geq 0} \hat{\mathbf{q}}_{1lg}^n \hbar^g,$$

where $\mathbf{q}_{110}^n = \mathbf{q}_{210}(\mathbf{n}_{10}, \cdot) = \mathbf{q}_{210} \circ_1 \mathbf{n}_{10}$ and $\mathbf{q}_{1lg}^n = \mathbf{q}_{210} \circ \mathbf{n}_{lg}$. The resulting IBL_{∞} -structure on $\hat{B}_{\text{cyc}}^* H_{\text{dR}}$, i.e., our chain model of equivariant string topology, is, in fact, a *quantum* coL_{∞} -algebra (\mathbf{q}_{1lg}^n) with *Drinfeld-compatible Lie bracket* \mathbf{q}_{210} . The weak homotopy equivalence of $(\hat{B}_{\text{cyc}}^* H_{\text{dR}}, \mathbf{q}_{110}^n)$ and $(C(L_{\mathbb{S}^1} M), \partial)$ inducing an isomorphism of the IBL -structure on homology is given by the composition $F \circ I^* : C(L_{\mathbb{S}^1} M) \rightarrow \hat{B}_{\text{cyc}}^* H_{\text{dR}}$, where

$$F = \mathbf{f}_{110} + \mathbf{f}_{210} \circ_1 \mathbf{m}_{10} + \frac{1}{2!} \mathbf{f}_{310} \circ_{1,1} (\mathbf{m}_{10}, \mathbf{m}_{10}) + \cdots$$

with $\mathbf{f}_{110} = \iota^*$ for $\iota : H_{\text{dR}} \simeq \mathcal{H} \hookrightarrow \Omega$, and where $\mathbf{f}_{k10} \circ_{1, \dots, 1} (\mathbf{m}_{10}, \dots, \mathbf{m}_{10})$ contributes by trivalent trees as in Figure 1.5. Note that in order to evaluate trees, we do not need the Schwartz kernel P and hence any pairing. The homotopy \mathcal{P} is enough because the graph is directed and we can distinguish inputs and outputs. On the other hand, evaluation of the 1-loop Feynman graph in Figure 1.6 contributing to \mathbf{q}_{120}^n , for instance, requires P , and hence $\langle \cdot, \cdot \rangle$. From Sullivan’s minimal model theory it is well-known that the homotopy type of the homotopy transferred A_{∞} -structure $(H_{\text{dR}}, (m_k))$ is a topological invariant which encodes the rational homotopy theory of a simply-connected manifold M . Our construction can be seen as being associated to the Poincaré DGA $(\Omega, d, \wedge, \langle \cdot, \cdot \rangle)$, i.e., a DGA whose homology is a Poincaré duality algebra. It is not clear to the author what kind of invariant is the weak homotopy type of the Poincaré DGA and what kind

1. Introduction

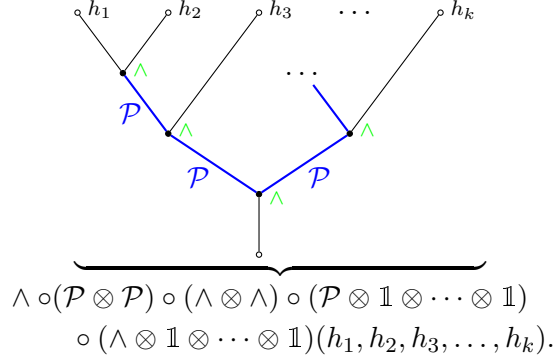


Figure 1.5.: Kontsevich-Soibelman evaluation of a decorated tree.

of invariant is the homotopy type of the twisted IBL_∞ -algebra on $\hat{B}_{\mathrm{cyc}}^* H_{\mathrm{dR}}$. However, if M is formal in the sense of DGA's, then M is formal in the sense of Poincaré DGA's, and we conjecture that it is formal also in the sense of IBL_∞ -algebras; i.e., the twisted IBL_∞ -algebra on $\hat{B}_{\mathrm{cyc}}^* H_{\mathrm{dR}}$ is homotopy equivalent to the canonical IBL -algebra.

Finally, note that from the theory of Koszul (pr)operads it is well-known that IBL is Koszul dual to Frob , i.e., $\mathrm{IBL}^\perp = \mathrm{Frob}$ and $\mathrm{Frob}^\perp = \mathrm{IBL}$, where Frob is the properad of Frobenius bialgebras. Therefore, it holds $\mathrm{IBL}_\infty = \Omega(\mathrm{Frob}^*)$, where Ω denotes the cobar construction and $*$ the linear dual coproperad; this precisely reflects our situation of having a Frobenius structure on H_{dR} (the coproduct is obtained from the product via dualization) and IBL_∞ -structure on $\hat{B}_{\mathrm{cyc}}^* H_{\mathrm{dR}} = \widehat{B^{\mathrm{cyc}} H_{\mathrm{dR}}^*}$, where $\hat{}$ denotes a suitable completion, since B^{cyc} is a cyclic version of the cobar construction.

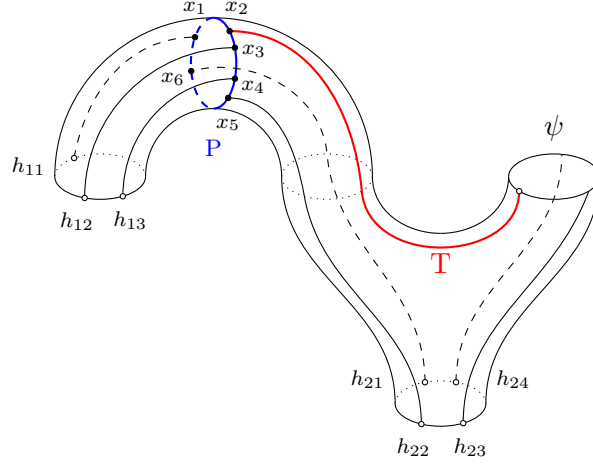
1.3. Relevant work of others

In [CJ02], there is a homotopy theoretical realization of string topology operations \mathfrak{m}_2 and \mathfrak{c}_2 . This is another approach to their rigorous definition.

In [CL07], they sketch a proof that symplectic field theory of the unit cotangent bundle of M and equivariant string topology of M are IBL_∞ -quasi-isomorphic via a map induced from evaluations at boundaries of holomorphic curves. String topology operations then arise naturally from the codimension 1 boundary structure of the moduli space of holomorphic curves with certain boundary conditions. In fact, a precise formulation of this correspondence was perhaps the main reason for Cieliebak & Latschev to think about the IBL_∞ -chain model of string topology.

An IBL_∞ -structure on the singular equivariant chain complex was proposed by [Sul05].

1. Introduction



$$= \sum_{a,b} \sum_{c=1}^4 \pm T^{ab} \psi(e_a h_{2,c+2} h_{2,c+3}) \left(\int_{x_1 x_2 x_3 x_4 x_5 x_6} P(x_1, x_2) P(x_2, x_3) P(x_3, x_4) \right. \\ \left. P(x_4, x_5) P(x_5, x_6) P(x_6, x_1) (h_{11}(x_1) h_{12}(x_3) h_{13}(x_4)) (e_b(x_2) h_{2,c}(x_6) h_{2,c+1}(x_5)) \right)$$

Figure 1.6.: A 1-loop diagram and its contribution to the twisted cobracket.

It is constructed using “diffusion intersection product” which realizes the intersection on the chain level. In fact, there is much more string operations coming from stacks of Sullivan diagrams than just the bracket and cobracket.

There is a recent construction of chain level non-equivariant string topology on the singular chain complex of LM using fat graphs [DPR15] and the diffusion intersection. It was shown in [CHV06] that there is a TQFT of string operations on the non-equivariant homology. The aforementioned reference constructs a chain model for this entire gadget.

There is an algebraic chain model for non-equivariant string topology in [Che12]. An algebraic chain model is also constructed in [Iri14].

Chain level string topology is required, e.g., in Fukaya’s work on Lagrangian submanifolds.

1.4. Summary of my results

- 1) The biggest part of the work was about setting up a formalism and deducing signs for a definition of the formal-pushforward Maurer-Cartan element, aka Chern-Simons Maurer-Cartan element, and the IBL_∞ -chain model in de Rham setting. Another big part was about understanding what is happening and discovering and

1. Introduction

formulating the structure and possible claims.

- 2) We compute the IBL_∞ -chain model for \mathbb{S}^n with $n \neq 2$ by finding an explicit Hodge propagator and computing Feynman integrals. In fact, for $n \geq 3$, all integrals which are relevant for the IBL_∞ -theory vanish.

A trick from [CM08] is based on modifying an abstract Hodge propagator to obtain special properties implying vanishing of the integrals. The author of this thesis was not aware of this trick and tried to compute integrals with an explicit propagator in spherical coordinates for around 3 years until he rediscovered a part of this trick himself. The interesting thing is that the discovery was made via explicit computations, and it was a coincidence that the constructed Hodge propagator satisfied the special properties.

- 3) Using the trick from [CM08], we generalize the previous computation to geometrically formal manifolds and show that the Feynman integrals vanish provided that $H_{\text{dR}}^1(M) = 0$. For a general manifold, all higher coproducts vanish unless M is a surface or a 3-manifold with $H_{\text{dR}}^1(M) \neq 0$. In fact, the homotopy type of the IBL_∞ -chain model for a manifold with $H_{\text{dR}}^1(M) = 0$ is determined by the tree-level perturbative Chern-Simons theory for a special Hodge propagator.
- 4) We formulate a conjecture that the IBL_∞ -chain model for formal manifolds with $H_{\text{dR}}^1(M) = 0$ is homotopy equivalent to the canonical dIBL-structure. We propose a strategy of proof based on Poincaré duality models.
- 5) We prove a proposition that the cyclic homology of a strictly unital A_∞ -algebra can be computed from its reduced version. We do it by rewriting the Loday's cyclic homology theory for DGA's.
- 6) We relate \mathfrak{q}_{210} to Gerstenhaber bracket and cyclization and \mathfrak{q}_{120} to the Schwarz's BV-operator and cyclic shuffle product.
- 7) We extend the MV-formalism to filtered MV-formalism and use it to construct a BV-formulation of the weak IBL_∞ -theory. This has the advantage that the exponentials are honest exponentials and honest maps. This will be useful for studying BV-chain complexes.
- 8) We formulate the composition at k -common channels \circ_k using "heart with veins" which appears in the iterated bialgebra compatibility condition.

1. Introduction

- 9) We understood the BV-formulation of IBL_∞ -theory with actions, effective action and quantum master equation.
- 10) We find the standard Hodge propagator for \mathbb{S}^2 up to constants and prove that it smoothly extends to blow-up.

Part I.

IBL-infinity chain model

2. Summary and results

An IBL_∞ -algebra is essentially a collection of multilinear operations \mathfrak{q}_{klg} with k inputs, l outputs and “genus” g satisfying certain relations; in particular, \mathfrak{q}_{110} is a boundary operator, and the pair $\mathfrak{q}_{210}, \mathfrak{q}_{120}$ induces the structure of an involutive Lie bialgebra on the homology of \mathfrak{q}_{110} . It was introduced in [CFL15] and applications to string topology, symplectic field theory and higher genus Lagrangian Floer theory were proposed.

This part of the thesis is an attempt to understand the application to *string topology*. The idea was to carry out some explicit computations according to the plan sketched in [CFL15, Section 13] and test the string topology conjecture (see below).

The following results from [CFL15, Corollary 11.9] are our starting point (precise definitions of all the notions will be given in the next chapters; IBL_∞ -algebras in Part I will be strict and filtered in the terminology of [CFL15]):

- (A) For a finite-dimensional cyclic cochain complex (V, Π, m_1) of degree $2 - n$, there is a canonical dIBL-structure $\mathfrak{p}_{110}, \mathfrak{p}_{210}, \mathfrak{p}_{120}$ of bidegree $(n - 3, 2)$ on the degree shifted dual cyclic bar complex

$$C(V) := B_{\text{cyc}}^* V[2 - n] \simeq \left(\bigoplus_{k \geq 1} (V[1]^{\otimes k} / \text{cyc})' \right) [2 - n],$$

where cyc stands for cyclic permutations with the Koszul sign, $'$ denotes the graded dual and $[\cdot]$ the degree shift. This structure is denoted by $\text{dIBL}(C(V))$.

- (B) Let $(\mathcal{H}, \Pi, m_1) \subset (V, \Pi, m_1)$ be a subcomplex such that the restriction of Π to $\mathcal{H}[1]$ is non-degenerate. We apply (A) to (\mathcal{H}, Π, m_1) to get the canonical dIBL-algebra $\text{dIBL}(C(\mathcal{H})) = (C(\mathcal{H}), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$. Suppose that $\pi : V[1] \rightarrow V[1]$ is a projection to $\mathcal{H}[1]$ which satisfies

$$\begin{aligned} \pi \circ m_1 &= m_1 \circ \pi \quad \text{and} \\ \Pi(\pi(v_1), v_2) &= \Pi(v_1, \pi(v_2)) \end{aligned}$$

for all $v_1, v_2 \in V[1]$, and let $\iota : \mathcal{H}[1] \rightarrow V[1]$ be the inclusion. A linear map

2. Summary and results

$\mathcal{P} : V[1] \rightarrow V[1]$ of degree -1 such that

$$\begin{aligned} m_1 \circ \mathcal{P} + \mathcal{P} \circ m_1 &= \iota \circ \pi - \mathbb{1}_{V[1]} \quad \text{and} \\ \Pi(\mathcal{P}(v_1), v_2) &= (-1)^{|v_1|} \Pi(v_1, \mathcal{P}(v_2)) \end{aligned} \tag{2.1}$$

for all $v_1, v_2 \in V[1]$ induces the IBL_∞ -homotopy equivalence

$$\mathfrak{f} = (\mathfrak{f}_{klg}) : \text{dIBL}(C(V)) \longrightarrow \text{dIBL}(C(\mathcal{H}))$$

such that $\mathfrak{f}_{110} : C(V)[1] \rightarrow C(\mathcal{H})[1]$ is the map given by the precomposition with ι in every component. We recall from [CFL15] that $\mathfrak{f}_{klg} : E_k C(V) \rightarrow E_l C(\mathcal{H})$ is a linear map between exterior powers.

The map \mathfrak{f}_{klg} is constructed as a sum of contributions coming from isomorphism classes of *ribbon graphs* (=: multigraphs with a cyclic ordering of half-edges at every internal vertex) with k internal vertices, l boundary components and genus g . To compute the contribution of a labeled ribbon graph Γ to the value

$$\mathfrak{f}_{klg}(\Psi_1 \otimes \cdots \otimes \Psi_k)(W_1 \otimes \cdots \otimes W_l)$$

for $\Psi_1, \dots, \Psi_k \in B_{\text{cyc}}^* V[3-n]$ and $W_1, \dots, W_l \in B^{\text{cyc}} \mathcal{H}[3-n]$, we decorate the i -th internal vertex of Γ with Ψ_i , external vertices lying on the i -th boundary component with components $v_{i1}, \dots, v_{is_i} \in V[1]$ of $W_i = s(v_{i1} \otimes \cdots \otimes v_{is_i} / \text{cyc})$, where s is a formal symbol of degree $n-3$, and internal edges with the Schwartz kernel P of \mathcal{P} with respect to Π . Decorated ribbon graphs are then evaluated in a consistent way to obtain real numbers (see Appendix A for an invariant formalism or [CFL15, Section 10] for a coordinate version of this construction).

We will also use the following results from [CFL15, Proposition 12.5 and Theorem 12.9] about deformations of IBL_∞ -algebras:

- (C) If in addition to (A) there is the product $m_2 : V[1] \otimes V[1] \rightarrow V[1]$ making (V, m_1, m_2) into a cyclic dga, then $(-1)^{n-2} m_2^+$ defines a canonical Maurer-Cartan element $\mathfrak{m} := (\mathfrak{m}_{10})$ for $\text{dIBL}(C(V))$. The twisted IBL_∞ -algebra is again a dIBL -algebra of bidegree $(n-3, 2)$; it is denoted by $\text{dIBL}^{\mathfrak{m}}(C(V))$ and satisfies

$$\begin{aligned} \text{dIBL}^{\mathfrak{m}}(C(V)) \\ = (C(V), \mathfrak{p}_{110}^{\mathfrak{m}} = \mathfrak{p}_{110} + \mathfrak{p}_{210} \circ_1 \mathfrak{m}_{10}, \mathfrak{p}_{210}^{\mathfrak{m}} = \mathfrak{p}_{210}, \mathfrak{p}_{120}^{\mathfrak{m}} = \mathfrak{p}_{120}). \end{aligned} \tag{2.2}$$

- (D) The IBL_∞ -morphism \mathfrak{f} from (B) can be used to pushforward \mathfrak{m} and obtain the

2. Summary and results

Maurer-Cartan element $\mathbf{n} = (\mathbf{n}_{lg})$ for $\mathrm{dIBL}(C(\mathcal{H}))$. The twist by \mathbf{n} is an IBL_∞ -algebra of bidegree $(n-3, 2)$; it is denoted by $\mathrm{dIBL}^{\mathbf{n}}(C(\mathcal{H}))$ and satisfies

$$\begin{aligned} \mathrm{dIBL}^{\mathbf{n}}(C(\mathcal{H})) \\ = (C(\mathcal{H}), \mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110} + \mathbf{q}_{210} \circ_1 \mathbf{n}_{10}, \mathbf{q}_{210}^{\mathbf{n}} = \mathbf{q}_{210}, \mathbf{q}_{120}^{\mathbf{n}} = \mathbf{q}_{120} \\ + \mathbf{q}_{210} \circ_1 \mathbf{n}_{20}, \text{ plus the higher operations } \mathbf{q}_{1lg}^{\mathbf{n}} = \mathbf{q}_{210} \circ_1 \mathbf{n}_{lg}). \end{aligned}$$

This IBL_∞ -algebra is IBL_∞ -homotopy equivalent to $\mathrm{dIBL}^{\mathbf{m}}(C(V))$ via the twisted IBL_∞ -morphism

$$\mathbf{f}^{\mathbf{m}} = (\mathbf{f}_{klg}^{\mathbf{m}}) : \mathrm{dIBL}^{\mathbf{m}}(C(V)) \longrightarrow \mathrm{dIBL}^{\mathbf{n}}(C(\mathcal{H})).$$

The pushforward Maurer-Cartan element $\mathbf{n} = \mathbf{f}_* \mathbf{m}$ can be expressed as a sum of contributions of isomorphism classes of *trivalent ribbon graphs* (m_2^+ has namely three inputs), where a labeled ribbon graph Γ is decorated with m_2^+ at internal vertices, with the components of the i -th argument of \mathbf{n}_{lg} , i.e., elements of $\mathcal{H}(V)[1]$, at the i -th boundary component and with P at internal edges. Note that whereas (A) – (C) can be formulated without completions, infinite sums appear in \mathbf{n}_{lg} , and hence filtration and completions necessarily come into play.

The application to string topology of an oriented closed manifold M of dimension n comes from studying generalizations of (A) – (D) to the infinite-dimensional cyclic dga $(\Omega^*(M), \Pi, m_1, m_2)$. Here $\Omega^*(M)$ is the de Rham complex of M and the maps $\Pi : \Omega(M)[1]^{\otimes 2} \rightarrow \mathbb{R}$, $m_1 : \Omega(M)[1] \rightarrow \Omega(M)[1]$ and $m_2 : \Omega(M)[1]^{\otimes 2} \rightarrow \Omega(M)[1]$ are defined for all $\eta, \eta_1, \eta_2 \in \Omega(M)$ as follows:

$$\text{de Rham cyc. dga} \left\{ \begin{array}{l} \Pi(\theta\eta_1, \theta\eta_2) := (-1)^{\eta_1} \int_M \eta_1 \wedge \eta_2, \\ m_1(\theta\eta) := \theta d\eta, \\ m_2(\theta\eta_1, \theta\eta_2) := (-1)^{\eta_1} \theta(\eta_1 \wedge \eta_2), \end{array} \right. \quad (2.3)$$

where d is the de Rham differential, \wedge the wedge product, θ a formal symbol of degree -1 and η_1 in the exponent denotes the form-degree of η_1 . By picking a Riemannian metric on M , we obtain the subcomplex of harmonic forms

$$(\mathcal{H}^*(M), \Pi, m_1 \equiv 0)$$

with the projection $\pi_{\mathcal{H}} : \Omega(M) \rightarrow \mathcal{H}(M)$ coming from the Hodge decomposition. This

2. Summary and results

cyclic cochain complex shall be taken as the subcomplex in (B).

From technical reasons stemming from the fact that the non-degenerate pairing Π on $\Omega(M)[1]$ is not perfect, one has to restrict the construction in (A) to the subspace $B_{\text{cyc}}^*\Omega(M)_\infty$ of elements with a smooth Schwartz kernel. Then (A) and (B) work in the setting of the so called Fréchet IBL $_\infty$ -algebras introduced in [CFL15, Section 13]. However, the element $\mathbf{m}_{10} \in B_{\text{cyc}}^*\Omega(M)[3-n]$, which translates into the *Chern-Simons term*

$$m_2^+(\theta\eta_1, \theta\eta_2, \theta\eta_3) := (-1)^{\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \eta_1, \eta_2, \eta_3 \in \Omega(M),$$

does not define the canonical Maurer-Cartan element \mathbf{m} in (C) directly because $m_2^+ \notin \hat{B}_{\text{cyc}}^*\Omega(M)_\infty$. This also means that one cannot use (D) to conclude the existence of the pushforward Maurer-Cartan element \mathbf{n} .

Nevertheless, it was proposed to define \mathbf{n} formally using the summation over trivalent ribbon graphs as in the finite-dimensional case. We call such \mathbf{n} a *Chern-Simons Maurer-Cartan element*. In order to compute the contribution of a labeled trivalent ribbon graph Γ with k internal vertices, l boundary components and genus g to the value

$$\mathbf{n}_{lg}(\Omega_1 \otimes \cdots \otimes \Omega_l),$$

where $\Omega_i = s\omega_i$ for $\omega_1, \dots, \omega_l \in B^{\text{cyc}}\mathcal{H}(M)$, one starts by decorating internal vertices with integration variables x_1, \dots, x_k on the k -fold product $M \times \cdots \times M$, external vertices on the i -th boundary component with the components $\alpha_{i1}, \dots, \alpha_{is_i} \in \mathcal{H}(M)[1]$ of ω_i and internal edges with the Hodge propagator P . In this setting, P becomes the Schwartz kernel of \mathcal{P} in the sense of pseudo-differential operators; this P is necessarily singular at the diagonal Δ , so that we have only $P \in \Omega^{n-1}(M \times M \setminus \Delta)$. One then takes the wedge product of all forms in the decorated graph in the order and with the sign deduced from the labeling of Γ and computes the integral over x_1, \dots, x_k . Similar integrals appear in *perturbative Chern-Simons quantum field theory*.

Because of the singularity of P at Δ , the integrand described above is smooth only on the k -th configuration space of M . It is not clear that all the integrals converge and that the resulting \mathbf{n}_{lg} are well-defined and satisfy the Maurer-Cartan equation. The idea of work in progress [CVona] of K. Cieliebak and E. Volkov is to use iterated spherical blow-ups of the diagonals to resolve the singularities and obtain integrals of smooth forms on compact manifolds with corners; this guarantees integrability. The Maurer-Cartan equation for $\mathbf{n} = (\mathbf{n}_{lg})$ is then proven with the help of Stokes' formula and by showing that the contributions of hidden codimension-1 faces cancel. This method is similar to the

2. Summary and results

method from [AS91] and [AS93], where Feynman integrals of perturbative Chern-Simons theory were considered.

Having \mathfrak{n} , the twisted IBL_∞ -algebra $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(M)))$, which can be equivalently written as $\mathrm{dIBL}^{\mathfrak{n}}(C(H_{\mathrm{dR}}(M)))$ using the Hodge isomorphism $\mathcal{H}(M) \simeq H_{\mathrm{dR}}(M)$, should satisfy the following conjecture:

String topology conjecture (Conjecture 1.12 in [CFL15]). *Let M be a closed oriented manifold of dimension n and $H_{\mathrm{dR}}(M)$ its de Rham cohomology. Then there exists an IBL_∞ -structure on (a suitable version of) $B_{\mathrm{cyc}}^* H_{\mathrm{dR}}[2-n]$ whose homology equals the cyclic cohomology of the de Rham complex of M .*

The idea is that the \mathbb{S}^1 -equivariant homology of the free loop space $H_*^{\mathbb{S}^1}(LM)$ is isomorphic to a version of Connes' cyclic cohomology of the de Rham algebra $H_\lambda^*(\Omega^*(M))$, at least for simply-connected M . The precise relation will be established in yet another work in progress [CVona] of K. Cieliebak and E. Volkov using a chain-map coming from a cyclic version of Chen's iterated integrals. Now, a suitable degree shift of $H_\lambda^*(\Omega^*(M))$ is isomorphic to the homology of the boundary operator $\mathfrak{q}_{110}^{\mathfrak{m}}$ of the only formally defined dIBL -algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(\Omega(M)))$, which is according to (D) (formally) quasi-isomorphic to $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(M)))$ via the twisted IBL_∞ -morphism $\mathfrak{f}^{\mathfrak{m}}$.

The space $H_*^{\mathbb{S}^1}(LM)$ is equipped with an IBL -structure coming from the Chas-Sullivan string bracket \mathfrak{m}_2 and string cobracket \mathfrak{c}_2 ; these operations were defined geometrically on suitably transverse smooth chains in [CS99] and [CS02], respectively.¹ The natural question is: How is the IBL -structure $\mathfrak{m}_2, \mathfrak{c}_2$ related to the IBL -structure $\mathfrak{q}_{210}^{\mathfrak{n}}, \mathfrak{q}_{120}^{\mathfrak{n}}$ induced on $H_*^{\mathbb{S}^1}(LM)$ via the isomorphism from the string topology conjecture? The *extended string topology conjecture* asserts that these structures agree, and hence the operations $\mathfrak{q}_{210}^{\mathfrak{n}}, \mathfrak{q}_{120}^{\mathfrak{n}}$ defined on cyclic cochains provide a *chain model* for $\mathfrak{m}_2, \mathfrak{c}_2$. Based on our observations and explicit computations, we formulate an up-to-date version of the string topology conjecture for simply-connected manifolds (see Conjecture 4.5.1).

A large part of this text consists of setting up the algebraic base for the work with $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(M)))$. In addition to repeating the theory from [CFL15] in a slightly different formalism, we also include the following topics:

- A formula for the partial composition \circ_s in terms of operations of the canonical associative bialgebra on the symmetric algebra (Definition 3.2.2); formulas for $\mathfrak{q}_{klg}^{\mathfrak{n}}$ (Proposition 3.4.4).

¹In fact, \mathfrak{c}_2 is geometrically defined only on the homology relative to constant loops and \mathfrak{m}_2 does not always restrict to it.

2. Summary and results

- Definition of the cyclic cohomology of A_∞ -algebras (Definition 3.3.9) and its relation to the homology of \mathfrak{q}_{110}^n (Proposition 3.4.6); definitions of the reduced versions (Definitions 3.3.12, 3.4.7 and 3.4.8) and their relation to the unreduced versions (Propositions 3.4.10 and 3.3.13).
- A coordinate-free framework for the evaluation of labeled ribbon graphs (Definition A.1.1 and Proposition A.1.2); formal analogy of the finite-dimensional and the de Rham case which we use to obtain signs for the definition of \mathfrak{n} (Proposition A.2.1).
- Definition of the Hodge propagator (Definition 4.2.4) and of the Chern-Simons Maurer-Cartan element \mathfrak{n} (Definition 4.3.6).

Our first result is an explicit computation of $\mathrm{dIBL}^n(C(H_{\mathrm{dR}}(\mathbb{S}^n)))$ by finding a particular Hodge propagator and showing that all integrals which contribute to \mathfrak{n} vanish for $n \geq 3$; for $n = 1$, there is a non-vanishing integral whose value we compute (see Section 5.1); for $n = 2$, the existence of a non-vanishing integral remains open.

Theorem A (Explicit computation for \mathbb{S}^n). *Consider the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Define $\mathbf{1} := \theta \mathbf{1}$, $\mathbf{v} := \theta \mathrm{Vol} \in H_{\mathrm{dR}}(\mathbb{S}^n)[1]$, where Vol is the volume form, $\mathbf{1}$ the constant one and θ a formal symbol of degree -1 . There exists a Hodge propagator \mathbf{P} such that the following holds. For the homology of the twisted boundary operator \mathfrak{q}_{110}^n , we have:*

$$\begin{aligned} \mathbb{H}^n(C(H_{\mathrm{dR}}(\mathbb{S}^n)))[1] &:= H(\hat{\mathbf{B}}_{\mathrm{cyc}}^* H_{\mathrm{dR}}(\mathbb{S}^n)[3-n], \mathfrak{q}_{110}^n) \\ &= \begin{cases} \langle \mathbf{sv}^{i*}, \mathbf{s1}^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \geq 3 \text{ odd,} \\ \langle \mathbf{sv}^{2i-1*}, \mathbf{s1}^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \text{ even,} \\ \langle \mathbf{s} \sum_{k=1}^{\infty} c_k \mathbf{v}^{k*}, \mathbf{s1}^{2j-1*} \mid c_k \in \mathbb{R}, j \geq 1 \rangle & \text{for } n = 1. \end{cases} \end{aligned}$$

Here $\langle \cdot \rangle$ denotes the linear span over \mathbb{R} , $*$ the dual and \mathbf{s} is a formal symbol of degree $n-3$. The product \mathfrak{q}_{210}^n vanishes on \mathbb{H}^n except for the following relations for $n \geq 3$ odd

$$\mathfrak{q}_{210}^n(\mathbf{s1}^* \otimes \mathbf{sv}^{k*}) = \mathfrak{q}_{210}^n(\mathbf{sv}^{k*} \otimes \mathbf{s1}^*) = -(k-1)\mathbf{v}^{k-1*}$$

and the following relations for $n = 1$:

$$\mathfrak{q}_{210}^n(\mathbf{s1}^* \otimes \mathbf{s} \sum_{k=1}^{\infty} c_k \mathbf{v}^{k*}) = -\mathbf{s} \sum_{k=1}^{\infty} k c_{k+1} \mathbf{v}^{k*}.$$

The coproduct \mathfrak{q}_{120}^n as well as all higher operations \mathfrak{q}_{1lg}^n vanish on \mathbb{H}^n in every dimension n . For \mathbb{S}^1 , we have $\mathfrak{q}_{120}^n \neq \mathfrak{q}_{120}$ on the chain level; i.e., the twisting is non-trivial. For

2. Summary and results

$n \neq 2$, all higher operations vanish on the chain level.

If we mod out $s1^{2j-1*}$, i.e., if we consider the point-reduced version, then, after dropping s , the results agree with the string topology of M relative to one constant loop and with Chas-Sullivan operations. The only exception is $M = \mathbb{S}^1$. This supports the string topology conjecture for simply-connected manifolds and provides a counterexample for non-simply connected manifolds.

Our second result generalizes the previous explicit computation and shows that in many cases, the twists with \mathbf{n} and \mathbf{m} coincide. Its proof is a combination of facts from Section 4.4.

Theorem B (Triviality of the twist with \mathbf{n} on the chain level). *Let M be a closed oriented n -manifold. There exists a Hodge propagator P , the so called special Hodge propagator, such that the following holds for the twisted IBL_∞ -structure $\text{dIBL}^{\mathbf{n}}(C(H_{\text{dR}}(M)))$:*

- (1) *For the basic operations $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{210} \circ_1 \mathbf{n}_{10}$, $\mathbf{q}_{210}^{\mathbf{n}} = \mathbf{q}_{210}$, $\mathbf{q}_{120}^{\mathbf{n}} = \mathbf{q}_{120} + \mathbf{q}_{210} \circ_1 \mathbf{n}_{20}$, we have:*
 - a) *If $H_{\text{dR}}^1(M) = 0$, then $\mathbf{n}_{20} = 0$, and hence $\mathbf{q}_{120}^{\mathbf{n}} = \mathbf{q}_{120}$.*
 - b) *If M is geometrically formal, then $\mathbf{n}_{10} = \mathbf{m}_{10}$, and hence $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110}^{\mathbf{m}}$. (In fact, if in addition $H_{\text{dR}}^1(M) = 0$, then $\mathbf{n} = \mathbf{m}$, at least for $n \neq 2$.)*
- (2) *For the higher operations $\mathbf{q}_{1lg}^{\mathbf{n}} = \mathbf{q}_{210} \circ_1 \mathbf{n}_{20}$ with $(l, g) \neq (1, 0), (2, 0)$, we have $\mathbf{n}_{lg} = 0$, and hence $\mathbf{q}_{1lg}^{\mathbf{n}} = 0$ with the possible exception of surfaces and 3-manifolds with $H_{\text{dR}}^1(M) \neq 0$.*

The following are some topics for future work:

- (1) We would like to find out which field theory has P as a propagator and the trivalent ribbon graphs with harmonic forms as Feynman graphs. Note that due to the polarization $P = \sum_{k=0}^{n-1} P^{(k)}(x, y)$, the theory has a field in every degree fields. This is in contrast to the Chern-Simons theory for $n = 3$, whose field is a 1-form. Some actions were suggested to the author by Dr. Puzzio and Prof. A. Cattaneo.
- (2) We would like to improve Theorem B by showing that the higher operations for \mathbb{S}^2 vanish. If this is the case, then the statement that all higher operations vanish for every manifold M with $H_{\text{dR}}^1(M) = 0$ is true.
- (3) For a formal simply-connected manifold M , we would like to investigate whether $\text{dIBL}^{\mathbf{n}}(C(H_{\text{dR}}(M)))$ and $\text{dIBL}^{\mathbf{m}}(C(H_{\text{dR}}(M)))$ are homotopy equivalent as IBL_∞ -algebras. If not, we would like to understand the obstruction.

2. Summary and results

- (4) We would like to compute $\mathrm{dIBL}^n(C(H_{\mathrm{dR}}(M)))$ for surfaces Σ_g with $g \geq 1$ and formulate a string topology conjecture for non-simply connected manifolds.
- (5) We would like to know whether the Schwartz kernel P_{std} of $\mathcal{P}_{\mathrm{std}} = -d^* \Delta^{-1}$ (the so called standard Hodge propagator), where d^* is the codifferential and Δ the Hodge-de Rham Laplacian, extends smoothly to a blow-up. If yes, then it is a canonical special Hodge propagator, and hence the statement of Theorem B holds.
- (6) We would like to define a generalization of an IBL_∞ -algebra — a weak, non-reduced IBL_∞ -algebra with a gauge group — and understand its precise relation to perturbative Chern-Simons theory within the BV-formalism.
- (7) Identify the theory from which the integrals come from, i.e., write down an action.

In the end, let us summarize some existing work on IBL_∞ -algebras which helped us to understand IBL_∞ -algebras in broader context: In [MS11], they find an IBL_∞ -structure in open-closed string field theory. In [DJP17], they view IBL_∞ -algebras as algebras over the cobar construction of the closed Frobenius properad. In [MV15], they consider IBL_∞ -algebras as a particular case of BV_∞ - or, more generally, MV -algebras.

3. Algebraic structures

In Section 3.1, we recall weight-grading (Definition 3.1.1), Koszul sign (Definition 3.1.2), degree shift (Definition 3.1.3), filtrations (Definition 3.1.8) and completions (Definition 3.1.9). We prove the Künneth formula for completed symmetric cohomology (Proposition 3.1.13).

In Section 3.2, we review basics of IBL_∞ -algebras from [CFL15]. We define the exterior algebra EC over a graded vector space C as the symmetric algebra S over $C[1]$ (Definition 3.2.1) and use the operations μ and Δ of the structure of an associative bialgebra on $S(C[1])$ to give explicit formulas for the partial compositions \circ_{h_1, \dots, h_k} (Definition 3.2.2). We use the compositions to define the notion of an IBL_∞ -algebra (q_{klg}) on C (Definition 3.2.4), a Maurer-Cartan element (n_{lg}) (Definition 3.2.6) and twisted operations (q_{klg}^n) (Definition 3.2.7). We mention that an IBL-algebra according to our definition is an odd degree shift of a classical IBL-algebra (Proposition 3.2.5). We define the induced IBL-structure on homology (Definition 3.2.8), briefly discuss the BV-formalism (Remark 3.2.9) and mention weak IBL_∞ -algebras (Remark 3.2.10). Finally, we summarize the situation for twisted dIBL-algebras (Proposition 3.2.11) and briefly discuss higher operations (Remark 3.2.12).

In Section 3.3, we define the (weight-reduced) dual cyclic bar-complex $B_{\text{cyc}}^* V$ of a graded vector space V (Definition 3.3.1) and introduce some notation (Notation 3.3.3). We then summarize some facts about the completions $\hat{B}_{\text{cyc}}^* V$ and $\hat{E}_k B_{\text{cyc}}^* V$ (Proposition 3.3.6). We define the notion of a cyclic A_∞ -structure on V (Definition 3.3.7) and its Hochschild and cyclic cohomology (Definition 3.3.9). We recall strict units and strict augmentations (Definition 3.3.10), define the reduced dual cyclic bar complex $B_{\text{cyc}, \text{red}}^* V$ (Definition 3.3.12) and sketch a proof of the fact that the cyclic cohomology is a direct sum of the reduced cyclic cohomology and the cyclic cohomology of the ground field (Proposition 3.3.13). We relate our version of the cyclic cohomology for dga's to the classical version from [Lod92] (Proposition 3.3.14). We also show that the reduced spaces for a simply connected and connected V are complete (Proposition 3.3.15).

In Section 3.4, we review the construction of the canonical dIBL-structure $\text{dIBL}(C(V))$ (Definition 3.4.1) and the canonical Maurer-Cartan element m (Definition 3.4.2) starting

3. Algebraic structures

from a cyclic dga (V, Π, m_1, m_2) . We give formulas for the operations (q_{llg}^n) of the IBL_∞ -algebra $\text{dIBL}^n(C(V))$ twisted by a Maurer-Cartan element \mathfrak{n} (Proposition 3.4.4). We consider the A_∞ -structure induced on V by \mathfrak{n}_{10} (Definition 3.4.5) and relate its cyclic cohomology to the homology of q_{l10}^n (Proposition 3.4.6). We define the reduced canonical dIBL -algebra $\text{dIBL}(C_{\text{red}}(V))$ (Definition 3.4.7) and the notion of a strictly reduced Maurer-Cartan element (Definition 3.4.8). The twisted IBL_∞ -structure then splits into the reduced part and the part generated by 1^{i*} , which we can explicitly compute (Proposition 3.4.10).

3.1. Gradings, degree shifts and completions

We will work with vector spaces over \mathbb{R} , possibly infinite-dimensional, graded by the degree $d \in \mathbb{Z}$ and the weight $k \in \mathbb{N}_0$.

Definition 3.1.1 (Weight-graded vector spaces). *A graded vector space is a vector space W together with a collection of subspaces $W^d \subset W$ for all $d \in \mathbb{Z}$ such that*

$$W = \bigoplus_{d \in \mathbb{Z}} W^d.$$

Elements of W^d are called homogenous of degree d ; given $w \in W^d$, we denote the degree of w by $|w| := d$.

A linear map of graded vector spaces $f : W_1 \rightarrow W_2$ is called homogenous of degree $|f| \in \mathbb{Z}$ if it holds

$$f(W_1^d) \subset W_2^{d+|f|} \quad \text{for all } d \in \mathbb{Z}. \quad (3.1)$$

A weight-graded vector space is a graded vector space W together with a collection of subspaces $W_k^d \subset W^d$ for all $k \in \mathbb{N}_0$ and $d \in \mathbb{Z}$ such that

$$W^d = \bigoplus_{k \in \mathbb{N}_0} W_k^d \quad \text{for all } d \in \mathbb{Z}.$$

We define the weight- k component of W by

$$W_k := \bigoplus_{d \in \mathbb{Z}} W_k^d \quad \text{for all } k \in \mathbb{N}_0.$$

If $W_0^d = 0$ for all $d \in \mathbb{Z}$, we say that W is weight-reduced. We define the weight-reduced

3. Algebraic structures

subspace \overline{W} of a weight-graded vector space W by

$$\overline{W} := \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}} W_k^d.$$

The superscript $*$ usually denotes the dual in the category we work in, e.g., chain complexes. For a (weight-)graded vector space W , we also introduce the following notation to avoid confusion:

$$\begin{aligned} W^{*\text{lin}} &:= \{\psi : W \rightarrow \mathbb{R} \text{ linear}\} \quad \dots \quad \text{linear dual,} \\ W^{*\text{g}} &:= \bigoplus_{d \in \mathbb{Z}} \prod_{k \in \mathbb{N}_0} W_k^{d*} \quad \dots \quad \text{graded dual,} \\ W^{*\text{wg}} &:= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}_0} W_k^{d*} \quad \dots \quad \text{weight-graded dual.} \end{aligned} \tag{3.2}$$

Here, $W^{*\text{lin}}$ is a vector space, $W^{*\text{g}}$ a graded vector space and $W^{*\text{wg}}$ a weight-graded vector space. The grading convention from (3.2) is the cohomological grading convention, which differs from the convention (3.1) for maps $f : W \rightarrow \mathbb{R}$ by degree reversal (for this, see Definition 3.1.3).

We identify $W^{*\text{g}}$ with the subspace of $W^{*\text{lin}}$ generated by homogenous maps and $W^{*\text{wg}}$ with the subspace of $W^{*\text{lin}}$ generated by maps which are non-zero only on finitely many W_k^d ; hence, under this identification, we have the tower of inclusions

$$W^{*\text{wg}} \subset W^{*\text{g}} \subset W^{*\text{lin}}.$$

Definition 3.1.2 (Koszul sign). Let $k \geq 1$, and let $\sigma \in \mathbb{S}_k$ be a permutation on k elements, i.e., a bijection of the set $\{1, \dots, k\}$. For $i = 1, \dots, k$, let a_i and b_i be graded symbols of degrees $|a_i|$ and $|b_i|$, respectively. We denote by

$$\varepsilon(\sigma, a) \quad \text{and} \quad \varepsilon(a, b)$$

the Koszul signs of the transformations

$$a_1 \dots a_k \mapsto a_{\sigma_1^{-1}} \dots a_{\sigma_k^{-1}} \quad \text{and} \quad a_1 \dots a_k b_1 \dots b_k \mapsto a_1 b_1 \dots a_k b_k,$$

respectively. Here $\sigma_i^{-1} := \sigma^{-1}(i)$. The Koszul sign is computed by permuting the left-hand side to the right-hand side via transpositions of two adjacent elements such that whenever we transpose two graded symbols, e.g., $a_i \longleftrightarrow a_j$, we multiply with $(-1)^{|a_i||a_j|}$.

We emphasize that the Koszul sign depends only on the initial and the final order of

3. Algebraic structures

the graded symbols and not on the sequence of transpositions; this is easy to show.

Definition 3.1.3 (Degree shift and grading reversal). *Let $A \in \mathbb{Z}$. The degree shift by A is a functor which associates to a graded vector space W the graded vector space $W[A]$ with degree- d components*

$$W[A]^d := W^{d+A} \quad \text{for all } d \in \mathbb{Z}.$$

There is the canonical degree shift morphism

$$W \longrightarrow W[A] \tag{3.3}$$

of degree $-A$ mapping W^d identically to $W[A]^{d-A}$. We view this morphism as multiplication from the left with a formal symbol s_A of degree $|s_A| = -A$, so that (3.3) can be written as $w \in W \mapsto s_A w \in W[A]$.

Given graded vector spaces W_1, W_2 and constants $A_1, A_2 \in \mathbb{Z}$, we associate to a morphism $f : W_1 \rightarrow W_2$ its degree shift $f : W_1[A_1] \rightarrow W_2[A_2]$ by defining¹

$$f(s_{A_1} w) := s_{A_2} f(w) \quad \text{for all } w \in W_1. \tag{3.4}$$

Notice that if $f : W_1 \rightarrow W_2$ has degree $|f|$, then $f : W_1[A_1] \rightarrow W_2[A_2]$ has degree $|f| + A_1 - A_2$.

The grading reversal r is a functor which associates to a graded vector space W the graded vector space $r(W)$ with

$$r(W)^d := W^{-d} \quad \text{for all } d \in \mathbb{Z}.$$

There is the canonical morphism $W \rightarrow r(W)$ mapping W^d identically to W^{-d} for every $d \in \mathbb{Z}$. The degree reversal of a morphism $f : W_1 \rightarrow W_2$ is the morphism $f : r(W_1) \rightarrow r(W_2)$ defined by conjugating f with the canonical morphism. If $|f|$ is the degree of $f : W_1 \rightarrow W_2$, then $-|f|$ is the degree of $f : r(W_1) \rightarrow r(W_2)$.

In our main reference [CFL15], they view W and $W[A]$ as one vector space with two different gradings $\deg(\cdot)$ and $|\cdot|$, respectively; these are related by

$$|w| = \deg(w) - A \quad \text{for all homogenous } w \in W.$$

On the other hand, we think of W and $W[A]$ as of two different graded vector spaces

¹This convention is not optimal; see Remark 3.1.4.

3. Algebraic structures

and never use the same symbol for an element $w \in W$ and its degree shift $s_A w \in W[A]$. It allows us to use just one notation $|\cdot|$ for the gradings on both W and $W[A]$. However, in order to preserve compatibility with [CFL15], we will also sometimes use the notation $\deg(w)$ (in the exponent just $(-1)^w$) for the degrees on W .

For graded vector spaces W_1, \dots, W_k and constants $A_1, \dots, A_k \in \mathbb{Z}$, we identify

$$W_1[A_1] \otimes \dots \otimes W_k[A_k] \simeq (W_1 \otimes \dots \otimes W_k)[A_1 + \dots + A_k]$$

using the *Koszul convention for the tensor product*; for homogenous elements $w_1 \in W_1, \dots, w_k \in W_k$, it reads

$$\begin{aligned} s_{A_1} w_1 \otimes \dots \otimes s_{A_k} w_k &= \varepsilon(s_A, w) \underbrace{s_{A_1} \dots s_{A_k}}_{=: s_{A_1 + \dots + A_k}} w_1 \otimes \dots \otimes w_k. \end{aligned} \quad (3.5)$$

If $A_1 = \dots = A_k =: A$ is fixed in the context, which is our usual case, we omit the subscript A and write just s .

In the case of the multilinear map $f : W_1 \otimes \dots \otimes W_k \rightarrow V_1 \otimes \dots \otimes V_l$, the combination of (3.4) and (3.5) gives for $f : W_1[A_1] \otimes \dots \otimes W_k[A_k] \rightarrow V_1[B_1] \otimes \dots \otimes V_l[B_l]$ the following:

$$f(s_{A_1 + \dots + A_k} w_1 \otimes \dots \otimes w_k) = s_{B_1 + \dots + B_l} f(w_1 \otimes \dots \otimes w_k). \quad (3.6)$$

Remark 3.1.4 (Why is this sign convention bad?). Let us illustrate that (3.6) is not compatible with the following standard Koszul rule:

$$(K) : \quad (f_1 \otimes f_2)(w_1 \otimes w_2) = (-1)^{|f_2||w_1|} f_1(w_1) \otimes f_2(w_2).$$

On one hand, we get

$$\begin{aligned} (f_1 \otimes f_2)(s^2 w_1 \otimes w_2) &\stackrel{(3.6)}{=} s^2 (f_1 \otimes f_2)(w_1 \otimes w_2) \\ &\stackrel{(K)}{=} (-1)^{|f_2||w_1|} s^2 f_1(w_1) \otimes f_2(w_2) \\ &\stackrel{(3.5)}{=} (-1)^{|f_2||w_1| + A(|f_1| + |w_1|)} s f_1(w_1) \otimes s f_2(w_2). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} (f_1 \otimes f_2)(s^2 w_1 \otimes w_2) &\stackrel{(3.5)}{=} (-1)^{A|w_1|} (f_1 \otimes f_2)(s w_1 \otimes s w_2) \\ &\stackrel{(K)}{=} (-1)^{A|w_1| + |f_2|(A + |w_1|)} f_1(s w_1) \otimes f_2(s w_2) \\ &\stackrel{(3.6)}{=} (-1)^{A|w_1| + |f_2|(A + |w_1|)} s f_1(w_1) \otimes s f_2(w_2). \end{aligned}$$

3. Algebraic structures

The results differ by $(-1)^{A(|f_1|+|f_2|)}$. Therefore, we can not use (K) to identify the tensor product $\text{Hom}(W_1, V_1) \otimes \text{Hom}(W_2, V_2)$ with a subspace of $\text{Hom}(W_1 \otimes W_2, V_1 \otimes V_2)$ in general. We will rather define an ad-hoc pairing in the case where we need it (see Definition 3.3.4).

Another caveat is that in the case of the tensor product, the degree shift by A_1 followed by the degree shift by A_2 is not the same as the degree shift by $A_1 + A_2$. Indeed, we compute

$$\begin{aligned} (s_{A_1+A_2} w_1) \otimes (s_{A_1+A_2} w_2) &= (s_{A_2} s_{A_1} w_1) \otimes (s_{A_2} s_{A_1} w_2) \\ &= (-1)^{A_2(A_1+|w_1|)} s_{A_2}^2 (s_{A_1} w_1) \otimes (s_{A_1} w_2) \\ &= (-1)^{A_2 A_1 + (A_1+A_2)|w_1|} s_{A_2}^2 s_{A_1}^2 (w_1 \otimes w_2) \\ &= (-1)^{A_2 A_1 + (A_1+A_2)|w_1|} s_{2(A_1+A_2)} (w_1 \otimes w_2), \end{aligned}$$

which differs by $(-1)^{A_1 A_2}$ from the direct degree shift by $A_1 + A_2$. Therefore, we have to always remember the vector spaces which we started with and the sequence of degree shifts.

Note that we also have the unnatural “ $s_{A_1} s_{A_2} = s_{A_1+A_2} = s_{A_2} s_{A_1}$ ” due to (3.5). \triangleleft

Remark 3.1.5 (Is there a better sign convention?). The author originally respected the Koszul rule for the algebra with formal symbols and considered the following map $s_*^l \bar{s}^{k*} f : W[A]^{\otimes k} \rightarrow V[A]^{\otimes l}$ as the degree shift of $f : W^{\otimes k} \rightarrow V^{\otimes l}$:

$$(s_*^l \bar{s}^{k*} f)(s^k w_1 \otimes \cdots \otimes w_k) = (-1)^{k|f|A + \frac{1}{2}k(k-1)A} s^l f(w_1 \otimes \cdots \otimes w_k). \quad (3.7)$$

Here \bar{s} denotes the “inverse” of s with $|\bar{s}| = -|s|$, $s_*^l f = s^l \circ f$ is the post-composition, $\bar{s}^{k*} f = (-1)^{kA|f|} f \circ \bar{s}^k$ the pre-composition, and the sign $\varepsilon(s, \bar{s}) = (-1)^{\frac{1}{2}k(k-1)A}$ comes from the “collision” $\bar{s}_1 \dots \bar{s}_k s_1 \dots s_k \mapsto \bar{s}_1 s_1 \dots \bar{s}_k s_k$.

However, the author did not manage to reprove the theory in [CFL15] using (3.7) (because of too many “external” signs appearing and a problem with disconnected graphs). A motivation to try a different sign convention was to explain some artificial signs in [CFL15] and formulate their coordinate constructions invariantly in order to generalize them to the “continuous” de Rham case.

It might be possible to deduce a “universal” sign convention “respecting” the Koszul rules by considering the category of chain complexes and graded morphisms \mathcal{C} as the category enriched in the closed monoidal category of chain complexes and chain maps of degree 0. One can then define the enriched degree shift functor $s_A : \mathcal{C} \rightarrow \mathcal{C}$, embed $\mathcal{C}^{\otimes k} \subset \mathcal{C}$ using (K) and study enriched natural transformations in the algebra of functors consisting

3. Algebraic structures

of tensor products and compositions of s_A , $\text{Hom}(\cdot, \cdot)$ and the dual $*$. The question is whether the sign rules for degree shifts, in particular (3.7), are uniquely determined by the requirements on enriched functors and enriched natural transformations.

Another idea for how to specify the degree shift convention (3.7) is the observation that maps $f : W^{\otimes k} \rightarrow W^{\otimes l}$ under consideration usually form an algebra $\mathcal{P} \rightarrow \text{End}_W$ of a certain PROP \mathcal{P} over W ; here End_W denotes the endomorphism PROP. Suppose that there is a natural notion of the degree shift $\mathcal{P}[A]$ of a PROP \mathcal{P} and a canonical degree shift morphism $\mathcal{P} \rightarrow \mathcal{P}[A]$. The PROPs $(\text{End}_W)[A]$ and $\text{End}_{W[A]}$ are better to be isomorphic, and any intelligent degree shift convention $\psi_{k,l} : \text{Hom}(W^{\otimes k}, W^{\otimes l})[A(k-l)] \rightarrow \text{Hom}(W[A]^{\otimes k}, W[A]^{\otimes l})$ should induce an isomorphism of the PROPs $(\text{End}_W)[A]$ and $\text{End}_{W[A]}$. \triangleleft

Definition 3.1.6 (Standard action of permutations). *For $k \geq 1$ and $\sigma \in \mathbb{S}_k$ ($:=$ the group of permutations on k elements), we define the standard action of σ on $W^{\otimes k}$ by*

$$\sigma(w_1 \otimes \cdots \otimes w_k) := \varepsilon(\sigma, w) w_{\sigma_1^{-1}} \otimes \cdots \otimes w_{\sigma_k^{-1}} \quad (3.8)$$

for all homogenous $w_1, \dots, w_k \in W$.

Notice that the i -th vector is permuted to the σ_i -th place — this is the “active” convention for permutations.

Definition 3.1.7 (Symmetric algebra). *Let $T(V) := \bigoplus_{k \geq 0} V^{\otimes k}$ be the tensor algebra over a graded vector space V . The symmetric algebra over V is defined by $S(V) := \bigoplus_{k \geq 0} S_k(V)$, where*

$$S_k(V) := V^{\otimes k} / \sum_{\sigma \in \mathbb{S}_k} \text{im}(\mathbb{1} - \sigma) \quad (=:\mathbb{S}_k\text{-coinvariants}).$$

It is a weight-graded vector space with components denoted by $(S_k V)^d$ for all $d \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. Note that $S_0 V = \mathbb{R}$ has degree 0 by definition. Consider the canonical projection

$$\begin{aligned} \pi : T(V) &\longrightarrow S(V) \\ v_1 \otimes \cdots \otimes v_k &\longmapsto v_1 \cdots v_k. \end{aligned}$$

If $v_i \in V$ are homogenous, we call $v_1 \cdots v_k$ a generating word; we have

$$v_1 \cdots v_k = \varepsilon(\sigma, v) v_{\sigma_1^{-1}} \cdots v_{\sigma_k^{-1}} \quad \text{for every } \sigma \in \mathbb{S}_k.$$

3. Algebraic structures

Let $\iota : S(V) \rightarrow T(V)$ be the section of π defined by

$$\iota(v_1 \cdots v_k) := \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \varepsilon(\sigma, v) v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_k^{-1}}.$$

We use it to identify $S(V)$ with the subspace of symmetric tensors

$$\iota(S_k(V)) = \bigcap_{\sigma \in \mathbb{S}_k} \ker(1 - \sigma) \subset T_k(V) \quad (=:\mathbb{S}_k\text{-invariants}).$$

Definition 3.1.8 (Filtrations). *Let W be a graded vector space. A filtration of W is a collection of linear subspaces $\mathcal{F}^\lambda W \subset W$ for $\lambda \in \mathbb{R}$ such that we have either*

- $\mathcal{F}^{\lambda_1} W \subset \mathcal{F}^{\lambda_2} W$ for all $\lambda_1 \leq \lambda_2$ \iff increasing filtration, or
- $\mathcal{F}^{\lambda_1} W \supset \mathcal{F}^{\lambda_2} W$ for all $\lambda_1 \leq \lambda_2$ \iff decreasing filtration.

We will assume that our filtrations are graded in the following sense:

$$\forall \lambda \in \mathbb{R} : \quad \mathcal{F}^\lambda W = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^\lambda W^d, \quad \text{where} \quad \mathcal{F}^\lambda W^d := \mathcal{F}^\lambda W \cap W^d.$$

A filtration $\mathcal{F}^\lambda W$ is called:

$$\begin{aligned} \text{exhaustive} & \iff \bigcup_{\lambda \in \mathbb{R}} \mathcal{F}^\lambda W = W; \\ \text{Hausdorff} & \iff \bigcap_{\lambda \in \mathbb{R}} \mathcal{F}^\lambda W = 0; \\ \mathbb{Z}\text{-gapped} & \iff \mathcal{F}^\lambda W = \mathcal{F}^{[\lambda]} W \text{ for all } \lambda \in \mathbb{R}; \\ \text{bounded from below} & \iff \exists \lambda \in \mathbb{R} : \mathcal{F}^\lambda W = 0; \\ \text{bounded from above} & \iff \exists \lambda \in \mathbb{R} : \mathcal{F}^\lambda W = W. \end{aligned}$$

Given a graded vector space W filtered by a \mathbb{Z} -gapped filtration $\mathcal{F}^\lambda W$, we associate to it the bi-graded vector space

$$\text{gr}(W) = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{\lambda \in \mathbb{Z}} \text{gr}(W)_\lambda^d$$

3. Algebraic structures

called the graded module whose components are given as follows:²

$$\forall d, \lambda \in \mathbb{Z} : \quad \text{gr}(W)_\lambda^d := \begin{cases} \mathcal{F}^\lambda W^d / \mathcal{F}^{\lambda-1} W^d & \text{for increasing } \mathcal{F}^\lambda W, \\ \mathcal{F}^\lambda W^d / \mathcal{F}^{\lambda+1} W^d & \text{for decreasing } \mathcal{F}^\lambda W. \end{cases}$$

We naturally extend a filtration over degree shifts, graded duals, direct sums, tensor products and symmetric products as follows:

$$\begin{aligned} \mathcal{F}^\lambda W[A]^d &:= \mathcal{F}^\lambda W^{d+A}, \\ \mathcal{F}^\lambda (W^{*\mathfrak{g}})^d &:= \{\psi \in W^{d*} \mid \psi|_{\mathcal{F}^\lambda W} = 0\}, \\ \mathcal{F}^\lambda \left(\bigoplus_{i \in I} W_i \right)^d &:= \bigoplus_{i \in I} \mathcal{F}^\lambda W_i^d, \\ \mathcal{F}^\lambda (W_1 \otimes \cdots \otimes W_k)^d &:= \bigoplus_{\substack{d_1, \dots, d_k \in \mathbb{Z} \\ d_1 + \dots + d_k = d}} \sum_{\substack{\lambda_1, \dots, \lambda_k \in \mathbb{R} \\ \lambda_1 + \dots + \lambda_k = \lambda}} \mathcal{F}^{\lambda_1} W_1^{d_1} \otimes \cdots \otimes \mathcal{F}^{\lambda_k} W_k^{d_k}, \\ \mathcal{F}^\lambda (S_k V)^d &:= \pi(\mathcal{F}^\lambda (V^{\otimes k})^d), \end{aligned}$$

where $\pi : T(V) \rightarrow S(V)$ is the canonical projection. The ground field \mathbb{R} is filtered by the trivial filtration:

$$\mathcal{F}^\lambda \mathbb{R} := \begin{cases} \mathbb{R} & \lambda \leq 0, \\ 0 & \lambda > 0. \end{cases} \quad (3.9)$$

If (W, ∂) is a filtered chain complex, we filter the homology as follows:

$$\forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda H_d(W, \partial) := \{\alpha \in H_d(C, \partial) \mid \exists w \in \alpha : w \in \mathcal{F}^\lambda W^d\}.$$

Definition 3.1.9 (Completions). Let W be a graded vector space filtered by a decreasing filtration $\mathcal{F}^\lambda W$. The filtration degree of $w \in W$ is defined by

$$\|w\| := \sup\{\lambda \in \mathbb{R} \mid w \in \mathcal{F}^\lambda W\}.$$

The filtration degree of a linear map $f : W_1 \rightarrow W_2$ is defined by

$$\|f\| := \sup\{\lambda \in \mathbb{R} \mid \|f(w)\| \geq \|w\| + \lambda \ \forall w \in W_1\}.$$

²The definition is made in such a way that if r is the functor which reverses λ , i.e., $r(\mathcal{F})^\lambda = \mathcal{F}^{-\lambda}$, then it holds $r(\text{gr}(\mathcal{F})) = \text{gr}(r(\mathcal{F}))$.

3. Algebraic structures

We say that the filtration degree is finite if $\|f\| > -\infty$. Note that $\|0\| = \infty$.

The completion of W is the graded vector space

$$\hat{W} := \bigoplus_{d \in \mathbb{Z}} \hat{W}^d,$$

where for all $d \in \mathbb{Z}$ we define

$$\hat{W}^d := \left\{ \sum_{i=0}^{\infty} w_i \mid \forall i \in \mathbb{N}_0 : w_i \in W^d; \|w_i\| \rightarrow \infty \text{ as } i \rightarrow \infty \right\} / \sim.$$

Here $\sum_{i=0}^{\infty} w_i \sim \sum_{i=0}^{\infty} w'_i$ if and only if $\|\sum_{i=0}^n (w_i - w'_i)\| \rightarrow \infty$ as $n \rightarrow \infty$.³ The completion \hat{W} is canonically filtered by the filtration $\mathcal{F}^\lambda \hat{W}$ defined as follows:

$$\forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda \hat{W}^d := \left\{ \sum_{i=0}^{\infty} w_i \in \hat{W}^d \mid \forall i \in \mathbb{N}_0 : w_i \in \mathcal{F}^\lambda W^d \right\}.$$

We denote the completion of $W_1 \otimes \cdots \otimes W_k$ by $W_1 \hat{\otimes} \cdots \hat{\otimes} W_k$ and the completion of $S_k V$ by $\hat{S}_k V$.

A map $f : W_1 \rightarrow W_2$ of finite filtration degree extends continuously to a linear map $f : \hat{W}_1 \rightarrow \hat{W}_2$; this continuous extension is defined by

$$f\left(\sum_{i=0}^{\infty} w_i\right) := \sum_{i=0}^{\infty} f(w_i) \quad \text{for all } \sum_{i=0}^{\infty} w_i \in \hat{W}.$$

Remark 3.1.10 (Completed tensor product). Using Proposition 3.1.11 below, one can show that the *completed tensor product* $\hat{\otimes}$ is associative and that $W_1 \hat{\otimes} W_2 \simeq \hat{W}_1 \hat{\otimes} \hat{W}_2$. By refining this argument, one can show that $\hat{S}_k V \simeq \hat{S}_k \hat{V}$ for any $k \in \mathbb{N}$. \triangleleft

A weight-graded vector space W is canonically *filtered by weights*:

$$\forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda W^d := \bigoplus_{k \leq \lambda} W_k^d. \quad (3.10)$$

This filtration is \mathbb{Z} -gapped, exhaustive, Hausdorff, increasing and bounded from below. The induced filtration on the graded dual $W^{*\text{g}}$ is \mathbb{Z} -gapped, Hausdorff, decreasing and bounded from above (and thus automatically exhaustive). It holds $\text{gr}(W) \simeq W$, and it is

³In fact, \hat{W} is the inverse limit $\varprojlim_{\lambda}^{\text{gr}} (W / \mathcal{F}^\lambda W)$ in the category of graded vector spaces and \hat{W}^d the inverse limit $\varprojlim_{\lambda} (W^d / \mathcal{F}^\lambda W^d)$ in the category of vector spaces. As a side-remark, if we forget the grading on W , we might also consider $\varprojlim_{\lambda} (W / \mathcal{F}^\lambda W)$, which would be a vector space containing \hat{W} as a subspace.

3. Algebraic structures

easy to see from (3.2) that the canonical map $W^{*\text{wg}} \rightarrow W^{*\text{g}}$ induces the isomorphism

$$\widehat{W^{*\text{wg}}} \simeq W^{*\text{g}}.$$

We also see that the condition

$$(WG0) : \quad \forall d \in \mathbb{Z} \exists J \subset \mathbb{N}_0, |J| < \infty \forall k \in \mathbb{N}_0 \setminus J : \quad W_k^d = 0$$

is equivalent to $W^{*\text{wg}} = W^{*\text{g}}$.

A useful tool to compare completions is the following proposition:

Proposition 3.1.11 ([Fre17, Proposition 7.3.7], Isomorphism criterion). *Let W_1 and W_2 be graded vector spaces filtered by \mathbb{Z} -gapped filtrations which are decreasing and bounded from above. Suppose that $f : W_1 \rightarrow W_2$ is a filtration preserving homogenous linear map. Then the continuous extension $f : \hat{W}_1 \rightarrow \hat{W}_2$ is an isomorphism if and only if the induced map $f : \text{gr}(W_1) \rightarrow \text{gr}(W_2)$ is an isomorphism.*

Proof. The implication from the right to the left is obtained from the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{gr}(W_1)_\lambda & \hookrightarrow & W_1/\mathcal{F}^{\lambda+1}W_1 & \twoheadrightarrow & W_1/\mathcal{F}^\lambda W_1 \longrightarrow 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ 0 & \longrightarrow & \text{gr}(W_2)_\lambda & \hookrightarrow & W_2/\mathcal{F}^{\lambda+1}W_2 & \twoheadrightarrow & W_2/\mathcal{F}^\lambda W_2 \longrightarrow 0 \end{array}$$

by induction using the definition of \hat{W} as the inverse limit of $W/\mathcal{F}^\lambda W$ (see Footnote 3 on page 32). The other implication follows from $\mathcal{F}^\lambda \hat{W}/\mathcal{F}^{\lambda-1} \hat{W} \simeq \mathcal{F}^\lambda W/\mathcal{F}^{\lambda-1} W$. \square

For a graded vector space W filtered by a \mathbb{Z} -gapped filtration, consider the following conditions:

$$\begin{aligned} (WG1) : \quad & \forall \lambda \in \mathbb{Z} \exists I \subset \mathbb{Z}, |I| < \infty \forall d \in \mathbb{Z} \setminus I : \quad \text{gr}(W)_\lambda^d = 0, \\ (WG2) : \quad & \forall d, \lambda \in \mathbb{Z} : \quad \dim(\text{gr}(W)_\lambda^d) < \infty. \end{aligned} \tag{3.11}$$

Lemma 3.1.12 (Completion of symmetric powers of the graded dual). *Let W be a graded vector space filtered by an exhaustive \mathbb{Z} -gapped filtration $\mathcal{F}^\lambda W$ which is increasing and bounded from below. If (WG1) & (WG2) are satisfied, then the natural map $S_k(W^{*\text{g}}) \rightarrow (S_k W)^{*\text{g}}$ induces the isomorphism*

$$\hat{S}_k(W^{*\text{g}}) \simeq (S_k W)^{*\text{g}} \quad \text{for every } k \in \mathbb{N}.$$

Note that we filter graded duals by the induced filtration from Definition 3.1.8.

3. Algebraic structures

Proof. The natural map $S_k(W^{*\mathfrak{g}}) \rightarrow (S_k W)^{*\mathfrak{g}}$ is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space $(S_k W)^{*\mathfrak{g}}$ is already complete (the dual space $W^{*\mathfrak{g}}$ is complete, provided that the filtration of W is exhaustive), and thus we obtain the map $\hat{S}_k(W^{*\mathfrak{g}}) \rightarrow (S_k W)^{*\mathfrak{g}}$. According to Proposition 3.1.11, this map is an isomorphism if and only if the induced map $\text{gr}(S_k(W^{*\mathfrak{g}})) \rightarrow \text{gr}((S_k W)^{*\mathfrak{g}})$ is. This is shown by the following computation (the maps involved are natural in at least one direction):

$$\begin{aligned}
\frac{\mathcal{F}^\lambda(W^{\otimes k * \mathfrak{g}})^d}{\mathcal{F}^{\lambda+1}(W^{\otimes k * \mathfrak{g}})^d} &\simeq \frac{\mathcal{F}^\lambda(W^{\otimes k})^{d*}}{\mathcal{F}^{\lambda+1}(W^{\otimes k})^{d*}} \simeq \left(\frac{\mathcal{F}^{\lambda+1}(W^{\otimes k})^d}{\mathcal{F}^\lambda(W^{\otimes k})^d} \right)^* \\
&\simeq \left(\frac{\bigoplus_{|\vec{d}|=d} \sum_{|\vec{\lambda}|=\lambda+1} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}}{\bigoplus_{|\vec{d}|=d} \sum_{|\vec{\lambda}|=\lambda} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \left(\bigoplus_{|\vec{d}|=d} \frac{\sum_{|\vec{\lambda}|=\lambda+1} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}}{\sum_{|\vec{\lambda}|=\lambda} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \left(\bigoplus_{|\vec{d}|=d} \bigoplus_{|\vec{\lambda}|=\lambda} \frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \left(\bigoplus_{|\vec{\lambda}|=\lambda} \bigoplus_{|\vec{d}|=d} \frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\stackrel{\mathbb{Z}\text{-gapped}}{\& \text{ bounded below}} \rightarrow \simeq \bigoplus_{|\vec{\lambda}|=\lambda} \bigoplus_{|\vec{d}|=d} \left(\frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\stackrel{(WG2)}{\& (WG1)} \rightarrow \simeq \bigoplus_{|\vec{\lambda}|=\lambda} \bigoplus_{|\vec{d}|=d} \left(\frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \right)^* \otimes \dots \otimes \left(\frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \bigoplus_{|\vec{d}|=d} \bigoplus_{|\vec{\lambda}|=\lambda} \frac{\mathcal{F}^{\lambda_1}(W^{*\mathfrak{g}})^{d_1}}{\mathcal{F}^{\lambda_1+1}(W^{*\mathfrak{g}})^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k}(W^{*\mathfrak{g}})^{d_k}}{\mathcal{F}^{\lambda_k+1}(W^{*\mathfrak{g}})^{d_k}} \\
&\simeq \frac{\mathcal{F}^\lambda(W^{*\mathfrak{g} \otimes k})^d}{\mathcal{F}^{\lambda+1}(W^{*\mathfrak{g} \otimes k})^d}.
\end{aligned}$$

In fact, this computation shows that $\hat{T}_k(W^{*\mathfrak{g}}) \simeq (T_k W)^{*\mathfrak{g}}$. The conclusion for S_k follows by checking that the maps above are S_k -equivariant. \square

Given a chain complex (W, ∂) , the boundary operator ∂ induces the boundary operator $\partial_k : W^{\otimes k} \rightarrow W^{\otimes k}$ for all $k \in \mathbb{N}$; for all $w_1, \dots, w_k \in W$, it is defined by

$$\partial_k(w_1 \otimes \dots \otimes w_k) := \sum_{i=1}^k (-1)^{|w_1| + \dots + |w_{i-1}|} w_1 \otimes \dots \otimes \partial w_i \otimes \dots \otimes w_k. \quad (3.12)$$

3. Algebraic structures

The map ∂_k is clearly \mathbb{S}_k -equivariant, and thus induces the boundary operator $\partial_k : S_k W \rightarrow S_k W$.

Proposition 3.1.13 (Künneth formula for completed symmetric cohomology). *Let (W, ∂) be a \mathbb{Z} -graded chain complex over \mathbb{R} filtered by an exhaustive \mathbb{Z} -gapped filtration $\mathcal{F}^\lambda W$ which is increasing and bounded from below. Consider the dual cochain complex $(W^{*\mathbb{g}}, d := \partial^*)$. Suppose that d has finite filtration degree, so that $d_k : S_k(W^{*\mathbb{g}}) \rightarrow S_k(W^{*\mathbb{g}})$ extends continuously to $d_k : \hat{S}_k(W^{*\mathbb{g}}) \rightarrow \hat{S}_k(W^{*\mathbb{g}})$ for every $k \in \mathbb{N}$. If (WG1) & (WG2) are satisfied, then the natural map $S_k H(W^{*\mathbb{g}}, d) \rightarrow H(\hat{S}_k(W^{*\mathbb{g}}), d_k)$ induces the isomorphism*

$$\hat{S}_k H(W^{*\mathbb{g}}, d) \simeq H(\hat{S}_k(W^{*\mathbb{g}}), d_k) \quad \text{for all } k \in \mathbb{N}.$$

Proof. The natural map $S_k H(W^{*\mathbb{g}}, d) \rightarrow H(\hat{S}_k W^{*\mathbb{g}}, d_k)$ is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space $H(\hat{S}_k W^{*\mathbb{g}}, d_k)$ is already complete (the homology of a complete space is complete), and hence we obtain the map $\hat{S}_k H(W^{*\mathbb{g}}, d) \rightarrow H(\hat{S}_k W^{*\mathbb{g}}, d_k)$. The following facts are easy to verify:

- (1) The isomorphism from Lemma 3.1.12 is an isomorphism of cochain complexes

$$(\hat{S}_k W^{*\mathbb{g}}, d_k) \simeq ((S_k W)^{*\mathbb{g}}, \partial_k^*).$$

- (2) If the filtration on W satisfies (WG1) and (WG2), then the filtration on $H(W)$ also satisfies (WG1) and (WG2), respectively. Consequently, Lemma 3.1.12 holds for symmetric powers of $H(W, \partial)^{*\mathbb{g}}$ as well.
- (3) The Künneth formula $H(W^{\otimes k}) \simeq H(W)^{\otimes k}$ implies $H(S_k W) \simeq S_k H(W)$ for any \mathbb{Z} -graded chain complex W over \mathbb{R} .
- (4) We have $(H(W))^{*\mathbb{g}} \simeq H(W^{*\mathbb{g}})$ over \mathbb{R} by the universal coefficient theorem.

Now, we compute

$$\begin{aligned} H(\hat{S}_k W^{*\mathbb{g}}, d_k) &\simeq H((S_k W)^{*\mathbb{g}}, \partial_k^*) \simeq H(S_k W, \partial_k)^{*\mathbb{g}} \simeq (S_k H(W, \partial))^{*\mathbb{g}} \\ &\quad \begin{matrix} \uparrow (1) & & \uparrow (4) & & \uparrow (3) \end{matrix} \\ &\simeq \hat{S}_k(H(W, \partial)^{*\mathbb{g}}) \simeq \hat{S}_k H(W^{*\mathbb{g}}, d). \\ &\quad \begin{matrix} \uparrow (2) & & \uparrow (4) \end{matrix} \end{aligned}$$

This proves the proposition. □

3.2. Basics of IBL-infinity-algebras

Definition 3.2.1 (Exterior algebra). *Given a graded vector space C over \mathbb{R} , we define the exterior algebra over C by*

$$EC := S(C[1]).$$

The weight- k component is denoted by $E_k C$ and the weight-reduced part by $\bar{E}C$. If C is in addition filtered, then $E_k C$ is filtered by the induced filtration and its completion is denoted by $\hat{E}_k C$.

We have the concatenation product $\mu : EC \otimes EC \rightarrow EC$ and the shuffle coproduct $\delta : EC \rightarrow EC \otimes EC$ defined by

$$\begin{aligned} \mu(c_{11} \dots c_{1k} \otimes c_{21} \dots c_{2k'}) &:= c_{11} \dots c_{1k} c_{21} \dots c_{2k'} \quad \text{and} \\ \delta(c_1 \dots c_k) &:= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = k}} \sum_{\sigma \in \mathbb{S}_{k_1, k_2}} \varepsilon(\sigma, c) c_{\sigma_1}^{-1} \dots c_{\sigma_{k_1}}^{-1} \otimes c_{\sigma_{k_1+1}}^{-1} \dots c_{\sigma_{k_1+k_2}}^{-1} \end{aligned}$$

for all homogenous c_{ij} , $c_i \in C[1]$ and $k, k' \geq 0$, respectively, where $\mathbb{S}_{k_1, k_2} \subset \mathbb{S}_{k_1+k_2}$ denotes the set of shuffle permutations with blocks of lengths k_1 and k_2 . These operations satisfy the relations of an *associative bialgebra* (see [LV12]):

$$\text{Assoc. bialg.} \quad \begin{cases} \mu \circ (\mathbb{1} \otimes \mu) = \mu \circ (\mu \otimes \mathbb{1}), \\ (\mathbb{1} \otimes \delta) \circ \delta = (\delta \otimes \mathbb{1}) \circ \delta, \\ \delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{1} \otimes \tau \otimes \mathbb{1}) \circ (\delta \otimes \delta). \end{cases} \quad (3.13)$$

Here $\tau : C_1 \otimes C_2 \rightarrow C_2 \otimes C_1$, $c_1 \otimes c_2 \mapsto (-1)^{|c_1||c_2|} c_2 \otimes c_1$ denotes the *twist map*.

We will use the bialgebra calculus ($\text{:= relations (3.13)}$) to write down explicit formulas for the operations \circ_{h_1, \dots, h_r} which were briefly introduced in [CFL15]; these operations take symmetric maps f_1, \dots, f_r and connect h_1, \dots, h_r of their outputs to the inputs of a symmetric map f in all possible ways, so that the result, which we denote by $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r)$, becomes a symmetric map again.

Definition 3.2.2 (Partial compositions). *Let C be a graded vector space. For $i, j \geq 0$, we denote by*

$$\begin{aligned} \pi_i : EC &\longrightarrow E_i C, \quad \iota_i : E_i C \longrightarrow EC, \\ \mathbb{1}_i : E_i C &\longrightarrow E_i C, \quad \tau_{i,j} : E_i C \otimes E_j C \longrightarrow E_j C \otimes E_i C \end{aligned}$$

the components of the canonical projection π , the canonical inclusion ι , the identity $\mathbb{1}$

3. Algebraic structures

and the twist map τ , respectively. We also set

$$\delta_{i,j} := (\pi_i \otimes \pi_j) \circ \delta \circ \iota_{i+j} \quad \text{and} \quad \mu_{i,j} := \pi_{i+j} \circ \mu \circ (\iota_i \otimes \iota_j).$$

For $k', k_1, l', l_1 \geq 0$, let $f : E_{k'}C \rightarrow E_{l'}C$ and $f_1 : E_{k_1}C \rightarrow E_{l_1}C$ be linear maps, and let $0 \leq h \leq \min(k', l_1)$. We set

$$k := k' + k_1 - h \quad \text{and} \quad l := l' + l_1 - h$$

and define the composition of f and f_1 at h common outputs to be the linear map $f \circ_h f_1 : E_kC \rightarrow E_lC$ given by

$$\begin{aligned} f \circ_h f_1 &:= \mu_{l', l_1 - h} \circ (f \otimes \mathbb{1}_{l_1 - h}) \circ (\mu_{h, k' - h} \otimes \mathbb{1}_{l_1 - h}) \circ (\mathbb{1}_h \otimes \tau_{l_1 - h, k' - h}) \\ &\quad \circ (\delta_{h, l_1 - h} \otimes \mathbb{1}_{k' - h}) \circ (f_1 \otimes \mathbb{1}_{k' - h}) \circ \delta_{k_1, k' - h}. \end{aligned} \quad (3.14)$$

More generally, we define the composition of $f : E_{k'} \rightarrow E_{l'}$ with $r \geq 1$ linear maps $f_i : E_{k_i} \rightarrow E_{l_i}$ with $k_i, l_i \geq 0$ for $i = 1, \dots, r$ at $0 \leq h_i \leq l_i$ common outputs such that $h := h_1 + \dots + h_r \leq k'$ as follows. We set

$$k := k' + k_1 + \dots + k_r - h \quad \text{and} \quad l := l' + l_1 + \dots + l_r - h$$

and define $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) : E_kC \rightarrow E_lC$ by

$$\begin{aligned} &f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) \\ &:= \mu \circ (f \otimes \mathbb{1}) \circ (\mu \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \tau) \\ &\quad \circ ([(\mu^{(r)} \otimes \mu^{(r)}) \circ (F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r}) \circ \sigma_r \circ \delta^{\otimes r}] \otimes \mathbb{1}) \\ &\quad \circ (f_1 \otimes \dots \otimes f_r \otimes \mathbb{1}) \circ \delta^{(r+1)}, \end{aligned} \quad (3.15)$$

where we have:

- The operation $\mu^{(r)}$ is the “product with r inputs” and the operation $\delta^{(r)}$ the “co-product with r outputs”; they are defined by

$$\begin{aligned} \mu^{(r)} &:= \mu(\mathbb{1} \otimes \mu) \dots (\mathbb{1}^{\otimes r-2} \otimes \mu), \quad \mu^{(1)} := \mathbb{1}, \\ \delta^{(r)} &:= (\mathbb{1}^{\otimes r-2} \otimes \delta) \dots (\mathbb{1} \otimes \delta) \delta, \quad \delta^{(1)} := \mathbb{1}. \end{aligned}$$

- $F_{h_1, \dots, h_r} := (\iota_{h_1} \pi_{h_1}) \otimes \dots \otimes (\iota_{h_r} \pi_{h_r})$.

3. Algebraic structures

- The permutation $\sigma_r \in \mathbb{S}_{2r}$ is given by

$$\sigma_r : (1, 2, 3, 4, \dots, 2r-1, 2r) \mapsto (1, r+1, 2, r+2, \dots, r, 2r).$$

- The symbols f and f_i inside the formula denote the trivial extensions of f and f_i , respectively; we extend a linear map $f : E_{k'}C \rightarrow E_{l'}C$ trivially to $f : EC \rightarrow EC$ by defining $f(E_iC) = 0$ for $i \neq k'$.

Remark 3.2.3 (On partial compositions). (i) Defining $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) : E_kC \rightarrow E_lC$ using (3.15) makes sense because the right hand side is a trivial extension of its component $E_kC \rightarrow E_lC$. In fact, all μ, δ, π, ι in (3.15) can be replaced with $\mu_{i,j}, \delta_{i,j}, \pi_i, \iota_i$ for unique i, j , so that trivial extensions do not have to be used at all. In this way, it can be seen that (3.14) is indeed a special case of (3.15).

(ii) If $h = k' = l_1$, then $f \circ_h f_1 = f \circ f_1$.

(iii) It holds $f \circ_0 f_1 = (-1)^{|f||f_1|} f_1 \circ_0 f$ and

$$f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) = \varepsilon(\sigma, f) f \circ_{h_{\sigma_1^{-1}}, \dots, h_{\sigma_r^{-1}}} (f_{\sigma_1^{-1}}, \dots, f_{\sigma_r^{-1}}).$$

(iv) Consider the (“non-trivial”) extension $\hat{f} := \mu(f \otimes \mathbb{1})\delta : EC \rightarrow EC$ and the symmetric product $f_1 \odot \dots \odot f_r := \mu^{(r)}(f_1 \otimes \dots \otimes f_r)\delta^{(r)} : EC \rightarrow EC$. The following formulas from [CFL15] hold:

$$\begin{aligned} f \circ_{h_1, \dots, h_{r-1}, 0} (f_1, \dots, f_r) &= f \circ_{h_1, \dots, h_{r-1}} (f_1, \dots, f_{r-1}) \odot f_r, \\ \hat{f} \circ \hat{f}_1 &= \sum_{h=0}^{\min(k', l_1)} \widehat{f \circ_h f_1}, \\ \hat{f} \circ (f_1 \odot \dots \odot f_r) &= \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k'}} f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r). \end{aligned} \tag{3.16}$$

We also have the “weak associativity”

$$\sum_{\substack{0 \leq h_2 \leq \min(f_3^-, f_2^+) \\ 0 \leq h_1 \leq \min(f_1^+, f_2^- + f_3^- - h_2) \\ h_1 + h_2 = h}} f_1 \circ_{h_1} (f_2 \circ_{h_2} f_3) = \sum_{\substack{0 \leq h_1 \leq \min(f_1^+, f_2^-) \\ 0 \leq h_2 \leq \min(f_1^+ + f_2^+ - h_1, f_3^-) \\ h_1 + h_2 = h}} (f_1 \circ_{h_1} f_2) \circ_{h_2} f_3 \tag{3.17}$$

for every $0 \leq h \leq \min(k_1 + k_2 + k_3, l_1 + l_2 + l_3)$, where f^+ denotes the number of inputs and f^- the number of outputs of f . The weak associativity of \circ_h can be proven using

3. Algebraic structures

the associativity of $\hat{\circ}$ and the second relation of (3.16).

(v) We refer to Section D.4 of Appendix D for a thorough treatment of partial compositions. We show there that \circ_{h_1, \dots, h_r} can be defined on maps on any connected weight-graded bialgebra using natural bilinear operations \square_A on polynomials in the convolution product. In Proposition D.4.4 in the appendix, we prove the relations above. \triangleleft

If C is filtered by a decreasing filtration, then the bialgebra operations extend continuously to

$$\begin{aligned} \mu : \hat{E}_{k_1} C \hat{\otimes} \hat{E}_{k_2} C &\longrightarrow \hat{E}_{k_1+k_2} C \quad \text{and} \\ \delta : \hat{E}_k C &\longrightarrow \bigoplus_{\substack{l_1, l_2 \geq 0 \\ l_1 + l_2 = k}} \hat{E}_{l_1} C \hat{\otimes} \hat{E}_{l_2} C \end{aligned}$$

for all $k_1, k_2, k \in \mathbb{N}_0$ because they preserve the filtration degree (see [Fre17] for a similar construction). Next, if $f_1 : \hat{E}_{k_1} C \rightarrow \hat{E}_{l_1} C$ and $f_2 : \hat{E}_{k_2} C \rightarrow \hat{E}_{l_2} C$ have finite filtration degrees, then $f_1 \otimes f_2 : \hat{E}_{k_1} C \otimes \hat{E}_{k_2} C \rightarrow \hat{E}_{l_1} C \otimes \hat{E}_{l_2} C$ has finite filtration degree too, and hence it extends continuously to $f_1 \otimes f_2 : \hat{E}_{k_1} C \hat{\otimes} \hat{E}_{k_2} C \rightarrow \hat{E}_{l_1} C \hat{\otimes} \hat{E}_{l_2} C$. Using these facts, we can canonically extend Definition 3.2.2 to maps $f : \hat{E}_{k'} C \rightarrow \hat{E}_{l'} C$ and $f_i : \hat{E}_{k_i} C \rightarrow \hat{E}_{l_i} C$ of finite filtration degrees. The resulting map $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) : \hat{E}_k C \rightarrow \hat{E}_l C$ will have finite filtration degree too. Moreover, the formulas in Remark 3.2.3 will still hold.

We will now rephrase the definitions of an IBL_∞ -algebra, a Maurer-Cartan element and twisted operations from [CFL15] in terms of \circ_{h_1, \dots, h_r} .

Definition 3.2.4 (IBL_∞ -algebra). *Let C be a graded vector space equipped with a decreasing filtration, and let $d \in \mathbb{Z}$ and $\gamma \geq 0$ be fixed constants. An IBL_∞ -algebra of bidegree (d, γ) on C is a collection of linear maps $\mathbf{q}_{klg} : \hat{E}_k C \rightarrow \hat{E}_l C$ for all $k, l \geq 1, g \geq 0$ which are homogenous, of finite filtration degree and satisfy the following conditions:*

- 1) $|\mathbf{q}_{klg}| = -2d(k + g - 1) - 1$.
- 2) $\|\mathbf{q}_{klg}\| \geq \gamma \chi_{klg}$, where $\chi_{klg} := 2 - 2g - k - l$.
- 3) The IBL_∞ -relations hold: for all $k, l \geq 1, g \geq 0$, we have

$$\sum_{h=1}^{g+1} \sum_{\substack{k_1, k_2, l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ k_1 + k_2 = k + h \\ l_1 + l_2 = l + h \\ g_1 + g_2 = g + 1 - h}} \mathbf{q}_{k_2 l_2 g_2} \circ_h \mathbf{q}_{k_1 l_1 g_1} = 0. \quad (3.18)$$

We denote a given IBL_∞ -algebra structure on C by $\text{IBL}_\infty(C)$; i.e., we write $\text{IBL}_\infty(C) = (C, (\mathbf{q}_{klg}))$.

3. Algebraic structures

If $\mathfrak{q}_{klg} \equiv 0$ for all $(k, l, g) \neq (1, 1, 0), (2, 1, 0), (1, 2, 0)$, then we call $\text{IBL}_\infty(C)$ a dIBL-algebra and denote it by $\text{dIBL}(C)$. If in addition $\mathfrak{q}_{110} \equiv 0$, then we have an IBL-algebra $\text{IBL}(C)$.

If the operations on the completed exterior powers $\hat{E}_k C$ arise as continuous extensions of operations $\mathfrak{q}_{klg} : E_k C \rightarrow E_l C$, then we call the IBL_∞ -algebra completion-free and denote C together with the operations $\mathfrak{q}_{klg} : E_k C \rightarrow E_l C$ by $\text{IBL}_\infty^0(C)$.

The acronym IBL stands for an *involutive Lie bialgebra*. It follows namely from the IBL_∞ -relations (3.18) that for $\text{IBL}(C) = (C, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ the following holds:

$$\text{Lie bialg.} \quad \begin{cases} 0 = \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{210} & \leftarrow \text{Jacobi id.} \\ 0 = \mathfrak{q}_{120} \circ_1 \mathfrak{q}_{120} & \leftarrow \text{co-Jacobi id.} \\ 0 = \mathfrak{q}_{120} \circ_1 \mathfrak{q}_{210} + \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{120} & \leftarrow \text{Drinfeld id.} \\ 0 = \mathfrak{q}_{210} \circ_2 \mathfrak{q}_{120} & \leftarrow \text{Involutivity} \end{cases}$$

The acronym dIBL stands for a *differential involutive Lie bialgebra* — an involutive Lie bialgebra together with a differential (a boundary operator in our case) such that the bracket and cobracket are chain maps.

Proposition 3.2.5 (Odd degree shift of an IBL-algebra). *Let $(C, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ be an IBL-algebra of degree d from Definition 3.2.4, and let $\tilde{\mathfrak{q}}_{210} : C^{\otimes 2} \rightarrow C$ and $\tilde{\mathfrak{q}}_{120} : C \rightarrow C^{\otimes 2}$ be the linear maps defined by*

$$\begin{aligned} \theta \tilde{\mathfrak{q}}_{210}(x_1 \otimes x_2) &:= \mathfrak{q}_{210}(\pi(\theta^2 x_1 \otimes x_2)) \quad \text{and} \\ \theta^2 \tilde{\mathfrak{q}}_{120}(x) &:= \iota(\mathfrak{q}_{120}(\theta x)) \end{aligned} \tag{3.19}$$

for all $x_1, x_2, x \in C$, where $\iota : S_2(C[1]) \rightarrow C[1]^{\otimes 2}$ is the section of $\pi : C[1]^{\otimes 2} \rightarrow S_2(C[1])$ from Definition 3.1.7 and θ is a formal symbol of degree $|\theta| = -1$. Then the degrees satisfy

$$|\tilde{\mathfrak{q}}_{210}| = |\mathfrak{q}_{210}| - 1 = -2d - 2 \quad \text{and} \quad |\tilde{\mathfrak{q}}_{120}| = |\mathfrak{q}_{120}| + 1 = 0,$$

the operations $\tilde{\mathfrak{q}}_{210}$ and $\tilde{\mathfrak{q}}_{120}$ are graded antisymmetric, i.e., we have

$$\tilde{\mathfrak{q}}_{210} \circ \tau = -\tilde{\mathfrak{q}}_{210} \quad \text{and} \quad \tau \circ \tilde{\mathfrak{q}}_{120} = -\tilde{\mathfrak{q}}_{120}$$

3. Algebraic structures

for the twist map τ , and the relations

$$\begin{aligned} 0 &= \tilde{\mathbf{q}}_{210} \circ (\tilde{\mathbf{q}}_{210} \otimes \mathbb{1}) \circ (\mathbb{1}^{\otimes 3} + t_3 + t_3^2), \\ 0 &= (\mathbb{1}^{\otimes 3} + t_3 + t_3^2) \circ (\tilde{\mathbf{q}}_{120} \otimes \mathbb{1}) \circ \tilde{\mathbf{q}}_{120}, \\ 0 &= x_1 \cdot \tilde{\mathbf{q}}_{120}(x_2) - (-1)^{x_1 x_2} x_2 \cdot \tilde{\mathbf{q}}_{120}(x_1) - \tilde{\mathbf{q}}_{120}(\tilde{\mathbf{q}}_{210}(x_1, x_2)), \\ 0 &= \tilde{\mathbf{q}}_{210} \circ \tilde{\mathbf{q}}_{120}, \end{aligned}$$

hold for all $x_1, x_2 \in C$. Here, $t_3 \in \mathbb{S}_3$ denotes the cyclic permutation with $t_3(1) = 2$ acting on $C^{\otimes 3}$, and we define

$$x \cdot (y_1 \otimes y_2) := \tilde{\mathbf{q}}_{210}(x, y_1) \otimes y_2 + (-1)^{x y_1} y_1 \otimes \tilde{\mathbf{q}}_{210}(x, y_2)$$

for all $x, y_1, y_2 \in C$.

Proof. The proof is a lengthy but straightforward computation. \square

Consider the *sign-action* of \mathbb{S}_k on $C^{\otimes k}$ given by $\sigma \mapsto \bar{\sigma}$, where

$$\bar{\sigma}(c_1 \otimes \cdots \otimes c_k) := (-1)^\sigma \varepsilon(\sigma, c) c_{\sigma_1^{-1}} \otimes \cdots \otimes c_{\sigma_k^{-1}}$$

for all $c_1, \dots, c_k \in C$ and $\sigma \in \mathbb{S}_k$. We define

$$\Lambda C := \bigoplus_{k=0}^{\infty} \Lambda_k C \quad \text{with} \quad \Lambda_k C := C^{\otimes k} / \sim,$$

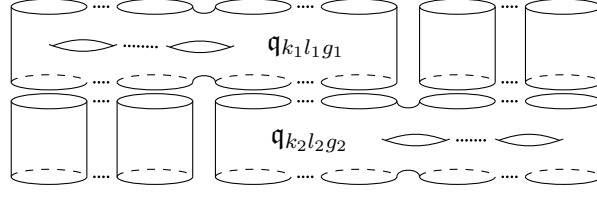
where $c \sim \bar{\sigma}(c)$ for all $c \in C^{\otimes k}$ and $\sigma \in \mathbb{S}_k$. It is easy to see that the degree-shift map $\theta^{\otimes k} : c_1 \otimes \cdots \otimes c_k \in C^{\otimes k} \mapsto \varepsilon(c, \theta)(\theta c_1) \otimes \cdots \otimes (\theta c_k) \in C[1]^{\otimes k}$ is equivariant with respect to the sign-action of \mathbb{S}_k on $C^{\otimes k}$ and the standard action of \mathbb{S}_k on $C[1]^{\otimes k}$ for all k , and thus it induces an isomorphism of vector spaces ΛC and EC . We use the following notation:

$$\begin{array}{ccc} \mathbf{q}_{klg} : \hat{\mathbf{E}}_k C & \longrightarrow & \hat{\mathbf{E}}_l C \\ \downarrow \theta^{\otimes k} & & \downarrow \theta^{\otimes l} \\ \tilde{\mathbf{q}}_{klg} : \hat{\Lambda}_k C & \longrightarrow & \hat{\Lambda}_l C. \end{array}$$

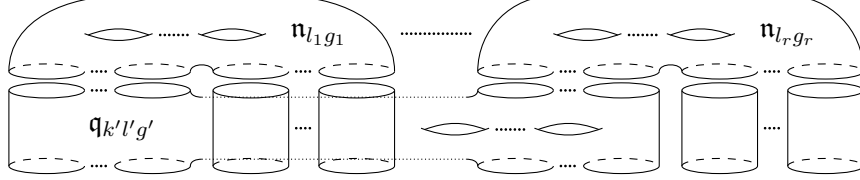
In fact, \mathbf{q}_{klg} and $\tilde{\mathbf{q}}_{klg}$ are related precisely by the degree shift (3.6).

Definition 3.2.6 (Maurer-Cartan element). *A Maurer-Cartan element for an IBL_∞ -algebra $\text{IBL}_\infty(C)$ from Definition 3.2.4 is a collection $\mathbf{n} := (\mathbf{n}_{lg})_{l \geq 1, g \geq 0}$ of elements $\mathbf{n}_{lg} \in \hat{\mathbf{E}}_l C$ which are homogenous, of finite filtration degree and satisfy the following conditions:*

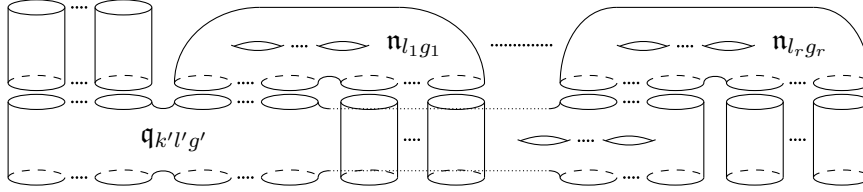
3. Algebraic structures



(a) The term $q_{k_2 l_2 g_2} \circ_h q_{k_1 l_1 g_1}$ in the IBL_∞ -equation (3.18).



(b) The term $q_{k' l' g'} \circ_{h_1, \dots, h_r} (n_{l_1 g_1}, \dots, n_{l_r g_r})$ in the Maurer-Cartan equation (3.20). We remark that the contour of the surface corresponding to $q_{k' l' g'}$ starts on the left and continues to the right along the dotted line behind the two trivial cylinders.



(c) The term $q_{k' l' g'} \circ_{l_1, \dots, l_r} (n_{l_1 g_1}, \dots, n_{l_r g_r})$ in the twisted operation (3.21). The remark to Figure (b) applies too.

Figure 3.1.: Graphical representation of compositions appearing in Definitions 3.2.4, 3.2.6 and 3.2.7 as gluing of connected Riemannian surfaces. The figure is to be read from the top to the bottom, the empty cylinder represents the identity, and the resulting surface must be connected. We emphasize that the gluing is not associative (c.f., weak associativity (3.17)).

$$1) \quad |n_{lg}| = -2d(g-1).$$

$$2) \quad \|n_{lg}\| \geq \gamma \chi_{0lg} \text{ with } > \text{ for } (l, g) = (1, 0), (2, 0) \text{ (see Definition 3.2.4 for } \chi_{klg}).$$

3) The Maurer-Cartan equation holds: for all $l \geq 1, g \geq 0$, we have

$$\sum_{r \geq 1} \frac{1}{r!} \sum_{\substack{l', k', l_1, \dots, l_r \geq 1 \\ g', g_1, \dots, g_r \geq 0 \\ h_1, \dots, h_r \geq 1 \\ l_1 + \dots + l_r + l' - k' = l \\ g_1 + \dots + g_r + g' + k' = g + r \\ h_1 + \dots + h_r - k' = 0}} q_{k' l' g'} \circ_{h_1, \dots, h_r} (n_{l_1 g_1}, \dots, n_{l_r g_r}) = 0, \quad (3.20)$$

where we view n_{lg} as a linear map $n_{lg} : \hat{E}_0 C = \mathbb{R} \rightarrow \hat{E}_l C$ with $n_{lg}(1) = n_{lg}$.

3. Algebraic structures

Definition 3.2.7 (Twisted operations). *In the setting of Definition 3.2.6, the twisted operations $\mathfrak{q}_{klg}^{\mathfrak{n}} : \hat{E}_k C \rightarrow \hat{E}_l C$ for $k, l \geq 1, g \geq 0$ are defined by*

$$\mathfrak{q}_{klg}^{\mathfrak{n}} = \sum_{r \geq 0} \frac{1}{r!} \sum_{\substack{k', l', l_1, \dots, l_r \geq 1 \\ g', g_1, \dots, g_r \geq 0 \\ h_1, \dots, h_r \geq 1 \\ l_1 + \dots + l_r + l' - k' = l - k \\ g_1 + \dots + g_r + g' + k' = g + r + k \\ h_1 + \dots + h_r - k' = -k}} \mathfrak{q}_{k'l'g'} \circ_{h_1, \dots, h_r} (\mathfrak{n}_{l_1 g_1}, \dots, \mathfrak{n}_{l_r g_r}). \quad (3.21)$$

In [CFL15, Proposition 9.3], they prove that $(\mathfrak{q}_{klg}^{\mathfrak{n}})_{k, l \geq 1, g \geq 0}$ is again an IBL_{∞} -algebra of bidegree (d, γ) on C — the twisted IBL_{∞} -algebra. We denote it by $\text{IBL}_{\infty}^{\mathfrak{n}}(C)$.

Let (\mathfrak{q}_{klg}) be an IBL_{∞} -algebra on C . The boundary operator $\mathfrak{q}_{110} : C[1] \rightarrow C[1]$ induces the boundary operator $\partial_k : E_k C \rightarrow E_k C$ for every $k \in \mathbb{N}$ (see (3.12)). Because of the finite filtration degree, ∂_k continuously extends to $\partial_k : \hat{E}_k C \rightarrow \hat{E}_k C$. The following is easy to see using (3.14):

$$\begin{aligned} \mathfrak{q}_{klg} \circ_1 \mathfrak{q}_{110} &= \mathfrak{q}_{klg} \circ \partial_k, \\ \mathfrak{q}_{110} \circ_1 \mathfrak{q}_{klg} &= \partial_l \circ \mathfrak{q}_{klg}. \end{aligned}$$

Because \mathfrak{q}_{klg} are odd ($:=$ have odd degree), we have

$$\begin{aligned} [\partial, \mathfrak{q}_{klg}] &:= \partial_l \circ \mathfrak{q}_{klg} - (-1)^{|\partial||\mathfrak{q}_{klg}|} \mathfrak{q}_{klg} \circ \partial_k \\ &= \partial_l \circ \mathfrak{q}_{klg} + \mathfrak{q}_{klg} \circ \partial_k \\ &= \mathfrak{q}_{110} \circ_1 \mathfrak{q}_{klg} + \mathfrak{q}_{klg} \circ_1 \mathfrak{q}_{110}. \end{aligned}$$

With this notation, the IBL_{∞} -relations (3.18) for $\mathfrak{q}_{210} : \hat{E}_2 C \rightarrow \hat{E}_1 C$ and $\mathfrak{q}_{120} : \hat{E}_1 C \rightarrow \hat{E}_2 C$ become $[\partial, \mathfrak{q}_{210}] = 0$ and $[\partial, \mathfrak{q}_{120}] = 0$, respectively. If moreover the canonical maps $E_k H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \rightarrow H(\hat{E}_k C, \partial_k)$ induce the isomorphisms $\hat{E}_k H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \simeq H(\hat{E}_k C, \partial_k)$ for $k = 1, 2$, e.g., when Proposition 3.1.13 holds, then we obtain the maps

$$\mathfrak{q}_{210} : \hat{E}_2 H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \rightarrow \hat{E}_1 H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \quad \text{and} \quad \mathfrak{q}_{120} : \hat{E}_1 H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \rightarrow \hat{E}_2 H(\hat{C}, \tilde{\mathfrak{q}}_{110}),$$

and $(H(\hat{C}, \tilde{\mathfrak{q}}_{110}), \mathfrak{q}_{210}, \mathfrak{q}_{120})$ becomes an IBL -algebra according to Definition 3.2.4 — the induced IBL -algebra on homology.

Definition 3.2.8 (Homology). *We define the homology of an IBL_{∞} -algebra $\text{IBL}_{\infty}(C)$ by*

$$\mathbb{H}(C)[1] := H(\hat{C}[1], \mathfrak{q}_{110}).$$

It is a graded vector space with the induced filtration. If \mathfrak{n} is a Maurer-Cartan element

3. Algebraic structures

for $\text{IBL}_\infty(C)$, we denote by $\mathbb{H}^n(C)$ the homology of $\text{IBL}_\infty^n(C)$.

Remark 3.2.9 (BV-formalism). Consider the weight-reduced exterior algebra $\bar{E}C$. Let $\hat{E}C[[\hbar]]$, resp. $\hat{E}C((\hbar))$ be the spaces of formal power, resp. Laurent series in the variable \hbar of degree $|\hbar| = 2d$ with coefficients in $\bar{E}C$ completed with respect to a suitable completion. Operations of an IBL_∞ -algebra on C can be encoded in a degree -1 operator $\Delta : \hat{E}C[[\hbar]] \rightarrow \hat{E}C[[\hbar]]$ called the BV_∞ -operator, while the data of a Maurer-Cartan element (\mathbf{n}_l) give rise to an operator $e^\mathbf{n} : \hat{E}C[[\hbar]] \rightarrow \hat{E}C((\hbar))$ — the multiplication with the exponential of \mathbf{n} with respect to \odot . These operators are given by

$$\Delta := \sum_{i \geq 0} \Delta_{i+1} \hbar^i \quad \text{and} \quad e^\mathbf{n} := \sum_{j \in \mathbb{Z}} (e^\mathbf{n})_j \hbar^j,$$

where the maps $\Delta_i, (e^\mathbf{n})_j : \hat{E}C \rightarrow \hat{E}C$ for $i \geq 1, j \in \mathbb{Z}$ are defined by

$$\begin{aligned} \Delta_i &:= \sum_{\substack{k \geq 1, g \geq 0 \\ k+g=i}} \sum_{l \geq 1} \hat{\mathbf{q}}_{klg} \quad \text{and} \\ (e^\mathbf{n})_j &:= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{g_1, \dots, g_r \geq 0 \\ g_1 + \dots + g_r = j}} \sum_{\substack{l_1, \dots, l_r \geq 1}} \mathbf{n}_{l_1 g_1} \odot \dots \odot \mathbf{n}_{l_r g_r}. \end{aligned}$$

It can be shown that the IBL_∞ -relations (3.18) and the Maurer-Cartan equation (3.20) are equivalent to

$$\Delta \circ \Delta = 0 \quad \text{and} \quad \Delta \circ e^\mathbf{n} = 0, \tag{3.22}$$

respectively, and that the BV_∞ -operator $\Delta^\mathbf{n}$ for the twisted IBL_∞ -structure $(\mathbf{q}_{klg}^\mathbf{n})$ satisfies

$$\Delta^\mathbf{n} = e^{-\mathbf{n}} \Delta(e^\mathbf{n} \cdot) \tag{3.23}$$

These facts were shown in [CFL15] using (3.16).⁴

There are some things which are unclear in the above. We do it properly in the Appendix about filtrations. \triangleleft

Remark 3.2.10 (Weak IBL_∞ -algebras). A possible generalization of the IBL_∞ -theory above is to allow $k = 0$ and $l = 0$, so that EC must be used instead of $\bar{E}C$ in Remark 3.2.9. Such structures would be called *weak IBL_∞ -algebras*. \triangleleft

In our application in string topology, a canonical dIBL -algebra $\text{dIBL}(C)$ with a natural Maurer-Cartan element \mathbf{n} are given, and we want to study $\text{dIBL}^\mathbf{n}(C)$; in particular, we

⁴One has to check that the compositions (3.22) and (3.23) are well-defined and pick a suitable completion $\hat{E}C$ so that all the constructions work.

3. Algebraic structures

are interested in $\mathbb{H}^n(C)$, $\text{IBL}(\mathbb{H}^n(C))$ and possible higher operations on $\mathbb{H}^n(C)$ induced by \mathfrak{q}_{klg}^n (these are not chain maps in general). The following proposition summarizes some observations in this situation:

Proposition 3.2.11 (Twist of a dIBL-algebra). *Let $\text{dIBL}(C) = (C, \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ be a dIBL-algebra, and let $\mathfrak{n} = (\mathfrak{n}_l)$ be a Maurer-Cartan element. The Maurer-Cartan equation (3.20) reduces to the following:*

$$0 = \mathbf{q}_{110} \circ_1 \mathbf{n}_{lg} + \mathbf{q}_{120} \circ_1 \mathbf{n}_{l-1,g} + \mathbf{q}_{210} \circ_2 \mathbf{n}_{l+1,g-1} \\ + \frac{1}{2} \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l+1 \\ g_1 + g_2 = g}} \mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{l_1 g_1}, \mathbf{n}_{l_2 g_2}) \quad \forall l \geq 1, g \geq 0.$$

In particular, the “lowest” equation is given by⁵

$$(l, g) = (1, 0) : \quad \mathbf{q}_{110}(\mathbf{n}_{10}) + \frac{1}{2}\mathbf{q}_{210}(\mathbf{n}_{10}, \mathbf{n}_{10}) = 0. \quad (3.24)$$

This can be visualized as

$$0 = \begin{array}{c} \text{n}_{10} \\ \text{q}_{110} \end{array} + \frac{1}{2} \begin{array}{cc} \text{n}_{10} & \text{n}_{10} \\ & \text{q}_{210} \end{array}.$$

The twisted IBL_∞ -algebra $\text{dIBL}^n(C)$ consists of the operations \mathfrak{q}_{110}^n , \mathfrak{q}_{210}^n and \mathfrak{q}_{120}^n , which we call the basic operations, and of the operations \mathfrak{q}_{1lg}^n for the pairs $(l, g) \in \mathbb{N} \times \mathbb{N}_0 \setminus \{(1, 0), (2, 0)\}$, which we call the higher operations. These operations are given by

$$\begin{aligned}\mathbf{q}_{110}^{\mathbf{n}} &= \mathbf{q}_{110} + \mathbf{q}_{210} \circ_1 \mathbf{n}_{10}, \\ \mathbf{q}_{210}^{\mathbf{n}} &= \mathbf{q}_{210}, \\ \mathbf{q}_{120}^{\mathbf{n}} &= \mathbf{q}_{120} + \mathbf{q}_{210} \circ_1 \mathbf{n}_{20}, \\ \mathbf{q}_{1lg}^{\mathbf{n}} &= \mathbf{q}_{210} \circ_1 \mathbf{n}_{lg}.\end{aligned}$$

⁵In [CFL15, Definition 2.4.], they define a partial ordering on the signatures (k, l, g) .

3. Algebraic structures

This can be visualized as

$$\begin{aligned}
 \mathfrak{q}_{110}^{\mathfrak{n}} &= \text{Diagram 1} + \text{Diagram 2}, \\
 \mathfrak{q}_{210}^{\mathfrak{n}} &= \text{Diagram 3}, \\
 \mathfrak{q}_{120}^{\mathfrak{n}} &= \text{Diagram 4} + \text{Diagram 5}, \\
 \mathfrak{q}_{1lg}^{\mathfrak{n}} &= \text{Diagram 6}.
 \end{aligned}$$

The IBL_∞ -relations satisfied by $(\mathfrak{q}_{klg}^{\mathfrak{n}})$ read for all $l \geq 1, g \geq 0$ as follows:

$$\begin{aligned}
 (3, 1, 0) : \quad 0 &= \mathfrak{q}_{210}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{210}^{\mathfrak{n}}, \\
 (2, l, g) : \quad 0 &= \mathfrak{q}_{1lg}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{210}^{\mathfrak{n}} + \mathfrak{q}_{210}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{1lg}^{\mathfrak{n}}, \\
 (1, l, g) : \quad 0 &= \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l + 1 \\ g_1 + g_2 = g}} \mathfrak{q}_{1l_1g_1}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{1l_2g_2}^{\mathfrak{n}} + \mathfrak{q}_{210}^{\mathfrak{n}} \circ_2 \mathfrak{q}_{1, l+1, g-1}^{\mathfrak{n}}.
 \end{aligned} \tag{3.25}$$

We call the relations for $(k, l, g) = (1, 1, 0), (2, 1, 0), (1, 2, 0), (3, 1, 0), (1, 3, 0), (2, 2, 0), (1, 1, 1)$ basic relations because they contain all compositions of basic operations. In the

3. Algebraic structures

order above, they read:

$$\begin{aligned}
0 &= \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{110}^n, \\
0 &= \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{210}^n + \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{110}^n, \\
0 &= \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{120}^n + \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{110}^n, \\
0 &= \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{210}^n, & \leftarrow \text{Jacobi identity} \\
0 &= \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{120}^n + \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{130}^n + \mathfrak{q}_{130}^n \circ_1 \mathfrak{q}_{110}^n, & \leftarrow \text{co-Jacobi id. up to htpy.} \\
0 &= \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{210}^n + \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{120}^n, & \leftarrow \text{Drinfeld identity} \\
0 &= \mathfrak{q}_{210}^n \circ_2 \mathfrak{q}_{120}^n + \mathfrak{q}_{111}^n \circ_1 \mathfrak{q}_{110}^n + \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{111}^n. & \leftarrow \text{Involutivity up to htpy.}
\end{aligned}$$

The last four equations can be visualized as

The first equation shows a diagram with two cups labeled \mathfrak{q}_{210}^n and a cylinder, equated to a diagram with a single cup labeled \mathfrak{q}_{210}^n .

The second equation shows a diagram with two cups labeled \mathfrak{q}_{120}^n and a cylinder, plus a diagram with a cup labeled \mathfrak{q}_{110}^n and a cylinder, plus a diagram with a cup labeled \mathfrak{q}_{130}^n and two cylinders, equated to zero.

The third equation shows a diagram with two cups labeled \mathfrak{q}_{210}^n and a cylinder, plus a diagram with a cup labeled \mathfrak{q}_{120}^n and a cylinder, plus a diagram with a cup labeled \mathfrak{q}_{210}^n and a cylinder, equated to zero.

The fourth equation shows a diagram with two cups labeled \mathfrak{q}_{120}^n and a cylinder, plus a diagram with a cup labeled \mathfrak{q}_{110}^n and a cylinder, plus a diagram with a cup labeled \mathfrak{q}_{111}^n and a cylinder, equated to zero.

Proof. The proof is clear by specializing (3.18), (3.20) and (3.21). \square

Remark 3.2.12 (Higher operations). (i) We see from Proposition 3.2.11 that if $\mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{120}^n = 0$ and $\mathfrak{q}_{210}^n \circ_2 \mathfrak{q}_{120}^n = 0$, then $[\partial^n, \mathfrak{q}_{130}^n] = 0$ and $[\partial^n, \mathfrak{q}_{111}^n] = 0$, respectively, and hence the operations $\mathfrak{q}_{130}^n : \hat{E}_1 \mathbb{H}^n \rightarrow \hat{E}_3 \mathbb{H}^n$ and $\mathfrak{q}_{111}^n : \hat{E}_1 \mathbb{H}^n \rightarrow \hat{E}_1 \mathbb{H}^n$ are well-defined (provided

3. Algebraic structures

that the assumption of Definition 3.2.8 holds). Likewise, the higher operation \mathfrak{q}_{1lg}^n defines a map $\hat{E}_1\mathbb{H}^n \rightarrow \hat{E}_l\mathbb{H}^n$, provided that the following equation holds:

$$\mathfrak{q}_{210}^n \circ_2 \mathfrak{q}_{1,l+1,g-1}^n + \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l+1 \\ g_1 + g_2 = g \\ (l_i, g_i) \neq (1, 0)}} \mathfrak{q}_{1l_1g_1}^n \circ_1 \mathfrak{q}_{1l_2g_2}^n = 0.$$

This expression is just the left-over after subtracting the commutator $[\mathfrak{q}_{1lg}^n, \mathfrak{q}_{110}^n] = \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{1lg}^n + \mathfrak{q}_{1lg}^n \circ_1 \mathfrak{q}_{110}^n$ from (3.25).

(ii) In the genus-0 case, i.e., $\mathfrak{q}_{1lg}^n = 0$ whenever $g \geq 1$, relations (3.25) reduce to

$$\begin{aligned} 0 &= \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{210}, \\ 0 &= \mathfrak{q}_{1lg}^n \circ_1 \mathfrak{q}_{210} + \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{1lg}^n, \\ 0 &= \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l+1 \\ g_1 + g_2 = g}} \mathfrak{q}_{1l_1g_1}^n \circ_1 \mathfrak{q}_{1l_2g_2}^n. \end{aligned}$$

The first relation is the Jacobi identity for \mathfrak{q}_{210} , the second relation is a generalization of the Drinfeld identity to higher coproducts \mathfrak{q}_{1l0}^n , and the third relations are coL_∞ -relations for \mathfrak{q}_{1lg}^n . Therefore, allowing $g \geq 0$, we see that the twisted dIBL-algebra $\mathfrak{q}_{110}^n, \mathfrak{q}_{210}, (\mathfrak{q}_{1lg}^n)$ is, in fact, a *quantum coL_∞-algebra*.

◁

3.3. Dual cyclic bar complex and cyclic cohomology

Definition 3.3.1 (Bar complexes). *Let V be a graded vector space. The bar- and dual bar-complex of V are the weight-graded vector spaces defined by*

$$BV := \bar{T}(V[1]) \quad \text{and} \quad B^*V := (BV)^{*wg},$$

respectively, where $\bar{T}V := \bigoplus_{k=1}^{\infty} V^{\otimes k}$ is the weight-reduced tensor algebra. For every $k \in \mathbb{N}$, let $t_k \in \mathbb{S}_k$ be the cyclic permutation $t_k : (1, \dots, k) \mapsto (2, \dots, k, 1)$, so that for all $v_1, \dots, v_k \in V[1]$ we have

$$t_k(v_1 \otimes \dots \otimes v_k) = (-1)^{|v_k|(|v_1| + \dots + |v_{k-1}|)} v_k \otimes v_1 \otimes \dots \otimes v_{k-1}.$$

3. Algebraic structures

We set

$$t := \sum_{k=1}^{\infty} t_k : BV \longrightarrow BV.$$

The cyclic bar-complex is defined by

$$B^{\text{cyc}}V := BV / \text{im}(1 - t).$$

We denote the image of $v_1 \otimes \cdots \otimes v_k \in BV$ under the canonical projection $\pi : BV \rightarrow B^{\text{cyc}}V$ by $v_1 \dots v_k$. If $v_i \in V[1]$ are homogenous, then $v_1 \dots v_k$ is called a generating word; we have

$$v_1 \dots v_k = (-1)^{|v_k|(|v_1| + \cdots + |v_{k-1}|)} v_k v_1 \dots v_{k-1}.$$

We define the section $\iota : B^{\text{cyc}}V \rightarrow BV$ of π by

$$\iota(v_1 \dots v_k) := \frac{1}{k} \sum_{i=0}^{k-1} \underbrace{t_k^i}_{=: t_k \circ \cdots \circ t_k \text{ } i\text{-times}} (v_1 \otimes \cdots \otimes v_k)$$

and use it to identify $B^{\text{cyc}}V$ with the subspace $\text{im } \iota = \ker(1 - t) \subset BV$ consisting of cyclic symmetric tensors.

We define the dual cyclic bar-complex by

$$B_{\text{cyc}}^*V := \{\psi \in B^*V \mid \psi \circ t = \psi\}.$$

Remark 3.3.2 (Non-weight-reduced bar complex). In fact, our B_{cyc}^*V is weight-reduced. The non-weight-reduced version would be $B_{\text{cyc}}^*V \oplus \mathbb{R}$ with \mathbb{R} of degree 0. This might play a role in the theory of weak A_{∞} -algebras (μ_0 added; c.f., Definition 3.3.7), and it might also be possible to consider IBL_{∞} -algebras on non-weight-reduced cyclic cochains (c.f., Section 3.4). \triangleleft

Notice that $\psi \in B^*V$ is homogenous of degree $|\psi| \in \mathbb{Z}$ if and only if for all homogenous $v_1, \dots, v_k \in V[1]$ the following implication holds:

$$|v_1| + \cdots + |v_k| \neq |\psi| \implies \psi(v_1 \otimes \cdots \otimes v_k) = 0.$$

This is the cohomological grading convention.

Notation 3.3.3 (Degree shifts of bar complexes). Let $A \in \mathbb{Z}$. In the following, we write B_{cyc}^*V , but the convention applies to all complexes from Definition 3.3.1. We denote by

3. Algebraic structures

s_A and θ the formal symbols of degrees

$$|s_A| = -A \quad \text{and} \quad |\theta| = -1,$$

respectively. The degree shift $V \mapsto V[1]$ will be realized as multiplication with θ and the degree shift $B_{\text{cyc}}^* V \mapsto B_{\text{cyc}}^* V[A]$ as multiplication with s_A . In addition, the following notation will be used consistently:

- $\tilde{v} \in V \longleftrightarrow v = \theta \tilde{v} \in V[1]$

To clarify this, given $\tilde{v} \in V$, then v automatically means $v = \theta \tilde{v} \in V[1]$, and the other way round. Recall that the degree of $\tilde{v} \in V$ is denoted by $\deg(\tilde{v})$ or simply by \tilde{v} in the exponent, e.g., $(-1)^{\tilde{v}}$.

- $\psi \in B_{\text{cyc}}^* V \longleftrightarrow \Psi = s_A \psi \in B_{\text{cyc}}^* V[A]$.
- A generating word of $B^{\text{cyc}} V$ of weight k will be denoted by the symbol w and written as $w = v_1 \dots v_k$, where $v_i = \theta \tilde{v}_i \in V[1]$. A generating word of $E_k B^{\text{cyc}} V$ is an element $w_1 \dots w_k \in E_k B^{\text{cyc}} V$ such that each w_i is a generating word of $B^{\text{cyc}} V$.
- $w \in B^{\text{cyc}} V \longleftrightarrow W = s_A w \in B^{\text{cyc}} V[A]$.

We abbreviate

$$B_{\text{cyc}}^* V[A] := (B_{\text{cyc}}^* V)[A].$$

In contrast to this, we would write $B_{\text{cyc}}^*(V[A])$ for the dual cyclic bar-complex of $V[A]$. We also identify $(B_{\text{cyc}}^* V[A])[1] = B_{\text{cyc}}^* V[A+1]$ in $EB_{\text{cyc}}^* V[A]$.

Definition 3.3.4 (Pairing of tensor powers of bar complexes). *For every $A \in \mathbb{Z}$ and $k \in \mathbb{N}$, we define the pairing as follows:*

$$\begin{aligned} (B^* V[A])^{\otimes k} \otimes (BV[A])^{\otimes k} &\longrightarrow \mathbb{R} \\ (\Psi_1 \otimes \dots \otimes \Psi_k, W_1 \otimes \dots \otimes W_k) &\longmapsto \underbrace{\psi_1(w_1) \dots \psi(w_k)}_{(\Psi_1 \otimes \dots \otimes \Psi_k)(W_1 \otimes \dots \otimes W_k) :=} \end{aligned} \quad (3.26)$$

This means that we evaluate elements from the left-hand side on the elements from the right-hand side in this way without any signs (see the discussion in Remark 3.1.4). We extend the pairing by 0 if the number of Ψ_i 's and the number of W_i 's differ.

Remark 3.3.5 (Dual bar complex and dual of the bar complex). Because the pairing (3.26) is non-degenerate, we can embed the space on the left into the linear dual of the space on the right. From Definition 3.3.1 we have $B_{\text{cyc}}^* V \subset B^* V$, and $B^{\text{cyc}} V$ is identified with

3. Algebraic structures

$\text{im } \iota \subset BV$. Therefore, we can restrict (3.26) to obtain the pairing of B_{cyc}^*V and $B^{\text{cyc}}V$. It is easy to see that for any $\psi \in B_{\text{cyc}}^*V$ and any generating word $v_1 \dots v_k \in B^{\text{cyc}}V$, we have

$$\psi(v_1 \dots v_k) = \psi(v_1 \otimes \dots \otimes v_k).$$

The subspace of $(B^{\text{cyc}}V)^{*_{\text{lin}}}$ corresponding to B_{cyc}^*V is then precisely $(B^{\text{cyc}}V)^{*_{\text{wg}}}$.

More generally, for every $k \in \mathbb{N}$, the spaces $E_k B_{\text{cyc}}^*V$ and $E_k B^{\text{cyc}}V$ are embedded into $(B_{\text{cyc}}^*V[1])^{\otimes k}$ and $(B^{\text{cyc}}V[1])^{\otimes k}$, respectively, using ι and π from Definition 3.1.7. Therefore, the restriction of (3.26) gives the pairing of $E_k B_{\text{cyc}}^*V$ and $E_k B^{\text{cyc}}V$. It is easy to see that for any generating word $w_1 \dots w_k \in E_k B^{\text{cyc}}V$ and any $\psi_1 \dots \psi_k \in E_k B_{\text{cyc}}^*V$, we have

$$(\psi_1 \dots \psi_k)(w_1 \dots w_k) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \varepsilon(\sigma, w) \psi_1(w_{\sigma_1^{-1}}) \dots \psi_k(w_{\sigma_k^{-1}}).$$

The subspace of $(E_k B^{\text{cyc}}V)^{*_{\text{lin}}}$ corresponding to $E_k B_{\text{cyc}}^*V$ lies in $(E_k B^{\text{cyc}}V)^{*_{\text{wg}}}$; it is equal to $(E_k B^{\text{cyc}}V)^{*_{\text{wg}}}$, provided that V is finite-dimensional.⁶ \triangleleft

The weight-graded vector spaces BV and $B^{\text{cyc}}V$ are canonically filtered by the filtration by weights (3.10). Their weight-graded duals B^*V and B_{cyc}^*V are filtered by the dual filtrations and the exterior powers $E_k B^*V$ and $E_k B_{\text{cyc}}^*V$ by the induced filtration from Definition 3.1.8.

Proposition 3.3.6 (Completed dual cyclic bar complex). *Let V be a graded vector space and $A \in \mathbb{Z}$. The filtration of B_{cyc}^*V dual to the weight-filtration of $B^{\text{cyc}}V$ is \mathbb{Z} -gapped, Hausdorff, decreasing and bounded from above. Moreover, the following holds:*

$$\dim(V) < \infty \implies (WG1) \text{ \& } (WG2) \text{ are satisfied.}$$

*The same holds for the induced filtration of $E_k B_{\text{cyc}}^*V[A]$.*

In the sense of Remark 3.3.5, we have

$$\hat{B}_{\text{cyc}}^*V \simeq (B^{\text{cyc}}V)^{*_{\text{g}}} \quad \text{and} \quad \hat{E}_k B_{\text{cyc}}^*V[A] \subset (E_k B^{\text{cyc}}V[A+1])^{*_{\text{g}}},$$

where “=” holds if V is finite-dimensional.

*The filtration degree of $\Psi \in \hat{E}_m B_{\text{cyc}}^*V[A]$ satisfies*

$$\|\Psi\| = \min\{k \in \mathbb{N}_0 \mid \exists W \in (E_m B^{\text{cyc}}V[A])_k : \Psi(W) \neq 0\}.$$

Proof. The proof is clear. □

⁶The problem is that if $\dim(V) = \infty$, then $(V \otimes V)^* \neq V^* \otimes V^*$.

3. Algebraic structures

Definition 3.3.7 (Cyclic A_∞ -algebra). *A graded vector space V together with a pairing*

$$\Pi : V[1] \otimes V[1] \rightarrow \mathbb{R}$$

of degree $d \in \mathbb{Z}$ and a collection of homogenous linear maps

$$\mu_k : V[1]^{\otimes k} \rightarrow V[1] \quad \text{for } k \geq 1$$

is called a cyclic A_∞ -algebra of degree d if the following conditions are satisfied:

(1) *The pairing Π is non-degenerate and graded antisymmetric; i.e., we have*

$$\Pi(v_1, v_2) = (-1)^{1+|v_1||v_2|} \Pi(v_2, v_1) \quad \text{for all } v_1, v_2 \in V[1].$$

(2) *The degrees satisfy $|\mu_k| = 1$ for all $k \geq 1$.*

(3) *The A_∞ -relations are satisfied: for all $k \geq 1$, we have*

$$\sum_{\substack{k_1, k_2 \geq 1 \\ k_1 + k_2 = k+1}} \sum_{p=1}^{k_1} \mu_{k_1} \circ_1^p \mu_{k_2} = 0, \quad (3.27)$$

where for all $p = 1, \dots, k$ and $v_1, \dots, v_k \in V[1]$ we define

$$\begin{aligned} (\mu_{k_1} \circ_1^p \mu_{k_2})(v_1, \dots, v_k) &:= (-1)^{|v_1| + \dots + |v_{p-1}|} \mu_{k_1}(v_1, \dots, v_{p-1}, \\ &\quad \mu_{k_2}(v_p, \dots, v_{p+k_2-1}), v_{p+k_2}, \dots, v_k). \end{aligned}$$

(4) *The operations $\mu_k^+ : V[1]^{\otimes k+1} \rightarrow \mathbb{R}$ defined by*

$$\mu_k^+ := \Pi \circ (\mu_k \otimes \mathbb{1})$$

for all $k \geq 1$ are cyclic symmetric; i.e., we have

$$\mu_k^+ \circ t_{k+1} = \mu_k^+.$$

We denote by $\tilde{\Pi} : V \otimes V \rightarrow \mathbb{R}$ and $\tilde{\mu}_k : V^{\otimes k} \rightarrow \mathbb{R}$ the operations before the degree shift; i.e., for all $k \geq 1$ and $\tilde{v}_1, \dots, \tilde{v}_k \in V$ with $v_i = \theta \tilde{v}_i$, we have

$$\begin{aligned} \tilde{\Pi}(\tilde{v}_1, \tilde{v}_2) &:= (-1)^{\tilde{v}_1} \Pi(v_1, v_2) \quad \text{and} \\ \tilde{\mu}_k(\tilde{v}_1, \dots, \tilde{v}_k) &:= \varepsilon(\theta, \tilde{v}) \mu_k(v_1, \dots, v_k). \end{aligned}$$

3. Algebraic structures

We define $\tilde{\mu}_k^+ : V^{\otimes k+1} \rightarrow \mathbb{R}$ similarly.

If $\mu_k \equiv 0$ for all $k \geq 2$, then (V, Π, μ_1) is called a cyclic cochain complex. If $\mu_k \equiv 0$ for all $k \geq 3$, then (V, Π, μ_1, μ_2) is called a cyclic dga. We use the same terminology but omit “cyclic” if there is no pairing Π and 1) and 4) are thus irrelevant.

Remark 3.3.8 (A difference in sign conventions). Our definition of μ_k^+ differs from the definition of m_k^+ in [CFL15, Definition 12.1] by a sign. To compensate this, we have to add this artificial sign in the definitions of Maurer-Cartan elements later; e.g., in Definition 3.4.2 or in the formula (A.12). \triangleleft

Definition 3.3.9 (Cyclic (co)homology of A_∞ -algebras). Let $\mathcal{A} = (V, (\mu_k))$ be an A_∞ -algebra. For every $k \geq 1$, we consider the maps $b'_k, R_k : V[1]^{\otimes k} \rightarrow BV$ given by

$$\begin{aligned} b'_k &:= \sum_{j=1}^k \sum_{i=0}^{k-j} t_{k-j+1}^i \circ (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^{-i} \quad \text{and} \\ R_k &:= \sum_{j=2}^k \sum_{i=1}^{j-1} (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^i, \end{aligned} \tag{3.28}$$

respectively, and define the following maps $BV \rightarrow BV$:

$$b' := \sum_{k=1}^{\infty} b'_k, \quad R := \sum_{k=2}^{\infty} R_k \quad \text{and} \quad b := b' + R.$$

We denote by $b^* : \hat{B}^*V = (BV)^{*g} \rightarrow \hat{B}^*V$ the dual map to $b : BV \rightarrow BV$. The following holds:⁷

$$|b| = 1 \quad (|b^*| = -1), \quad b \circ b = 0 \quad \text{and} \quad b(1-t) = (1-t)b'. \tag{3.29}$$

From the last equation we see that b restricts to $B^{\text{cyc}}V = BV/\text{im}(1-t)$. We define the following graded vector spaces:

$$\begin{aligned} D(V) &:= r(BV)[1], & D^*(V) &:= r(\hat{B}^*V)[1], \\ D^\lambda(V) &:= r(B^{\text{cyc}}V)[1], & D_\lambda^*(V) &:= r(\hat{B}_{\text{cyc}}^*V)[1]. \end{aligned}$$

For instance, we have

$$D_\lambda^q(V) = r(\hat{B}_{\text{cyc}}^*V)^{q+1} = (\hat{B}_{\text{cyc}}^*V)^{-q-1} \quad \text{for all } q \in \mathbb{Z}.$$

Then $(D(V), b)$ and $(D^\lambda(V), b)$ are chain complexes and $(D^*(V), b^*)$ and $(D_\lambda^*(V), b^*)$

⁷The facts (3.29) are generally known in some form (see [Mes16] or [Laz03]).

3. Algebraic structures

the dual cochain complexes, respectively. We define the following (co)homologies:

$$\begin{aligned} \mathrm{HH}(\mathcal{A}; \mathbb{R}) &:= \mathrm{H}(D(V), \mathrm{b}), & \mathrm{HH}^*(\mathcal{A}; \mathbb{R}) &:= \mathrm{H}(D^*(V), \mathrm{b}^*), \\ \mathrm{H}^\lambda(\mathcal{A}; \mathbb{R}) &:= \mathrm{H}(D^\lambda(V), \mathrm{b}), & \mathrm{H}_\lambda^*(\mathcal{A}; \mathbb{R}) &:= \mathrm{H}(D_\lambda^*(V), \mathrm{b}^*). \end{aligned}$$

We call HH the Hochschild homology and H^λ the cyclic homology of the A_∞ -algebra \mathcal{A} . We call HH^* the Hochschild cohomology and H_λ^* the cyclic cohomology of \mathcal{A} .

For a dga $\mathcal{A} = (V, \mu_1, \mu_2)$, we have for all $v_1, \dots, v_k \in V[1]$ the formula

$$\begin{aligned} \mathrm{b}(v_1 \dots v_k) &= \sum_{i=1}^k (-1)^{|v_1|+\dots+|v_{i-1}|} v_1 \dots \mu_1(v_i) \dots v_k \\ &\quad + \sum_{i=1}^{k-1} (-1)^{|v_1|+\dots+|v_{i-1}|} v_1 \dots \mu_2(v_i, v_{i+1}) \dots v_k \\ &\quad + (-1)^{|v_k|(|v_1|+\dots+|v_{k-1}|)} \mu_2(v_k, v_1) v_2 \dots v_{k-1}. \end{aligned}$$

Definition 3.3.10 (Strict units and strict augmentations). *Let $\mathcal{A} = (V, (\mu_k))$ be an A_∞ -algebra. A non-zero homogenous element $1 \in V[1]$ with $|1| = -1$ is called a strict unit for \mathcal{A} if the following holds:*

$$\begin{aligned} \mu_2(1, v) &= (-1)^{|v|+1} \mu_2(v, 1) = v & \forall v \in V[1], \\ \mu_k(v_1, \dots, v_{i-1}, 1, v_{i+1}, \dots, v_k) &= 0 & \forall k \neq 2, 1 \leq i \leq k, v_j \in V[1]. \end{aligned}$$

The pair $(\mathcal{A}, 1)$ is called a strictly unital A_∞ -algebra.

A strictly unital A_∞ -algebra $(\mathcal{A}, 1)$ is called strictly augmented if it is equipped with a linear map $\varepsilon : V[1] \rightarrow \mathbb{R}[1]$ which satisfies

$$\varepsilon(1_V) = 1_{\mathbb{R}}, \quad \varepsilon \circ \mu_1 = 0 \quad \text{and} \quad \varepsilon \circ \mu_2 = \mu_2 \circ (\varepsilon \otimes \varepsilon),$$

where $1_{\mathbb{R}}$ is the strict unit for \mathbb{R} endowed with the standard multiplication. The map ε is called a strict augmentation.

If the homological dga $\mathrm{H}(\mathcal{A}) := (\mathrm{H}(V, \tilde{\mu}_1), \mu_1 \equiv 0, \mu_2)$ of \mathcal{A} is strictly unital and strictly augmented, then \mathcal{A} is called homologically unital and homologically augmented, respectively. A strictly unital and strictly augmented cochain complex $(V, \mu_1, 1, \varepsilon)$ is called just augmented.

We denote by $u : \mathbb{R}[1] \rightarrow V[1]$ the injective linear map defined by $u(1_{\mathbb{R}}) := 1_V$, and by $u^* : \mathrm{B}_{\mathrm{cyc}}^* V \rightarrow \mathrm{B}_{\mathrm{cyc}}^* \mathbb{R}$ and $\varepsilon^* : \mathrm{B}_{\mathrm{cyc}}^* \mathbb{R} \rightarrow \mathrm{B}_{\mathrm{cyc}}^* V$ the precompositions with $u^{\otimes k}$ and $\varepsilon^{\otimes k}$ in

3. Algebraic structures

every weight- k component, respectively.

Remark 3.3.11 (On units and augmentations). (i) A strict unit 1_V for \mathcal{A} induces an A_∞ -morphism $(u_k) : \mathbb{R} \rightarrow V$ given by $u_1(1_{\mathbb{R}}) := 1_V$ and $u_k \equiv 0$ for all $k \geq 2$. A (general) augmentation of $(\mathcal{A}, 1_V)$ is by definition any A_∞ -morphism $(\varepsilon_k) : V \rightarrow \mathbb{R}$ such that $(\varepsilon_k) \circ (u_k) = \mathbb{1}$ as A_∞ -morphisms (see [Kel99]). Strict augmentations are precisely the maps ε_1 coming from augmentations (ε_k) with $\varepsilon_k \equiv 0$ for all $k \geq 2$.

(ii) As for $(V, \mu_1, 1, \varepsilon)$, we need the chain map ε to provide the splitting of the short exact sequence of chain complexes

$$0 \longrightarrow \mathbb{R}[1] \xrightarrow{u} V[1] \twoheadrightarrow \text{coker}(u) \longrightarrow 0,$$

$$\quad \quad \quad \nwarrow \varepsilon$$

so that we get $H(V) \simeq H_{\text{red}}(V) \oplus \mathbb{R}$, where $H_{\text{red}}(V) := H(\text{coker}(u))$. If (V, μ_1) is non-negatively graded and we are given an injective chain map $u : \mathbb{R}[1] \rightarrow V[1]$ ($=$: the classical augmentation), then one can show that such ε always exists. \triangleleft

Definition 3.3.12 (Reduced dual cyclic bar complex). *Let $(\mathcal{A}, 1)$ be a strictly unital A_∞ -algebra. Consider the injection $\iota_1 : BV \rightarrow BV$, $v_1 \otimes \cdots \otimes v_k \mapsto 1 \otimes v_1 \otimes \cdots \otimes v_k$. We define the reduced dual cyclic bar-complex by*

$$B_{\text{cyc,red}}^* V := \{\psi \in B_{\text{cyc}}^* V \mid \psi \circ \iota_1 = 0\}.$$

Under the assumption of strict unitality, b^ preserves $B_{\text{cyc,red}}^* V$, and hence we can consider the reduced cyclic cochain complex*

$$D_{\lambda,\text{red}}^*(V) := r(\hat{B}_{\text{cyc,red}}^* V)[1]$$

and define the reduced cyclic cohomology of \mathcal{A} by

$$H_{\lambda,\text{red}}^*(\mathcal{A}; \mathbb{R}) := H(D_{\lambda,\text{red}}^*(V), b^*).$$

Proposition 3.3.13 (Reduction to the reduced cyclic cohomology). *Let $\mathcal{A} = (V, (\mu_k))$ be an A_∞ -algebra with a strict unit 1 and a strict augmentation ε . Then the inclusions $B_{\text{cyc,red}}^* V$, $\varepsilon^*(B_{\text{cyc}}^* \mathbb{R}) \subset B_{\text{cyc}}^* V$ induce the decomposition*

$$H_{\lambda}^*(\mathcal{A}; \mathbb{R}) \simeq H_{\lambda,\text{red}}^*(\mathcal{A}; \mathbb{R}) \oplus H_{\lambda}^*(\mathbb{R}; \mathbb{R}).$$

3. Algebraic structures

Here we have

$$H_\lambda^q(\mathbb{R}; \mathbb{R}) = \begin{cases} \langle 1^{q+1*} \rangle & \text{for } q \geq 0 \text{ even,} \\ 0 & \text{for } q > 0 \text{ odd and } q < 0, \end{cases}$$

where $1^{i*} : \mathbb{R}[1]^{\otimes i} \rightarrow \mathbb{R}$ is defined by $1^{i*}(1^i) := 1$.

Sketch of the proof. The maps $\varepsilon^* : D_\lambda(\mathbb{R}) \rightarrow D_\lambda(V)$ and $u^* : D_\lambda(V) \rightarrow D_\lambda(\mathbb{R})$ are chain maps with $u^* \circ \varepsilon^* = \mathbb{1}$. Therefore, we have the sequence of cochain complexes

$$0 \longrightarrow D_{\lambda, \text{red}}(V) \hookrightarrow D_\lambda(V) \xrightarrow{u^*} D_\lambda(\mathbb{R}) \longrightarrow 0, \quad (3.30)$$

$\xleftarrow{\varepsilon^*}$

which is exact everywhere except for the middle, and where ε^* is a splitting map. The idea of [Lod92] is to replace these cochain complexes with quasi-isomorphic bicomplexes consisting of normalized Hochschild cochains $\bar{D}(V)$ such that the sequence becomes exact. The work then reduces to proving that $\bar{D}(V)$ computes $\text{HH}(\mathcal{A}; \mathbb{R})$; a variant of this result for A_∞ -algebras was proven in [Laz03]. \square

We will now compare our version of the cyclic cohomology of a dga (V, μ_1, μ_2) to a version based on [Lod92, Section 5.3.2]. Let $\tilde{b}, \tilde{\delta} : \bar{T}V \rightarrow \bar{T}V$ be the linear maps defined for all $\tilde{v}_1, \dots, \tilde{v}_k \in V$ by

$$\begin{aligned} \tilde{b}(\tilde{v}_1 \otimes \dots \otimes \tilde{v}_k) &:= \sum_{i=1}^{k-1} (-1)^{i-1} \tilde{v}_1 \otimes \dots \otimes \tilde{\mu}_2(\tilde{v}_i, \tilde{v}_{i+1}) \otimes \dots \otimes \tilde{v}_k \\ &\quad + (-1)^{k-1+\tilde{v}_k(\tilde{v}_1+\dots+\tilde{v}_{k-1})} \tilde{\mu}_2(\tilde{v}_k, \tilde{v}_1) \otimes \tilde{v}_2 \otimes \dots \otimes \tilde{v}_{k-1}, \\ \tilde{\delta}(\tilde{v}_1 \otimes \dots \otimes \tilde{v}_k) &:= \sum_{i=1}^k (-1)^{\tilde{v}_1+\dots+\tilde{v}_{i-1}} \tilde{v}_1 \otimes \dots \otimes \tilde{\mu}_1(\tilde{v}_i) \otimes \dots \otimes \tilde{v}_k. \end{aligned}$$

For all $q \geq 0$, we define

$$\tilde{D}_q(V) := \bigoplus_{\substack{k \geq 1 \\ d \in \mathbb{Z} \\ k-d=q+1}} (V^{\otimes k})^d$$

and $\tilde{\partial} : \tilde{D}_{q+1}(V) \rightarrow \tilde{D}_q(V)$ by

$$\tilde{\partial}(\tilde{v}_1 \dots \tilde{v}_k) = \tilde{b}(\tilde{v}_1 \dots \tilde{v}_k) + (-1)^{k+1} \tilde{\delta}(\tilde{v}_1 \dots \tilde{v}_k).$$

It can be checked that $\tilde{\partial} \circ \tilde{\partial} = 0$ and $\tilde{\partial}(\text{im}(1 - \tilde{t})) \subset \text{im}(1 - \tilde{t})$, so that $\tilde{\partial}$ induces a

3. Algebraic structures

boundary operator on the chain complexes

$$\tilde{D}(V) := \bigoplus_{q \in \mathbb{Z}} \tilde{D}_q(V) \quad \text{and} \quad \tilde{D}^\lambda(V) := \tilde{D}(V) / \text{im}(1 - \tilde{t}).$$

Here, we have $\tilde{t}(\tilde{v}_1 \cdots \tilde{v}_k) := (-1)^{k+|\tilde{v}_k|(|\tilde{v}_1|+\cdots+|\tilde{v}_{k-1}|)} \tilde{v}_k \tilde{v}_1 \cdots \tilde{v}_{k-1}$. We call $(\tilde{D}(V), \tilde{\partial})$ the *classical Hochschild complex* and $(\tilde{D}^\lambda(V), \tilde{\partial})$ the *classical cyclic complex* of the dga (V, μ_1, μ_2) . The chain complex $(\tilde{D}(V), \tilde{\partial})$ is the total complex of the bicomplex

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \downarrow \\ \leftarrow & (V^{\otimes 3})^1 & \xleftarrow{\tilde{\delta}} & (V^{\otimes 3})^0 & \xleftarrow{\tilde{\delta}} & (V^{\otimes 3})^{-1} & \leftarrow \\ & \downarrow \tilde{b} & & \downarrow \tilde{b} & & \downarrow \tilde{b} & \\ \leftarrow & (V^{\otimes 2})^1 & \xleftarrow{-\tilde{\delta}} & (V^{\otimes 2})^0 & \xleftarrow{-\tilde{\delta}} & (V^{\otimes 2})^{-1} & \leftarrow \\ & \downarrow \tilde{b} & & \downarrow \tilde{b} & & \downarrow \tilde{b} & \\ \leftarrow & V^1 & \xleftarrow{\tilde{\delta}} & V^0 & \xleftarrow{\tilde{\delta}} & V^{-1} & \leftarrow \end{array},$$

which differs from the bicomplex [Lod92, Equation (5.3.2.1)] by the reversed grading and by the fact that it lies in the whole upper half-plane and not just in the first quadrant. Their convention for a dga is namely $|\tilde{\mu}_1| = -1$, whereas ours is $|\tilde{\mu}_1| = 1$, and they consider \mathbb{N}_0 -grading, whereas we have \mathbb{Z} -grading.

Proposition 3.3.14 (The classical case). *Let $\mathcal{A} = (V, \mu_1, \mu_2)$ be a dga. Then the degree shift map*

$$\begin{aligned} U : \tilde{D}_q(V) &\longrightarrow D_q(V), \\ \tilde{v}_1 \otimes \cdots \otimes \tilde{v}_k &\longmapsto \varepsilon(\theta, \tilde{v}) v_1 \otimes \cdots \otimes v_k, \end{aligned}$$

where we denote $v_i = \theta \tilde{v}_i$, is an isomorphism of the chain complexes $(\tilde{D}(V), \tilde{\partial}) \simeq (D(V), b)$ and $(\tilde{D}^\lambda(V), \tilde{\partial}) \simeq (D^\lambda(V), b)$, respectively.

Proof. First of all, for the degrees holds $|\tilde{\mu}_j| = 2 - j$ for every $j \geq 1$. For every $j, k, l \geq 1$ such that $j + l \leq k + 1$ and for every $\tilde{v}_1, \dots, \tilde{v}_k \in V$, we compute

$$\begin{aligned} &[U^{-1}(\mathbb{1}^{l-1} \otimes \mu_j \otimes \mathbb{1}^{k-j-l+1})U](\tilde{v}_1 \cdots \tilde{v}_k) \\ &= (-1)^{l-1+(j-2)(\tilde{v}_1+\cdots+\tilde{v}_{l-1}+k-l-j+1)} \tilde{v}_1 \cdots \tilde{v}_{l-1} \tilde{\mu}_j(\tilde{v}_l \cdots \tilde{v}_{l+j-1}) \tilde{v}_{l+j} \cdots \tilde{v}_k, \\ &[U^{-1}t_k U](\tilde{v}_1 \cdots \tilde{v}_k) = (-1)^{k-1} \tilde{v}_1 \cdots \tilde{v}_k, \end{aligned}$$

where we use the Koszul convention $(f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{|f_2||v_1|} f_1(v_1) \otimes f_2(v_2)$. Using

3. Algebraic structures

this, we obtain

$$U^{-1}b'_k U = \sum_{j=1}^k \sum_{i=0}^{k-1} (-1)^{i+j(i+k+1)} t_{k-j+1}^i (\tilde{\mu}_j \otimes \mathbb{1}^{k-j}) t_k^{-i} \quad \text{and}$$

$$U^{-1}R_k U = \sum_{j=1}^k \sum_{i=1}^{j-1} (-1)^{(i+j)(k+1)} (\tilde{\mu}_j \otimes \mathbb{1}^{k-j}) t_k^i.$$

It is now easy to check that $U^{-1} \circ b \circ U = \tilde{\partial}$.

If $k \in \mathbb{N}$ is a weight and $d \in \mathbb{Z}$ a degree such that $k - d - 1 = q$ for some $q \in \mathbb{Z}$, we have schematically $U : (k, d) \mapsto (k, d - k) = (k, -q - 1)$. Therefore, U preserves the grading of chain complexes. This finishes the proof. \square

Proposition 3.3.15 (Reduced cochains are complete in 0, 1-connected case). *Suppose that $V = \bigoplus_{d \geq 0} V^d$ is a non-negatively graded vector space with $V^0 = \langle 1 \rangle$ for some $1 \in V$ ($\because V$ is connected) and $V^1 = 0$ ($\because V$ is simply-connected). Then for all $m \geq 1$, we have*

$$\hat{E}_m B_{\text{cyc,red}}^* V = E_m B_{\text{cyc,red}}^* V.$$

Proof. Let $\bar{V} := \bigoplus_{d \geq 2} V^d$. We clearly have $B_{\text{cyc,red}}^* V \simeq B_{\text{cyc}}^* \bar{V}$. Since $\bar{V}[1]$ is positively graded, we have $(B\bar{V})_k^d = 0$ whenever $k > d$. Therefore, a map $\Psi \in \hat{E}_m \bar{V}$, which is non-zero only on finitely many homogenous components of $B^{\text{cyc}} V[1]^{\otimes m}$, will be non-zero only on finitely many weights. This implies that $\Psi \in E_m \bar{V}$. \square

Remark 3.3.16 (Universal coefficient theorem). We have

$$BV = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k=1}^{\infty} (V[1]^{\otimes k})^d \quad \text{and} \quad B^* V = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k=1}^{\infty} (V[1]^{\otimes k})^{d*},$$

and hence

$$(BV)^{*_{\mathfrak{g}}} = \bigoplus_{d \in \mathbb{Z}} \prod_{k=1}^{\infty} (V[1]^{\otimes k})^{d*} = \bigoplus_{d \in \mathbb{Z}} \widehat{(B^* V)^d} = \hat{B}^* V.$$

Therefore, $(D_{\lambda}^*(V), b^*)$ is dual to $(D^{\lambda}(V), b)$ as a chain complex. Now, because we work over \mathbb{R} , the universal coefficient theorem gives

$$H_{\lambda}^q(\mathcal{A}, b^*) \simeq [H_q^{\lambda}(\mathcal{A}, b)]^* \quad \text{for all } q \in \mathbb{Z}.$$

Suppose that we have found closed homogenous elements $(w_i)_{i \in I} \subset D^{\lambda}(V)$ for some index set I which induce a basis of $H^{\lambda}(\mathcal{A}; \mathbb{R})$. For every $i \in I$, we define the linear map

3. Algebraic structures

$w_i^* : D^\lambda(V) \rightarrow \mathbb{R}$ by prescribing

$$w_i^*(w_j) = \delta_{ij} \quad \text{for all } j \in I$$

and $w_i^* \equiv 0$ on $\text{im } b$ and on a complement of $\ker(b)$ in $D^\lambda(V)$. Then $(w_i^*)_{i \in I} \subset D_\lambda^*(V)$ are closed homogenous elements which generate linearly independent cohomology classes in $H_\lambda^*(\mathcal{A}; \mathbb{R})$; if we denote $I_q := \{i \in I \mid w_i \in C_q^\lambda(V)\}$, then we can write

$$H_\lambda^q(\mathcal{A}; \mathbb{R}) = \left\{ \sum_{i \in I_q} \alpha_i w_i^* \mid \alpha_i \in \mathbb{R} \right\} \quad \text{for all } q \in \mathbb{Z}. \quad \triangleleft$$

3.4. Canonical dIBL-structure on cyclic cochains

In this section, we will consider a finite-dimensional cyclic dga (V, Π, m_1, m_2) of degree $2 - n$ for some $n \in \mathbb{N}$. This means that for all $v_1, v_2, v_3 \in V[1]$, the following relations holds:

$$\text{cyc. dga} \left\{ \begin{array}{l} \Pi(v_1, v_2) = (-1)^{1+|v_1||v_2|} \Pi(v_2, v_1), \\ m_1(m_1(v_1)) = 0, \\ m_1^+(v_1, v_2) = (-1)^{|v_1||v_2|} m_1^+(v_2, v_1), \\ m_1(m_2(v_1, v_2)) = -m_2(m_1(v_1), v_2) \\ \quad - (-1)^{|v_1|} m_2(v_1, m_1(v_2)), \\ m_2(m_2(v_1, v_2), v_3) = (-1)^{|v_1|+1} m_2(v_1, m_2(v_2, v_3)), \\ m_2^+(v_1, v_2, v_3) = (-1)^{|v_3|(|v_1|+|v_2|)} m_2^+(v_3, v_1, v_2). \end{array} \right. \quad \begin{array}{l} \text{cyc.} \\ \text{cochain} \\ \text{complex} \end{array} \quad (3.31)$$

The facts (A) and (C) from the Introduction apply, and we get the canonical dIBL-algebra $\text{dIBL}(\text{B}_{\text{cyc}}^* V[2 - n])$ of bidegree $(n - 3, 2)$ and the canonical Maurer-Cartan element $\mathfrak{m} = (\mathfrak{m}_{10})$. We will denote

$$C(V) := \text{B}_{\text{cyc}}^* V[2 - n]$$

and call it the space of *cyclic cochains on V* . If V is fixed, we will write just C .

Definition 3.4.1 (The canonical dIBL-algebra). *Let (V, Π, m_1) be a cyclic cochain complex of degree $2 - n$ which is finite-dimensional. Let $(e_0, \dots, e_m) \subset V[1]$ be a basis of $V[1]$, and let (e^0, \dots, e^m) be the dual basis with respect to Π ; this means that*

$$\Pi(e_i, e^j) = \delta_{ij} \quad \text{for all } i, j = 0, \dots, m.$$

3. Algebraic structures

We define the tensor $T = \sum_{i,j=0}^m T^{ij} e_i \otimes e_j \in V[1]^{\otimes 2}$ by⁸

$$T^{ij} = (-1)^{|e_i|} \Pi(e^i, e^j) \quad \text{for all } i, j = 0, \dots, m. \quad (3.32)$$

The canonical dIBL-algebra on $C(V)$ is the quadruple

$$\text{dIBL}(C(V)) := (C(V), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}),$$

where the operations $\mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}$ are defined for all $\psi, \psi_1, \psi_2 \in \hat{B}_{\text{cyc}}^* V$ and generating words $w = v_1 \dots v_k, w_1 = v_{11} \dots v_{1k_1}, w_2 = v_{21} \dots v_{2k_2} \in B^{\text{cyc}} V$ with $k, k_1, k_2 \geq 1$ as follows:

- The dIBL-boundary operator $\mathfrak{q}_{110} : \hat{E}_1 C \rightarrow \hat{E}_1 C$ of degree $|\mathfrak{q}_{110}| = -1$ is defined by

$$\mathfrak{q}_{110}(s\psi)(sw) := s \sum_{i=1}^k (-1)^{|v_1| + \dots + |v_{i-1}|} \psi(v_1 \dots v_{i-1} m_1(v_i) v_{i+1} \dots v_k). \quad (3.33)$$

- The product $\mathfrak{q}_{210} : \hat{E}_2 C \rightarrow \hat{E}_1 C$ of degree $|\mathfrak{q}_{210}| = -2(n-3) - 1$ is written schematically as

$$\mathfrak{q}_{210}(s^2\psi_1 \otimes \psi_2)(sw) := \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{|e_j||w^1|} T^{ij} \psi_1(e_i w^1) \psi_2(e_j w^2)$$

and defined “algorithmically” as follows:

For every cyclic permutation $\sigma \in \mathbb{S}_k$, consider the tensor

$$\sigma(w) := \varepsilon(\sigma, w) v_{\sigma_1^{-1}} \otimes \dots \otimes v_{\sigma_k^{-1}},$$

and split it into two parts w^1 and w^2 of possibly zero length such that $v_{\sigma_1^{-1}} \otimes \dots \otimes v_{\sigma_k^{-1}} = w^1 \otimes w^2$. Feed w^1 and w^2 into ψ_1 and ψ_2 preceded by e_i and e_j , respectively, and multiply the result with the sign $(-1)^{|e_j||w^1|}$, which is the Koszul sign to order

$$e_i e_j w^1 w^2 \mapsto e_i w^1 e_j w^2.$$

Finally, sum over all $\sigma \in \mathbb{S}_k$, all splittings of $\sigma(w)$ and all indices $i, j = 0, \dots, m$. The sign $\varepsilon(\sigma, w)$ is denoted by $\varepsilon(w \mapsto w^1 w^2)$ to indicate the splitting.

⁸See Appendix A for the invariant meaning of T as the Schwartz kernel of $\pm \mathbb{1}$.

3. Algebraic structures

- The coproduct $\mathfrak{q}_{120} : \hat{E}_1 C \longrightarrow \hat{E}_2 C$ of degree $|\mathfrak{q}_{120}| = -1$ is written schematically as

$$\begin{aligned} & \mathfrak{q}_{120}(s\psi)(s^2 w_1 \otimes w_2) \\ &= \frac{1}{2} \sum \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|e_j||w_1^1|} T^{ij} \psi(e_i w_1^1 e_j w_2^1) \end{aligned}$$

and defined “algorithmically” as follows:

For all cyclic permutations $\sigma \in \mathbb{S}_{k_1}$ and $\mu \in \mathbb{S}_{k_2}$, denote $w_1^1 := \sigma(w_1)$ and $w_2^1 := \mu(w_2)$ and let $\varepsilon(w_1 \mapsto w_1^1)$ and $\varepsilon(w_2 \mapsto w_2^1)$ be the corresponding Koszul signs, respectively. Feed w_1^1 and w_2^1 into ψ in the indicated order interleaved by e_i and e_j and multiply the result with the sign $(-1)^{|e_j||w_1^1|}$, which is the Koszul sign to order

$$e_i e_j w_1^1 w_2^1 \mapsto e_i w_1^1 e_j w_2^1.$$

Finally, sum over all $\sigma \in \mathbb{S}_{k_1}$, $\mu \in \mathbb{S}_{k_2}$ and all indices $i, j = 0, \dots, m$.

The operations are extended continuously to the completion.

Definition 3.4.2 (Canonical Maurer-Cartan element). *Let (V, Π, m_1, m_2) be a finite-dimensional cyclic dga of degree $2 - n$. The canonical Maurer-Cartan element \mathfrak{m} for $\text{dIBL}(C(V))$ consists of only one element $\mathfrak{m}_{10} \in \hat{E}_1 C$ of degree $|\mathfrak{m}_{10}| = 2(n - 3)$ which is defined by*

$$\mathfrak{m}_{10}(sv_1 v_2 v_3) := (-1)^{n-2} \mu_2^+(v_1, v_2, v_3) \quad \text{for all } v_1, v_2, v_3 \in V[1]$$

on the weight-three component of $B^{\text{cyc}} V[3 - n]$ and extended by 0 to other weight- k components.

Remark 3.4.3 (On canonical dIBL-structure). (i) Elements of the completion $\hat{C}(V)$ which are not in $C(V)$ will be called *long cyclic cochains*. Because there are no infinite sums in Definition 3.4.1, $\text{dIBL}(C)$ is completion-free. Clearly, the twist $\text{dIBL}^n(C)$ remains completion-free as long as $\mathfrak{n}_{lg} \in E_l C$ for all l, g .

(ii) The constructions of \mathfrak{q}_{210} and \mathfrak{q}_{120} do not depend on the choice of a basis and can be rephrased in terms of summation over ribbon graphs (see Example A.1.5).

(iii) According to Proposition 3.3.6, the filtration on $C(V)$ satisfies (WG1) & (WG2), and hence the IBL-structures $\text{IBL}(\mathbb{H}(C))$ and $\text{IBL}(\mathbb{H}^m(C))$ are well-defined (see Definition 3.2.8). \triangleleft

3. Algebraic structures

Proposition 3.4.4 (Formulas for twisted operations). *Let $\text{dIBL}(C(V))$ be the canonical dIBL-algebra for a finite-dimensional cyclic cochain complex (V, Π, m_1) of degree $2 - n$, and let $\mathbf{n} = (\mathbf{n}_{lg})$ be a Maurer-Cartan element. Then for all $l \geq 1$, $g \geq 0$, $\Psi \in \hat{B}_{\text{cyc}}^* V[3 - n]$ and generating words $W_1, \dots, W_l \in B^{\text{cyc}} V[3 - n]$, we have*

$$\begin{aligned} & [(\mathbf{q}_{210} \circ_1 \mathbf{n}_{lg})(\Psi)](W_1 \otimes \dots \otimes W_l) \\ &= \sum_{j=1}^l \sum \varepsilon' \varepsilon(w_j \mapsto w_j^1 w_j^2) T^{ab} \Psi(se_a w_j^1) \mathbf{n}_{lg}(W_1 \otimes \dots \otimes W_{j-1} \otimes (se_b w_j^2) \\ & \quad \otimes W_{j+1} \otimes \dots \otimes W_l), \end{aligned} \quad (3.34)$$

where the sum without limits is the sum in Definition 3.4.1 for \mathbf{q}_{210} and ε' is the Koszul sign of the following operation:

$$\begin{aligned} & (se_a e_b) W_1 \dots W_{j-1} (sw_j^1 w_j^2) W_{j+1} \dots W_l \\ & \mapsto (se_a w_j^1) W_1 \dots W_{j-1} (se_b w_j^2) W_{j+1} \dots W_l. \end{aligned}$$

In particular, for $l = 1$, $g \geq 0$ and $W \in B^{\text{cyc}} V[3 - n]$, we have

$$(\mathbf{q}_{210} \circ_1 \mathbf{n}_{1g})(W) = (-1)^{n-3} \sum T^{ab} \varepsilon(w \mapsto w^1 w^2) \mathbf{n}_{1g}(se_a w^1) \psi(e_b w^2), \quad (3.35)$$

and for $l = 2$, $g \geq 0$ and $W_1, W_2 \in B^{\text{cyc}} V[3 - n]$, we have

$$\begin{aligned} & [(\mathbf{q}_{210} \circ_1 \mathbf{n}_{2g})(\Psi)](W_1 \otimes W_2) \\ &= (-1)^{(n-3)(|\Psi|+1)} \left[\sum T^{ab} \varepsilon(w_1 \mapsto w_1^1 w_1^2) (-1)^{|e_b||w_1^1|} \Psi(se_a w_1^1) \right. \\ & \quad \mathbf{n}_{20}(se_b w_1^2 \otimes W_2) + (-1)^{|W_1||W_2|} \sum T^{ab} \varepsilon(w_2 \mapsto w_2^1 w_2^2) \\ & \quad \left. (-1)^{|e_b||w_2^1|} \Psi(se_a w_2^1) \mathbf{n}_{20}(se_b w_2^2 \otimes W_1) \right]. \end{aligned} \quad (3.36)$$

Proof. Let us first discuss the completions. Given $\mathbf{n}_{lg} \in \hat{E}_l C$, we can write it as $\mathbf{n}_{lg} = \sum_{i=1}^{\infty} \Phi_1^i \dots \Phi_l^i$ with generating words $\Phi_1^i \dots \Phi_l^i \in E_l C$ of weights approaching ∞ . The canonical extension of \circ_h to maps with finite filtration degree commutes with convergent infinite sums, and hence we have $\mathbf{q}_{klg} \circ_h \mathbf{n}_{lg} = \sum_{i=1}^{\infty} \mathbf{q}_{klg} \circ_h (\Phi_1^i \dots \Phi_l^i)$. Therefore, it suffices to prove the formulas for generating words $\Phi_1^i \dots \Phi_l^i \in E_l C$.

From (3.14), we get for every $\Psi, \Phi_1, \dots, \Phi_l \in C$ the equation

$$[\mathbf{q}_{210} \circ_1 (\Phi_1 \dots \Phi_l)](\Psi) = \sum_{i=1}^l (-1)^{|\Phi_i|(|\Phi_1| + \dots + |\Phi_{i-1}|)} \mathbf{q}_{210}(\Psi, \Phi_i) \Phi_1 \dots \hat{\Phi}_i \dots \Phi_l,$$

3. Algebraic structures

where $\Phi_1 \cdots \Phi_l$ on the left-hand-side is considered as a map $E_0 C = \mathbb{R} \rightarrow E_l C$ mapping 1 to $\Phi_1 \cdots \Phi_l$. For $W_1, \dots, W_l \in B^{\text{cyc}} V[3-n]$ and $\sigma \in \mathbb{S}_l$, we use

$$[\sigma(\Phi_1 \otimes \cdots \otimes \Phi_l)](W_1 \otimes \cdots \otimes W_l) = (\Phi_1 \otimes \cdots \otimes \Phi_l)[\sigma^{-1}(W_1 \otimes \cdots \otimes W_l)]$$

and Definition 3.4.1 to get

$$\begin{aligned} & ([q_{210} \circ_1 (\Phi_1 \cdots \Phi_l)](\Psi))(W_1 \otimes \cdots \otimes W_l) = \\ &= \sum_{i=1}^l (-1)^{|\Phi_i|(|\Phi_1|+\cdots+|\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in \mathbb{S}_l} \varepsilon(\sigma^{-1}, W) [q_{210}(\Psi, \Phi_i)](W_{\sigma_1}) \\ & \quad \Phi_1(W_{\sigma_2}) \cdots \hat{\Phi}_i(\emptyset) \cdots \Phi_l(W_{\sigma_l}) \\ &= \sum_{i=1}^l (-1)^{|\Phi_i|(|\Phi_1|+\cdots+|\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in \mathbb{S}_l} \varepsilon(\sigma^{-1}, W) (-1)^{|\mathbf{s}| \Psi} \\ & \quad \sum \varepsilon(w_{\sigma_1} \mapsto w_{\sigma_1}^1 w_{\sigma_1}^2) (-1)^{|e_b| |w_{\sigma_1}^1|} T^{ab} \Psi(e_a w_{\sigma_1}^1) \Phi_i(e_b w_{\sigma_1}^2) \\ & \quad \Phi_1(W_{\sigma_2}) \cdots \hat{\Phi}_i(\emptyset) \cdots \Phi_l(W_{\sigma_l}) \\ &=: (*), \end{aligned}$$

where $\hat{\Phi}_i(\emptyset)$ means omission of the corresponding term. Consider the bijection

$$\begin{aligned} I : \{1, \dots, l\} \times \mathbb{S}_l &\longrightarrow \{1, \dots, l\} \times \mathbb{S}_l \\ (i, \sigma) &\longmapsto \left(j := \sigma_1, \mu := \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & l \\ \sigma_2 & \dots & \sigma_i & \sigma_1 & \sigma_{i+1} & \dots & \sigma_l \end{pmatrix} \right). \end{aligned}$$

Given $(i, \sigma) \in \{1, \dots, l\} \times \mathbb{S}_l$ and $b \in \{1, \dots, m\}$, let $(j, \mu) := I(i, \sigma)$ and

$$W' := W_1 \otimes \cdots \otimes W_{j-1} \otimes (se_b w_j^2) \otimes W_{j+1} \otimes \cdots \otimes W_l.$$

Suppose that $(\Phi_1 \otimes \cdots \otimes \Phi_l)(W') \neq 0$. We compute the Koszul sign $\varepsilon(\mu^{-1}, W')$ in the following way:

$$\begin{aligned} W' &\mapsto (-1)^{(|w_j^1|+|e_b|+|W_j|)(|W_1|+\cdots+|W_{j-1}|)} (se_b w_j^2) W_1 \dots \hat{W}_j \dots W_l \\ &\mapsto \underbrace{(-1)^{(|w_j^1|+|e_b|)(|W_1|+\cdots+|W_{j-1}|)}}_{=:\varepsilon_1} \varepsilon(\sigma^{-1}, W) (se_b w_j^2) W_{\sigma_2} \dots W_{\sigma_l} \\ &\mapsto \underbrace{\varepsilon_1 \varepsilon(\sigma^{-1}, W) (-1)^{|\Phi_i|(|\Phi_1|+\cdots+|\Phi_{i-1}|)}}_{=\varepsilon(\mu^{-1}, W')} W_{\sigma_2} \dots W_{\sigma_i} \underbrace{(se_b w_j^2) W_{\sigma_{i+1}} \dots W_{\sigma_l}}_{=W'_{\mu_1} \dots W'_{\mu_l}}. \end{aligned}$$

3. Algebraic structures

Using this, we can rewrite (*) as

$$\begin{aligned}
(*) &= (-1)^{|s||\Psi|} \sum_{j=1}^l \sum \varepsilon(w_j \mapsto w_j^1 w_j^2) (-1)^{|e_b||w_j^1|} T^{ab} \Psi(e_a w_j^1) \\
&\quad \varepsilon_1 \frac{1}{l!} \sum_{\mu \in \mathbb{S}_l} \varepsilon(\mu^{-1}, W') \Phi_1(W'_{\mu_1}) \dots \Phi_l(W'_{\mu_l}) \\
&= \sum_{j=1}^l \sum \varepsilon(w_j \mapsto w_j^1 w_j^2) (-1)^{|s||\Psi|+|e_b||w_j^1|+(|w_j^1|+|e_b|)(|W_1|+\dots+|W_{j-1}|)} T^{ab} \\
&\quad \Psi(se_a w_j^1) (\Phi_1 \dots \Phi_l)(W_1 \otimes \dots \otimes W_{j-1} \otimes (se_b w_j^2) \otimes W_{j+1} \otimes \dots \otimes W_l).
\end{aligned}$$

Finally, we use

$$T^{ab} \neq 0 \implies |e_a| + |e_b| = n - 2$$

to write

$$\begin{aligned}
|s||\Psi| &= |s|(|w_j^1| + |e_a|) = (n - 3)(|w_j^1| + n - 2 - |e_b|) \\
&= |s|(|w_j^1| + |e_b|) \pmod{2},
\end{aligned}$$

and the formula (3.34) follows.

As for (3.35), we first compute ε' for $l = 1$ as follows:

$$\begin{aligned}
\ln_{-1}(\varepsilon') &= |w_1||e_b| + (|e_b| + |w_1|)|s| = |w^1||w^2| + |s||e_b| \\
&\quad \uparrow \\
&\quad 2(n-3)=|n_{10}|=|s|+|e_b|+|w^2| \\
&= |w^1||w^2| + |e_a||e_b| \pmod{2}. \\
&\quad \uparrow \\
&\quad |e_a|+|e_b|=|s|+1
\end{aligned}$$

Using this, we obtain

$$\begin{aligned}
[(q_{210} \circ_1 n_{1g})(\Psi)](W) &= \sum \varepsilon' \varepsilon(w \mapsto w^1 w^2) T^{ab} \Psi(se_a w^1) n_{1g}(se_b w^2) \\
&= (-1)^{|s|} \sum \varepsilon(w \mapsto w^2 w^1) T^{ba} n_{1g}(se_b w^2) \Psi(se_a w^1), \\
&\quad \uparrow \\
&\quad T^{ab}=(-1)^{|s|+|e_a||e_b|} T^{ba} \\
\varepsilon(w \mapsto w^1 w^2) &= (-1)^{|w^1||w^2|} \varepsilon(w \mapsto w^2 w^1)
\end{aligned}$$

which implies (3.35).

The proof of (3.36) is a combination of the same arguments. \square

We will now relate homology of the twisted boundary operator q_{110}^n to cohomology of an A_∞ -algebra on V induced by n_{10} .

Definition 3.4.5 (A_∞ -operations and compatible Maurer-Cartan element). *Suppose that (V, Π, m_1) is a finite-dimensional cyclic cochain complex of degree $2 - n$, and let*

3. Algebraic structures

$\mathbf{n} = (\mathbf{n}_{lg})$ be a Maurer-Cartan element for $\text{dIBL}(C(V))$. We define the operations $\mu_k : V[1]^{\otimes k} \rightarrow V[1]$ for all $k \geq 1$ by

$$\mu_k(v_1, \dots, v_k) := (-1)^{n-3} \sum_{i,j} T^{ij} \mathbf{n}_{10}(se_i v_1 \dots v_k) e_j$$

for all $v_1, \dots, v_k \in V[1]$, where T^{ij} is the matrix from Definition 3.4.1.

If (V, Π, m_1, m_2) is in addition a cyclic dga and \mathbf{m} the canonical Maurer-Cartan element for $\text{dIBL}(C(V))$, then we say that \mathbf{n} is compatible with \mathbf{m} if

$$\mathbf{n}_{10}(sv_1 v_2 v_3) = \mathbf{m}_{10}(sv_1 v_2 v_3) \quad \text{for all } v_1, v_2, v_3 \in V[1].$$

Proposition 3.4.6 (Twisted boundary operator $\mathbf{q}_{110}^{\mathbf{n}}$ and A_∞ -cyclic cohomology). *In the setting of Definition 3.4.5, the triple $\mathcal{A}_{\mathbf{n}}(V) := (V, \Pi, (\mu_k))$ is a cyclic A_∞ -algebra. We always have $\mu_1 = m_1$, and if \mathbf{n} is compatible with \mathbf{m} for a cyclic dga (V, Π, m_1, m_2) , then also $\mu_2 = m_2$.*

The following holds for the homologies:

$$\mathbb{H}^n(C(V)) = r(H_\lambda^*(\mathcal{A}_{\mathbf{n}}(V); \mathbb{R}))[3 - n].$$

Proof. First of all, according to Definition 3.2.6 we must have $\|\mathbf{n}_{10}\| > 2$, and hence

$$\mathbf{n}_{10}(sv_1 v_2) = \mathbf{n}_{10}(sv_1) = 0 \quad \text{for all } v_1, v_2 \in V[1].$$

This implies $\mu_1 = m_1$.

Now, let e_0, \dots, e_m be a basis of $V[1]$ and let e^0, \dots, e^m be the dual basis with respect to Π . For all $k \geq 2$ and $v_1, \dots, v_k \in V[1]$, we compute the following:

$$\begin{aligned} & \Pi(\mu_k(v_1, \dots, v_k), v_{k+1}) \\ &= (-1)^{n-3} \sum_{i,j} (-1)^{|e_i|} \Pi(e^i, e^j) \mathbf{n}_{10}(se_i v_1 \dots v_k) \Pi(e_j, v_{k+1}) \\ &= (-1)^{n-3} \sum_i (-1)^{|e_i|} \mathbf{n}_{10}(se_i v_1 \dots v_k) \Pi(e^i, v_{k+1}) \\ & \quad \uparrow \\ & \forall v \in V[1]: \sum_j \Pi(v, e^j) e_j = v \\ &= (-1)^{n-3} \sum_i (-1)^{|e_i|(n-3)} \mathbf{n}_{10}(sv_1 \dots v_k e_i) \Pi(e^i, v_{k+1}) \\ & \quad \uparrow \\ & (|v|_1 + \dots + |v_k|) |e_i| = \\ & (|\mathbf{n}_{10}| + |s| + |e_i|) |e_i| = (|s| + 1) |e_i| \end{aligned}$$

3. Algebraic structures

$$\begin{aligned}
&= (-1)^{n-2} \sum_i \mathbf{n}_{10}(sv_1 \dots v_k e_i) \Pi(v_{k+1}, e^i) \\
&\quad \uparrow \\
&\quad 1+|v_{k+1}||e^i|=1+(|e|^i+2-n)|e^i| \\
&\quad =1+(3-n)|e^i|=1+(3-n)|e_i| \\
&= (-1)^{n-2} \mathbf{n}_{10}(sv_1 \dots v_{k+1}).
\end{aligned}$$

Therefore, we have

$$\mathbf{n}_{10} = (-1)^{n-2} \sum_{k \geq 2} \mu_k^+.$$

In this case, [CFL15, Proposition 12.3] asserts that the A_∞ -relations (3.27) for $(\mu_k)_{k \geq 1}$ are equivalent to the “lowest” Maurer-Cartan equation (3.24) for \mathbf{n}_{10} . The degree condition $|\mu_k| = 1$ and the cyclic symmetry of μ_k^+ are easy to check. Therefore, $\mathcal{A}_n(V)$ is a cyclic A_∞ -algebra.

As for the compatibility with \mathbf{m} , we have for all $v_1, v_2 \in V[1]$ the following:

$$\begin{aligned}
m_2(v_1, v_2) &= \sum_i \Pi(e_i, m_2(v_1, v_2)) e^i \\
&= \sum_{i,j} (-1)^{|e_i|} T^{ij} \Pi(e_i, m_2(v_1, v_2)) e_j \\
&\quad \uparrow \\
&\quad T^{ij} = (-1)^{|e_i|} \Pi(e^i, e^j) \\
&= \sum_{i,j} (-1)^{1+(n-2)|e_i|} T^{ij} \underbrace{\Pi(m_2(v_1, v_2), e_i)}_{= (-1)^{n-2} \mathbf{m}_{10}(sv_1 v_2 e_i)} e_j \\
&\quad \uparrow \\
\Pi(v_1, v_2) &= (-1)^{1+(n-3)|v_1|} \Pi(v_2, v_1) = (-1)^{n-2} \mathbf{m}_{10}(sv_1 v_2 e_i) \\
&= (-1)^{n-3} \sum_{i,j} T^{ij} \mathbf{m}_{10}(se_i v_1 v_2) e_j \\
&\quad \uparrow \\
(|v_1|+|v_2|)|e_i| &= (|\mathbf{m}_{10}| - |s| - |e_i|)|e_i| \\
&= (n-2)|e_i| \\
&= \mu_2(v_1, v_2).
\end{aligned}$$

We will now clarify the relation to the cyclic cohomology of $\mathcal{A}_n(V)$. Recall from Proposition 3.2.11 that $\mathbf{q}_{110}^n(\Psi) = \mathbf{q}_{110}(\Psi) + \mathbf{q}_{210}(\mathbf{n}_{10}, \Psi)$ for $\Psi \in \hat{\mathbf{B}}_{\text{cyc}}^* V[3-n]$, where the first term is given by (3.33) and the second by (3.35). Consider now \mathbf{b}'^k and R^k from (3.28), whose sum gives the Hochschild boundary operator \mathbf{b} . Using the cyclic symmetry, we can rewrite a summand of \mathbf{b}'^k for $j = 1, \dots, k$ and $i = 0, \dots, k-j$ applied

3. Algebraic structures

to a generating word $v_1 \dots v_k \in B^{\text{cyc}}V$ as follows:

$$\begin{aligned}
& [t_{k-j+1}^i \circ (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^{-i}](\underbrace{v_1 \dots v_k}_{=:w}) = \\
& = (-1)^{|v_1|+\dots+|v_i|} v_1 \dots v_i \mu_j(v_{i+1} \dots v_{i+j}) v_{i+j+1} \dots v_k \quad (3.37) \\
& = \varepsilon(w \mapsto w^1 w^2) \mu_j(\underbrace{v_{i+1} \dots v_{i+j}}_{=:w^1}) \underbrace{v_{i+j+1} \dots v_k v_1 \dots v_i}_{=:w^2}
\end{aligned}$$

Clearly, summing (3.37) over $j = 1$ and $i = 0, \dots, k-1$ gives the dual to \mathfrak{q}_{110} . For $j = 2, \dots, k$, we can write (3.37) as

$$(-1)^{n-3} \sum_{i,j} \varepsilon(w \mapsto w^1 w^2) T^{ij} \mathfrak{n}_{10}(se_i w^1) e_j w^2.$$

Therefore, the sum over $j = 2, \dots, k$ and $i = 0, \dots, k-j$ gives the part of the dual to $\mathfrak{q}_{210}(\mathfrak{n}_{10}, \Psi)$ corresponding to the cyclic permutations $\sigma \in \mathbb{S}_k$ with $\sigma_1 = 1, j+1, \dots, k$. The rest, i.e., the cyclic permutations with $\sigma_1 = 2, \dots, j$, is obtained analogously from the summands $(\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^i$ of R^k for $j = 2, \dots, k$ and $i = 1, \dots, j-1$. We conclude that $\mathfrak{q}_{110}^n : \hat{B}_{\text{cyc}}^* V[3-n] \rightarrow \hat{B}_{\text{cyc}}^* V[3-n]$ is a degree shift of $b^* : \hat{B}_{\text{cyc}}^* V \rightarrow \hat{B}_{\text{cyc}}^* V$. As for the gradings, we have:

$$\begin{aligned}
r(D_\lambda(V))[3-n]^i &= r(D_\lambda(V))^{i+3-n} = (D_\lambda(V))^{-i-3+n} = (\hat{B}_{\text{cyc}}^* V)^{i+3-n-1} \\
&= \hat{B}_{\text{cyc}}^* V[2-n]^i.
\end{aligned}$$

This finishes the proof. \square

We will now turn to units and augmentations.

Definition 3.4.7 (Reduced canonical dIBL-algebra). *Let $(V, \Pi, m_1, 1, \varepsilon)$ be an augmented cyclic cochain complex of degree $2-n$ from Definition 3.3.10. We define the space of reduced cyclic cochains on V by*

$$C_{\text{red}}(V) := B_{\text{cyc,red}}^* V[2-n].$$

We define the reduced canonical dIBL-algebra by

$$\text{dIBL}(C_{\text{red}}(V)) := (C_{\text{red}}(V), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}),$$

where $\mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}$ are restrictions of the operations of $\text{dIBL}(C(V))$.

3. Algebraic structures

Definition 3.4.8 (Strictly reduced Maurer-Cartan element). *In the setting of Definition 3.4.7, we call a Maurer-Cartan element $\mathbf{n} = (\mathbf{n}_{lg})$ for $\mathrm{dIBL}(C(V))$ strictly reduced if $\mathbf{n}_{lg} \in \hat{E}_l C_{\mathrm{red}}(V)$ for all $(l, g) \neq (1, 0)$ and if the A_∞ -algebra $(\mathcal{A}_{\mathbf{n}}(V), \mathbf{1}, \varepsilon)$ induced by \mathbf{n}_{10} is strictly unital and strictly augmented. Given a strictly reduced Maurer-Cartan element \mathbf{n} , we can define the twisted IBL_∞ -algebra*

$$\mathrm{dIBL}^{\mathbf{n}}(C_{\mathrm{red}}(V)) := (C_{\mathrm{red}}(V), (\mathbf{q}_{klg}^{\mathbf{n}})),$$

where $\mathbf{q}_{klg}^{\mathbf{n}}$ are the restrictions of the operations of $\mathrm{dIBL}^{\mathbf{n}}(C(V))$. We denote the homology of $\mathrm{dIBL}^{\mathbf{n}}(C_{\mathrm{red}})$ by $\mathbb{H}^{\mathbf{n}}(C_{\mathrm{red}})$ or $\mathbb{H}^{\mathbf{n}, \mathrm{red}}(C)$.⁹

Remark 3.4.9 (On strictly reduced Maurer-Cartan element). (i) We can say that the IBL_∞ -algebra $\mathrm{dIBL}^{\mathbf{n}}(C_{\mathrm{red}})$ is a *subalgebra* of $\mathrm{dIBL}^{\mathbf{n}}(C)$, which means that the inclusion $C_{\mathrm{red}} \hookrightarrow C$ induces the following commutative diagram for all $k, l \geq 1, g \geq 0$:

$$\begin{array}{ccc} \hat{E}_k C & \xrightarrow{\mathbf{q}_{klg}^{\mathbf{n}}} & \hat{E}_l C \\ \uparrow & & \uparrow \\ \hat{E}_k C_{\mathrm{red}} & \xrightarrow{\mathbf{q}_{klg}^{\mathbf{n}}} & \hat{E}_l C_{\mathrm{red}}. \end{array}$$

We denote this fact by $\mathrm{dIBL}^{\mathbf{n}}(C_{\mathrm{red}}) \subset \mathrm{dIBL}^{\mathbf{n}}(C)$.

(ii) The canonical Maurer-Cartan element \mathbf{m} of a strictly augmented strictly unital dga $(V, m_1, m_2, \mathbf{1}, \varepsilon)$ is strictly reduced (this follows from Proposition 3.4.6).

(iii) In the situation of Definition 3.4.8, we denote

$$\bar{V}[1] := \ker(\varepsilon),$$

so that $V = \bar{V} \oplus \langle 1 \rangle$. We use the canonical projection $\pi : V \rightarrow \bar{V}$ to identify $\hat{B}_{\mathrm{cyc}}^* \bar{V} \xrightarrow{\sim} \hat{B}_{\mathrm{cyc}, \mathrm{red}}^* V$ via the componentwise pullback π^* . In this way, we obtain the IBL_∞ -algebras $\mathrm{dIBL}(C(\bar{V}))$ and $\mathrm{dIBL}^{\mathbf{n}}(C(\bar{V}))$, which are isomorphic to $\mathrm{dIBL}(C_{\mathrm{red}}(V))$ and $\mathrm{dIBL}^{\mathbf{n}}(C_{\mathrm{red}}(V))$, respectively. \triangleleft

In the following list, we sum up our main reasons for considering units, augmentations and reduced Maurer-Cartan elements. Suppose that we are in the situation of Definition 3.4.8, then:

⁹The latter option suggests that it might be possible to define the reduced homology with the induced IBL -algebra even if \mathbf{n} is not strictly reducible, e.g., if $(\mathcal{A}_{\mathbf{n}}(V), \mathbf{1}, \varepsilon)$ is only homologically unital and augmented.

3. Algebraic structures

- Proposition 3.3.13 implies the splitting

$$\mathbb{H}^n(C)[1] = \mathbb{H}^n(C_{\text{red}})[1] \oplus \langle s_1^{2q-1*} \mid q \in \mathbb{N} \rangle. \quad (3.38)$$

Here $1^{i*} \in B_{\text{cyc}}^* V$ is the componentwise pullback ε^* of $1^{i*} \in B_{\text{cyc}}^*(\mathbb{R})$. To get this, we used

$$\mathbb{H}^n(C_{\text{red}}) = r(H_{\lambda, \text{red}}^*(\mathcal{A}_n))[3 - n],$$

which can be seen by redoing the proof of Proposition 3.4.6 with reduced cochains.

- The subalgebra $\text{dIBL}^n(C_{\text{red}}) \subset \text{dIBL}^n(C)$ induces the subalgebra

$$\text{IBL}(\mathbb{H}^n(C_{\text{red}})) \subset \text{IBL}(\mathbb{H}^n(C)),$$

and any higher operation \mathfrak{q}_{1lg}^n which induces a map $\hat{E}_1\mathbb{H}(C) \rightarrow \hat{E}_l\mathbb{H}(C)$ induces a map $\hat{E}_1\mathbb{H}(C_{\text{red}}) \rightarrow \hat{E}_l\mathbb{H}(C_{\text{red}})$ as well.

- If V is non-negatively graded, connected and simply-connected, then we have $\hat{E}_k C_{\text{red}} \simeq E_k C_{\text{red}}$ for all $k \in \mathbb{N}_0$ by Proposition 3.3.15, and hence $\text{dIBL}^n(C_{\text{red}})$ is completion-free.

Proposition 3.4.10 (Operations on units). *Suppose that $(V, \Pi, m_1, 1, \varepsilon)$ is a finite-dimensional augmented cyclic cochain complex of degree $2 - n$ such that $n \geq 1$, and let \mathfrak{n} be a strictly reduced Maurer-Cartan element for $\text{dIBL}(C(V))$. The following relations are the only relations containing 1^{i*} which may be non-zero on the homology $\mathbb{H}^n(C)$:*

For all $\Psi \in C_{\text{red}}(V)$ and $l \geq 1$, $g \geq 0$, we have

$$\begin{aligned} \mathfrak{q}_{210}(s_1^* \otimes \Psi) &= (-1)^{(n-2)|\Psi|} \mathfrak{q}_{210}(\Psi \otimes s_1^*) = (-1)^{n-2} \Psi \circ \iota_v \quad \text{and} \\ \mathfrak{q}_{1lg}^n(s_1^*) &= -\mathfrak{n}_{lg} \circ \iota_v, \end{aligned}$$

where ι_v is defined as follows:

- The element $v \in V[1]$ is the unique vector such that $\Pi(1, v) = 1$ and $v \perp \bar{V}[1]$ with respect to Π . Note that $|v| = n - 1$ and that such v always exists due to non-degeneracy.
- We start by defining $\iota_v : B^{\text{cyc}} V \rightarrow B^{\text{cyc}} V$ by

$$\iota_v(v_1 \dots v_k) := \sum_{i=1}^k (-1)^{|v|(|v_1| + \dots + |v_{i-1}|)} v_1 \dots v_{i-1} v v_i \dots v_k$$

3. Algebraic structures

for all generating words $v_1 \dots v_k \in B^{\text{cyc}}V$. Next, for all $k \geq 1$, we define $\iota_v : (B^{\text{cyc}}V)^{\otimes k} \rightarrow (B^{\text{cyc}}V)^{\otimes k}$ by

$$\begin{aligned} \iota_v(w_1 \otimes \dots \otimes w_k) &:= (-1)^{|v|k} \sum_{j=1}^k (-1)^{|v|(|w_1| + \dots + |w_{j-1}|)} w_1 \otimes \dots \otimes w_{j-1} \\ &\quad \otimes \iota_v(w_j) \otimes w_{j+1} \otimes \dots \otimes w_k \end{aligned}$$

for all generating words $w_1, \dots, w_k \in B^{\text{cyc}}V$. Finally, we take the degree shift $\iota_v : (B^{\text{cyc}}V[3-n])^{\otimes k} \rightarrow (B^{\text{cyc}}V[3-n])^{\otimes k}$ according to the degree shift convention (3.4).

Proof. Pick a basis (e_0, \dots, e_m) of $V[1]$ such that $e_0 = 1$ and $\bar{V}[1] = \langle e_1, \dots, e_m \rangle$. If (e^0, \dots, e^m) is the dual basis, then we have $v = e^0$. We will often use the following relation:

$$\sum_{j=0}^m T^{1j} e_j = \sum_{j=0}^m (-1)^{|1|} \Pi(v, e^j) e_j = -v. \quad (3.39)$$

We consider only those generating words $w = v_1 \dots v_k$ of $B^{\text{cyc}}V$ with either $v_i \in \bar{V}$ for each i (shortly $w \in B^{\text{cyc}}\bar{V}$) or $v_i = 1$ for each i with k odd (i.e., $w = 1^{2j-1}$ for some j). Let w_1, \dots, w_l with $w_j = v_{j1} \dots v_{jk_j}$ denote such generating words. Clearly, if $\Phi \in \hat{E}_l C(V)$ is a \mathfrak{q}_{110}^n -closed element which vanishes on all $w_1 \otimes \dots \otimes w_l$, then (3.38) implies that $[\Phi] = 0$ in $\hat{E}_l \mathbb{H}(C)$.

For $\Psi \in C_{\text{red}}(V)$ and $q \geq 1$ odd, we compute using (3.39) the following:

$$\begin{aligned} \mathfrak{q}_{210}(s^2 1^{q*} \otimes \psi)(sw) &= \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{(n-1)|w^1|} T^{1j} 1^{q*}(1w^1) \psi(e_j w^2) \\ &= - \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{(n-1)|w^1|} 1^{q*}(1w^1) \psi(v w^2) \\ &=: (*). \end{aligned}$$

Now, in order to get $(*) \neq 0$, we need either $q = 1$ and $w \in B^{\text{cyc}}\bar{V}$, in which case

$$\begin{aligned} (*) &= - \sum \varepsilon(w \mapsto \underbrace{w^1}_{=\emptyset} w^2) \psi(v w^2) \\ &= - \sum_{j=1}^k (-1)^{|v|(|v_1| + \dots + |v_{j-1}|)} \psi(v_1 \dots v_{j-1} v v_{j+1} \dots v_k) \\ &= -(\psi \circ \iota_v)(w) = (-1)^{n-2} (\Psi \circ \iota_v)(w), \end{aligned}$$

3. Algebraic structures

or $q > 1$ odd and $w = 1^{q-1}$, in which case

$$\begin{aligned}
 (*) &= \sum \varepsilon(w \mapsto w^1 \underbrace{w^2}_{=\emptyset}) 1^{q*} (1^q) \psi(v) \\
 &= \psi(v) \sum_{j=1}^{q-1} (-1)^j \\
 &= 0.
 \end{aligned}$$

Next, because $n \geq 1$, we get $T^{11} = 0$, and hence

$$\mathfrak{q}_{120}(1^{q*}) = 0 \quad \text{for all } q \in \mathbb{N}$$

on the chain level. Therefore, we have $\mathfrak{q}_{1lg}^n = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg}$ for all $l \geq 1, g \geq 0$, and using Proposition 3.4.4 and (3.39), we obtain

$$\begin{aligned}
 &[(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg})(1^{q*})](W_1 \otimes \cdots \otimes W_l) \\
 &= - \sum_{j=1}^l \sum \varepsilon' \varepsilon(w_j \mapsto w_j^1 w_j^2) 1^{q*} (1 w_j^1) \mathfrak{n}_{lg}(W_1 \otimes \cdots \otimes W_{j-1} \otimes (sv w_j^2) \\
 &\quad \otimes W_{j+1} \otimes \cdots \otimes W_l) \\
 &=: (**).
 \end{aligned}$$

In order to get $(**) \neq 0$, we need either $q = 1$ and $w_j \in B^{\text{cyc}} \bar{V}$ for all j , in which case

$$\begin{aligned}
 (**) &= - \sum_{j=1}^l \sum_{i=1}^{k_j} (-1)^{|v|(|W_1|+\cdots+|W_{j-1}|+|s|)} (-1)^{|v|(|v_1|+\cdots+|v_{j-1}|)} \mathfrak{n}_{lg}(W_1 \otimes \cdots \\
 &\quad \otimes W_{j-1} \otimes (sv_1 \dots v_{i-1} v v_i \dots v_{k_j}) \otimes W_{j+1} \otimes \cdots \otimes W_l) \\
 &= -(\mathfrak{n}_{lg} \circ \iota_v)(W_1 \otimes \cdots \otimes W_l),
 \end{aligned}$$

or $q > 1$ odd and $w_j = 1^{q-1}$ for some j , in which case

$$\begin{aligned}
 (**) &= - \sum_{\substack{1 \leq j \leq l \\ w_j = 1^{q-1}}} \varepsilon' \left(\sum \varepsilon(w_j \mapsto w_j^1 \underbrace{w_j^2}_{=\emptyset}) 1^{q*} (1 w_j^1) \right) \mathfrak{n}_{lg}(W_1 \otimes \cdots \\
 &\quad \otimes W_{j-1} \otimes (sv) \otimes W_{j+1} \otimes \cdots \otimes W_l) \\
 &= - \sum_{\substack{1 \leq j \leq l \\ w_j = 1^{q-1}}} \varepsilon' \left(\underbrace{\sum_{i=1}^{q-1} (-1)^i}_{=0} \right) \mathfrak{n}_{lg}(W_1 \otimes \cdots \otimes W_{j-1} \otimes (sv) \otimes W_{j+1} \otimes \\
 &\quad \cdots \otimes W_l) \\
 &= 0.
 \end{aligned}$$

3. Algebraic structures

The only relation left to check is

$$\mathfrak{q}_{210}(s_1^{q_1*}, s_1^{q_1*}) = 0 \quad \text{for all } q_1, q_2 \in \mathbb{N}.$$

However, this is easy to see, and the proof is done. □

4. Chern-Simons Maurer-Cartan element and string topology

In Section 4.1, we consider the cyclic dga's $\Omega(M)$, $H_{\text{dR}}(M)$ and $\mathcal{H}(M)$ for a closed oriented n -manifold M (Proposition 4.1.2) and apply the theory from Section 3.4 to the last two, which are finite-dimensional.

In Section 4.2, we define the admissible Hodge propagator P (Definition 4.2.4). It is a primitive to the Schwartz kernel H of the harmonic projection $\pi_{\mathcal{H}}$ (see Proposition 4.2.7) outside the diagonal and extends smoothly to the spherical blow-up of the diagonal. These ideas come from an early version of [CVona]. We consider conditions (P1)–(P5) on a linear operator \mathcal{P} and its Schwartz kernel P (see p. 83) and show that P satisfying all these conditions always exists (Proposition 4.2.10). We also mention the standard Hodge propagator P_{std} (see (4.15)), which might be a canonical Hodge propagator satisfying (P1)–(P5).

In Section 4.3, we review ribbon graphs, labelings, compatibility of the order and orientation of internal edges, and the edge and vertex order from [CFL15] (Definitions 4.3.1, 4.3.3, 4.3.4 and 4.3.5). We then define \mathbf{n} as a signed sum of integrals of products of Hodge propagators and harmonic forms which are associated to labeled trivalent ribbon graphs (Definition 4.3.6). We do not show that these integrals converge and that \mathbf{n} satisfies the Maurer-Cartan equation, but we do show all other properties of a Maurer-Cartan element (Lemma 4.3.7 and Proposition 4.3.10). We define the Y -graph, trees, circular graphs, vertices of types A , B , C and their contributions A_{α_1, α_2} , B_{α} , C , respectively (Definitions 4.3.8 and 4.3.11).

In Section 4.4, we observe that vanishing of some special vertices in the graphs implies $\mathbf{n}_{lg} = \mathbf{m}_{lg}$. For example, if all graphs, except for the Y -graph, with $\mathbf{1}$ at an external vertex vanish, which holds if P satisfies (P4) and (P5) (Proposition 4.4.2), then all higher operations $\mathbf{q}_{lg}^{\mathbf{n}}$ vanish on the chain level in dimensions $n > 3$ (Proposition 4.4.1). Next, if all graphs with an A -vertex vanish, then $\mathbf{n}_{10} = \mathbf{m}_{10}$, and hence $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110}^{\mathbf{m}}$ (Proposition 4.4.3). We show that $\mathbf{n} = \mathbf{m}$ for simply-connected geometrically formal manifolds with $n \neq 2$ (Proposition 4.4.5). Using the results of [CVona], we argue that the

4. Chern-Simons Maurer-Cartan element and string topology

chain complexes of \mathfrak{q}_{110}^m and \mathfrak{q}_{110}^n are quasi-isomorphic provided M is simply-connected and formal (Proposition 4.4.7).

In Section 4.5, we recall basic facts about the Chas-Sullivan operations m_2 and c_2 on the \mathbb{S}^1 -equivariant homology of the free loop space and formulate a version of the string topology conjecture for simply-connected manifolds (Conjecture 4.5.1).

4.1. Canonical dIBL-structures on cyclic cochains of de Rham cohomology

Let M be an oriented closed Riemannian manifold of dimension n . We consider the following graded vector spaces:

$$\begin{aligned}\Omega^*(M) & \dots \text{ smooth de Rham forms,} \\ \mathcal{H}^*(M) & \dots \text{ harmonic forms,} \\ H_{\text{dR}}^*(M) & \dots \text{ de Rham cohomology.}\end{aligned}$$

Since M is fixed, we often write just Ω , \mathcal{H} and H_{dR} . We consider the Hodge decomposition $\Omega = \mathcal{H} \oplus \text{im } d \oplus \text{im } d^*$, where d is the de Rham differential and d^* the codifferential. We call the corresponding projection

$$\pi_{\mathcal{H}} : \Omega^*(M) \longrightarrow \mathcal{H}^*(M) \tag{4.1}$$

the *harmonic projection* and the induced isomorphism $\pi_{\mathcal{H}} : H_{\text{dR}} \rightarrow \mathcal{H}$ mapping a cohomology class into its unique harmonic representative the *Hodge isomorphism*.

Notation 4.1.1 (Updated notation for bar complexes). *We use Notation 3.3.3 for $V = \Omega$, \mathcal{H} , H_{dR} and $A = n - 3$ with the following changes:*

$$\tilde{v} \sim \eta \in V, \quad v \sim \alpha \in V[1], \quad w \sim \omega \in B^{\text{cyc}}V, \quad W \sim \Omega \in B^{\text{cyc}}V[n - 3].$$

We use the formal symbols s and θ with $|s| = n - 3$ and $|\theta| = -1$, so that $\alpha = \theta\eta$ and $\Omega = s\omega$.

Proposition 4.1.2 (De Rham cyclic dga's). *Let M be an oriented closed Riemannian manifold of dimension n . The quadruple $(\Omega(M), \Pi, m_1, m_2)$ with the operations from (2.3)*

4. Chern-Simons Maurer-Cartan element and string topology

is a cyclic dga of degree $2 - n$. For the operations before the degree shift, we have

$$\begin{aligned}\tilde{m}_1(\eta_1) &= d\eta_1, \\ \tilde{m}_2(\eta_1, \eta_2) &= \eta_1 \wedge \eta_2, \\ \tilde{\Pi}(\eta_1, \eta_2) &= \int_M \eta_1 \wedge \eta_2 =: (\eta_1, \eta_2),\end{aligned}$$

where d is the de Rham differential, \wedge the wedge product and $\tilde{\Pi}$ the intersection pairing. The operations restrict to $H_{dR}(M)$ and make $(H_{dR}(M), \Pi, m_1 \equiv 0, m_2)$ into a cyclic dga. If we define $\mu_1 \equiv 0$ and

$$\mu_2(\alpha_1, \alpha_2) := \pi_{\mathcal{H}}(m_2(\alpha_1, \alpha_2)) \quad \text{for all } \alpha_1, \alpha_2 \in \mathcal{H}(M)[1], \quad (4.2)$$

then $(\mathcal{H}(M), \Pi, \mu_1, \mu_2)$ is a cyclic dga as well, and $\pi_{\mathcal{H}} : H_{dR} \rightarrow \mathcal{H}$ is an isomorphism of cyclic dga's. All three dga's Ω , H_{dR} and \mathcal{H} are strictly unital and strictly augmented with the unit $1 := \theta 1 \in \Omega[1]$, where 1 is the constant one.

Proof. The relations (3.31) follow from the classical properties of d and \wedge and from the Stokes' theorem for oriented closed manifolds. The Poincaré duality asserts that (\cdot, \cdot) is non-degenerate on H_{dR} and \mathcal{H} , and thus they are cyclic dga's as well. The fact that $\pi_{\mathcal{H}} : H_{dR} \rightarrow \mathcal{H}$ is an isomorphism of vector spaces follows from the Hodge theory. As for compatibility with the product, given $\eta_1, \eta_2 \in \mathcal{H}$, then $\eta_1 \wedge \eta_2$ is closed, and since $\ker d = \mathcal{H} \oplus \text{im } d$, we see that $\pi_{\mathcal{H}}(\eta_1 \wedge \eta_2) = \eta_1 \wedge \eta_2 + d\eta$ for some $\eta \in \Omega$ is a harmonic representative of the cohomology class $[\eta_1 \wedge \eta_2] = [\eta_1] \wedge [\eta_2]$. Unitality is obvious, and the construction of an augmentation map clear. Note that a strict augmentation for $\Omega(M)$ is the evaluation at a point, for instance. \square

The facts (A) and (C) from the Introduction apply to the cyclic dga's \mathcal{H} and H_{dR} (not to Ω because it is infinite-dimensional!), and we get the canonical dIBL-algebras $\text{dIBL}(C(\mathcal{H}))$ and $\text{dIBL}(C(H_{dR}))$ of bidegrees $(n-3, 2)$ with the canonical Maurer-Cartan element $\mathfrak{m} = (\mathfrak{m}_{10})$. The Hodge isomorphism induces an isomorphism of these dIBL-algebras, and hence we can use \mathcal{H} and H_{dR} interchangeably. We have $\mathfrak{q}_{110} \equiv 0$, and hence $\text{dIBL}(C(\mathcal{H}))$ is, in fact, an IBL-algebra. However, we will denote it by dIBL and call it a dIBL algebra as a reminder of the canonical dIBL-structure. The canonical Maurer-Cartan element \mathfrak{m} satisfies

$$\mathfrak{m}_{10}(s\alpha_1\alpha_2\alpha_3) = (-1)^{n-2+\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \alpha_1, \alpha_2, \alpha_3 \in \mathcal{H}[1]. \quad (4.3)$$

4. Chern-Simons Maurer-Cartan element and string topology

We get the canonical twisted dIBL-algebra $\text{dIBL}^{\mathfrak{m}}(C(\mathcal{H}))$ from (2.2) with, in general, non-trivial boundary operator $\mathfrak{q}_{110}^{\mathfrak{m}}$ whose homology is the cyclic homology of H_{dR} up to degree shifts.

4.2. Hodge propagator

We will use fiberwise integration and spherical blow-ups, which we now recall.

Definition 4.2.1 (Fiberwise integration). *Let $\text{pr} : E \rightarrow B$ be a smooth oriented fiber bundle with an oriented fiber F over an oriented manifold B with $\partial B = \emptyset$. We orient E as $F \times B$. Let $\Omega_c(B)$ denote the space of forms with compact support and $\Omega_{\text{cv}}(E)$ the space of forms with compact vertical support. For any $\kappa \in \Omega_{\text{cv}}(E)$, let $\int^F \kappa \in \Omega(B)$ be the unique smooth form such that*

$$\int_E \kappa \wedge \text{pr}^* \eta = \int_B \left(\int^F \kappa \right) \wedge \eta \quad \text{for all } \eta \in \Omega_c(B).$$

Definition 4.2.2 (Spherical blow-up). *Let X be a smooth n -dimensional manifold and $Y \subset X$ a smooth k -dimensional submanifold. The blow-up of X at Y is as a set defined by*

$$\text{Bl}_Y X := X \setminus Y \sqcup P^+ NY,$$

where $P^+ NY$ is the real oriented projectivization of the normal bundle NY of Y in X . This means that $P^+ NY$ is the quotient of $\{v \in NY \mid v \neq 0\}$ by the relation $v \sim av$ for all $a \in (0, \infty)$. The blow-down map is defined by

$$\begin{aligned} \pi : \text{Bl}_Y X &\longrightarrow X \\ p \in X \setminus Y &\longmapsto p, \\ [v]_p \in P^+ NY &\longmapsto p. \end{aligned}$$

In the following, we will equip the blow-up with the structure of a smooth manifold with boundary such that its interior becomes diffeomorphic to $X \setminus Y$ via the blow-down map and the boundary becomes $P^+ NY$. Consider an adapted chart (U, ψ) for Y in X

4. Chern-Simons Maurer-Cartan element and string topology

with $\psi(U) = \mathbb{R}^n$ and $\psi(U \cap Y) = \{(0, y) \mid y \in \mathbb{R}^k\}$. It induces the bijection

$$\begin{aligned}\tilde{\psi} : \text{Bl}_{U \cap Y} U &\longrightarrow [0, \infty) \times \mathbb{S}^{n-k-1} \times \mathbb{R}^k \\ p \in U \setminus Y &\longmapsto \left(|\pi_1 \psi(p)|, \frac{\pi_1 \psi(p)}{|\pi_1 \psi(p)|}, \pi_2 \psi(p) \right), \\ [v] \in P^+ N_p Y &\longmapsto \left(0, \frac{\pi_1 d\psi(v)}{|\pi_1 d\psi(v)|}, \pi_2 \psi(p) \right),\end{aligned}$$

where π_1 and π_2 are the canonical projections to the factors of $\mathbb{R}^{n-k} \times \mathbb{R}^k$. Notice that we have the canonical inclusion $\text{Bl}_{U \cap Y} U \subset \text{Bl}_Y X$. It can be checked that for any two overlapping adapted charts (U_1, ψ_1) and (U_2, ψ_2) , the transition function $\tilde{\psi}_1 \circ \tilde{\psi}_2^{-1}$ is a diffeomorphism of manifolds with boundary. Therefore, we can use the charts $(\text{Bl}_{U \cap Y} U, \tilde{\psi})$ to define a smooth atlas on $\text{Bl}_Y X$. If X is oriented, we orient $\text{Bl}_Y X$ so that π restricts to an orientation preserving diffeomorphism of the interior.

An important fact is that if X is compact, then $\text{Bl}_Y X$ is compact.

We are interested in the case when $X = M \times M$ for an oriented closed manifold M and $Y = \Delta := \{(m, m) \mid m \in M\}$ is the *diagonal*. Given a chart $\varphi : U \rightarrow \mathbb{R}^n$ on M , the following is a smooth chart on $\text{Bl}_\Delta(M \times M)$:

$$\begin{aligned}\tilde{\varphi} : \text{Bl}_\Delta(U \times U) &\longrightarrow [0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n \\ (x, y) \in (U \times U) \setminus \Delta &\longmapsto (r, w, u) := \left(\frac{1}{2} |\varphi(x) - \varphi(y)|, \frac{\varphi(x) - \varphi(y)}{|\varphi(x) - \varphi(y)|}, \right. \\ &\quad \left. \frac{1}{2} (\varphi(x) + \varphi(y)) \right), \\ [(v, -v)]_{(x, x)} &\longmapsto \left(0, \frac{d\varphi_x(v)}{|d\varphi_x(v)|}, \varphi(x) \right).\end{aligned}\tag{4.4}$$

The inverse relations for $r > 0$ read

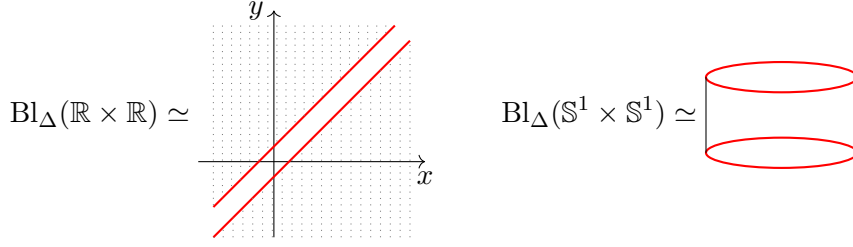
$$\varphi(x) = u + wr \quad \text{and} \quad \varphi(y) = u - wr.$$

We will denote by M_i the i -th factor of $M \times M$; i.e., we will write $M \times M = M_1 \times M_2$. We denote the corresponding projection by pr_i . We define $\widetilde{\text{pr}}_i := \text{pr}_i \circ \pi$, where $\pi : \text{Bl}_\Delta(M \times M) \rightarrow M \times M$ is the blow-down map. We also identify $(M \times M) \setminus \Delta$ with the interior of $\text{Bl}_\Delta(M \times M)$ via π .

Example 4.2.3 (Product minus open thickening of diagonal). We have the following

4. Chern-Simons Maurer-Cartan element and string topology

examples:



In fact, the manifold $\text{Bl}_\Delta(M \times M)$ is always diffeomorphic to $M \times M$ minus an open thickening of the diagonal. The blow-down map is therefore an essential part of the blow-up construction. \triangleleft

The map $\widetilde{\text{pr}}_2 : \text{Bl}_\Delta(M \times M) \rightarrow M_2$ is an oriented fiber bundle with fiber $\text{Bl}_*(M_1)$, which is the blow-up of M_1 at a point (we shall assume that M is connected). The fiberwise integration along $\widetilde{\text{pr}}_2$ will be denoted by $\int^{\text{Bl}_*M_1}$.

Definition 4.2.4 (Hodge propagator). *Let M be an oriented closed n -dimensional Riemannian manifold. Consider the harmonic projection $\pi_{\mathcal{H}}$ from (4.1), and let $\iota_{\mathcal{H}} : \mathcal{H}(M) \hookrightarrow \Omega(M)$ be the inclusion. A smooth $(n-1)$ -form P on $(M \times M) \setminus \Delta$ is called an admissible Hodge propagator if the following conditions are satisfied:*

- (1) *The form P admits a smooth extension to $\text{Bl}_\Delta(M \times M)$. More precisely, the pullback $(\pi|_{\text{int}})^*P$ along the blow-down map restricted to the interior is a restriction of a smooth form on $\text{Bl}_\Delta(M \times M)$. We denote this form by P again by uniqueness.*
- (2) *The operator $\mathcal{P} : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$ defined by*

$$\mathcal{P}(\eta) := \int^{\text{Bl}_*M_1} P \wedge \widetilde{\text{pr}}_1^* \eta \quad \text{for all } \eta \in \Omega(M) \quad (4.5)$$

satisfies

$$d \circ \mathcal{P} + \mathcal{P} \circ d = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1}. \quad (4.6)$$

Any homogenous linear operator $\mathcal{P} : \Omega^(M) \rightarrow \Omega^{*-1}(M)$ satisfying (4.6) will be called a Hodge homotopy.*

- (3) *For the twist map $\tau : M \times M \rightarrow M \times M$ defined by $(x, y) \mapsto (y, x)$, the following symmetry property holds:*

$$\tau^*P = (-1)^n P. \quad (4.7)$$

4. Chern-Simons Maurer-Cartan element and string topology

Remark 4.2.5 (On Hodge propagator). (i) Given a homogenous linear operator $\mathcal{P} : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$, if there is a $P \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M))$ such that (4.5) holds, then it is unique.

(ii) Because $\tau : M \times M \rightarrow M \times M$ preserves Δ , it extends to a diffeomorphism $\tilde{\tau}$ of $\text{Bl}_\Delta(M \times M)$. The condition (4.7) is then equivalent to $\tilde{\tau}^*\tilde{P} = (-1)^n\tilde{P}$ for the extension \tilde{P} of P to $\text{Bl}_\Delta(M \times M)$. We denote both extensions by τ and P , respectively.

(iii) Using the intersection pairing (\cdot, \cdot) , we have

$$\begin{aligned} (\mathcal{P}(\eta_1), \eta_2) &= \int_{\text{Bl}_\Delta(M \times M)} P \wedge \widetilde{\text{pr}}_1^* \eta_1 \wedge \widetilde{\text{pr}}_2^* \eta_2 \\ &= (-1)^n \int_{\text{Bl}_\Delta(M \times M)} \tau^* P \wedge \widetilde{\text{pr}}_2^* \eta_1 \wedge \widetilde{\text{pr}}_1^* \eta_2 \end{aligned}$$

and

$$\begin{aligned} (\eta_1, \mathcal{P}(\eta_2)) &= (-1)^{\eta_1(\eta_2-1)} (\mathcal{P}(\eta_2), \eta_1) \\ &= (-1)^{\eta_1} \int_{\text{Bl}_\Delta(M \times M)} P \wedge \widetilde{\text{pr}}_2^* \eta_1 \wedge \widetilde{\text{pr}}_1^* \eta_2 \end{aligned}$$

for all $\eta_1, \eta_2 \in \Omega(M)$. This implies the following:

$$\begin{aligned} \tau^* P = (-1)^n P \quad \Longleftrightarrow \quad (\mathcal{P}(\eta_1), \eta_2) &= (-1)^{\eta_1} (\eta_1, \mathcal{P}(\eta_2)) \\ &\quad \forall \eta_1, \eta_2 \in \Omega(M). \end{aligned} \tag{4.8}$$

(iv) Because $\text{Bl}_\Delta(M \times M)$ is compact, $P \in \Omega(\text{Bl}_\Delta(M \times M))$ induces an L^1 -integrable form on $M \times M$.

(v) In the original article [CFL15], Hodge propagator is called ‘‘Green kernel’’. However, this term is standardly used to denote the Schwartz kernel of the generalized inverse of an elliptic pseudo-differential operator, e.g., of the Laplacian Δ . Because this caused confusion in discussions, we decided to change it to Hodge propagator. This turned out to appear already in [CMR15]. This is how we discovered [CM08], where this problematic was also treated, and where we took the trick in the proof of Proposition 4.2.10 from. \triangleleft

We will now prove three propositions which will allow us to rewrite (4.6) equivalently as a differential equation for P on $M \times M \setminus \Delta$.

Proposition 4.2.6 (Identities for fiberwise integration). *In the situation of Definition 4.2.1, assume that F has a boundary ∂F . We orient ∂F using $T_p F = N(p) \oplus T_p \partial F$ for $p \in \partial F$, where N is an outward pointing normal vector field. The following formulas hold for all $\kappa \in \Omega_{\text{cv}}(E)$ and $\eta \in \Omega_c(B)$:*

4. Chern-Simons Maurer-Cartan element and string topology

- The projection formula

$$\int^F (\kappa \wedge \pi^* \eta) = \left(\int^F \kappa \right) \wedge \eta,$$

- Stokes' formula

$$(-1)^F d \int^F \kappa = \int^F d\kappa - \int^{\partial F} \kappa,$$

where F in the exponent denotes the dimension of F .

Proof. The projection formula is proven by a straightforward calculation from the definition.

As for Stokes' formula, we get the oriented fiber bundle $\partial E \rightarrow B$ with fiber ∂F by restricting an oriented trivialization of E . There are two orientations on ∂E — as the total space of $\partial E \rightarrow B$ and as the boundary of E . They agree due to our orientation convention. Using standard Stokes' theorem, we get

$$\begin{aligned} (-1)^F \int_B d \left(\int^F \kappa \right) \wedge \eta &= (-1)^{\kappa+1} \int_B \left(\int^F \kappa \right) \wedge d\eta \\ &= (-1)^{\kappa+1} \int_E \kappa \wedge d\pi^* \eta \\ &= \int_E (d\kappa \wedge \pi^* \eta - d(\kappa \wedge \pi^* \eta)) \\ &= \int_B \left(\int^F d\kappa \right) \wedge \eta - \int_{\partial E} \kappa \wedge \pi^* \eta \\ &= \int_B \left(\int^F d\kappa - \int^{\partial F} \kappa \right) \wedge \eta. \end{aligned}$$

This proves the proposition. □

In what comes next, we will view the canonical projection $\text{pr}_2 : M_1 \times M_2 \rightarrow M_2$ as an oriented fiber bundle such that the orientation of the total space agrees with the product orientation. The fiberwise integration for this bundle will be denoted by \int^{M_1} .

Proposition 4.2.7 (Schwartz kernel of the harmonic projection). *Let M be an oriented closed n -dimensional Riemannian manifold. Let ν_1, \dots, ν_m be a homogenous basis of $\mathcal{H}(M)$ which is orthonormal with respect to the L^2 -inner product*

$$(\eta_1, \eta_2)_{L^2} := \int_M \eta_1 \wedge * \eta_2 \quad \text{for } \eta_1, \eta_2 \in \Omega(M),$$

4. Chern-Simons Maurer-Cartan element and string topology

where $*$ denotes the Hodge star. The smooth form $H \in \Omega^n(M \times M)$ defined by

$$H := \sum_{i=1}^m (-1)^{n\nu_i} \text{pr}_1^*(\nu_i) \wedge \text{pr}_2^*(\nu_i) \quad (4.9)$$

satisfies the following properties:

(a) For all $\eta \in \Omega(M)$, we have

$$\pi_{\mathcal{H}}(\eta) = \int^{M_1} H \wedge \text{pr}_1^* \eta.$$

(b) The form H is closed and Poincaré dual to $\Delta \subset M \times M$.

(c) The following symmetry condition is satisfied:

$$\tau^* H = (-1)^n H. \quad (4.10)$$

Proof. (a) For the purpose of the proof, we denote $\mathcal{H}(\eta) := \int^{M_1} H \wedge \text{pr}_1^* \eta$. For every $k = 1, \dots, m$, we use the projection formula to compute

$$\begin{aligned} \mathcal{H}(\nu_k) &= \sum_{i=1}^m (-1)^{\nu_i n + \nu_i \nu_k} \int^{M_1} \text{pr}_1^*(\nu_i \wedge \nu_k) \wedge \text{pr}_2^*(\nu_i) \\ &= \sum_{i=1}^m (-1)^{\nu_i(n+\nu_k) + \nu_k(n+\nu_i)} \left(\int_M \nu_k \wedge \nu_i \right) \nu_i \\ &= \nu_k. \end{aligned}$$

It is easy to see that $\mathcal{H}(\eta) \in \mathcal{H}(M)$ for all $\eta \in \Omega(M)$. Therefore, \mathcal{H} is a projection to $\mathcal{H}(M)$. Relations $\mathcal{H}(d\eta) = \mathcal{H}(d^*\eta) = 0$ for all $\eta \in \Omega(M)$ follow from the second line of the computation above with ν_k replaced by $d\eta$ and $d^*\eta$ using that $\text{im } d^* \oplus \text{im } d$ is L^2 -orthogonal to $\mathcal{H}(M)$. We see that $\mathcal{H} = \pi_{\mathcal{H}}$.

(b) Using $d \circ \mathcal{H} = \mathcal{H} \circ d = 0$ and Stokes' theorem, we get

$$\int^{M_1} dH \wedge \text{pr}_1^* \eta = (-1)^n d\mathcal{H}(\eta) - \mathcal{H}(d\eta) = 0 \quad \text{for all } \eta \in \Omega(M).$$

It follows that $dH = 0$. Using the Künneth formula, we can write a given $\kappa \in \Omega(M \times M)$

4. Chern-Simons Maurer-Cartan element and string topology

with $d\kappa = 0$ as $\kappa = \text{pr}_1^* \eta_1 \wedge \text{pr}_2^* \eta_2 + d\eta$ for some $\eta_1, \eta_2 \in \mathcal{H}(M)$ and $\eta \in \Omega(M)$. Then

$$\begin{aligned} \int_{M \times M} H \wedge \kappa &= \int_{M \times M} H \wedge \text{pr}_1^* \eta_1 \wedge \text{pr}_2^* \eta_2 \\ &= \int_M \mathcal{H}(\eta_1) \wedge \eta_2 \\ &= \int_M \eta_1 \wedge \eta_2 = \int_{\Delta} \kappa. \end{aligned}$$

This shows that H is Poincaré dual to Δ .

(c) It follows from the Hodge decomposition that

$$(\pi_{\mathcal{H}}(\eta_1), \eta_2) = (\pi_{\mathcal{H}}(\eta_1), \pi_{\mathcal{H}}(\eta_2)) = (\eta_1, \pi_{\mathcal{H}}(\eta_2)) \quad \text{for all } \eta_1, \eta_2 \in \Omega(M). \quad (4.11)$$

As in (iii) of Remark 4.2.5, one shows that this is equivalent to (4.10). \square

Proposition 4.2.8 (Differential condition). *Let M be an oriented closed n -dimensional Riemannian manifold. For $P \in \Omega^{n-1}(\text{Bl}_{\Delta}(M \times M))$, the following claims are equivalent:*

(1) *The operator $\mathcal{P} : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$ defined by $\mathcal{P}(\eta) := \int^{\text{Bl}_* M_1} P \wedge \widetilde{\text{pr}}_1^* \eta$ for $\eta \in \Omega(M)$ is a Hodge homotopy.*

(2) *It holds*

$$dP = (-1)^n H \quad \text{on } (M \times M) \setminus \Delta. \quad (4.12)$$

Proof. Before we begin, note that (4.12) is equivalent to the equation $d\tilde{P} = (-1)^n \pi^* H$ on $\text{Bl}_{\Delta}(M \times M)$ for the extension \tilde{P} of P ; we denote \tilde{P} by P and $\pi^* H$ by H by uniqueness.

We will first prove $2) \implies 1)$. Using Stokes' formula, we get for every $\eta \in \Omega(M)$ the following:

$$\begin{aligned} d\mathcal{P}(\eta) &= d \int^{\text{Bl}_* M_1} P \wedge \widetilde{\text{pr}}_1^* \eta \\ &= (-1)^n \left(\int^{\text{Bl}_* M_1} d(P \wedge \widetilde{\text{pr}}_1^* \eta) - \int^{\partial \text{Bl}_* M_1} P \wedge \widetilde{\text{pr}}_1^* \eta \right) \\ &= \pi_{\mathcal{H}}(\eta) - \mathcal{P}(d\eta) + \int^{\partial \text{Bl}_* M_1} (-1)^{n+1} P \wedge \widetilde{\text{pr}}_1^* \eta. \end{aligned}$$

Since $\widetilde{\text{pr}}_1 = \widetilde{\text{pr}}_2$ on $\partial \text{Bl}_{\Delta}(M \times M)$, we get with the help of the projection formula the following:

$$\int^{\partial \text{Bl}_* M_1} P \wedge \widetilde{\text{pr}}_1^* \eta = \int^{\partial \text{Bl}_* M_1} P \wedge \widetilde{\text{pr}}_2^* \eta = \left(\int^{\partial \text{Bl}_* M_1} P \right) \wedge \eta.$$

4. Chern-Simons Maurer-Cartan element and string topology

We will show that the 0-form $\int^{\partial \text{Bl}_* M_1} \text{P}$ is constant $(-1)^n$. Stokes' formula implies

$$\int^{\partial \text{Bl}_* M_1} \text{P} = \int^{\text{Bl}_* M_1} d\text{P} = (-1)^n \int^{\text{Bl}_* M_1} \text{H}.$$

Using that H is Poincaré dual to Δ , we get for every $\eta \in \Omega^n(M)$ the following:

$$\begin{aligned} \int_M \left(\int^{\text{Bl}_* M_1} \text{H} \right) \wedge \eta &= \int_{\text{Bl}_\Delta(M \times M)} \text{H} \wedge \widetilde{\text{pr}}_2^* \eta \\ &= \int_{M \times M} H \wedge \text{pr}_2^* \eta \\ &= \int_\Delta \text{pr}_2^* \eta = \int_M 1 \wedge \eta. \end{aligned}$$

The implication follows.

We will now prove $1) \implies 2)$. Assume that (4.6) holds and that P extends smoothly to the blow-up. Denote

$$K := (-1)^n d\text{P} - \text{H} \quad \text{and} \quad L := -1 + \int^{\partial \text{Bl}_*(M_1)} (-1)^n \text{P}.$$

Notice that L is a function on M . From the previous computations, we deduce that

$$\int^{\text{Bl}_*(M_1)} K \wedge \widetilde{\text{pr}}_1^* \eta = L\eta \quad \text{for all } \eta \in \Omega(M),$$

and hence

$$\int_{\text{Bl}_\Delta(M \times M)} K \wedge \widetilde{\text{pr}}_1^*(\eta_1) \wedge \widetilde{\text{pr}}_2^*(\eta_2) = \int_M L\eta_1 \wedge \eta_2 \quad \text{for all } \eta_1, \eta_2 \in \Omega(M).$$

If $K(x, y) \neq 0$ for some $(x, y) \in (M \times M) \setminus \Delta$, we can choose η_1, η_2 with disjoint supports such that the left-hand side is non-zero. This is a contradiction. Consequently, we have $K \equiv 0$. \square

In general, the *Schwartz kernel* of a linear operator $\mathcal{P} : \Omega(M) \rightarrow \Omega(M)$ is a distributional form P on $M \times M$ which satisfies¹

$$\mathcal{P}(\eta)(x) = \int_{y \in M_1} \text{P}(y, x) \eta(y) \quad \text{for all } \eta \in \Omega(M) \text{ and } x \in M_2.$$

We consider the following conditions on \mathcal{P} and P :

¹We may consider such class of \mathcal{P} 's, e.g., pseudo-differential operators, such that P exists and is unique (c.f., the well-known Schwartz kernel theorem).

4. Chern-Simons Maurer-Cartan element and string topology

- (G1) The Schwartz kernel P of \mathcal{P} is a restriction of a smooth form on $\text{Bl}_\Delta(M \times M)$.
- (G2) $d \circ \mathcal{P} + \mathcal{P} \circ d = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1}$.
- (G3) $(\mathcal{P}(\eta_1), \eta_2) = (-1)^{\eta_1}(\eta_1, \mathcal{P}(\eta_2))$ for all $\eta_1, \eta_2 \in \Omega(M)$.
- (G4) $\mathcal{P} \circ \iota_{\mathcal{H}} = 0, \pi_{\mathcal{H}} \circ \mathcal{P} = 0$.
- (G5) $\mathcal{P} \circ \mathcal{P} = 0$.

Clearly, (P1)–(P3) are equivalent to P being a Hodge propagator from Definition 4.2.4. Conditions (P4) and (P5) play a crucial role in the vanishing results for the Chern-Simons Maurer-Cartan element \mathfrak{n} in Section 4.4 — the more conditions are satisfied, the more vanishing we get.

The following lemma will be used in the proof of the upcoming proposition.

Lemma 4.2.9. *Let $\mathcal{P}_1, \mathcal{P}_2$ be two linear operators $\Omega(M) \rightarrow \Omega(M)$ with Schwartz kernels $P_1, P_2 \in \Omega(\text{Bl}_\Delta(M \times M))$. Then $\mathcal{P} := \mathcal{P}_1 \circ \mathcal{P}_2$ is a smoothing operator, i.e., its Schwartz kernel P is a smooth form on $M \times M$.*

Proof. It holds $P(x_1, x_2) = \pm \int_x P_2(x_1, x) P_1(x, x_2)$. The lemma follows from properties of convolution. \square

A version of the following proposition can be found in [CM08].

Proposition 4.2.10 (Existence of special Hodge propagator). *Every oriented closed Riemannian manifold M admits a special Hodge propagator.*

Proof of Proposition 4.2.10. Because H is Poincaré dual to Δ , we have for any closed $\kappa \in \Omega_c((M \times M) \setminus \Delta)$ the following:

$$\int_{(M \times M) \setminus \Delta} H \wedge \kappa = \int_{M \times M} H \wedge \kappa = \int_{\Delta} \kappa = 0.$$

Poincaré duality for non-compact oriented manifolds (see [BT82]) implies that H is exact on $(M \times M) \setminus \Delta$. Because a manifold with boundary is homotopy equivalent to its interior, the restriction of the blow-down map induces an isomorphism $\pi^* : H_{\text{dR}}((M \times M) \setminus \Delta) \rightarrow H_{\text{dR}}(\text{Bl}_\Delta(M \times M))$. It follows that $(-1)^n \pi^* H$ admits a primitive $P \in \Omega(\text{Bl}_\Delta(M \times M))$. According to Proposition 4.2.8, the corresponding \mathcal{P} satisfies (P1) and (P2).

If we define

$$\tilde{P} := \frac{1}{2}(P + (-1)^n \tau^* P) \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M)),$$

4. Chern-Simons Maurer-Cartan element and string topology

then \tilde{P} satisfies $\tau^* \tilde{P} = (-1)^n \tilde{P}$ and is still a primitive to $(-1)^n \pi^* H$. Proposition 4.2.8 and (4.8) imply that the corresponding \mathcal{P} satisfies (P1)–(P3).

Given \mathcal{P} satisfying (P1)–(P3), we will now show that we can arrange (P4). Let us define

$$\tilde{\mathcal{P}} := (\mathbb{1} - \pi_{\mathcal{H}}) \circ \mathcal{P} \circ (\mathbb{1} - \pi_{\mathcal{H}}),$$

where we view $\pi_{\mathcal{H}}$ as a map $\pi_{\mathcal{H}} : \Omega \rightarrow \mathcal{H} \subset \Omega$. Then $\tilde{\mathcal{P}}$ is a Hodge homotopy because

$$d \circ \tilde{\mathcal{P}} + \tilde{\mathcal{P}} \circ d = (\mathbb{1} - \pi_{\mathcal{H}}) \circ (d \circ \mathcal{P} + \mathcal{P} \circ d) \circ (\mathbb{1} - \pi_{\mathcal{H}}) = \mathbb{1} - \pi_{\mathcal{H}}.$$

Using (4.11) and (4.8), we see that $\tilde{\mathcal{P}}$ satisfies (P3). Using the intersection pairing and Proposition 4.2.7, we can write

$$\pi_{\mathcal{H}}(\eta) = \sum_{i=1}^m (-1)^{(n+\eta)\nu_i} (*\nu_i, \eta) \nu_i \quad \text{for all } \eta \in \Omega(M),$$

and hence we have for all $\eta_1, \eta_2 \in \Omega(M)$ the following:

$$\begin{aligned} (\mathcal{P}(\pi_{\mathcal{H}}(\eta_1)), \eta_2) &= \sum_{i=1}^m (-1)^{(n+\eta_1)\nu_i} (*\nu_i, \eta_1) (\mathcal{P}(\nu_i), \eta_2) \\ &= \sum_{i=1}^m (-1)^{(n+1)\nu_i} \int_{M \times M} \text{pr}_1^(*\nu_i) \wedge \text{pr}_2^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_1^*(\eta_1) \wedge \text{pr}_2^*(\eta_2). \end{aligned}$$

It follows that the Schwartz kernel of $\mathcal{P} \circ \pi_{\mathcal{H}}$ is the smooth form

$$\mathcal{K}_{\mathcal{P} \circ \pi_{\mathcal{H}}} := \sum_{i=1}^m (-1)^{(n+1)\nu_i} \text{pr}_1^(*\nu_i) \wedge \text{pr}_2^*(\mathcal{P}(\nu_i)).$$

Moreover, if we replace \mathcal{P} with $\pi_{\mathcal{H}} \circ \mathcal{P}$, we get the smooth Schwartz kernel $\mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}}$ of $(\pi_{\mathcal{H}} \circ \mathcal{P}) \circ \pi_{\mathcal{H}}$. In the same way, but now using in addition (4.8), we can write

$$\begin{aligned} (\pi_{\mathcal{H}}(\mathcal{P}(\eta_1)), \eta_2) &= (-1)^{\eta_1} (\eta_1, \mathcal{P}(\pi_{\mathcal{H}}(\eta_2))) = (-1)^{\eta_1 \eta_2} (\mathcal{P}(\pi_{\mathcal{H}}(\eta_2)), \eta_1) \\ &= \sum_{i=1}^m (-1)^{\eta_1 \eta_2 + (n+1)\nu_i} \int_{M \times M} \text{pr}_1^(*\nu_i) \wedge \text{pr}_2^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_1^*(\eta_2) \wedge \text{pr}_2^*(\eta_1) \\ &= \sum_{i=1}^m (-1)^{(n+1)\nu_i + n} \int_{M \times M} \text{pr}_2^(*\nu_i) \wedge \text{pr}_1^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_1^*(\eta_1) \wedge \text{pr}_2^*(\eta_2), \end{aligned}$$

where in the last equality we pulled back the integral along the twist map. It follows

4. Chern-Simons Maurer-Cartan element and string topology

that the Schwartz kernel of $\pi_{\mathcal{H}} \circ \mathcal{P}$ is the smooth form

$$\mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P}} := \sum_{i=1}^m (-1)^{\nu_i} \text{pr}_1^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_2^*(\nu_i).$$

The Schwartz kernel of $\tilde{\mathcal{P}} = \mathcal{P} - \pi_{\mathcal{H}} \circ \mathcal{P} - \mathcal{P} \circ \pi_{\mathcal{H}} + \pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}$ is then

$$\tilde{\mathcal{P}} = \mathcal{P} - \pi^* \mathcal{K}_{\mathcal{P} \circ \pi_{\mathcal{H}}} - \pi^* \mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P}} + \pi^* \mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}},$$

which is a smooth form on $\text{Bl}_{\Delta}(M \times M)$. Therefore, $\tilde{\mathcal{P}}$ satisfies (P1)–(P4).

Given \mathcal{P} satisfying (P1)–(P4), we will show that we can arrange (P5). The trick from [CM08] is to define

$$\tilde{\mathcal{P}} = \mathcal{P} d\mathcal{P}.$$

Applying (P1) and (P2) repeatedly, we compute

$$\begin{aligned} d\mathcal{P}\mathcal{P}\mathcal{P}d &= d\mathcal{P}\mathcal{P} - d\mathcal{P}\mathcal{P}d\mathcal{P} = d\mathcal{P}\mathcal{P} - d\mathcal{P}\mathcal{P} + d\mathcal{P}d\mathcal{P}\mathcal{P} \\ &= d\mathcal{P}\mathcal{P} - dd\mathcal{P}\mathcal{P}\mathcal{P} = d\mathcal{P}\mathcal{P} = \mathcal{P} - \mathcal{P}d\mathcal{P}, \end{aligned} \tag{4.13}$$

and hence

$$\tilde{\mathcal{P}} = \mathcal{P} - d\mathcal{P}\mathcal{P}\mathcal{P}d.$$

Clearly, $\tilde{\mathcal{P}}$ satisfies (P1) and (P2). As for (P3), we compute

$$(\eta_1, \tilde{\mathcal{P}}\eta_2) = (-1)^{\eta_1} (\mathcal{P}\eta_1, d\mathcal{P}\eta_2) = (d\mathcal{P}\eta_1, \mathcal{P}\eta_2) = (-1)^{\eta_1} (\tilde{\mathcal{P}}\eta_1, \eta_2).$$

As for (P5), we have

$$\begin{aligned} \tilde{\mathcal{P}}\tilde{\mathcal{P}} &= \mathcal{P}d\mathcal{P}(\mathcal{P}d)\mathcal{P} = \mathcal{P}d\mathcal{P}\mathcal{P} - \mathcal{P}d(\mathcal{P}d)\mathcal{P}\mathcal{P} \\ &= \mathcal{P}d\mathcal{P}\mathcal{P} - \mathcal{P}d\mathcal{P}\mathcal{P} + \mathcal{P}dd\mathcal{P}\mathcal{P} = 0. \end{aligned} \tag{4.14}$$

In order to show (P4), we have to compute the Schwartz kernel of $d\mathcal{P}\mathcal{P}\mathcal{P}d$. By Lemma 4.2.9, the Schwartz kernel T of $T := \mathcal{P}\mathcal{P}\mathcal{P}$ is a smooth form on $M \times M$. Therefore, Stokes' formula without the boundary term applies, and we get

$$(dTd)\eta = d \int^{M_1} T \wedge d\pi_1^*(\eta) = \int^{M_1} dT \wedge d\pi_1^*(\eta) = (-1)^T \int^{M_1} d_1 dT \wedge \pi_1^*(\eta).$$

4. Chern-Simons Maurer-Cartan element and string topology

Here $d_1 : \Omega(M \times M) \rightarrow \Omega(M \times M)$ is the operator defined in local coordinates by

$$d_1(f(x, y) dx^I dy^J) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x, y) dx^i dx^I dy^J.$$

It follows that the Schwartz kernel \tilde{P} of $\tilde{\mathcal{P}}$ satisfies

$$\tilde{P} = P + (-1)^n d_1 dT$$

and is a smooth $(n-1)$ -form on $\text{Bl}_\Delta(M \times M)$. Conditions (P1)–(P5) are satisfied. \square

Remark 4.2.11 (Property (P5) in dimensions 1 and 2). In dimension 1, every operator of degree -1 satisfies (P5) from degree reasons. In dimension 2, every operator satisfying (P1) and (P2) satisfies (P5) as well, which follows from (4.13) and (4.14). \triangleleft

Remark 4.2.12 (The standard Hodge propagator). Consider the Hodge-de Rham Laplacian $\Delta = d \circ d^* + d^* \circ d : \Omega(M) \rightarrow \Omega(M)$ and its “Green operator” \mathcal{G} of degree 0 (see (v) of Remark 4.2.5 for the collision of terminology) which was defined in [War83, Definition 6.9] by

$$\mathcal{G} := (\Delta|_{\mathcal{H}(M)^\perp})^{-1} \circ \pi_{\mathcal{H}(M)^\perp},$$

where \perp denotes the L^2 -orthogonal complement. We introduce the *standard Hodge homotopy* by

$$\mathcal{P}_{\text{std}} := -d^* \circ \mathcal{P}_\Delta. \quad (4.15)$$

Using the properties of \mathcal{G} , d and d^* , one can show that \mathcal{P}_{std} satisfies (P2)–(P5).

As for (P1), the author was able to show it for flat manifolds ($:=$ locally isometric to \mathbb{R}^n) by transforming the following formula inspired by [Har04] to blow-up coordinates and explicitly computing the integral and limit:

$$P_{\text{std}} = -\lim_{t \rightarrow 0} \int_t^\infty \frac{1}{2} d^* K_\tau d\tau,$$

where $K_t(x, y) = \sum_i (-1)^{ne_i} e^{-\lambda_i t} (*e_i)(x) \wedge e_i(y)$ is the heat kernel of Δ and e_i the L^2 -orthonormal eigenbasis of Δ with eigenvalues λ_i (the signs come from our convention for fiberwise integration, c.f., (4.9)). \triangleleft

4.3. Chern-Simons Maurer-Cartan element

We first recall ribbon graphs and their labelings based on [CFL15].

4. Chern-Simons Maurer-Cartan element and string topology

Definition 4.3.1 (Ribbon graph). A graph Γ is a quadruple $(V, H, \mathcal{V}, \mathcal{E})$, where V is a finite set of vertices, H a finite set of half-edges, $\mathcal{V} : H \rightarrow V$ the “vertex map” and $\mathcal{E} : H \rightarrow H$ with $\mathcal{E} \circ \mathcal{E} = \text{id}$ and without fixed points the “edge map”. The preimage $\mathcal{E}^{-1}(h_1) = \{h_1, h_2\}$ for some $h_1, h_2 \in H$ is called an edge; the set of edges is denoted by E . We assume that the graphs are connected, i.e., that for any $v_1, v_2 \in V$ there exists a path in E connecting v_1 to v_2 .

A ribbon graph is a graph Γ together with a free transitive action $\mathbb{Z}_{d(v)} \curvearrowright \mathcal{V}^{-1}(v)$ for every $v \in V$, where

$$d(v) := |\mathcal{V}^{-1}(v)|$$

is the valency of v . We denote by $\mathcal{N} : H \rightarrow H$ the bijection induced by $1 \in \mathbb{Z}_{d(v)}$ for every $v \in V$.

For a ribbon graph Γ , consider the set of sequences $(h_n)_{n \in \mathbb{Z}} \subset H$ such that the following conditions holds:

$$\forall n \in \mathbb{Z} : \quad h_{n+1} = \begin{cases} \mathcal{E}(h_n) & n \text{ even,} \\ \mathcal{N}(h_n) & n \text{ odd.} \end{cases}$$

Two such sequences $(h_n)_{n \in \mathbb{Z}}$ and $(h'_n)_{n \in \mathbb{Z}}$ are equivalent if and only if there exist $n_0, n'_0 \in \mathbb{Z}$ both even or both odd such that $h_{n_0} = h'_{n'_0}$. An equivalence class $[(h_n)_{n \in \mathbb{Z}}]$ is called a boundary (or a boundary component) of Γ . The set of boundaries of Γ is denoted by $\partial\Gamma$.

An IE ribbon graph is a ribbon graph Γ together with the decomposition $V = V_{\text{int}} \sqcup V_{\text{ext}}$ into internal and external vertices V_{int} and V_{ext} , respectively, such that $d(v) = 1$ for all $v \in V_{\text{ext}}$. This decomposition induces the decomposition $E = E_{\text{int}} \sqcup E_{\text{ext}}$, where an edge e is internal if it connects two internal vertices and is external otherwise. We allow only graphs with at least one internal vertex. We often identify an external vertex with its unique adjacent half-edge or the unique adjacent external edge; we call either of these an external leg. For any $b \in \partial\Gamma$, we define the valency of b by

$$s(b) := |\mathcal{V}(b) \cap V_{\text{ext}}|,$$

where $\mathcal{V}(b) = \{\mathcal{V}(h_n) \mid n \in \mathbb{Z}\}$. We also have the free transitive $\mathbb{Z}_{s(b)}$ -action on $\mathcal{V}(b) \cap V_{\text{ext}}$ mapping $v \in \mathcal{V}(b) \cap V_{\text{ext}}$ to the next external vertex in the sequence $(\mathcal{V}(h_n))_{n \in \mathbb{Z}}$. We will denote this action by \mathcal{N} again.

We say that an IE ribbon graph Γ is reduced if $s(b) \geq 1$ for all $b \in \partial\Gamma$.

4. Chern-Simons Maurer-Cartan element and string topology

The following letters will be used to denote the numerical invariants of a graph:

k	\dots	the number of internal vertices,
s	\dots	$\text{---}^{\#}\text{---}$ external vertices.
l	\dots	$\text{---}^{\#}\text{---}$ boundary components,
e	\dots	$\text{---}^{\#}\text{---}$ internal edges.

Moreover, we define the genus $g \in \mathbb{N}_0$ so that the following Euler formula holds:

$$k - e + l = 2 - 2g. \quad (4.16)$$

We denote by RG_{klg} the set of isomorphism classes of connected IE ribbon graphs with fixed k, l, g . We let $\overline{\text{RG}}_{klg} \subset \text{RG}_{klg}$ be the subset of reduced graphs. For $m \in \mathbb{N}_0$, we denote by $\text{RG}_{klg}^{(m)} \subset \text{RG}_{klg}$ the set of isomorphism classes of connected IE ribbon graphs with all internal vertices m -valent, i.e., with

$$d(v) = m \quad \text{for all } v \in V_{\text{int}}.$$

The notation $\Gamma \in \text{RG}_{klg}$ means that Γ is a representative of an equivalence class $[\Gamma] \in \text{RG}_{klg}$.

Remark 4.3.2 (On ribbon graphs). (i) An m -valent ribbon graph with $m \geq 2$ has a unique decomposition $V = V_{\text{int}} \sqcup V_{\text{ext}}$, and hence we can omit writing IE.

(ii) In this text, we will use only reduced ribbon graphs. Non-reduced ribbon graphs may play a role in the extension of the theory of $\text{dIBL}^n(C(\mathcal{H}))$ to non-reduced cyclic cochains or in the weak IBL_{∞} -theory (see Remarks 3.2.10 and 3.3.2). \triangleleft

Definition 4.3.3 (Labeling). A labeling of an IE ribbon graph Γ is the triple $L = (L_1, L_2, L_3)$, where L_i have the following meanings:

- The symbol L_1 represents an ordering of internal vertices ($=: L_1^v$), and of boundary components ($=: L_1^b$). Given L_1 , we write $V_{\text{ext}} = \{v_1, \dots, v_k\}$, $\partial\Gamma = \{b_1, \dots, b_l\}$ and denote

$$d_i := d(v_i) \quad \text{and} \quad s_j := s(b_j).$$

- The symbol L_2 represents an ordering and orientation of internal edges. Given L_2 , we write $E_{\text{int}} = \{e_1, \dots, e_e\}$ and $e_i = \{h_{i,1}, h_{i,2}\}$ for $h_{i,1}, h_{i,2} \in H$.
- The symbol L_3 represents an ordering of half-edges at every internal vertex ($=: L_3^v$) and of external vertices at every boundary component ($=: L_3^b$), both compatible

4. Chern-Simons Maurer-Cartan element and string topology

with the ribbon structure ($\text{:= the } \mathbb{Z}_m\text{-actions}$). Given L_3 , we write $\mathcal{V}^{-1}(\mathbf{v}) = \{h_{\mathbf{v},1}, \dots, h_{\mathbf{v},d(\mathbf{v})}\}$ and $\mathcal{V}(\mathbf{b}) \cap V_{\text{ext}} = \{v_{\mathbf{b},1}, \dots, v_{\mathbf{b},s(\mathbf{b})}\}$ with $\mathcal{N}(h_{\mathbf{v},i}) = h_{\mathbf{v},i+1}$ and $\mathcal{N}(v_{\mathbf{b},j}) = v_{\mathbf{b},j+1}$ for all i, j , respectively.

We sometimes call L_i partial labelings and L a full labeling. A ribbon graph Γ together with a labeling L is called a labeled ribbon graph.

Given a ribbon graph Γ , one can construct an oriented surface with boundary Σ_Γ — the thickening of Γ — in the obvious way and a closed oriented surface $\hat{\Sigma}_\Gamma$ by gluing oriented disks to the oriented boundaries of Σ_Γ . If partial labelings L_1 and L_2 are given, we obtain the following chain complex with oriented chain groups (vector spaces over \mathbb{R}):

$$C_2 := \langle b_1, \dots, b_l \rangle \xrightarrow{\partial_2} C_1 := \langle e_1, \dots, e_e \rangle \xrightarrow{\partial_1} C_0 := \langle v_1, \dots, v_k \rangle. \quad (4.17)$$

Here b_i stands for the oriented disc glued to the i -th boundary component of Σ_Γ and now being mapped into $\hat{\Sigma}_\Gamma$, e_i stands for the 1-simplex in $\hat{\Sigma}_\Gamma$ corresponding to the i -th internal edge, v_i stands for the 0-simplex in $\hat{\Sigma}_\Gamma$ corresponding to the i -th internal vertex, and the boundary map ∂ is the “geometric” boundary operator. The homology of this chain complex is isomorphic to the singular homology $H(\hat{\Sigma}) := H(\hat{\Sigma}_\Gamma; \mathbb{R})$.

The orientation of C_i ($\text{:= the order of generators in (4.17)}$) induces naturally an orientation of $H(\hat{\Sigma}_\Gamma)$. The construction from [CFL15, Appendix A] is as follows. We pick complements H_i of $\text{im}(\partial_{i+1})$ in $\ker(\partial_i)$ and complements V_i of $\ker(\partial_i)$ in C_i and write

$$C_2 = V_2 \oplus H_2 \xrightarrow{\partial_2} C_1 = V_1 \oplus H_1 \oplus \text{im}(\partial_2) \xrightarrow{\partial_1} C_0 = \text{im}(\partial_1) \oplus H_0.$$

We orient V_i arbitrarily and transfer the orientation to $\text{im}(\partial_i)$ via $\partial_i : V_i \xrightarrow{\sim} \text{im}(\partial_i)$. Then, assuming the direct sum orientation, orienting H_i is equivalent to orienting C_i , and we obtain the orientation of $H_i(\hat{\Sigma}_\Gamma)$ via the canonical projection $\pi : H_i \xrightarrow{\sim} H_i(\hat{\Sigma}_\Gamma) = \ker(\partial_i) / \text{im}(\partial_{i+1})$. This construction does not depend on the choices of complements and orientations of V_i .

Definition 4.3.4 (Compatibility of L_1 and L_2). *Given a ribbon graph Γ with partial labelings L_1 and L_2 , we say that L_2 is compatible with L_1 if the orientation on $H(\hat{\Sigma}_\Gamma)$ induced by (4.17) agrees with the canonical orientation*

$$H(\hat{\Sigma}_\Gamma) = \langle v_1 + \dots + v_k \rangle \oplus H_1(\hat{\Sigma}_\Gamma) \oplus \langle b_1 + \dots + b_l \rangle,$$

where $H_1(\hat{\Sigma}_\Gamma)$ is oriented using the canonical symplectic intersection form.

4. Chern-Simons Maurer-Cartan element and string topology

Given a labeled IE ribbon graph Γ , the set of half-edges adjacent to internal vertices $\mathcal{V}^{-1}(V_{\text{int}})$ can be ordered in two ways corresponding to writing

$$2e + (s_1 + \cdots + s_l) = d_1 + \cdots + d_k.$$

This leads to the following definition.

Definition 4.3.5 (Edge order and vertex order). *For a labeled IE ribbon graph Γ , we define the following two orders on the set of half-edges H :*

- Edge order: *The first $2e$ half-edges $h_{i,j}^e$ are the ones from internal edges; they are ordered according to L_2 . They are followed by blocks of s_1, \dots, s_l half-edges $h_{i,j}^b$ which come from the boundary components $i = 1, \dots, l$, respectively, and which are ordered according to L_3^b inside the blocks. Schematically, we have*

$$(h_{1,1}^e h_{1,2}^e) \cdots (h_{e,1}^e h_{e,2}^e) (h_{1,1}^b \cdots h_{1,s_1}^b) \cdots (h_{l,1}^b \cdots h_{l,s_l}^b).$$

- Vertex order: *It consists of blocks of d_1, \dots, d_k half-edges $h_{i,j}^v$ which come from internal vertices $1, \dots, k$, and which are ordered according to L_3^v inside the blocks. Schematically, we have*

$$(h_{1,1}^v \cdots h_{1,d_1}^v) \cdots (h_{k,1}^v \cdots h_{k,d_k}^v).$$

We denote by $\sigma_L \in \mathbb{S}_{|H|}$ the permutation from the edge to the vertex order which is constructed such that the i -th half-edge in the edge order is the same as the $\sigma_L(i)$ -th half-edge in the vertex order.

From now on, we will consider only reduced trivalent ribbon graphs $\overline{\text{RG}}_{klg}^{(3)}$ with $k, l \geq 1, g \geq 0$. We will often use the equation

$$2e + s = 3k. \tag{4.18}$$

Definition 4.3.6 (Chern-Simons Maurer-Cartan element). *Let M be an oriented closed Riemannian manifold, and let $P \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M))$ be an admissible Hodge propagator from Definition 4.2.4. The Chern-Simons Maurer-Cartan element \mathfrak{n} , or formal pushforward Maurer-Cartan element, is the collection of*

$$\mathfrak{n}_{lg} \in \hat{E}_l C(\mathcal{H}(M)) \quad \text{for all } l \geq 1, g \geq 0$$

4. Chern-Simons Maurer-Cartan element and string topology

defined on generating words $\omega_i = \alpha_{i1} \dots \alpha_{is_i} \in \text{B}^{\text{cyc}}\mathcal{H}(M)$, where $\alpha_{ij} = \theta\eta_{ij}$ with $\eta_{ij} \in \mathcal{H}(M)$ for $s_i \geq 1$ and $i = 1, \dots, l$, by the formula

$$\begin{aligned} \mathbf{n}_{lg}(\mathbf{s}^l \omega_1 \otimes \dots \otimes \omega_l) \\ := \frac{1}{l!} \sum_{[\Gamma] \in \overline{\text{RG}}_{klg}^{(3)}} \frac{1}{|\text{Aut}(\Gamma)|} (-1)^{s(k,l)+P(\omega)} \sum_{L_1, L_3^b} (-1)^{\sigma_L} I(\sigma_L), \end{aligned} \quad (4.19)$$

which we explain as follows:

- The second sum is over all partial labelings L_1 and L_3^b of a representative Γ of $[\Gamma]$. In every summand, we complete L_1 and L_3^b to a full labeling $L = (L_1, L_2, L_3)$ by picking an arbitrary L_3^v and an arbitrary L_2 compatible with L_1 .
- Suppose that Γ and L_1 are admissible with respect to the input $\omega_1, \dots, \omega_l$; this means that Γ has l boundary components and that the i -th boundary component has valency s_i for every $i = 1, \dots, l$. In this case, denoting $\sigma = \sigma_L$, we define

$$\begin{aligned} I(\sigma_L) := \int_{x_1, \dots, x_k} P(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \dots P(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) \\ \eta_{11}(x_{\xi(\sigma_{2e+1})}) \dots \eta_{ls_l}(x_{\xi(\sigma_{2e+s})}), \end{aligned} \quad (4.20)$$

where $\xi : \{1, \dots, 3k\} \rightarrow \{1, \dots, k\}$ is the function defined by

$$\xi(3j-2) = \xi(3j-1) = \xi(3j) := j$$

for all $j = 1, \dots, k$, $s = s_1 + \dots + s_l$, $\eta(x_i)$ denotes the pullback of η along the canonical projection $\pi_i : M^{\times k} \rightarrow M$ to the i -th component M_i , $P(x_i, x_j)$ denotes the pullback of P along $\pi_i \times \pi_j : M^{\times k} \rightarrow M_i \times M_j$, and \int_{x_1, \dots, x_k} denotes the integral of an nk -form over k copies of M .

If Γ and L_1 are not admissible, then we set $I(\sigma_L) := 0$.

- $s(k, l) := k + kl(n-1) + \frac{1}{2}k(k-1)n \pmod{2}$.
- $P(\omega) := \sum_{i=1}^l \sum_{j=1}^{s_i} (s - s_1 - \dots - s_{i-1} - j) \eta_{ij} \pmod{2}$.

In order to show that \mathbf{n}_{lg} is well-defined and that the collection (\mathbf{n}_{lg}) satisfies Definition 3.2.6 for $\text{dIBL}(C(\mathcal{H}(M)))$, there are several things to check:

- (1) The integral $I(\sigma_L)$ converges.
- (2) The sums are finite.

4. Chern-Simons Maurer-Cartan element and string topology

- (3) The product $(-1)^{\sigma_L} I(\sigma_L)$ is independent of the choice of L_3^v and L_2 compatible with L_1 .
- (4) The sum over labelings is independent of the chosen representative Γ in an isomorphism class from $\overline{\text{RG}}_{klg}^{(3)}$.
- (5) The map $\mathbf{n}_{lg} : \text{B}^{\text{cyc}}\mathcal{H}(M)[3-n]^{\otimes l} \rightarrow \mathbb{R}$ is graded symmetric on permutations of its inputs $s\omega_i$.
- (6) The map \mathbf{n}_{lg} is graded symmetric on cyclic permutations of the components α_{ij} of each ω_i .
- (7) The degree condition 1) from Definition 3.2.6 holds with $d = n - 3$.
- (8) The filtration-degree condition 2) from Definition 3.2.6 holds with $\gamma = 2$.
- (9) The Maurer-Cartan equation (3.20) holds.

Conditions 1) and 9) will be proven in [CVona] using the theory of iterated blow-ups. In this text, we will take 1) and 9) for granted.

Lemma 4.3.7. *Assuming 1), the conditions 2) – 8) hold.*

Proof. As for 2), the fixed input $\omega_1, \dots, \omega_l$ fixes the number s of external vertices of Γ by admissibility. Expressing e from (4.16) and plugging it in (4.18) gives

$$k = s + 2l + 4g - 4. \quad (4.21)$$

We see that all parameters are fixed. Now, there is only finitely many elements with fixed s in $\overline{\text{RG}}_{klg}^{(3)}$, and each of them has only finitely many labelings. Therefore, the sums are finite.

As for 3), we have to consider the orientation of the complex (4.17). Clearly, if two L_2 's are compatible with L_1 , then they differ by an even number of the following operations: a transposition of two internal edges or a change of the orientation of an internal edge. The former operation introduces no sign in $(-1)^{\sigma_L}$ but generates the sign $(-1)^{n-1}$ in $I(\sigma_L)$ from swapping the corresponding P's. The latter operation induces the sign -1 in $(-1)^{\sigma_L}$ and the sign $(-1)^n$ in $I(\sigma_L)$ from the symmetry $P(x, y) = (-1)^n P(y, x)$. Because the overall signs in $(-1)^{\sigma_L} I(\sigma_L)$ are the same, an even number of these operations preserves $(-1)^{\sigma_L} I(\sigma_L)$. This implies the independence of an L_2 compatible with L_1 . A change in L_3^v produces no sign in $(-1)^{\sigma_L}$ because every internal vertex is trivalent and a cyclic permutation of an odd number of elements is even. The integral $I(\sigma_L)$

4. Chern-Simons Maurer-Cartan element and string topology

remains unchanged because the change in σ_L is compensated by the composition with ξ . Independence of the choice of L_3^v follows.

As for 4), every isomorphism of ribbon graphs $\Gamma \rightarrow \Gamma'$ induces the bijection $L \mapsto L'$ of compatible labelings such that $\sigma_L = \sigma_{L'}$ (L' is the “pushforward” labeling). The independence of the choice of a representative of $[\Gamma]$ follows.

As for 5), let $\mu \in \mathbb{S}_l$ be a permutation of the inputs $s\omega_1, \dots, s\omega_l$. The set of graphs which admit an admissible labeling is the same for both $\mathbf{n}_{lg}(s^l\omega_1 \otimes \dots \otimes \omega_l)$ and $\mathbf{n}_{lg}(s^l\omega_{\sigma_1^{-1}} \otimes \dots \otimes \omega_{\sigma_l^{-1}})$; we will pick one such Γ and study the admissible labelings L and L' , respectively. We write $\eta_i = \eta_{i1} \dots \eta_{is_i}$ and $\Omega_i = s\omega_i$ for all i, j , and denote by $I'(\sigma_{L'})$ the integral in the definition of $\mathbf{n}_{lg}(s^l\omega_{\mu_1^{-1}} \otimes \dots \otimes \omega_{\mu_l^{-1}})$. Let $\tilde{\mu} \in \mathbb{S}_{3k}$ be the permutation which acts as the identity on $1, \dots, 2e$ and as the block permutation determined by μ on $2e+1, \dots, 2e+s$ divided into l blocks of lengths s_1, \dots, s_l . For any $\sigma \in \mathbb{S}_{3k}$, we have

$$\begin{aligned} I'(\sigma) &= \int_{x_1, \dots, x_k} P(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \dots P(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) \\ &\quad \eta_{\mu_1^{-1}1}(x_{\xi(\sigma_{2e+1})}) \dots \eta_{\mu_l^{-1}s_{\mu_l^{-1}}}(x_{\xi(\sigma_{2e+s})}) \\ &= \varepsilon(\mu, \eta) \int_{x_1, \dots, x_k} P(x_{\xi((\sigma \circ \tilde{\mu})_1)}, x_{\xi((\sigma \circ \tilde{\mu})_2)}) \dots P(x_{\xi((\sigma \circ \tilde{\mu})_{2e-1})}, x_{\xi((\sigma \circ \tilde{\mu})_{2e})}) \\ &\quad \eta_{11}(x_{\xi((\sigma \circ \tilde{\mu})_{2e+1})}) \dots \eta_{ls_l}(x_{\xi((\sigma \circ \tilde{\mu})_{2e+s})}) \\ &= \varepsilon(\mu, \eta) I(\sigma \circ \tilde{\mu}). \end{aligned}$$

The precomposition with $\tilde{\mu}$ corresponds to a bijection $(L_1, L_3^b) \mapsto (L'_1, L_3^{b'})$ of partial labelings for $\mathbf{n}_{lg}(s^l\omega_1 \dots \omega_l)$ and $\mathbf{n}_{lg}(s^l\omega_{\mu_1^{-1}} \otimes \dots \otimes \omega_{\mu_l^{-1}})$, respectively. However, if L_2 is compatible with L_1 , then in order to get an L'_2 compatible with L'_1 , the labeling L_2 has to be altered by as many operations of switching two internal edges or changing the orientation of an internal edge as there are transpositions in μ . We explained in the proof of 3) that this produces the sign $(-1)^{(n-1)\mu}$ in $(-1)^{\sigma_{L'}} I(\sigma_{L'})$. Therefore, after the choice of compatible L_2 and L'_2 , we have

$$(-1)^{\sigma_{L'}} I'(\sigma_{L'}) = (-1)^{(n-1)\mu} (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta) (-1)^{\sigma_L} I(\sigma_L).$$

If we view η as $\eta_{11} \dots \eta_{ls_l}$, we can understand $(-1)^{P(\omega)}$ as the Koszul sign $\varepsilon(\theta, \eta)$. Similarly, we write $(-1)^{P(\mu(\omega))} = \varepsilon(\theta, \mu(\eta))$, where we first view η as $\eta_1 \otimes \dots \otimes \eta_l$ to apply μ and then as the list of components η_{ij} to compute the Koszul sign (this is a little ambiguity in our notation). If we denote by $\bar{\mu}$ the permutation of $1, \dots, s$ permuting the l blocks of lengths s_1, \dots, s_l according to μ , then $\bar{\mu}$ has the same sign as $\tilde{\mu}$, and the decomposition

4. Chern-Simons Maurer-Cartan element and string topology

of $\varepsilon(\theta, \mu(\eta))$ into the moves

$$\begin{aligned} \theta_1 \dots \theta_s \eta_{\mu_1^{-1}1} \dots \eta_{\mu_l^{-1}s_{\mu_l^{-1}}} &\xrightarrow{(1)} \theta_{\bar{\mu}_1} \dots \theta_{\bar{\mu}_s} \eta_{11} \dots \eta_{ls_l} \xrightarrow{(2)} \theta_{\bar{\mu}_1} \eta_{11} \dots \theta_{\bar{\mu}_s} \eta_{ls_l} \\ &\xrightarrow{(3)} \theta_1 \eta_{\mu_1^{-1}1} \dots \theta_s \eta_{\mu_l^{-1}s_l} \end{aligned}$$

shows that

$$(-1)^{P(\mu(\omega))} = \underbrace{(-1)^{\tilde{\mu}} \varepsilon(\mu, \eta)}_{(1)} \underbrace{(-1)^{P(\omega)}}_{(2)} \underbrace{\varepsilon(\mu, \omega)}_{(3)}.$$

Using this, we write

$$(-1)^{P(\mu(\omega))} (-1)^{\sigma_{L'}} I'(\sigma_{L'}) = \varepsilon(\mu, \omega) (-1)^{(n-1)\mu} (-1)^{P(\omega)} (-1)^{\sigma_L} I(\sigma_L),$$

and compute

$$\begin{aligned} \mathbf{n}_{lg}(\Omega_{\mu_1^{-1}} \otimes \dots \otimes \Omega_{\mu_l^{-1}}) &= \varepsilon(\mu(s), \mu(\omega)) \mathbf{n}_{lg}(s^l \omega_{\mu_1^{-1}} \otimes \dots \otimes \omega_{\mu_l^{-1}}) \\ &= \varepsilon(\mu(s), \mu(\omega)) (-1)^{|s|\mu} \varepsilon(\mu, \omega) \mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \omega_l) \\ &= \underbrace{\varepsilon(\mu(s), \mu(\omega))}_{(1)} \underbrace{(-1)^{|s|\mu} \varepsilon(\mu, \omega)}_{(2)} \underbrace{\varepsilon(s, \omega)}_{(3)} \mathbf{n}_{lg}(s\omega_1 \otimes \dots \otimes s\omega_l) \\ &= \varepsilon(\mu, \Omega) \mathbf{n}_{lg}(\Omega_1 \otimes \dots \otimes \Omega_l). \end{aligned}$$

We used $|s| = n - 1 \bmod 2$, and the last equality follows from the decomposition of $\varepsilon(\mu, \Omega)$ into the moves

$$\begin{aligned} s_1 \omega_1 \dots s_l \omega_l &\xrightarrow{(3)} s_1 \dots s_l \omega_1 \dots \omega_l \xrightarrow{(2)} s_{\mu_1^{-1}} \dots s_{\mu_l^{-1}} \omega_{\mu_1^{-1}} \dots \omega_{\mu_l^{-1}} \\ &\xrightarrow{(1)} s_{\mu_1^{-1}} \omega_{\mu_1^{-1}} \dots s_{\mu_l^{-1}} \omega_{\mu_l^{-1}}. \end{aligned}$$

This proves the symmetry of \mathbf{n}_{lg} .

As for 6), fix an $i = 1, \dots, l$ and let $\mu \in \mathbb{S}_{s_i}$ be a cyclic permutation permuting the components of $\omega_i = \alpha_{i1} \dots \alpha_{is_i}$. Similarly to the previous case, we denote by $\tilde{\mu}$ the corresponding permutation of $1, \dots, 3k$ and get a bijection $(L_1, L_3^b) \mapsto (L'_1 = L_1, L_3^{b'})$ of admissible labelings of a given graph Γ for $\mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \alpha_{i1} \dots \alpha_{is_i} \otimes \dots \otimes \omega_l)$ and $\mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \alpha_{i\mu_1^{-1}} \dots \alpha_{i\mu_{s_i}^{-1}} \otimes \dots \otimes \omega_l)$, respectively. This time, there is no change in

4. Chern-Simons Maurer-Cartan element and string topology

L_1 , and thus we can take $L'_2 = L_2$, producing no sign. Therefore, we have

$$(-1)^{\sigma_{L'}} I'(\sigma_{L'}) = (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta_i) (-1)^{\sigma_L} I(\sigma_L),$$

where $\varepsilon(\mu, \eta_i)$ comes from permuting the forms in $I'(\sigma_{L'})$. Further, we deduce

$$(-1)^{P(\mu(\omega))} = (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta_i) (-1)^{P(\omega)} \varepsilon(\mu, \omega_i),$$

and hence

$$\begin{aligned} \mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \alpha_{i\mu_1^{-1}} \cdots \alpha_{i\mu_{s_i}^{-1}} \otimes \cdots \otimes \omega_l) \\ = \varepsilon(\mu, \omega_i) \mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \alpha_{i1} \cdots \alpha_{is_i} \otimes \cdots \otimes \omega_l). \end{aligned}$$

This shows the symmetry of \mathbf{n}_{lg} on cyclic permutations of the components of ω_i .

As for 7), suppose that $\mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \omega_l) \neq 0$, and let D denote the total form-degree of the input $\eta_{11}, \dots, \eta_{ls_l} \in \mathcal{H}(M)$; i.e., we define

$$D := \deg(\eta_{11}) + \cdots + \deg(\eta_{1s_1}) + \cdots + \deg(\eta_{l1}) + \cdots + \deg(\eta_{ls_l}).$$

Clearly, we must have

$$nk = (n-1)e + D, \tag{4.22}$$

where the left-hand side is the dimension of $M^{\times k}$ and the right-hand side the form-degree of the integrand of $I(\sigma_L)$. If we plug in e from (4.16) and k from (4.21), we get

$$\begin{aligned} D &= nk - (n-1)e \\ &= nk - (n-1)(k+l+2g-2) \\ &= k - (n-1)(l+2g-2) \\ &= s + 2l + 4g - 4 - (n-1)(l+2g-2) \\ &= s - (n-3)(l+2g-2). \end{aligned}$$

It follows that

$$|\mathbf{n}_{lg}| = |s^l| + |\omega_1| + \cdots + |\omega_l| = l(n-3) + D - s = -2(n-3)(g-1).$$

This is exactly the degree from Definition 3.2.6.

As for 8), if $\mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \omega_l) \neq 0$, then

$$s = k - 2l - 4g - 4 \geq 1 + 2(2 - 2g - l) = 1 + 2\chi_{0lg},$$

4. Chern-Simons Maurer-Cartan element and string topology

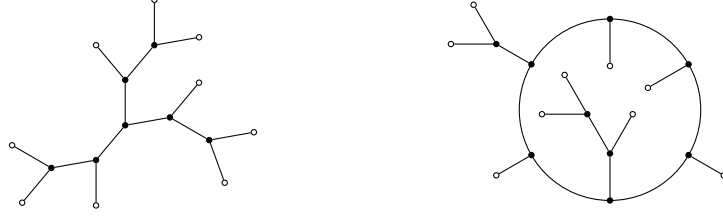


Figure 4.1.: A tree and a circular graph. Internal vertices are denoted with a full dot and external vertices with an empty dot.



Figure 4.2.: The Y -graph and an O_6 -graph.

and hence $\mathbf{n}_{lg} \in \mathcal{F}^{1+2\chi_{0lg}} \hat{\mathbf{E}}_l C$ for the filtration induced from the dual of the filtration of $B^{\text{cyc}} \mathcal{H}$ by weights. Therefore, we get

$$\|\mathbf{n}_{lg}\| \geq 1 + 2\chi_{0lg} > 2\chi_{0lg} \quad \text{for all } l \geq 1, g \geq 0.$$

This finishes the proof. \square

Definition 4.3.8 (Vertices of types A, B, C and some special graphs). *Let $\Gamma \in \text{RG}_{klg}^{(3)}$ be a trivalent ribbon graph and v its internal vertex. We say that v is of type A, B or C if it is connected to precisely 1, 2 or 3 internal vertices, respectively (see Figure 4.3). The graph Γ is called (see Figures 4.1 and 4.2):*

- a tree if $[\Gamma] \in \text{RG}_{k10}$ for some $k \geq 1$;
- circular if $[\Gamma] \in \text{RG}_{k20}$ for some $k \geq 1$;
- the Y -graph is the unique tree with $k = 1$;
- an O_k -graph if Γ is circular with k internal vertices and no A-vertex.

We denote the Y -graph simply by Y .

Remark 4.3.9 (On A, B, C vertices and special graphs). We observe the following:

- (i) A trivalent graph $\Gamma \neq Y$ has each internal vertex of type A, B or C.

4. Chern-Simons Maurer-Cartan element and string topology

(ii) The term \mathbf{n}_{10} is a sum over trees, and the term \mathbf{m}_{10} is the contribution of the Y -graph to \mathbf{n}_{10} (see Proposition 4.3.10 below). The term \mathbf{n}_{20} is a sum over circular graphs. \triangleleft

We will also denote by A, B, C the numbers of internal vertices of the corresponding type. Under the change of variables

$$\begin{aligned} s &= 2A + B, \\ e &= B + \frac{1}{2}A + \frac{3}{2}C, \\ k &= A + B + C, \end{aligned} \tag{4.23}$$

the trivalent formula (4.18) becomes trivial and the Euler formula (4.16) becomes

$$C - A = 2l - 4 + 4g. \tag{4.24}$$

Proposition 4.3.10 (Chern-Simons Maurer-Cartan element). *The collection $\mathbf{n} = (\mathbf{n}_{lg})$ from Definition 4.3.6 is a Maurer-Cartan element for $\mathrm{dIBL}(\mathcal{H}(M))$ which is compatible with \mathbf{m} . In particular, the A_∞ -algebra $\mathcal{H}(M)_{\mathbf{n}}$ is homologically unital and augmented.*

Proof. The fact that \mathbf{n} is a Maurer-Cartan element follows from Lemma 4.3.7 assuming 1) and 9) from [CVona].

As for the compatibility with \mathbf{m} , the only graph contributing to $\mathbf{n}_{10}(s\alpha_1\alpha_2\alpha_3)$ is the Y -graph with $k = 1$. The group $\mathrm{Aut}(Y)$ consists of three rotations, and there is only one possible L_1 , no L_2 and three L_3^b . In Definition 4.3.6, we get $s(1, 1) = n - 2$, $(-1)^{\sigma_L} = 1$ because a cyclic permutation of an odd number of elements is even, and also $P(\alpha_1\alpha_2\alpha_3) = \eta_2$. Finally, we compute

$$\begin{aligned} \mathbf{n}_{10}(s\alpha_1\alpha_2\alpha_3) &= \frac{1}{3}(-1)^{n-2+\eta_2} \sum_{L_3^b} \int_x \alpha_1(x_{\xi(\sigma_1)}) \alpha_2(x_{\xi(\sigma_2)}) \alpha_3(x_{\xi(\sigma_3)}) \\ &= (-1)^{n-2+\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \\ &= \mathbf{m}_{10}(s\alpha_1\alpha_2\alpha_3). \end{aligned} \quad \square$$

Definition 4.3.11 (Contributions of A, B, C vertices). *Consider an internal vertex of type A, B or C as in Figure 4.3. We define the following smooth forms on $M, M^{\times 2}$ and*

4. Chern-Simons Maurer-Cartan element and string topology

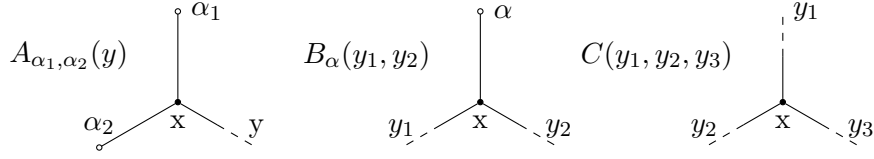


Figure 4.3.: Trivalent vertices of types A , B and C with the corresponding forms A_{α_1, α_2} , B_α and C , respectively.

$M^{\times 3}$, respectively:²

$$\begin{aligned} A_{\alpha_1, \alpha_2}(y) &:= \int_x P(y, x) \eta_1(x) \eta_2(x), \\ B_\alpha(y_1, y_2) &:= \int_x P(y_1, x) P(x, y_2) \eta(x), \\ C(y_1, y_2, y_3) &:= \int_x P(x, y_1) P(x, y_2) P(x, y_3). \end{aligned}$$

4.4. Results about vanishing of Chern-Simons Maurer-Cartan element

In the situation of Definition 4.3.6, let $\Gamma \in \overline{\text{RG}}_{klg}^{(3)}$ be a reduced trivalent ribbon graph, $L = (L_1, L_2, L_3)$ its labeling, x_i the integration variable associated to the i -th internal vertex, $P(x_i, x_j)$ an admissible Hodge propagator on the oriented internal edge between x_i and x_j , and $\alpha_{ij} \in \mathcal{H}(M)[1]$ the harmonic form on the j -th external vertex on the i -th boundary component. Recall that we denote by $\omega_i = s\alpha_{i1} \dots \alpha_{is_i}$ the i -th input of \mathfrak{n}_{lg} and by D the total form-degree of all inputs.

By saying “a graph vanishes” we mean that $I(\sigma_L) = 0$ in the given context.

Proposition 4.4.1 (Vanishing of graphs with 1). *In the setting of Definition 4.3.6, suppose that the following condition is satisfied:*

(V₁) *Every graph $\Gamma \in \overline{\text{RG}}_{klg}^{(3)}$, $\Gamma \neq Y$ which has $1 = \theta 1 \in \mathcal{H}(M)[1]$ at an external vertex vanishes.*

Then \mathfrak{n} is strictly reduced, and the following holds depending on the dimension n :

- (a) *For $n > 3$: All graphs which are not trees or circular vanish. Therefore, $\mathfrak{n}_{lg} = 0$ for all $(l, g) \neq (1, 0), (2, 0)$, and it follows that all higher operations $\mathfrak{q}_{1lg}^{\mathfrak{n}}$ vanish on the chain level.*

²The definitions can be made precise in local coordinates. Smoothness of A_{α_1, α_2} is clear, smoothness of B_α follows from Lemma 4.2.9, and smoothness of C can be shown by a similar argument.

4. Chern-Simons Maurer-Cartan element and string topology

- (b) For $n = 3$: A tree vanishes unless all η_1, \dots, η_s are one-forms. Therefore, $\mathbf{n}_{10}(s\alpha_1 \dots \alpha_s) \neq 0$ implies $\deg(\eta_i) = 1$ for all i .
- (c) For $n < 3$: All trees except for Y vanish. Therefore, we have $\mathbf{n}_{10} = \mathbf{m}_{10}$, and consequently $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110}^{\mathbf{m}}$.

Moreover, we have

- (d) A circular graph vanishes unless all $\eta_{11}, \dots, \eta_{2s_2}$ are one-forms. Therefore, $\mathbf{n}_{20}(s^2\alpha_{11} \dots \alpha_{1s_1} \otimes \alpha_{21} \dots \alpha_{2s_2}) \neq 0$ implies $\deg(\eta_{ij}) = 1$ for all i, j .

In addition to (V_1) , suppose that $H_{\text{dR}}^1(M) = 0$. Then:

- (e) All circular graphs vanish. Therefore, we have $\mathbf{n}_{20} = 0$, and consequently $\mathbf{q}_{120}^{\mathbf{n}} = \mathbf{q}_{120}$.
- (f) For $n \leq 6$: All trees except for Y vanish. Therefore, we have $\mathbf{n}_{10} = \mathbf{m}_{10}$, and consequently $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110}^{\mathbf{m}}$.

Proof. The proof is just combinatorics with D . Suppose that a trivalent ribbon graph $\Gamma \neq Y$ does not vanish on the input $\omega_1, \dots, \omega_l$. Because all external vertices of Γ are adjacent to an A -vertex or a B -vertex, the assumption (V_1) implies $D \geq s$, where s is the total number of external vertices. A combination of (4.22) and (4.18) yields

$$nk - (n-1)e = D \geq s = 3k - 2e \iff (n-3)k \geq (n-3)e.$$

- (a) For $n > 3$, we get $k \geq e$, which implies that Γ is either a tree or a circular graph.
- (b) If Γ is a tree, then $s = k + 2$ and $e = k - 1$. From (4.22) we get

$$D = nk - (n-1)(k-1) = k + n - 1. \tag{4.25}$$

Now D is the sum of $s = k + 2$ form-degrees $\deg(\eta_{ij}) > 0$, and hence (4.25) for $n = 3$ implies that $\deg(\eta_{ij}) = 1$ for all i, j .

- (c) For $n < 3$, we get $e \geq k$, which implies that Γ is not a tree.
- (d) If Γ is a circular graph, then $e = k = s$, and we get using (4.22) that

$$D = nk - (n-1)k = k.$$

Here D is the sum of $s = k$ form-degrees $\deg(\eta_{ij}) > 0$, and hence $\deg(\eta_{ij}) = 1$ for all i, j .

4. Chern-Simons Maurer-Cartan element and string topology

We will now assume, in addition, that $\mathcal{H}^1(M) \simeq H_{\text{dR}}^1(M) = 0$.

(e) We must have $D \geq 2s$, which is in contradiction with $D = s$ for a circular graph. Therefore, $\mathbf{n}_{20} = 0$.

(f) Finally, for a tree $\Gamma \neq Y$, we have

$$k + n - 1 = D \geq 2s = 2(k + 2) \iff n - 5 \geq k.$$

This finishes the proof of the proposition. \square

Proposition 4.4.2 (Special Hodge propagator). *In the setting of Definition 4.3.6, suppose that the Hodge propagator \mathcal{P} is special. Then the condition (V_1) , and hence Proposition 4.4.1 holds.*

Proof. It is easy to see that $A_{\alpha_1, \alpha_2} = \mathcal{P}(\eta_1 \wedge \eta_2)$ for all $\alpha_1, \alpha_2 \in \mathcal{H}(M)[1]$, and that $-B_1$ is the Schwartz kernel of $\mathcal{P} \circ \mathcal{P}$. Therefore, (P4) and (P5) imply $A_{\alpha_1, 1} = 0$ and $B_1 = 0$, respectively.

As for the integral $I(\sigma_L)$, one has to apply the Fubini theorem in order to integrate out single vertices $A_{\alpha_1, 1}$ and B_1 . This step relies on L^1 -integrability of the integrand which follows from [CVona] (the integrand comes from a smooth form on a compact manifold with corners). \square

Proposition 4.4.3 (Vanishing of A -vertices). *In the setting of Definition 4.3.6, suppose that the following condition is satisfied:*

(V_A) Every graph with an A -vertex vanishes.

Then we have $\mathbf{n}_{10} = \mathbf{m}_{10}$, and the only contribution to $\mathbf{n}_{20}(s^2 \alpha_{11} \dots \alpha_{1s_1} \otimes \alpha_{21} \dots \alpha_{2s_2})$ comes from O_k -graphs with $k = s_1 + s_2 = D$.

Proof. The only trees and circular graphs which are not excluded by the assumption are the Y -graph and O_k -graphs, respectively (the external branches contract). The condition on form-degrees is obtained as in the proof of Proposition 4.4.1.

To argue that $I(\sigma_L) = 0$, we again need L^1 -integrability as in the proof of Proposition 4.4.2. \square

Remark 4.4.4 (Integrability for trees). Given a tree, we can start at a leaf and write $I(\sigma_L)$ as an iterative integral of contributions A_{α_1, α_2} for $\alpha_1, \alpha_2 \in \Omega(M)$. These are smooth forms, and hence integrability is guaranteed. Therefore, the result $\mathbf{n}_{10} = \mathbf{m}_{10}$ is independent of the convergence results from [CVona]. \triangleleft

4. Chern-Simons Maurer-Cartan element and string topology

Proposition 4.4.5 (1-connected geometrically formal manifolds). *Let M be a geometrically formal n -manifold and P a special Hodge propagator (it exists by Proposition 4.2.10). If $H_{\text{dR}}^1(M) = 0$, then the following holds:*

($n \neq 2$) All $Y \neq \Gamma \in \overline{\text{RG}}_{klg}$ with $k, l \geq 1, g \geq 0$ vanish, and hence $\mathfrak{n} = \mathfrak{m}$.

($n = 2$) All $Y \neq \Gamma \in \overline{\text{RG}}_{kl0}$ with $k, l \geq 1$ vanish, and hence $\mathfrak{n}_{l0} = \mathfrak{m}_{l0}$ for all $l \geq 1$.

Proof. Given $\eta_1, \eta_2 \in \mathcal{H}$, geometric formality implies $\eta_1 \wedge \eta_2 \in \mathcal{H}$, and hence $A_{\alpha_1, \alpha_2} = \mathcal{P}(\eta_1 \wedge \eta_2) = 0$. We see that (V_1) and (V_A) are satisfied, and hence the implications of Propositions 4.4.1 and 4.4.3 hold. The claim for $n > 3$ follows.

As for $n = 3$, Poincaré duality implies $H_{\text{dR}}^2(M; \mathbb{R}) = 0$. Therefore, the total form-degree D satisfies $D = nB$, where B is the number of B -vertices. We see using (4.23) that (4.22) is equivalent to

$$B + \frac{1}{2}(3 - n)C = D = nB \iff (n - 1)B = \frac{1}{2}(3 - n)C. \quad (4.26)$$

It follows that $B = 0$, and hence all reduced graphs vanish.

As for $n = 2$, we get from (4.26) and (4.24) that $B \geq l$ is equivalent to $g \geq 1$. \square

Remark 4.4.6 (A_∞ -homotopy transfer). In [CVona], it will be shown that the A_∞ -algebra $\mathcal{H}(M)_\mathfrak{n} = (\mathcal{H}(M), (\mu_k))$ induced by \mathfrak{n}_{10} agrees with the A_∞ -algebra obtained by the A_∞ -homotopy transfer

$$\left(\begin{array}{c} \Omega(M) \\ m_1, m_2 \end{array} \right) \rightsquigarrow \left(\begin{array}{c} \mathcal{H}(M) \\ \mu_1 \equiv 0, \mu_2 = \pi_{\mathcal{H}} m_2 (\iota_{\mathcal{H}}, \iota_{\mathcal{H}}), \mu_3, \dots \end{array} \right)$$

using the homotopy retract (see [Val12])

$$\mathcal{P} \left(\begin{array}{c} \curvearrowright \\ (\Omega(M), m_1) \end{array} \right) \xrightleftharpoons[\iota_{\mathcal{H}}]{\pi_{\mathcal{H}}} (\mathcal{H}(M), m_1 \equiv 0).$$

The operation μ_k of the transferred A_∞ -structure is computed as a sum over planar trees with a root and k leaves decorated by $\iota_{\mathcal{H}}$ at the leaves, $\pi_{\mathcal{H}}$ at the root and \mathcal{P} at the internal edges (see [Aka07]). The result of [CVona] is plausible because the part of \mathfrak{n}_{10} contributing to μ_k is a sum over trivalent ribbon trees with $k + 1$ leaves.

In [CVona], they will also show that $\iota_1 := \iota_{\mathcal{H}} : \mathcal{H} \rightarrow \Omega$ extends to an A_∞ -quasi-isomorphism $(\iota_k)_{k \geq 1}$ from $(\mathcal{H}, (\mu_k))$ to (Ω, m_1, m_2) . The induced chain map on the

4. Chern-Simons Maurer-Cartan element and string topology

dual cyclic bar complexes is then the map f_{110}^m coming from the IBL_∞ -theory in the Introduction. \triangleleft

Proposition 4.4.7 (Twisted boundary operator for formal manifolds). *In the setting of Definition 4.3.6, suppose that M is formal in the sense of rational homotopy theory. Then there is a quasi-isomorphism*

$$\mathfrak{h}_{110} : (\hat{B}_{\text{cyc}}^* H_{\text{dR}}(M)[3-n], \mathfrak{q}_{110}^m) \longrightarrow (\hat{B}_{\text{cyc}}^* \mathcal{H}(M)[3-n], \mathfrak{q}_{110}^n).$$

Proof. Formality of M is equivalent to the existence of a zig-zag of quasi-isomorphisms of dga's (see [Val12])

$$(H_{\text{dR}}(M), m_1 \equiv 0, m_2) \rightsquigarrow \bullet \quad \cdots \quad \bullet \leftarrow (\Omega(M), m_1, m_2).$$

Because a dga-quasi-isomorphism has a homotopy inverse in the category of A_∞ -algebras, we get a direct A_∞ -quasi-isomorphism

$$(g_k) : (\Omega(M), m_1, m_2) \rightsquigarrow (H_{\text{dR}}(M), m_1 \equiv 0, m_2).$$

Precomposing with (ι_k) from Remark 4.4.6, we get the A_∞ -isomorphism

$$(h_k) : (\mathcal{H}(M), (\mu_k)) \rightsquigarrow (H_{\text{dR}}(M), m_1 \equiv 0, m_2).$$

This induces the quasi-isomorphism \mathfrak{h}_{110} of the corresponding cyclic cochain complexes. \square

Remark 4.4.8 (On formality). Geometrically formal manifolds include \mathbb{S}^n , $\mathbb{C}P^n$ and Lie groups (see [Kot00]). Any geometrically formal manifold is formal. Every simply-connected manifold of dimension at most 6 is formal (see [Mil79]). \triangleleft

4.5. Conjectured relation to string topology

Given a smooth connected oriented n -dimensional manifold M , we consider the equivariant homology of the free loop space $LM := \{\gamma : \mathbb{S}^1 \rightarrow M \text{ continuous}\}$ with respect to the reparametrization action of \mathbb{S}^1 . It is defined as the singular homology of the Borel construction

$$L_{\mathbb{S}^1} M := E\mathbb{S}^1 \times_{\mathbb{S}^1} LM := (E\mathbb{S}^1 \times LM)/\mathbb{S}^1,$$

4. Chern-Simons Maurer-Cartan element and string topology

where $E\mathbb{S}^1 = \mathbb{S}^\infty \rightarrow B\mathbb{S}^1 = \mathbb{CP}^\infty$ is a model for the universal bundle for \mathbb{S}^1 , and we quotient out the diagonal action. We denote this homology by

$$H^{\mathbb{S}^1}(LM) := H(L_{\mathbb{S}^1}M).$$

Recall that $(E\mathbb{S}^1 \times LM)/\mathbb{S}^1$ is the homotopically correct quotient replacing LM/\mathbb{S}^1 , which we prefer to use because the diagonal action of \mathbb{S}^1 on $E\mathbb{S}^1$ is free, and hence $E\mathbb{S}^1 \times LM \rightarrow (E\mathbb{S}^1 \times LM)/\mathbb{S}^1$ is a principal \mathbb{S}^1 -bundle (in contrast to the pathological map $LM \rightarrow LM/\mathbb{S}^1$). Recall also that $E\mathbb{S}^1$ is contractible.

The “geometric versions” of the homologies were defined in [CS99] as the degree shifts

$$\mathbb{H}(LM) := H(LM)[n] \quad \text{and} \quad \mathcal{H}(LM) := H^{\mathbb{S}^1}(LM)[n].$$

There is the *loop product* $\bullet : \mathbb{H}(LM)^{\otimes 2} \rightarrow \mathbb{H}(LM)$ of degree 0 which makes $\mathbb{H}(LM)$ into a graded commutative associative algebra. There is also the *loop coproduct* $\tau : \bar{\mathbb{H}}(LM) \rightarrow \bar{\mathbb{H}}(LM)^{\otimes 2}$ of degree $1 - 2n$ which is graded cocommutative and coassociative and is a derivation of \bullet . The geometric construction of \bullet and τ on transverse smooth chains in LM was described in [CS99] and [Bas11], respectively. Here, the symbol $\bar{\mathbb{H}}(LM)$ stands for the degree shifted relative homology

$$\bar{\mathbb{H}}(LM) := H(LM, M)[n]$$

with respect to constant loops $M \hookrightarrow LM$. The geometric construction of τ does not work on the whole $\mathbb{H}(LM)$ because of the phenomenon of “vanishing of small loops” depicted in [CL07, Figure 4, p. 13].

The projection $E\mathbb{S}^1 \times LM \rightarrow L_{\mathbb{S}^1}M$ is an \mathbb{S}^1 -principal bundle and thus induces a Gysin sequence. This sequence written using the geometric versions reads

$$\dots \longrightarrow \mathbb{H}_i \xrightarrow{\mathcal{E}} \mathcal{H}_i \xrightarrow{\cap c} \mathcal{H}_{i-2} \xrightarrow{\mathcal{M}} \mathbb{H}_{i-1} \longrightarrow \dots, \quad (4.27)$$

where the map \mathcal{M} adds a marked point in each string in a family in all possible positions, the map \mathcal{E} erases the marked point of each string in a family, $c \in H_{\mathbb{S}^1}^2(LM)$ is the Euler class of the circle bundle and \cap the cap product.

The *string bracket* $\tilde{m}_2 : \mathcal{H}(LM)^{\otimes 2} \rightarrow \mathcal{H}(LM)$ and the *string cobracket* $\tilde{c}_2 : \bar{\mathcal{H}}(LM) \rightarrow \bar{\mathcal{H}}(LM)^{\otimes 2}$ are defined by

$$\tilde{m}_2 := \mathcal{E} \circ \bullet \circ \mathcal{M}^{\otimes 2} \quad \text{and} \quad \tilde{c}_2 := \mathcal{E}^{\otimes 2} \circ \nu \circ \mathcal{M}.$$

4. Chern-Simons Maurer-Cartan element and string topology

Here, the symbol $\bar{\mathcal{H}}(LM)$ stands for the degree shifted relative \mathbb{S}^1 -equivariant homology

$$\bar{\mathcal{H}}(LM) := \underbrace{H^{\mathbb{S}^1}(E\mathbb{S}^1 \times_{\mathbb{S}^1} LM, E\mathbb{S}^1 \times_{\mathbb{S}^1} M)[n]}_{=: \bar{H}^{\mathbb{S}^1}(LM)}.$$

Because $|\mathcal{M}| = 1$ and $|\mathcal{E}| = 0$, we have for all $\xi \in \bar{\mathcal{H}}(LM)$ and $\xi_1, \xi_2 \in \mathcal{H}$ the relations

$$\begin{aligned} \tilde{\mathfrak{m}}_2(\xi_1, \xi_2) &= (-1)^{|\xi_1|} \mathcal{E}(\mathcal{M}(\xi_1) \bullet \mathcal{M}(\xi_2)), \\ \tilde{\mathfrak{c}}_2(\xi) &= \sum \mathcal{E}(\nu^1) \otimes \mathcal{E}(\nu^2), \end{aligned} \tag{4.28}$$

where we write $\nu(\mathcal{M}(\xi)) = \sum \nu^1 \otimes \nu^2$. The operations $\tilde{\mathfrak{m}}_2$ and $\tilde{\mathfrak{c}}_2$ have degrees 2 and $2-2n$ with respect to the grading on $\mathcal{H}(LM)$, respectively. In fact, we will consider $\tilde{\mathfrak{m}}_2$ and $\tilde{\mathfrak{c}}_2$ given by (4.28) as operations on the even degree shift $H^{\mathbb{S}^1}(LM)[2-n] = \mathcal{H}(LM)[2-2n]$, which have degrees $2(2-n)$ and 0, respectively. The symbols \mathfrak{m}_2 and \mathfrak{c}_2 will denote their degree shifts to $H^{\mathbb{S}^1}(LM)[3-n]$, which have degrees of an IBL-algebra from Definition 3.2.4.

In work in progress [CVonb], they consider the map

$$I_{\lambda,*} : H_{-\bullet,-1}^{\lambda}(\Omega(M)) \longrightarrow H_{\mathbb{S}^1}^{\bullet}(LM; \mathbb{R})$$

defined on the chain level as a cyclic version of Chen's iterated integrals I_{λ} . Recall that $H_{-\bullet,-1}^{\lambda}(\Omega) = H_{\bullet}(\text{B}^{\text{cyc}}\Omega, \mathfrak{b})$, where $\mathfrak{b} : \text{B}\Omega = \bigoplus_{k \geq 1} \Omega[1]^{\otimes k} \rightarrow \text{B}\Omega$ is the Hochschild differential of the de Rham dga (Ω, m_1, m_2) , and the grading on $H^{\lambda}(\Omega)$ was chosen such that $H_{\bullet}^{\lambda}(\Omega) \simeq H_{\bullet}^{\lambda, \text{cl}}(\Omega)$ for the classical cyclic homology of a dga (see Section 3.3). They prove in [CVonb] that if M is simply-connected, then the map $I_{\lambda,*}$ induces an isomorphism $H_{-\bullet,-1}^{\lambda, \text{red}}(\Omega(M)) \simeq H_{\mathbb{S}^1, \text{red}}^{\bullet}(LM)$, where

$$H_{\mathbb{S}^1, \text{red}}(LM) := H_{\mathbb{S}^1}(E\mathbb{S}^1 \times_{\mathbb{S}^1} LM, E\mathbb{S}^1 \times_{\mathbb{S}^1} \{x_0\})$$

is the *reduced \mathbb{S}^1 -equivariant cohomology* with respect to a base point $x_0 \in M$ (the constant loop at x_0). Dualizing their map, we obtain the isomorphism

$$H_{\lambda, \text{red}}^{-\bullet-1}(\Omega(M)) \simeq H_{\bullet}^{\mathbb{S}^1, \text{red}}(LM; \mathbb{R}). \tag{4.29}$$

Suppose from now on that M is closed. Pick a Riemannian metric and an admissible Hodge propagator $P \in \Omega^{n-1}(\text{Bl}_{\Delta}(M \times M))$. We will assume that P is special, i.e., that it satisfies (P1)–(P5) from Section 4.2, so that the Chern-Simons Maurer-Cartan element \mathfrak{n} is strictly reduced, and hence the twisted reduced IBL $_{\infty}$ -algebra $\text{dIBL}^{\mathfrak{n}}(C_{\text{red}}(\mathcal{H}))$ and

4. Chern-Simons Maurer-Cartan element and string topology

the induced IBL-algebra $\text{IBL}(\mathbb{H}^{n,\text{red}}(C(\mathcal{H})))$ are well-defined. Recall that $\mathbb{H}^n_\bullet(C(\mathcal{H})) = H_{n-3-\bullet}^\lambda(\mathcal{H}_n)$, where \mathcal{H}_n is the A_∞ -algebra on \mathcal{H} twisted by \mathfrak{n}_{10} . From [CVona], we have

$$H_\lambda(\mathcal{H}(M)_n) \simeq H_\lambda(\Omega(M)). \quad (4.30)$$

A combination of (4.29) and (4.30) gives the following version of the string topology conjecture from the Introduction.

Conjecture 4.5.1 (String topology conjecture for simply-connected manifold). *Let M be an oriented closed manifold of dimension n . There is a chain map*

$$(C^{\text{sing}}(\text{L}_{\mathbb{S}^1}M; \mathbb{R}), \partial) \longrightarrow (\hat{\text{B}}_{\text{cyc}}^* \mathcal{H}(M), \mathfrak{q}_{110}^n),$$

where C^{sing} denotes the (smooth) singular chain complex and ∂ the standard boundary operator, which, if M is simply-connected, satisfies the following:

- It induces an isomorphism $H^{\mathbb{S}^1, \text{red}}(\text{LM}; \mathbb{R})[2-n] \simeq \mathbb{H}^{n, \text{red}}(C(\mathcal{H}(M)))$.
- It intertwines \mathfrak{m}_2 on $H^{\mathbb{S}^1}(\text{LM}; \mathbb{R})$ and \mathfrak{q}_{210} .
- The pullback of \mathfrak{q}_{120}^n to $H^{\mathbb{S}^1, \text{red}}(\text{LM}; \mathbb{R})$ is compatible with \mathfrak{c}_2 on $\bar{H}^{\mathbb{S}^1}(\text{LM}; \mathbb{R})$ under the morphism induced by the inclusion $(\text{LM}, x_0) \rightarrow (\text{LM}, M)$.

Remark 4.5.2 (On string topology conjecture). (i) The conjecture can be interpreted as follows. There is an IBL-structure on $H^{\mathbb{S}^1, \text{red}}(\text{LM}; \mathbb{R})$ compatible with Chas-Sullivan operations, and the IBL_∞ -algebra $\text{dIBL}^n(C_{\text{red}}(\mathcal{H}(M)))$ is its chain model.

(ii) The loop coproduct τ is geometrically defined only on $\bar{H}^{\mathbb{S}^1}(\text{LM})$; the conjecture thus provides an extension of \mathfrak{c}_2 to $H^{\mathbb{S}^1, \text{red}}(\text{LM})$. In [Bas11], it is shown that the geometric definition of τ can be extended to $H(\text{LM})$ for manifolds with zero Euler characteristic, i.e., $\chi(M) = 0$. This extension depends on the choice of a non-vanishing vector field on M . By homotopy invariance (see (v) below), our extension of \mathfrak{c}_2 should not depend on the admissible Hodge propagator P .

(iii) The loop product \bullet is geometrically defined on $H(\text{LM})$; however, it does not always induce an associative product on $H^{\text{red}}(\text{LM}) = H(\text{LM}, x_0)$. Indeed, the examples of \mathbb{T}^2 (see [Bas11]) and \mathbb{S}^3 (see [CS99]) show that $H(x_0; \mathbb{R}) \subset H(\text{LM}; \mathbb{R})$ is not an ideal with respect to \bullet . By [Tam10], this does not happen when $\chi(M) \neq 0$, and hence, in this case, \bullet restricts to $H(\text{LM}, x_0; \mathbb{R})$.

4. Chern-Simons Maurer-Cartan element and string topology

(iv) The computation for \mathbb{S}^n with $n \geq 2$ and the computation for \mathbb{CP}^n in Section 5 support the conjecture. The computation for \mathbb{S}^1 in Section 5.3 provides a counterexample for non-simply-connected M . Surfaces of genus $g \geq 1$ should be considered.

(v) We expect that if M_1 and M_2 are homotopy equivalent, then the IBL_∞ -algebras $\mathrm{dIBL}^{\mathfrak{n}_1}(C(\mathrm{H}_{\mathrm{dR}}(M_1)))$ and $\mathrm{dIBL}^{\mathfrak{n}_2}(C(\mathrm{H}_{\mathrm{dR}}(M_2)))$ are IBL_∞ -homotopy equivalent. \triangleleft

5. Explicit computations

In Section 5.1, we solve the differential equation for the Hodge propagator P for \mathbb{S}^n (Proposition 5.1.2) using the Relative Poincaré Lemma (Lemma 5.1.1). In the rest of the section, we will be showing that P is admissible (Proposition 5.1.10); the most work is to show that P extends smoothly to the blow-up (Proposition 5.1.9). Another Hodge propagator for \mathbb{S}^1 can be obtained in an alternative simple way by writing $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and there are nice geometric formulas for P for \mathbb{S}^2 (Example 5.1.3).

In Section 5.2, we use P from Section 5.1 to compute the Chern-Simons Maurer-Cartan element \mathbf{n} for \mathbb{S}^n (Proposition 5.2.10). We first prove that the condition (V_1) from Proposition 4.4.1 is satisfied (Lemma 5.2.1) and then perform combinatorics with degrees to show vanishing of some more integrals (Proposition 5.2.2). In fact, all the integrals vanish for \mathbb{S}^n with $n \geq 3$, and the only non-vanishing integrals for \mathbb{S}^1 are the O_k -graphs with even k . We compute these integrals explicitly together with all signs and combinatorial coefficients required to obtain \mathbf{n}_{20} (Lemmas 5.2.5, 5.2.6, 5.2.7 and 5.2.8). There might be some non-vanishing integrals associated to reduced graphs for \mathbb{S}^2 as well as some non-vanishing integrals associated to graphs without external vertices for \mathbb{S}^3 ; however, the simplest examples vanish (Remarks 5.2.3 and 5.2.4).

In the remaining Sections 5.3 and 5.4, we compute $\text{IBL}(\mathbb{H}^n(C(\mathcal{H}(M))))$ and the higher operations $\mathbf{q}_{1lg}^{\mathbf{n}}$ on \mathbb{H}^n for $M = \mathbb{S}^n, \mathbb{CP}^n$. As soon as we argue that $\mathbf{n}_{10} = \mathbf{m}_{10}$ due to geometric formality, the computation of $\mathbb{H}^m(C(\mathcal{H}(\mathbb{S}^n)))$ and $\mathbb{H}^m(C(\mathcal{H}(\mathbb{CP}^n)))$ is an easy exercise in cyclic homology. The operations for \mathbb{S}^{2m} and \mathbb{CP}^n vanish for degree reasons (Remark 5.3.1). Therefore, the integrals from Section 5.2 help only in the case of \mathbb{S}^{2m-1} . We compare our results to Chas-Sullivan string topology from [Bas11] and confirm Conjecture 4.5.1 for \mathbb{S}^n with $n \geq 2$ and for \mathbb{CP}^n .

5.1. Construction of a Hodge propagator for spheres

The standard Riemannian volume form on the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the restriction of the following closed form on $\mathbb{R}^{n+1} \setminus \{0\}$:

$$\text{Vol}(x) := \frac{1}{|x|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}.$$

Here $\widehat{dx_i}$ means that dx_i is omitted. We denote the Riemannian volume of \mathbb{S}^n by

$$V := \int_{\mathbb{S}^n} \text{Vol}.$$

The n -form H from Proposition 4.2.7 reads

$$H = \frac{1}{V} (\text{pr}_1^* \text{Vol} + (-1)^n \text{pr}_2^* \text{Vol}).$$

According to Proposition 4.2.8, the equation which we want to solve reads

$$dP = \frac{1}{V} ((-1)^n \text{pr}_1^* \text{Vol} + \text{pr}_2^* \text{Vol}). \quad (5.1)$$

We denote

$$\tilde{G} := VP \quad \text{and} \quad \tilde{H} := VH.$$

The following lemma will be used to construct a solution to (5.1).

Lemma 5.1.1 (Relative Poincaré Lemma). *Let M be a smooth oriented manifold and $\psi : [0, 1] \times M \rightarrow M$ a smooth map. Consider the operator $T : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$ defined by*

$$T(\eta) := \int^{[0,1]} \psi^* \eta \quad \text{for all } \eta \in \Omega(M),$$

where we integrate along the fiber of the oriented fiber bundle $\text{pr}_2 : [0, 1] \times M \rightarrow M$. Then we have

$$d \circ T + T \circ d = \psi_1^* - \psi_0^*.$$

Proof. Stokes' formula from Proposition 4.2.6 gives

$$d \int^{[0,1]} \psi^* \eta = - \left(\int^{[0,1]} d\psi^* \eta - \int^{\partial[0,1]} \psi^* \eta \right) = - \int^{[0,1]} \psi^* d\eta + \psi_1^* \eta - \psi_0^* \eta$$

for all $\eta \in \Omega(M)$. □

5. Explicit computations

Proposition 5.1.2 (Solution to (5.1)). *For all $(x, y) \in (\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$, let*

$$P(x, y) := (-1)^n \sum_{k=0}^{n-1} g_k(x, y) \omega_k(x, y), \quad (5.2)$$

where

$$g_k(x, y) := \int_0^1 \frac{t^k (t-1)^{n-1-k}}{(2t(t-1)(1+x \cdot y) + 1)^{\frac{n+1}{2}}} dt \quad (5.3)$$

and

$$\omega_k(x, y) := \frac{1}{k!} \frac{1}{(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n+1}} (-1)^\sigma x^{\sigma_1} y^{\sigma_2} dx^{\sigma_3} \dots dx^{\sigma_{2+k}} dy^{\sigma_{3+k}} \dots dy^{\sigma_{n+1}}. \quad (5.4)$$

The form (5.2) is a smooth solution to (5.1) on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$.

Proof. Define the set

$$N := (\mathbb{R}_{\neq 0}^{n+1} \times \mathbb{R}_{\neq 0}^{n+1}) \setminus \{(x, ax) \mid x \in \mathbb{R}^{n+1}, a > 0\}.$$

It is an open thickening of $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta$. Consider the smooth deformation retraction

$$\begin{aligned} \psi : [0, 1] \times N &\longrightarrow N \\ (t, x, y) &\longmapsto \psi_t(x, y) := (x, (1-t)y - tx) \end{aligned}$$

with

$$\psi_0(x, y) = (x, y) \quad \text{and} \quad \psi_1(x, y) = (x, -x) \quad \text{for all } (x, y) \in N.$$

The retraction is depicted in Figure 5.1. Denote by $A : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, $x \mapsto -x$ the antipodal map. It is easy to see that

$$A^* \text{Vol} = (-1)^{n+1} \text{Vol},$$

and hence

$$\psi_1^* \tilde{H} = \psi_1^* \text{pr}_1^* \text{Vol} + (-1)^n \psi_1^* \text{pr}_2^* \text{Vol} = \text{pr}_1^* \text{Vol} + (-1)^n \text{pr}_1^* A^* \text{Vol} = 0.$$

Define

$$P := (-1)^{n+1} \int_{[0,1]} \psi^* H. \quad (5.5)$$

Let $T : \Omega^*(N) \rightarrow \Omega^{*-1}(N)$ be the cochain homotopy from Lemma 5.1.1 associated to ψ .

5. Explicit computations

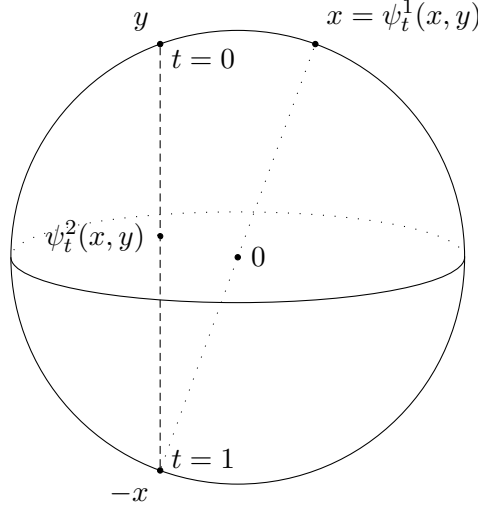


Figure 5.1.: Retraction $\psi_t = (\psi_t^1, \psi_t^2)$. A point of $\mathbb{S}^n \times \mathbb{S}^n$ is visualized as a pair of points on \mathbb{S}^n .

Because $dH = 0$, we get

$$dP = (-1)^{n+1} dT(H) = (-1)^{n+1} (dT + Td)H = (-1)^{n+1} (\psi_1^* - \psi_0^*)H = (-1)^n H.$$

For every $i = 1, \dots, n+1$, we have

$$\psi^*(dx^i) = dx^i \quad \text{and} \quad \psi^*(dy^i) = (1-t) dy^i - t dx^i - (y^i + x^i) dt.$$

We compute

$$\begin{aligned} (-1)^{n+1} \int_{[0,1]} \psi^* \tilde{H} &= - \int_{[0,1]} \psi^* \text{pr}_2^* \text{Vol} \\ &= \int_{[0,1]} \sum_{i=1}^{n+1} (-1)^i \frac{((1-t)y^i - tx^i)}{|(1-t)y - tx|^{n+1}} \psi^*(dy^1 \dots \widehat{dy^i} \dots dy^{n+1}) \\ &= \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} (x^i y^j - y^i x^j) \int_{[0,1]} \frac{dt \psi^*(dy^1 \dots \widehat{dy^i} \dots \widehat{dy^j} \dots dy^{n+1})}{|(1-t)y - tx|^{n+1}} \\ &= (-1)^n \sum_{k=0}^{n-1} \left(\int_0^1 \frac{t^k (t-1)^{n-1-k}}{|(1-t)y - tx|^{n+1}} dt \right) \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} (x^i y^j - y^i x^j) \\ &\quad \sum_{\substack{\sigma: \{1, \dots, n-1\} \rightarrow \{1, \dots, \hat{i}, \dots, \hat{j}, \dots, n+1\} \\ \sigma_1 < \dots < \sigma_k \\ \sigma_{k+1} < \dots < \sigma_{n-1}}} (-1)^\sigma dx^{\sigma_1} \dots dx^{\sigma_k} dy^{\sigma_{k+1}} \dots dy^{\sigma_{n-1}}. \end{aligned}$$

5. Explicit computations

The formulas (5.3) and (5.4) are obtained from this by writing

$$|(1-t)y - tx|^2 = 2t(t-1)(1+x \cdot y) + 1$$

in the denominator of the integrand and by simple combinatorics in the form part, respectively. Smoothness of P on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ follows from the expression (5.5). \square

Note that g_k are smooth functions on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$.

Example 5.1.3 (for \mathbb{S}^1 and \mathbb{S}^2). (a) Let

$$\alpha : (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta \rightarrow (0, 2\pi)$$

be the smooth function assigning to a pair $(x, y) \in (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta$ the counterclockwise angle from x to y . Let $\alpha_1, \alpha_2 \in [0, 2\pi)$ be such that $x = \cos(\alpha_1)e_1 + \sin(\alpha_1)e_2$ and $y = \cos(\alpha_2)e_1 + \sin(\alpha_2)e_2$ for the standard Euclidean basis e_1, e_2 of \mathbb{R}^2 . It is easy to see that

$$\alpha(x, y) = \begin{cases} \alpha_2 - \alpha_1 & \text{if } \alpha_1 < \alpha_2, \\ \alpha_2 - \alpha_1 + 2\pi & \text{if } \alpha_1 > \alpha_2. \end{cases}$$

Therefore, we get

$$d\alpha = d\alpha_2 - d\alpha_1 = -2\pi H \quad \text{on } (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta.$$

On the other hand, we can compute P from (5.2) as follows. Using the substitution $u = 2t - 1$, we get for all $x, y \in \mathbb{S}^1$ with $x \neq \pm y$ the following:

$$\begin{aligned} g_0(x, y) &= \int_0^1 \frac{dt}{2t(t-1)(1+x \cdot y) + 1} = \frac{1}{1-x \cdot y} \int_{-1}^1 \frac{du}{\frac{1+x \cdot y}{1-x \cdot y} u^2 + 1} \\ &= \frac{2}{\sqrt{1-(x \cdot y)^2}} \arctan\left(\sqrt{\frac{1+x \cdot y}{1-x \cdot y}}\right) \\ &= \frac{\pi - \arccos(x \cdot y)}{\sqrt{1-(x \cdot y)^2}} = \frac{\pi - \arccos(x \cdot y)}{|x^1 y^2 - x^2 y^1|} = \frac{\pi - \alpha(x, y)}{x^1 y^2 - x^2 y^1}. \end{aligned}$$

The third from last equality can be obtained by trigonometric considerations and the second from last equality by an algebraic manipulation with the denominator. We will explain the last equality. Consider the matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

representing the counterclockwise rotation by $\frac{\pi}{2}$. The function $\arccos : (-1, 1) \rightarrow (0, \pi)$

5. Explicit computations

satisfies

$$\arccos(x \cdot y) = \begin{cases} \alpha(x, y) & \text{if } y \cdot Rx > 0, \\ 2\pi - \alpha(x, y) & \text{if } y \cdot Rx < 0. \end{cases}$$

The last equality becomes clear when we notice that $x^1 y^2 - x^2 y^1 = y \cdot Rx$.

Finally, we have $\omega_0(x, y) = x^1 y^2 - x^2 y^1$, and hence

$$2\pi G(x, y) = -g_0(x, y)\omega_0(x, y) = \alpha(x, y) - \pi = \pi - \alpha(y, x).$$

(b) For $n = 2$, we get the formulas

$$\begin{aligned} g_0(x, y) &= -g_1(x, y) = \frac{1}{x \cdot y - 1} \quad \text{and} \\ \omega_0(x, y) &= (x^2 y^3 - x^3 y^2) dy^1 + (x^3 y^1 - x^1 y^3) dy^2 + (x^1 y^2 - x^2 y^1) dy^3 \\ &= \sum_{i=1}^3 (x \times y)^i dy^i. \end{aligned}$$

The formula for $\omega_1(x, y)$ is obtained from the formula for $\omega_0(x, y)$ by replacing dy with dx . \triangleleft

Consider the diagonal action of the orthogonal group $O(n+1)$ on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ by matrix multiplication.

Proposition 5.1.4 (Symmetries of P). *Consider P from Proposition 5.1.2. For all $R \in O(n+1)$, we have*

$$R^*P = (-1)^R P,$$

where $(-1)^R = \det(R)$. Moreover, if τ denotes the twist map, then

$$\tau^*P = (-1)^n P.$$

Proof. We will use the thickening N , the antipodal map A and the expression (5.5) for P from the proof of Proposition 5.1.2.

It is easy to check that both τ and R preserve N . Let $\tilde{\tau}$ and \tilde{R} be the isomorphisms of the fiber bundle $\text{pr}_2 : [0, 1] \times N \rightarrow N$ given by

$$\tilde{\tau}(t, x, y) := (1 - t, y, x) \quad \text{and} \quad \tilde{R}(t, x, y) := (t, Rx, Ry)$$

for all $(t, x, y) \in [0, 1] \times N$. Then $\tilde{\tau}$ covers τ and \tilde{R} covers R . A simple computation directly from Definition 4.2.1 shows that the fiberwise integration commutes with the

5. Explicit computations

pullback along a bundle morphism if the bundle map and the base map are both either orientation preserving or reversing. In our case, we have

$$(-1)^{\tau+\tilde{\tau}} = -1 \quad \text{and} \quad (-1)^{R+\tilde{R}} = 1.$$

Using this and the equation

$$\text{pr}_2 \circ \psi \circ \tilde{\tau} = A \circ \text{pr}_2 \circ \psi,$$

we get firstly

$$\begin{aligned} \tau^* \int^{[0,1]} \psi^* \tilde{H} &= - \int^{[0,1]} \tilde{\tau}^* \psi^* \text{pr}_2^* \text{Vol} \\ &= - \int^{[0,1]} \psi^* \text{pr}_2^* A^* \text{Vol} \\ &= (-1)^n \int^{[0,1]} \psi^* \text{pr}_2^* \text{Vol} \\ &= (-1)^n \int^{[0,1]} \psi^* \tilde{H} \end{aligned}$$

and secondly

$$R^* \int^{[0,1]} \psi^* H = \int^{[0,1]} \tilde{R}^* \psi^* H = \int^{[0,1]} \psi^* R^* H = (-1)^{n+1} \int^{[0,1]} \psi^* H.$$

This proves the proposition. □

Both diffeomorphisms R and τ preserve Δ , and hence they extend to diffeomorphisms of $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$. If also P extends, then the statement of Proposition 5.1.4 holds for P on $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$.

In the rest of the section, we will be proving that P extends smoothly to $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$. This is a local problem at the boundary, where we introduce the following radial coordinates. Define the set

$$X := \{(r, \eta, x) \in [0, \infty) \times \mathbb{S}^n \times \mathbb{S}^n \mid \eta \cdot x = 0\},$$

and let $\kappa : X \longrightarrow \text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ be the map defined by

$$\kappa(r, \eta, x) := \begin{cases} \left(x, \frac{x + r\eta}{|x + r\eta|}\right) \in (\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta & \text{for } r > 0, \\ [(-\eta, \eta)] \in P^+ N_{(x,x)} \Delta & \text{for } r = 0. \end{cases}$$

5. Explicit computations

For the upcoming computations, it is convention to define the map $\gamma : \mathbb{R} \rightarrow (-1, 1)$ by

$$\gamma(r) := \frac{r}{\sqrt{1+r^2}+1} \quad \text{for all } r \in \mathbb{R}.$$

It is a diffeomorphism with inverse $r = \frac{2\gamma}{1-\gamma^2}$.

Lemma 5.1.5 (Parametrization of the collar neighborhood). *The subset $X \subset \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a submanifold with boundary, and the map $\kappa : X \rightarrow \text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ is an embedding onto a neighborhood of $\partial \text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$.*

Proof. The set X is a Cartesian product of $[0, \infty)$ and a regular level set; therefore, it is a submanifold with boundary. The inclusion $\mathbb{S}^n \times \mathbb{S}^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ induces an embedding of manifolds with boundary $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n) \subset \text{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. Consider the global chart $\tilde{\mathbb{I}} : \text{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \rightarrow [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^{n+1}$ from (4.4) induced by the identity. We have

$$\begin{aligned} Y &:= \tilde{\mathbb{I}}(\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)) \\ &= \{(\tilde{r}, w, u) \in [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^{n+1} \mid |u|^2 + \tilde{r}^2 = 1, w \cdot u = 0\}, \end{aligned}$$

where we denote r on Y by \tilde{r} in order to distinguish it from r on X . It suffices to prove the claim for the map $\mu := \tilde{\mathbb{I}} \circ \kappa : X \rightarrow Y$. For $(r, \eta, x) \in X$, we compute

$$\mu(r, \eta, x) = \left(\frac{\gamma}{\sqrt{1+\gamma^2}}, \frac{1}{\sqrt{1+\gamma^2}}(\gamma x - \eta), \frac{1}{1+\gamma^2}(x + \gamma\eta) \right).$$

This formula defines a smooth map of $\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. It is a local diffeomorphism because its Jacobian is non-vanishing:

$$|\text{D}\mu| = \frac{\partial \tilde{r}}{\partial r} \left(\frac{\partial w}{\partial \eta} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} \frac{\partial u}{\partial \eta} \right)^{n+1} = (-1)^{n+1} (1+\gamma^2)^{-\frac{n+4}{2}} \frac{\partial \gamma}{\partial r}.$$

Moreover, the map μ is injective, maps X into Y and ∂X onto ∂Y . The claim follows. \square

Consider the action of $O(n+1)$ on X defined by

$$R \cdot (r, \eta, x) := (r, R\eta, Rx) \quad \text{for all } (r, \eta, x) \in X \text{ and } R \in O(n+1).$$

Via κ , this agrees with the diagonal action of $O(n+1)$ on $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$. Denote

$$\text{P}' := \kappa^* \text{P} \in \Omega^{n-1}(\text{Int}(X)).$$

5. Explicit computations

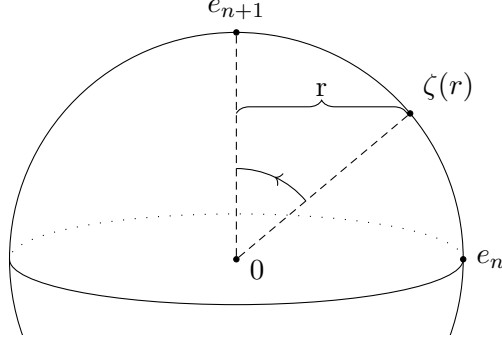


Figure 5.2.: The curve $\zeta := \kappa \circ \zeta'$ is given by $\zeta(r) = (e_{n+1}, \frac{e_{n+1} + r e_n}{|e_{n+1} + r e_n|})$ for $r > 0$.

From Proposition 5.1.4 we get

$$R^*P' = (-1)^R P' \quad \text{for all } R \in O(n+1). \quad (5.6)$$

Consider the smooth curve (see Figure 5.2)

$$\begin{aligned} \zeta' : [0, \infty) &\longrightarrow X \\ r &\longmapsto (r, e_n, e_{n+1}). \end{aligned}$$

We have the following lemma.

Lemma 5.1.6 (Smooth extension along the curve). *The form P' extends smoothly to X if and only if the map $P' \circ \zeta' : (0, \infty) \rightarrow \Lambda^{n-1}T^*X$ extends smoothly to the interval $[0, \infty)$.*

Proof. As for the non-trivial implication, let $(0, \eta_0, x_0) \in X$ be a boundary point. Pick vectors $v_1, \dots, v_{n-1} \in \mathbb{R}^{n+1}$ so that the vectors $v_1, \dots, v_{n-1}, \eta_0, x_0$ are linearly independent, and define the set

$$U := \{(r, \eta, x) \in X \mid v_1, \dots, v_{n-1}, \eta, x \text{ are linearly independent}\}.$$

It is an open neighborhood of $(0, \eta_0, x_0)$ in X . Applying the Gram-Schmidt orthogonalization to $v_1, \dots, v_{n-1}, \eta, x$, we find a smooth map $R : U \rightarrow O(n+1)$ such that

$$R(r, \eta, x) \cdot (r, \eta, x) = (r, e_n, e_{n+1}) \quad \text{for all } (r, \eta, x) \in U.$$

The equation (5.6) implies

$$P'(r, \eta, x) = (-1)^R R(r, \eta, x)^* (P'(r, e_n, e_{n+1})) \quad \text{for all } (r, \eta, x) \in \text{Int}(U),$$

5. Explicit computations

where $R(r, \eta, x)^* : \Lambda^* T^* X \rightarrow \Lambda^* T^* X$ is the smooth cotangential map which is induced by the diffeomorphism $R(r, \eta, x) : X \rightarrow X$, and which maps the fiber over $z \in X$ to the fiber over $R(r, \eta, x)^{-1}z$. By the assumption, all maps in the composition are smooth in their arguments. The lemma follows. \square

Lemma 5.1.7 (Local expression at the boundary). *On the interval $(0, \infty)$, we have*

$$\tilde{P}' \circ \zeta' = (-1)^{n+1} (1 + \gamma^2)^{-\frac{n-1}{2}} \sum_{k=0}^{n-1} \gamma^{n-k} (h_k \circ \gamma) (\nu_k \circ \zeta'),$$

where the functions $h_k : (0, 1) \rightarrow \mathbb{R}$ are defined by

$$h_k(\gamma) := \int_{-1}^1 \frac{(u + \gamma^2)^k (u - 1)^{n-1-k}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du \quad \text{for all } \gamma \in (0, 1)$$

and the forms $\nu_k \in \Omega(X)$ are defined by

$$\nu_k(r, x, \eta) := \frac{1}{k!(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n-1}} (-1)^\sigma dx^{\sigma_1} \dots dx^{\sigma_k} d\eta^{\sigma_{k+1}} \dots d\eta^{\sigma_{n-1}}.$$

Proof. We start with the following formula from the proof of Proposition 5.1.2:

$$\tilde{P} = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} (x^i y^j - y^i x^j) \int_{[0,1]} \frac{dt \psi^*(dy^1 \dots \widehat{dy^i} \dots \widehat{dy^j} \dots dy^{n+1})}{|(1-t)y - tx|^{n+1}}.$$

We restrict to the points $(x, y) = \kappa(r, e_n, e_{n+1})$ with $r > 0$. There, we have

$$\begin{aligned} x^1 &= \dots = x^n = 0, \quad x^{n+1} = 1, \\ y^1 &= \dots = y^{n-1} = 0, \quad y^n = \frac{2\gamma}{1 + \gamma^2}, \quad y^{n+1} = \frac{1 - \gamma^2}{1 + \gamma^2}. \end{aligned}$$

Under the substitution $u = 2t - 1$, we get

$$|(1-t)y - tx|^2 = \frac{4t(t-1)}{1 + \gamma^2} + 1 = \frac{u^2 + \gamma^2}{1 + \gamma^2}.$$

5. Explicit computations

We make the following preliminary computations:

$$\begin{aligned} x^i y^j - y^i x^j &= 0 \quad \text{for } 1 \leq i \leq n-1 \text{ and } i < j \leq n+1, \\ x^n y^{n+1} - y^n x^{n+1} &= -\frac{2\gamma}{1+\gamma^2}, \\ \kappa^*(dy^i) &= \frac{1}{1+\gamma^2} ((1-\gamma^2) dx^i + 2\gamma d\eta^i) \quad \text{for } 1 \leq i \leq n-1. \end{aligned}$$

We plug these in the formula for \tilde{G} and get

$$\begin{aligned} \tilde{P}'(\zeta'(r)) &= 2\gamma(1+\gamma^2)^{\frac{n-1}{2}} \int_{[0,1]} dt \frac{\prod_{i=1}^{n-1} ((1-t)\kappa^*(dy^i) - t dx^i)}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} \\ &= (-1)^{n+1} \gamma(1+\gamma^2)^{-\frac{n-1}{2}} \int_{[-1,1]} du \frac{\prod_{i=1}^{n-1} ((u+\gamma^2) dx^i + \gamma(u-1) d\eta^i)}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} \\ &= (-1)^{n+1} (1+\gamma^2)^{-\frac{n-1}{2}} \sum_{k=0}^{n-1} \gamma^{n-k} \left(\int_{-1}^1 \frac{(u+\gamma^2)^k (u-1)^{n-1-k}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du \right) \nu_k. \end{aligned}$$

The lemma follows. \square

Lemma 5.1.8 (Integrals depending on parameter). *Let $n \in \mathbb{N}$, and let $l = 0, 1, \dots, n-1$. The function $F_{n,l} : (0, \infty) \rightarrow \mathbb{R}$ defined by*

$$F_{n,l}(t) := \int_{-1}^1 \frac{t^{n-l} u^l}{(u^2 + t^2)^{\frac{n+1}{2}}} du \quad \text{for all } t \in (0, \infty) \quad (5.7)$$

extends smoothly to \mathbb{R} .

Proof. We have

$$F_{1,0}(t) = 2 \arctan\left(\frac{1}{t}\right) = \pi - 2 \arctan(t) \quad \text{for all } t \in (0, \infty).$$

The right-hand side is a smooth function on \mathbb{R} .

For $n \geq 2$, we deduce the recursive formula

$$F_{n,0}(t) = \frac{1}{n-1} \left((n-2) F_{n-2,0}(t) + \frac{2t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}} \right).$$

If l is odd, then $F_{n,l} \equiv 0$ for all n because the integrand of (5.7) is odd as a function of u .

5. Explicit computations

For $n \geq 3$ and even $2 \leq l \leq n-1$, we deduce yet another recursive formula

$$F_{n,l}(t) = \frac{1}{n-l} \left((l-1)F_{n,l-2}(t) - \frac{2t^{n-l}}{(1+t^2)^{\frac{n-1}{2}}} \right).$$

The claim for all $F_{n,l}$ follows by induction. \square

Proposition 5.1.9 (Smooth extension to the boundary). *The form P from (5.2) extends smoothly to $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$.*

Proof. According to Lemmas 5.1.5 and 5.1.6, it suffices to show that the curve $P' \circ \zeta' : (0, \infty) \rightarrow \Lambda^{n-1}T^*X$ extends smoothly to $[0, \infty)$. Lemma 5.1.7 gives an expression for $P' \circ \zeta'$ as a linear combination of smooth forms $\nu_k \in \Omega^{n-1}(X)$ with coefficients $\gamma^{n-k}(h_k \circ \gamma)$ for $k = 0, \dots, n-1$ multiplied by the overall coefficient $(-1)^n(1+\gamma^2)^{-\frac{n-1}{2}}$. We expand

$$\gamma^{n-k}(h_k \circ \gamma) = \sum_{a=0}^k \sum_{b=0}^{n-1-k} (-1)^{n-1-k-b} \binom{k}{a} \binom{n-1-k}{b} \int_{-1}^1 \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du$$

and notice that we can write

$$\int_{-1}^1 \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du = \gamma^{k-a+b} (F_{n,a+b} \circ \gamma)$$

for the function $F_{n,l}$ from (5.7) with $l := a+b$. Because $0 \leq l \leq n-1$, Lemma 5.1.8 asserts that $F_{n,l}$ extends smoothly to $[0, \infty)$. Because $k-a+b \geq 0$, the entire coefficient at ν_k extends smoothly to $[0, \infty)$ for every $k = 0, \dots, n-1$. The lemma follows. \square

We summarize our results in the following proposition:

Proposition 5.1.10 (for \mathbb{S}^n). *The form P from (5.2) defines a for \mathbb{S}^n satisfying Definition 4.2.4. Moreover, we have the symmetries*

$$\begin{aligned} R^*P &= (-1)^R P \quad \text{for all } R \in O(n+1) \text{ and} \\ \tau^*P &= (-1)^n P. \end{aligned}$$

Proof. The proposition is a summary of Propositions 5.1.2, 5.1.4 and 5.1.9. \square

Remark 5.1.11 (Better notation due to R. Bryant, see [Bry18]). Pick an oriented basis e_1, \dots, e_{n+1} of \mathbb{R}^{n+1} as generators of the exterior algebra $\Lambda^*(\mathbb{R}^{n+1})$, and view x, y, dx, dy as $\Lambda^*(\mathbb{R}^{n+1})$ -valued forms on \mathbb{R}^{n+1} . For example, we view x as the map $x \in \mathbb{R}^{n+1} \mapsto \sum_{i=1}^{n+1} x^i e_i \in \Lambda^1(\mathbb{R}^{n+1})$ and dx as the map $x \in \mathbb{R}^{n+1} \mapsto \sum_{i=1}^{n+1} (dx_i)_x e_i \in \Lambda^1(\mathbb{R}^{n+1})$. There

5. Explicit computations

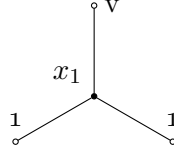


Figure 5.3.: The Y-graph for \mathbb{S}^n .

is a natural wedge product on the space of $\Lambda^*(\mathbb{R}^{n+1})$ -valued forms. If ω is a top-form, we denote by $[\omega]$ the coefficient of ω at $e_1 \wedge \cdots \wedge e_{n+1}$. Then it holds

$$\omega_k(x, y) = \frac{1}{k!} \frac{1}{(n-1-k)!} [x \wedge y \wedge (dx)^k \wedge (dy)^{n-1-k}].$$

Note that if we view e_i as odd variables, then $[\cdot]$ corresponds to the odd integration $\int De(\cdot)$. It would be interesting to know whether this notation simplifies some proofs, especially if Lemma 5.2.1 can be deduced from abstract algebraic facts or rules valid for odd integration. \triangleleft

5.2. Computation of Chern-Simons Maurer-Cartan element for spheres

We recall from Definition 4.3.6 that the Chern-Simons Maurer-Cartan element \mathbf{n} is computed as a sum over trivalent ribbon graphs decorated with the admissible Hodge Propagator P at internal edges, integration variables x_i at internal vertices and, in the case of \mathbb{S}^n , with 1 or v at external vertices.

The canonical Maurer-Cartan element \mathbf{m} is the contribution of the Y-graph (see Figure 5.3), and it is easy to see that

$$\mathbf{m}_{10}(\text{sv}11) = (-1)^n \mathbf{m}_{10}(\text{s}1\text{v}1) = \mathbf{m}_{10}(\text{s}11\text{v}) = (-1)^{n-2}.$$

Throughout this section, we will be in the setting of Definition 4.3.6. In particular, $\Gamma \in \text{RG}_{klg}^{(3)}$ is a ribbon graph, L its compatible labeling admissible with respect to an input $\omega_1, \dots, \omega_l$ and $I(\sigma_L)$ the corresponding integral.

Lemma 5.2.1 (Condition (V_1) holds). *For the Hodge Propagator P from (5.2) for \mathbb{S}^n , every graph $\Gamma \neq Y$ with 1 at an external vertex vanishes.*

Proof. The only contribution of an A -vertex which does not vanish for degree reasons is

$$A_{v,1}(y) = \int_x P(x, y) \text{Vol}(x).$$

5. Explicit computations

From the symmetry of P and Vol under the action of $O(n+1)$, we get

$$R^* A_{v,1} = (-1)^R A_{v,1} \quad \text{for all } R \in O(n+1).$$

Therefore, it suffices to check that $A_{v,1}(e_1) = 0$, where e_1, \dots, e_{n+1} denotes the standard basis of \mathbb{R}^{n+1} . Evaluation of (5.4) at (x, e_1) gives

$$\omega_0(x, e_1) = \frac{1}{(n-1)!} \sum_{\substack{\sigma \in \mathbb{S}_{n+1} \\ \sigma_2=1}} (-1)^\sigma x^{\sigma_1} dy^{\sigma_3} \dots dy^{\sigma_{n+1}}.$$

Therefore, we get

$$\begin{aligned} A_{v,1}(e_1) &= (-1)^n \int_x g_0(x \cdot e_1) \omega_0(x, e_1) \text{Vol}(x) \\ &= \sum_{j=2}^{n+1} (-1)^{n+j+1} \left(\int_x g_0(x^1) x^j \text{Vol}(x) \right) dy^2 \dots \widehat{dy^j} \dots dy^{n+1}, \end{aligned}$$

where we view g_0 as a function of $x \cdot y$. For every $j = 2, \dots, n+1$, consider the orientation reversing diffeomorphism

$$\begin{aligned} I_j : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ (x^1, \dots, x^{n+1}) &\longmapsto (x^1, \dots, -x^j, \dots, x^{n+1}). \end{aligned}$$

Then we have

$$\int_x g_0(x^1) x^j \text{Vol}(x) = - \int_x I_j^* (g_0(x^1) x^j \text{Vol}(x)) = - \int_x g_0(x^1) (-x^j) (-\text{Vol}(x)),$$

and it follows that $A_{v,1}(e_1) = 0$.

Let us now consider the contribution of a B -vertex with 1:

$$B_1(y, z) = \int_x P(y, x) P(x, z) = (-1)^n \int_x P(y, x) P(x, z).$$

For $n = 1$, the degree of $P(y, x) P(x, z)$ is 0, and hence $B_1 = 0$ trivially. Suppose that $n \geq 2$. As in the case of $A_{v,1}$, we get that

$$R^* B_1 = (-1)^R B_1 \quad \text{for all } R \in O(n+1).$$

5. Explicit computations

Therefore, it suffices to check that $B_1(e_1, c_1 e_1 + c_2 e_2) = 0$ for all $(c_1, c_2) \in \mathbb{S}^1$. We have

$$B_1(e_1, c_1 e_1 + c_2 e_2) = (-1)^n \sum_{a=1}^{n-1} \int_x g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) \omega_a(x, e_1) \omega_{n-a}(x, c_1 e_1 + c_2 e_2).$$

We will show that for every $a = 1, \dots, n-1$ we can write

$$\mu_a(x) := \omega_a(x, e_1) \omega_{n-a}(x, c_1 e_1 + c_2 e_2) = \left(\sum_{i=3}^{n+1} \pm x^i \text{Vol}(x) \right) \eta_a(y, z) \quad (5.8)$$

for some form $\eta_a(y, z)$. Then, using the same argument as for $A_{v,1}$, we will have

$$\begin{aligned} & \int_x g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) x^i \text{Vol}(x) \\ &= - \int_x I_i^*(g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) x^i \text{Vol}(x)) \\ &= - \int_x g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) (-x^i) (-\text{Vol}(x)) \end{aligned}$$

for all $3 \leq i \leq n+1$, and hence $B_1(e_1, c_1 e_1 + c_2 e_2) = 0$.

In order to show (5.8), we have to study the product of ω_i 's. From (5.4) we get

$$\begin{aligned} & \omega_a(x, y) \omega_{n-a}(x, z) \\ &= \frac{1}{a!(n-1-a)!(n-a)!(a-1)!} \sum_{\sigma, \mu \in \mathbb{S}_{n+1}} (-1)^{\sigma+\mu} x^{\sigma_1} x^{\mu_1} y^{\sigma_2} z^{\mu_2} \\ & \quad dx^{\sigma_3} \dots dx^{\sigma_{2+a}} dx^{\mu_3} \dots dx^{\mu_{2+n-a}} dy^{\sigma_{3+a}} \dots dy^{\sigma_{n+1}} \\ & \quad dz^{\mu_{3+n-a}} \dots dz^{\mu_{n+1}}. \end{aligned} \quad (5.9)$$

In order to simplify this expression, we decompose $\sigma \in \mathbb{S}_{n+1}$ as

$$\sigma = \sigma^5 \circ \sigma^4 \circ \sigma^3 \circ \sigma^2 \circ \sigma^1, \quad (5.10)$$

where $\sigma^1, \dots, \sigma^5 \in \mathbb{S}_{n+1}$ are permutations defined as follows:

- The permutation σ^1 is a shuffle permutation $\sigma^1 \in \mathbb{S}_{2+a, n-a-1}$ such that its first block denoted by $\sigma^1(1) = (\sigma_1^1, \dots, \sigma_{2+a}^1)$ is equal to the ordered set $\{\sigma_1, \dots, \sigma_{2+a}\}$. The second block $\sigma^1(2)$ is then the ordered set $\{\sigma_{3+a}, \dots, \sigma_{n+1}\}$, which will be denoted by J_σ .
- The permutation σ^2 acts on the block $\sigma^1(1)$ by moving σ_2 in front. We denote the

5. Explicit computations

new block $\sigma^1(1) \setminus \{\sigma_2\}$ by I_σ , so that we can write $\sigma^2 : \sigma^1(1) \mapsto (\sigma_2, I_\sigma)$.

- The permutation σ^3 acts on the block I_σ by moving σ_1 in front. Together with the previous step we get $\sigma^1(1) \mapsto (\sigma_2, \sigma_1, I_\sigma \setminus \{\sigma_1\})$.
- The permutation σ^4 is a transposition of σ_1 and σ_2 .
- The permutation σ^5 is determined by the pair $(\sigma^{51}, \sigma^{52}) \in \mathbb{S}_a \times \mathbb{S}_{n-1-a}$ of permutations σ^{51} and σ^{52} acting on blocks $I_\sigma \setminus \{\sigma_1\}$ and J_σ to get $(\sigma_3, \dots, \sigma_{2+a})$ and $(\sigma_{3+a}, \dots, \sigma_{n+1})$, respectively.

We define the decomposition μ^1, \dots, μ^5 for $\mu \in \mathbb{S}_{n+1}$ from (5.9) analogously with a replaced by $n - a$. Using (5.10), the product (5.9) can be written as

$$\begin{aligned} & \frac{1}{a!(n-1-a)!(n-a)!(a-1)!} \sum_{\substack{\sigma^1, \dots, \sigma^5 \\ \mu^1, \dots, \mu^5}} (-1)^{\sigma^1 + \dots + \sigma^5 + \mu^1 + \dots + \mu^5} x^{\sigma^1} x^{\mu^1} y^{\sigma^2} z^{\mu^2} \\ & \quad dx^{\sigma^{51}(I_\sigma \setminus \{\sigma_1\})} dx^{\mu^{51}(I_\mu \setminus \{\mu_1\})} dy^{\sigma^{52}(J_\sigma)} dz^{\mu^{52}(J_\mu)} \\ & = - \sum_{\sigma^1, \mu^1} (-1)^{\sigma^1 + \mu^1} \left(\sum_{\sigma^2, \mu^2} (-1)^{\sigma^2 + \mu^2} \sum_{\sigma^3, \mu^3} (-1)^{\sigma^3 + \mu^3} x^{\sigma^1} x^{\mu^1} y^{\sigma^2} z^{\mu^2} \right. \\ & \quad \left. dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}} \right) dy^{J_\sigma} dz^{J_\mu}, \end{aligned}$$

where -1 comes from $(-1)^{\sigma^4}$ and σ^5 is compensated by permutations of forms. For fixed σ^1 and μ^1 , consider the coefficient at $dy^{J_\sigma} dz^{J_\mu}$ in the brackets. If we evaluate it at $y = e_1, z = c_1 e_1 + c_2 e_2$, we get

$$\begin{aligned} & \overbrace{\sum_{\substack{\sigma^3, \mu^3 \\ \sigma_2=1 \\ \mu_2=1}} (-1)^{\sigma^3 + \mu^3} x^{\sigma^1} x^{\mu^1} dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}}} =: \text{I} \\ & + (-1)^{\mu^2} c_2 \overbrace{\sum_{\substack{\sigma^3, \mu^3 \\ \sigma_2=1 \\ \mu_2=2}} (-1)^{\sigma^3 + \mu^3} x^{\sigma^1} x^{\mu^1} dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}}} =: \text{II}, \end{aligned}$$

where $(-1)^{\mu^2} = -1$ if and only if $1 \in I_\mu$.

More generally, for multiindices $I_1, I_2 \subset \{1, \dots, n+1\}$ of lengths $a+1$ and $n-a+1$,

5. Explicit computations

respectively, consider the sum

$$S(I_1, I_2) := \sum_{\substack{i_1 \in I_1 \\ i_2 \in I_2}} (-1)^{(i_1, I_1) + (i_2, I_2)} x^{i_1} x^{i_2} dx^{I_1 \setminus \{i_1\}} dx^{I_2 \setminus \{i_2\}}, \quad (5.11)$$

where (i_j, I_j) is the number of transpositions required to move i_j in front of I_j . The following implication holds:

$$S(I_1, I_2) \neq 0 \implies 1 \leq |I_1 \cap I_2| \leq 2.$$

We distinguish the two cases left:

Case $I_1 \cap I_2 = \{i, j\}$ with $i < j$: We get

$$\begin{aligned} S(I_1, I_2) &= (-1)^{(i, I_1) + (j, I_2)} x^i x^j dx^{I_1 \setminus \{i\}} dx^{I_2 \setminus \{j\}} \\ &\quad + (-1)^{(j, I_1) + (i, I_2)} x^j x^i dx^{I_1 \setminus \{j\}} dx^{I_2 \setminus \{i\}} \\ &= (-1)^{(i, I_1) + (j, I_2) + (j, I_1) + 1 + (i, I_2)} x^i x^j dx^j dx^{I_1 \setminus \{i, j\}} \\ &\quad dx^i dx^{I_2 \setminus \{i, j\}} + (-1)^{(j, I_1) + (i, I_2) + (i, I_1) + (j, I_2) + 1} x^i x^j \\ &\quad dx^i dx^{I_1 \setminus \{i, j\}} dx^j dx^{I_2 \setminus \{i, j\}} \\ &= \pm(-1 + 1) x^i x^j dx^i dx^{I_1 \setminus \{i, j\}} dx^j dx^{I_2 \setminus \{i, j\}}, \end{aligned}$$

where in the last step we switched $dx^i \leftrightarrow dx^j$ in the first summand. Therefore, it holds $S(I_1, I_2) = 0$.

Case $I_1 \cap I_2 = \{i\}$: We must have $I_1 \cup I_2 = \{1, \dots, n+1\}$. A non-zero summand in (5.11) has either $i_1 = i$ and $i_2 \in I_2$, in which case

$$I_1 \setminus \{i_1\} \cup I_2 \setminus \{i_2\} = \{1, \dots, \widehat{i_2}, \dots, n+1\},$$

or $i_2 = i$ and $i_1 \in I_1$ with $i_1 \neq i$, in which case

$$I_1 \setminus \{i_1\} \cup I_2 \setminus \{i_2\} = \{1, \dots, \widehat{i_1}, \dots, n+1\}.$$

Indices i_2 from the first case and i_1 from the second case constitute $\{1, \dots, n+1\}$. Therefore, for some signs \pm , we can write

$$S(I_1, I_2) = x^i \sum_{j=1}^{n+1} \pm x^j dx^1 \dots \widehat{dx^j} \dots dx^{n+1}.$$

5. Explicit computations

We will prove that the signs alternate, and hence $S(I_1, I_2) = \pm x^i \text{Vol}(x)$. Suppose that $j, j+1 \in I_1$ for some $j \in \{1, \dots, n\}$. The two summands in (5.11) with $(i_1, i_2) = (j, i)$ and $(i_1, i_2) = (j+1, i)$, respectively, give

$$\begin{aligned} & (-1)^{(j, I_1) + (i, I_2)} x^j x^i \, dx^{I_1 \setminus \{j\}} \, dx^{I_2 \setminus \{i\}} + (-1)^{(j+1, I_1) + (i, I_2)} x^{j+1} x^i \\ & \quad \quad \quad dx^{I_1 \setminus \{j+1\}} \, dx^{I_2 \setminus \{i\}} \\ &= (-1)^{(i, I_2)} x^j x^i \, dx^{j+1} \, dx^{I_1 \setminus \{j, j+1\}} \, dx^{I_2 \setminus \{i\}} \\ & \quad + (-1)^{1 + (i, I_2)} x^{j+1} x^i \, dx^j \, dx^{I_1 \setminus \{j, j+1\}} \, dx^{I_2 \setminus \{i\}} \\ &= (-1)^{(i, I_2)} x^i (x^j \, dx^{j+1} - x^{j+1} \, dx^j) \, dx^{I_1 \setminus \{j, j+1\}} \, dx^{I_2 \setminus \{i\}}. \end{aligned}$$

The signs clearly alternate. A symmetric argument holds when $j, j+1 \in I_2$. Now assume that $j \in I_1$ and $j+1 \in I_2$. The two summands in (5.11) which have $(i_1, i_2) = (j, i)$ and $(i_1, i_2) = (i, j+1)$, respectively, give

$$\begin{aligned} & (-1)^{(j, I_1) + (i, I_2)} x^j x^i \, dx^{I_1 \setminus \{j\}} \, dx^{I_2 \setminus \{i\}} + (-1)^{(i, I_1) + (j+1, I_2)} x^i x^{j+1} \\ & \quad \quad \quad dx^{I_1 \setminus \{i\}} \, dx^{I_2 \setminus \{j+1\}} \\ &= (-1)^{(j, I_1) + (i, I_1 \setminus \{j\}) + (a+1)} x^j x^i \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2} \\ & \quad + (-1)^{(i, I_1) + (j+1, I_2) + (j, I_1 \setminus \{i\})} x^i x^{j+1} \, dx^j \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}} \\ &= (-1)^{(j, I_1) + (i, I_1 \setminus \{j\}) + (j+1, I_2)} x^i x^j \, dx^{j+1} \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}} \\ & \quad + (-1)^{1 + (j, I_1) + (i, I_1 \setminus \{j\}) + (j+1, I_2)} x^i x^{j+1} \, dx^j \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}} \\ &= (-1)^{(j, I_1) + (i, I_1 \setminus \{j\}) + (j+1, I_2)} x^i (x^j \, dx^{j+1} - x^{j+1} \, dx^j) \\ & \quad \quad \quad dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}}. \end{aligned}$$

The signs alternate again. A symmetric argument holds for $j \in I_2$ and $j+1 \in I_1$.

Back to the original problem, we have $I = S(I_\sigma, I_\mu)$ with $I_\sigma, I_\mu \subset \{2, \dots, n+1\}$. It follows that the first case applies, and hence $I = 0$. We have $II = S(I_\sigma, I_\mu)$ with $I_\sigma \subset \{2, \dots, n+1\}$ and $I_\mu \subset \{1, \widehat{2}, \dots, n+1\}$. It follows that either the first case or the second case with $i \geq 3$ applies. This proves (5.8). Consequently, we get $B_1 = 0$ also for $n \geq 2$.

The last paragraph of the proof of Proposition 4.4.2 finishes the proof. \square

We summarize the consequences in the following proposition. The main argument is the same as in the proof of Proposition (4.4.5).

Proposition 5.2.2 (Vanishing of graphs for \mathbb{S}^n). *Consider \mathbb{S}^n with the Hodge Propagator (5.2). Only the following trivalent ribbon graphs $\Gamma \neq Y$ do not necessarily vanish:*

5. Explicit computations

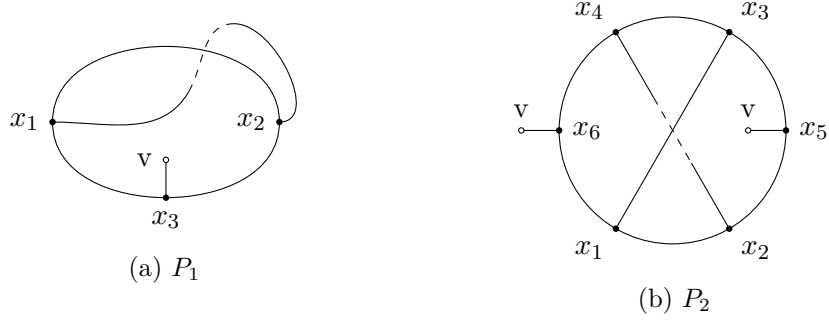


Figure 5.4.: Vanishing graphs P_1 and P_2 for \mathbb{S}^2 .

$(n = 1)$: The O_k -graph with $k \in 2\mathbb{N}$ internal vertices of type B with v at the external vertex (see Figure 5.7).

$(n = 2)$: It must hold $A = 0$, $C = 2B$ and all B vertices must have v at the external vertex. Moreover, if Γ is reduced, it must have $g \geq 1$.

$(n = 3)$: There is no external vertex and $4 \mid C$ holds.

$(n > 3)$: All graphs vanish.

Proof. Lemma 5.2.1 implies that $A = 0$ and that the total form-degree D satisfies $D = nB$. Therefore, we get from (4.26) the following: for $n > 3$ there is neither a B -vertex nor a C -vertex; for $n = 3$, there is no B -vertex; for $n = 2$, we have $C = 2B$; and for $n = 1$, there is no C -vertex.

Consider the pullback of $I(\sigma_L)$ along the (multi)diagonal action of an $R \in O(n+1)$ with $\det(R) = -1$ on $(\mathbb{S}^n)^{\times k}$. We get schematically

$$\int_{(\mathbb{S}^n)^{\times k}} \text{P}^e \text{Vol}^s = (-1)^{k+e+s} \int_{(\mathbb{S}^n)^{\times k}} G^e \text{Vol}^s.$$

Therefore, $k + e + s$ has to be even. If we plug-in from (4.23), we get

$$k + e + s = \begin{cases} 3B & \text{for } n = 1, \\ 8B & \text{for } n = 2, \\ \frac{5}{2}C & \text{for } n = 3. \end{cases}$$

A non-vanishing reduced graph must have $B \geq l$. For $n = 2$, so that $C = 2B$, the formula (4.24) gives $g \geq 1$. \square

5. Explicit computations

Remark 5.2.3 (Graphs for \mathbb{S}^2). The simplest possibly non-vanishing graph for \mathbb{S}^2 has $A = 0$, $B = 1$, $C = 2$. If it is reduced, we must have $l = g = 1$, and hence it will contribute to \mathfrak{n}_{11} . Up to an isomorphism, there is only one such graph, which we denote by P_1 (see Figure 5.4). However, we see that the pair of internal vertices x_1 and x_2 is connected by two edges, which implies that $P_1 = 0$. Indeed, $P(x, y)$ has odd degree, and hence we have¹

$$P(x, y)P(y, x) = P(x, y)^2 = 0$$

by the symmetry on the pullback along the twist map. It follows that $\mathfrak{n}_{11} = 0$.

The second simplest possibly non-vanishing reduced graph is the graph P_2 from Figure 5.4. Let

$$\begin{aligned} \eta(x_1, x_2, x_3, x_4, x_5) &:= P(x_1, x_2)P(x_1, x_3)P(x_4, x_2)P(x_4, x_3)P(x_3, x_5) \\ &\quad P(x_2, x_5)\text{Vol}(x_5) \end{aligned}$$

denote the form in the integrand coming from the part of the graph on the right-hand side of the vertical axis going through x_1, x_4 . If $\tau_{1,4}$ denotes the exchange of x_1 and x_4 , then clearly $\tau_{1,4}^*\eta = \eta$ because the graph is symmetric with respect to the horizontal axis going through x_5, x_6 . Using this, we compute

$$\begin{aligned} &\int_{x_1, x_2, x_3, x_4, x_5, x_6} v(x_6)P(x_1, x_6)P(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5) \\ &= \int_{\tau_{1,4}(x_1, x_2, x_3, x_4, x_5, x_6)} \tau_{1,4}^*(v(x_6)P(x_1, x_6)P(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5)) \\ &= \int_{x_4, x_2, x_3, x_1, x_5, x_6} v(x_6)P(x_4, x_6)P(x_1, x_6)\eta(x_4, x_2, x_3, x_1, x_5) \\ &= - \int_{x_1, x_2, x_3, x_4, x_5, x_6} v(x_6)P(x_1, x_6)P(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5), \end{aligned}$$

where the minus sign comes from switching the first two P 's. We see that P_2 vanishes. The other variants with x_5 moved on the edge x_3, x_4 and x_2, x_4 vanish by a similar argument using the compositions $\tau_{1,3} \circ \tau_{5,6}$ and $\tau_{1,2} \circ \tau_{5,6}$, respectively. We conclude that $\mathfrak{n}_{21} = 0$, and hence $\mathfrak{q}_{121}^{\mathfrak{n}} = 0$.

We sum up some general observations about the integrals for \mathbb{S}^2 :

- We have $B_1 \neq 0$ and $C \neq 0$ for the corresponding forms.

¹We recall from Section 4.3 that the notation $P(x_i, x_j)$ means $(\pi_i \times \pi_j)^*P$ and not just the evaluation at (x_i, x_j) .

5. Explicit computations

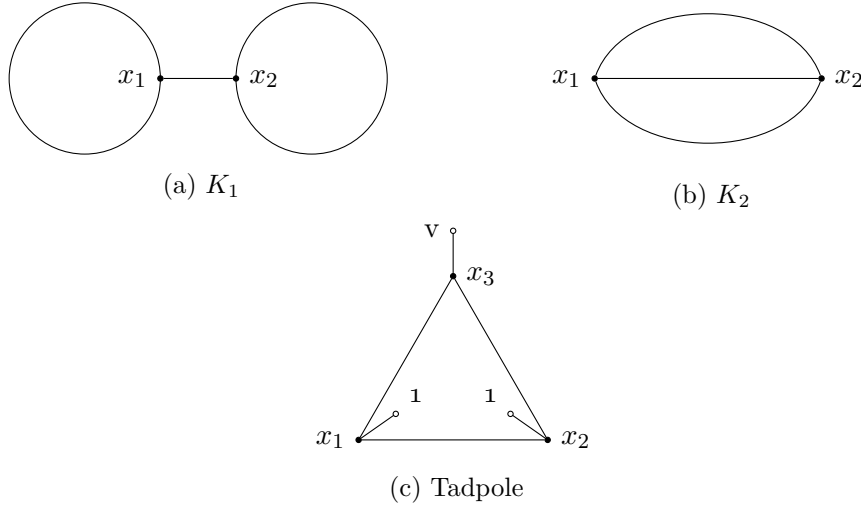


Figure 5.5.: Graphs K_1 and K_2 from the Chern-Simons theory and the tadpole graph with $(l, g) = (2, 0)$ for $n = 3$.

- We have the multiplication formula (c.f., Example 5.1.3)

$$\omega_1(x, y)\omega_1(x, z) = x \cdot (y \times z)\text{Vol}(x).$$

- The number $(-1)^{\sigma_L}I(\sigma_L)$ does not depend on the choice of L_1 provided a compatible L_2 is chosen.
- It holds $\sum_{L_3^b}(-1)^{\sigma_L}I(\sigma_L) = 0$ whenever there is a boundary component with even number of v 's.
- If there is a B -vertex x such that the underlying graph (after forgetting the ribbon structure) is symmetric on the reflection along an axis going through x , then $I(\sigma_L) = 0$. \triangleleft

Remark 5.2.4 (Graphs for \mathbb{S}^3). For \mathbb{S}^3 , we consider the non-reduced graphs K_1 and K_2 and the tadpole graph from Figure 5.5. The graphs K_1 and K_2 appear in the definition of the Chern-Simons topological invariant in [Koh02] (with a gauge group). The corresponding integrals from our theory vanish “algebraically”, i.e., at the level of wedge products of ω_i . Indeed, every summand in K_1 contains

$$\omega_a(x_1, x_1) = 0 \quad \text{for some } a = 0, 1, 2,$$

5. Explicit computations

and, for degree reasons, the form part of K_2 can contain only

$$\omega_1(x_1, x_2)^3 = 0 \quad \text{or} \quad \omega_0(x_1, x_2)\omega_1(x_1, x_2)\omega_2(x_1, x_2) = 0.$$

The tadpole graph contains only

$$\omega_2(x_1, x_3)\omega_1(x_1, x_2)\omega_2(x_2, x_3) = 0. \quad \triangleleft$$

Equations in Remarks 5.2.3 and 5.2.4 were checked by the computer. The program for Wolfram Mathematica 10.4 will be made available at [Háj18].

We will now compute n_{20} for \mathbb{S}^1 , which according to Proposition 5.2.2 consists only of contributions from the O_k -graphs with k even. By analogy with the finite dimensional case (see Appendix A), we expect that the number $(-1)^{\sigma_L} I(\sigma_L)$ does not depend on L . All inputs are namely the same and the degrees even, i.e., $|m_2^+| = -2$, $|\theta^2 P| = -2$ and $|v| = 0$.

We fix $s_1, s_2 \geq 1$ such that $k = s_1 + s_2$ is even and make the ansatz

$$n_{20}(\text{sv}^{s_1} \otimes \text{sv}^{s_2}) := \varepsilon(s_1, s_2) C(s_1, s_2) I(k),$$

where $I(k)$ is the integral

$$\frac{1}{V^k} \int_{x_1, \dots, x_k} G(x_1, x_2) \cdots G(x_{k-1}, x_k) G(x_k, x_1) \text{Vol}(x_1) \cdots \text{Vol}(x_k), \quad (5.12)$$

$\varepsilon(s_1, s_2)$ a sign and $C(s_1, s_2)$ a combinatorial coefficient to be determined.

We fix a circle in the plane with k points (= internal vertices) and denote by $O(s_1, s_2)$ the set of ribbon graphs constructed by attaching external legs from which s_1 points in the interior and s_2 in the exterior, or the other way round, so that $O(s_1, s_2) = O(s_2, s_1)$ (see Figure 5.7). Recall that the ribbon structure is induced from the counterclockwise orientation of the plane. It is easy to see that all graphs in $O(s_1, s_2)$ admit a labeling which is admissible with respect to $\text{sv}^{s_1} \otimes \text{sv}^{s_2}$, and that $O(s_1, s_2)$ contains a representative of every such O_k -graph.

Lemma 5.2.5 (Integral for the O_k -graph for \mathbb{S}^1). *For every even $k \geq 2$, the integral $I(k)$ is equal to*

$$(-1)^{k+1} \frac{1}{2^k} \sum_{i=2,4,\dots,k} \frac{i}{(i+1)!} \sum_{\substack{i_1+\dots+i_r=k-i \\ i_1,\dots,i_r \in 2\mathbb{N}, r \in \mathbb{N}}} (-1)^r \frac{1}{(i_1+1)! \cdots (i_r+1)!}. \quad (5.13)$$

5. Explicit computations

Proof. Denote $\bar{P}(x, y) := -2\pi P$. For all $k, l \geq 1$, we consider the more general integral

$$I(k, l) := \int_{x_1, \dots, x_k} \bar{P}(x_1, x_2) \cdots \bar{P}(x_{k-1}, x_k) \bar{P}(x_k, x_1)^l \text{Vol}(x_1) \cdots \text{Vol}(x_k).$$

Taking the pullback along $(x_1, x_2, \dots, x_{k-1}, x_k) \mapsto (x_k, x_{k-1}, \dots, x_2, x_1)$ and using the antisymmetry of $\bar{P}(x, y)$, we get $I(k, l) = 0$ whenever $k + l$ is even. We will compute $I(k, 1)$ for $k \in 2\mathbb{N}$ from a recursive relation which arises from successive integration.

For the recursion step, we need to evaluate the integral

$$\int_y \bar{P}(x, y) \bar{P}(y, z)^l \text{Vol}(y)$$

for fixed $(x, z) \in (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta$. Pick the chart $g : \mathbb{S}^1 \setminus \{z\} \rightarrow (-\pi, \pi)$ defined by

$$g(y) = \bar{P}(y, z) = \pi - \alpha(y, z) \quad \text{for } y \in \mathbb{S}^1 \setminus \{z\},$$

where the angle α was defined in Example 5.1.3. It holds $dg(y) = \text{Vol}(y)$ and

$$\bar{P}(x, y) = \begin{cases} \bar{P}(x, z) - g(y) - \pi & \text{for } -\pi < g(y) < \bar{P}(x, z), \\ \bar{P}(x, z) - g(y) + \pi & \text{for } \bar{P}(x, z) < g(y) < \pi. \end{cases}$$

We compute

$$\begin{aligned} \int_y \bar{P}(x, y) \bar{P}(y, z)^l \text{Vol}(y) &= \int_{-\pi}^{\pi} (\bar{P}(x, z) - g) g^l dg - \pi \int_{-\pi}^{\bar{P}(x, z)} g^l dg \\ &\quad + \pi \int_{\bar{P}(x, z)}^{\pi} g^l dg \\ &= \frac{2\pi}{l+1} \begin{cases} \pi^l \bar{P}(x, z) - \bar{P}(x, z)^{l+1} & \text{for } l \text{ even,} \\ \frac{\pi^{l+1}}{l+2} - \bar{P}(x, z)^{l+1} & \text{for } l \text{ odd.} \end{cases} \end{aligned}$$

From now on, \int will stand for the Riemannian integral, i.e., $\int f := \int f \text{Vol}$ for a function f . We compute

$$\begin{aligned} I(2, l) &= \int_{x_1, x_2} \bar{P}(x_1, x_2) \bar{P}(x_2, x_1)^l = - \int_{yz} \bar{P}(y, z)^{l+1} = -2\pi \int_{-\pi}^{\pi} g^{l+1} dg \\ &= \begin{cases} 0 & \text{for } l \text{ even,} \\ -\frac{4\pi^{l+3}}{l+2} & \text{for } l \text{ odd.} \end{cases} \end{aligned}$$

5. Explicit computations

For $k \geq 4$ even and l odd, we compute

$$\begin{aligned}
I(k, l) &= \frac{2\pi}{l+1} \int_{x_1, \dots, x_{k-1}} \bar{P}(x_1, x_2) \cdots \bar{P}(x_{k-2}, x_{k-1}) \\
&\quad \left(\frac{\pi^{l+1}}{l+2} - \bar{P}(x_{k-1}, x_1)^{l+1} \right) \\
&= \frac{-4\pi^2}{(l+1)(l+2)} \int_{x_1, \dots, x_{k-2}} \bar{P}(x_1, x_2) \cdots \bar{P}(x_{k-3}, x_{k-2}) \\
&\quad \left(\pi^{l+1} \bar{P}(x_{k-2}, x_1) - \bar{P}(x_{k-2}, x_1)^{l+2} \right) \\
&= \frac{4\pi^2}{(l+1)(l+2)} (-\pi^{l+1} I(k-2, 1) + I(k-2, l+2)).
\end{aligned}$$

For the second equality, we used $\int_{x_1} \bar{P}(x_1, x_2) = 0$ to show that the term multiplied by $\frac{\pi^{l+1}}{l+2}$ vanishes. It follows that

$$\begin{aligned}
I(k, 1) &= \frac{(2\pi)^{k-2}}{(k-1)!} I(2, k-1) - \sum_{l=2,4,\dots,k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l, 1) \\
&= -\frac{k(2\pi^2)^k}{(k+1)!} - \sum_{l=2,4,\dots,k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l, 1) \quad \text{for all } k = 2, 4, \dots
\end{aligned}$$

This is a recursive equation of the form $a_k = c_k + \sum_{l=1}^{k-1} d_{k-l} a_l$. Its solution is $a_k = \sum_{i=1}^k c_i D_{k-i}$ with $D_0 := 1$ and $D_i = \sum d_{i_1} \cdots d_{i_r}$, where we sum over all $r = 1, \dots, i$ and $i_1, \dots, i_r \in \mathbb{N}$ such that $i_1 + \cdots + i_r = i$. Therefore, we get

$$I(k, 1) = -(2\pi^2)^k \sum_{i=2,4,\dots,k} \frac{i}{(i+1)!} \sum_{\substack{i_1+\dots+i_r=k-i \\ i_1,\dots,i_r \in 2\mathbb{N}, r \in \mathbb{N}}} (-1)^r \frac{1}{(i_1+1)! \cdots (i_r+1)!}.$$

The result has to be multiplied by $(-1)^k (2\pi)^{-2k}$ in order to get $I(k)$. \square

Lemma 5.2.6 (Independence of labeling). *The summand $(-1)^{\sigma_L} I(\sigma_L)$ in the definition of $\mathfrak{n}_{20}(\mathfrak{sv}^{s_1} \otimes \mathfrak{sv}^{s_2})$ for \mathbb{S}^1 is independent of the choice of $\Gamma \in O(s_1, s_2)$ and its labeling L which is compatible and admissible with respect to the input.*

Proof. Pick $\Gamma \in O(s_1, s_2)$ and its admissible labeling L . Let L' be an other admissible labeling of Γ . We distinguish the following situations:

- Suppose that L and L' differ by a permutation μ in L_3^b . A similar argument as in the proof of Lemma 4.3.7 shows that $(-1)^{\sigma_{L'}} = (-1)^\mu (-1)^{\sigma_L}$ and $I(\sigma_{L'}) = (-1)^\mu I(\sigma_L)$, where the sign in the integral comes from the permutation of Vol's, which have form-degree 1. Hence $(-1)^{\sigma_{L'}} I(\sigma_{L'}) = (-1)^{\sigma_L} I(\sigma_L)$.

5. Explicit computations

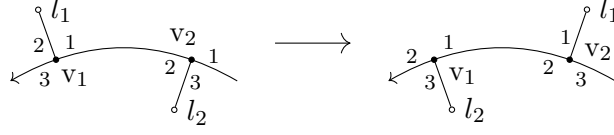


Figure 5.6.: Swapping adjacent legs.

- Suppose that the boundaries are permuted, i.e., that L and L' differ in L_1^b . Notice that $s_1 = s_2$ because otherwise one of L or L' would not be admissible. The sign from changing L_1^b cancels as in the previous case.
- Suppose that L and L' differ in L_2 . It was explained in the proof of Lemma 4.3.7 that a single change of L_2 induces the sign $(-1)^{n-1} = 1$ in $(-1)^{\sigma_L} I(\sigma_L)$.
- A cyclic permutation in L_3^v induces a sign neither in $(-1)^{\sigma_L}$ nor in $I(\sigma_L)$.
- A permutation μ in L_1^v induces $(-1)^\mu$ in $(-1)^{\sigma_L}$ and a change in $I(\sigma_L)$, which can be realized by taking the pullback along $\mu : (x_1, \dots, x_k) \mapsto (x_{\mu_1}, \dots, x_{\mu_k})$. However, the sign of the Jacobian is $(-1)^\mu$, which cancels the sign from $(-1)^{\sigma_L}$.

Next, we prove the independence of $\Gamma \in O(s_1, s_2)$. Let L be an admissible and compatible labeling of Γ . Pick two adjacent internal vertices with external legs pointing to different regions, i.e., one to the interior of the circle and the other to the exterior. Suppose that the vertices are labeled by v_1 and v_2 and the legs by l_1 and l_2 , respectively. Let $\Gamma' \in O(s_1, s_2)$ be the graph with the two legs turned inside out (see Figure 5.6). We can construct an admissible and compatible labeling L' of Γ' by making the following changes to L : The new leg at v_1 will be labeled by l_2 and the new leg at v_2 by l_1 . The cyclic orderings at v_1 and v_2 , respectively, have to be modified by a transposition in order to get compatibility with the new ribbon structure. All other labelings can be copied from L . In total, we get

$$(-1)^{\sigma_L - \sigma_{L'}} = -1.$$

This sign is compensated by swapping the one-forms in $I(\sigma_L)$:

$$\text{Vol}(x_{v_1}) \dots \text{Vol}(x_{v_2}) \longleftrightarrow \text{Vol}(x_{v_2}) \dots \text{Vol}(x_{v_1}).$$

The independence of $\Gamma \in O(s_1, s_2)$ follows from the fact that we can span the entire $O(s_1, s_2)$ by repeating the swap-of-legs operation. \square

5. Explicit computations

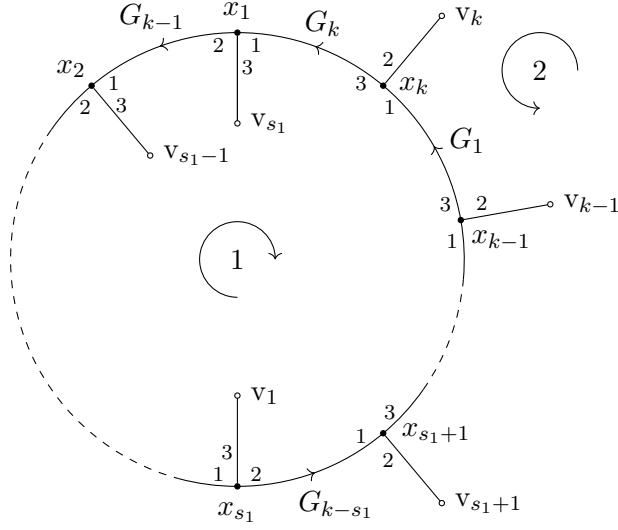


Figure 5.7.: The graph Γ^* with the labeling L^* . It can be checked that L_1 and L_2 are compatible.

Lemma 5.2.7 (Sign). *We have*

$$\varepsilon(s_1, s_2) = (-1)^{s_1+1}.$$

Proof. By Lemma 5.2.6, in order to compute $(-1)^{\sigma_L} I(\sigma_L)$, we can pick $\Gamma^* \in O(s_1, s_2)$ and its admissible and compatible labeling L^* from Figure 5.7. We abbreviate $\sigma_0 = \sigma_{L^*}$. The corresponding integral (4.20) reads

$$I(\sigma_0) = \frac{1}{V^k} \int_{x_1, \dots, x_k} G(x_{k-1}, x_k) \cdots G(x_1, x_2) G(x_k, x_1) \text{Vol}(x_{s_1}) \cdots \text{Vol}(x_1) \\ \text{Vol}(x_{s_1+1}) \cdots \text{Vol}(x_k).$$

It differs from $I(k)$ from (5.12) in the order of P's and Vol's. A reordering produces the sign

$$(-1)^{\frac{1}{2}s_1(s_1-1)}.$$

We will compute $(-1)^{\sigma_0}$ by ordering half-edges from the edge order back to the vertex order while looking at Figure 5.7. The steps are as follows:

- Transpose half-edges at internal vertices so that the first half-edge goes inside the vertex and the third outside with respect to the counterclockwise orientation. This gives $(-1)^a$.

5. Explicit computations

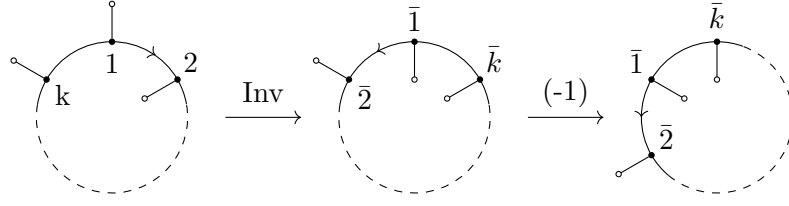


Figure 5.8.: The mirror isomorphism $M : 1 \dots k \mapsto \bar{k} \dots \bar{1}$ is a composition of the inversion and the counterclockwise rotation by one place.

- Permute external legs so that v_i is at x_i for all $i = 1, \dots, k$. This gives

$$(-1)^{\frac{1}{2}s_1(s_1-1)}.$$

- Permute internal edges so that P_i starts at the third half-edge of x_i and ends at the first half-edge of x_{i+1} for all $i = 1, \dots, k-1$. This does not produce any sign as swapping of two P 's requires two transpositions.
- At this point, we have the permutation

$$\begin{pmatrix} 1 & 2 & \dots & 2(e-1)-1 & 2(e-1) & 2e-1 & 2e & 2e+1 & \dots & 3k \\ 3 & 4 & \dots & 3k-3 & 3k-2 & 3k & 1 & 2 & \dots & 3k-1 \end{pmatrix}.$$

We interpret the last line as $P_1 \dots P_k v_1 \dots v_k$ and permute it to the sequence $v_1 P_1 v_2 P_2 \dots v_k P_k$, which does not produce any sign. We end up with

$$\sigma'_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & 3k-1 & 3k \\ 2 & 3 & 4 & \dots & 3k & 1 \end{pmatrix}.$$

It is now easy to see that

$$(-1)^{\sigma'_0} = (-1)^{3k-1}.$$

In total, we get

$$(-1)^{\sigma_0} = (-1)^{s_1 + \frac{1}{2}s_1(s_1-1) + k + 1}.$$

As for the other signs in Definition 4.3.6, we have $s(k, l) = k + \frac{1}{2}k(k-1)$ and $P(v^k) = \frac{1}{2}k(k-1)$. There is no sign from $s^2 v^{s_1} \otimes v^{s_2} = s v^{s_1} \otimes s v^{s_2}$ since $|s| = -2$. Multiplying everything together, we get $\varepsilon(s_1, s_2)$. \square

5. Explicit computations

Lemma 5.2.8 (Combinatorial coefficient). *It holds*

$$C(s_1, s_2) = \frac{1}{2}ak! \binom{k-1}{s_1}.$$

Proof. Every isomorphism of ribbon graphs Γ and Γ' from $O(s_1, s_2)$ is a composition of the clockwise rotation (r) for $r \in \mathbb{Z}$ and the mirror operation M defined as follows: If $1, \dots, k$ label internal vertices in the clockwise direction starting from the north-pole, then the result of M is $\bar{k}, \dots, \bar{1}$, where \bar{i} means that the external leg is reversed (see Figure 5.8). These operations satisfy

$$(r+k) = (r), (r)(-r) = \mathbb{1}, M^2 = \mathbb{1}, (r)M = M(-r),$$

and hence generate a group G which is isomorphic to the dihedral group $\mathbb{Z}_k \rtimes \mathbb{Z}_2$. The orbit space $O(s_1, s_2)/G$ is in 1 : 1 correspondence with isomorphism classes of admissible O_k -graphs and $\text{Aut}(\Gamma)$ is in 1 : 1 correspondence with $\text{Stab}(\Gamma)$. From the orbit-stabilizer formula, we get

$$\begin{aligned} \sum_{\substack{[\Gamma] \text{ admiss.} \\ O_k\text{-graph}}} \frac{1}{|\text{Aut}(\Gamma)|} &= \sum_{[\Gamma] \in O(s_1, s_2)/G} \frac{1}{|\text{Stab}(\Gamma)|} = \sum_{\Gamma \in O(s_1, s_2)} \frac{1}{|\text{Orb}(\Gamma)| |\text{Stab}(\Gamma)|} \\ &= \frac{|O(s_1, s_2)|}{|G|} = \frac{1}{2k} \binom{k}{s_1} \times \begin{cases} 1 & \text{for } s_1 = s_2, \\ 2 & \text{for } s_1 \neq s_2. \end{cases} \end{aligned}$$

The two cases are compensated in the sum over labelings: For $s_1 = s_2$, both labelings L_1^b are admissible, and hence we get the factor 2.

Next, we multiply by $k!s_1(k-s_1)$, which is the number of L_1^v and L_3^b . There is also the factor $\frac{1}{l!} = \frac{1}{2}$. Multiplying everything together, we get $C(s_1, s_2)$. \square

Before we summarize the results of our computations (see Proposition 5.2.10 below), we show directly that \mathbf{n} is a Maurer-Cartan element.

Lemma 5.2.9 (Maurer-Cartan equation for \mathbb{S}^n). *Consider \mathbb{S}^n with the Hodge Propagator from (5.2). The collection (\mathbf{n}_{lg}) satisfies the Maurer-Cartan equation (3.20) for $\text{dIBL}(C(\mathcal{H}(\mathbb{S}^n)))$.*

Proof. We will show that for every $l \geq 1, g \geq 0$ all summands in the relation corresponding to (l, g) vanish. The summands for $(l, g) = (1, 0)$ are $\mathbf{q}_{110}(\mathbf{n}_{10})$ and $\frac{1}{2}\mathbf{q}_{210}(\mathbf{n}_{10}, \mathbf{n}_{10})$, and the summand for $(l, g) = (2, 0)$ is $\mathbf{q}_{120}(\mathbf{n}_{10})$. The first term vanishes trivially as $\mathbf{q}_{110} = 0$,

5. Explicit computations

while the other two terms vanish by [CFL15, Proposition 12.5] because $\mathbf{n}_{10} = \mathbf{m}_{10}$ is the canonical Maurer-Cartan element. For $(l, g) \neq (1, 0)$, we have the following four situations:

$\mathbf{q}_{210} \circ_2 \mathbf{n}_{lg}$, $l \geq 2$: Let $\Psi = \Psi_1 \cdots \Psi_l \in E_l C$ be a summand of \mathbf{n}_{lg} . From Proposition 5.2.2 it follows that the summands can be chosen such that $\Psi_1, \dots, \Psi_l \in B_{\text{cyc,red}}^* \mathcal{H}(\mathbb{S}^n)[3-n]$, i.e., such that Ψ_i evaluates to 0 whenever 1 is a part of its argument. From Definition 3.2.2, we compute

$$\mathbf{q}_{210} \circ_2 (\Psi_1 \cdots \Psi_l) = \sum_{\sigma \in \mathbb{S}_{2,l-2}} \varepsilon(\sigma, \Psi) \mathbf{q}_{210}(\Psi_{\sigma_1^{-1}} \cdot \Psi_{\sigma_2^{-1}}) \cdot \Psi_{\sigma_3^{-1}} \cdots \Psi_{\sigma_l^{-1}}.$$

We clearly have $\mathbf{q}_{210}(\Psi_{\sigma_1^{-1}} \cdot \Psi_{\sigma_2^{-1}}) = 0$ because \mathbf{q}_{210} feeds 1 into one of its inputs. It follows that $\mathbf{q}_{210} \circ_2 \mathbf{n}_{lg} = 0$.

$\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{lg_1} \odot \mathbf{n}_{lg_2})$, $(l_i, g_i) \neq (1, 0)$: A similar argument as above.

$\mathbf{q}_{120} \circ_1 \mathbf{n}_{lg}$, $(l, g) \neq (1, 0)$: A similar argument as above using that \mathbf{q}_{120} also feeds 1 into its input.

$\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{10} \odot \mathbf{n}_{lg})$, $(l, g) \neq (1, 0)$: As in the case of $\mathbf{q}_{210} \circ_2 \mathbf{n}_{lg}$, suppose that $\Psi_1, \dots, \Psi_l \in B_{\text{cyc,red}}^* \mathcal{H}(\mathbb{S}^n)[3-n]$. Recall that we write $\Omega_i = s\omega_i \in B^{\text{cyc}} \mathcal{H}(\mathbb{S}^n)[3-n]$ and $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_l$. From Definition 3.2.2, we compute

$$\begin{aligned} & [\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{10} \odot \Psi)](\Omega_1 \otimes \cdots \otimes \Omega_l) \\ &= \left[\sum_{i=1}^l (-1)^{|\Psi_i|(|\Psi_1| + \cdots + |\Psi_{i-1}|)} \mathbf{q}_{210}(\mathbf{n}_{10} \Psi_i) \Psi_1 \cdots \widehat{\Psi_i} \cdots \Psi_l \right] \\ & \quad (\Omega_1 \otimes \cdots \otimes \Omega_l) \\ &= \sum_{\substack{\mu \in \mathbb{S}_l \\ i=1, \dots, l}} \frac{1}{l!} (-1)^{|\Psi_i|(|\Psi_1| + \cdots + |\Psi_{i-1}|)} \varepsilon(\mu, \Omega) (\mathbf{q}_{210}(\mathbf{n}_{10} \cdot \Psi_i) \otimes \Psi_1 \otimes \cdots \\ & \quad \widehat{\Psi_i} \cdots \otimes \Psi_l) (\Omega_{\mu_1^{-1}} \otimes \cdots \otimes \Omega_{\mu_l^{-1}}). \end{aligned}$$

For every $i = 1, \dots, l$, we have

$$\begin{aligned} \mathbf{q}_{210}(\mathbf{n}_{10} \cdot \Psi_i)(\Omega) &= \mathbf{q}_{210}(\mathbf{n}_{10} \otimes \Psi_i)(\Omega) \\ &= - \sum \varepsilon(\omega \mapsto \omega^1 \omega^2) [(-1)^{(n-1)|\omega^1|} \mathbf{n}_{10}(s_1 \omega^1) \Psi_i(s\nu \omega^2) \\ & \quad + (-1)^{|\omega^1|} \mathbf{n}_{10}(s\nu \omega^1) \Psi_i(s_1 \omega^2)] \\ &= - \sum \varepsilon(\omega \mapsto \omega^1 \omega^2) (-1)^{(n-1)|\omega^1|} \mathbf{n}_{10}(s_1 \omega^1) \Psi_i(s\nu \omega^2). \end{aligned}$$

5. Explicit computations

This can be non-zero only if $\omega = 1v^{s-1}$ for some $s \geq 2$ (up to a cyclic permutation). For this input, we get

$$\begin{aligned} & \mathbf{q}_{210}(\mathbf{n}_{10} \cdot \Psi_i)(s1v^{s-1}) \\ &= - [\varepsilon(1v^{s-1} \mapsto 1v^{s-1})\mathbf{n}_{10}(s11v)\Psi_i(sv^{s-1}) \\ & \quad + \varepsilon(1v^{s-1} \mapsto v1v^{s-2})\mathbf{n}_{10}(s1v1)\Psi_i(sv^{s-1})] \\ &= (-1)^{n-3}[1 + (-1)^{ns+s-1}]\Psi_i(sv^{s-1}). \end{aligned}$$

The prefactor in brackets is 0 for n odd or s even, whereas $v^{s-1} = 0$ for n even and s odd. Therefore, we have $\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{10} \odot \mathbf{n}_{lg}) = 0$. \square

Proposition 5.2.10 (Chern-Simons Maurer-Cartan element for \mathbb{S}^n). *Consider the round sphere \mathbb{S}^n with the Hodge Propagator (5.2). The Chern-Simons Maurer-Cartan element \mathbf{n} is a strictly reduced Maurer-Cartan element for $\text{dIBL}(C(\mathcal{H}(\mathbb{S}^n)))$ which satisfies*

$$\mathbf{n}_{10} = \mathbf{m}_{10} \quad \text{for all } n \in \mathbb{N}$$

plus the following properties depending on the dimension:

($n = 1$): It holds $\mathbf{n}_{lg} = 0$ for all $l \geq 1, g \geq 0$ such that $(l, g) \neq (1, 0), (2, 0)$; the only non-trivial relation for \mathbf{n}_{20} is

$$\mathbf{n}_{20}(sv^{s_1} \otimes sv^{s_2}) = (-1)^{s_1+1} \frac{1}{2} s_1 k! \binom{k-1}{s_1} I(k), \quad (5.14)$$

where $s_1, s_2 \geq 1$ are such that $k = s_1 + s_2$ is even, and $I(k)$ is given by (5.13).

($n = 2$): It holds $\mathbf{n}_{l0} = 0$ for all $l \geq 2$. We also have $\mathbf{n}_{11} = 0$.

($n \geq 3$): It holds $\mathbf{n}_{lg} = 0$ for all $l \geq 1, g \geq 0$ such that $(l, g) \neq (1, 0)$.

Notice that $\mathbf{n}_{20} \notin E_2 C(\mathcal{H}(\mathbb{S}^1))$, i.e. \mathbf{n}_{20} is a long cochain, because it is non-zero in infinitely many weights.

5.3. Twisted IBL-infinity-structure for spheres

Let e_0, e_1 be the basis of $\mathcal{H}(\mathbb{S}^n)[1]$ defined by

$$e_0 := 1 := \theta 1, \quad e_1 := v := \frac{1}{V} \theta \text{Vol}.$$

5. Explicit computations

The degrees satisfy

$$|1| = -1, \quad |v| = n - 1.$$

The matrix of the pairing Π with respect to the basis e_0, e_1 reads

$$\Pi = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

The dual basis e^0, e^1 to e_0, e_1 with respect to Π is thus

$$e^0 = v, \quad e^1 = (-1)^n 1.$$

It follows that the matrix (T^{ij}) from (3.32) satisfies

$$(T^{ij}) = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We clearly have

$$\hat{B}_{\text{cyc,red}}^* \mathcal{H}(\mathbb{S}^1) = \left\{ \sum_{k=1}^{\infty} c_k v^{k*} \mid c_k \in \mathbb{R} \right\},$$

where v^{k*} is the dual to the cyclic word $v^k = v \dots v$ of length k . Observe that the cyclic symmetry gives

$$v^i = (-1)^{(n-1)(i-1)} v^i \quad \text{for all } i \geq 1.$$

Therefore, $v^{i*} = 0$ holds if both n and i are even.

For $n \geq 2$, the vector space $\mathcal{H}(\mathbb{S}^n)$ is connected and simply-connected, and Proposition 3.3.15 implies that there are no long reduced cyclic cochains (i.e., we have only finite sums of v^{k*} 's).

The product $\mu_2 : \mathcal{H}[1]^{\otimes 2} \rightarrow \mathcal{H}[1]$ from (4.2) has the following matrix with respect to the basis $1, v$:

$$\mu_2 = \begin{pmatrix} 1 & v \\ (-1)^n v & 0 \end{pmatrix}.$$

Because $\mu_2(v, v) = 0$, we get

$$\mathbb{H}^m(C_{\text{red}}(\mathcal{H}(\mathbb{S}^n)))[1] = \begin{cases} \langle sv^{i*} \mid i \geq 1 \rangle & \text{for } n \geq 3 \text{ odd,} \\ \langle sv^{2i-1*} \mid i \geq 1 \rangle & \text{for } n \text{ even,} \\ \{s \sum_{k=1}^{\infty} c_k v^{k*} \mid c_k \in \mathbb{R}\} & \text{for } n = 1. \end{cases}$$

5. Explicit computations

Because we are in the strictly unital and strictly augmented case, we obtain

$$\mathbb{H}^m(C)[1] = \begin{cases} \langle \text{sv}^{i*}, \text{s}1^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \geq 3 \text{ odd,} \\ \langle \text{sv}^{2i-1*}, \text{s}1^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \text{ even,} \\ \langle \text{s} \sum_{k=1}^{\infty} c_k \text{v}^{k*}, \text{s}1^{2j-1*} \mid c_k \in \mathbb{R}, j \geq 1 \rangle & \text{for } n = 1. \end{cases} \quad (5.15)$$

The canonical IBL-operations can be written as

$$\begin{aligned} \mathfrak{q}_{210}(\text{s}^2\psi_1 \otimes \psi_2)(\text{s}\omega) &= - \sum \varepsilon(\omega \mapsto \omega^1\omega^2)[(-1)^{(n-1)|\omega^1|}\psi_1(e_0\omega^1) \\ &\quad \psi_2(e_1\omega^2) + (-1)^{|\omega_1|}\psi_1(e_1\omega^1)\psi_2(e_0\omega^2)], \\ \mathfrak{q}_{120}(\text{s}\psi)(\text{s}^2\omega_1 \otimes \omega_2) &= - \frac{1}{2} \sum \varepsilon(\omega_1 \mapsto \omega_1^1)\varepsilon(\omega_2 \mapsto \omega_2^1)[(-1)^{(n-1)|\omega_1^1|} \\ &\quad \psi(e_0\omega_1^1 e_1\omega_2^1) + (-1)^{|\omega_1^1|}\psi(e_1\omega_1^1 e_0\omega_2^1)] \end{aligned}$$

for all $\psi, \psi_1, \psi_2 \in \hat{\mathcal{B}}_{\text{cyc}}^* \mathcal{H}$ and generating words $\omega, \omega_1, \omega_2 \in \text{B}^{\text{cyc}} \mathcal{H}$. For all $k, k_1, k_2 \geq 1$, we have

$$\mathfrak{q}_{210}((\text{sv}^{k_1*}) \cdot (\text{sv}^{k_2*})) = 0 \quad \text{and} \quad \mathfrak{q}_{120}(\text{sv}^{k*}) = 0$$

because both \mathfrak{q}_{210} and \mathfrak{q}_{120} feed 1 into their inputs. For the *canonically twisted reduced IBL-algebra*, this implies the following:

$$\text{IBL}(\mathbb{H}^m(C_{\text{red}})) = (\mathbb{H}^m(C_{\text{red}}), \mathfrak{q}_{210} \equiv 0, \mathfrak{q}_{120} \equiv 0) \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 3.4.10, the only possibly non-zero relation of $\text{IBL}(\mathbb{H}^m(C))$ is

$$\begin{aligned} \mathfrak{q}_{210}(\text{s}1^* \otimes \text{sv}^{k*}) &= (-1)^{n-2} \text{s}(\text{v}^{k*} \circ \iota_{\text{v}}) \\ &= (-1)^{n-2} \left(\sum_{i=1}^{k-1} (-1)^{i|\text{v}|} \text{sv}^{k-1*} \right) = \begin{cases} -(k-1) \text{sv}^{k-1*} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

The reason for 0 for even n is that either k is odd, in which case $\sum_{i=1}^{k-1} (-1)^i = 0$, or k is even, in which case $\text{v}^{k*} = 0$. Therefore, for the *canonically twisted IBL-algebra*, we have

$$\text{IBL}(\mathbb{H}^m(C)) = (\mathbb{H}^m(C), \mathfrak{q}_{210}, \mathfrak{q}_{120} \equiv 0) \quad \text{for all } n \in \mathbb{N},$$

where $\mathbb{H}^m(C)$ is given by (5.15) and \mathfrak{q}_{210} satisfies the following:

(n even): $\mathfrak{q}_{210} \equiv 0$.

5. Explicit computations

($n \geq 3$ odd): The non-trivial relations are

$$\mathfrak{q}_{210}(s_1^* \otimes sv^{k*}) = \mathfrak{q}_{210}(sv^{k*} \otimes s_1^*) = -(k-1)v^{k-1*} \quad \text{for } k \geq 2.$$

($n = 1$): The non-trivial relations are

$$\mathfrak{q}_{210}\left(s_1^* \otimes s \sum_{k=1}^{\infty} c_k v^{k*}\right) = -s \sum_{k=1}^{\infty} k c_{k+1} v^{k*} \quad \text{for } c_k \in \mathbb{R}.$$

Recall that the twist by \mathfrak{m} does not produce any higher operation $\mathfrak{q}_{1lg}^{\mathfrak{m}}$.

We will now consider $\text{dIBL}^n(C(\mathcal{H}(\mathbb{S}^n)))$. Recall that $\mathfrak{q}_{110}^{\mathfrak{n}} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$, $\mathfrak{q}_{210}^{\mathfrak{n}} = \mathfrak{q}_{210}$ and $\mathfrak{q}_{120}^{\mathfrak{n}} = \mathfrak{q}_{120} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$. By Proposition 5.2.10, we have $\mathfrak{n}_{10} = \mathfrak{m}_{10}$ for all $n \in \mathbb{N}$ and $\mathfrak{n}_{20} = 0$ for all $n \geq 2$. It follows that $\mathfrak{q}_{110}^{\mathfrak{n}} = \mathfrak{q}_{110}^{\mathfrak{m}}$ for all $n \in \mathbb{N}$ and that the only non-trivial twist may occur in $\mathfrak{q}_{120}^{\mathfrak{n}}$ for \mathbb{S}^1 . Using (3.36), we get for all $\psi \in \hat{B}_{\text{cyc}}^* \mathcal{H}(\mathbb{S}^n)$ and generating words $\omega_1, \omega_2 \in B^{\text{cyc}} \mathcal{H}(\mathbb{S}^n)$ the following:

$$\begin{aligned} & (\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s\psi)(s\omega_1 \otimes s\omega_2) \\ &= (-1)^{n-2} \left[\sum \varepsilon(\omega_1 \mapsto \omega_1^1 \omega_1^2) \psi(1\omega_1^1) \mathfrak{n}_{20}(sv\omega_1^2 \otimes s\omega_2) \right. \\ & \quad \left. + (-1)^{(n-3+|\omega_1|)(n-3+|\omega_2|)} \sum \varepsilon(\omega_2 \mapsto \omega_2^1 \omega_2^2) \psi(1\omega_2^1) \right. \\ & \quad \left. \mathfrak{n}_{20}(sv\omega_2^2 \otimes s\omega_1) \right]. \end{aligned} \tag{5.16}$$

In this paragraph, we suppose that $n = 1$ and compute $\mathfrak{q}_{120}^{\mathfrak{n}}$. Clearly, $(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(sv^{k*}) = 0$ for all $k \geq 1$ since 1 is fed into v^{k*} . A non-zero evaluation of $(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s_1^{k*})$ for some $k \geq 1$ odd is possible only on $s_1^{k-1}v^{k_1} \otimes sv^{k_2}$ for $k_1, k_2 \geq 0$ (up to a transposition of arguments and their cyclic permutation). If $k > 1$, only the first summand of (5.16) contributes, and we get

$$\begin{aligned} & (\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s_1^{k*})(s_1^{k-1}v^{k_1} \otimes sv^{k_2}) \\ &= (-1)^{n-2} \sum \varepsilon(1^{k-1}v^{k_1} \mapsto \omega_1 \omega_2) 1^{k*}(1\omega_1) \mathfrak{n}_{20}(sv\omega_2 \otimes sv^{k_2}) \\ &= (-1)^{n-2} 1^{k*}(11^{k-1}) \mathfrak{n}_{20}(svv^{k_1} \otimes sv^{k_2}) \\ &= -\mathfrak{n}_{20}(sv^{k_1+1} \otimes sv^{k_2}). \end{aligned}$$

According to Proposition 5.2.10, this is non-zero if and only if $k_1 + k_2$ is odd. It follows that

$$\mathfrak{q}_{120}^{\mathfrak{n}} \neq \mathfrak{q}_{120}^{\mathfrak{m}} = \mathfrak{q}_{120} \quad \text{on the chain level for } \mathbb{S}^1.$$

However, the chains $s_1^{k-1}v^{k_1} \otimes sv^{k_2}$ for $k > 1$ do not survive to the homology (c.f., (5.15)).

5. Explicit computations

The only possibility is thus $k = 1$. In this case, both summands of (5.16) contribute, and using (5.14), we get for all $k_1, k_2 \geq 1$ the following:

$$\begin{aligned}
& (\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s_1^*)(\mathrm{sv}^{k_1} \otimes \mathrm{sv}^{k_2}) \\
&= (-1)^{n-2} \left[\sum \varepsilon(v^{k_1} \mapsto v^0 v^{k_1})_1^*(1) \mathfrak{n}_{20}(\mathrm{sv}^{k_1+1} \otimes \mathrm{sv}^{k_2}) \right. \\
&\quad \left. + (-1)^{(n-3+k_1(n-1))(n-3+k_2(n-1))} \sum \varepsilon(v^{k_2} \mapsto v^0 v^{k_2})_1^*(1) \right. \\
&\quad \left. \mathfrak{n}_{20}(\mathrm{sv}^{k_2+1} \otimes \mathrm{sv}^{k_1}) \right] \\
&= -k_1 \mathfrak{n}_{20}(\mathrm{sv}^{k_1+1} \otimes \mathrm{sv}^{k_2}) - k_2 \mathfrak{n}_{20}(\mathrm{sv}^{k_2+1} \otimes \mathrm{sv}^{k_1}) \\
&= -\frac{1}{2}(k_1 + k_2 + 1)! I(k_1 + k_2 + 1) \left[(-1)^{k_1} k_1 (k_1 + 1) \binom{k_1 + k_2}{k_1 + 1} \right. \\
&\quad \left. + (-1)^{k_2} k_2 (k_2 + 1) \binom{k_1 + k_2}{k_2 + 1} \right] \\
&= -\frac{1}{2}(k_1 + k_2 + 1)! k_1 k_2 \binom{k_1 + k_2}{k_1} \underbrace{I(k_1 + k_2 + 1) [(-1)^{k_1} + (-1)^{k_2}]}_{=:(*)}.
\end{aligned}$$

Denoting $k := k_1 + k_2 + 1$, we have that $(-1)^{k_1} + (-1)^{k_2} = 0$ for k even and $I(k) = 0$ for k odd. Therefore, $(*) = 0$ for any $k_1, k_2 \geq 1$. This implies that

$$\mathfrak{q}_{120}^{\mathfrak{n}} = \mathfrak{q}_{120}^{\mathfrak{m}} = \mathfrak{q}_{120} \quad \text{on the homology for } \mathbb{S}^1.$$

We conclude that the *twisted IBL-algebra* satisfies

$$\mathrm{IBL}(\mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^n)))) = \mathrm{IBL}(\mathbb{H}^{\mathfrak{m}}(C(\mathcal{H}(\mathbb{S}^n)))) \quad \text{for all } n \in \mathbb{N}.$$

As for the *higher twisted operations*, combining Propositions 3.2.11 and 5.2.10, we see that for \mathbb{S}^n with $n \in \mathbb{N} \setminus \{2\}$ all higher operations $\mathfrak{q}_{1lg}^{\mathfrak{n}}$ vanish already on the chain level. For $n = 2$, we have that $\mathfrak{q}_{1l0}^{\mathfrak{n}} = 0$ for all $l \geq 3$ and $\mathfrak{q}_{111}^{\mathfrak{n}} = 0$ on the chain level. However, we did not prove that all higher operations vanish on the chain level. As for the operations induced on the homology, the graded vector space $\mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^2)))$ is concentrated in even degrees and $\mathfrak{q}_{1lg}^{\mathfrak{n}}$ are odd (see Definition 3.2.4). Therefore, all higher operations vanish also on $\mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^2)))$.

The string topology $\mathrm{H}^{\mathbb{S}^1}(\mathbb{S}^n)$ and the string operations \mathfrak{m}_2 and \mathfrak{c}_2 were computed in [Bas11] for all $n \in \mathbb{N}$. We review their results and basic ideas below:

We will consider *even spheres* first. The minimal model for the Borel construction $\mathrm{L}_{\mathbb{S}^1} \mathbb{S}^{2m}$ for $m \in \mathbb{N}$ is denoted by $\Lambda^{\mathbb{S}^1}(2, m)$ — it is the free graded commutative dga

5. Explicit computations

(=:cdga) over \mathbb{R} generated by homogenous vectors x_1, y_1, x_2, y_2, u of degrees

$$|x_1| = 2m, \quad |y_1| = 2m - 1, \quad |x_2| = 4m - 1, \quad |y_2| = 2(2m - 1), \quad |u| = 2,$$

whose differential d satisfies

$$dy_1 = 0, \quad dx_1 = y_1 u, \quad dy_2 = -2x_1 y_1, \quad dx_2 = x_1^2 + y_2 u.$$

The minimal model for the loop space LS^{2m} is the dga $\Lambda(2, m)$ which is obtained from $\Lambda^{\mathbb{S}^1}(2, m)$ by setting $u = 0$. A computation (see [Bas11, Theorem 3.6]) gives the following for all $m \in \mathbb{N}$:

$$\begin{aligned} H^*(LS^{2m}; \mathbb{R}) &\simeq H(\Lambda(2, m), d) = \langle y_2^i x_1 - 2iy_1 x_2 y_2^{i-1}, y_1 y_2^j, 1 \mid i, j \in \mathbb{N}_0 \rangle, \\ H_{\mathbb{S}^1}^*(LS^{2m}; \mathbb{R}) &\simeq H(\Lambda^{\mathbb{S}^1}(2, m), d) = \langle y_1 y_2^i, u^j \mid i, j \in \mathbb{N}_0 \rangle, \end{aligned} \tag{5.17}$$

where $y_2^0 := u^0 := 1$ is the unit in $\Lambda^{\mathbb{S}^1}(2, m)$ and $\langle \cdot \rangle$ denotes the linear span over \mathbb{R} . Clearly, the cohomology groups are degree-wise finite-dimensional, and hence, using the universal coefficient theorem, they are isomorphic to the corresponding homology groups. We can thus identify $H(LS^{2m}; \mathbb{R})$ and $H^{\mathbb{S}^1}(LS^{2m}; \mathbb{R})$ with the vector spaces on the right hand side of (5.17). We have $H_{2k}^{\mathbb{S}^1} = \langle u^k \rangle$ for all $k \in \mathbb{N}_0$, and hence the multiplication with u induces an isomorphism $H_{2k}^{\mathbb{S}^1} \simeq H_{2k+2}^{\mathbb{S}^1}$. This corresponds to the cap product with the Euler class in (4.27), and exactness of the sequence implies $\mathcal{M}(H_{2k}^{\mathbb{S}^1}) = \mathcal{E}(H_{2k}^{\mathbb{S}^1}) = 0$. Using this and degree considerations, we get $\mathfrak{m}_2 = \mathfrak{c}_2 = 0$.

We will now consider *odd spheres* with $n \geq 3$. The minimal model for $L_{\mathbb{S}^1} \mathbb{S}^{2m+1}$ for $m \in \mathbb{N}$ is denoted simply by $\Lambda(x, y, u)$ — it is the free cdga on homogenous vectors x, y, u of degrees

$$|x| = 2m + 1, \quad |y| = 2m, \quad |u| = 2,$$

such that

$$dx = yu, \quad dy = du = 0.$$

We get immediately

$$\begin{aligned} H^*(LS^{2m+1}; \mathbb{R}) &\simeq \langle x^i, y^j \mid i, j \in \mathbb{N}_0 \rangle, \\ H_{\mathbb{S}^1}^*(LS^{2m+1}; \mathbb{R}) &\simeq \langle y^i, u^j \mid i, j \in \mathbb{N}_0 \rangle, \end{aligned}$$

and we can again identify H and $H^{\mathbb{S}^1}$ with the vector spaces on the right hand side. Clearly, $H_{2k-1}^{\mathbb{S}^1} = 0$ for all $k \in \mathbb{N}$, and hence $\mathfrak{m}_2 = \mathfrak{c}_2 = 0$ for degree reasons (the operations are odd).

We will now consider *the circle* \mathbb{S}^1 . For every $i \in \mathbb{Z}$, let $\alpha_i : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and $\theta_i : \mathbb{S}^1 \rightarrow LS^1$

5. Explicit computations

be the maps defined by

$$\alpha_i(z) := z^i \quad \text{and} \quad \theta_i(w) := w\alpha_i \quad \text{for all } w, z \in \mathbb{S}^1 \subset \mathbb{C}.$$

By examining the equivariant homology of connected components of LS^1 containing α_i separately as in [Bas11, Section 2.1.4], we get

$$\begin{aligned} H(\text{LS}^1; \mathbb{R}) &= \langle \alpha_i, \theta_j \mid i, j \in \mathbb{Z} \rangle, \\ H^{\mathbb{S}^1}(\text{LS}^1; \mathbb{R}) &= \langle u^i, \theta_0 u^j, \alpha_k \mid i, j \in \mathbb{N}_0, k \in \mathbb{Z} \setminus \{0\} \rangle, \end{aligned}$$

where u corresponds to the Euler class and

$$|u| = 2, \quad |\theta_i| = 1, \quad |\alpha_i| = 0$$

are the degrees in the singular chain complex. On [Bas11, p. 21] they show that the string cobracket \mathfrak{c}_2 is 0 and that all non-trivial relations for the string bracket $\mathfrak{m}_2 : H^{\mathbb{S}^1}(\text{LS}^1)[2]^{\otimes 2} \rightarrow H^{\mathbb{S}^1}(\text{LS}^1)[2]$ are the following:

$$\mathfrak{m}_2(s\alpha_k, s\alpha_{-k}) = k^2 s\theta_0 \quad \forall k \in \mathbb{N}.$$

We will now compare the reduced IBL-structures motivated by Conjecture 4.5.1. The point-reduced versions $H^{\mathbb{S}^1, \text{red}}(\text{LS}^n)$ for $n \geq 2$ are obtained from $H^{\mathbb{S}^1}(\text{LS}^n)$ by deleting u^i . We have the following isomorphisms of graded vector spaces:

$$\begin{aligned} \mathbb{H}^n(C_{\text{red}}(\mathcal{H}(\mathbb{S}^n)))[1] &\longrightarrow H^{\mathbb{S}^1, \text{red}}(\text{LS}^n)[3-n] \\ \text{sv}^i &\longmapsto sy^i && \text{for } n > 1 \text{ odd,} \\ \text{sv}^{2i+1} &\longmapsto sy_1 y_2^i && \text{for } n \text{ even.} \end{aligned}$$

Because all operations are trivial, it induces the isomorphism

$$\text{IBL}(\mathbb{H}^n(C_{\text{red}}(\mathcal{H}(\mathbb{S}^n)))) \simeq \text{IBL}(H^{\mathbb{S}^1, \text{red}}(\text{LS}^n)[2-n]) \quad \text{for } n \geq 2.$$

For $n = 1$, the reduced homology is seemingly different.

Remark 5.3.1 (Triviality for degree reasons). The graded vector spaces

$$H^{\mathbb{S}^1}(\text{LS}^{2m-1})[3-n] \quad \text{and} \quad \mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^{2m}))) [1]$$

are concentrated in even degrees, and so any IBL_∞ -structure must be trivial for degree

5. Explicit computations

reasons. On the other hand, the graded vector spaces

$$H^{\mathbb{S}^1}(\mathcal{L}\mathbb{S}^{2m}) \quad \text{and} \quad \mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^{2m-1})))[1]$$

have both even and odd degrees, and hence an additional argument is needed to prove vanishing of the IBL-structure. This is not the case of the reduced homology, which is again concentrated in even degree. \triangleleft

5.4. Homology of twisted IBL-infinity structure for complex projective space

Let $K \in \Omega^2(\mathbb{CP}^n)$ be the Fubini–Study Kähler form on \mathbb{CP}^n (see [Huy04, Examples 3.1.9]). The powers of K are harmonic,² and we get easily

$$\mathcal{H}(\mathbb{CP}^n) = \langle 1, K, \dots, K^n \rangle.$$

We denote the Riemannian volume of \mathbb{CP}^n by

$$V := \int_{\mathbb{CP}^n} \frac{1}{n!} K^n.$$

Consider the basis e_0, \dots, e_n of $\mathcal{H}(\mathbb{CP}^n)[1]$ defined for all $i = 0, \dots, n$ by

$$e_i := \frac{k^i}{(n!V)^{\frac{i}{n}}}, \quad \text{where} \quad k^i := \theta K^i.$$

The matrix of the pairing Π from (2.3) with respect to the basis e_0, \dots, e_n reads:

$$(\Pi^{ij}) = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

The basis e^0, \dots, e^n dual to e_0, \dots, e_n with respect to Π thus satisfies

$$e^i = e_{n-i} \quad \text{for all } i = 0, \dots, n.$$

²This follows by induction on the power of K using the fact that, on a general Kähler manifold M , the Lefschetz operator $L : \Omega(M) \rightarrow \Omega(M)$ defined by $L(\eta) := \eta \wedge K$ for all $\eta \in \Omega(M)$ commutes with the Hodge–de Rham Laplacian Δ (see [Huy04, Chapter 3]).

5. Explicit computations

Therefore, the following holds for the matrix (T^{ij}) from (3.32):

$$(T^{ij}) = -(\Pi^{ij}).$$

For all $1 \leq i, j, k \leq n$, we have

$$\mu_2(e_i, e_j) = e_{i+j} \quad \text{and} \quad \mathbf{m}_{10}(se_i e_j e_k) = \delta_{i+j+k, n}.$$

For $\psi, \psi_1, \psi_2 \in \hat{B}_{\text{cyc}}^* \mathcal{H}$ and generating words $\omega, \omega_1, \omega_2 \in B^{\text{cyc}} \mathcal{H}$, we have

$$\begin{aligned} \mathfrak{q}_{210}(s^2 \psi_1 \otimes \psi_2)(s\omega) &= - \sum_{i=0}^n \sum \varepsilon(\omega \mapsto \omega^1 \omega^2) (-1)^{|\omega^1|} \psi_1(e_i \omega^1) \psi_2(e_{n-i} \omega^2), \\ \mathfrak{q}_{120}(s\psi)(s^2 \omega_1 \otimes \omega_2) &= - \sum_{i=0}^n \sum \varepsilon(\omega_1 \mapsto \omega_1^1) \varepsilon(\omega_2 \mapsto \omega_2^1) (-1)^{|\omega_1^1|} \psi(e_i \omega_1^1 e_{n-i} \omega_2^1). \end{aligned}$$

The cyclic homology of $\mathcal{H}(\mathbb{CP}^n)$ is that of the truncated polynomial algebra

$$A := \mathbb{R}[x]/(x^{n+1}) \quad \text{with} \quad \deg(x) = 2.$$

The computation of $H^{\lambda, \text{cl}}(A)$ for $|x| = 0$ over a field is the goal of [Lod92, Exercise 4.1.8.] or [Wei94, Exercise 9.1.1]. The case of $|x| = d$ can be solved by taking suitable degree shifts in the proposed projective resolution which is used to compute $\text{HH}(A)$. Unfortunately, using a non-canonical projective resolution, we lose the concrete form of the cyclic cycles and obtain just the following result:

For all $i = 1, \dots, n$ and $k \in \mathbb{N}_0$, there are cycles $\tilde{t}_{2k+1, i} \in \tilde{D}_q(A)$ of weights $2k + 1$ and degrees $d(i + (n + 1)k)$ which form a basis of $H^{\lambda, \text{cl}}(A)$. We apply the degree shift $U : \tilde{D}(A) \rightarrow D(A)$ from Proposition 3.3.14 to get the generators

$$t_{w, i} := U(\tilde{t}_{w, i}) \in D^\lambda(\mathcal{H}(\mathbb{CP}^n))$$

of weights w and degrees $2i + (w - 1)n - 1$, so that

$$H^\lambda(\mathcal{H}(\mathbb{CP}^n)) = \langle t_{w, i}, 1^w \mid w \in \mathbb{N} \text{ odd}, i = 1, \dots, n \rangle.$$

By the universal coefficient theorem we have $H_\lambda^* = (H^\lambda)^{*g}$ with respect to the grading by the degree. Given $d \in \mathbb{Z}$, the equation $d = 2i + (w - 1)n - 1$ has only finitely many solution $(w, i) \in \mathbb{N} \times \{1, \dots, n\}$, and hence we get

$$\mathbb{H}^m(C(\mathcal{H}(\mathbb{CP}^n))) = \langle st_{w, i}^*, s1^{w*} \mid w \in \mathbb{N} \text{ odd}, i = 1, \dots, n \rangle, \quad (5.18)$$

5. Explicit computations

where $t_{w,i}^*$ and $1^{w*} \in B_{\text{cyc}}^* \mathcal{H}$ are the duals to $t_{w,i}$ and 1^w , respectively (see Remark 3.3.16). Notice that both $|st_{w,i}^*|$ and $|s1^{w*}|$ are even since $|s| = 2n - 3$.

Because \mathbb{CP}^n is geometrically formal, Proposition 4.4.5 implies that $\mathfrak{n}_{10} = \mathfrak{m}_{10}$. Because $\mathbb{H}^{\mathfrak{m}}(C)$ is concentrated in even degrees and because a general IBL_{∞} -operation \mathfrak{q}_{klg} is odd, all operations vanish on the homology. Therefore, for the *twisted IBL-algebras* we have

$$\text{IBL}(\mathbb{H}^{\mathfrak{n}}(C)) = \text{IBL}(\mathbb{H}^{\mathfrak{m}}(C)) = (\mathbb{H}^{\mathfrak{m}}(C), \mathfrak{q}_{210} \equiv 0, \mathfrak{q}_{120} \equiv 0),$$

where $\mathbb{H}^{\mathfrak{m}}(C)$ is given by (5.18).

According to [Bas11, Section 3.1.2], the minimal model for the Borel construction $L_{\mathbb{S}^1} \mathbb{CP}^n$ is the cdga $\Lambda^{\mathbb{S}^1}(n+1, 1)$, which is freely generated (over \mathbb{R}) by the homogenous vectors x_1, x_2, y_1, y_2, u of degrees

$$|x_1| = 2, \quad |x_2| = 2n + 1, \quad |y_1| = 1, \quad |y_2| = 2n, \quad |u| = 2,$$

and whose differential d satisfies

$$dy_1 = 0, \quad dx_1 = y_1 u, \quad dy_2 = -(n+1)x_1^n y_1, \quad dx_2 = x_1^{n+1} + y_2 u.$$

By [Bas11, Theorem 3.6], the string cohomology $H_{\mathbb{S}^1}^*(L\mathbb{CP}^n; \mathbb{R}) \simeq H(\Lambda^{\mathbb{S}^1}(n+1, 1), d)$ satisfies for all $m \in \mathbb{N}_0$ the following:

$$H_{\mathbb{S}^1}^m(L\mathbb{CP}^n; \mathbb{R}) = \begin{cases} \langle u^j \rangle & \text{if } m = 2j, \\ \langle y_1 y_2^p x_1^q \mid 0 \leq q \leq n-1, p \geq 0; q + np = j \rangle & \text{if } m = 2j + 1. \end{cases}$$

The right-hand side can be identified with $H^{\mathbb{S}^1}(L\mathbb{CP}^n; \mathbb{R})$ by the universal coefficient theorem. According to [Bas11, Proposition 3.7], we have $\mathfrak{m}_2 = 0$ and $\mathfrak{c}_2 = 0$. We conclude that the map

$$\begin{aligned} \mathbb{H}^{\mathfrak{n}}(C_{\text{red}}(\mathcal{H}(\mathbb{CP}^n)))[1] &\longrightarrow H^{\mathbb{S}^1, \text{red}}(L\mathbb{CP}^n; \mathbb{R})[3-n] \\ st_{2k+1,l}^* &\longmapsto sy_1 y_2^k x_1^{l-1} \quad \text{for } k \geq 0 \text{ and } l = 1, \dots, n \end{aligned}$$

induces an isomorphism of IBL-algebras

$$\text{IBL}(\mathbb{H}^{\mathfrak{n}}(C_{\text{red}}(\mathcal{H}(\mathbb{CP}^n)))) \simeq \text{IBL}(H^{\mathbb{S}^1, \text{red}}(L\mathbb{CP}^n; \mathbb{R})[3-n]).$$

Part II.

Follow-up topics

6. IBL-infinity formality and Poincaré duality models

In this chapter, we study the following question:

Question 6.0.1. *Let M be a connected oriented closed n -manifold which is formal in the sense of rational homotopy theory. Let \mathfrak{m} be the canonical and \mathfrak{n} the formal pushforward Maurer-Cartan element for $\mathrm{dIBL}(C(H_{\mathrm{dR}}(M)))$. Are $\mathrm{dIBL}^{\mathfrak{m}}(C(H_{\mathrm{dR}}(M)))$ and $\mathrm{dIBL}^{\mathfrak{n}}(C(H_{\mathrm{dR}}(M)))$ homotopy equivalent as IBL_{∞} -algebras?*

We remind that Theorem B in Part I states that for a geometrically formal manifold M with $H_{\mathrm{dR}}^1(M) = 0$, one can pick a special Hodge propagator P_{std} such that $\mathfrak{n} = \mathfrak{m}$ at least for $n \neq 2$.¹ Question 6.0.1 asks for a generalization of Theorem B. In the upcoming sections, we will propose a strategy to answer Question 6.0.1 in the case of $H_{\mathrm{dR}}^1(M) = 0$, when the expected answer is “yes”. Our main tool is the theory of Poincaré duality models from [LS07]. Nevertheless, we also study DGA’s of Hodge type and their small subalgebras from [Fio+19], which, as we think, naturally fit in the picture.

In Section 6.1, we consider DGA’s and define the notions of an orientation and cyclic structure (Definition 6.1.1), non-degeneracy and Poincaré duality (Definition 6.1.6), degenerate subspace and non-degenerate quotient (Definition 6.1.8), Hodge decomposition and Hodge pair (Definition 6.1.9) and Hodge and small subalgebra from [Fio+19] (Definition 6.1.14). We prove that in some cases orientations and cyclic structures are in one-to-one correspondence (Proposition 6.1.4) and that being of Hodge type is equivalent to acyclicity of the degenerate subspace (Proposition 6.1.13). We describe the small subalgebra of [Fio+19] in terms of Kontsevich-Soibelman-like evaluations of rooted binary trees with internal edges labeled with the identity or the standard Hodge homotopy (Proposition 6.1.15). We study what happens when we take small subalgebras and non-degenerate quotients iteratively (Proposition 6.1.17). We conjecture that the small subalgebra is an image of the Sullivan minimal model (Conjecture 6.1.22), and if this is the case, we prove that the non-degenerate quotient of the small subalgebra is independent

¹We did not prove that the homotopy class of $\mathrm{dIBL}^{\mathfrak{n}}(C(H_{\mathrm{dR}}(M)))$ does not depend on P_{std} , but we expect so.

6. IBL-infinity formality and Poincaré duality models

of the Hodge decomposition up to a pairing preserving isomorphism (Conjecture 6.1.23). We finish with some open questions (Questions 6.1.24). We also prove several lemmas (Lemmas 6.1.12, 6.1.18, 6.1.19 and 6.1.20) and make several remarks (Remarks 6.1.2, 6.1.5, 6.1.7, 6.1.11 and 6.1.16) which might be useful for future investigations.

In Section 6.2, we define differential Poincaré duality algebras² and Poincaré DGA's (Definition 6.2.1), i.e., DGA's whose homology is a Poincaré duality algebra. We consider Poincaré duality models (Definition 6.2.3). We argue that a Poincaré DGA is formal if and only if it is formal as a DGA (Proposition 6.2.4). We check that the proofs of the existence and uniqueness of Poincaré duality models from [LS07] still work when quasi-isomorphisms preserving orientation are requested (Propositions 6.2.5 and 6.2.6). We discuss minimal models (Remark 6.2.8) and conjecture that small Poincaré duality models can be thought of as minimal models for PDGA's (Corollary 6.2.9). We finish with some open questions (Questions 6.2.10).

In Section 6.3, we formulate “functoriality” of the canonical dIBL^m construction on differential Poincaré duality algebras up to IBL_∞ -homotopy (Conjectures 6.3.1 and 6.3.2). We define IBL_∞ -formality of a PDGA (Definition 6.3.3). We conjecture that IBL_∞ -formality is implied by DGA-formality (Conjecture 6.3.4). We discuss another relevant notions of formality (Remark 6.3.5). For the de Rham complex of a smooth manifold, we conjecture that the IBL_∞ -algebra on cyclic cochains of de Rham cohomology twisted by the Chern-Simons Maurer-Cartan element is homotopy equivalent to the dIBL -algebra on cyclic cochains of the non-degenerate quotient of the small subalgebra of the de Rham complex twisted by the canonical Maurer-Cartan element (Conjecture 6.3.6). Assuming that the conjectures hold, we obtain a positive answer to Question 6.0.1 when $H_{\mathrm{dR}}^1(M) = 0$ (Corollary 6.3.7).

6.1. Orientation, Poincaré duality, Hodge decomposition

In this section, a DGA V is a triple (V, d, \wedge) , where

- $V = \bigoplus_{n \in \mathbb{Z}} V^n$ is a \mathbb{Z} -graded vector space,
- $d : V \rightarrow V$ a differential of degree 1, and
- $\wedge : V \otimes V \rightarrow V$ an associative product of degree 0 such that d and \wedge satisfy the Leibnitz identity $d(v_1 \wedge v_2) = dv_1 \wedge v_2 + (-1)^{v_1} v_1 \wedge dv_2$ for all homogenous $v_1, v_2 \in V$.

²The same as cyclic DGA's for unital commutative DGA's up to a degree shift and the correspondence of orientations and cyclic structures

6. IBL-infinity formality and Poincaré duality models

In other words, we consider general, possibly non-unital and non-commutative, \mathbb{Z} -graded DGA's. We denote by $\deg(v)$ the degree of a homogenous element $v \in V$ and write $(-1)^v$ in the exponent.

Definition 6.1.1 (Orientations and cyclic structures). *Let (V, d) be a \mathbb{Z} -graded cochain complex. We define the following:*

- An orientation in degree n is a linear function $\circ : V^n \rightarrow \mathbb{R}$ such that

$$\circ \neq 0 \quad \text{and} \quad \circ \circ d = 0.$$

In other words, it is a surjective chain map $\circ : (V, d) \rightarrow (\mathbb{R}[-n], 0)$.

- A cyclic structure of degree n is a homogenous bilinear form $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ of degree $-n$ as a map such that for all homogenous $v_1, v_2 \in V$, we have

$$\langle dv_1, v_2 \rangle = (-1)^{1+v_1v_2} \langle dv_2, v_1 \rangle. \quad (6.1)$$

A cyclic structure on a DGA (V, d, \wedge) satisfies in addition

$$\langle v_1 \wedge v_2, v_3 \rangle = (-1)^{v_3(v_1+v_2)} \langle v_3 \wedge v_1, v_2 \rangle \quad (6.2)$$

for all homogenous $v_1, v_2, v_3 \in V$.

Analogously, one can define a cyclic structure on an A_∞ -algebra.

Remark 6.1.2 (Cyclic DGA). A cyclic DGA (V, Π, μ_1, μ_2) of degree n from Definition 3.3.7 in Part I is the same as a DGA equipped with a cyclic structure $\langle \cdot, \cdot \rangle$ of degree n which is symmetric, i.e.,

$$\langle v_1, v_2 \rangle = (-1)^{v_1v_2} \langle v_2, v_1 \rangle$$

for all homogenous $v_1, v_2 \in V$, and non-degenerate (see Definition 6.1.6). The correspondence is via the degree shift

$$\begin{aligned} \mu_1(\theta v) &= \theta d(v), \\ \mu_2(\theta v_1, \theta v_2) &= (-1)^{v_1} \theta(v_1 \wedge v_2), \\ \Pi(\theta v_1, \theta v_2) &= (-1)^{v_1} \langle v_1, v_2 \rangle, \end{aligned}$$

where θ is a formal symbol of degree -1 . From this reason, we sometimes call a DGA equipped with a non-degenerate symmetric cyclic structure a cyclic DGA, although there are degree shifts involved. \triangleleft

6. IBL-infinity formality and Poincaré duality models

The homology $H(V) := H(V, d)$ of a DGA (V, d, \wedge) is also a DGA with the induced product \wedge and with zero differential. Given an orientation o or a cyclic structure $\langle \cdot, \cdot \rangle$ on V , we define the maps $o^H : H(V) \rightarrow \mathbb{R}$ and $\langle \cdot, \cdot \rangle^H : H(V) \otimes H(V) \rightarrow \mathbb{R}$ for all closed $h_1, h_2 \in V$ by

$$\begin{aligned} o^H([h_1]) &:= o(h_1) \text{ and} \\ \langle [h_1], [h_2] \rangle^H &:= \langle h_1, h_2 \rangle, \end{aligned} \tag{6.3}$$

respectively, where $[\cdot]$ denotes the cohomology class. It is easy to see that $\langle \cdot, \cdot \rangle^H$ is a cyclic structure on $H(V)$ and that o^H is an orientation on $H(V)$ provided that $o|_{\ker d} \neq 0$.

Proposition 6.1.3 (Orientation on homology). *We have the following:*

- (a) *Let (V, d) be a \mathbb{Z} -graded cochain complex, and let $n \in \mathbb{Z}$. Given an orientation $\tilde{o} : H^n(V) \rightarrow \mathbb{R}$, there is an orientation $o^V : V^n \rightarrow \mathbb{R}$ such that $\tilde{o} = o^H$. If $dV^n = 0$, then we have the correspondence*

$$\text{Orientations on } V \text{ in degree } n \stackrel{1:1}{\simeq} \text{Orientations on } H(V) \text{ in degree } n.$$

- (b) *If (V_1, d_1, o_1) and (V_2, d_2, o_2) are \mathbb{Z} -graded cochain complexes oriented in degree n with $dV_1 \simeq dV_2 = 0$, then a chain map $f : V_1 \rightarrow V_2$ preserves the orientation if and only if the induced map $f_* : H(V_1) \rightarrow H(V_2)$ preserves the induced orientation.*

Proof. (a) We define $o^V(v) := \tilde{o}([v])$ for all closed $v \in V$ and extend it by 0 to a complement of $\ker d$ in V . It is obvious that $o^H = \tilde{o}$. If $dV^n = 0$, any complement is trivial and o^V is uniquely determined by o^H .

- (b) Let $v \in V_1^n$. Because $dv = 0$, we have

$$o_2(f(v)) = o_2^H([f(v)]) = o_2^H(f_*[v]) = o_1^H([v]) = o_1(v).$$

This finishes the proof. □

Suppose that 1 is a unit for (V, d, \wedge) , i.e., $1 \in V^0$, $d1 = 0$ and $1 \wedge v = v \wedge 1 = v$ for all $v \in V$, and let $\langle \cdot, \cdot \rangle$ be a cyclic structure on V . For all homogenous $v_1, v_2 \in V$, we have

$$\begin{aligned} \langle v_1, v_2 \rangle &= \langle v_1 \wedge 1, v_2 \rangle \\ &= (-1)^{v_1 v_2} \langle 1 \wedge v_2, v_1 \rangle \\ &= (-1)^{v_1 v_2} \langle v_2, v_1 \rangle, \end{aligned}$$

and hence $\langle \cdot, \cdot \rangle$ is automatically symmetric.

6. IBL-infinity formality and Poincaré duality models

Recall that a DGA is called commutative if $v_1 \wedge v_2 = (-1)^{v_1 v_2} v_2 \wedge v_1$ for all homogenous $v_1, v_2 \in V$. Commutativity of a DGA and symmetricity of a general cyclic structure on it seem to be unrelated.

Proposition 6.1.4 (Correspondence of orientations and cyclic structures on DGA's). *Let (V, d, \wedge) be a DGA. Then the following holds:*

(a) *If \wedge is commutative, then any orientation \circ in degree n induces a cyclic structure $\langle \cdot, \cdot \rangle$ of degree n which is given for all homogenous $v_1, v_2 \in V$ by*

$$\langle v_1, v_2 \rangle := \begin{cases} \circ(v_1 \wedge v_2) & \text{if } \deg(v_1) + \deg(v_2) = n, \\ 0 & \text{otherwise.} \end{cases} \quad (6.4)$$

(b) *If 1 is a unit, then any non-zero cyclic structure $\langle \cdot, \cdot \rangle$ of degree n induces an orientation \circ in degree n by defining*

$$\circ(v) := \begin{cases} \langle v, 1 \rangle & \text{for } v \in V^n, \\ 0 & \text{otherwise.} \end{cases} \quad (6.5)$$

(c) *For a unital commutative DGA $(V, d, \wedge, 1)$, formulas (6.4) and (6.5) define the correspondence*

$$\text{Orientations in degree } n \stackrel{1:1}{\cong} \text{Non-zero cyclic structures of degree } n.$$

Proof. (a) Using the Leibnitz identity, properties of an orientation and commutativity, we check that

$$\begin{aligned} \langle dv_1, v_2 \rangle &= \circ(dv_1 \wedge v_2) \\ &= \circ(d(v_1 \wedge v_2) - (-1)^{v_1} v_1 \wedge dv_2) \\ &= (-1)^{1+v_1} \circ(v_1 \wedge dv_2) \\ &= (-1)^{1+v_1 v_2} \circ(dv_2 \wedge v_1) \\ &= (-1)^{1+v_1 v_2} \langle dv_2, v_1 \rangle \end{aligned}$$

6. IBL-infinity formality and Poincaré duality models

and

$$\begin{aligned}\langle v_1 \wedge v_2, v_3 \rangle &= o(v_1 \wedge v_2 \wedge v_3) \\ &= (-1)^{v_3(v_1+v_2)} o(v_3 \wedge v_1 \wedge v_2) \\ &= (-1)^{v_3(v_1+v_2)} \langle v_3 \wedge v_1, v_2 \rangle\end{aligned}$$

for all homogenous $v_1, v_2, v_3 \in V$.

(b) For all $v \in V$, we have

$$o(dv) = \langle dv, 1 \rangle = -\langle d1, v \rangle = 0.$$

From $\langle \cdot, \cdot \rangle \neq 0$ it follows that there are homogenous $v_1, v_2 \in V$ with $\deg(v_1) + \deg(v_2) = n$ such that $\langle v_1, v_2 \rangle \neq 0$. Then $v_1 \wedge v_2 \in V^n$ and

$$\begin{aligned}o(v_1 \wedge v_2) &= \langle v_1 \wedge v_2, 1 \rangle \\ &= \langle 1 \wedge v_1, v_2 \rangle \\ &= \langle v_1, v_2 \rangle \neq 0.\end{aligned}$$

Therefore, o is an orientation on V .

(c) This is a combination of (a) and (c) plus the uniqueness, which is easy to check. \square

Remark 6.1.5 (Volume forms). If $H^n(V) \simeq \mathbb{R}$, then orientations $o : H^n(V) \rightarrow \mathbb{R}$ and elements $0 \neq [\text{Vol}] \in H^n(V)$ called *volume forms* are in one-to-one correspondence via

$$o([\text{Vol}]) = 1.$$

A consequence is the following: Suppose that (V_1, d_1, o_1) and (V_2, d_2, o_2) are cochain complexes oriented in degree n which satisfy

$$H^n(V_1) \simeq H^n(V_2) \simeq \mathbb{R} \quad \text{and} \quad d_1 V_1^n \simeq d_2 V_2^n = 0, \quad (6.6)$$

so that Proposition 6.1.3 applies. Then a chain map $f : V_1 \rightarrow V_2$ preserves orientation if and only if the induced map $f_* : H^n(V_1) \rightarrow H^n(V_2)$ maps $[\text{Vol}_1]$ to $[\text{Vol}_2]$. In the category of unital commutative DGA's satisfying (6.6), so that also Proposition 6.1.4 holds, if the orientations come from cyclic structures, then f preserves cyclic structure if and only if f_* preserves volume form. \triangleleft

6. IBL-infinity formality and Poincaré duality models

Definition 6.1.6 (Non-degeneracy and Poincaré duality). *A symmetric homogenous bilinear form (\equiv : pairing) $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ of degree $-n$ (as a map) on a graded vector space V is called non-degenerate if for every $v \in V$, the following implication holds:*

$$\langle v, w \rangle = 0 \quad \text{for all } w \in V \quad \implies \quad v = 0.$$

We say that $\langle \cdot, \cdot \rangle$ satisfies Poincaré duality if the map $\flat : V \rightarrow V^{*\mathfrak{g}}$ (graded dual) defined by

$$\flat(v)(w) := \langle v, w \rangle \quad \text{for all } v, w \in V$$

is a graded isomorphism of graded vector spaces (it has degree $-n$ as a map).

Remark 6.1.7 (On non-degeneracy and Poincaré duality). (i) Clearly, Poincaré duality implies non-degeneracy. If the degree k component V^k of V is finite-dimensional for every $k \in \mathbb{Z}$ — we say that V is of *finite type* — then the opposite is true as well. Note that Poincaré duality does not imply that V is of finite type since there are infinite-dimensional vector spaces with $V^* \simeq V$.

(ii) If V is non-negatively graded, then non-degeneracy of $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ implies $V = V^0 \oplus \cdots \oplus V^n$. Therefore, non-negatively graded vector spaces of finite type which admit a non-degenerate homogenous bilinear form are finite-dimensional.

(iii) The relation of non-degeneracy and Poincaré duality for V and for $H(V)$ is not obvious. For instance, the de Rham complex $(\Omega(M), d, \wedge)$ of an oriented closed n -manifold M with the orientation $\int : \Omega^n(M) \rightarrow \mathbb{R}$ is a non-degenerate oriented DGA whose induced cyclic structure does not satisfy Poincaré duality. On the other hand, the induced structure on homology $H(\Omega(M))$ satisfies Poincaré duality. \triangleleft

An analog of the following definition is used in [Fio+19] and also in [LS07] (under the name the “set of orphans”).

Definition 6.1.8 (Degenerate subspace and non-degenerate quotient). *Given a symmetric pairing $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ on a graded vector space V , we define the degenerate subspace $V^\perp \subset V$ by*

$$V^\perp := \{v \in V \mid \langle w, v \rangle = 0 \text{ for all } w \in V\}.$$

If (V, d, \wedge) is a DGA and $\langle \cdot, \cdot \rangle$ a cyclic structure, then (6.1) implies that V^\perp is a differential graded ideal in V , and thus we obtain the short exact sequence of DGA's

$$0 \longrightarrow V^\perp \xhookrightarrow{\iota} V \xrightarrow{\pi^\mathcal{Q}} \mathcal{Q}(V) := V/V^\perp \longrightarrow 0, \quad (6.7)$$

6. IBL-infinity formality and Poincaré duality models

where ι is the inclusion and $\pi^{\mathcal{Q}}$ the canonical projection. We call the DGA $\mathcal{Q}(V)$ together with the induced non-degenerate cyclic structure $\langle \cdot, \cdot \rangle^{\mathcal{Q}}$ such that $\langle \pi^{\mathcal{Q}}(\cdot), \pi^{\mathcal{Q}}(\cdot) \rangle^{\mathcal{Q}} = \langle \cdot, \cdot \rangle$ the non-degenerate quotient.

It was observed in [Fio+19] that the question whether (V^{\perp}, d) is acyclic, and hence $\pi^{\mathcal{Q}}$ is a quasi-isomorphism, turns out to be related to the existence of Hodge decomposition.

Definition 6.1.9 (Hodge decomposition). *A cochain complex (V, d) with a symmetric cyclic structure $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ is of Hodge type if there exist subspaces $\mathcal{H} \subset \ker d$ and $C \subset V$ such that*

$$V = \ker d \oplus C, \quad \ker d = \operatorname{im} d \oplus \mathcal{H} \quad \text{and} \quad C \perp \mathcal{H} \oplus C, \quad (6.8)$$

where \perp denotes the relation of being perpendicular with respect to $\langle \cdot, \cdot \rangle$. Such decomposition is called a Hodge decomposition. We call \mathcal{H} the harmonic subspace and C the coexact part.

Given a Hodge decomposition, we define the standard Hodge homotopy $\mathcal{P}_{\text{std}} : V \rightarrow V$ by

$$\mathcal{P}_{\text{std}} := \begin{cases} -(d|_C)^{-1} & \text{on } \operatorname{im} d, \\ 0 & \text{on } \mathcal{H} \oplus C. \end{cases}$$

Then we have $d\mathcal{P}_{\text{std}} = -\pi_{\operatorname{im} d}$, $\mathcal{P}_{\text{std}}d = -\pi_C$, and hence

$$[d, \mathcal{P}_{\text{std}}] = d\mathcal{P}_{\text{std}} + \mathcal{P}_{\text{std}}d = \pi_{\mathcal{H}} - \mathbb{1}.$$

We call $(\mathcal{H}, \mathcal{P}_{\text{std}})$ the Hodge pair associated to the Hodge decomposition (6.8).

Proposition 6.1.10 (Non-deg., fin. type implies Hodge type). *Any cochain complex of finite type with a non-degenerate symmetric cyclic structure is of Hodge type.*

Proof. This is [CFL15, Lemma 11.1], and the proof uses formal Hodge theory. \square

Remark 6.1.11 (Harmonic subspaces). In the situation of Proposition 6.1.10, it was shown in [Fio+19, Remark 2.6] that for any section $s : H(V) \rightarrow \ker d$ of the canonical projection $\pi : \ker d \rightarrow H(V)$, there is a coexact part C such that $V = \operatorname{im} s \oplus \operatorname{im} d \oplus C$ is a Hodge decomposition. From this reason, we call any $\operatorname{im} s$ a harmonic subspace. \triangleleft

The following lemma will be used in the proof of Proposition 6.1.13.

6. IBL-infinity formality and Poincaré duality models

Lemma 6.1.12 (Complement of acyclic subcomplex over \mathbb{R}). *Let $f : V_1 \rightarrow V_2$ be an injective chain map of cochain complexes (V_1, d_1) and (V_2, d_2) over \mathbb{R} such that (V_1, d_1) is acyclic. Then there is a chain map $g : V_2 \rightarrow V_1$ such that $g \circ f = \mathbb{1}$.³*

Proof. For every $i \in \mathbb{Z}$, consider the diagram

$$\begin{array}{ccc} \ker d_1^i \oplus C_1^i & \xrightarrow{f^i} & \ker d_2^i \oplus C_2^i \\ \downarrow d_1^i & & \downarrow d_2^i \\ \ker d_1^{i+1} \oplus C_1^{i+1} & \xrightarrow{f^{i+1}} & \ker d_2^{i+1} \oplus C_2^{i+1}, \end{array}$$

where C_1^i is a complement of $\ker d_1^i$ in V_1^i and C_2^i is a complement of $\ker d_2^i$ in V_2^i . With respect to this decomposition, we write

$$\begin{aligned} f^i &= \begin{pmatrix} f_{11}^i & f_{12}^i \\ f_{21}^i & f_{22}^i \end{pmatrix}, & g^i &= \begin{pmatrix} g_{11}^i & g_{12}^i \\ g_{21}^i & g_{22}^i \end{pmatrix}, \\ d_1^i &= \begin{pmatrix} 0 & d_1^i \\ 0 & 0 \end{pmatrix}, & d_2^i &= \begin{pmatrix} 0 & d_2^i \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The assumption $H(V_1) = 0$ implies that d_1^i is an isomorphism. The fact that f is a chain map translates to

$$d_2^i f_{21}^i = 0, \quad f_{21}^{i+1} d_1^i = 0, \quad f_{11}^{i+1} d_1^i = d_2^i f_{22}^i. \quad (6.9)$$

From the second relation and surjectivity of d_1^i , we get that $f_{21}^{i+1} = 0$. Now, f_{11}^{i+1} has to be injective because it is the only possibly non-zero part of f on $\ker d_1^{i+1}$. From the third relation of (6.9) and the fact that d_1^i is injective, we get that f_{22}^i is injective as well. Relations (6.9) hold also for g with d_1 and d_2 switched. In particular, we have $g_{21}^i = 0$. The relation $g \circ f = \mathbb{1}$ translates using $f_{21}^i = g_{21}^i = 0$ to

$$g_{11}^i f_{11}^i = \mathbb{1}, \quad g_{11}^i f_{12}^i + g_{12}^i f_{22}^i = 0, \quad g_{22}^i f_{22}^i = \mathbb{1}. \quad (6.10)$$

Because d_1^i is an isomorphism, the last equation is equivalent to

$$\mathbb{1} = d_1^i g_{22}^i f_{22}^i (d_1^i)^{-1} = g_{11}^{i+1} d_2^i f_{22}^i (d_1^i)^{-1} = g_{11}^{i+1} f_{11}^{i+1} d_1^i (d_1^i)^{-1} = g_{11}^{i+1} f_{11}^{i+1}.$$

We see that g can be constructed as follows. For all $i \in \mathbb{Z}$, let g_i^{11} be an arbitrary

³This lemma can be used to prove that over \mathbb{R} , every surjective quasi-isomorphism is a deformation retraction and every injective quasi-isomorphism is a section of a deformation retraction.

6. IBL-infinity formality and Poincaré duality models

left inverse of f_i^{11} . Set $g_{22}^i := (d_1^i)^{-1} g_{11}^{i+1} d_2^i$ and $g_i^{21} := 0$. Finally, g_{12}^i has to be chosen such that the second equation of (6.10) is satisfied. This is possible since we can first define g_{12}^i on $\text{im } f_{22}^i$ because f_{22}^i injective and then extend it by 0 to a complement. \square

Claim (b) of the following proposition corresponds to [Fio+19, Lemma 2.8]. Claim (c) was suggested by Prof. Hồng Vân Lê via e-mail correspondence.

Proposition 6.1.13 (Hodge decomposition and acyclicity of V^\perp). *Let (V, d) be a cochain complex with a symmetric cyclic structure $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$. If V is of Hodge type, then the following implications hold:*

(a) *If $\langle \cdot, \cdot \rangle$ is non-degenerate, then $\langle \cdot, \cdot \rangle^H$ is non-degenerate.*

(b) *If $\langle \cdot, \cdot \rangle^H$ is non-degenerate, then (V^\perp, d) is acyclic.*

Moreover, the following reverse implication holds:

(c) *If V is of finite type and (V^\perp, d) is acyclic, then V is of Hodge type.*

Proof. (a) Let $V = \text{im } d \oplus \mathcal{H} \oplus C$ be a Hodge decomposition. Then $\text{im } d \oplus C \subset \mathcal{H}^\perp$, and hence

$$\langle d\eta + b + c, b' \rangle = \langle b, b' \rangle \quad \text{for all } \eta \in V, c \in C \text{ and } b, b' \in \mathcal{H}.$$

The claim follows easily. Notice that having a Hodge decomposition, it holds

$$\langle \cdot, \cdot \rangle|_{\mathcal{H} \otimes \mathcal{H}} \text{ non-degenerate} \iff \text{im } d \oplus C = \mathcal{H}^\perp.$$

(b) Consider a Hodge decomposition as above, and let $v \in V^\perp \cap \ker d$ be a non-zero vector. Suppose that $v \notin \text{im } d$. Then $[v] \neq 0$ in $H(V)$, and hence there is a $b \in \mathcal{H}$ such that $\langle v, b \rangle \neq 0$ by non-degeneracy of $\langle \cdot, \cdot \rangle^H$. This is a contradiction with $v \in V^\perp$. Therefore, it holds $V^\perp \cap \ker d = V^\perp \cap \text{im } d$. In particular, there is an $\eta \in C$ such that $v = d\eta$. Now, for any $\eta' \in C$, $b \in \mathcal{H}$ and $c \in C$, we have using $C \perp \mathcal{H} \oplus C$ and $v \in V^\perp$ the following:

$$\begin{aligned} \langle d\eta' + b + c, \eta \rangle &= \langle d\eta', \eta \rangle \\ &= (-1)^{1+\eta\eta'} \langle d\eta, \eta' \rangle \\ &= (-1)^{1+\eta\eta'} \langle v, \eta' \rangle \\ &= 0. \end{aligned}$$

Therefore, it holds $\eta \in V^\perp$, and we have shown that $V^\perp \cap \text{im } d = dV^\perp$. The claim follows.

6. IBL-infinity formality and Poincaré duality models

(c) Because $V^\perp \subset V$ is an acyclic subcomplex and we work over \mathbb{R} , there is a complementary subcomplex $Z \subset V$; i.e., $dZ \subset Z$ and $V = V^\perp \oplus Z$. This follows from Lemma 6.1.12 by setting $V^\perp = \text{im } f$ and $Z = \ker g$. Now, the restriction of $\langle \cdot, \cdot \rangle$ to Z is non-degenerate, and Proposition 6.1.10 provides its Hodge decomposition $Z = dZ \oplus \mathcal{H} \oplus D$. Let $E \subset V^\perp$ be a graded vector space which is complementary to dV^\perp in V^\perp ; i.e., $V^\perp = dV^\perp \oplus E$. It is easy to check that $V = \text{im } d \oplus \mathcal{H} \oplus C$ with $C := D \oplus E$ is a Hodge decomposition. \square

The following notions were taken from [Fio+19].

Definition 6.1.14 (Hodge subalgebra and small subalgebra). *Consider a DGA (V, d, \wedge) with a symmetric cyclic structure $\langle \cdot, \cdot \rangle$. Suppose that it admits a Hodge decomposition with the Hodge pair $(\mathcal{H}, \mathcal{P}_{\text{std}})$. A Hodge subalgebra is a differential graded subalgebra $W \subset V$ which satisfies*

$$\mathcal{H} \subset W \quad \text{and} \quad \mathcal{P}_{\text{std}} W \subset W.$$

We denote the smallest Hodge subalgebra of V by $\mathcal{S}(V)$ and call it the small subalgebra. We stress that the definition of $\mathcal{S}(V)$ depends on $(\mathcal{H}, \mathcal{P}_{\text{std}})$!

The following is a version of [Fio+19, Proposition 3.3] which generalizes to the non-simply-connected case (see (iii) of Remark 6.1.16 below for the comparison).

Proposition 6.1.15 (Description of small subalgebra). *Consider the situation of Definition 6.1.14. The small subalgebra $\mathcal{S}(V)$ is generated as a graded vector space by Kontsevich-Soibelman-like evaluations of rooted binary trees with $k \geq 1$ leaves labeled with homogenous elements of \mathcal{H} , interior vertices labeled with \wedge and interior edges labeled either with \mathcal{P}_{std} or with $\mathbb{1}$.*

Proof. We abbreviate $\mathcal{S} := \mathcal{S}(V)$, denote the set of labeled trees by \mathcal{T} and denote the vector space generated by evaluations of elements of \mathcal{T} by $\langle \mathcal{T} \rangle$.

Clearly, we have $\langle \mathcal{T} \rangle \subset \mathcal{S}$. For “=”, it suffices to check that for any $T, T_1, T_2 \in \mathcal{T}$, it holds $dT, \mathcal{P}_{\text{std}} T, T_1 \wedge T_2 \in \langle \mathcal{T} \rangle$, i.e., that $\langle \mathcal{T} \rangle$ is a Hodge subalgebra.

As for dT , we imagine d propagating from the root to the leaves. When it encounters $L \wedge R$, where L stands for the left and R for the right sub-branch, it duplicates the tree and continues propagating in L and R , respectively (we take the sum of the two copies in the end). This is justified by the Leibnitz identity $d(L \wedge R) = dL \wedge R + (-1)^L L \wedge dR$. When it encounters \mathcal{P}_{std} , it triples the tree and either goes past \mathcal{P}_{std} and continues propagating, or exchanges \mathcal{P}_{std} for $\mathbb{1}$ and stops, or exchanges \mathcal{P}_{std} for $\pi_{\mathcal{H}}$ and stops. This is justified by $d\mathcal{P}_{\text{std}} = -\mathcal{P}_{\text{std}}d - \mathbb{1} + \pi_{\mathcal{H}}$. If it encounters $\mathbb{1}$, nothing happens and it keeps propagating. If it reaches a leaf with $h \in \mathcal{H}$, then the corresponding tree evaluates to 0. We see that we always obtain an element of $\langle \mathcal{T} \rangle$.

6. IBL-infinity formality and Poincaré duality models

As for $\mathcal{P}_{\text{std}}T$, we have either $\mathcal{P}_{\text{std}}T = 0$ if the interior edge adjacent to the root ($=$: the root edge) is labeled with \mathcal{P}_{std} , or $\mathcal{P}_{\text{std}}T$ is a new tree in \mathcal{T} which arises from T by replacing $\mathbb{1}$ by \mathcal{P}_{std} on the root edge.

As for $T_1 \wedge T_2$, using $\mathbb{1} = \pi_{\mathcal{H}} - d\mathcal{P}_{\text{std}} - \mathcal{P}_{\text{std}}d$ and the Leibnitz identity, we get

$$\begin{aligned} T_1 \wedge T_2 &= \pi_{\mathcal{H}}(T_1 \wedge T_2) - (d\mathcal{P}_{\text{std}})(T_1 \wedge T_2) - (\mathcal{P}_{\text{std}}d)(T_1 \wedge T_2) \\ &= \pi_{\mathcal{H}}(T_1 \wedge T_2) - d(\mathcal{P}_{\text{std}}(T_1 \wedge T_2)) - \mathcal{P}_{\text{std}}(dT_1 \wedge T_2) - (-1)^{T_1} \mathcal{P}_{\text{std}}(T_1 \wedge dT_2). \end{aligned}$$

We see that $T_1 \wedge T_2 \in \langle \mathcal{T} \rangle$. \square

Remark 6.1.16 (On Hodge subalgebra and small subalgebra). (i) Any Hodge subalgebra W inherits the Hodge decomposition

$$W = \mathcal{H} \oplus dW \oplus \mathcal{P}_{\text{std}}W \tag{6.11}$$

with the Hodge pair $(\mathcal{H}, \mathcal{P}_{\text{std}}|_W)$.

(ii) We will be in the situation of Definition 6.1.14 and use the notation of the proof of Proposition 6.1.15. In addition, we suppose that V is non-negatively graded. Having established $\mathcal{S} = \langle \mathcal{T} \rangle$, consider the Hodge decomposition (6.11) for $W = \mathcal{S}$. Let $\bar{\mathcal{S}} = \bigoplus_{k \geq 1} \mathcal{S}^k$ denote the reduced part of \mathcal{S} , and let $\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle$ denote the graded vector space generated by products $v_1 \wedge v_2$ for $v_1, v_2 \in \bar{\mathcal{S}}$. We have

$$\begin{aligned} \mathcal{P}_{\text{std}}\langle \mathcal{T} \rangle &= \langle \{T \in \mathcal{T} \text{ with } \mathcal{P}_{\text{std}} \text{ on the root edge}\} \rangle \\ &= \mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle \end{aligned}$$

and

$$\begin{aligned} d\langle \mathcal{T} \rangle &= d(\pi_{\mathcal{H}} - d\mathcal{P}_{\text{std}} - \mathcal{P}_{\text{std}}d)\langle \mathcal{T} \rangle \\ &= d\mathcal{P}_{\text{std}}d\langle \mathcal{T} \rangle \\ &\subset d\mathcal{P}_{\text{std}}\langle \mathcal{T} \rangle \\ &\subset d\mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle. \end{aligned}$$

It holds even $d\langle \mathcal{T} \rangle = d\mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle$ due to (6.11). We can now write (6.11) as

$$\mathcal{S}^k = \mathcal{H}^k \oplus d\mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle^k \oplus \mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle^{k+1}. \tag{6.12}$$

If $H^1 = 0$, this agrees with the formula from [Fio+19, Proposition 3.3]. In this case, it

6. IBL-infinity formality and Poincaré duality models

holds $\mathcal{S}^1 = 0$, and hence $\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle^{k+1}$ depends only on \mathcal{S}^i for $i < k$. Therefore, we can compute \mathcal{S}^k from (6.12) inductively starting with $\mathcal{S}^0 = \langle 1 \rangle$.

(iii) The previous remark implies the following: Let (V, d, \wedge) be a non-negatively graded DGA with cyclic structure $\langle \cdot, \cdot \rangle$ of Hodge type. Suppose that $H^1(V) = 0$. If $H(V)$ is of finite type, then so is $\mathcal{S}(V)$.

(iv) The subalgebra $\mathcal{S}(V)$ resolves V ; i.e., there is a DGA-quasi-isomorphism $\mathcal{S}(V) \hookrightarrow V$ (the inclusion). Consider the smallest subalgebra $\langle \mathcal{H} \rangle^\wedge$ containing \mathcal{H} ; similarly as $\mathcal{S}(V)$, it is generated as graded vector space by evaluations of rooted binary trees with $\mathbb{1}$ on interior edges. The DGA $\langle \mathcal{H} \rangle^\wedge$ also resolves V . The advantage of $\mathcal{S}(V)$ over $\langle \mathcal{H} \rangle^\wedge$ is that it is of Hodge type and is thus quasi-isomorphic to its non-degenerate quotient $\mathcal{Q}(\mathcal{S}(V))$. Therefore, we have the diagram of pairing preserving DGA-quasi-isomorphisms

$$V \longleftarrow \mathcal{S}(V) \longrightarrow \mathcal{Q}(\mathcal{S}(V)).$$

If V is non-negatively graded and $\mathcal{S}(V)$ is of finite type, e.g., as in (iii), then $\mathcal{Q}(\mathcal{S}(V))$ is finite-dimensional. \triangleleft

Proposition 6.1.17 (Properties of \mathcal{S} and \mathcal{Q}). *Let (V, d, \wedge) be a DGA with a symmetric cyclic structure $\langle \cdot, \cdot \rangle$. Suppose that it admits a Hodge decomposition with Hodge pair $(\mathcal{H}, \mathcal{P}_{\text{std}})$. Then $\mathcal{S}(V)$ and $\mathcal{Q}(V)$ admit Hodge decompositions with the Hodge pairs $(\mathcal{H}, \mathcal{P}_{\text{std}}^\mathcal{S} := \mathcal{P}_{\text{std}}|_{\mathcal{S}(V)})$ and $(\pi^\mathcal{Q}(\mathcal{H}), \mathcal{P}_{\text{std}}^\mathcal{Q})$, respectively, where $\pi^\mathcal{Q} : V \rightarrow \mathcal{Q}(V)$ is the canonical projection and $\mathcal{P}_{\text{std}}^\mathcal{Q}$ the unique map on $\mathcal{Q}(V)$ satisfying $\mathcal{P}_{\text{std}}^\mathcal{Q} \circ \pi^\mathcal{Q} = \pi^\mathcal{Q} \circ \mathcal{P}_{\text{std}}$. Furthermore, with respect to the induced Hodge decompositions, we have*

$$\mathcal{S}(\mathcal{S}(V)) = \mathcal{S}(V), \quad \mathcal{Q}(\mathcal{Q}(V)) = \mathcal{Q}(V) \quad \text{and} \quad \mathcal{S}(\mathcal{Q}(\mathcal{S}(V))) = \mathcal{Q}(\mathcal{S}(V)). \quad (6.13)$$

Proof. The fact that $\mathcal{S} := \mathcal{S}(V)$ admits a Hodge decomposition with Hodge pair $(\mathcal{H}, \mathcal{P}_{\text{std}}^\mathcal{S})$ was stated in (i) of Remark 6.1.16 and is easy to check.

Let us prove that $\mathcal{Q} := \mathcal{Q}(V)$ has a Hodge decomposition with Hodge pair $(\pi^\mathcal{Q}(\mathcal{H}), \mathcal{P}_{\text{std}}^\mathcal{Q})$. Because $\pi^\mathcal{Q}$ is a quasi-isomorphism, $\pi^\mathcal{Q}(\mathcal{H})$ is a harmonic subspace of \mathcal{Q} , and we have $\ker d^\mathcal{Q} = \pi^\mathcal{Q}(\mathcal{H}) \oplus \text{im } d^\mathcal{Q}$. Let $c \in C$ with $dc \in V^\perp$ such that $\pi^\mathcal{Q}(c) \in \ker d^\mathcal{Q} \cap \pi^\mathcal{Q}(C)$. From the cyclicity of $\langle \cdot, \cdot \rangle$ with respect to d and from $C \perp \mathcal{H} \oplus C$, it follows that $c \in V^\perp$, and thus $\pi^\mathcal{Q}(c) = 0$. Together with surjectivity of $\pi^\mathcal{Q}$ this implies that $\mathcal{Q} = \ker d^\mathcal{Q} \oplus \pi^\mathcal{Q}(C)$. From $\langle \pi^\mathcal{Q}(\cdot), \pi^\mathcal{Q}(\cdot) \rangle^\mathcal{Q}$, we see that $\pi^\mathcal{Q}(C) \perp \pi^\mathcal{Q}(\mathcal{H}) \oplus \pi^\mathcal{Q}(C)$. Therefore, $\mathcal{Q} = \pi^\mathcal{Q}(\mathcal{H}) \oplus \text{im } d^\mathcal{Q} \oplus \pi^\mathcal{Q}(C)$ is a Hodge decomposition, and it is easy to see that its standard Hodge homotopy $\mathcal{P}_{\text{std}}^\mathcal{Q}$ satisfies $\mathcal{P}_{\text{std}}^\mathcal{Q} \circ \pi^\mathcal{Q} = \pi^\mathcal{Q} \circ \mathcal{P}_{\text{std}}$. This defines $\mathcal{P}_{\text{std}}^\mathcal{Q}$ uniquely because $\pi^\mathcal{Q}$ is surjective.

6. IBL-infinity formality and Poincaré duality models

As for the relations (6.13), the first two are clear. The third can be seen as follows. Since $\pi^{\mathcal{Q}} : V \rightarrow \mathcal{Q}$ is a DGA-morphism mapping the harmonic subspaces isomorphically onto each other and commuting with d and \mathcal{P}_{std} , the assignments $Y \subset V \mapsto \pi^{\mathcal{Q}}(Y) \subset \mathcal{Q}$ and $Z \subset \mathcal{Q} \mapsto (\pi^{\mathcal{Q}})^{-1}(Z) \subset V$ preserve Hodge subalgebras. Therefore, if $Z \subset \mathcal{Q}(\mathcal{S})$ is a Hodge subalgebra, then $(\pi^{\mathcal{Q}})^{-1}(Z) \subset \mathcal{S}$ is a Hodge subalgebra. It holds even $(\pi^{\mathcal{Q}})^{-1}(Z) = \mathcal{S}$ by minimality of \mathcal{S} , and hence $Z = \mathcal{Q}(\mathcal{S})$ by surjectivity of $\pi^{\mathcal{Q}}$. \square

Notice that solely the fact that $\langle \cdot, \cdot \rangle$ is non-degenerate on V does not imply that $\mathcal{S}(V) \simeq V$. A counterexample is $V = \Omega(M)$, where M is a compact oriented manifold with $H_{\text{dR}}^1(M) = 0$; indeed, by (iii) of Remark 6.1.16, $\mathcal{S}(\Omega(M))$ is of finite type because $H_{\text{dR}}(M)$ is, but $\Omega(M)$ is not of finite type.

A natural question is how does $\mathcal{Q}(\mathcal{S}(V))$ depend on the chosen Hodge pair and how does it behave under quasi-isomorphisms. We hope that the following lemmas will be useful in answering this question.

Lemma 6.1.18 (Kernel of pairing-preserving morphism). *Let V_1 and V_2 be vector spaces with symmetric bilinear forms $\langle \cdot, \cdot \rangle_1 : V_1 \otimes V_1 \rightarrow \mathbb{R}$ and $\langle \cdot, \cdot \rangle_2 : V_2 \otimes V_2 \rightarrow \mathbb{R}$, respectively. Let $f : V_1 \rightarrow V_2$ be a linear map such that*

$$\langle v_1, v_2 \rangle_1 = \langle f(v_1), f(v_2) \rangle_2 \quad \text{for all } v_1, v_2 \in V_1. \quad (6.14)$$

Then it holds

$$\ker f \subset V_1^\perp \quad \text{and} \quad f(V_1^\perp) \subset f(V_1)^\perp.$$

In particular, the following statements are true:

- (a) *If $\langle \cdot, \cdot \rangle_1$ is non-degenerate, then f is injective.*
- (b) *If $\langle \cdot, \cdot \rangle_2$ is non-degenerate and f is surjective, then $\ker f = V_1^\perp$.*

Proof. Clear. \square

Lemma 6.1.19 (Injectivity on domain with non-degenerate orientation). *Let (V_1, d_1, \wedge_1) be a non-negatively graded commutative DGA with an orientation \circ_1 in degree n such that the induced cyclic structure is non-degenerate. Let (V_2, d_2, \wedge_2) be any DGA, and let $f : V_1 \rightarrow V_2$ be a morphism of DGA's. Then injectivity of $f_* : H^n(V_1) \rightarrow H^n(V_2)$ implies injectivity of $f : V_1 \rightarrow V_2$.*

Proof. Firstly, injectivity of a homogenous map is equivalent to degree-wise injectivity. Secondly, because V_1 is non-negatively graded and the cyclic structure of degree n is

6. IBL-infinity formality and Poincaré duality models

non-degenerate, we have $V_1 = V_1^0 \oplus \cdots \oplus V_1^n$. Now, suppose that $v \in V_1^k$ for some $k = 0, \dots, n$ satisfies $f(v) = 0$. For any $w \in V^{n-k}$, the product $v \wedge w$ lies in V^n , and thus $d(v \wedge w) = 0$. We compute

$$f_*[v \wedge w] = [f(v \wedge w)] = [f(v) \wedge f(w)] = 0,$$

and hence $v \wedge w = d\eta$ for some $\eta \in V_1^{n-1}$ by injectivity of f_* . Consequently, we have

$$o_1(v \wedge w) = o_1(d\eta) = 0,$$

and hence $v = 0$ by non-degeneracy of o_1 . □

Lemma 6.1.20 (Small subalgebra of cyclic DGA and quasi-iso.). *Let $(V_1, d_1, \wedge_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, d_2, \wedge_2, \langle \cdot, \cdot \rangle_2)$ be finite-dimensional non-negatively graded unital commutative DGA's with non-degenerate cyclic structures of degree n . Let $f : V_1 \rightarrow V_2$ be a DGA-morphism such that $f_* : (H(V_1), o_1^H) \rightarrow (H(V_2), o_2^H)$ is an isomorphism. Then a Hodge decomposition of V_1 with Hodge pair $(\mathcal{H}_1, \mathcal{P}_{\text{std}}^1)$ induces a Hodge decomposition of V_2 with Hodge pair $(f(\mathcal{H}_1), \mathcal{P}_{\text{std}}^2)$, where $\mathcal{P}_{\text{std}}^2$ satisfies $\mathcal{P}_{\text{std}}^2 \circ f = f \circ \mathcal{P}_{\text{std}}^1$. Consequently, f induces an isomorphism $\mathcal{S}(V_1) \simeq \mathcal{S}(V_2)$.*

Proof. Because f_* preserves orientation and it holds $V_1^{n+1} \simeq V_2^{n+1} = 0$, Proposition 6.1.3 implies that f preserves cyclic structure and hence is injective by Lemma 6.1.18. Let $V_1 = \mathcal{H}_1 \oplus \text{im } d_1 \oplus C_1$ be a Hodge decomposition. From the injectivity of f , it follows that

$$f(V_1) = f(\mathcal{H}_1) \oplus df(V_1) \oplus f(C_1)$$

and that the restriction of $\langle \cdot, \cdot \rangle_2$ to $f(V_1) \otimes f(V_1)$ is non-degenerate. Because V_1 is of finite type, $f(V_1)$ is of finite type, and so non-degeneracy implies Poincaré duality $f(V_1)^{*g} \simeq f(V_1)$. It follows that

$$V_2 = f(V_1) \oplus f(V_1)^\perp.$$

Cyclicity of d_2 with respect to $\langle \cdot, \cdot \rangle_2$ implies that $f(V_1)^\perp \subset V_2$ is a subcomplex. Because f is a quasi-isomorphism, we have $H(V_1) \simeq H(f(V_1)) \simeq H(V_2)$; because the homology is additive, we have $H(V_2) \simeq H(f(V_1)) \oplus H(f(V_1)^\perp) \simeq H(V_2) \oplus H(f(V_1)^\perp)$; finally, because $H(V_2)$ is of finite type, we have $H(f(V_1)^\perp) = 0$. Because $V_2 = f(V_1) \oplus f(V_1)^\perp$ and $\langle \cdot, \cdot \rangle_2$ is non-degenerate, its restriction to $f(V_1)^\perp$ is non-degenerate too. As V_2 and hence $f(V_1)^\perp$ is of finite type, Proposition 6.1.10 gives a Hodge decomposition $f(V_1)^\perp = df(V_1)^\perp \oplus C_2'$.

6. IBL-infinity formality and Poincaré duality models

It is easy to check that

$$V_2 = f(\mathcal{H}_1) \oplus \underbrace{(df(V_1) \oplus df(V_1)^\perp)}_{= \operatorname{im} d_2} \oplus \underbrace{(f(C_1) \oplus C'_2)}_{=: C_2}$$

is a Hodge decomposition of V_2 . The corresponding standard Hodge homotopy clearly satisfies $\mathcal{P}_{\text{std}}^2 \circ f = f \circ \mathcal{P}_{\text{std}}^1$, and it holds $\mathcal{P}_{\text{std}}^2(f(V_1)) \subset f(V_1)$, so that $f(V_1)$ is a Hodge subalgebra. By injectivity, it follows that $\mathcal{S}(V_1) \simeq \mathcal{S}(V_2)$. \square

Proposition 6.1.21 (Small subalgebra in geometrically formal case). *Let (V, d, \wedge) be a non-negatively graded unital commutative DGA with a cyclic structure $\langle \cdot, \cdot \rangle$ of Hodge type such that the induced cyclic structure on $H(V)$ is non-degenerate. Suppose that there is DGA-quasi-isomorphism $f : H(V) \rightarrow V$ preserving orientation on homology. Then there is a Hodge decomposition of V with $\mathcal{H} = \operatorname{im} f$ and f induces an isomorphism $H(V) \simeq \mathcal{Q}(\mathcal{S}(V))$.*

Proof. By Remark 6.1.11, there is a Hodge decomposition with $\mathcal{H} = \operatorname{im} f$. Therefore, we can restrict the codomain of f and obtain the DGA-quasi-isomorphism $\tilde{f} : H(V) \rightarrow \mathcal{S}(V) \rightarrow \mathcal{Q}(\mathcal{S}(V))$ which preserves orientation on homology. Lemma 6.1.20 implies that there is a Hodge decomposition of V such that \tilde{f} induces an isomorphism $\mathcal{S}(H(V)) \simeq \mathcal{S}(\mathcal{Q}(\mathcal{S}(V)))$. However, $\mathcal{S}(H(V)) = H(V)$ and $\mathcal{S}(\mathcal{Q}(\mathcal{S}(V))) = \mathcal{Q}(\mathcal{S}(V))$ by Proposition 6.1.17. \square

The following conjecture is based on a remark of Prof. Hồng Vân Lê, communicated in an e-mail correspondence, that “the small subalgebra is the image of a minimal model”.

Conjecture 6.1.22 (Hodge-Sullivan minimal model). *Let (V, d, \wedge) be a non-negatively graded unital commutative DGA with a cyclic structure $\langle \cdot, \cdot \rangle$ of Hodge type. Suppose that V is of finite type, $H^0(V) = \mathbb{R}$ and $H^1(V) = 0$. Then for any Hodge decomposition of V , there is a Sullivan minimal model $\Lambda U \rightarrow V$ (see [FOT08] or Remark 6.2.8 later) whose image contains a Hodge subalgebra.*

Idea of proof. The author tried to modify the well-known inductive construction of $\Lambda U \rightarrow V$ to achieve that its image is \mathcal{P}_{std} -invariant; he observed that vanishing of higher homology groups might be required. In any case, it sufficient (and necessary) to prove that the image contains at least the trees from Proposition 6.1.15. \square

If Conjecture 6.1.22 is true, then we have the following:

6. IBL-infinity formality and Poincaré duality models

Conjecture 6.1.23 (Uniqueness of non-degenerate small subalgebra). *Let (V, d, \wedge) be a non-negatively graded unital commutative DGA with a cyclic structure $\langle \cdot, \cdot \rangle$ of degree n of Hodge type. Suppose that $H(V)$ is of finite type, $H^0(V) = \mathbb{R}$ and $H^1(V) = 0$. Suppose, in addition, that $dV^n = 0$ and $d(\Lambda U)^n = 0$ for the Sullivan minimal model ΛU of V . Then for any two Hodge decompositions with small subalgebras $\mathcal{S}_1(V)$ and $\mathcal{S}_2(V)$, there is a pairing-preserving isomorphism $\mathcal{Q}(\mathcal{S}_1(V)) \simeq \mathcal{Q}(\mathcal{S}_2(V))$.*

Proof. Conjecture 6.1.22 applied to \mathcal{S}_1 , resp. \mathcal{S}_2 delivers Sullivan minimal models $f_1 : \Lambda U_1 \rightarrow \mathcal{S}_1$, resp. $f_2 : \Lambda U_2 \rightarrow \mathcal{S}_2$ whose images contain Hodge subalgebras. Because \mathcal{S}_i are small, f_i are forced to be surjective. Considering the compositions $\Lambda U \xrightarrow{f_1} \mathcal{S}_1 \hookrightarrow V$ and $\Lambda U \xrightarrow{f_2} \mathcal{S}_2 \hookrightarrow V$, the uniqueness of the Sullivan minimal model of V (see [FOT08, Theorem 2.24]), provides, after composing f_2 with an isomorphism $\Lambda U := \Lambda U_1 \simeq \Lambda U_2$, a diagram

$$\begin{array}{ccccc}
 & & f_1 & \twoheadrightarrow & \mathcal{S}_1 \longrightarrow \mathcal{Q}_1 \\
 & \nearrow & & \nwarrow & \\
 \Lambda U & & & & V \\
 & \searrow & & \swarrow & \\
 & & f_2 & \twoheadrightarrow & \mathcal{S}_2 \longrightarrow \mathcal{Q}_2
 \end{array} \tag{6.15}$$

which is commutative up to a homotopy of DGA's.

We now construct a cyclic structure $\langle \cdot, \cdot \rangle^{\Lambda U}$ on ΛU which is preserved by both f_1 and f_2 (\mathcal{S}_i and \mathcal{Q}_i are equipped with the cyclic structures induced from V). Proposition 6.1.4 applies, and hence we can use orientations and cyclic structures interchangeably. By homotopy commutativity of (6.15), the maps $\Lambda U \xrightarrow{f_1} \mathcal{S}_1 \hookrightarrow V$ and $\Lambda U \xrightarrow{f_2} \mathcal{S}_2 \hookrightarrow V$ induce the same isomorphism on homology, which we can use to pullback the orientation \mathfrak{o}^H on $H(V)$ to an orientation $\tilde{\mathfrak{o}}$ on $H(\Lambda U)$. Now, all maps of (6.15) preserve orientation on homology. Because $dV^n = 0$ and $d(\Lambda U)^n = 0$, Proposition 6.1.3 implies that $\tilde{\mathfrak{o}}$ defines a unique orientation $\mathfrak{o}^{\Lambda U}$ on ΛU and guarantees that if a map preserves orientation on homology, then it preserves orientation on chain-level too. This gives the desired $\langle \cdot, \cdot \rangle^{\Lambda U}$.

In the last step, we denote $f_i^{\mathcal{Q}} := \pi_i^{\mathcal{Q}} \circ f_i$ in (6.15) and write down the following diagram with pairing-preserving DGA-quasi-isomorphisms:

$$\begin{array}{ccc}
 & (\Lambda U, \langle \cdot, \cdot \rangle^{\Lambda U}) & \\
 f_1^{\mathcal{Q}} \swarrow & & \searrow f_2^{\mathcal{Q}} \\
 (\mathcal{Q}_1, \langle \cdot, \cdot \rangle_1^{\mathcal{Q}}) & & (\mathcal{Q}_2, \langle \cdot, \cdot \rangle_2^{\mathcal{Q}})
 \end{array} \tag{6.16}$$

6. IBL-infinity formality and Poincaré duality models

Claim (b) of Lemma 6.1.18 implies that $\ker f_1^{\mathcal{Q}} = \ker f_2^{\mathcal{Q}} = (\Lambda U)^\perp$, and hence

$$\mathcal{Q}_1 \simeq \underbrace{\Lambda U / (\Lambda U)^\perp}_{= \mathcal{Q}(\Lambda U)} \simeq \mathcal{Q}_2.$$

This finishes the proof. □

Questions 6.1.24. (i) Given a cochain complex (V, d) with a symmetric pairing $\langle \cdot, \cdot \rangle$, which conditions have to be satisfied by the maps $\pi_{\mathcal{H}}, \mathcal{P}_{\text{std}} : V \rightarrow V$ so that $V = \ker \pi_{\mathcal{H}} \oplus \text{im } d \oplus \text{im } \mathcal{P}_{\text{std}}$ is a Hodge decomposition with Hodge pair $(\ker \pi_{\mathcal{H}}, \mathcal{P}_{\text{std}})$? This would characterize Hodge decompositions in terms of Hodge pairs.

(ii) Give an example of two Hodge decompositions whose small subalgebras are not isomorphic.

(iii) Is Conjecture 6.1.22 true and can we drop the assumption $d(\Lambda U)^n = 0$ in Conjecture 6.1.23?

6.2. Poincaré DGA's and Poincaré duality models

In this section, we restrict to non-negatively graded unital commutative DGA's, which we denote by $\text{CDGA}_1^{\geq 0}$. In this case, the notions of orientation and of cyclic structure agree by Proposition 6.1.4.

We modify and combine definitions from [Fio+19] and [LS07] as follows.

Definition 6.2.1 (Dif. Poincaré duality algebra, PDGA and formality). *A differential Poincaré duality algebra of degree n is a $\text{CDGA}_1^{\geq 0}(V, d, \wedge)$ of finite type with orientation \mathbf{o} in degree n such that the induced pairing on V satisfies Poincaré duality. If $d = 0$, we call it just Poincaré duality algebra.*

A Poincaré DGA (shortly PDGA) of degree n is a $\text{CDGA}_1^{\geq 0}(V, d, \wedge)$ together with an orientation $\mathbf{o}^H : H(V) \rightarrow \mathbb{R}$ of degree n which makes $H(V)$ into a Poincaré duality algebra.

A morphism of PDGA's $(V_1, d_1, \wedge_1, \mathbf{o}_1^H)$ and $(V_2, d_2, \wedge_2, \mathbf{o}_2^H)$ is a DGA-morphism $f : V_1 \rightarrow V_2$ such that the induced map $f_ : H(V_1) \rightarrow H(V_2)$ preserves orientation, i.e., it holds $\mathbf{o}_2^H \circ f_* = \mathbf{o}_1^H$.*

A quasi-isomorphism (or weak equivalence), of PDGA's is a morphism of PDGA's $f : V_1 \rightarrow V_2$ such that $f_ : H(V_1) \rightarrow H(V_2)$ is an isomorphism of oriented DGA's.*

Two PDGA's are weakly homotopy equivalent (or isomorphic in the homotopy category) if they are connected by a zig-zag of PDGA-quasi-isomorphisms.

6. IBL-infinity formality and Poincaré duality models

A PDGA is formal if it is weakly homotopy equivalent as a PDGA to its homology.

It follows from Proposition 6.1.4 that a differential Poincaré duality algebra according to Definition 6.2.1 is precisely a cyclic DGA from Part I. In particular, it is finite dimensional. However, if we relax finite type, unitality or commutativity, we obtain a different notion.

Remark 6.2.2 (Frobenius algebra). A differential Poincaré duality algebra, resp. a cyclic DGA, is precisely a finite-dimensional symmetric dg-Frobenius algebras from [Val12, p. 13] or [CHV06, Theorem 1.1]. \triangleleft

Definition 6.2.3 (Poincaré duality model). A Poincaré duality model of a PDGA (V, d, \wedge, o^H) is a differential Poincaré duality algebra $(\mathcal{M}, d^{\mathcal{M}}, \wedge^{\mathcal{M}}, o^{\mathcal{M}})$ which is weakly homotopy equivalent to V as a PDGA.

A Poincaré duality model is small if $\mathcal{Q}(\mathcal{S}(\mathcal{M})) \simeq \mathcal{M}$ for every Hodge decomposition.

The definition of a Poincaré duality model in [LS07] requires only weak homotopy equivalence of DGA's.

Note that free algebras, e.g., the Sullivan minimal model ΛU , usually fail to be Poincaré duality models because they are non-zero in degrees $> n$ which prevents non-degeneracy in degree n .

Consider the functor from PDGA to DGA which forgets the orientation on homology. We have the following trivial yet surprising observation.

Proposition 6.2.4 (PDGA-formality is the same as DGA-formality). A Poincaré DGA (V, d, \wedge, o^H) is formal (as a PDGA) if and only if it is formal as a DGA.

Proof. The “only if” part is clear.

As for the “if” part, let

$$V \longleftarrow \bullet \cdots \bullet \longrightarrow H(V) \tag{6.17}$$

be a weak homotopy equivalence of DGA's. Denote by $f : H(V) \rightarrow H(V)$ the isomorphism on homology induced by (6.17) from the left to the right. We adjoin $H(V)$ to the right of (6.17) to obtain the homotopy

$$V \longleftarrow \bullet \cdots \bullet \longrightarrow H(V) \xrightarrow{f^{-1}} H(V) \tag{6.18}$$

whose induced map on homology from the left to the right is the identity. Therefore, we can orient homologies of the inner nodes of (6.18) so that all maps preserve orientation on homology. \square

The following is mostly [LS07, Theorem 1.1]. We just have to make sure that orientations are preserved.

6. IBL-infinity formality and Poincaré duality models

Proposition 6.2.5 (Existence of Poincaré duality model for $H^1 = 0$). *A Poincaré DGA (V, d, \wedge, o^H) with $H^0 = \mathbb{R}$ and $H^1 = 0$ admits a Poincaré duality model $(\mathcal{M}, d^{\mathcal{M}}, \wedge^{\mathcal{M}}, o^{\mathcal{M}})$.*

Proof. If o^H comes from a pairing on V which is of Hodge type, then we can take $\mathcal{M} = \mathcal{Q}(\mathcal{S}(V))$. The weak homotopy equivalence of PDGA's looks like

$$\begin{array}{ccc} & \mathcal{S}(V) & \\ \swarrow & & \searrow \\ \mathcal{Q}(\mathcal{S}(V)) & & V. \end{array} \quad (6.19)$$

If V is not of Hodge type, we proceed as follows. Let n be the degree of the orientation on $H(V)$. If $n \leq 6$, then V is formal as a DGA by [Mil79], and we can take $\mathcal{M} = H(V)$ by Proposition 6.2.4. The weak homotopy equivalence of PDGA's looks like

$$\begin{array}{ccc} & \Lambda U & \\ \swarrow & & \searrow \\ H(V) & & V, \end{array} \quad (6.20)$$

where ΛU is the Sullivan minimal model of V . The Sullivan minimal model $W := \Lambda U$ is of finite type and satisfies $W^0 = \mathbb{R}$, $W^1 = 0$ and $dW^2 = 0$. Suppose that $n \geq 7$. Under these assumptions on W and n , a certain extension of W , which we denote by $E(W)$, together with an orientation, which we denote by o^E , is constructed in [LS07, Section 4] inductively by adding generators to W . This extension is of finite type, the inclusion $W \hookrightarrow E(W)$ is a quasi-isomorphism of DGA's and $E(W)_{\perp}$ is acyclic. Moreover, we checked that $W \hookrightarrow E(W)$ preserves orientation on homology. By (c) of Proposition 6.1.13, $E(W)$ is of Hodge type, and hence we can take $\mathcal{M} = \mathcal{Q}(E(\Lambda U))$. The weak homotopy equivalence of PDGA's looks like

$$\begin{array}{ccc} & \Lambda U & \\ \swarrow & & \searrow \\ \mathcal{Q}(E(\Lambda U)) & & V. \end{array} \quad (6.21)$$

This proves the proposition. □

The following is mostly [LS07, Theorem 7.1]. Again, we just have to make sure that the orientations match.

Proposition 6.2.6 (“Uniqueness” of Poincaré duality model). *Let $(V_1, d_1, \wedge_1, o_1)$ and $(V_2, d_2, \wedge_2, o_2)$ be differential Poincaré duality algebras of degree n which are weakly*

6. IBL-infinity formality and Poincaré duality models

homotopy equivalent as PDGA's. Suppose that $H^0(V_i) \simeq \mathbb{R}$, $H^1(V_i) \simeq H^2(V_i) \simeq H^3(V_i) = 0$, $V_i^0 = \mathbb{R}$ and $V_i^1 \simeq V_i^2 = 0$ for $i = 1, 2$. Suppose that $n \geq 7$. Then there is a differential Poincaré duality algebra $(V_3, d_3, \wedge_3, o_3)$ and quasi-isomorphisms of PDGA's

$$\begin{array}{ccc} & V_3 & \\ \nearrow & & \nwarrow \\ V_1 & & V_2. \end{array} \quad (6.22)$$

Proof. By the assumption, there is $k \geq 1$ and a zig-zag of PDGA-quasi-isomorphisms

$$V_1 \longleftarrow Z_1 \longrightarrow Z_2 \longleftarrow Z_3 \longrightarrow Z_4 \longleftarrow \cdots \longleftarrow Z_k \longrightarrow V_2. \quad (6.23)$$

Consider the Sullivan minimal model $\Lambda U \rightarrow Z_2$ and use the Lifting Lemma [FOT08, Lemma 2.15] to construct DGA-quasi-isomorphisms $\Lambda U \rightarrow Z_1$ and $\Lambda U \rightarrow Z_3$ such that the diagram

$$\begin{array}{ccccc} & \Lambda U & & & \\ & \swarrow \downarrow \searrow & & & \\ Z_1 & \longrightarrow & Z_2 & \longleftarrow & Z_3 \end{array}$$

commutes up to homotopy of DGA's. It is easy to see that there is an orientation on $H(\Lambda U)$ such that all morphisms preserve orientation on homology. Therefore, we can replace the segment $V_1 \longleftarrow Z_1 \longrightarrow Z_2 \longleftarrow Z_3 \longrightarrow Z_4$ in (6.23) by $V_1 \longleftarrow \Lambda U \longrightarrow Z_4$. Repeating this process, we can shorten (6.23) to

$$\begin{array}{ccc} & \Lambda U & \\ f_1 \swarrow & & \searrow f_2 \\ V_1 & & V_2, \end{array} \quad (6.24)$$

where f_1 and f_2 are PDGA-quasi-isomorphisms. In order to take $\mathcal{Q}(E(\Lambda U))$ and obtain a zig-zag with three terms, we have to revert the arrows in (6.24). The trick from [LS07] is the following:

Consider the relative minimal model of the multiplication $\mu : \Lambda U \otimes \Lambda U \rightarrow \Lambda U$; from [FOT08, Example 2.48], it is given by

$$\begin{array}{ccc} \Lambda U \otimes \Lambda U & \xrightarrow{\mu} & \Lambda U \\ & \searrow i & \uparrow p \\ & M(\mu) := \Lambda U \otimes \Lambda U \otimes \Lambda(U[1]), & \end{array} \quad (6.25)$$

6. IBL-infinity formality and Poincaré duality models

where i is the inclusion into the first two factors, which is a cofibration, and p is a surjective quasi-isomorphism. Let $\iota_i : \Lambda U \rightarrow \Lambda U \otimes \Lambda U$ for $i = 1, 2$ be the inclusions to the first and the second factor, respectively. Because $\mu \circ \iota_i = \mathbb{1}$ and p is a quasi-isomorphism, the maps $i \circ \iota_i : \Lambda U \rightarrow M(\mu)$ for $i = 1, 2$ are quasi-isomorphisms. Moreover, it is easy to see that $H(M(\mu))$ inherits an orientation such that p_* and $(i \circ \iota_i)_*$ are orientation preserving. To transfer this situation to V_i , we use the diagram

$$\begin{array}{ccc}
 \Lambda U & \xrightarrow{f_i} & V_1 \\
 \downarrow \iota_i & & \downarrow \iota_i^V \\
 \Lambda U \otimes \Lambda U & \xrightarrow{f_1 \otimes f_2} & V_1 \otimes V_2 \\
 \downarrow i & & \downarrow g_2 \\
 M(\mu) & \xrightarrow{g_1} & \tilde{V}_3 := M(\mu) \otimes_{\Lambda U \otimes \Lambda U} (V_1 \otimes V_2),
 \end{array} \tag{6.26}$$

where $\iota_i^V : V_i \rightarrow V_1 \otimes V_2$ for $i = 1, 2$ are inclusions. The lower square, i.e., the maps g_1, g_2 and the DGA \tilde{V}_3 , is a pushout diagram (see [LO15, Example 1.4]). According to [nLa19], the model category of $\text{CDGA}_1^{\geq 0}$ is proper, and hence pushouts along cofibrations preserve quasi-isomorphisms. Therefore, $f_1 \otimes f_2$ being a quasi-isomorphism implies that g_1 is a quasi-isomorphism. We push the orientation to $H(\tilde{V}_3)$ via g_{1*} . Since $i \circ \iota_i, g_1$ and f_i are quasi-isomorphisms preserving orientation on homology, it follows that $h_i := g_2 \circ \iota_i^V : V_i \rightarrow V_3$ are quasi-isomorphisms preserving orientation of homology as well. It holds

$$\tilde{V}_3 \simeq \Lambda U[1] \otimes V_1 \otimes V_2$$

as algebras. It follows that \tilde{V}_3 is of finite type and $\tilde{V}_3^0 = \mathbb{R}$, $\tilde{V}_3^1 = 0$ and $\tilde{V}_3^2 = \tilde{V}_1^0 \otimes \tilde{V}_2^0 \otimes U^3 = 0$ hold due to the additional assumptions. Therefore, the conditions for an application of E are satisfied, and we can set

$$V_3 := \mathcal{Q}(E(\tilde{V}_3)).$$

This finishes the proof. □

Conjecture 6.2.7 (PDGA-version of conjecture of [LS07]). *Proposition 6.2.6 holds under the weaker assumptions that $H^0(V_1) \simeq H^0(V_2) = \mathbb{R}$ and $H^0(V_1) \simeq H^0(V_2) = 0$. Moreover, it holds for any n .*

We would like to define a “minimal Poincaré duality model”. We motivate this notion in the following remark.

6. IBL-infinity formality and Poincaré duality models

Remark 6.2.8 (Model and minimal model). We shall understand models and minimal models in terms of model categories and their homotopy categories.

Let us illustrate this on Sullivan models. A Sullivan DGA is a free graded commutative algebra ΛU over a positively graded vector space U which admits a well-ordered homogenous basis (v_α) such that $dv_\alpha \in \langle v_\beta \mid \beta < \alpha \rangle^\wedge$ ($:=$ the subalgebra of ΛU generated by the v_β 's) for all α . A Sullivan DGA is called minimal if $\text{im } d \subset \Lambda_{\geq 2} U$ ($:=$ the set of decomposable elements).

According to [nLa19], the category $\text{CDGA}_1^{\geq 0}$ is a model category with weak equivalences being DGA-quasi-isomorphisms, fibrations being degree-wise surjective DGA-morphisms and cofibrations being retracts of relative Sullivan algebras (see [FOT08, Proposition 2.22 and Proposition 2.28]). Cofibrant objects are then precisely Sullivan algebras.

So, the homotopy extension property holds already for Sullivan DGA's. To see the role of minimality, we shall descent to the homotopy category. The homotopy category is constructed from a model category by localizing morphisms at weak equivalences. An isomorphism in the homotopy category, called weak homotopy equivalence, corresponds to a zig-zag of weak equivalences. If V_1 is weakly homotopy equivalent to V_2 , we say that V_2 is a model of V_1 . If there is a weak equivalence $V_2 \rightarrow V_1$, we say that V_2 is a resolution of V_1 . We understand minimality as a condition which is in each weak homotopy equivalence class satisfied by at most one object up to isomorphism in the model category. If minimal models exist, they form a skeleton of the homotopy category. This is precisely the case of $\text{CDGA}_1^{\geq 0}$ and minimal Sullivan algebras. Indeed, by [FOT08, Theorem 2.24], every connected $\text{CDGA}_1^{\geq 0}$ is resolved by a minimal Sullivan algebra. Next, by [FOT08, Proposition 2.26], a DGA-morphism lifts to resolutions by Sullivan algebras, and by [FOT08, Corollary 2.13], quasi-isomorphic minimal Sullivan DGA's are isomorphic. Finally, this implies that weakly homotopy equivalent minimal Sullivan algebras are isomorphic, and hence are minimal in the sense above.

As another example, for an operad (or properad) \mathcal{O} , one wants to construct a dg-operad \mathcal{O}_∞ which is a quasi-free resolution of \mathcal{O} (see [Val12]). Quasi-free means that after forgetting the differential, the operad \mathcal{O}_∞ is free over \mathbb{S} -bimodules. This is similar to Sullivan models which are free over vector spaces. For quadratic operads, \mathcal{O}_∞ is often constructed as the cobar construction Ω of the Koszul dual cooperad \mathcal{O}^i . This is the case of A_∞ , L_∞ or of the properad IBL_∞ . The differential on $\Omega \mathcal{O}^i$ is the extension of the decomposition on \mathcal{O}^i to a derivative, and hence it has decomposable image (c.f., the explicit formula [Pek18, Formula (2)]). This is similar to minimal Sullivan models. It would be interesting to know whether \mathcal{O}_∞ can be constructed using the same inductive method of “killing” and “adding” generators of homology as Sullivan minimal models.

6. IBL-infinity formality and Poincaré duality models

Finally, let us note that in [CR19], they use the inductive method to construct (minimal) models of \mathcal{O} -algebras for a wide class of operads \mathcal{O} . Note that cyclic A_∞ -algebras can be formulated in the language of cyclic operads. However, in the case of PDGA's, we have the non-degenerate pairing on homology and an operadic description is not clear. \triangleleft

Consider small Poincaré duality models. Let (V, d, \wedge, o^H) be a PDGA with $H^0(V) = \mathbb{R}$ and $H^1(V) = 0$. Proposition 6.2.5 gives a Poincaré duality model \mathcal{M} . Proposition 6.1.17 asserts that $\mathcal{Q}(\mathcal{S}(\mathcal{M}))$ is small and Proposition 6.1.13 asserts that it is weakly homotopy equivalent to \mathcal{M} and hence to V as a PDGA. Therefore, small Poincaré duality models exist. Let \mathcal{M}_1 and \mathcal{M}_2 be two small Poincaré duality models of V . Pick any of their Hodge decompositions. Proposition 6.2.6, more generally Conjecture 6.2.7, gives the diagram (6.22), where $V_1 = \mathcal{M}_1$, $V_2 = \mathcal{M}_2$ and V_3 are differential Poincaré duality algebras and the maps, which we denote by $f_1 : V_1 \rightarrow V_3$ and $f_2 : V_2 \rightarrow V_3$, are PDGA-quasi-isomorphisms. Lemma 6.1.20 asserts that there are Hodge decompositions of V_3 with small subalgebras $\mathcal{S}_1(V_3)$ and $\mathcal{S}_2(V_3)$ such that f_1 and f_2 induce pairing preserving isomorphisms $\mathcal{S}(V_1) \simeq \mathcal{S}_1(V_3)$ and $\mathcal{S}(V_2) \simeq \mathcal{S}_2(V_3)$, respectively. Conjecture 6.1.23 asserts that $\mathcal{Q}(\mathcal{S}_1(V_3)) \simeq \mathcal{Q}(\mathcal{S}_2(V_3))$. In total, we have

$$\mathcal{M}_1 \simeq \mathcal{Q}(\mathcal{S}(\mathcal{M}_1)) \simeq \mathcal{Q}(\mathcal{S}_1(V_3)) \simeq \mathcal{Q}(\mathcal{S}_2(V_3)) \simeq \mathcal{Q}(\mathcal{S}(\mathcal{M}_2)) \simeq \mathcal{M}_2$$

as differential Poincaré duality algebras.

Corollary 6.2.9 (Small Poincaré duality models are minimal). *Small Poincaré duality models exist when $H^0 = \mathbb{R}$ and $H^1 = 0$ and are unique up to an isomorphism provided that the conjectures from Sections 6.1 and 6.2 are true.*

Questions 6.2.10. (i) Is it possible to prove Proposition 6.2.6 for $n \leq 6$ when formality holds in an easier way?

(ii) Can we describe a model category structure on Poincaré DGA's ?

(iii) Sullivan's inductive construction gives a minimal model which resolves V . The construction of Proposition 6.2.5 does not resolve V in any case (6.19), (6.20) and (6.21). Is it possible to resolve a PDGA by a Poincaré duality model? It is clearly possible in the geometrically formal case, i.e., when there is a DGA-quasi-isomorphism $H(V) \rightarrow V$.

6.3. Consequences for IBL-infinity theory

Consider the construction $\mathrm{dIBL}^m(C(\cdot))$ on cyclic DGA's (= differential Poincaré duality algebras) from Section 3.4 in Part I. We first need the following technical result.

6. IBL-infinity formality and Poincaré duality models

Conjecture 6.3.1 (Functoriality of $\mathrm{dIBL}^{\mathfrak{m}}$ -construction up to homotopy). *Consider differential Poincaré duality algebras $(V_1, d_1, \wedge_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, d_2, \wedge_2, \langle \cdot, \cdot \rangle_2)$, and let $f : V_2 \rightarrow V_1$ be a pairing-preserving DGA-quasi-isomorphism. Then there is an IBL_{∞} -quasi-isomorphism $\mathfrak{g} : \mathrm{dIBL}(C(V_1)) \rightarrow \mathrm{dIBL}(C(V_2))$ with $\mathfrak{g}_{110} = f^*$.*

Idea of proof. Lemma 6.1.18 implies that f is injective, and it follows from [CFL15, Theorem 11.3] that there is an IBL_{∞} -quasi-isomorphism $\mathfrak{f} : \mathrm{dIBL}(C(V_1)) \rightarrow \mathrm{dIBL}(C(V_2))$ with $\mathfrak{f}_{110} = f^*$. Because f preserves \wedge , the map f^* is a quasi-isomorphism also with respect to the Hochschild differential $\mathfrak{q}_{110}^{\mathfrak{m}}$. We hope to use the obstruction theory from [CFL15, Section 3] to find \mathfrak{g} or prove that $\mathfrak{f}_* \mathfrak{m}^{V_1}$ and \mathfrak{m}^{V_2} , where \mathfrak{m}^{V_i} are the canonical Maurer-Cartan elements for $\mathrm{dIBL}(C(V_i))$ for $i = 1, 2$, are gauge equivalent Maurer-Cartan elements. The Homological Perturbation Lemma mentioned in Section 8.3 might give an alternative proof in the BV-formalism. \square

For any PDGA $(V, d, \wedge, \mathrm{o}^H)$ with $H^0(V) = \mathbb{R}$ and $H^1(V) = 0$, we can pick its (small) Poincaré duality model $\mathcal{M}(V)$ and associate to it the dIBL -algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}(V)))$. This extends the domain of $\mathrm{dIBL}^{\mathfrak{m}}(C(\cdot))$ to PDGA's; however, removing choices requires us to relax the codomain to IBL_{∞} -homotopy equivalence classes.

Conjecture 6.3.2 (Extension of $\mathrm{dIBL}^{\mathfrak{m}}$ to PDGA's). *For any PDGA $(V, d, \wedge, \mathrm{o}^H)$ with $H^0(V) = \mathbb{R}$ and $H^1(V) = 0$, the IBL_{∞} -homotopy class of $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}(V)))$ does not depend on the Poincaré duality model $\mathcal{M}(V)$ of V .*

Proof. Let $\mathcal{M}_1(V)$ and $\mathcal{M}_2(V)$ be two Poincaré duality models of V . By Proposition 6.2.6, more generally Conjecture 6.2.7, there is a differential Poincaré duality algebra V_3 and PDGA-quasi-isomorphisms $f_1 : \mathcal{M}_1(V) \rightarrow V_3$ and $f_2 : \mathcal{M}_2(V) \rightarrow V_3$. They are injective and pairing preserving by previous results. Conjecture 6.3.1 gives IBL_{∞} -quasi-isomorphisms $\mathfrak{f}_1 : \mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_1(V))) \rightarrow \mathrm{dIBL}^{\mathfrak{m}}(C(V_3))$ and $\mathfrak{f}_2 : \mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_2(V))) \rightarrow \mathrm{dIBL}^{\mathfrak{m}}(C(V_3))$ extending f_1^* and f_2^* , respectively. Application of [CFL15, Theorem 1.2] (existence of IBL_{∞} -homotopy inverses for IBL_{∞} -quasi-isomorphisms) finishes the proof. \square

If we had the notion of a minimal Poincaré duality model $\mathcal{M}_0(V)$, e.g., the small Poincaré duality model provided that Corollary 6.2.9 holds, we could associate to a weak homotopy equivalence class of a PDGA $(V, d, \wedge, \mathrm{o}^H)$ a canonical dIBL -algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_0(V)))$ up to an IBL_{∞} -isomorphism.⁴

⁴In order to think about functors, we would first have to find a canonical construction of an IBL_{∞} -morphism induced by a PDGA-morphism of cyclic DGA's. The construction of [CFL15, Section 11] applies only to quasi-isomorphisms, and even if we pick the standard Hodge propagator P_{std} , it

6. IBL-infinity formality and Poincaré duality models

Definition 6.3.3 (IBL $_{\infty}$ -formality). *We say that a differential Poincaré duality algebra $(V, d, \wedge, \langle \cdot, \cdot \rangle)$ is IBL $_{\infty}$ -formal if $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$ and $\mathrm{dIBL}^{\mathfrak{m}}(C(H(V)))$ are IBL $_{\infty}$ -homotopy equivalent.*

Conjecture 6.3.4 (DGA-formality implies IBL $_{\infty}$ -formality). *Let (V, d, \wedge) be a PDGA with $H^0(V) = \mathbb{R}$ and $H^1(V) = 0$ which is formal as a DGA. Then any its Poincaré duality model $\mathcal{M}(V)$ is IBL $_{\infty}$ -formal.*

Proof. By Proposition 6.2.4, DGA-formality is equivalent to PDGA-formality. Therefore, $H(V)$ is a Poincaré duality model of V . The rest follows from Conjecture 6.3.2. \square

Note that due to the existence of homotopy inverses of quasi-isomorphisms in the IBL $_{\infty}$ -category, being IBL $_{\infty}$ -homotopy equivalent is equivalent to being weakly equivalent as IBL $_{\infty}$ -algebras (existence of a direct quasi-isomorphism), and this is equivalent to being weakly IBL $_{\infty}$ -homotopy equivalent (existence of zig-zag of quasi-isomorphisms).

Remark 6.3.5 (IBL $_{\infty}$ -formality of V and $C(V)$). Let \mathcal{O} be a dg-operad. If V is an \mathcal{O} -algebra, then $H(V)$ is an \mathcal{O} -algebra with zero differential. We say that V is \mathcal{O} -formal if V and $H(V)$ are weakly homotopy equivalent as \mathcal{O} -algebras; i.e., if there is a zig-zag of \mathcal{O} -quasi-isomorphisms between V and $H(V)$.

Consider DGA and A_{∞} . If a DGA V is formal, then it is also A_{∞} -formal because DGA is a subcategory of A_{∞} .⁵ The converse is also true by the rectification procedure (see [Idr18]). Therefore, DGA-formality and A_{∞} -formality of a DGA V are equivalent. This should be true for any suitable (pro)perad \mathcal{O} and its quasi-free resolution \mathcal{O}_{∞} .

In the spirit above, we can speak about dIBL -, resp. IBL $_{\infty}$ -formality of the dIBL -algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$ on $C(V)$, which is about the existence of zig-zags of dIBL -, resp. IBL $_{\infty}$ -quasi-isomorphisms between $C(V)$ and $H(C(V)) := H(C(V), \mathfrak{q}_{110}^{\mathfrak{m}})$. These notions are most likely equivalent as in the case of DGA- and A_{∞} -formality.

On the other hand, IBL $_{\infty}$ -formality of a differential Poincaré duality algebra V from Definition 6.3.3 is a different notion because V is not an IBL $_{\infty}$ -algebra.

We conclude that the following three types of “formalities” might be interesting:

- 1) DGA-formality of a PDGA V ,
- 2) IBL $_{\infty}$ -formality of a PDGA V (resp. of its Poincaré duality model),
- 3) IBL $_{\infty}$ -formality of $C(V)$ as the dIBL -algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$.

still depends on the choice of the Hodge decomposition. Therefore, the best we can hope for is an assignment of PDGA-morphisms to IBL $_{\infty}$ -homotopy classes of IBL $_{\infty}$ -morphisms.

⁵Note that A_{∞} , similarly as IBL $_{\infty}$, admits homotopy inverses of quasi-isomorphisms, and hence A_{∞} -formality is equivalent to the existence of a direct A_{∞} -quasi-isomorphism $V \rightarrow H(V)$.

6. IBL-infinity formality and Poincaré duality models

Except for the fact that (1) may imply (2), we do not have any other guesses. Notions (1)–(3) should also be interpreted geometrically in the context of the String Topology Conjecture. \triangleleft

Let M be a connected oriented closed n -manifold with $H_{\text{dR}}^1(M) = 0$. We know that $\Omega(M)$ is of Hodge type and that a Riemannian metric on M induces a canonical Hodge decomposition $\Omega(M) = \mathcal{H} \oplus \text{im } d \oplus \text{im } d^*$, where \mathcal{H} is the subspace of harmonic forms and d^* the codifferential. So, in this case, we have a canonical small Poincaré duality model

$$\mathcal{M}(\Omega(M)) = \mathcal{Q}(\mathcal{S}(\Omega(M))).$$

Due to $H_{\text{dR}}^1(M) = 0$, it is of finite type and hence finite-dimensional. Therefore, $\text{dIBL}^{\mathfrak{m}}(C(\mathcal{M}(\Omega(M))))$ is well-defined. Recall that $(\Omega, d, \wedge, \int)$ is not of finite type and $\text{dIBL}^{\mathfrak{m}}(C(\cdot))$ is not well-defined; that was the reason to define the Chern-Simons, aka formal pushforward, Maurer-Cartan element \mathfrak{n} for $\text{dIBL}(C(H_{\text{dR}}(M)))$ directly as a summation over Feynman integrals. The idea was that $\text{dIBL}^{\mathfrak{n}}(C(H_{\text{dR}}(M)))$ is “homotopy equivalent” to “ $\text{dIBL}^{\mathfrak{m}}(C(\Omega(M)))$ ”.

We conjecture the following:

Conjecture 6.3.6 ($\text{dIBL}^{\mathfrak{m}}(C(\cdot))$ on small model of $\Omega(M)$). *Let M be a connected oriented closed Riemannian n -manifold with $H_{\text{dR}}^1(M) = 0$, and let \mathfrak{n} be the formal pushforward Maurer-Cartan element for an admissible Hodge propagator P . Then the IBL_{∞} -algebras*

$$\text{dIBL}^{\mathfrak{n}}(C(H_{\text{dR}}(M))) \quad \text{and} \quad \text{dIBL}^{\mathfrak{m}}(C(\mathcal{Q}(\mathcal{S}(\Omega(M)))))$$

are IBL_{∞} -homotopy equivalent.

Comment on proof. The proof shall require going into integrals and analytical results from [CVona]. It might be reasonable to try to show that $\mathfrak{f}_*\mathfrak{m}$ and \mathfrak{n} are gauge equivalent Maurer-Cartan elements for $\text{dIBL}(C(H_{\text{dR}}(M)))$. \square

Corollary 6.3.7 (Formality conjecture). *In the situation of Conjecture 6.3.6, suppose that M is formal. Then $\text{dIBL}^{\mathfrak{n}}(C(H_{\text{dR}}(M)))$ is IBL_{∞} -homotopy equivalent to $\text{dIBL}^{\mathfrak{m}}(C(H_{\text{dR}}(M)))$.*

Together with the String Topology Conjecture 4.5.1 this gives a canonical chain model for the equivariant Chas-Sullivan string topology for formal M with $H_{\text{dR}}^1(M) = 0$.

7. Standard Hodge propagator

In this chapter, we tackle the question whether the standard Hodge propagator P_{std} extends smoothly to the blow-up.

In Section 7.1, we recall the Schwartz kernel theorem (Proposition 7.1.1) and define the Schwartz form of an operator between sections of the exterior bundle of the cotangent bundle (Proposition 7.1.2 and Definition 7.1.3). We mention that the Schwartz form of a pseudo-differential operator is smooth outside of the diagonal (Proposition 7.1.4). On the example of the identity (Example 7.1.5) we illustrate that the smooth part does not always determine the pseudo-differential operator. We recall basic facts about regularity of the Laplace Green kernel and standard Hodge propagator (Proposition 7.1.6). We formulate the problem of smooth extension to the blow-up (Definition 7.1.7) and ask whether it is satisfied by G and P_{std} .

In Section 7.2, we study uniqueness of the Hodge homotopy (Proposition 7.2.1). We would like to characterize the standard Hodge propagator as a unique primitive to the harmonic kernel satisfying certain properties. We give some ideas how to do it. We also illustrate that one has to be careful with blow-ups (Proposition 7.2.2).

In Section 7.3, we consider the heat form approximation. We define the heat form and sum up its computational properties (Proposition 7.3.1). We write down formulas for the heat form approximations (Proposition 7.3). We postulate that the standard Hodge propagator is the codifferential of the Laplacian Green kernel with respect to one variable (Proposition 7.3.3).

In Section 7.4, we study the standard Hodge propagator on \mathbb{R}^n . We recall the well-known formulas for the heat form and the Green form on \mathbb{R}^n (Proposition 7.4.1). We compute the standard Hodge propagator in two ways — as a coderivative of the Green form and as an integral using heat form approximation (Proposition 7.4.2 for $n \geq 2$ and Example 7.4.3 for $n = 1$).

In Section 7.5, we study the standard Hodge propagator on \mathbb{S}^n . We compute it explicitly for \mathbb{S}^1 (Example 7.5.1). Next, we study the structure of the space of Hodge propagators on \mathbb{S}^n (Proposition 7.5.2). We show that the Hodge propagator for \mathbb{S}^n constructed in Part I is coexact (Proposition 7.5.3). Finally, using the previous results, we prove that

7. Standard Hodge propagator

the standard Hodge propagator for \mathbb{S}^2 extends smoothly to the blow-up and give an explicit formula with two unknown constants (Proposition 7.5.4).

7.1. Schwartz form and smooth extension to blow-up

For a smooth vector bundle $\pi : E \rightarrow M$, let $\mathcal{D}(E)$ be the space of “test sections of E ”, and let $\mathcal{D}'(E^*)$ be the space of distributions. Let $\langle \cdot, \cdot \rangle : \mathcal{D}'(E^*) \otimes \mathcal{D}(E) \rightarrow \mathbb{R}$ be the natural pairing. See [Alb12] for detailed definitions.

Proposition 7.1.1 (Schwartz kernel theorem, see [Alb12, Theorem 12]). *Let E_1 and E_2 be smooth vector bundles over smooth manifolds M_1 and M_2 , respectively. Then there is a one-to-one correspondence between continuous operators $T : \mathcal{D}(E_1) \rightarrow \mathcal{D}'(E_2)$ and elements $K_T \in \mathcal{D}'(E_1 \boxtimes E_2^*)$ which is given by the equation*

$$\langle Ts_1, \psi_2 \rangle = \langle K_T, s_1 \boxtimes \psi_2 \rangle \quad \text{for all } s_1 \in \mathcal{D}(E_1) \text{ and } \psi_2 \in \mathcal{D}(E_2^*).$$

The distributional section K_T is called the Schwartz kernel of T .

An L^1_{loc} -integrable function $k : M \times M \rightarrow \text{Hom}(E_1, E_2) = E_1^* \boxtimes E_2$ on an oriented Riemannian manifold M defines the distribution $K \in \mathcal{D}'(E_1 \boxtimes E_2^*)$ by

$$\langle K, s_1 \boxtimes \psi_2 \rangle = \int_{x_1, x_2} \langle k(x_1, x_2) s_1(x_1), \psi_2(x_2) \rangle \text{Vol}(x_1) \text{Vol}(x_2)$$

In the case of exterior bundles, we introduce an equivalent notion of a Schwartz form. This name was proposed by Dr. A. Hermann in a discussion in Potsdam.

Proposition 7.1.2 (Schwartz kernel and Schwartz form). *In the setting of Proposition 7.1.1, suppose that $M_1 = M_2 = M$ is a smooth oriented Riemannian manifold and $E_1 = E_2 = \Lambda T^*M$. We consider the isomorphism of vector bundles $\Psi : \Lambda T^*M \boxtimes (\Lambda T^*M)^* \rightarrow \Lambda T^*(M \times M)$ which is for every $x_1, x_2 \in M$ given by*

$$\begin{aligned} \Psi : \Lambda T^*_{x_1} M \otimes (\Lambda T^*_{x_2} M)^* &\longrightarrow \Lambda T^*_{(x_1, x_2)}(M \times M) \\ \omega_1 \otimes \xi_2 &\longmapsto \omega_1 \wedge \sharp \xi_2. \end{aligned}$$

Here, $\sharp : \Lambda T^*M \rightarrow (\Lambda T^*M)^*$ denotes the musical isomorphism with respect to the natural pointwise inner product on ΛT^*M . We obtain the isomorphism

$$\begin{aligned} \Psi_* : \Omega'(M \times M) &\longrightarrow \mathcal{D}'(E_1^* \boxtimes E_2) \\ \Omega_T &\longmapsto K_T, \end{aligned}$$

7. Standard Hodge propagator

where Ω' denotes the space of de Rham currents.

If T is homogenous of degree $|T|$, then the degree of Ω_T satisfies

$$\deg(\Omega_T) = \dim(M) + |T|.$$

If K_T is represented by an L^1_{loc} -integrable section of $\text{Hom}(E_1, E_2) \simeq E_1^* \boxtimes E_2$, then Ω_T is represented by an L^1_{loc} -integrable form, and the other way round. In this case, for all $\omega_1 \in \Omega(M)$, $\omega_2 \in \Omega_c(M)$, we have

$$\begin{aligned} \int_{x_2} ((T\omega_1)(x_2)) [\omega_2(x_2)] &= \int_{x_1, x_2} K_T(x_1, x_2) [\omega_1(x_1)] \wedge \text{Vol}(x_1) \wedge \omega_2(x_2) \\ &= \int_{x_1, x_2} \Omega_T(x_1, x_2) \wedge \omega_1(x_1) \wedge \omega_2(x_2). \end{aligned} \quad (7.1)$$

Proof. Straightforward computations similar to Example 7.1.5 below. \square

Definition 7.1.3 (Schwartz form). *The current $\Omega_T \in \Omega'(M \times M)$ from Proposition 7.1.2 is called the Schwartz form of T .*

We will consider pseudo-differential operators $T : \Omega(M) \rightarrow \Omega(M)$ on a Riemannian manifold M ; this class of operators generalizes differential operators and contains generalized inverses of elliptic operators (see [Alb12] and [Hör07] for thorough treatments).

Proposition 7.1.4 (Schwartz form of pseudo-differential operators). *Let $T : \Omega(M) \rightarrow \Omega(M)$ be a pseudo-differential operator on a smooth oriented Riemannian manifold M . Then the Schwartz form Ω_T restricts to a smooth form on $M \times M \setminus \Delta$.*

Proof. Use methods of [Hör07]. \square

Example 7.1.5 (Schwartz form of $\mathbb{1}$). The Schwartz form of the identity $\mathbb{1} : \Omega(\mathbb{R}^n) \rightarrow \Omega(\mathbb{R}^n)$ reads

$$\Omega_{\mathbb{1}}(x, y) = \delta(x - y)(dx^1 - dy^1) \cdots (dx^n - dy^n), \quad (7.2)$$

where δ denotes the Dirac delta function on \mathbb{R}^n centered at 0. In order to prove this, we start by rewriting

$$\begin{aligned} (dx^1 - dy^1) \cdots (dx^n - dy^n) &= \sum_I (-1)^{|I|} \varepsilon(I^c I \mapsto [n]) dx^{I^c} \wedge dy^I \\ &= \sum_I (-1)^{n|I|} (*dx^I) \wedge dy^I. \end{aligned}$$

Here, the sum is over all multiindices $I \subset \{1, \dots, n\}$, and we use that $*(dx^I) = \varepsilon(I, I^c) dx^{I^c}$, where $\varepsilon(I, I^c)$ denotes the sign to order II^c to $\{1, \dots, n\}$. Now, for any

7. Standard Hodge propagator

$\omega \in \Omega_c(\mathbb{R}^n)$, which we write as $\omega(x) = \sum_K \omega_K(x) dx^K$, we compute using $dx^K \wedge *(dx^I) = (dx^K, dx^I) \text{Vol}(x) = \delta^{KI} \text{Vol}(x)$ the following:

$$\begin{aligned}
& \int_x \delta(x-y)(dx^1 - dy^1) \cdots (dx^n - dy^n) \omega(x) \\
&= \sum_{I,K} \int_x \delta(x-y) (-1)^{n|I|} \omega_K(x) (*dx^I) dy^I dx^K \\
&= \sum_{I,K} \int_x \delta(x-y) (-1)^{n|I|+n|K|} \delta(x-y) \omega_K(x) dx^K *(dx^I) dy^I \\
&= \sum_I \int_x \delta(x-y) \omega_I(x) \text{Vol}(x) dy^I \\
&= \omega(y).
\end{aligned}$$

This shows (7.1), and (7.2) follows. Notice that

$$\Omega_{\mathbb{1}}|_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta} = 0,$$

and thus the smooth part of $\Omega_{\mathbb{1}}$ does not recover the data of the operator $\mathbb{1}$. \triangleleft

We consider the Green operator \mathcal{G} for the Laplacian Δ (see [War83]) and the standard Hodge propagator \mathcal{P}_{std} . They are both pseudo-differential operators, and it holds $\mathcal{P}_{\text{std}} = \pm d^* \mathcal{G}$. We will study their Schwartz forms G and P_{std} , which are called the Green kernel and the standard Hodge propagator, respectively.

Proposition 7.1.6 (Basic facts about G and P_{std}). *The Green kernel G represents an L^2 -integrable form. The standard Hodge propagator P_{std} represent an L^1 -integrable form.*

Proof. The fact that G is L^2 is an exercise in [War83]. The fact that P_{std} defines an L^1 -integrable form on $M \times M$ for a compact manifold M was proved in [Har04] using the heat kernel approximation (see the next section). \square

A consequence of Proposition 7.1.6 is that the Schwartz forms G and P_{std} are determined by their smooth restrictions to $M \times M \setminus \Delta$. This follows from (7.1) because the integral does not depend on sets of zero measure. Therefore, we will write $G, P_{\text{std}} \in \Omega^{n-1}(M \times M \setminus \Delta)$.

Definition 7.1.7 (Smooth extension to the blow-up). *Let M be a smooth manifold, and let $\pi : \text{Bl}_{\Delta}(M \times M) \rightarrow M \times M$ be the Melrose blow-up of $M \times M$ at the diagonal Δ (this*

7. Standard Hodge propagator

is another name for the spherical blow-up in Part I). Consider the blow-up diagram

$$\begin{array}{ccc} & & \text{Bl}_\Delta(M \times M) \\ & \nearrow \tilde{\iota} & \downarrow \pi \\ M \times M \setminus \Delta & \xhookrightarrow{\iota} & M \times M, \end{array}$$

where ι is the inclusion and $\tilde{\iota}$ its unique smooth lift — the embedding of the interior. We say that a smooth form $\omega \in \Omega(M \times M \setminus \Delta)$ extends smoothly to the blow-up if there is a smooth form $\tilde{\omega} \in \Omega(\text{Bl}_\Delta(M \times M))$ such that

$$\tilde{\iota}^* \tilde{\omega} = \omega.$$

Note that the extension, if it exists, is necessarily unique.

Question 7.1.8 (Smooth extension of standard propagator). *Do G and P_{std} extend smoothly to the blow-up?*

7.2. Uniqueness of Hodge propagator

Proposition 7.2.1 (Uniqueness of Hodge homotopy). *Let M be a closed oriented manifold, and let $\Omega(M) = \text{im } d \oplus \text{im } d^* \oplus \mathcal{H}$ be its Hodge decomposition. Any linear map $\mathcal{P} : \Omega(M) \rightarrow \Omega(M)$ satisfying*

$$d \circ \mathcal{P} + \mathcal{P} \circ d = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1} \tag{7.3}$$

can be expressed as

$$\mathcal{P} = \begin{cases} -(d|_{\text{im } d^*})^{-1} + R_1 & \text{on } \text{im } d, \\ (d|_{\text{im } d^*})^{-1} R_1 d + R_2 & \text{on } \text{im } d^*, \\ R_3 & \text{on } \mathcal{H}, \end{cases} \tag{7.4}$$

where $d|_{\text{im } d^*} : \text{im } d^* \rightarrow \text{im } d$ is an isomorphism, for some linear maps $R_1 : \text{im } d \rightarrow \text{im } d$, $R_2 : \text{im } d^* \rightarrow \ker d$ and $R_3 : \mathcal{H} \rightarrow \ker d$. Moreover, the following facts are equivalent for an operator \mathcal{P} satisfying (7.3):

- (1) $\mathcal{P} = \mathcal{P}_{\text{std}}$ is the standard Hodge homotopy,
- (2) $R_1 = R_2 = R_3 = 0$,

7. Standard Hodge propagator

(3) $\text{im } \mathcal{P} \subset \text{im } d^*$,

Proof. Suppose first that $\omega \in \text{im } d$. Plugging it in (7.3), we get that $\mathcal{P}\omega$ has to satisfy

$$d\mathcal{P}\omega = \omega.$$

We see that

$$\mathcal{P}\omega \in \eta + \ker d,$$

where η is the unique coexact form with $d\eta = \omega$. In other words, η is the preimage of ω under the isomorphism $d : \text{im } d^* \rightarrow \text{im } d$, and we can write

$$\mathcal{P}\omega = (d|_{\text{im } d^*})^{-1}\omega + R_1\omega, \quad (7.5)$$

where $R_1 : \text{im } d \rightarrow \ker d$.

Suppose now that $\omega \in \text{im } d^*$. Using (7.3) and (7.5), we obtain

$$d\mathcal{P}\omega = \omega - \underbrace{(d|_{\text{im } d^*})^{-1}d\omega}_{=0} - R_1d\omega,$$

where the two terms cancel because ω is the unique coexact primitive to $d\omega$. Notice that this equation restricts R_1 to $R_1 : \text{im } d \rightarrow \text{im } d$. Similarly as in the first case, we obtain

$$\mathcal{P}\omega = -(d|_{\text{im } d^*})^{-1}R_1d\omega + R_2\omega,$$

where $R_2 : \text{im } d^* \rightarrow \ker d$.

If $\omega \in \mathcal{H}$, then (7.3) gives

$$d\mathcal{P}\omega = 0,$$

and hence $\mathcal{P}\omega = R_3\omega$ for some $R_3 : \mathcal{H} \rightarrow \ker d$.

Therefore, (7.3) implies (7.4). The other direction clearly holds as well.

As for the equivalent facts, because $\ker d \perp \text{im } d^*$ (with respect to the L^2 -inner product), it is clear from (7.4) that $\text{im}(\mathcal{P}) \subset \text{im}(d^*)$ is equivalent to $R_1 = R_2 = R_3 = 0$. Therefore, if there is a \mathcal{P} satisfying (7.3) and (3), then it is unique; it must be $\mathcal{P}_{\text{std}} = -d^*\mathcal{G}$. \square

We see that the standard Hodge homotopy \mathcal{P}_{std} can be characterized as the unique Hodge homotopy with coexact image.

We would like to use the equation

$$d\mathcal{P}_{\text{std}} = \pm H \quad \text{on } (M \times M) \setminus \Delta \quad (7.6)$$

7. Standard Hodge propagator

and characterize P_{std} as its unique coexact solution. This is probably not enough and additional assumptions on the asymptotic behavior near Δ are needed.

Another idea is to lift the equation (7.6) to the blow-up and study primitives to π^*H on $\text{Bl}_\Delta(M \times M)$. The advantage is that $\text{Bl}_\Delta(M \times M)$ is a compact manifold with boundary, and hence for a given Riemannian metric, we have the Hodge decomposition with boundary conditions. However, the pull-back metric g along $\pi : \text{Bl}_\Delta(M \times M) \rightarrow M \times M$ is singular at the boundary. We would have to approximate it with a family of metrics g_t in a correct way, use Hodge theory to find a d^* -primitive η_t to a unique coexact d -primitive P_t to π^*H with certain boundary conditions, and finally prove that P_t converges to a solution of (7.6) on the interior and η_t to its d^* -primitive.

Natural differential operators on $\Omega(M \times M)$ do not always pull-back to differential operators on $\Omega(\text{Bl}_\Delta(M \times M))$ via the blow-down map. The total differential d does, but the operators $\mathbb{1} \otimes d_y$ and $d_x \otimes \mathbb{1}$ do not. We suppose that none of d^* , $\mathbb{1} \otimes d_y^*$ and $d_x^* \otimes \mathbb{1}$ does. We illustrate the consequences on the following seemingly pathological example.

Proposition 7.2.2 (Pathological example). *For any $n \in \mathbb{N}$, there exists a smooth form $\eta \in \Omega(\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n))$ with compact vertical support with respect to the fiber bundle $\tilde{\pi}_2 = \text{pr}_2 \circ \pi : \text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow \mathbb{R}$ such that $d_y \eta = 0$ on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$ but $d\tilde{\pi}_{2*}\omega \neq 0$.*

If $\eta \in \Omega(\text{Bl}_\Delta(\mathbb{R} \times \mathbb{R}))$ is as above and satisfies in addition $\tau^\eta = \pm\eta$, then $d_y \eta = 0$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ implies $d\tilde{\pi}_{2*}\eta = 0$.*

Proof. Let us start with $n = 1$. Let $f : \mathbb{R} \times \mathbb{R} \setminus \Delta \rightarrow \mathbb{R}$ be a function such that

1. $f : \mathbb{R} \times \mathbb{R} \setminus \Delta \rightarrow \mathbb{R}$ is smooth.
2. It holds $\frac{\partial f}{\partial y}(x, y) = 0$ for all $(x, y) \in \mathbb{R} \times \mathbb{R} \setminus \Delta$.
3. For every $y \in \mathbb{R}$, the function $f(\cdot, y)$ has compact support in \mathbb{R} .

Clearly, (1) and (2) implies

$$f(x, y) = \begin{cases} f^-(x) & \text{for } x < y, \\ f^+(x) & \text{for } x > y, \end{cases} \quad \text{for all } (x, y) \in \mathbb{R} \times \mathbb{R} \setminus \Delta, \quad (7.7)$$

where $f^+, f^- : \mathbb{R} \rightarrow \mathbb{R}$ are smooth functions. It is easy to see that (3) implies the existence of $x^+, x^- \in \mathbb{R}$ such that

$$f^+(x) = 0 \quad \text{for all } x > x^+ \quad \text{and} \quad f^-(x) = 0 \quad \text{for all } x < x^-.$$

7. Standard Hodge propagator

We consider the form

$$\eta(x, y) = f(x, y) dx \quad \text{on } \mathbb{R} \times \mathbb{R} \setminus \Delta. \quad (7.8)$$

Recall that in general, a form $\eta(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ is restriction of a smooth form on $\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$ if and only if the form $(\Phi^*\eta)(r, \omega, u)$ for the diffeomorphism $\Phi : (0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ given by $\Phi(r, \omega, u) = (u + r\omega, u)$ is restriction of a smooth form on $[0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$. In our case, we have

$$(\Phi^*\eta)(r, u) = \begin{cases} f^+(u + r)(du + dr) & \text{on } D^+ := [0, \infty) \times \{1\} \times \mathbb{R}, \\ f^-(u - r)(du - dr) & \text{on } D^- := [0, \infty) \times \{-1\} \times \mathbb{R}, \end{cases}$$

where we used the fact that $\mathbb{S}^0 = \{\pm 1\}$ and splitted the domain in two connected components. We see that $\Phi^*\eta$ extends smoothly to $[0, \infty) \times \mathbb{S}^0 \times \mathbb{R}$ in the obvious way. We denote the extension by $\widetilde{\Phi^*\eta}$ and the induced extension of η to $\text{Bl}_\Delta(\mathbb{R} \times \mathbb{R})$ by $\tilde{\eta}$. It holds $\tilde{\Phi}^*\tilde{\eta} = \widetilde{\Phi^*\eta}$ under the extended diffeomorphism $\tilde{\Phi} : [0, \infty) \times \mathbb{S}^0 \times \mathbb{R} \rightarrow \text{Bl}_\Delta(\mathbb{R} \times \mathbb{R})$. The fiberwise integral along the smooth oriented fiber bundle $\tilde{\pi}_2 : \text{Bl}_\Delta(\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}$ transforms under $\tilde{\Phi}$ to the fiberwise integral along $\tilde{p}_3 : [0, \infty) \times \mathbb{S}^0 \times \mathbb{R} \rightarrow \mathbb{R}$, and we get for all $y = u \in \mathbb{R}$ by the definition

$$(\tilde{\pi}_{3*}\tilde{\eta})(y) = \tilde{p}_{3*}(\widetilde{\Phi^*\eta})(u) = \int_0^\infty (f^+(u + r) + f^-(u - r)) dr.$$

The algebraic sign of the second term was canceled by the geometric sign coming from different orientations of $[0, \infty)$ in $[0, \infty) \times \{0\}$ and in $[0, \infty) \times \{1\}$. We think of $\tilde{\Phi}$ as of an isomorphism of fiber bundles \tilde{p}_3 and $\tilde{\pi}_2$ covering the identity, which explains the notation $y = u$. The fiberwise integration along $\tilde{p}_3 : [0, \infty) \times \mathbb{S}^0 \times \mathbb{R} \rightarrow \mathbb{R}$ reduces to Lebesgue integration of a smooth function g on $[0, \infty) \times \mathbb{S}^0 \times \mathbb{R}$ with respect to (r, ω) . Clearly, we can permute any differential operator acting on u with the integral. In our

7. Standard Hodge propagator

case, we use d and obtain

$$\begin{aligned}
(d\tilde{\pi}_{3*}\tilde{\eta})(y) &= (d\tilde{p}_{3*}\widetilde{\Phi^*\eta})(u) \\
&= \frac{\partial}{\partial u} \left(\int_0^\infty (f^+(u+r) + f^-(u-r)) dr \right) du \\
&= \left(\int_0^\infty (f^+)'(u+r) + (f^-)'(u-r) dr \right) du \\
&= \left(\int_u^\infty (f^+)'(z) dz + \int_{-\infty}^u (f^-)'(z) dz \right) du \\
&= (f^-(u) - f^+(u)) du
\end{aligned}$$

This is zero precisely when f extends continuously to $\mathbb{R} \times \mathbb{R}$.

A general form on $\mathbb{R} \times \mathbb{R} \setminus \Delta$ is

$$\eta(x, y) = f_0(x, y) + f_1(x, y) dx + f_2(x, y) dy + f_3(x, y) dx dy.$$

Imposing the condition $d_y \eta = 0$, we get (7.7) for both f_0 and f_1 . Imposing $\tau^* \omega = -\omega$, we get $f_3(x, y) = f_3(y, x)$, $f_2(x, y) = -f_1(y, x)$ and $f_0(x, y) = -f_0(y, x)$. Now, requiring that η has compact vertical support in x implies $f_0 = f_1 \equiv 0$ because of (7.7), and hence also $f_2 \equiv 0$. However, the form $f_3(x, y) dx dy$ satisfies $d_y \eta = 0$ and $d\tilde{\pi}_{2*} = 0$ trivially.

For any $n \geq 1$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, consider

$$\eta(x, y) := \frac{\lambda(x)}{|x - y|^n} \sum_{i=1}^n (-1)^{i+1} (x^i - y^i) dx^1 \cdots dx^n dy^1 \cdots \widehat{dy^i} \cdots dy^n,$$

where $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth bump function. Then η extends smoothly to $\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$, it holds $d_y \eta(x, y) = 0$, but

$$d \int_x \eta(x, y) = \text{Vol}(\mathbb{S}^n) \lambda(y) dy^1 \cdots dy^n.$$

This finishes the proof. □

7.3. Approximation using heat form

Given an oriented Riemannian manifold (M, g) , consider the *heat form*

$$K_t(x, y) = \sum (-1)^{kn} e^{-\lambda_i t} (\star e_i)(x) \wedge e_i(y), \quad (7.9)$$

7. Standard Hodge propagator

where (e_i) are eigenvectors of Δ and λ_i the corresponding eigenvalues. It is equivalently a solution of the equation $\Delta K_t(x, y) = -\frac{\partial}{\partial t} K_t(x, y)$ with certain boundary condition.

Proposition 7.3.1 (Properties of the heat kernel). *Let M be an oriented Riemannian manifold. The heat form $K_t(x, y)$ is smooth on $M \times M \times (0, \infty)$ and satisfies*

$$dK_t = 0, \quad \tau^* K_t = (-1)^n K_t \quad \text{and} \quad \frac{1}{2} \Delta K_t = \Delta_x K_t = -\frac{\partial}{\partial t} K_t.$$

Proof. Straightforward computation. □

Proposition 7.3.2 (Approximation using heat form). *Let M be an oriented Riemannian manifold and $K_t(x, y)$ the heat form. For all $(t, x, y) \in ([0, \infty) \times M \times M) \setminus \{0\} \times \Delta =: D(K)$, we define*

$$\begin{aligned} G_t(x, y) &:= \int_t^\infty K_\tau(x, y) d\tau \quad \text{and} \\ P_t(x, y) &:= (-1)^{n+1} \int_t^\infty (\mathbb{1} \hat{\otimes} d_y^*) K_\tau(x, y) d\tau. \end{aligned} \tag{7.10}$$

Then:

- (a) *The forms G_t and P_t are smooth on $D(K)$, the point-wise limits G' and P' exist, and it holds $G_t \xrightarrow{t} G'$ and $P_t \xrightarrow{t} P'$ in $C_{loc}^\infty(M \times M \setminus \Delta)$.*
- (b) *On $D(K)$, the following relations hold:*

$$\begin{aligned} dG_t &= 0 & P_t &= (-1)^{n+1} \frac{1}{2} d^* G_t \\ \Delta G_t &= K_t - H & dP_t &= (-1)^n (H - K_t) \\ \tau^* G_t &= (-1)^n G_t & \tau^* P_t &= (-1)^n P_t. \end{aligned}$$

It follows that $G' = G$ is the (Laplace) Green form and $P' = P_{\text{std}}$ the standard Hodge propagator.

Proof. The formal computation is clear. An honest proof uses the standard heat kernel estimates. □

Proposition 7.3.3 (P_{std} is codifferential of G). *Let M be an oriented Riemannian manifold, and let $G \in \Omega^n(M \times M \setminus \Delta)$ be the Green form. Then the standard Hodge propagator P_{std} satisfies*

$$P_{\text{std}}(x, y) = (-1)^{n+1} (\mathbb{1} \otimes d_y^*) G(x, y),$$

7. Standard Hodge propagator

where $\mathbb{1} \otimes d_y^* : \Omega^\bullet(M \times M) \rightarrow \Omega^{\bullet-1}(M \times M)$ is the differential operator defined in local coordinates by commuting d^* over the first factor with the Koszul sign and applying it to the second factor.

Proof. As for the signs, $(-1)^n$ comes from $T\mathcal{G}\omega(y) = (-1)^{nT} \int_x (\mathbb{1} \otimes T_y)G(x, y)\omega(x)$ with $T = d^*$ and -1 from $\mathcal{P}_{\text{std}} = -d^*\mathcal{G}$. The rest can be proven using the heat kernel approximation and standard heat kernel estimates. There is an other method using the asymptotic expansion of G , which was shown to the author by Prof. Christian Bär. \square

7.4. Standard Hodge propagator for Euclidean space

We will use Propositions 7.3.3 and 7.3.2 to prove in two ways that \mathcal{P}_{std} for \mathbb{R}^n extends smoothly to the blow-up. We will start with the following well known formulas.

Proposition 7.4.1 (Green form and heat kernel for \mathbb{R}^n). *The Green form for \mathbb{R}^n , which can be equivalently characterized as the unique solution $G \in \Omega^n(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta)$ of*

$$\Delta_y G(x, y) = \delta(x - y)(dx^1 - dy^1) \cdots (dx^n - dy^n), \quad (7.11)$$

where $\delta(x - y)$ is the Dirac delta function, satisfies

$$G(x, y) = \begin{cases} \frac{1}{(n-2)\text{Vol}(\mathbb{S}^{n-1})} \frac{1}{|x-y|^{n-2}} (dx^1 - dy^1) \cdots (dx^n - dy^n) & \text{for } n \geq 3, \\ -\frac{1}{2\pi} \ln |x-y| (dx^1 - dy^1)(dx^2 - dy^2) & \text{for } n = 2. \end{cases}$$

The heat kernel for \mathbb{R}^n , i.e., the solution of $\Delta_y Q_t(x, y) = -\frac{\partial}{\partial t} Q_t(x, y)$, is given by

$$Q_t(x, y) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right) (dx^1 - dy^1) \cdots (dx^n - dy^n). \quad (7.12)$$

Proof. See [BGV04] \square

Proposition 7.4.2 (Standard Hodge propagator for \mathbb{R}^n). *The standard Hodge propagator for \mathbb{R}^n with $n \geq 2$ satisfies*

$$\mathcal{P}_{\text{std}}(x, y) = \frac{(-1)^{n+1}}{\text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n (-1)^{i-1} \frac{x^i - y^i}{|x-y|^n} (dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n)$$

and extends smoothly to $\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$.

7. Standard Hodge propagator

Proof using the Green form. Consider the formulas from Proposition 7.4.1. Recall the formula for the codifferential

$$d^* f(x) dx^I = - \sum_{i \in I} \varepsilon(i, I) \frac{\partial f(x)}{\partial x^i} dx^{I \setminus \{i\}}. \quad (7.13)$$

For $n \geq 3$, we compute

$$\begin{aligned} & (\mathbb{1} \otimes d_y^*) G(x, y) \\ &= \frac{1}{(n-2)\text{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|} (\mathbb{1} \otimes d_y^*) \left(\frac{1}{|x-y|^{n-2}} (*dx^I) \wedge dy^I \right) \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|+|I|+n+1} \sum_{i \in I} \frac{x^i - y^i}{|x-y|^n} \varepsilon(i, I) (*dx^I) \wedge dy^{I \setminus \{i\}} \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|+|I|+n+1} \sum_{i \in I} \frac{x^i - y^i}{|x-y|^n} \varepsilon(i, I) \varepsilon(I, I^c) dx^{I^c} \wedge dy^{I \setminus \{i\}} \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n \frac{x^i - y^i}{|x-y|^n} \sum_{J \subset \{1, \dots, \hat{i}, \dots, n\}} (-1)^{n|J|+|J|} \varepsilon(i, J) \varepsilon(J \cup \{i\}, J^c \setminus \{i\}) \\ & \quad dx^{J^c \setminus \{i\}} \wedge dy^J \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n (-1)^{i-1} \frac{x^i - y^i}{|x-y|^n} (dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n). \end{aligned}$$

In the last step, we used that

$$\varepsilon(i, J) \varepsilon(J \cup \{i\}, J^c \setminus \{i\}) = (-1)^{i-1} \varepsilon(J, J^c \setminus \{i\})$$

and

$$\begin{aligned} & dx^{J^c \setminus \{i\}} \wedge dy^J \\ &= (-1)^{|J|} \varepsilon(J^c \setminus \{i\}, J) [(dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n)]_{J^c \setminus \{i\}, J} \\ &= (-1)^{n|J|+|J|} [(dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n)]_{J^c \setminus \{i\}, J}, \end{aligned}$$

where $[\cdot]_{I_1, I_2}$ denotes the part of the product which picks the first variable at positions I_1 and the second at positions I_2 . The computation for $n = 2$ gives the same result, and the formula for P_{std} is justified by Proposition 7.3.3.

We will now study whether P_{std} extends smoothly to the blow-up. Consider the polar

7. Standard Hodge propagator

coordinates in one variable

$$\begin{aligned}\phi : [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (r, \omega, u) &\longmapsto (u + r\omega, u).\end{aligned}$$

There is a unique smooth map $\tilde{\phi}$ which fits in the blow-up diagram

$$\begin{array}{ccc} & & \text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n) \\ & \nearrow \tilde{\phi} & \downarrow \pi \\ [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R}^n \times \mathbb{R}^n,\end{array}$$

and it is a diffeomorphism of manifolds with boundary. We denote by

$$\phi_0 := \phi|_{(0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n} : (0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$$

the restriction of ϕ to the interior. The commutative diagram

$$\begin{array}{ccc} [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n & \xrightarrow{\tilde{\phi}} & \text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n) \\ \uparrow & & \uparrow \tilde{\iota} \\ (0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n & \xrightarrow{\phi_0} & \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \end{array}$$

shows that we can equivalently study whether the form

$$\mathbf{P}_{\text{pol}} := \phi_0^* \mathbf{P}_{\text{std}}$$

7. Standard Hodge propagator

admits a smooth extension $\widetilde{P_{\text{pol}}}$ to $[0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$. We have

$$\begin{aligned}
P_{\text{pol}}(r, \omega, u) &= \phi_0^*(d_y^* G)(r, \omega, u) \\
&= \frac{(-1)^{n+1}}{r^{n-1} \text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n (-1)^{i-1} \omega^i (r d\omega^1 + \omega^1 dr) \cdots \widehat{(r d\omega^i + \omega^i dr)} \cdots \\
&\quad (r d\omega^n + \omega^n dr) \\
&= \frac{(-1)^{n+1}}{r^{n-1} \text{Vol}(\mathbb{S}^{n-1})} \left(r^{n-1} \sum_{1 \leq i \leq n} (-1)^{i-1} \omega^i d\omega^1 \cdots \widehat{d\omega^i} \cdots d\omega^n \right. \\
&\quad \left. - r^{n-2} \sum_{1 \leq i < j \leq n} (-1)^{i+j} \omega^i \omega^j dr d\omega^1 \cdots \widehat{d\omega^i} \cdots \widehat{d\omega^j} \cdots d\omega^n \right. \\
&\quad \left. + r^{n-2} \sum_{1 \leq j < i \leq n} (-1)^{i+j} \omega^j \omega^i dr d\omega^1 \cdots \widehat{d\omega^j} \cdots \widehat{d\omega^i} \cdots d\omega^n \right) \\
&= (-1)^{n+1} \frac{\text{Vol}_{\mathbb{S}^{n-1}}(\omega)}{\text{Vol}(\mathbb{S}^{n-1})},
\end{aligned}$$

where the last two sums canceled. The result extends smoothly beyond $r = 0$ because it does not depend on r at all. \square

Proof using the heat kernel. Consider the polar coordinates with respect to the diagonal

$$\begin{aligned}
\varphi : [0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \\
(r, \omega, u) &\longmapsto (x, y) = (u + r\omega, u - r\omega).
\end{aligned}$$

There is a unique smooth map $\tilde{\varphi}$ which fits in the blow-up diagram

$$\begin{array}{ccc}
& & \text{Bl}_{\Delta}(\mathbb{R}^n \times \mathbb{R}^n) \\
& \nearrow \tilde{\varphi} & \downarrow \pi \\
[0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n & \xrightarrow{\varphi} & \mathbb{R}^n \times \mathbb{R}^n,
\end{array}$$

and it is a diffeomorphism of manifolds with boundary. The heat kernel (7.12) transforms under φ as

$$\begin{aligned}
Q_t^{\text{pol}}(r, \omega, u) &:= (\varphi^* Q_t)(r, \omega, u) \\
&= (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) 2^n r^{n-1} dr \text{Vol}(\omega) \\
&= (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) r^{n-1} dr \text{Vol}(\omega).
\end{aligned}$$

7. Standard Hodge propagator

It follows from Proposition 7.3.2 that for $(r, \omega, u) \in (0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$ we have

$$P_{\text{pol}}(r, \omega, u) = (-1)^{n+1} \frac{1}{2} \lim_{t \rightarrow 0} \int_t^\infty d^{*\text{pol}} Q_\tau^{\text{pol}}(r, \omega, u) d\tau, \quad (7.14)$$

where $d^{*\text{pol}}$ is the codifferential on $\mathbb{R}^n \times \mathbb{R}^n$ computed with respect to the pullback of the product Riemannian metric. If $g : \mathbb{R}^n \otimes \mathbb{R}^n \rightarrow \mathbb{R}$ is an inner product, then the pullback Riemannian metric satisfies

$$g^{\text{pol}} := \varphi^*(g \oplus g)(r, \omega, u) = 2 \begin{pmatrix} g(\omega, \omega) & 0 & 0 \\ 0 & r^2 g & 0 \\ 0 & 0 & g \end{pmatrix}; \quad (7.15)$$

it is degenerate at the boundary $r = 0$. Next, it is easy to check that the conformal transformation $g \mapsto \lambda g$ on vectors induces the conformal transformation $g \mapsto \frac{1}{\lambda^k} g$ on k -forms. Thus, if $\|\cdot\|^{\text{pol}}$ denotes the point-wise norm with respect to g^{pol} for the standard Euclidean metric, then

$$\|\text{dr}\|^{\text{pol}} = 2^{-\frac{1}{2}}, \quad \|\text{Vol}(u)\|^{\text{pol}} = 2^{-\frac{n}{2}}, \quad \|\text{Vol}(\omega)\|^{\text{pol}} = 2^{-\frac{n-1}{2}} r^{-(n-1)},$$

and hence

$$\text{Vol}^{\text{pol}}(r, \omega, u) = 2^n r^{n-1} \text{dr Vol}(\omega) \text{Vol}(u).$$

In order to get the Hodge star $*^{\text{pol}}$, we compute

$$\begin{aligned} \text{dr Vol}(\omega) \wedge *^{\text{pol}}(\text{dr Vol}(\omega)) &\stackrel{!}{=} \|\text{dr Vol}(\omega)\|^{\text{pol}^2} \text{Vol}^{\text{pol}}(r, \omega, u) \\ &= r^{-(n-1)} \text{dr Vol}(\omega) \text{Vol}(u), \\ \text{dr Vol}(u) \wedge *^{\text{pol}}(\text{dr Vol}(u)) &\stackrel{!}{=} \|\text{dr Vol}(u)\|^{\text{pol}^2} \text{Vol}^{\text{pol}}(r, \omega, u) \\ &= \frac{1}{2} r^{n-1} \text{dr Vol}(\omega) \text{Vol}(u). \end{aligned}$$

Using the product structure (7.15), it follows that

$$\begin{aligned} *^{\text{pol}}(\text{dr Vol}(\omega)) &= r^{-(n-1)} \text{Vol}(u) \quad \text{and} \\ *^{\text{pol}}(\text{dr Vol}(u)) &= \frac{1}{2} r^{n-1} \text{Vol}(\omega). \end{aligned}$$

7. Standard Hodge propagator

Recalling the definition of the codifferential $d^* \alpha = (-1)^{n(k-1)+1} * d * \alpha$, we compute

$$\begin{aligned} (*^{\text{pol}} d *^{\text{pol}})(Q_t^{\text{pol}}(r, \omega, u)) &= *^{\text{pol}} d \left((\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) \text{Vol}(u) \right) \\ &= *^{\text{pol}} \left(\frac{-2r}{t} (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) dr \text{Vol}(u) \right) \\ &= -\frac{1}{t} (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) r^n \text{Vol}(\omega), \end{aligned}$$

and because the product dimension is even, we obtain

$$d^{\text{pol}} Q_t^{\text{pol}}(r, \omega, u) = \frac{1}{t} (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) r^n \text{Vol}(\omega).$$

In order to integrate this according to (7.14), we will make use of the Γ -function $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$. Then, we have

$$\begin{aligned} (-1)^{n+1} P_{\text{pol}}(r, \omega, u) &= \frac{1}{2} \lim_{t \rightarrow 0} \int_t^\infty d^{\text{pol}} Q_\tau^{\text{pol}}(r, \omega, u) d\tau \\ &= \frac{1}{2} \pi^{-\frac{n}{2}} r^n \left(\lim_{t \rightarrow 0} \int_t^\infty \tau^{-\frac{n}{2}-1} \exp\left(-\frac{r^2}{\tau}\right) d\tau \right) \text{Vol}(\omega) \\ &= \frac{1}{2} \pi^{-\frac{n}{2}} r^n \left(\lim_{t \rightarrow 0} \int_{\frac{r^2}{t}}^0 \left(\frac{r^2}{z}\right)^{-\frac{n}{2}-1} \exp(-z) (-z^{-2} r^2 dz) \right) \text{Vol}(\omega) \\ &= \frac{1}{2} \pi^{-\frac{n}{2}} \left(\lim_{t \rightarrow 0} \int_0^{\frac{r^2}{t}} z^{\frac{n}{2}-1} \exp(-z) dz \right) \text{Vol}(\omega) \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \text{Vol}(\omega) \\ &= \frac{\text{Vol}(\omega)}{\text{Vol}(\mathbb{S}^{n-1})}, \end{aligned}$$

which recovers the formula in the proof of Proposition 7.4.2. □

Example 7.4.3 (The case $n = 1$). It is easy to check directly that

$$P(x, y) := \theta(x - y),$$

where θ is the Heavyside step function, is the Schwartz form of a Hodge homotopy \mathcal{P} . Indeed, for any smooth $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support, we have

$$\mathcal{P}(f dx)(y) = \int_x P(x, y) f(x) dx = \int_y^\infty f(x) dx,$$

7. Standard Hodge propagator

and hence $d \circ \mathcal{P} = -\mathbb{1}$. It is also easy to check that the following is the Laplace Green form:

$$G(x, y) = -\frac{1}{2}|x - y|(dx - dy).$$

Indeed, using $\text{sgn}(x) = 2\theta(x) - 1$, we get

$$\frac{\partial^2}{\partial y^2} \frac{1}{2}|x - y| = \frac{\partial}{\partial y} \frac{1}{2} \text{sgn}(y - x) = \delta(y - x) = \delta(x - y),$$

which shows (7.11). We carefully compute

$$\begin{aligned} (\mathbb{1} \otimes d_y^*)G(x, y) &= d_y^* \left(\frac{1}{2}|x - y| dy \right) = -\frac{1}{2} \frac{\partial}{\partial y} |x - y| = -\frac{1}{2}(2\theta(y - x) - 1) \\ &= -\theta(y - x) + \frac{1}{2} = \theta(x - y) - \frac{1}{2} \end{aligned}$$

and apply Proposition 7.3.3 to get

$$P_{\text{std}}(x, y) = \theta(x - y) - \frac{1}{2}.$$

In particular, we see that the sign agrees with the direct computation of the Hodge propagator above. Smoothness on the blow-up is for $n = 1$ equivalent to smoothness on the closures of connected components of $\mathbb{R} \times \mathbb{R} \setminus \Delta$; this is readily satisfied. \triangleleft

7.5. Standard Hodge propagator for 1- and 2-sphere

In this section, we denote the Hodge propagator for \mathbb{S}^n constructed in Part I by P_{art} . We would like to use P_{art} to study the standard Hodge propagator P_{std} for \mathbb{S}^n .

For \mathbb{S}^1 , we can compute K_t and hence P_{std} explicitly.

Example 7.5.1 (P_{std} for \mathbb{S}^1). We write $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and use the coordinate $x \in [0, 2\pi)$. Because \mathbb{S}^1 is flat, we have

$$\Delta = -\frac{\partial^2}{\partial x^2}.$$

Solving the eigenvalue problem $\Delta \omega = \lambda \omega$ for $\omega \in \Omega(\mathbb{S}^1)$ and $\lambda \in \mathbb{R}$, we get $\lambda \in \{0, n^2 \mid n \in \mathbb{N}\}$ and the corresponding eigenvectors

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \cos(nx) dx, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \sin(nx) dx \mid n \in \mathbb{N} \right\},$$

7. Standard Hodge propagator

which we normalized in the L^2 -norm. Plugging in (7.9), we get

$$\begin{aligned} K_t(x_1, x_2) &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} (\cos(nx_1) \cos(nx_2) + \sin(nx_1) \sin(nx_2)) (dx_1 - dx_2) \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \cos(nx_1 - nx_2) (dx_1 - dx_2). \end{aligned}$$

Applying the product codifferential, we get

$$\begin{aligned} d^* K_t(x_1, x_2) &= \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \left[-\frac{\partial}{\partial x_1} \cos(nx_1 - nx_2) + \frac{\partial}{\partial x_2} \cos(nx_1 - nx_2) \right] \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} n \sin(nx_1 - nx_2). \end{aligned}$$

Finally, the integration gives

$$\begin{aligned} P_{\text{std}}(x_1, x_2) &= \frac{1}{2} \int_0^{\infty} d^* K_t(x_1, x_2) dt \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\infty} e^{-n^2 t} dt \right) n \sin(nx_1 - nx_2) \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n(x_1 - x_2))}{n} \\ &= \frac{1}{2\pi} \begin{cases} \pi - (x_1 - x_2) & x_1 > x_2, \\ -\pi - (x_1 - x_2) & x_1 < x_2. \end{cases} \\ &= \frac{1}{2\pi} (\alpha(x_1, x_2) - \pi). \end{aligned}$$

This is precisely P_{art} from Part I. \triangleleft

For $n \geq 2$, an explicit formula for any of P_{std} , G or K_t seems to be unknown. For \mathbb{S}^2 , the formula for P_{std} on functions was derived by A. Hermann.

Our idea to study P_{std} via P_{art} is to examine the uniqueness of Hodge propagators. Consider the Schwartz form $H(x, y) = \frac{1}{V} (\text{Vol}(x) + (-1)^n \text{Vol}(y))$ of the harmonic projection for \mathbb{S}^n . Let $C_2(\mathbb{S}^n) := \mathbb{S}^n \times \mathbb{S}^n \setminus \Delta$ denote the configuration space, and let

$$\mathcal{S}_n := \{G \in \Omega^{n-1}(C_2(\mathbb{S}^n)) \mid dG = (-1)^n H\} \quad (7.16)$$

be the space of primitives to $(-1)^n H$. We know that $P_{\text{std}} \in \mathcal{S}_n$. The following holds.

7. Standard Hodge propagator

Proposition 7.5.2 (The space of primitives to H for \mathbb{S}^n). *Let*

$$V_n := \begin{cases} \Omega^{n-2}(C_2(\mathbb{S}^n))/d\Omega^{n-3}(C_2(\mathbb{S}^n)) & \text{for } n \geq 3, \\ \Omega^0(C_2(\mathbb{S}^n))/\mathbb{R} & \text{for } n = 2, \end{cases}$$

where $\mathbb{R} \subset \Omega^0(C_2(\mathbb{S}^n))$ denotes the constants. The action $\rho : V_n \times \mathcal{S}_n \rightarrow \mathcal{S}_n$, $(\lambda, G) \mapsto G + d\lambda$ of the additive group V_n on \mathcal{S}_n defines the structure of an affine space on \mathcal{S}_n for $n \geq 2$. If we require $SO(n+1)$ or $(-1)^n \tau^*$ -invariance, then the same holds with Ω replaced by the correspondingly invariant forms.

Proof. We have to check that the action ρ is free and transitive. For $G_1, G_2 \in \mathcal{S}_n$, the difference $\eta := G_1 - G_2$ is a closed $(n-1)$ -form; it is exact because $C_2(\mathbb{S}^n)$ is homotopy equivalent to \mathbb{S}^n . A primitive λ_1 is an $n-2$ form. If λ_2 is another primitive, then $\lambda_1 - \lambda_2$ is closed, and hence it is a constant for $n = 2$ and an exact form for $n \geq 3$. Therefore, \mathcal{S}_n is an affine space over V_n . As for the invariance, we can average a primitive of an invariant form over $SO(n+1)$ or take $\frac{1}{2}(\mathbb{1} + (-1)^n \tau^*)$. \square

Note that $\mathcal{S}_1 \simeq \mathbb{R}$ by adding the constant and that all functions on $C_2(\mathbb{S}^1)$ are coexact. Therefore, P_{std} for \mathbb{S}^1 can not be characterized as a unique coexact solution of the differential equation for the Hodge propagator.

Proposition 7.5.3 (Coexactness of artificial Hodge propagator). *The Hodge propagator $P_{\text{art}} \in \Omega^{n-1}(\mathbb{S}^n \times \mathbb{S}^n \setminus \Delta)$ constructed in Part I is coexact for every $n \in \mathbb{N}$.*

Proof. First of all, we rewrite

$$\begin{aligned} \omega_k(x, y) &= \frac{1}{k!(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n+1}} x^{\sigma_1} y^{\sigma_1} dx^{\sigma_3} \dots dx^{\sigma_{2+k}} dy^{\sigma_{3+k}} \dots dy^{\sigma_{n+1}} \\ &= (-1)^k \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} \iota_x(dx^I) \wedge \underbrace{\iota_y *^{\mathbb{R}^{n+1}}}_{*^{\mathbb{S}^n}}(dy^I). \end{aligned}$$

Recall the formulas $d^* \alpha = (-1)^{d(k-1)+1} * d * \alpha$ and $** \alpha = (-1)^{k(n-k)} \alpha$ for $\alpha \in \Omega^k(M)$,

7. Standard Hodge propagator

where $d = \dim(M)$. For all $(x, y) \in \mathbb{S}^n \times \mathbb{S}^n \setminus \Delta$, we compute

$$\begin{aligned}
d_y^* G(x, y) &= d_y^* \left(\sum_{k=0}^{n-1} (-1)^k g_k(x \cdot y) \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} (\iota_x dx^I) \wedge *^{\mathbb{S}^n} (dy^I) \right) \\
&= \sum_{k=0}^{n-1} \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} (\iota_x dx^I) \wedge d_y^* (g_k(x \cdot y) *^{\mathbb{S}^n} (dy^I)) \\
&= \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} (\iota_x dx^I) \wedge *^{\mathbb{S}^n} d_y (g_k(x \cdot y) dy^I) \\
&\quad \uparrow \\
&\quad d_y^* = (-1)^{n(n-k)+1} *^{\mathbb{S}^n} d *^{\mathbb{S}^n} \\
&\quad *^{\mathbb{S}^n} *^{\mathbb{S}^n} = (-1)^{(k+1)(n-k-1)} \mathbb{1} \\
&\quad \text{tot. sign} = (-1)^k \\
&= \sum_{k=0}^{n-1} (-1)^k g'_k(x \cdot y) \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} \sum_{i \in I} \sum_{j \in \{1, \dots, n+1\} \setminus I} \varepsilon(i, I) \varepsilon(j, I) x^i x^j \\
&\quad \underbrace{dx^{I \setminus \{i\}} \wedge *^{\mathbb{S}^n} (dy^{I \cup \{j\}})}_{=0} \\
&= 0.
\end{aligned}$$

The cancellation occurs because the summand (I, i, j) contains the same terms as the summand $(I' = I \setminus \{i\} \cup j, j, i)$, and the signs satisfy

$$\varepsilon(j, I') \varepsilon(i, I') = -\varepsilon(i, I) \varepsilon(j, I).$$

We have

$$H_{n-1}(\Omega(\mathbb{S}^n \times \mathbb{S}^n \setminus \Delta), d^*) \simeq H_{\text{dR}}^{n+1}(\mathbb{S}^n \times \mathbb{S}^n \setminus \Delta) = 0,$$

and hence any coclosed $(n-1)$ -form is coexact. \square

Proposition 7.5.4 (Smooth extension to the blow-up for \mathbb{S}^2). *The standard Hodge propagator for \mathbb{S}^2 extends smoothly to the blow-up.*

Proof. Let $G_1, G_2 \in \mathcal{S}_2$ be two $\text{SO}(3)$ -symmetric solutions. Proposition 7.5.2 asserts that there is a smooth $\text{SO}(3)$ -symmetric function $\alpha : C_2(\mathbb{S}^2) \rightarrow \mathbb{R}$ such that $G_1 - G_2 = d\lambda$. Because $\text{SO}(3)$ acts on \mathbb{S}^2 transitively, there is a smooth function $f : [-1, 1] \rightarrow \mathbb{R}$ such that

$$\lambda(x, y) = f(x \cdot y) \quad \text{for all } (x, y) \in C_2(\mathbb{S}^2).$$

7. Standard Hodge propagator

Note that one can let f explode at 1 and obtain Hodge propagators which do not extend smoothly to the blow-up. Let us assume, in addition, that $G_1 - G_2$ is coexact. We obtain

$$0 = d^*(G_1 - G_2) = d^*d\lambda = \Delta\lambda.$$

Therefore, λ is a harmonic function on $C_2(\mathbb{S}^2)$. Denoting $B(x, y) := x \cdot y$, we can write $\lambda = f \circ B$, which implies

$$\Delta(f \circ B) = f''\|\nabla B\|^2 + f' \Delta B.$$

The computation of $\|\nabla B\|$ and ΔB is straightforward and we will do it for any $n \in \mathbb{N}$. If $\tilde{f} : \mathbb{S}^n \rightarrow \mathbb{R}$ is a smooth function and $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is defined by

$$f(x) := \tilde{f}\left(\frac{x}{|x|}\right) \quad \text{for all } x \in \mathbb{R}^{n+1} \setminus \{0\},$$

then

$$\Delta^{\mathbb{S}^n} \tilde{f} = (\Delta^{\mathbb{R}^{n+1}} f)|_{\mathbb{S}^n} \quad \text{and} \quad \nabla^{\mathbb{S}^n} \tilde{f} = (\nabla^{\mathbb{R}^{n+1}} f)|_{\mathbb{S}^n}.$$

We compute

$$\begin{aligned} \Delta_x^{\mathbb{R}^{n+1}} \left(\frac{x}{|x|} \cdot y \right) &= \sum_{i=1}^{n+1} -\frac{\partial}{\partial x^i} \left(\frac{y^i}{|x|} - \frac{x^i}{|x|^3} x \cdot y \right) \\ &= \sum_{i=1}^{n+1} \frac{x^i y^i}{|x|^3} - 3 \frac{x^i x^i}{|x|^5} x \cdot y + \frac{x \cdot y}{|x|^3} + \frac{x^i y^i}{|x|^3} \\ &= 0, \\ \nabla_x^{\mathbb{R}^{n+1}} \left(\frac{x}{|x|} \cdot y \right) &= \sum_{i=1}^{n+1} \left(\frac{y^i}{|x|} - \frac{x^i}{|x|^3} x \cdot y \right) \frac{\partial}{\partial x^i} \quad \text{and} \\ \|\nabla(x \cdot y)\|^2 &= \left\| \sum_{i=1}^{n+1} (y^i - (x \cdot y)x^i) \frac{\partial}{\partial x^i} + \sum_{i=1}^{n+1} (x^i - (x \cdot y)y^i) \frac{\partial}{\partial y^i} \right\|^2 \\ &= \sum_{i=1}^{n+1} (y^i - (x \cdot y)x^i)^2 + (x^i - (x \cdot y)y^i)^2 \\ &= 1 - 2(x \cdot y)^2 + (x \cdot y)^2 + 1 - 2(x \cdot y)^2 + (x \cdot y)^2 \\ &= 2(1 - (x \cdot y)^2). \end{aligned}$$

Therefore, $\Delta B = 0$, $\|\nabla B(x_1, x_2)\|^2 = 2(1 - (x_1 \cdot x_2)^2)$, and we arrive to the equation

$$2(1 - u^2)f''(u) = 0 \quad \text{for all } u \in [-1, 1)$$

7. *Standard Hodge propagator*

and a smooth function $f : [-1, 1) \rightarrow \mathbb{R}$. The only solution is a linear function, and it must hold

$$\lambda(x_1, x_2) = A(x_1 \cdot x_2) + B \quad \text{for some } A, B \in \mathbb{R}.$$

Now, B comes from a smooth function on $\mathbb{S}^n \times \mathbb{S}^n$, and hence λ extends smoothly to the blow-up. \square

If we determine the constants A and B in the proof of Proposition 7.5.4, then we get a formula for \mathcal{P}_{std} for \mathbb{S}^2 with \mathcal{P}_{art} .

8. BV-formalism for IBL-infinity theory on cyclic cochains

In this chapter, we formulate the theory of dIBL-algebras on cyclic cochains of an odd symplectic vector space and their twisting with Maurer-Cartan elements in BV-formalism for quantum field theories. The fields are cyclic Hochschild chains and the BV-operator comes from the algebraic string bracket and cobracket. This point of view will hopefully help to understand conceptually the relation between string topology, Chern-Simons theory, symplectic field theory and string field theory.

In Section 8.1, we summarize relevant IBL_∞ -theory from [CFL15].

In Section 8.2, we define the canonical string BV-operator on the space of observables and the total action; it consists of the free part and of the interaction part which corresponds to the Maurer-Cartan element. We argue that the quantum master equation is satisfied for the canonical (Chern-Simons) Maurer-Cartan element and that the string BV-operator twisted by the action corresponds to the twisted dIBL-algebra (Proposition 8.2.1). In general, we show that the action satisfies the quantum master equation if and only if its interaction part encodes a Maurer-Cartan element and that the twistings in both formalisms agree (Proposition 8.2.2).

In Section 8.3, we sketch how to apply ideas from [DJP18] for quantum L_∞ -algebras to IBL_∞ -algebras. We ask whether their formulas for the effective action and the path integral which come from the Homological Perturbation Lemma (and Wick's Theorem) agree with the formulas from [CFL15] with summations over ribbon graphs (Question 8.3.1).

8.1. Summary of IBL-infinity theory for cyclic DGA

Let (V, Π, m_1, m_2) be a cyclic DGA of degree n of finite type. We have:

- Cyclic cochains $C(V) = \hat{B}_{\text{cyc}}^* V[2 - n]$,
- Canonical dIBL-algebra $\mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}$ on $C(V)$ denoted by $\text{dIBL}(C(V))$,

8. BV-formalism for IBL-infinity theory on cyclic cochains

- Canonical Maurer-Cartan element $\mathfrak{m} = (\mathfrak{m}_{10})$,
- Twisted dIBL-algebra $\mathfrak{q}_{110}^{\mathfrak{m}}, \mathfrak{q}_{210}, \mathfrak{q}_{120}$ on $C(V)$ denoted by $\text{dIBL}^{\mathfrak{m}}(C(V))$.

Recall from Appendix C that, up to signs and permutation of variables, \mathfrak{q}_{210} is the cyclization $^+$ of the Gerstenhaber bracket $[\cdot, \cdot]$ and \mathfrak{q}_{120} fits in $\Delta_{\text{cyc}} = \mu_{\text{cyc}} \circ \cdot$ where Δ_{cyc} is an extension of the Schwarz's BV-operator on polynomial functions on V to $C(V)$ and μ_{cyc} is the cyclic shuffle coproduct.

Let (V', Π', m'_1, m'_2) be another cyclic DGA of degree n of finite type, and let

$$\mathcal{P} \begin{array}{c} \longrightarrow \\ \circlearrowleft \end{array} (V, m_1) \xrightleftharpoons[\iota]{\pi} (V', m'_1) \quad (8.1)$$

be a deformation retraction. We have the following:

- IBL $_{\infty}$ -morphism $\mathfrak{f} : \text{dIBL}(C(V)) \rightarrow \text{dIBL}(C(V'))$ with components $\mathfrak{f}_{klg} : \hat{\mathbb{E}}_k C(V) \rightarrow \hat{\mathbb{E}}_l C(V')$ such that

$$\mathfrak{f}_{110} = \iota^* : (C(V), \hat{\mathfrak{q}}_{110}) \longrightarrow (C(V'), \hat{\mathfrak{q}}'_{110})$$

is a quasi-isomorphism. The number $\mathfrak{f}_{klg}(s^k \psi_1 \otimes \cdots \otimes \psi_k)(s^l \omega_1 \otimes \cdots \otimes \omega_l)$ for $\psi_1, \dots, \psi_k \in \hat{\mathbb{B}}_{\text{cyc}}^* V$ and $\omega_1 = \omega_{11} \cdots \omega_{1s_1}, \dots, \omega_l = \omega_{l1} \cdots \omega_{ls_l} \in \mathbb{B}^{\text{cyc}} V'$, where s is the formal symbol of degree $n-3$, is computed via summation over ribbon graphs with interior vertices decorated with ψ_1, \dots, ψ_k , interior edges decorated with the propagator — the Schwartz kernel \mathbb{P} of \mathcal{P} — and exterior vertices decorated with ω_{ij} at the i -th boundary component and with j respecting the orientation. We will refer to such graphs shortly as *Feynman graphs*.

- Pushforward Maurer-Cartan element $\mathfrak{n} := \mathfrak{f}_* \mathfrak{m}$ with components $\mathfrak{n}_{lg} \in \hat{\mathbb{E}}_l C(V)$. The number $\mathfrak{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \omega_l)$ is computed via summation over Feynman graphs as above with trivalent internal vertices \mathfrak{m}_{10} .
- Twisted IBL $_{\infty}$ -morphism $\mathfrak{f}^{\mathfrak{m}} : \text{dIBL}^{\mathfrak{m}}(C(V)) \rightarrow \text{dIBL}^{\mathfrak{n}}(C(V'))$ with components $\mathfrak{f}_{klg}^{\mathfrak{m}} : \hat{\mathbb{E}}_k C(V) \rightarrow \hat{\mathbb{E}}_l C(V')$ such that

$$\begin{aligned} \mathfrak{f}_{110}^{\mathfrak{m}} : (C(V), \hat{\mathfrak{q}}_{110}^{\mathfrak{m}}) &\longrightarrow (C(V'), \hat{\mathfrak{q}}_{110}^{\mathfrak{n}}) \\ &= \iota^* + \mathfrak{f}_{210} \circ_1 \mathfrak{m}_{10} + \frac{1}{2!} \mathfrak{f}_{310} \circ_{1,1} (\mathfrak{m}_{10}, \mathfrak{m}_{10}) + \cdots \end{aligned}$$

is a quasi-isomorphism. The number $\mathfrak{f}_{klg}(s^k \psi_1 \otimes \cdots \otimes \psi_k)(s^l \omega_1 \otimes \cdots \otimes \omega_l)$ is computed via summation over Feynmann graphs as above with interior vertices ψ_1, \dots, ψ_k

and \mathfrak{m}_{10} .

Recall that an IBL_∞ -quasi-isomorphism is automatically an IBL_∞ -homotopy equivalence, which has been proven in [CFL15, Theorem 1.2] via obstruction theory. Obstruction theory also gives the existence of a minimal model of any IBL_∞ -algebra, which is the content of [CFL15, Theorem 1.3]. What is not proven yet but what we suspect is true is the following:

- (1) For a different choice of the deformation retraction (8.1), the corresponding IBL_∞ -morphisms \mathfrak{f} and \mathfrak{f}' are IBL_∞ -homotopic.
- (2) If ι is in addition a morphism of Poincaré DGA's (perhaps simply-connected), then $\text{dIBL}^{\mathfrak{m}}(C(V))$ and $\text{dIBL}^{\mathfrak{m}'}(C(V'))$ are IBL_∞ -homotopy equivalent (perhaps even the Maurer-Cartan elements \mathfrak{m}' and $\mathfrak{f}_*\mathfrak{m}$ are gauge equivalent).

8.2. BV-formulation for dIBL-algebra on cyclic cochains

In the setting of Section 8.1, we define

$$B(V) := B^{\text{cyc}}V[3 - n] \quad \text{and} \quad \mathcal{F}(B(V)) := \hat{E}C(V)((\hbar)) = EC(V) \hat{\otimes} \mathbb{K}((\hbar)).$$

We call $\mathcal{F}(B(V))$ the *space of functions on $B(V)$* . This makes sense because $C(V) \subset (B^{\text{cyc}}V[2 - n])^{*\mathfrak{g}}$, $EC(V) = S(C(V)[1])$ and the symmetric algebra of the dual can be viewed as polynomial functions. In contrast to [DJP18], we do not have any canonical odd symplecticform on $B(V)$, and hence there is no Schwarz's BV-operator on $\mathcal{F}(B(V))$. However, we have the following BV-operators from [CFL15]:

$$\begin{aligned} \Delta_0 &:= \hat{\mathfrak{q}}_{120} + \hbar \hat{\mathfrak{q}}_{210}, \\ \Delta &:= \hat{\mathfrak{q}}_{110} + \Delta_0, \\ \Delta^{\mathfrak{m}} &:= \widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{10}} + \Delta. \end{aligned}$$

The first BV-operator is canonical for an odd symplectic vector space (V, Π) and we call it the *string BV-operator*. The second BV-operator is canonical for a cyclic cochain complex (V, Π, m_1) and the third for a cyclic DGA (V, Π, m_1, m_2) (in both cases, (V, Π) is an odd symplectic vector space).

For Δ , we consider the associated Gerstenhaber bracket $\{\cdot, \cdot\} : \mathcal{F}(B(V))^{\otimes 2} \rightarrow \mathcal{F}(B(V))$, which is for all $\Psi_1, \Psi_2 \in \mathcal{F}(B(V))$ given by

$$\{\Psi_1, \Psi_2\} := (-1)^{|\Psi_1|} (\Delta(\Psi_1 \Psi_2) - \Delta(\Psi_1) \Psi_2 - (-1)^{|\Psi_1|} \Psi_1 \Delta(\Psi_2)).$$

8. BV-formalism for IBL-infinity theory on cyclic cochains

It is easy to see that

$$\{\Psi_1, \Psi_2\} = (-1)^{1+(|\Psi_1|+1)(|\Psi_2|+1)} \{\Psi_2, \Psi_1\}.$$

If $\{\cdot, \cdot\}_0$ and $\{\cdot, \cdot\}^{\mathfrak{m}}$ are the Gerstenhaber brackets for Δ_0 and $\Delta^{\mathfrak{m}}$, respectively, then it holds

$$\{\cdot, \cdot\}_0 = \{\cdot, \cdot\}^{\mathfrak{m}} = \{\cdot, \cdot\}$$

because Δ , Δ_0 and $\Delta^{\mathfrak{m}}$ differ only by differential operators of order ≤ 1 .

Consider the cyclizations $m_1^+, m_2^+ : B^{\text{cyc}}V \rightarrow \mathbb{R}$ defined for all $v_1, v_2, v_3 \in V[1]$ by

$$m_1^+(v_1, v_2) := \Pi(m_1(v_1), v_2) \quad \text{and} \quad m_2^+(v_1, v_2, v_3) := \Pi(m_2(v_1, v_2), v_3).$$

They have degree $3 - n$ as maps, thus degree $n - 3$ as elements of B_{cyc}^*V with the cohomological grading, and thus $sm_1^+, sm_2^+ \in \mathcal{F}(B(V))$ have degree $2(n - 3)$. They are “linear functions” in the sense that $sm_1^+, sm_2^+ \in \hat{E}_1C(V)$. We define the *total action*

$$S := \underbrace{(-1)^{n-2}(sm_1^+)\hbar^{-1}}_{=: S_{\text{free}}} + \underbrace{(-1)^{n-2}(sm_2^+)\hbar^{-1}}_{=: S_{\text{int}}} \in \mathcal{F}(B(V)) \quad (8.2)$$

and call S_{free} the *free action* and S_{int} the *interaction*. The total action is linear and has degree 0. As a function, we can write

$$S(sb) = (-1)^{n-2}m_1^+(b)\hbar^{-1} + (-1)^{n-2}m_2^+(b)\hbar^{-1} \quad \text{for } b \in B^{\text{cyc}}V.$$

Recall that $\mathfrak{m}_{10} = (-1)^{n-2}sm_2^+$ is the canonical Maurer-Cartan element.

Proposition 8.2.1 (BV-action for canonical Maurer-Cartan element). *Let (V, Π, m_1, m_2) be a cyclic DGA of degree n of finite type, and let $\Delta_0 = \hat{\mathbf{q}}_{120} + \hbar\hat{\mathbf{q}}_{210}$ be the string BV-operator on $\mathcal{F}(B(V))$. The total action $S \in \mathcal{F}(B(V))$ from (8.2) satisfies the quantum master equation*

$$\Delta_0 S + \frac{1}{2}\{S, S\}_0 = 0, \quad (8.3)$$

and it holds

$$\Delta^{\mathfrak{m}}(f) = \Delta_0(f) + \{S, f\}_0 \quad \text{for all } f \in \mathcal{F}(C(V)), \quad (8.4)$$

where $\Delta^{\mathfrak{m}} = \hat{\mathbf{q}}_{110}^{\mathfrak{m}} + \hat{\mathbf{q}}_{120} + \hbar\hat{\mathbf{q}}_{210}$ is the BV-operator associated to the twisted dIBL-algebra $\text{dIBL}^{\mathfrak{m}}(C(V))$.

Proof. Let $f \in \mathcal{F}(B(V))$, and consider the left multiplication $L_f : \mathcal{F}(B(V)) \rightarrow \mathcal{F}(B(V))$

8. BV-formalism for IBL-infinity theory on cyclic cochains

given by $g \mapsto fg$. Using formulas from Appendix D, we compute

$$\begin{aligned}\hat{\mathbf{q}}_{210} \circ L_f &= \hat{\mathbf{q}}_{210}(f \odot \mathbb{1}) \\ &= \mathbf{q}_{210} \circ_{0,2} (f, \mathbb{1}) + \mathbf{q}_{210} \circ_{1,1} (f, \mathbb{1}) + \mathbf{q}_{210} \circ_{2,0} (f, \mathbb{1}) \\ &= (-1)^{|f|} L_f \circ \hat{\mathbf{q}}_{210} + \widehat{\mathbf{q}_{210} \circ_1 f} + L_{\hat{\mathbf{q}}_{210}(f)},\end{aligned}$$

where we view f as a linear map $\mathcal{F}(B(V)) \rightarrow \mathcal{F}(B(V))$ which maps the constant 1 to f and vanishes on the reduced part. Therefore, it holds

$$\begin{aligned}\{f, \cdot\}_0 &= \hbar(\hat{\mathbf{q}}_{210} \circ L_f - L_{\hat{\mathbf{q}}_{210}(f)} - (-1)^{|f|} L_f \circ \hat{\mathbf{q}}_{210}) \\ &= \hbar \widehat{\mathbf{q}_{210} \circ_1 f}.\end{aligned}$$

We compute

$$\begin{aligned}&(\mathbf{q}_{210} \circ_1 (sm_1^+))(s\psi)(\omega) \\ &= \mathbf{q}_{210}(sm_1^+, s\psi)(\omega) \\ &= (-1)^{|s|} \mathbf{q}_{210}(s^2 m_1^+, \psi) \\ &= (-1)^{|s|} \sum \varepsilon(\omega \rightarrow \omega^1 \omega^2) (-1)^{|e_j||\omega^1|} T^{ij} m_1^+(e_i \omega^1) \psi(e_j \omega^2) \\ &\quad \uparrow \\ &\quad T^{ij} = (-1)^{|s|+|e_i||e_j|} T^{ji} \\ &\quad T^{ji} = (-1)^{|e_j|} \Pi(e^j, e^i) \\ &\quad \Pi(e^j, e^i) \neq 0 \implies |e_j| + |e_i| = |s| + 1 \\ &= \sum \varepsilon(\omega \rightarrow \omega^1 \omega^2) (-1)^{|e_j|(|s|+1+|\omega^1|)+|e^j||\omega^1|} \Pi(m_1(\omega^1), e^j) \psi(e_j \omega^2) \\ &\quad \uparrow \\ &\quad \sum_i \Pi(v, e^i) e_i = v \\ &\quad m_1^+(e^j \omega^1) = (-1)^{|e^j||\omega^1|} m_1^+(\omega^1 e^j) \\ &= (-1)^{|s|+1} \sum \varepsilon(\omega^1 \omega^2) \psi(m_1(\omega^1) \omega^2) \\ &\quad \uparrow \\ &\quad \Pi(m_1(\omega^1), e^j) \neq 0 \implies 1 + |\omega^1| + |e^j| = |s| + 1 \\ &\quad |e^j| + |e_j| = |s| + 1 \\ &= (-1)^{|s|+1} \sum_{i=1}^k (-1)^{|\omega_1|+\dots+|\omega_{i-1}|} \psi(\omega_1 \dots m_1(\omega_i) \dots \omega_k) \\ &= (-1)^{|s|+1} \mathbf{q}_{110}(s\psi)(s\omega).\end{aligned}\tag{8.5}$$

Therefore, we have

$$\{S_{\text{free}}, \cdot\}_0 = \hat{\mathbf{q}}_{110}.\tag{8.6}$$

Since $m_1 \circ m_1 = 0$ and since m_1^+ is cyclically symmetric, it holds

$$\{S_{\text{free}}, S_{\text{free}}\}_0 = (-1)^{n-2} \hbar^{-1} \mathbf{q}_{110}(sm_1^+) = 0.$$

8. BV-formalism for IBL-infinity theory on cyclic cochains

Because $m_1^+ \in B_{\text{cyc}}^* V$ has weight 2, i.e., it vanishes on words of length 3 and more, \mathbf{q}_{120} decreases the weight by 2, and $B_{\text{cyc}}^* V$ is reduced, i.e., its weight-0 part is 0, we have

$$\Delta_0 S_{\text{free}} = (-1)^{n-2} \mathbf{q}_{120}(sm_1^+) = 0.$$

We now compute

$$\begin{aligned} \{S, S\}_0 &= \{S_{\text{free}}, S_{\text{free}}\}_0 + 2\{S_{\text{free}}, S_{\text{int}}\}_0 + \{S_{\text{int}}, S_{\text{int}}\}_0 \\ &= 2\{S_{\text{free}}, S_{\text{int}}\}_0 + \{S_{\text{int}}, S_{\text{int}}\}_0 \\ &= 2\hbar^{-1} \left(\mathbf{q}_{110}(\mathbf{m}_{10}) + \frac{1}{2} \mathbf{q}_{210}(\mathbf{m}_{10}, \mathbf{m}_{10}) \right) \end{aligned} \tag{8.7}$$

and

$$\Delta_0 S = \hbar^{-1} \mathbf{q}_{120}(\mathbf{m}_{10}). \tag{8.8}$$

The right-hand side of (8.7) vanishes due to [CFL15, Proposition 12.3], and the right-hand side of (8.8) vanishes by the discussion preceding [CFL15, Proposition 12.5] (for degree reasons). Therefore, S satisfies the quantum master equation (8.3). The twisting equation (8.4) is also clear. \square

The inclusion of \hbar^{-1} into S is the convention from [CFL15]. After the transformation $\Delta_0 \rightarrow \Delta'_0 = \Delta_0$, $S \rightarrow S' = \hbar S$, we get the well-known equations

$$\hbar \Delta'_0 S' + \frac{1}{2} \{S', S'\}'_0 = 0 \quad \text{and} \quad \hbar \Delta'^m = \hbar \Delta'_0 + \{S', \cdot\}'_0.$$

Proposition 8.2.2 (BV-action for general Maurer-Cartan element). *Let (V, Π, m_1, m_2) be a cyclic DGA of degree n of finite type, and let $\Delta_0 = \hat{\mathbf{q}}_{120} + \hbar \hat{\mathbf{q}}_{210}$ be the string BV-operator on $\mathcal{F}(B(V))$. Let $\mathbf{m}_{lg} \in \hat{\mathbf{E}}_l C(V)$ for all $l, g \geq 0$ satisfy the degree and filtration degree conditions of a Maurer-Cartan element for an IBL_∞ -algebra of bidegree $(n-3, 2)$. Then $\mathbf{m} = (\mathbf{m}_{lg})$ satisfies the Maurer-Cartan equation for $\text{dIBL}(C(V))$ if and only if the total action*

$$S := S_{\text{free}} + \sum_{l, g \geq 0} \mathbf{m}_{lg} \hbar^{g-1} \in \mathcal{F}(B(V)) \tag{8.9}$$

satisfies the quantum master equation (8.3). The BV_∞ -operator Δ^m of the twisted IBL_∞ -algebra $\text{dIBL}^m(C(V))$ is given by (8.4). The elements \mathbf{m}_{0g} appear neither in (8.3) nor in (8.4).

8. BV-formalism for IBL-infinity theory on cyclic cochains

Proof. Using $\Delta_0 S_{\text{free}} = \{S_{\text{free}}, S_{\text{free}}\}_0 = 0$, we compute

$$\begin{aligned}
\Delta_0 S + \frac{1}{2}\{S, S\}_0 &= \Delta_0 S_{\text{int}} + \{S_{\text{free}}, S_{\text{int}}\}_0 + \frac{1}{2}\{S_{\text{int}}, S_{\text{int}}\}_0 \\
&= \sum_{l,g \geq 0} (\hat{q}_{120}(\mathbf{m}_{lg})\hbar^{g-1} + \hat{q}_{210}(\mathbf{m}_{lg})\hbar^g + \hat{q}_{110}(\mathbf{m}_{lg})\hbar^{g-1}) \\
&\quad + \frac{1}{2} \sum_{l_1, l_2, g_1, g_2 \geq 0} \widehat{q_{210} \circ_1 \mathbf{m}_{l_1 g_1}(\mathbf{m}_{l_2 g_2})} \hbar^{g_1 + g_2 - 1} \\
&= \sum_{l,g \geq 0} (\iota_l \pi_l) \left(q_{120} \circ_1 \mathbf{m}_{l-1,g} + q_{210} \circ_2 \mathbf{m}_{l+1,g-1} + q_{110} \circ_1 \mathbf{m}_{lg} \right. \\
&\quad \left. + \frac{1}{2} \sum_{\substack{l_1, l_2, g_1, g_2 \geq 0 \\ l_1 + l_2 - 1 = l \\ g_1 + g_2 = g}} q_{210} \circ_{1,1}(\mathbf{m}_{l_1 g_1}, \mathbf{m}_{l_2 g_2}) \right) \hbar^{g-1}.
\end{aligned}$$

Comparing to Proposition 3.2.11 in Part I, we see that the (l, g) -components for $l \geq 1$, $g \geq 0$ give precisely the Maurer-Cartan equation in $\text{dIBL}(C(V))$. We compute

$$\begin{aligned}
\Delta_0 + \{S, \cdot\}_0 &= \hat{q}_{120} + \hbar \hat{q}_{210} + \hat{q}_{110} + \sum_{l,g \geq 0} \widehat{q_{210} \circ_1 \mathbf{m}_{lg}} \hbar^g \\
&= (\hat{q}_{110} + \widehat{q_{210} \circ_1 \mathbf{m}_{10}}) + (\hat{q}_{120} + \widehat{q_{210} \circ_1 \mathbf{m}_{20}}) + (\widehat{q_{210} \circ_1 \mathbf{m}_{30}}) + \cdots \\
&\quad + [(\hat{q}_{210}) + (\widehat{q_{210} \circ_1 \mathbf{m}_{11}}) + (\widehat{q_{210} \circ_1 \mathbf{m}_{21}}) + \cdots] \hbar \\
&\quad + \sum_{l \geq 0, g \geq 2} (\widehat{q_{210} \circ_1 \mathbf{m}_{lg}}) \hbar^g \\
&= (\hat{q}_{110}^{\mathbf{m}}) + (\hat{q}_{120}^{\mathbf{m}}) + (\hat{q}_{130}^{\mathbf{m}}) + \cdots \\
&\quad + [(\hat{q}_{210}^{\mathbf{m}}) + (\hat{q}_{111}^{\mathbf{m}}) + (\hat{q}_{121}^{\mathbf{m}}) + \cdots] \hbar \\
&\quad + \sum_{l \geq 0, g \geq 2} (\hat{q}_{1lg}^{\mathbf{m}}) \hbar^g
\end{aligned}$$

and see that this is indeed the BV_∞ -operator $\Delta^{\mathbf{m}}$ of $\text{dIBL}^{\mathbf{m}}(C(V))$. □

8.3. Homotopy transfer and effective action

In [DJP18], they consider multilinear operations $l_{lg} : V^{\otimes l} \rightarrow \mathbb{R}$ for $l \geq 1$, $g \geq 0$ on an odd symplectic vector space V and write down an action $S \in \mathcal{F}(V)$ in the form of (8.9) with \mathbf{m}_{lg} replaced by l_{lg}^+ . We remark that their “vertices” are $l_{lg}^+(v, \dots, v)$ for a “field” $v \in V$, whereas ours are $\mathbf{m}_{lg}(v_1, \dots, v_l)$ for a “string of fields” $v_1 \cdots v_l \in B(V)$. They consider the Schwarz’s canonical BV-operator on $\mathcal{F}(V)$ from [Sch92] and show that S satisfies the quantum master equation if and only if (l_{lg}) satisfy the relations of a

8. BV-formalism for IBL-infinity theory on cyclic cochains

quantum L_∞ -algebra.¹ On the other hand, we have the string BV-operator on $\mathcal{F}(B(V))$ and our action (8.9) satisfies the quantum master equation if and only if $\mathfrak{m} = (\mathfrak{m}_{lg})$ is a Maurer-Cartan element for $\mathrm{dIBL}(C(V))$. Next, they consider a deformation retract (\equiv : DR) as in (8.1) and obtain explicit formulas for the homotopy transferred quantum L_∞ -algebra on V' together with all maps and homotopies via the Homological Perturbation Lemma (\equiv : HPL). In what follows, we will sketch how to apply their construction to IBL_∞ -algebras. We stress that the details have NOT been done yet!

A DR (8.1) induces a DR

$$K_C \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} (C(V), \mathfrak{q}_{110}) \xrightleftharpoons[I_C]{P_C} (C(V'), \mathfrak{q}'_{110}),$$

which further induces a DR

$$K \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} (\mathcal{F}(B(V)), \hat{\mathfrak{q}}_{110}) \xrightleftharpoons[I]{P} (\mathcal{F}(B(V')), \hat{\mathfrak{q}}'_{110}). \quad (8.10)$$

In [DJP18, Remark 3], they write down a formula for K on $\mathcal{F}(V)$ given \mathcal{P} using a “tensor trick” due to Eilenberg Mac-Lane; the same method may apply to get K_C from \mathcal{P} on $C(V)$ and K on $\mathcal{F}(B(V))$ from K_C in our case. With such K ’s, one can take P_C and P , resp. I_C and I to be the natural extensions of ι^* , resp. π^* . Note that any surjective quasi-isomorphism over \mathbb{R} is a deformation retraction, but their formula is explicit and preserves special DR’s, i.e., DR’s satisfying $\mathcal{P}^2 = \mathcal{P}\iota = \pi\mathcal{P} = 0$ (\equiv : SDR). The crucial idea of [DJP18] translated to our situation is to view Δ and $\Delta^{\mathfrak{m}}$ as perturbations of $\{S_{\text{free}}, \cdot\} = \hat{\mathfrak{q}}_{110}$, which we denote by $\delta^{(1)}$ and $\delta^{(2)}$, respectively, and apply the HPL from [Cra04]:

For $i = 1, 2$, suppose that $\delta^{(i)}$ is “small”, i.e., that $(\mathbb{1} - \delta^{(i)}K)$ is invertible, and consider the maps

$$\begin{aligned} \Delta^{(i)} &:= \hat{\mathfrak{q}}'_{110} + P(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}I = \hat{\mathfrak{q}}'_{110} + P\delta^{(i)}I + P\delta^{(i)}K\delta^{(i)}I + \dots, \\ I^{(i)} &:= I + K(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}I = I + K\delta^{(i)}I + K\delta^{(i)}K\delta^{(i)}I + \dots, \\ P^{(i)} &:= P + P(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}K = P + P\delta^{(i)}K + P\delta^{(i)}K\delta^{(i)}K + \dots, \\ K^{(i)} &:= K + K(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}K = K + K\delta^{(i)}K + K\delta^{(i)}K\delta^{(i)}K + \dots. \end{aligned}$$

¹This is equivalent to the notion of a loop homotopy algebra from [Mar97] and to string brackets in closed string field theory from [Zwi92].

8. BV-formalism for IBL-infinity theory on cyclic cochains

The HPL asserts that if (8.10) is an SDR, then the following are SDR's as well:

$$\begin{array}{ccc}
 K \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & (\mathcal{F}(B(V)), \hat{\mathbf{q}}_{110}) \xrightleftharpoons[I]{P} & (\mathcal{F}(B(V')), \hat{\mathbf{q}}'_{110}) \quad \delta^{(1)} = \Delta_0 \\
 K^{(1)} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & (\mathcal{F}(B(V)), \Delta) \xrightleftharpoons[I^{(1)}]{P^{(1)}} & (\mathcal{F}(B(V')), \Delta^{(1)}) \\
 K^{(2)} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & (\mathcal{F}(B(V)), \Delta^{\mathfrak{m}}) \xrightleftharpoons[I^{(2)}]{P^{(2)}} & (\mathcal{F}(B(V')), \Delta^{(2)}) \quad \delta^{(2)} = \Delta_0 + \{S_{\text{int}}, \cdot\}
 \end{array}$$

$\delta^{(1)} = \Delta_0$
 $\delta^{(2)} = \Delta_0 + \{S_{\text{int}}, \cdot\}$

One defines the *effective action*

$$W := \log(P^{(1)}(e^{S_{\text{int}}})) \in \mathcal{F}(B(V'))$$

and the *path integral*

$$Z := L_{e^{-W}} \circ P^{(1)} \circ L_{e^{S_{\text{int}}}} : \mathcal{F}(B(V)) \rightarrow \mathcal{F}(B(V')).$$

Under certain circumstances, the following formulas hold:

$$\Delta^{(1)} = P \Delta_0 I, \quad \Delta^{(2)} = \Delta^{(1)} + \{W, \cdot\}^{(1)} \quad \text{and} \quad P^{(2)} = Z. \quad (8.11)$$

This is proven in [DJP18] on $\mathcal{F}(V)$ when $V' = \mathcal{H}$ is a “harmonic” subspace in a Hodge decomposition $V = \mathcal{H} \oplus C$ into odd symplectic subspaces, π and ι are the canonical projection and inclusion, respectively, the homotopy \mathcal{P} is such that $(\pi, \iota, \mathcal{P})$ is an SDR, and the homotopy K was constructed from \mathcal{P} via the tensor trick. Since we deal with the string BV-operator on $\mathcal{F}(B(V))$, we can not talk about this “symplectic compatibility” and the proof of (8.11) might be based on another arguments.

Question 8.3.1 (HPL and IBL_∞). *If one picks an SDR of V onto a harmonic subspace \mathcal{H} as above (basically equivalent to the setting of [CFL15, Section 11]) and constructs K_C and K in a particular way, can one achieve that (8.11) and the following identities hold?*

$$\Delta^{(1)} = \Delta'_0, \quad \Delta^{(2)} = \Delta^{\mathfrak{m}}, \quad W = S_{\mathfrak{f}_* \mathfrak{m}}, \quad P^{(1)} = e^{\mathfrak{f}}, \quad P^{(2)} = e^{\mathfrak{f}^{\mathfrak{m}}}.$$

Here, $S_{\mathfrak{f}_* \mathfrak{m}}$ denotes the action (8.9) for the Maurer-Cartan element $\mathfrak{f}_* \mathfrak{m}$.

Remark 8.3.2 (On BV-formalism for IBL_∞). (i) As summarized in [DJP18, Section 5],

8. BV-formalism for IBL-infinity theory on cyclic cochains

given a BV-action S , there are various approaches to obtain W and Z as summations over Feynman graphs (see [Mne17] for the stationary phase formula approach).

(ii) The appearance of Feynman graphs can be explained from the proof of [DJP18, Theorem 2], where they show that in the special setting above, it holds

$$P^{(1)} = P e^{D_P}$$

for an order ≤ 2 differential operator D_P which “connects” two legs with the propagator P (obtained by “rising one index” of \mathcal{P} using the odd symplectic form). This is reminiscent of the Wick’s Theorem for the (formal) perturbative expansion of the path integral (the classical approach to quantum field theories).

(iii) Having a BV-formulation of the IBL_∞ -theory, it is intriguing to compare it to [MS11], where certain IBL_∞ -structures are considered in the context of open-closed string field theory. \triangleleft

Part III.

Appendices

A. Evaluation of labeled ribbon graphs

In this appendix, we define the algebraic propagator P and the graph pairing $\langle \cdot, \cdot \rangle_\Gamma^P$ (Definition A.1.1), which encapsulates the contribution of a ribbon graph Γ to the map $f_{klg} : (B_{\text{cyc}}^* V)^{\otimes k} \rightarrow (B_{\text{cyc}}^* V)^{\otimes l}$ defined as a sum of contributions of ribbon graphs (Proposition A.1.2). Such maps were already defined in [CFL15, Section 11] using coordinates; here we use an invariant framework inspired by [Mne17]. As an example, we work out in details expressions for the canonical dIBL-operations \mathfrak{q}_{210} and \mathfrak{q}_{120} (Example A.1.5). We also explain the technicality of identifying symmetric maps with maps on symmetric powers (Remark A.1.3).

Next, we define the notion of an algebraic Schwartz kernel (Definition A.1.4) and show that the matrix (T^{ij}) from Definition 3.4.1 in Part I corresponds to the Schwartz kernel of the identity $\mathbb{1}$ up to a sign. Assuming that the Hodge propagator P from Definition 4.2.4 in Part I is algebraic, we deduce the signs in Definition 4.3.6 in Part I using the formula from [CFL15, Remark 12.10] for the genuine pushforward Maurer-Cartan element \mathfrak{n} in the finite-dimensional case. Establishing the formal analogy between the de Rham case and the finite-dimensional case is our main application of the invariant framework. Finally, we sketch how to obtain signs for the Fréchet dIBL-structure on $\Omega(M)$ (Remark A.2.2).

Throughout this appendix, we will use Notation 3.3.3 in Part I without further remarks.

A.1. Finite dimensional case

Definition A.1.1 (Propagator & graph pairing). *Let V be a graded vector space. The tensor $P \in V[1]^{\otimes 2}$ is called an (algebraic) propagator if it satisfies the following symmetry condition:*

$$\tau(P) = (-1)^{|P|} P. \quad (\text{A.1})$$

The map τ is the twist map defined by $\tau(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1$ for all $v_1, v_2 \in V[1]$.

For a ribbon graph $\Gamma \in \overline{\text{RG}}_{klg}$ and its labeling L , consider the permutation σ_L from Definition 4.3.5 in Part I. It acts on tensor powers according to Definition 3.1.6 in Part I

A. Evaluation of labeled ribbon graphs

and thus defines the map

$$\sigma_L : (V[1]^{\otimes 2})^{\otimes e} \otimes V[1]^{\otimes s_1} \otimes \cdots \otimes V[1]^{\otimes s_l} \longrightarrow V[1]^{\otimes d_1} \otimes \cdots \otimes V[1]^{\otimes d_k},$$

where d_i and s_i are the valencies of internal vertices $1, \dots, k$ and boundary components $1, \dots, l$, respectively, and e is the number of internal edges. We extend σ_L by 0 to other combinations of tensor powers. The graph pairing

$$\langle \cdot, \cdot \rangle_{\Gamma}^P : (B_{\text{cyc}}^* V)^{\otimes k} \otimes (B^{\text{cyc}} V)^{\otimes l} \longrightarrow \mathbb{R}$$

is defined for all $\psi_1, \dots, \psi_k \in B_{\text{cyc}}^* V$ and generating words $w_i = v_{i1} \dots v_{im_i}$ with $v_{ij} \in V[1]$ for $m_i \in \mathbb{N}$ and $i = 1, \dots, l$ by the following formula:

$$\begin{aligned} & \langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^P \\ &:= \sum_{L_1, L_3^b} (\psi_1 \otimes \cdots \otimes \psi_k) (\sigma_L(P^{\otimes e} \otimes (v_{11} \otimes \cdots \otimes v_{1m_1}) \otimes \cdots \\ & \quad \otimes (v_{l1} \otimes \cdots \otimes v_{lm_l}))), \end{aligned}$$

where we use the pairing from Definition 3.3.4 in Part I and in every summand an L_2 compatible with L_1 and an L_3^v are chosen arbitrarily to get a full labeling L of Γ . The graph pairing extends to $\langle \cdot, \cdot \rangle_{\Gamma}^P : \bar{\text{TB}}_{\text{cyc}}^* V \otimes \bar{\text{TB}}^{\text{cyc}} V \rightarrow \mathbb{R}$.

Proposition A.1.2. *In the setting of Definition A.1.1, we denote $w = w_1 \otimes \cdots \otimes w_l$ and $\psi = \psi_1 \otimes \cdots \otimes \psi_k$ and have the following:*

- (a) *The number $\psi(\sigma_L(P^{\otimes e} \otimes w))$ does not depend on the choice of L_3^v and an L_2 compatible with L_1 . Moreover, $\langle \cdot, \cdot \rangle_{\Gamma}^P$ does not depend on the representative of $[\Gamma] \in \overline{\text{RG}}_{klg}$.*
- (b) *If V is finite-dimensional, then for every $k, l \geq 1, g \geq 0$ there is a unique linear map*

$$f_{klg} : (B_{\text{cyc}}^* V)^{\otimes k} \rightarrow (B_{\text{cyc}}^* V)^{\otimes l}$$

such that

$$\begin{aligned} & f_{klg}(\psi_1 \otimes \cdots \otimes \psi_k)(w_1 \otimes \cdots \otimes w_l) \\ &= \frac{1}{l!} \sum_{[\Gamma] \in \overline{\text{RG}}_{klg}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^P. \end{aligned}$$

- (c) *The following holds for the map f_{klg} from b):*

A. Evaluation of labeled ribbon graphs

- It is homogenous of degree

$$|f_{klg}| = -|P|(k + l - 2 + 2g). \quad (\text{A.2})$$

- The filtration degree satisfies

$$\|f_{klg}\| \geq -2(k + l - 2 + 2g). \quad (\text{A.3})$$

- For all $\eta \in \mathbb{S}_l$ and $\mu \in \mathbb{S}_k$, we have

$$\eta \circ f_{klg} \circ \mu = (-1)^{|P|(\eta+\mu)} f_{klg}. \quad (\text{A.4})$$

Proof. (a) Let us denote by \bar{i} and ij the operations on L_2 given by $e_i \mapsto -e_i$ and $e_i \leftrightarrow e_j$, respectively. An even number of these operations does not change the orientation of the complex (4.17) in Part I. Their effect in σ_L acting on $P^{\otimes e} \otimes w$ is

$$\bar{i} : P_i \mapsto \tau(P_i) = (-1)^{|P|} P_i \quad \text{and} \quad ij : P_i \dots P_j \mapsto (-1)^{|P|} P_j \dots P_i.$$

Therefore, an even number of them does not change $\sigma_L(P^{\otimes e} \otimes w)$. This proves the independence of the choice of a compatible L_2 . The independence of the choice of L_3^v is clear since ψ_i are cyclic symmetric.

An isomorphism of ribbon graphs $\eta : \Gamma \rightarrow \Gamma'$ induces the map of compatible labelings $L \mapsto L' = \eta_* L$ such that $\sigma_L = \sigma_{L'}$. The independence of the choice of a representative of $[\Gamma]$ follows.

(b) Suppose that $\psi = \psi_1 \otimes \dots \otimes \psi_k$ with $\psi_i \in (\mathbf{B}_{\text{cyc}}^* V)_{r_i}^{c_i}$, where $r_i \in \mathbb{N}$ and $c_i \in \mathbb{Z}$ for $i = 1, \dots, k$. A general element of $(\mathbf{B}_{\text{cyc}}^* V)^{\otimes k}$ is then a finite linear combination of such ψ 's.

First of all, let us argue that the sum $\sum_{\overline{\text{RG}}_{klg}}$ is finite. The number of internal edges e is fixed from the Euler formula (4.16) in Part I. Therefore, the number of contributing graphs $(V_{\text{int}}, E_{\text{int}})$ is finite. In order to bound the number of external vertices, we notice that $d_1 = r_1, \dots, d_k = r_k$ must hold for $\psi(\sigma_L(P^{\otimes e} \otimes w))$ to be non-zero. Therefore, the sum is finite.

We now have the linear functional

$$f_{klg}(\psi) := \frac{1}{l!} \sum_{[\Gamma] \in \overline{\text{RG}}_{klg}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi \mid \cdot \rangle_{\Gamma}^P : (\mathbf{B}^{\text{cyc}} V)^{\otimes l} \rightarrow \mathbb{R}$$

A. Evaluation of labeled ribbon graphs

and need to show that $f_{klg}(\psi) \in (B_{\text{cyc}}^* V)^{\otimes l} \subset (B^{\text{cyc}} V)^{\otimes l*}$. Because V has finite dimension, the weight-filtration of $B^{\text{cyc}} V$ satisfies (WG1) & (WG2) (see (3.11) and Proposition 3.3.6 in Part I), and hence we have

$$(B_{\text{cyc}}^* V)^{\otimes l} = (B^{\text{cyc}} V)^{*_{\text{wg}} \otimes l} = ((B^{\text{cyc}} V)^{\otimes l})^{*_{\text{wg}}}$$

for the weight-graded duals. Therefore, it suffices to show that $f_{klg}(\psi)$ vanishes on all but finitely many degrees and weights of $(B^{\text{cyc}} V)^{\otimes k}$. However, the relation $f_{klg}(\psi)(w) \neq 0$ for a generating word $w \in (B^{\text{cyc}} V)^{\otimes k}$ implies

$$\begin{aligned} |w| &= |\psi| - e|P| \quad \text{and} \\ k(w) &= k(\psi) - 2e, \end{aligned} \tag{A.5}$$

where k denotes the weight, and hence $f_{klg}(\psi) \in (B_{\text{cyc}}^* V)^{\otimes l}$ indeed holds.

(c) The formulas (A.2) and (A.3) follow from (A.5) and from (4.16) in Part I.

As for the symmetry (A.4), suppose that L and L' are compatible labelings of the same graph Γ such that L'_1 differs from L_1 by a permutation $\mu \in \mathbb{S}_k$ of internal vertices and a permutation $\eta \in \mathbb{S}_l$ of boundary components. Viewing μ and η as block permutations in the vertex and edge order, respectively, we get

$$\sigma_{L'}(P^{\otimes e} \otimes w) = (-1)^{|P|(\eta+\mu)} \mu(\sigma_L(P^{\otimes e} \otimes \eta(w))).$$

The sign comes from the difference of L_2 and L'_2 which compensates the change of the orientation of (4.17) in Part I caused by μ and η . \square

Given $\mu \in \mathbb{S}_k$ and $\psi = \psi_1 \otimes \cdots \otimes \psi_k \in (B_{\text{cyc}}^* V)^{\otimes k}$, it is easy to see that

$$\varepsilon(\mu, \Psi) = \varepsilon(\mu(s), \mu(\psi)) \varepsilon(\mu, s) \varepsilon(s, \psi) \varepsilon(\mu, \psi),$$

where $\Psi = (s\psi_1) \otimes \cdots \otimes (s\psi_k) \in (B_{\text{cyc}}^* V[A])^{\otimes k}$ and $\varepsilon(\mu, s) = (-1)^{|s|\mu}$. If $A = -|P|$, then we get from (A.4) that the degree shift $\mathfrak{f}_{klg} : (B_{\text{cyc}}^* V[A])^{\otimes k} \rightarrow (B_{\text{cyc}}^* V[A])^{\otimes l}$ has the following symmetries:

$$\forall \mu \in \mathbb{S}_k, \eta \in \mathbb{S}_l : \quad \eta \circ \mathfrak{f}_{klg} \circ \mu = \mathfrak{f}_{klg}. \tag{A.6}$$

Note that the degrees satisfy

$$|\mathfrak{f}_{klg}| = |f_{klg}| + (k - l)A. \tag{A.7}$$

A. Evaluation of labeled ribbon graphs

Remark A.1.3 (Symmetric maps versus maps on symmetric powers). In the situation above, we define \tilde{f}_{klg} as the unique map such that the solid lines of the following diagram commute:

$$\begin{array}{ccc} (B_{\text{cyc}}^* V[A])^{\otimes k} & \xrightarrow{f_{klg}} & (B_{\text{cyc}}^* V[A])^{\otimes l} \\ \downarrow \scriptstyle \begin{array}{c} \swarrow \iota \\ \pi \end{array} & & \downarrow \scriptstyle \begin{array}{c} \pi \\ \swarrow \iota \end{array} \\ S_k B_{\text{cyc}}^* V[A] & \xrightarrow{\tilde{f}_{klg}} & S_l B_{\text{cyc}}^* V[A]. \end{array}$$

The symmetry condition (A.6) provides the existence of \tilde{f}_{klg} and implies commutativity of the dotted diagram as well. Moreover, for all $\psi_1, \dots, \psi_k \in B_{\text{cyc}}^* V$ and $w_1, \dots, w_l \in B^{\text{cyc}} V$, we have

$$\tilde{f}_{klg}(s^k \psi_1 \cdots \psi_k)(s^l w_1 \cdots w_l) = f_{klg}(s^k \psi_1 \otimes \cdots \otimes \psi_k)(s^l w_1 \otimes \cdots \otimes w_l),$$

where we use the pairing from Definition 3.3.4 in Part I. We denote \tilde{f}_{klg} again by f_{klg} . \triangleleft

Definition A.1.4 (Algebraic Schwartz kernel). *Let V be a graded vector space and $\Pi : V \otimes V \rightarrow \mathbb{R}$ a non-degenerate pairing on V . We extend Π to a non-degenerate pairing $\Pi : V^{\otimes k} \otimes V^{\otimes l} \rightarrow \mathbb{R}$ for $k \geq 1$ by setting*

$$\Pi(v_{11} \otimes \cdots \otimes v_{1k}, v_{21} \otimes \cdots \otimes v_{2l}) := \varepsilon(v_1, v_2) \Pi(v_{11}, v_{21}) \cdots \Pi(v_{1k}, v_{2k})$$

for all $v_{11}, \dots, v_{1k}, v_{21}, \dots, v_{2l} \in V$, where ε is the Koszul sign (see Definition 3.1.2 in Part I). For $k = 0$, we let $\Pi : \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R}$ be the multiplication on \mathbb{R} .

For $k, l \geq 0$, we say that $\mathcal{K}_L \in V^{\otimes k+l}$ is the algebraic Schwartz kernel of a linear operator $L : V^{\otimes k} \rightarrow V^{\otimes l}$ if the following is satisfied:

$$\forall w_1 \in V^{\otimes k}, w_2 \in V^{\otimes l} : \quad \Pi(L(w_1), w_2) = \Pi(\mathcal{K}_L, w_1 \otimes w_2). \quad (\text{A.8})$$

We usually omit writing “algebraic” if it is clear from the context (i.e., if we do not consider any “extensions” of $V^{\otimes k}$).

In the situation of Definition A.1.4, let $(e_i) \subset V$ be a basis and (e^i) its dual basis such that $\Pi(e_i, e^j) = \delta_{ij}$. We define the coordinates $K_L^{ij} \in \mathbb{R}$ and $L^{ij} \in \mathbb{R}$ by

$$\mathcal{K}_L = \sum_{i,j} K_L^{ij} e_i \otimes e_j \quad \text{and} \quad L^{ij} := \Pi(L(e^i), e^j).$$

From (A.8) we have

$$\mathcal{K}_L^{ij} = (-1)^{(|L|+1)(|\Pi|+|e_i|)} L^{ij} \quad \text{for all } i, j. \quad (\text{A.9})$$

A. Evaluation of labeled ribbon graphs

From now on, we will be in the situation of (A) and (B) in the Introduction; in particular, we put $V[1]$ in place of V in Definition A.1.4. Let $\mathcal{K}_1 \in V[1]^{\otimes 2}$ be the Schwartz kernel of the identity $\mathbb{1} : V[1] \rightarrow V[1]$ and $\mathcal{K}_{\mathcal{P}} \in V[1]^{\otimes 2}$ the Schwartz kernel of the cochain homotopy $\mathcal{P} : V[1] \rightarrow V[1]$. From (A.9), we get

$$\mathcal{K}_{\mathcal{P}}^{ij} = \mathcal{P}^{ij} \quad \text{and} \quad \mathcal{K}_1^{ij} = (-1)^{|e_i| + |\Pi|} \Pi(e^i, e^j) \quad \text{for all } i, j.$$

We see that the tensor $T = \sum_{i,j} T^{ij} e_i \otimes e_j$ from (3.32) in Part I can be expressed as

$$T = (-1)^{n-2} \mathcal{K}_1.$$

This is the invariant meaning of T . Note that the degrees satisfy

$$|T| = n - 2 \quad \text{and} \quad |\mathcal{K}_{\mathcal{P}}| = n - 3.$$

The assumption (2.1) in Part I on \mathcal{P} is equivalent to graded antisymmetry of the bilinear form $\mathcal{P}^+ := \Pi \circ (\mathcal{P} \otimes \mathbb{1}) : V[1]^{\otimes 2} \rightarrow \mathbb{R}$. This is further equivalent to

$$\tau(\mathcal{K}_{\mathcal{P}}) = (-1)^{|\mathcal{K}_{\mathcal{P}}|} \mathcal{K}_{\mathcal{P}}.$$

Therefore, $\mathcal{K}_{\mathcal{P}}$ satisfies (A.1), and hence it can be used as a propagator for the construction of f_{klg} for every $k, l \geq 1, g \geq 0$. We have from (A.4) that the degree shift $\mathfrak{f}_{klg} : (B_{\text{cyc}}^* V[3 - n])^{\otimes k} \rightarrow (B_{\text{cyc}}^* V[3 - n])^{\otimes l}$ is symmetric. Moreover, using (A.2), (A.3) and (A.7), we obtain

$$\begin{aligned} |\mathfrak{f}_{klg}| &= -2d(k + g - 1), \\ \|\mathfrak{f}_{klg}\| &\geq \gamma(2 - 2g - k - l), \end{aligned}$$

where $(d, \gamma) = (n - 3, 2)$. These are the degree and filtration conditions on an IBL_{∞} -morphism from [CFL15, Definition 2.8 and (8.3)]. In fact, our $\mathfrak{f} = (\mathfrak{f}_{klg})_{k,l \geq 1, g \geq 0}$ is precisely the IBL_{∞} -homotopy from [CFL15, Theorem 11.3].

Graded antisymmetry of Π is equivalent to

$$\tau(T) = (-1)^{|T|+1} T.$$

Visibly, T does not satisfy (A.1). Nevertheless, we can still use it to define f_{210} and f_{120} since the corresponding graphs Γ (see Figure A.1) have only one internal edge e , and, for a given L_1 , there is a unique compatible L_2 determined by the orientation of e (see Example A.1.5 for the compatibility condition). As for the symmetry of the resulting maps,

A. Evaluation of labeled ribbon graphs

a transposition of internal vertices or boundary components in (4.17) in Part I can be compensated only by $e \mapsto -e$, which produces $(-1)^{|T|+1}$ (c.f. the proof of Proposition A.1.2 (a)). Therefore, if we shift the degrees by $A = -|T| + 1 = n - 3$, we obtain symmetric maps $\mathfrak{q}_{210} : (B_{\text{cyc}}^* V[A])^{\otimes 2} \rightarrow B_{\text{cyc}}^* V[A]$ and $\mathfrak{q}_{120} : B_{\text{cyc}}^* V[A] \rightarrow (B_{\text{cyc}}^* V[A])^{\otimes 2}$. We show in Example A.1.5 below that these operations agree with those defined in Definition 3.4.1 in Part I.

Example A.1.5 (The canonical dIBL-operations). We have

$$\begin{aligned} f_{210}(\psi_1 \otimes \psi_2)(w) &= \frac{1}{1!} \sum_{[\Gamma] \in \overline{\text{RG}}_{210}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi_1 \otimes \psi_2 \mid w \rangle_{\Gamma}^P \quad \text{and} \\ f_{120}(\psi)(w_1 \otimes w_2) &= \frac{1}{2!} \sum_{[\Gamma] \in \overline{\text{RG}}_{120}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi \mid w_1 \otimes w_2 \rangle_{\Gamma}^P. \end{aligned} \tag{A.10}$$

We parametrize RG_{210} by the ribbon graphs Γ_{k_1, k_2} with $1 \leq k_1 \leq k_2$ and RG_{120} by the ribbon graphs Γ^{s_1, s_2} with $0 \leq s_1 \leq s_2$; these graphs are depicted in Figure A.1. We have $\overline{\text{RG}}_{210} = \text{RG}_{210} \setminus \{[\Gamma_{1,1}]\}$ and $\overline{\text{RG}}_{120} = \text{RG}_{120} \setminus \{[\Gamma^{0,0}], [\Gamma^{0,1}]\}$. We also have

$$|\text{Aut}(\Gamma_{k_1, k_2})| = \begin{cases} 1 & \text{if } k_1 \neq k_2, \\ 2 & \text{if } k_1 = k_2, \end{cases}$$

and likewise for Γ^{s_1, s_2} . We fix labelings L_3^v and parametrize L_3^b by $c = 1, \dots, k_1 + k_2 - 2$ for Γ_{k_1, k_2} and by $c_1 = 1, \dots, s_1$ and $c_2 = 1, \dots, s_2$ for Γ^{s_1, s_2} as it is indicated in Figure A.1.

There are two possible labelings L_1^v for Γ_{k_1, k_2} and two possible labelings L_1^b for Γ^{s_1, s_2} ; this is the only freedom in choosing a full labeling L because L_3 is fixed and L_2 is just the orientation of the single internal edge, which is uniquely determined by L_1 . For both Γ_{k_1, k_2} and Γ^{s_1, s_2} , we will denote the two possible full labelings by L^1 and L^2 . They can be depicted as follows:

	Γ_{k_1, k_2}	Γ^{s_1, s_2}
L^1	$1 \rightarrow 2$	$2 \leftarrow 1$
L^2	$\begin{array}{c} \circlearrowleft 1 \end{array} \quad 2$	$\begin{array}{c} \circlearrowright 2 \end{array} \quad 1$

(A.11)

Let us check that the indicated L_1 and L_2 are compatible. For the complexes $C_2 \rightarrow$

A. Evaluation of labeled ribbon graphs

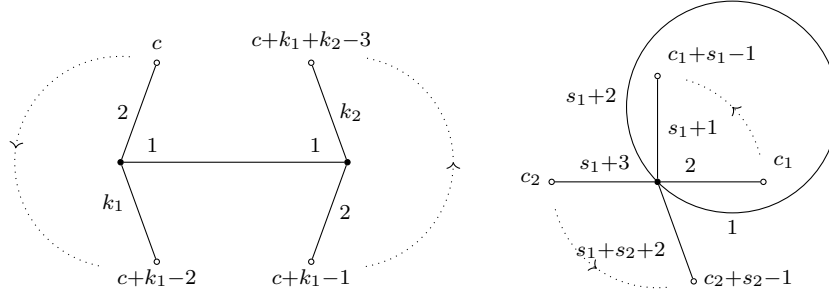


Figure A.1.: Graphs Γ_{k_1, k_2} and Γ^{s_1, s_2} with fixed labelings L_3 .

$C_1 \rightarrow C_0$ from (4.17) in Part I, we have the following:

$$\begin{aligned} \Gamma_{k_1, k_2} : \quad & \langle b \rangle \xrightarrow{\partial_2=0} \langle e \rangle \xrightarrow{\partial_1} \langle v_2 - v_1 \rangle \oplus \langle v_1 + v_2 \rangle, \\ \Gamma^{s_1, s_2} : \quad & \langle b_1 - b_2 \rangle \oplus \langle b_1 + b_2 \rangle \xrightarrow{\partial_2} \langle e \rangle \xrightarrow{\partial_1=0} \langle v \rangle. \end{aligned}$$

As for Γ_{k_1, k_2} , the basis $v_2 - v_1, v_1 + v_2$ of C_0 is positively oriented with respect to the basis v_2, v_1 . Therefore, e has to be oriented such that $\partial_1 e = v_2 - v_1$; i.e., it is a path from v_1 to v_2 . As for Γ^{s_1, s_2} , the basis $b_1 - b_2, b_1 + b_2$ of C_2 is positively oriented with respect to b_1, b_2 . Therefore, e has to be oriented such that $e = \partial_2(b_1 - b_2)$. Recall that we orient the boundary of a 2-simplex by the “outer normal first” convention. We conclude that the labelings from (A.11) are indeed compatible.

As for f_{210} , the permutations $\sigma_1 := \sigma_{L^1}$ and $\sigma_2 := \sigma_{L^2}$ corresponding to the labelings L^1 and L^2 , respectively, read

$$\begin{aligned} \sigma_1 &= \left(\begin{array}{cc|ccc} 1 & 2 & \dots & c+2 & \dots \\ 1 & k_1+1 & \dots & 2 & \dots \end{array} \right) \quad \text{and} \\ &\quad \underbrace{\hspace{10em}}_{k_1+k_1-2} \\ \sigma_2 &= \left(\begin{array}{cc|ccc} 1 & 2 & \dots & c+2 & \dots \\ 1 & k_2+1 & \dots & k_2+2 & \dots \end{array} \right). \\ &\quad \underbrace{\hspace{10em}}_{k_1+k_2-2} \end{aligned}$$

The underbracket marks the block which represents a cyclic permutation of the remaining indices. We see that

$$\begin{aligned} \sigma_1 : V^{\otimes 2} \otimes V^{\otimes s} &\longrightarrow V^{\otimes k_1} \otimes V^{\otimes k_2}, \quad e_i e_j w \longmapsto e_i w^1 e_j w^2, \\ \sigma_2 : V^{\otimes 2} \otimes V^{\otimes s} &\longrightarrow V^{\otimes k_2} \otimes V^{\otimes k_1}, \quad e_i e_j w \longmapsto e_i w^2 e_j w^1, \end{aligned}$$

where $w^1 = w_c \dots w_{c+k_1-2}$, $w^2 = w_{c+k_1-1} \dots w_{c+k_1+k_2-3}$ and $s := k_1 + k_2 - 2$. Defining

A. Evaluation of labeled ribbon graphs

$\tilde{w}^1 := w^2$ and $\tilde{w}^2 := w^1$, The Koszul sign of σ_2 can be written as

$$\varepsilon(w \mapsto w^1 w^2)(-1)^{|w^2||e_j|+|w^1||w^2|} = \varepsilon(w \mapsto \tilde{w}^1 \tilde{w}^2)(-1)^{|\tilde{w}^1||e_j|}.$$

We use these facts to rewrite (A.10) as follows:

$$\begin{aligned} f_{210}(\psi_1 \otimes \psi_2)(w) &= \\ &\sum_{1 \leq k_1 < k_2} \sum_{i,j} T^{ij} \left(\sum_{k(w^1)=k_1-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1||e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2) \right. \\ &+ \sum_{k(w^1)=k_2-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^2||e_j|+|w^1||w^2|} \psi_1(e_i w^2) \psi_2(e_j w^1) \Big) \\ &+ \sum_{1 < k_1 = k_2} \frac{1}{2} \left(\sum_{k(w^1)=k_1-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1||e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2) \right. \\ &+ \sum_{k(w^1)=k_2-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^2||e_j|+|w^1||w^2|} \psi_1(e_i w^2) \psi_2(e_j w^1) \Big) \\ &= \sum_{\substack{k_1, k_2 \geq 1 \\ k_1 + k_2 > 2}} \sum_{\substack{k(w^1)=k_1-1 \\ k(w^2)=k_2-1}} T^{ij} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1||e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2). \end{aligned}$$

This coincides with the formula from Definition 3.4.1 in Part I.

As for f_{120} , the permutations $\sigma_1 := \sigma_{L^1}$ and $\sigma_2 := \sigma_{L^2}$ corresponding to the labelings L^1 and L^2 , respectively, read

$$\begin{aligned} \sigma_1 &= \left(\begin{array}{cc|ccc} 1 & 2 & \dots & c_1 + 2 & \dots \\ 1 & s_1 + 2 & \dots & 2 & \dots \end{array} \middle| \underbrace{\dots}_{s_1} \underbrace{\dots}_{s_2} \right) \text{ and} \\ \sigma_2 &= \left(\begin{array}{cc|ccc} 1 & 2 & \dots & c_2 + 2 & \dots \\ s_1 + 2 & 1 & \dots & s_1 + 3 & \dots \end{array} \middle| \underbrace{\dots}_{s_2} \underbrace{\dots}_{s_1} \right), \end{aligned}$$

where the underbracketed blocks denote cyclic permutations of consecutive indices on the corresponding boundary component. We see that

$$\begin{aligned} \sigma_1 : V^{\otimes 2} \otimes V^{\otimes s_1} \otimes V^{\otimes s_2} &\longrightarrow V^{\otimes k}, & e_i e_j w_1 w_2 &\longmapsto e_i w_1^1 e_j w_2^1, \\ \sigma_2 : V^{\otimes 2} \otimes V^{\otimes s_2} \otimes V^{\otimes s_1} &\longrightarrow V^{\otimes k}, & e_i e_j w_1 w_2 &\longmapsto e_j w_2^1 e_i w_1^1, \end{aligned}$$

where w_i^1 denotes a cyclic permutation and $k := s_1 + s_2 + 2$. The Koszul sign of σ_2 can

A. Evaluation of labeled ribbon graphs

be written as

$$\begin{aligned} & (-1)^{|e_i||e_j|+|w_1||w_2|+|e_i||w_2|} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \\ & = (-1)^{(|e_i|+|w_1|)(|e_j|+|w_2|)+|w_1||e_j|} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1). \end{aligned}$$

We use this fact and the cyclic symmetry of ψ to rewrite (A.10) as follows:

$$\begin{aligned} & f_{120}(\psi)(w_1 \otimes w_2) \\ & = \sum_{0 \leq s_1 < s_2} \left(\delta_{\substack{k(w_1)=s_1 \\ k(w_2)=s_2}} \sum T^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|w_1||e_j|} \right. \\ & \quad \left. \psi(e_i w_1^1 e_j w_2^1) + \delta_{\substack{k(w_1)=s_2 \\ k(w_2)=s_1}} \sum T^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \right. \\ & \quad \left. (-1)^{|e_i||e_j|+|w_2||w_1|+|e_i||w_2|} \psi(e_j w_2^1 e_i w_1^1) \right) \\ & + \sum_{0 < s_1 = s_2} \delta_{\substack{k(w_1)=k(w_1)=s_1 \\ k(w_2)=k(w_2)=s_2}} \frac{1}{2} \left(\sum T^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \right. \\ & \quad \left. (-1)^{|w_1||e_j|} \psi(e_i w_1^1 e_j w_2^1) + \sum T^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \right. \\ & \quad \left. (-1)^{|e_i||e_j|+|w_2||w_1|+|e_i||w_2|} \psi(e_j w_2^1 e_i w_1^1) \right) \\ & = \sum_{\substack{s_1, s_2 \geq 0 \\ s_1 + s_2 > 0}} \delta_{\substack{k(w_1)=s_1 \\ k(w_2)=s_2}} \sum T^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|w_1||e_j|} \\ & \quad \psi(e_i w_1^1 e_j w_2^1). \end{aligned}$$

This coincides with the formula from Definition 3.4.1 in Part I. \triangleleft

A.2. De Rham case

We will now establish a formal analogy between the finite-dimensional and the de Rham case, which will explain the signs in Definition 4.3.6 in Part I.

The finite-dimensional case. Consider the situation of (A) – (D) in the Introduction. To recall briefly, we have a finite-dimensional cyclic dga (V, Π, m_1, m_2) and a subcomplex $\mathcal{H} \subset V$ such that there is a projection $\pi : V[1] \rightarrow \mathcal{H}[1]$ chain homotopic to $\mathbb{1}$ via a chain homotopy $\mathcal{P} : V[1] \rightarrow V[1]$. Using m_2 , one constructs the canonical Maurer-Cartan element \mathfrak{m} for $\mathrm{dIBL}(C(V))$. Using the algebraic Schwartz kernel $\mathcal{K}_{\mathcal{P}}$ of \mathcal{P} , one constructs the IBL_{∞} -quasi-isomorphism $\mathfrak{f} = (\mathfrak{f}_{klg}) : \mathrm{dIBL}(C(V)) \rightarrow \mathrm{dIBL}(C(\mathcal{H}))$. The Maurer-Cartan element \mathfrak{m} is then pushed forward along \mathfrak{f} to obtain the Maurer-Cartan element $\mathfrak{n} := \mathfrak{f}_* \mathfrak{m}$ for $\mathrm{dIBL}(C(\mathcal{H}))$ (see [CFL15, Lemma 9.5]). The formula for \mathfrak{n} given in [CFL15,

A. Evaluation of labeled ribbon graphs

Remark 12.10] reads

$$\begin{aligned} \mathfrak{n}_{lg}(s^l w_1 \otimes \cdots \otimes w_l) \\ = \frac{1}{l!} \sum_{[\Gamma] \in \overline{\text{RG}}_{klg}^{(3)}} \frac{1}{|\text{Aut}(\Gamma)|} (-1)^{k(n-2)} \langle (m_2^+)^{\otimes k}, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^{\mathcal{K}_{\mathcal{P}}}. \end{aligned} \quad (\text{A.12})$$

Here the artificial sign $(-1)^{k(n-2)}$ is added because our sign conventions for m_2^+ differ (see Remark 3.3.8 in Part I).

The de Rham case. We are in the setting of Definition 4.3.6 in Part I. To recall briefly, we have the cyclic dga $(\Omega(M), \Pi, m_1, m_2)$, the subspace of harmonic forms $\mathcal{H} \subset \Omega$, the harmonic projection $\pi_{\mathcal{H}} : \Omega \rightarrow \mathcal{H}$ and a Hodge propagator $P \in \Omega(\text{Bl}_{\Delta}(M \times M))$, which is the Schwartz kernel of a chain homotopy $\mathcal{P} : \Omega \rightarrow \Omega$ between $\pi_{\mathcal{H}}$ and $\mathbb{1}$. In analogy with the finite-dimensional case, the canonical Maurer-Cartan element (4.3) in Part I for $\text{dIBL}(\mathcal{H})$ satisfies $\mathfrak{m}_{10} = (-1)^{n-2} m_2^+$ with $m_2^+ = \Pi(m_2 \otimes \mathbb{1})$. Because $\dim(\Omega) = \infty$, Definition 3.4.1 in Part I does not give the canonical dIBL -structure on $C(\Omega)$, and hence we have neither \mathfrak{f} nor \mathfrak{n} in the standard sense.

In order to deduce the formal analogy, we embed $\Omega(M)^{\otimes 2}$ into $\Omega(\text{Bl}_{\Delta}(M \times M))$ using the external wedge product $(\eta_1, \eta_2) \mapsto \tilde{\pi}_1^* \eta_1 \wedge \tilde{\pi}_2^* \eta_2$ and suppose that the Hodge propagator P satisfies $P \in \Omega^{\otimes 2}$. This never happens, so what follows is just a formal computation.

Proposition A.2.1. *In the de Rham case, suppose that $P \in \Omega(M)^{\otimes 2}$. Then (A.12) reduces to (4.19) in Part I.*

Proof. Consider the intersection pairing $\tilde{\Pi}$ and its degree shift Π (see Proposition 4.1.2 in Part I). According to Definition A.1.4, they extend to pairings on $\Omega(M)^{\otimes k}$ and $\Omega(M)[1]^{\otimes k}$ for all $k \geq 1$, respectively. For all $\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22} \in \Omega(M)$, we have:

$$\begin{aligned} \Pi(\theta^2 \eta_{11} \otimes \eta_{12}, \theta^2 \eta_{21} \otimes \eta_{22}) \\ = (-1)^{\eta_{11} + \eta_{21}} \Pi(\theta \eta_{11} \otimes \theta \eta_{12}, \theta \eta_{21} \otimes \theta \eta_{22}) \\ = (-1)^{\eta_{11} + \eta_{21} + (1 + \eta_{12})(1 + \eta_{21})} \Pi(\theta \eta_{11}, \theta \eta_{21}) \Pi(\theta \eta_{12}, \theta \eta_{22}) \\ = (-1)^{1 + \eta_{12} \eta_{21}} \tilde{P}(\eta_{11}, \eta_{21}) \tilde{\Pi}(\eta_{12}, \eta_{22}) \\ = -\tilde{\Pi}(\eta_{11} \otimes \eta_{12}, \eta_{21} \otimes \eta_{22}). \end{aligned} \quad (\text{A.13})$$

One can also check that

$$\tilde{\Pi}(\eta_{11} \otimes \eta_{12}, \eta_{21} \otimes \eta_{22}) = \int_{x,y} \eta_{11}(x) \eta_{12}(y) \eta_{21}(x) \eta_{22}(y).$$

For the Hodge homotopy $\mathcal{P} : \Omega(M) \rightarrow \Omega(M)$ and its Hodge propagator $P \in \Omega(M)^{\otimes 2}$,

A. Evaluation of labeled ribbon graphs

we have the following:

$$\forall \eta_1, \eta_2 \in \Omega(M) : \quad \tilde{\Pi}(\mathcal{P}(\eta_1), \eta_2) = \int_{x,y} P(x, y) \eta_1(x) \eta_2(y) = \tilde{\Pi}(\mathcal{P}, \eta_1 \otimes \eta_2).$$

From this and (A.13), we obtain

$$\begin{aligned} \Pi(\mathcal{P}(\theta\eta_1), \eta_2) &= \Pi(\theta\mathcal{P}(\eta_1), \theta\eta_2) = (-1)^{1+\eta_1} \tilde{\Pi}(\mathcal{P}(\eta_1), \eta_2) \\ &= (-1)^{1+\eta_1} \tilde{\Pi}(\mathcal{P}, \eta_1 \otimes \eta_2) = (-1)^{\eta_1} \Pi(\theta^2\mathcal{P}, \theta^2\eta_1 \otimes \eta_2) \\ &= \Pi(\theta^2\mathcal{P}, \theta\eta_1 \otimes s\eta_2). \end{aligned}$$

Therefore, the element $\theta^2\mathcal{P} \in V[1]^{\otimes 2}$ corresponds to the Schwartz kernel $\mathcal{K}_{\mathcal{P}}$ of $\mathcal{P} : V[1] \rightarrow V[1]$. We write this correspondence as

$$\mathcal{K}_{\mathcal{P}} \in V[1]^{\otimes 2} \sim \theta^2\mathcal{P} \in \text{Bl}_{\Delta}(M \times M)[2].$$

Let us check that $\theta^2\mathcal{P}$ satisfies (A.1). First of all, if we embed $\Omega(M)^{\otimes k}$ into $\Omega(M^{\times k})$ using the external wedge product $\eta_1 \otimes \cdots \otimes \eta_k \mapsto \pi_1^* \eta_1 \wedge \cdots \wedge \pi_k^* \eta_k =: \eta_1(x_1) \wedge \cdots \wedge \eta_k(x_k)$, then for all $\eta_1, \dots, \eta_k \in \Omega(M)$ we have

$$\sigma(\eta_1 \otimes \cdots \otimes \eta_k)(x_1, \dots, x_k) = \eta_1(x_{\sigma_1}) \wedge \cdots \wedge \eta_k(x_{\sigma_k}),$$

where the action on the left-hand side is given by (3.8) in Part I. Now, the symmetry property (4.7) in Part I implies

$$\tau(\theta^2\mathcal{P}) = -\theta^2\tau^*(\mathcal{P}) = (-1)^{n+1}\theta^2\mathcal{P} = (-1)^{|\theta^2\mathcal{P}|}\theta^2\mathcal{P}.$$

Therefore, the symmetry condition (A.1) is indeed satisfied.

Let $\Gamma \in \overline{\text{RG}}_{klg}^{(3)}$, and let L be a labeling of Γ . We abbreviate $\sigma := \sigma_L \in \mathbb{S}_{3k}$. Given $\eta_{ij} \in \Omega(M)$ for $j = 1, \dots, s_i$ and $i = 1, \dots, l$, where s_i is the valency of the i -th boundary component, we set $\eta_i = \eta_{i1} \otimes \cdots \otimes \eta_{is_i}$, $\eta = \eta_1 \otimes \cdots \otimes \eta_l$, $\alpha_{ij} = \theta\eta_{ij}$, $\omega_i = \alpha_{i1} \otimes \cdots \otimes \alpha_{is_i}$ and $\omega = \omega_1 \otimes \cdots \otimes \omega_l$. We denote $s := s_1 + \cdots + s_l$, so that $3k = 2e + s$, where e is the number of internal edges. We have

$$\begin{aligned} (m_2^+)^{\otimes k}(\sigma((\theta^2\mathcal{P})^{\otimes e} \otimes \omega)) &= \varepsilon(\theta, \eta)(m_2^+)^{\otimes k}(\sigma((\theta^2\mathcal{P})^{\otimes e} \otimes \theta^s\eta)) \\ &= (-1)^{se(n-1)}\varepsilon(\theta, \eta)(m_2^+)^{\otimes k}(\sigma(\theta^{2e+s}\mathcal{P}^{\otimes e} \otimes \eta)) \\ &= \underbrace{(-1)^{\sigma+se(n-1)}\varepsilon(\theta, \eta)(m_2^+)^{\otimes k}(\theta^{2e+s}\sigma(\mathcal{P}^{\otimes e} \otimes \eta))}_{=:\varepsilon_1}, \end{aligned}$$

A. Evaluation of labeled ribbon graphs

where $\varepsilon(\theta, \eta)$ is the Koszul sign to order $\theta^s \eta_{11} \dots \eta_{ls_l} \mapsto \theta \eta_{11} \dots \theta \eta_{ls_l}$ and the operation $m_2^+ : \Omega(M)[1]^{\otimes 3} \rightarrow \mathbb{R}$ is given by $m_2^+ = \Pi(m_2 \otimes \mathbb{1})$. We denote $\kappa := P^{\otimes e} \otimes \eta = \kappa_1 \otimes \dots \otimes \kappa_{3k}$, $\kappa_i \in \Omega(M)[1]$ and compute

$$\begin{aligned}
& (m_2^+)^{\otimes k} (\theta^{3k} \sigma(\kappa)) \\
& = \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes k} (\theta^{3k} \kappa_{\sigma_1^{-1}} \otimes \cdots \otimes \kappa_{\sigma_{3k}^{-1}}) \\
& \quad \uparrow \\
& |m_2^+| = 3-n \\
& = (-1)^{\frac{1}{2}k(k-1)n} \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes k} (\theta^3 (\kappa_{\sigma_1^{-1}} \otimes \kappa_{\sigma_2^{-1}} \otimes \kappa_{\sigma_3^{-1}}) \otimes \cdots \\
& \qquad \qquad \qquad \otimes \theta^3 (\kappa_{\sigma_{3k-2}^{-1}} \otimes \kappa_{\sigma_{3k-1}^{-1}} \otimes \kappa_{\sigma_{3k}^{-1}})) \\
& \qquad \qquad \qquad \underbrace{\hspace{10em}}_{=:\varepsilon_2} \\
& = (-1)^{\frac{1}{2}k(k-1)n + \kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes k} ((\theta \kappa_{\sigma_1^{-1}} \otimes \theta \kappa_{\sigma_2^{-1}} \\
& \qquad \qquad \qquad \otimes \theta \kappa_{\sigma_3^{-1}}) \otimes \cdots \otimes (\theta \kappa_{\sigma_{3k-2}^{-1}} \otimes \theta \kappa_{\sigma_{3k-1}^{-1}} \otimes \theta \kappa_{\sigma_{3k}^{-1}})).
\end{aligned}$$

Next, using the formula (2) in Part I for m_2^+ , we get

$$\begin{aligned}
(m_2^+)^{\otimes k} & ((\theta \kappa_{\sigma_1^{-1}} \otimes \theta \kappa_{\sigma_2^{-1}} \otimes \theta \kappa_{\sigma_3^{-1}}) \otimes \cdots \otimes (\theta \kappa_{\sigma_{3k-2}^{-1}} \otimes \theta \kappa_{\sigma_{3k-1}^{-1}} \otimes \theta \kappa_{\sigma_{3k}^{-1}})) \\
&= (-1)^{\kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \left(\int_{x_1} \kappa_{\sigma_1^{-1}}(x_1) \kappa_{\sigma_2^{-1}}(x_1) \kappa_{\sigma_3^{-1}}(x_1) \right) \cdots \\
&\quad \left(\int_{x_k} \kappa_{\sigma_{3k-2}^{-1}}(x_k) \kappa_{\sigma_{3k-1}^{-1}}(x_k) \kappa_{\sigma_{3k}^{-1}}(x_k) \right) \\
&= (-1)^{\kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \int_{x_1, \dots, x_k} \kappa_{\sigma_1^{-1}}(x_1) \kappa_{\sigma_2^{-1}}(x_1) \kappa_{\sigma_3^{-1}}(x_1) \cdots \\
&\quad \kappa_{\sigma_{3k-2}^{-1}}(x_k) \kappa_{\sigma_{3k-1}^{-1}}(x_k) \kappa_{\sigma_{3k}^{-1}}(x_k) \\
&= \overbrace{(-1)^{\kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \varepsilon(\sigma, \kappa)}^{=: \varepsilon_3} \int_{x_1, \dots, x_k} \kappa_1(x_{\xi(\sigma_1)}) \kappa_2(x_{\xi(\sigma_2)}) \kappa_3(x_{\xi(\sigma_3)}) \cdots \\
&\quad \kappa_{3k-2}(x_{\xi(\sigma_{3k-2})}) \kappa_{3k-1}(x_{\xi(\sigma_{3k-1})}) \kappa_{3k}(x_{\xi(\sigma_{3k})}),
\end{aligned}$$

where $\xi(3j - 2) = \xi(3j - 1) = \xi(3j) = j$ for $j = 1, \dots, k$ (see Definition 4.3.6 in Part I).

In total, we have

$$\begin{aligned} & (m_2^+)^{\otimes k}(\sigma((\theta^2 \mathbf{P})^{\otimes e} \otimes \omega)) \\ &= \varepsilon_1 \varepsilon_2 \varepsilon_3 \int_{x_1, \dots, x_k} \mathbf{P}(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots \mathbf{P}(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) \\ & \quad \alpha_{11}(x_{\xi(\sigma_{2e+1})}) \cdots \alpha_{ls_l}(x_{\xi(\sigma_{2e+s})}), \end{aligned}$$

A. Evaluation of labeled ribbon graphs

where

$$\varepsilon_1 \varepsilon_2 \varepsilon_3 = (-1)^{\sigma + se(n-1) + \frac{1}{2}k(k-1)n} \varepsilon(\theta, \eta).$$

Using (4.16) and (4.18) in Part I and $\varepsilon(\theta, \eta) = (-1)^{P(\omega)}$, we get the total sign

$$(-1)^{k(n-2)} \varepsilon_1 \varepsilon_2 \varepsilon_3 = (-1)^{s(k,l) + \sigma + P(\omega)},$$

where $(-1)^{k(n-2)}$ is the artificial sign from (A.12). This proves the proposition. \square

Remark A.2.2 (Signs for the Fréchet dIBL-structure on $\Omega(M)$). In [CFL15, Section 13], they consider the weight-graded nuclear Fréchet space $B_{\text{cyc}}^* \Omega(M)_\infty \subset B_{\text{cyc}}^* \Omega(M)$ generated by $\varphi \in B_{\text{cyc}}^* \Omega(M)$ which have a smooth Schwartz kernel $k_\varphi \in \Omega(M^{\times k})$; they showed that there is a canonical Fréchet dIBL-structure on $B_{\text{cyc}}^* \Omega(M)_\infty[2-n]$. In order to deduce the signs, we can consider the subspace $B_{\text{cyc}}^* \Omega(M)_{\text{alg}} \subset B_{\text{cyc}}^* \Omega(M)_\infty$ generated by $\varphi \in B_{\text{cyc}}^* \Omega(M)$ with an algebraic Schwartz kernel $\mathcal{K}_\varphi \in \Omega(M)[1]^{\otimes k}$, rewrite (A.10) in terms of \mathcal{K}_φ and extend the obtained formulas to $B_{\text{cyc}}^* \Omega(M)_\infty$. \triangleleft

B. Reduced cyclic homology of A_∞ -algebras

In this appendix, we prove Proposition 3.3.13 in Part I. The idea from [Lod92] is to resolve cyclic (co)invariants degreewise and obtain certain bicomplexes with better properties.

In Section B.1, given a strictly unital A_∞ -algebra on a graded vector space V , we define the normalized and reduced Hochschild (co)chain complexes (Definition B.1.1) and prove the computational prerequisites CP1–CP4 (Lemmas B.1.4, B.1.5, B.1.6 and B.1.7); they are necessary for the development of the cyclic homology theory in the upcoming section. These prerequisites, like squaring to zero of the Hochschild differential, seem to be much harder computationally for A_∞ -algebras than for DGA's. Proofs of some of these relations in different formalisms appeared already in [Mes16] and [Laz03].

In Section B.2, we define Loday's and Connes' cyclic half-plane bicomplexes for A_∞ -algebras together with their normalized and reduced versions (Definition B.2.3). We then summarize some convergence results for spectral sequences associated to horizontal, vertical and diagonal filtrations (Proposition B.2.1). In a series of lemmas (Lemmas B.2.5, B.2.7, B.2.8 and B.2.9), we prove that some of the (co)homologies are isomorphic. These lemmas copy results for DGA's from [Lod92]; we just do them carefully for half-plane bicomplexes and more explicitly. There is a new phenomenon of long chains coming from completing the direct sum total complex; these long chains seem to disappear in homology if the degrees of V are bounded (Lemma B.2.6). Additionally, we point out some differences between first-quadrant and half-plane bicomplexes (Remark B.2.2) and mention the relation to mixed complexes (Remark B.2.4).

In Section B.3, we obtain short exact sequences for reduced Connes' bicomplexes (Lemma B.3.1), which replace, up to quasi-isomorphisms, the non-exact sequences for reduced cyclic Hochschild (co)chains. We summarize the isomorphisms of (co)homologies from Section B.2 (Figure B.3), finish the proof of Proposition 3.3.13 in Part I and formulate a few open question (Remark B.3.2).

B.1. Computational prerequisites

The heart of cyclic (co)homology theory, following [Lod92], are the following five *computational prerequisites* (CP):

CP0 (horizontal relations): $\ker N = \text{im}(\mathbb{1} - t)$, $\ker(\mathbb{1} - t) = \text{im } N$,

CP1 (vertical relations): $b \circ b = 0$, $b' \circ b' = 0$,

CP2 (vertical-horizontal relations): $b' \circ N = N \circ b$, $(\mathbb{1} - t) \circ b' = b \circ (\mathbb{1} - t)$;

and in the strictly unital case

CP3 (null-homotopy of the bar resolution): $b' \circ \iota_1 + \iota_1 \circ b' = \mathbb{1}$,

CP4 (contraction onto normalized chains): $\bar{p} : DV \rightarrow \bar{D}V$ is a quasi-isomorphisms.

The definitions of t , b and b' can be found in Section 3.3 in Part I. The new players are the *counting operator*

$$N := \sum_{k=1}^{\infty} \underbrace{\sum_{i=0}^{k-1} t_k^i}_{=: N_k} : DV \longrightarrow DV$$

and the projection $\bar{p} : DV \rightarrow \bar{D}V$ to normalized Hochschild chains — this we define below.

Definition B.1.1 (Normalized and reduced Hochschild complex). *Let $(V, (\mu_j), \mathbb{1})$ be a strictly unital A_{∞} -algebra. Let $\bar{V}[1] := V[1]/\langle \mathbb{1} \rangle$. We define the normalized Hochschild chain complex by*

$$\bar{D}V := \bigoplus_{k=0}^{\infty} V[1] \otimes \bar{V}[1]^{\otimes k}.$$

We define $\bar{p} : DV \rightarrow \bar{D}V$ to be the canonical projection $V[1] \rightarrow \bar{V}[1]$ for $k = 0$, and for $k \geq 1$, we define

$$\bar{p}|_{V[1]^{\otimes k}} := \mathbb{1} \otimes \underbrace{\bar{p} \otimes \cdots \otimes \bar{p}}_{k\text{-times}}.$$

For every $k \geq 1$, we define the operator $\iota_k : DV \rightarrow DV$ by inserting $\mathbb{1}$ at the k -th position of a tensor product, where the position $k = 1$ is in front; i.e., we have

$$\iota_1(v_1 \otimes \cdots \otimes v_i) = \mathbb{1} \otimes v_1 \otimes \cdots \otimes v_i \quad \text{for all } v_j \in V[1] \text{ and } i \geq j \geq 1.$$

We define the normalized Hochschild cochain complex by

$$\bar{D}^*V := \{\varphi \in D^*V \mid \varphi \circ \iota_i = 0 \text{ for all } i \geq 2\}.$$

B. Reduced cyclic homology of A-infinity-algebras

If $u : D\mathbb{R} \rightarrow DV$ is the unit map and $\bar{\iota} : \bar{D}^*V \rightarrow D^*V$ the inclusion, we define the reduced Hochschild chain and cochain complexes $D^{\text{red}}V$ and D_{red}^*V by

$$D^{\text{red}}V := \text{coker}(\bar{p} \circ u) \quad \text{and} \quad D_{\text{red}}^*V := \ker(u^* \circ \bar{\iota}), \quad \text{respectively.}$$

We denote by $p^{\text{red}} : DV \rightarrow D^{\text{red}}V$ and $\iota_{\text{red}} : D_{\text{red}}^*V \rightarrow D^*V$ the canonical projection and inclusion, respectively.

Remark B.1.2 (Some details on normalized and reduced complexes). Since 1 is a unit for μ_2 , we have

$$\begin{aligned} 0 &= (-1)^{|v_1|+\dots+|v_{k-1}|} (v_1 \cdots \mu_2(v_k, 1) + (-1)^{|v_k|} \mu_2(1, v_1) \cdots v_k) \quad \text{and} \\ 0 &= (-1)^{|v_1|+\dots+|v_{i-2}|} (v_1 \cdots \mu_2(v_{i-1}, 1) v_i \cdots v_k + (-1)^{|v_{i-1}|} v_1 \cdots v_{i-1} \mu_2(1, v_i) \cdots v_k) \end{aligned}$$

for all $i = 2, \dots, k$. This fact and strict unitality implies

$$\text{b}(\sum_{i \geq 2} \text{im } \iota_i) \subset \sum_{i \geq 2} \text{im } \iota_i = \ker \bar{p}.$$

Therefore, b induces a differential on $\bar{D}V$. Since $\bar{D}^*V = \{\varphi \in D^*V \mid \varphi(\sum_{i \geq 2} \text{im } \iota_i) = 0\}$, the dual b^* restricts to \bar{D}^*V . Clearly, both \bar{p} and $\bar{\iota}$ are chain maps, and they are compatible under the dualization from Definition 3.3.4 in Part I; i.e., $\bar{\iota} \simeq \bar{p}^*$ under $\bar{D}^*V \simeq (\bar{D}V)^{*g}$ and $D^*V \simeq (DV)^{*g}$, where *g denotes the graded dual.

As for the reduced complexes, u is a chain map, and thus \ker and coker are chain complexes. Again, it holds $D_{\text{red}}^*V \simeq (D^{\text{red}}V)^{*g}$ and $\iota_{\text{red}} \simeq p^{\text{red},*}$ under the dualization. \triangleleft

We will now prove CP1, CP2, CP3 and CP4 for strictly unital A_∞ -algebras. We do not prove CP0 because it is a standard fact which does not depend on the algebra we work with (see [Lod92]). A proof of CP1 in a slightly different notation and in a more general setting (coefficients in a bimodule) can also be found in [Mes16]. The proofs of CP2 and CP3 work in the same way as the proofs for DGA's from [Lod92]. The computation is just a little longer. As for CP4, we can not use the proof for DGA's from [Lod92, Proposition 1.6.5] anymore because we do not have a simplicial module; instead of this, we consider an explicit homotopy inspired by [Laz03], where CP4 is also proven in a slightly different notation.

We first introduce some notation which simplifies computations:

Definition B.1.3 (Notation). *For the cyclic permutation t_k^i ($:= i$ -times t_k), we define $c := k - i + 1$ and write*

$$v_c \cdots v_{c-1} := t_k^i(v_1 \otimes \cdots \otimes v_k).$$

B. Reduced cyclic homology of A-infinity-algebras

We compute indices modulo k and often omit writing the tensor product.

For every $i = 1, \dots, k$ and $j = 1, \dots, k - i + 1$, we define the closed bracket by

$$\begin{aligned} v_1 \cdots \overline{v_i \cdots v_{i+j-1}} \cdots v_k \\ := (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \otimes \cdots \otimes \mu_j(v_i \otimes \cdots \otimes v_{i+j-1}) \otimes \cdots \otimes v_k. \end{aligned} \quad (\text{B.1})$$

If we apply the closed bracket two-times, we write the first application as an underbracket and the second as an overbracket; for instance, we have

$$\begin{aligned} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \underbrace{v_{i_3} \cdots v_{i_4}} \cdots v_k \\ = (-1)^{|v_{i_1}| + \cdots + |v_{i_3-1}|} v_1 \otimes \cdots \otimes \mu_{j_2}(v_{i_1} \otimes \cdots \otimes v_{i_2}) \otimes \cdots \\ \otimes \mu_{j_1}(v_{i_3} \otimes \cdots \otimes v_{i_4}) \otimes \cdots \otimes v_k, \end{aligned}$$

where $j_1 = i_4 - i_3 + 1$, $j_2 = i_2 - i_1 + 1$. Clearly, the difference is only in the sign. We denote

$$v_1 \cdots v_{i-1} \bar{1} v_i \cdots v_k := \iota_i(v_1 \cdots v_k) = (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots v_{i-1} 1 v_i \cdots v_k.$$

If ι_i is composed with an other operation, we write $\underline{1}$ if the corresponding 1 was inserted first and $\bar{1}$ if it was inserted second. For example, we have

$$v_1 \underline{1} v_2 v_3 \bar{1} v_4 = \iota_5(\iota_2(v_1 v_2 v_3 v_4)).$$

For $j \geq 1$ and $1 \leq i_1 \leq i_2 \leq k$ with $i_2 - i_1 \geq j$, we define the open bracket as follows:

$$v_1 \cdots \overbrace{v_{i_1} \cdots v_{i_2}}^j \cdots v_k := \sum_{\substack{i_1 \leq i_3 \leq i_4 \leq i_2 \\ i_4 - i_3 = j}} v_1 \cdots v_{i_1} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_{i_2} \cdots v_k.$$

Using the notation above, it holds

$$\begin{aligned} b'(v_1 \cdots v_k) &= \sum_{1 \leq i_1 \leq i_2 \leq k} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots v_k = \sum_{j=1}^k \overbrace{v_1 \cdots v_k}^j, \\ R(v_1 \cdots v_k) &= \sum_{2 \leq c \leq k} \overline{v_c \cdots v_1} \cdots v_{c-1}, \end{aligned}$$

B. Reduced cyclic homology of A-infinity-algebras

and the A_∞ -relations simplify to

$$\sum_{1 \leq i_1 \leq i_2 \leq k} \overline{v_1 \cdots v_{i_1} \cdots v_{i_2} \cdots v_k} = \sum_{j=1}^k \overline{v_1 \cdots v_k}_j = 0. \quad (\text{B.2})$$

Because all signs are, in fact, Koszul signs for the symbols $\mu_{j_1}, \mu_{j_2}, v_1, \dots, v_k$, and because μ 's have odd degree, we have for every $1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq k$ the following relation:

$$v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k + v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots v_{i_3} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k = 0. \quad (\text{B.3})$$

Lemma B.1.4 (CP1). *For an A_∞ -algebra $(V, (\mu_j))$, it holds*

$$b' \circ b' = 0 \quad \text{and} \quad b \circ b = 0.$$

Proof. We write

$$b \circ b = (b' + R) \circ (b' + R) = b' \circ b' + b' \circ R + R \circ b' + R \circ R$$

and evaluate it on a tensor $v_1 \cdots v_k \in DV$. We claim that a summand of $b(b(v_1 \cdots v_k))$ coming from the subsequent application of the operations can be uniquely determined by the following data:

- the information whether it comes from $b' \circ b'$, $b' \circ R$, $R \circ b'$ or $R \circ R$;
- a cyclic permutation c of v_1, \dots, v_k ;
- positions of the under- and upperbracket.

The reason for this is that both b' and R produce only Koszul signs, and hence the total sign of a summand in $b(b(v_1 \cdots v_k))$ is the Koszul sign for the symbols $\mu_{j_1}, \mu_{j_2}, v_1, \dots, v_k$, which depends only on the start and final position of the symbols; this is precisely encoded in the data above.

B. Reduced cyclic homology of A-infinity-algebras

For $c = 1$, only $b' \circ b'$ contributes; however, using (B.2) and (B.3), we get

$$\begin{aligned}
 (b' \circ b')(v_1 \cdots v_k) &= \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq k} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2} \cdots v_{i_3} \cdots v_{i_4}} \cdots v_k \\
 &+ \sum_{\substack{1 \leq i_1 \leq i_2 \leq k-1 \\ i_2+1 \leq i_3 \leq i_4 \leq k}} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k \\
 &+ \sum_{\substack{2 \leq i_3 \leq i_4 \leq k \\ 1 \leq i_1 \leq i_2 \leq i_3-1}} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k \\
 &= 0.
 \end{aligned}$$

Let $c \geq 2$ and consider the summands from $b' \circ R$, $R \circ b'$ and $R \circ R$ based on $v_c \cdots v_{c-1}$. The contribution of $R \circ b'$ consists of the following three parts:

$$\begin{aligned}
 \text{I.} \quad & \sum_{\substack{c \leq i_1 \leq i_2 \leq k \\ 1 \leq i_4 \leq c-1}} \overline{v_c \cdots v_{i_1} \cdots v_{i_2} \cdots v_k v_1 \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}, \\
 \text{II.} \quad & \sum_{1 \leq i_1 \leq i_2 \leq i_4 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_1} \cdots v_{i_2} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}, \\
 \text{III.} \quad & \sum_{1 \leq i_4 < i_1 \leq i_2 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_4}} v_{i_4+1} \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots v_{c-1}.
 \end{aligned}$$

The contribution of $b' \circ R$ consists of the following two parts:

$$\begin{aligned}
 \text{IV.} \quad & \sum_{1 \leq i_2 \leq i_4 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_2} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}, \\
 \text{V.} \quad & \sum_{1 \leq i_2 < i_3 \leq i_4 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_2}} v_{i_2+1} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_{c-1}.
 \end{aligned}$$

The contribution of $R \circ R$ is:

$$\text{VI.} \quad \sum_{\substack{c < i_1 \leq k \\ 1 \leq i_2 \leq i_4 \leq c-1}} \overline{v_c \cdots v_{i_1} \cdots v_k v_1 \cdots v_{i_2} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}.$$

Using (B.3), it is easy to see that III cancels with V. The sum of the other terms I, II, IV, VI vanishes for fixed $1 \leq i_4 \leq c-1$ due to (B.2). \square

Lemma B.1.5 (CP2). *For an A_∞ -algebra $(V, (\mu_j))$, the following relations hold:*

- (a) $b' \circ N = N \circ b$,
- (b) $(\mathbb{1} - t) \circ b' = b \circ (\mathbb{1} - t)$.

B. Reduced cyclic homology of A-infinity-algebras

Proof. (a) We denote $z_j := \mu_j \otimes \mathbb{1}^{k-j}$ and omit writing the composition \circ . We consider the components

$$b'_k{}^j := \sum_{i=0}^{k-j} t_{k-j+1}^i z_j t_k^{-i} \quad \text{and} \quad R_k^j := \sum_{i=1}^{j-1} z_j t_k^i.$$

It holds

$$\begin{aligned} b'_k{}^j N_k &= \sum_{l=0}^{k-1} \sum_{i=0}^{k-j} t_{k-j+1}^i z_j t_k^{-i+l} \\ &= \sum_{l=0}^{k-1} \sum_{i=0}^{k-j} t_{k-j+1}^l t_{k-j+1}^{i-l} z_j t_k^{-(i-l)} \\ &= \sum_{u=1-k}^{k-j} \left(\sum_{l \in L_u} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u}, \end{aligned}$$

where $u := i - l$ and

$$L_u := \{l \in \{0, \dots, k-1\} \mid \exists i \in \{0, \dots, k-j\} : u = i - l\}.$$

We distinguish the cases

$$L_u = \begin{cases} \{0, \dots, k-j-u\} & \text{for } 0 \leq u \leq k-j, \\ \{-u, \dots, k-j-u\} & \text{for } 1-j \leq u \leq -1 \\ \{-u, \dots, k-1\} & \text{for } 1-k \leq u \leq -j \end{cases} \quad \text{and}$$

and denote the corresponding sums by I, II and III, respectively. It holds

$$\text{I} = \sum_{u=0}^{k-j} \left(\sum_{l=0}^{k-j-u} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u}$$

B. Reduced cyclic homology of A-infinity-algebras

and

$$\begin{aligned}
\text{III} &= \sum_{u=1-k}^{-j} \sum_{l=-u}^{k-1} t_{k-j+1}^l t_{k-j+1}^u z_j t_k^{-u} \\
&= \sum_{u=1-k}^{-j} \sum_{l=-u}^{k-1} t_{k-j+1}^l t_{k-j+1}^{u+k-j+1} z_j t_k^{-u-k} \\
&= \sum_{u=1}^{k-j} \sum_{l=k-u}^{k-1} t_{k-j+1}^{l-j+1} t_{k-j+1}^u z_j t_k^{-u} \\
&= \sum_{u=1}^{k-j} \left(\sum_{l=k-j-u+1}^{k-j} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u}.
\end{aligned}$$

Therefore, we have

$$\text{I} + \text{III} = \sum_{u=0}^{k-j} \left(\sum_{l=0}^{k-j} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u} = N_{k-j+1} b_k^j$$

Next, we have

$$\text{II} = \sum_{u=1-j}^{-1} \sum_{l=-u}^{k-j-u} t_{k-j+1}^l t_{k-j+1}^u z_j t_k^{-u} = \sum_{u=1}^{j-1} \left(\sum_{l=0}^{k-j} t_{k-j+1}^l \right) z_j t_k^u = N_{k-j+1} R_k^j.$$

We conclude that

$$b_k^j N_k = N_{k-j+1} b_k^j + N_{k-j+1} R_k^j = N_{k-j+1} b_k.$$

This proves the claim.

B. Reduced cyclic homology of A-infinity-algebras

(b) For every $k \geq 1$ and $1 \leq j \leq k$ we compute

$$\begin{aligned}
(\mathbb{1} - t)b_k^j &= b_k^j - \sum_{i=0}^{k-j} t_{k-j+1}^{i+1} z_j t_k^{-(i+1)} t_k \\
&= b_k^j - \sum_{i=1}^{k-j+1} t_{k-j+1}^i z_j t_k^{-i} t_k \\
&= b_k^j - \sum_{i=0}^{k-j} t_{k-j+1}^i z_j t_k^{-i} t_k + t_{k-j+1}^0 z_j t_k^{-0} t_k - t_{k-j+1}^{k-j+1} z_j t_k^{-k+j-1} t_k \\
&= b_k^j (\mathbb{1} - t_k) + z_j t_k - z_j t_k^j \\
&= b_k^j (\mathbb{1} - t_k) + \sum_{i=1}^{j-1} z_j t_k^i (\mathbb{1} - t_k) \\
&= (b_k^j + R_k^j) (\mathbb{1} - t_k).
\end{aligned}$$

This proves the claim. \square

Lemma B.1.6 (CP3). *For a strictly unital A_∞ -algebra $(V, (\mu_k), \mathbb{1})$, it holds*

$$b' \circ \iota_1 + \iota_1 \circ b' = \mathbb{1}.$$

Proof. For any $k \geq 1$ and $v_1, \dots, v_k \in V[1]$, we compute

$$\begin{aligned}
b' \iota_1(v_1 \dots v_k) + \iota_1 b'(v_1 \dots v_k) &= \overline{\mathbb{1} v_1} v_2 \dots v_k + \overline{\mathbb{1} v_1 \dots v_k} + \overline{\mathbb{1} v_1 \dots v_k} \\
&= v_1 \dots v_k.
\end{aligned}$$

This proves the claim. \square

Lemma B.1.7 (CP4). *Let $\mathcal{A} = (V, (\mu_k), \mathbb{1})$ be a strictly unital A_∞ -algebra. For all $k \geq 2$, we define $h_k : DV \rightarrow DV$ by*

$$h_k := \iota_k \circ b + b \circ \iota_k + \mathbb{1}.$$

Then the formulas

$$\begin{aligned}
s^* &:= s_2^* + s_3^* \circ h_2^* + s_4^* \circ h_3^* \circ h_2^* + \dots \quad \text{and} \\
h^* &:= \dots \circ h_k^* \circ \dots \circ h_2^*
\end{aligned}$$

define homogenous linear maps s^ and $h^* : D^*V \rightarrow D^*V$ of degrees 1 and 0, respectively.*

B. Reduced cyclic homology of A-infinity-algebras

The map h^* is a projection onto \bar{D}^*V , and the following homotopy relation holds:

$$s^* \circ b^* + b^* \circ s^* = h^* - \mathbb{1}. \quad (\text{B.4})$$

It implies that $\bar{\iota}$ and \bar{p} are quasi-isomorphisms.

Proof. We set $\bar{D}_{(1)}^*V := D^*V$, and for all $k \geq 2$, we define

$$\bar{D}_{(k)}^*V := \{\psi \in D^*V \mid \psi \circ \iota_i = 0 \text{ for all } i = 2, \dots, k\}.$$

We will show first that h_k^* restricts to a projection $\bar{D}_{(k-1)}^*V \rightarrow \bar{D}_{(k)}^*V$. Let $i \geq 1$ and $v_1, \dots, v_i \in V[1]$. We make the following computations:

(1) For $i < k - 1$, we have

$$(\iota_i b + b \iota_i)(v_1 \dots v_i) = 0$$

by the definition of $\iota_i k$ and by the fact that b does not increase weights.

(2) For $i = k - 1$, we have

$$\begin{aligned} & (\iota_i b + b \iota_i)(v_1 \dots v_i) \\ &= \frac{v_1 \dots v_i \bar{1}}{1} + \frac{1}{v_1 \dots v_i \underline{1}} + \sum_{j=2}^i \frac{j}{v_1 \dots v_i \underline{1}} + v_1 \dots \overline{v_i \underline{1}} + \underline{1} v_1 \dots v_i \\ &= \sum_{j=2}^i \frac{j}{v_1 \dots v_i \underline{1}}. \end{aligned}$$

Notice that $\bar{1}$ in the result is at positions $< k$.

B. Reduced cyclic homology of A-infinity-algebras

(3) For $i > k - 1$, we have

$$\begin{aligned}
& (\iota_i b + b \iota_i)(v_1 \dots v_i) \\
&= \sum_{j=1}^{i-k+2} \frac{v_1 \dots v_{k+j-2} \bar{1} v_{k+j-1} \dots v_i}{j} + \sum_{j=1}^{i-k+1} \frac{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i}{j} \\
&+ \sum_{m=1}^{i-k+1} \sum_{c=m+k-1}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2} \bar{1} v_{m+k-1} \dots v_{c-1} \\
&+ \sum_{j=1}^{k-1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i} + \sum_{j=1}^{i-k+1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i} \\
&+ v_1 \dots \overline{v_{k-1} \bar{1} v_k} \dots v_i + v_1 \dots v_{k-1} \bar{1} v_k \dots v_i \\
&+ \sum_{m=1}^{k-1} \sum_{c=k}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1} \bar{1} v_k \dots v_{c-1} \\
&= \sum_{j=1}^{i-k+2} \frac{v_1 \dots v_{k+j-2} \bar{1} v_{k+j-1} \dots v_i}{j} + \sum_{j=1}^{k-1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i} \\
&+ \sum_{m=1}^{i-k+1} \sum_{c=m+k-1}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2} \bar{1} v_{m+k-1} \dots v_{c-1} \\
&+ \sum_{m=1}^{k-1} \sum_{c=k}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1} \bar{1} v_k \dots v_{c-1} \\
&= \overbrace{\sum_{j=2}^{i-k+2} \frac{v_1 \dots v_{k+j-2} \bar{1} v_{k+j-1} \dots v_i}{j}}^{=:I} + \overbrace{\sum_{j=2}^{k-1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i}}^{=:II} \\
&+ \overbrace{\sum_{m=2}^{i-k+1} \sum_{c=m+k-1}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2} \bar{1} v_{m+k-1} \dots v_{c-1}}^{=:III} \\
&+ \overbrace{\sum_{m=2}^{k-1} \sum_{c=k}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1} \bar{1} v_k \dots v_{c-1}}^{=:IV}
\end{aligned}$$

Notice that $\bar{1}$ is at the k -th position in I and III, whereas at positions $< k$ in II and IV.

Let $k \geq 2$, and let $\psi \in \bar{D}_{(k-1)}^* V$. In order to show that $\iota_j^* h_k^* \psi = 0$ for $2 \leq j \leq k$, let $i \geq j$, and let $v_1, \dots, v_i \in V[1]$ be such that $v_j = 1$. Clearly, $\psi(\text{II}) = \psi(\text{IV}) = 0$ for any v 's. As for III, the vector $v_j = 1$ lies either inside μ_j with $j \geq 3$ or at a position $< k$. It follows

B. Reduced cyclic homology of A-infinity-algebras

that $\psi(\text{III}) = 0$. As for I, we write

$$\text{I} = \underbrace{\frac{v_1 \cdots v_k \bar{1} v_{k+1} \cdots v_i}{2}}_{\text{Ia}} + \underbrace{\sum_{j=3}^{i-k+2} \frac{v_1 \cdots v_{k+j-2} \bar{1} v_{k+j-1} \cdots v_i}{j}}_{\text{Ib}}.$$

It holds $\psi(\text{Ib}) = 0$. For $2 \leq j < k$, it holds

$$\begin{aligned} \psi(\text{Ia}) &= \psi(v_1 \cdots \underbrace{v_{j-1} \bar{1}}_{\text{Ia}} v_{j+1} \cdots v_k \bar{1} v_{k+1} \cdots v_i) \\ &\quad + \psi(v_1 \cdots v_{j-1} \bar{1} \underbrace{v_{j+1}}_{\text{Ia}} \cdots v_k \bar{1} v_{k+1} \cdots v_i) \\ &= 0, \end{aligned}$$

whereas for $j = k$, we have

$$\psi(v_1 \cdots \underbrace{v_{k-1} \bar{1}}_{\text{Ia}} v_{k+1} \cdots v_i) = -\psi(v_1 \cdots v_i).$$

It follows that

$$h_k^* \psi(v_1 \cdots v_i) = \psi(v_1 \cdots v_i) + \psi((\iota_i \bar{b} + b \iota_i)(v_1 \cdots v_i)) = 0. \quad (\text{B.5})$$

Therefore, we have $h_k^* \psi \in \bar{D}_{(k)}^* V$. If $\psi \in \bar{D}_{(k)}^* V$, then clearly $h_k^*(\psi) = \psi$. Consequently, h_k^* is a projection $h_k^* : \bar{D}_{(k-1)}^* V \rightarrow \bar{D}_{(k)}^* V$.

For $k \geq 2$, we define

$$\begin{aligned} {}^k s^* &:= \iota_2^* + \iota_3^* \circ h_2^* + \cdots + \iota_k^* \circ h_{k-1}^* \circ \cdots \circ h_2^*, \\ {}^k h^* &:= h_k^* \circ \cdots \circ h_2^*. \end{aligned}$$

Let $\mathcal{F}_{\mathbf{w}}^n DV = \bigoplus_{k=1}^n D_k V$ be the filtration of DV by weights. For all $k \geq n+1$ it holds

$${}^k h^* \psi \Big|_{\mathcal{F}_{\mathbf{w}}^n DV} = {}^{n+1} h^* \psi \quad \text{and} \quad {}^k s^* \psi \Big|_{\mathcal{F}_{\mathbf{w}}^n DV} = {}^{n+1} s^* \psi.$$

It follows that $h^*, s^* : D^* V \rightarrow D^* V$ are well-defined and that h^* is a projection onto $\bar{D}^* V$. Also, it suffices to prove (B.4) with ${}^k s^*$ and ${}^k h^*$ instead of s^* and h^* for each $k \geq 2$.

B. Reduced cyclic homology of A-infinity-algebras

For $k = 2$, it holds by definition. Suppose that (B.4) holds for some $k \geq 2$. Then

$$\begin{aligned}
 & b^* \circ ({}^{k+1}s^*) + ({}^{k+1}s^*) \circ \mathbb{1} \\
 &= b^* \circ ({}^k s^*) + ({}^k s^*) \circ b^* + b^* \circ \iota_{k+1}^* \circ h_k^* \circ \cdots \circ h_2^* + \iota_{k+1}^* \circ h_k^* \circ \cdots \circ h_2^* \circ b^* \\
 &= {}^k h^* - \mathbb{1} + (b^* \circ \iota_{k+1}^* + \iota_{k+1}^* \circ b^*) \circ h_k^* \circ \cdots \circ h_2^* \\
 &= {}^k h^* - \mathbb{1} + (h_{k+1}^* - \mathbb{1}) \circ h_k^* \circ \cdots \circ h_2^* \\
 &= {}^{k+1} h^* - \mathbb{1}.
 \end{aligned}$$

The lemma is finally proven. □

B.2. Homological algebra of bicomplexes

We will consider homological and cohomological half-plane bicomplexes, which we depict as

$$\begin{array}{ccccc}
 & \downarrow & & \downarrow & & \downarrow \\
 & B_{q,p+2} & \longleftarrow & B_{q+1,p+2} & \longleftarrow & B_{q+2,p+2} & \longleftarrow \\
 & \downarrow & & \downarrow & & \downarrow \\
 B : & B_{q,p+1} & \longleftarrow & B_{q+1,p+1} & \longleftarrow & B_{q+2,p+1} & \longleftarrow \\
 & \downarrow & & \downarrow & & \downarrow \\
 & B_{q,p} & \longleftarrow & B_{q+1,p} & \longleftarrow & B_{q+2,p} & \longleftarrow \\
 & \downarrow & & \downarrow & & \downarrow
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
 & B^{q,p+2} & \longrightarrow & B^{q+1,p+2} & \longrightarrow & B^{q+2,p+2} & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
 B^* : & B^{q,p+1} & \longrightarrow & B^{q+1,p+1} & \longrightarrow & B^{q+2,p+1} & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & B^{q,p} & \longrightarrow & B^{q+1,p} & \longrightarrow & B^{q+2,p} & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow &
 \end{array} ,$$

respectively. The standard convention is that the squares anticommute (see [Lod92])!

We consider the *total complexes* $(\text{Tot}_I(B), \partial)$ and $(\text{Tot}_{II}(B), \partial)$, where for all $q \in \mathbb{Z}$, the chain groups are defined by

$$(\text{Tot}_I B)_q := \bigoplus_{i+j=q} B_{i,j}, \quad \text{and} \quad (\text{Tot}_{II} B)_q := \prod_{i+j=q} B_{i,j},$$

B. Reduced cyclic homology of A -infinity-algebras

respectively, and where $\partial = \partial_v + \partial_h$ is the total boundary operator consisting of the vertical and horizontal boundary operators ∂_v and ∂_h , respectively. The homologies of Tot_I and Tot_{II} are denoted by HB and $H\hat{B}$, respectively. We proceed similarly in cohomology.

For each B and B^* , we consider the vertical and horizontal filtrations which are defined in such a way that they are preserved by all the arrows and that the k -th group contains the k -th column and the k -th row, respectively. More precisely, the vertical filtration $\mathcal{F}_v^k B$ of B consists of the columns $0, \dots, k$, whereas the vertical filtration $\mathcal{F}_v^k B^*$ of B^* consists of the columns $k+1, k+2, \dots$; the horizontal filtration $\mathcal{F}_h^k B$ of B consists of the rows $k, k-1, \dots$, whereas the horizontal filtration $\mathcal{F}_h^k B^*$ of B^* consists of the rows $k, k+1, \dots$, and so on.

In order to check that morphisms of bicomplexes induce quasi-isomorphisms of total complexes, we will use the techniques of spectral sequences. Because we work with half-plane bicomplexes, our spectral sequences do not lie in the first quadrant, as in [Wei94], and the notion of conditional convergence from [Boa98] comes in handy. In the following, we recall some basic theory and formulate a proposition about the convergence of some unbounded spectral sequences.

A *cohomological spectral sequence* is a collection E_r of bigraded vector spaces and differentials $d_r : E_r^{\bullet, \bullet} \rightarrow E_r^{\bullet+r, \bullet-r+1}$ for $r \in \mathbb{N}$ such that $E_{r+1} = H(E_r, d_r)$. Let (C^*, d) be a cochain complex with a decreasing filtration $\mathcal{F}^s C^*$ (it has to be graded and preserved by d). For every $s \in \mathbb{Z}$, the short exact sequence

$$0 \longrightarrow \mathcal{F}^{s+1} C^* \xrightarrow{i} \mathcal{F}^s C^* \xrightarrow{j} \text{gr}_s(C^*) := \mathcal{F}^s C^* / \mathcal{F}^{s+1} C^* \longrightarrow 0$$

induces the long exact sequence

$$\dots \longrightarrow H^\bullet(\mathcal{F}^{s+1} C^*) \xrightarrow{i} H^\bullet(\mathcal{F}^s C^*) \xrightarrow{j} H^\bullet(\text{gr}_s C^*) \xrightarrow{\delta} H^{\bullet+1}(\mathcal{F}^{s+1} C^*) \longrightarrow \dots,$$

which wraps into the exact couple of bigraded vector spaces

$$\begin{array}{ccc} A_1 := \bigoplus_{s \in \mathbb{Z}} H(\mathcal{F}^s C^*)[s] & \xrightarrow{\quad i \quad} & A_1 \\ & \swarrow \delta \quad \searrow j & \\ & E_1 := \bigoplus_{s \in \mathbb{Z}} H(\text{gr}_s C^*)[s] & \end{array}$$

This is the so called *geometric grading* convention.¹ We also define $A_1^s := H(\mathcal{F}^s C^*)$ and $E_1^s := H(\text{gr}_s C^*)$. By deriving this triangle (see, e.g., [Cie13]), one obtains a spectral

¹It is chosen such that $E_1^{sd} = H(B^{sd}, d_v)$ for the vertical filtration.

B. Reduced cyclic homology of A-infinity-algebras

sequence E_r associated to the filtration. One defines the E_∞ page (see [Boa98]) and studies the convergence of E_r to a filtered group G . In order to formulate this, one considers the limit $A^\infty := \lim_s A^s$, the colimit $A^{-\infty} := \operatorname{colim}_s A^s$ and the right derived module for the limit $RA^\infty := \lim_s^1 A^s$. We will use the following notions of convergence:

1. *Strong convergence to a filtered group G* $:\iff$ For each $s \in \mathbb{Z}$, we have $\operatorname{gr}_s G \simeq E_\infty^s / E_\infty^{s+1}$ and the filtration on G is exhaustive, Hausdorff and complete (i.e., $G^\infty = 0$, $G^{-\infty} = G$ and $RG^\infty = 0$ for $G^s := \mathcal{F}^s G$).
2. *Conditional convergence to the colimit $G := A^{-\infty}$* $:\iff$ It holds $A^\infty = 0$ and $RA^\infty = 0$.

Note that neither notion implies, in general, the other (see (b) of Remark B.2.2).

Proposition B.2.1 (Convergence of certain unbounded spectral sequences). *The following statements about convergence of spectral sequences hold:*

- (a) *For any \mathbb{Z} -graded cochain complex (C^*, d) with the canonical filtration $\mathcal{F}_{\text{can}}^k C^* := \bigoplus_{i \geq k} C^i$, the associated spectral sequence converges strongly and conditionally to the colimit $H(C^*)$.*
- (b) *The spectral sequence associated to the total complex $\operatorname{Tot}_I B^*$ of a cohomological half-plane bicomplex B^* with the filtration induced from the horizontal filtration $\mathcal{F}_h^k B^* = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \geq k} B^{ij}$ converges strongly to the colimit $H(\operatorname{Tot}_I B^*)$.*
- (c) *The spectral sequence associated to the total complex $\operatorname{Tot}_I B^*$ of a cohomological half-plane bicomplex B^* with the filtration induced from the diagonal filtration*

$$\mathcal{F}^k B^* = \bigoplus_{j-i > k} B^{ij} \oplus \bigoplus_{i \in \mathbb{N}_0} Z^{i, k+i},$$

where $Z^{ij} \subset B^{ij}$ is such that $d_h Z^{ij} = 0$ and $d_h B^{i-1j} \subset Z^{ij}$, converges strongly to the colimit $H(\operatorname{Tot}_I B^)$.*

- (d) *The spectral sequence associated the total complex $\operatorname{Tot}_{II} B^*$ of a cohomological half-plane bicomplex B^* with the filtration induced from the vertical filtration $\mathcal{F}_v^k B^* = \bigoplus_{i \geq k} \bigoplus_{j \in \mathbb{Z}} B^{ij}$ converges conditionally to the colimit $H(\operatorname{Tot}_{II} B^*)$.*

The following statements about morphisms of spectral sequences hold:

- (e) *Let f be a morphism of filtered complexes of types (a), (b) or (c). If it induces an isomorphism of E_r for some r , then it induces an isomorphism of the target groups.*

B. Reduced cyclic homology of A-infinity-algebras

(f) Let f be a morphism of filtered complexes of type (d) . If it induces an isomorphism of E_r for some r , then it induces an isomorphism of the target groups.

Proof. (a) Strong convergence follows from (b) by embedding C^* as the first column of B^* . Also, the computation there show that $A^{-\infty} = H(C^*)$ and $A^\infty = 0$. The condition $RA^\infty = 0$ is equivalent to the degreewise completeness of the filtration, which is true as the filtration is degreewise trivial. This shows the conditional convergence.

(b) Let us compute the first page with the geometrical bigrading:

$$\begin{aligned} E_1^{sd} &= H(\mathrm{gr}_s \mathrm{Tot}_I B^*, d_h)[s]^d \\ &= H^{d+s}(\mathrm{Tot}_I(s\text{-th row of } B^*), d_h) \\ &= H(B^{d,s}, d_h). \end{aligned}$$

We want to use the following theorem:

Theorem B.2.1 ([Boa98, Theorem 6.1]). Suppose that $E^s = 0$ for all $s > 0$ and $A^{-\infty} = 0$. Then the spectral sequence converges strongly to the colimit A^∞ .

The proof can be done degreewise (see [Rog19]), and it can be shown that Theorem B.2.1 generalizes appropriately under the weaker assumption of “exiting differentials”. This means that the pages occupy a half-plane and for any fixed (s, d) , all but finitely many differentials $d_r : E_r^{sd} \rightarrow E_r^{s+r, d-r+1}$ leave the half-plane (and thus vanish). In our case, E_r^{sd} occupy the half-plane $\{(s, d) \mid d \geq 0\}$, and because $d - r + 1 \rightarrow -\infty$ as $r \rightarrow \infty$, the condition of exiting differentials is satisfied.

We still have to check that $A^\infty = 0$ and compute $A^{-\infty}$. Because the colimit is an exact functor, it commutes with H , and we have

$$A^{-\infty} = \mathrm{colim}_s H(\mathcal{F}^s \mathrm{Tot}_I B^*) = H(\mathrm{colim}_s \mathcal{F}^s \mathrm{Tot}_I B^*) = H\left(\bigcup_s \mathcal{F}^s \mathrm{Tot}_I B^*\right) = H(\mathrm{Tot}_I B^*).$$

We used here that \mathcal{F} is exhaustive. The limit A^∞ can be represented as

$$A^\infty \simeq \left\{ ([a_s]) \in \prod_{s \in \mathbb{Z}} A^s \mid [a_{s+1}] \mapsto [a_s] \right\},$$

where $H(\mathcal{F}^{s+1} \mathrm{Tot}_I B^*) \rightarrow H(\mathcal{F}^s \mathrm{Tot}_I B^*)$ is induced by the inclusion $\mathcal{F}^{s+1} \mathrm{Tot}_I B^* \hookrightarrow \mathcal{F}^s \mathrm{Tot}_I B^*$. Pick $s_0 \in \mathbb{Z}$ and consider $[a_{s_0}] \in A^{s_0}$ with a fixed representative $a_{s_0} \in \mathcal{F}^{s_0} B^*$. Because the cohomological degrees of a_{s_0} are bounded, let's say that $d_0 \in \mathbb{Z}$ is an upper bound, and the filtration is degreewise bounded from below, there is an $s_1 \geq s_0$ such that

B. Reduced cyclic homology of A -infinity-algebras

$\mathcal{F}^s \text{Tot}_I B^* \cap (\text{Tot}_I B^*)^d = 0$ for all $s \geq s_1$ and $d \leq d_0$. Now, we have $[a_{s_1}] \mapsto [a_{s_0}]$, and hence $[a_{s_0}] = 0$. Because s_0 was arbitrary, we get $([a_s]) = 0$. Therefore, it holds $A^\infty = 0$.

Alternatively, a direct proof of (b) can be found in [Cen98].

(c) The first page reads

$$E_1^{sd} = H^{s+d}(\text{gr}_s \text{Tot}_I B^*) = H(B^{\lfloor \frac{d}{2} \rfloor, s + \lceil \frac{d}{2} \rceil}, d'),$$

where d' is the differential on $\text{gr} \text{Tot}_I B^*$. We see that E_r^{sd} occupy the half-plane $\{(s, d) \mid d \geq 0\}$, and hence the condition of exiting differentials is satisfied. The groups A^∞ and $A^{-\infty}$ are computed as in (b). The strong convergence is again implied by a generalization of Theorem B.2.1.

Alternatively, one can modify the direct proof for the horizontal filtration from [Cen98].

(d) We want to use the following theorem:

Theorem B.2.2 ([Boa98, Theorem 9.2]). Let C^* be a cochain complex filtered by an exhaustive, Hausdorff and complete filtration. Then the spectral sequence converges conditionally to $H(C^*)$.

We have

$$\mathcal{F}_v^k(\text{Tot}_I B^*)^d = \bigoplus_{i \geq k} B^{i, d-i},$$

and hence $\text{Tot}_{II} B^*$ is the completion of $\text{Tot}_I B^*$ with respect to \mathcal{F}_v . Hence, it is complete. Clearly, it is also Hausdorff and exhaustive, and we can apply Theorem B.2.2.

(e) This follows from (a), (b), (c) and the following theorem:

Theorem B.2.3 ([Boa98, Theorem 5.3]). Let $f : C^* \rightarrow \bar{C}^*$ be a morphism of filtered cochain complexes. Suppose that E_r converges strongly to a filtered group G and that \bar{E}_r converges (strongly) to a filtered group \bar{G} . If f induces an isomorphism $E_r \simeq \bar{E}_r$ for some r , then it induces an isomorphism $G \simeq \bar{G}$.

(f) We want to use the following theorem:

Theorem B.2.4 ([Boa98, Theorem 7.2]). Let $f : C^* \rightarrow \bar{C}^*$ be a morphism of filtered cochain complexes. Suppose that $E^s = \bar{E}^s = 0$ for all $s < 0$ and that the spectral sequences converge conditionally to the colimits $A^{-\infty}$ and $\bar{A}^{-\infty}$, respectively. If f induces isomorphisms $E_\infty \simeq \bar{E}_\infty$ and $RE_\infty \simeq R\bar{E}_\infty$, then it induces an isomorphism $A^\infty \simeq \bar{A}^\infty$.

B. Reduced cyclic homology of A-infinity-algebras

The first page for the vertical filtration reads:

$$\begin{aligned} E_1^{sd} &= H(\mathrm{gr}_s \mathrm{Tot}_{\mathrm{II}} B^*, d_v)[s]^d \\ &= H^{d+s}(\mathrm{Tot}_{\mathrm{II}}(s\text{-th column of } B^*), d_v) \\ &= H(B^{sd}, d_v). \end{aligned}$$

Therefore, the condition $E^s = \bar{E}^s = 0$ for all $s < 0$ is satisfied.² By (d), conditional convergence is guaranteed. Since both E_∞ and RE_∞ depend only on E_r for $r \geq r_0$ and any r_0 (see [Boa98, p. 16]), the rest of the assumptions of Theorem B.2.4 is fulfilled. \square

Remark B.2.2 (Differences to first-quadrant bicomplexes). (i) Given a half-plane bicomplex B , we have

$$(\mathrm{Tot}_{\mathrm{I}} B)^{*g} \simeq \mathrm{Tot}_{\mathrm{II}} B^*,$$

where $B^* = \bigoplus_{i,j} B_{ij}^*$ is the “pointwise dual” to B . This is why we have to consider homology and cohomology separately and can not just dualize the results.

(ii) The vertical filtration of B^* might not converge strongly to $H(\mathrm{Tot}_{\mathrm{I}} B^*)$. Indeed, let

$$B^* : \begin{array}{ccccc} & \mathbb{R} & & 0 & & 0 \\ & \uparrow \mathbb{1} & & & & \\ & \mathbb{R} & \xrightarrow{\mathbb{1}} & \mathbb{R} & & 0 \quad . \\ & & & \uparrow \mathbb{1} & & \\ & & & \mathbb{R} & \xrightarrow{\mathbb{1}} & \dots \\ & 0 & & & & \end{array}$$

Then $H(\mathrm{Tot}_{\mathrm{I}} B^*) = \mathbb{R}$ (the \mathbb{R} in the first column), but $E_1 = 0$ because every column is exact. Notice that $H(\mathrm{Tot}_{\mathrm{II}} B^*) = 0$. Taking 0’s instead of $\mathbb{1}$ ’s in the definition of B^* , we see that the horizontal filtration does not converge conditionally to $H(\mathrm{Tot}_{\mathrm{I}} B^*)$ because its filtration by A^s is incomplete. \triangleleft

We will work with the following bicomplexes.

Definition B.2.3 (Bicomplexes for cyclic (co)homology). *Let $\mathcal{A} = (V, (\mu_k))$ be an A_∞ -algebra. Loday’s cyclic bicomplexes are defined by*

²It is again possible to relax this assumption and prove B.2.4 when the condition of “entering differentials” is satisfied. This means that the pages occupy a half-plane and for any fixed (s, d) , all but finitely many differentials $d_r : E_r^{s-r, d+r-1} \rightarrow E_r^{s, d}$ start outside of the half-plane (and thus vanish). See [Rog19].

B. Reduced cyclic homology of A-infinity-algebras

$$\begin{array}{cccc}
 \downarrow & \downarrow & \downarrow & \downarrow \\
 D_2V & \xleftarrow{\mathbb{1}-t} D_2V & \xleftarrow{N} D_2V & \xleftarrow{\mathbb{1}-t} D_2V & \xleftarrow{N} D_2V \\
 \downarrow b & \downarrow -b' & \downarrow b & \downarrow -b' & \\
 CC(\mathcal{A}) : D_1V & \xleftarrow{\mathbb{1}-t} D_1V & \xleftarrow{N} D_1V & \xleftarrow{\mathbb{1}-t} D_1V & \xleftarrow{N} D_1V \\
 \downarrow b & \downarrow -b' & \downarrow b & \downarrow -b' & \\
 D_0V & \xleftarrow{\mathbb{1}-t} D_0V & \xleftarrow{N} D_0V & \xleftarrow{\mathbb{1}-t} D_0V & \xleftarrow{N} D_0V \\
 \downarrow b & \downarrow -b' & \downarrow b & \downarrow -b' &
 \end{array}$$

and

$$\begin{array}{ccccccc}
 b^*\uparrow & -b'^*\uparrow & & b^*\uparrow & -b'^*\uparrow & & \\
 D^2V & \xrightarrow{\mathbb{1}-t^*} D^2V & \xrightarrow{N^*} D^2V & \xrightarrow{\mathbb{1}-t^*} D^2V & \xrightarrow{N^*} D^2V & & \\
 b^*\uparrow & -b'^*\uparrow & & b^*\uparrow & -b'^*\uparrow & & \\
 CC^*(\mathcal{A}) : D^1V & \xrightarrow{\mathbb{1}-t^*} D^1V & \xrightarrow{N^*} D^1V & \xrightarrow{\mathbb{1}-t^*} D^1V & \xrightarrow{N^*} D^1V & & \\
 b^*\uparrow & -b'^*\uparrow & & b^*\uparrow & -b'^*\uparrow & & \\
 D^0V & \xrightarrow{\mathbb{1}-t^*} D^0V & \xrightarrow{N^*} D^0V & \xrightarrow{\mathbb{1}-t^*} D^0V & \xrightarrow{N^*} D^0V & & \\
 b^*\uparrow & -b'^*\uparrow & & b^*\uparrow & -b'^*\uparrow & &
 \end{array} .$$

Clearly, CC^* is the “pointwise” graded dual to CC and analogously for other bicomplexes we are going to define.

Let $\mathbf{1}$ be a strict unit for \mathcal{A} . We define the Connes’ operator $B : DV \rightarrow DV$ by

$$B := (\mathbb{1} - t) \circ \iota_1 \circ N. \quad (B.6)$$

Connes’ cyclic bicomplexes are defined by

$$\begin{array}{cccc}
 \downarrow b & \downarrow b & \downarrow b & \\
 D_2V & \xleftarrow{B} D_1V & \xleftarrow{B} D_0V & \xleftarrow{B} \\
 \downarrow b & \downarrow b & \downarrow b & \\
 BC(\mathcal{A}) : D_1V & \xleftarrow{B} D_0V & \xleftarrow{B} D_{-1}V & \xleftarrow{B} \\
 \downarrow b & \downarrow b & \downarrow b & \\
 D_0V & \xleftarrow{B} D_{-1}V & \xleftarrow{B} D_{-2}V & \xleftarrow{B} \\
 \downarrow b & \downarrow b & \downarrow b &
 \end{array}$$

and

B. Reduced cyclic homology of A-infinity-algebras

$$\begin{array}{ccccccc}
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 & D^2 V & \xrightarrow{B^*} & D^1 V & \xrightarrow{B^*} & D^0 V & \xrightarrow{B^*} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 BC^*(\mathcal{A}) : & D^1 V & \xrightarrow{B^*} & D^0 V & \xrightarrow{B^*} & D^{-1} V & \xrightarrow{B^*} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 & D^0 V & \xrightarrow{B^*} & D^{-1} V & \xrightarrow{B^*} & D^{-2} V & \xrightarrow{B^*} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow &
 \end{array}$$

We define the normalized Connes' operator $\bar{B} : \bar{D}V \rightarrow \bar{D}V$ by³

$$\bar{B} := \bar{p} \circ B = \bar{p} \circ \iota_1 \circ N. \quad (\text{B.7})$$

The normalized Connes' cyclic bicomplexes are defined by

$$\begin{array}{ccccccc}
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 & \bar{D}_2 V & \xleftarrow{\bar{B}} & \bar{D}_1 V & \xleftarrow{\bar{B}} & \bar{D}_0 V & \xleftarrow{\bar{B}} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 \overline{BC}(\mathcal{A}) : & \bar{D}_1 V & \xleftarrow{\bar{B}} & \bar{D}_0 V & \xleftarrow{\bar{B}} & \bar{D}_{-1} V & \xleftarrow{\bar{B}} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 & \bar{D}_0 V & \xleftarrow{\bar{B}} & \bar{D}_{-1} V & \xleftarrow{\bar{B}} & \bar{D}_{-2} V & \xleftarrow{\bar{B}} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} &
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 & \bar{D}^2 V & \xrightarrow{\bar{B}^*} & \bar{D}^1 V & \xrightarrow{\bar{B}^*} & \bar{D}^0 V & \xrightarrow{\bar{B}^*} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 \overline{BC}^*(\mathcal{A}) : & \bar{D}^1 V & \xrightarrow{\bar{B}^*} & \bar{D}^0 V & \xrightarrow{\bar{B}^*} & \bar{D}^{-1} V & \xrightarrow{\bar{B}^*} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 & \bar{D}^0 V & \xrightarrow{\bar{B}^*} & \bar{D}^{-1} V & \xrightarrow{\bar{B}^*} & \bar{D}^{-2} V & \xrightarrow{\bar{B}^*} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow &
 \end{array} .$$

The reduced Connes' cyclic bicomplexes BC^{red} and BC_{red}^* are defined by replacing $\bar{D}V$ and \bar{D}^*V by $D^{\text{red}}V$ and D_{red}^*V , respectively.

The coordinate $(0, 0)$ in the bicomplexes above always correspond to D^0V in the first column (bottom-left position in the figures).

³The definition does not depend on the chosen section of $\bar{p} : DV \rightarrow \bar{D}V$.

B. Reduced cyclic homology of A-infinity-algebras

Note that an other convention of drawing homological bicomplexes in the left half-plane and cohomological bicomplexes in the right half-plane might be more natural.

Remark B.2.4 (Mixed complexes). One can equivalently encode the data of a cohomological Connes bicomplex into that of a mixed complex (D^*, b^*, B^*) . In general, it is a graded vector space D^* with a differential b^* , $|b^*| = 1$ and a boundary operator B^* , $|B^*| = -1$ which anticommute. One introduces the formal symbol u of degree $|u| = 2$ and considers the polynomial ring $D^*[u]$ in u with values in D^* and the ring of power series $D^*[[u]]$ with the differential $b^* + B^*u$. Clearly, the former is quasi-isomorphic to $\text{Tot}_I(BC^*)$ and the latter to $\text{Tot}_{II}(BC^*)$ (columns of BC^* are indexed with non-negative powers of u). Altogether, there are seven versions $[u]$, $[u^{-1}]$, $[u, u^{-1}]$, $[[u^{-1}, u]$, $[u^{-1}, u]$, $[[u, u^{-1}]$ whose relation is studied in [CVonb]. Some of these might be related to periodic and negative versions of cyclic homology (see [Lod92]). \triangleleft

In the following proofs, we might not need the full strength of Proposition B.2.1 since the spectral sequences for the bicomplexes from Definition B.2.3 mostly collapse already on the second page (see question (iii) of Remark B.3.2).

Lemma B.2.5 (Loday's cyclic bicomplexes and cyclic homology). *Let $\mathcal{A} = (V, (\mu_k))$ be an A_∞ -algebra. The projection $p^\lambda : CC \rightarrow D^\lambda$ to the first column modulo $\text{im}(\mathbb{1} - t)$ is a chain map and induces an isomorphism $H(\widehat{CC}) \simeq H^\lambda(\mathcal{A})$. The inclusion $\iota_\lambda : D_\lambda^* \rightarrow CC^*$ into the first column is a chain map and induces an isomorphism $H_\lambda^*(\mathcal{A}) \simeq H(CC^*)$.*

Proof. The fact that p^λ and ι_λ are chain maps is obvious. We consider the horizontal filtration \mathcal{F}^h of CC^* . Because of (CP1), the rows are acyclic, and we see that the only non-zero terms of the first page of the corresponding spectral sequence are

$$E_1^{0d} = D^d V / \ker(\mathbb{1} - t^*).$$

The differential d_1 is easy to check to be b^* , and the inclusion ι_λ induces the isomorphism $D_\lambda^d \simeq D^d V / \ker(\mathbb{1} - t^*)$. Considering the canonical filtration on D_λ^* , claims (a), (b) and (e) of Proposition B.2.1 apply.

As for homology, we consider the degree reversed cochain complex $r(\text{Tot}_{II} CC)$ and the reversed filtration $r(\mathcal{F}^h)_s = \mathcal{F}_{-s}^h$. For the corresponding cohomological spectral sequence \tilde{E}_r , we have

$$\begin{aligned} \tilde{E}_1^{sd} &= H^{s+d}(r(\mathcal{F})_s r(\text{Tot}_{II} CC) / r(\mathcal{F})_{s+1} r(\text{Tot}_{II} CC)) \\ &= H_{-s-d}(\text{Tot}_{II}(-s\text{-th row of } CC)) \\ &= H(B_{-d, -s}, \partial_h). \end{aligned}$$

B. Reduced cyclic homology of A -infinity-algebras

Therefore, the only groups are $\tilde{E}^{s0} = D_{-s} / \text{im}(\mathbb{1} - t)$, and the spectral sequence converges conditionally to $r(H(\widehat{CC}))$. Clearly, ι_λ induces an isomorphism of the first pages, where on $r(D^\lambda)$ we consider the canonical filtration. Proposition B.2.1 and its proof finishes the argument. \square

Claim (a) of the following is similar to [CVonb, Lemma 2.12].

Lemma B.2.6 (No long chains in homology for bounded degrees). *Let $\mathcal{A} = (V, (\mu_k), 1)$ be a strictly unital A_∞ -algebra. Suppose that V has bounded degrees. Then the canonical inclusion $\text{Tot}_I \hookrightarrow \text{Tot}_{II}$ induces the following isomorphism:*

$$(a) \ H(CC(\mathcal{A})) \simeq H(\widehat{CC}(\mathcal{A})),$$

$$(b) \ H(\overline{BC}(\mathcal{A})) \simeq H(\widehat{\overline{BC}}(\mathcal{A})).$$

Proof. (a) We will denote $CC(\mathcal{A})$, $\overline{BC}(\mathcal{A})$ and DV simply by CC , \overline{BC} and D , respectively. Consider the (increasing) filtration \mathcal{F}_w of D by weights. We first prove the following subclaim:

Subclaim (Weight normalization). Let $c = (c_i)_{i=0}^\infty \in \text{Tot}_{IIk}(CC)$ be a closed chain of degree k ; i.e., for all $i \in \mathbb{N}_0$, we have $c_i \in D_{k-i}$, and the relations

$$bc_{2i} + (\mathbb{1} - t)c_{2i+1} = 0 \quad \text{and} \quad -b'c_{2i+1} + Nc_{2i+2} = 0$$

hold. Suppose that we are given $j \geq 1$ and $n_0 \in \mathbb{N}$ such that $c_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j+1}$. Then we can construct $\tilde{c}_j \in \mathcal{F}_w^{n_0} D_{k-j}$, $\tilde{c}_{j+1} \in D_{k-j-1}$ and $\tilde{z}_{j+1} \in D_{k-j}$ such that if we define

$$\tilde{c}_i := \begin{cases} \tilde{c}_j & \text{for } i = j, \\ \tilde{c}_{j+1} & \text{for } i = j + 1, \\ c_i & \text{otherwise} \end{cases} \quad \text{and} \quad z_i := \begin{cases} \tilde{z}_{j+1} & \text{for } i = j + 1, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{B.8})$$

then $\tilde{c} := (\tilde{c}_i)_{i=0}^\infty$ is a closed chain and $z := (z_i)_{i=0}^\infty$ satisfies $\partial z = c - \tilde{c}$. By repeating this procedure inductively, we obtain chains $c' \in \text{Tot}_{IIk}(CC)$ and $z' \in \text{Tot}_{IIk+1}(CC)$ such that $c'_i \in \mathcal{F}_w^{n_0} D_{k-i}$ for all $i \geq j$ and $c - c' = \partial z'$.

Proof of the Subclaim. We will assume that j is odd; the proof is analogous for j even with the roles of $(\mathbb{1} - t)$ and N , resp. b and $-b'$ switched. The situation is depicted in Figure B.1. Because $c_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j+1}$ and b does not increase weights, we have $bc_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j}$. Since c is closed, we have $(\mathbb{1} - t)c_j = -bc_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j}$. Therefore, there is a $\tilde{c}_j \in \mathcal{F}_w^{n_0} D_{k-j}$ such that $(\mathbb{1} - t)\tilde{c}_j = (\mathbb{1} - t)c_j$. As $c_j - \tilde{c}_j \in \ker(\mathbb{1} - t) = \text{im } N$,

B. Reduced cyclic homology of A -infinity-algebras

$$\begin{array}{ccc}
\overleftarrow{\mathbb{1}-t} & c_j, \tilde{c}_j \in D_{k-j} & \xleftarrow{N} \tilde{z}_{j+1} \in D_{j-k} \\
\downarrow -b' & & \downarrow b \\
& \xleftarrow{N} c_{j+1}, \tilde{c}_{j+1} \in D_{j-k-1} & \\
& & \downarrow b
\end{array}$$

Figure B.1.: Positions of element in CC for j odd.

we have $c_j - \tilde{c}_j = N\tilde{z}_{j+1}$ for some $\tilde{z}_{j+1} \in D_{k-j}$. We define $\tilde{c}_{j+1} := c_{j+1} - b\tilde{z}_{j+1}$. The following relations hold:

- I. $(\mathbb{1} - t)\tilde{c}_j = (\mathbb{1} - t)c_j$,
- II. $-b'\tilde{c}_j + N\tilde{c}_{j+1} = -b'\tilde{c}_j + Nc_{j+1} - Nb\tilde{z}_{j+1}$
 $= -b'\tilde{c}_j + Nc_{j+1} - b'N\tilde{z}_{j+1}$
 $= -b'\tilde{c}_j + Nc_{j+1} - b'(c_j - \tilde{c}_j)$
 $= -b'c_j + Nc_{j+1}$
 $= 0$,
- III. $b\tilde{c}_{j+1} = bc_{j+1}$,
- IV. $N\tilde{z}_{j+1} = c_j - \tilde{c}_j$,
- V. $b\tilde{z}_{j+1} = c_{j+1} - \tilde{c}_{j+1}$.

The relations I–III show that \tilde{c} is closed and the relations IV–V that $\partial z = c - \tilde{c}$.⁴

Starting with $c^1 := \tilde{c}$ and $z^1 := z$, we repeat the construction above to produce the telescopic sequence of homotopies

$$\begin{aligned}
c - c^1 &= \partial z^1 \\
c^1 - c^2 &= \partial z^2 \\
&\dots = \dots
\end{aligned}$$

such that $c_i^l \in \mathcal{F}_w^{n_0} D_{k-i}$ for all $j \leq i \leq j+l$. The limit chain $c' := \lim_{k \rightarrow \infty} c^k \in \text{Tot}_{\text{II}} CC$ has the property that $c_i' \in \mathcal{F}_w^{n_0} D_{k-i}$ for all $i \geq j$, and the sum homotopy $z' := \sum_{k=1}^{\infty} z^k \in \text{Tot}_{\text{II}} CC$ converges and satisfies $\partial z' = c - c'$. (Subclaim) \square

We will now prove surjectivity of the map on homology induced by the inclusion $\text{Tot}_{\text{I}} \hookrightarrow \text{Tot}_{\text{II}}$. Given $[c] \in H_k(\widehat{CC})$, using the Subclaim, we can assume that there is an

⁴In fact, $\partial\tilde{c} = 0$ follows from $\partial c = 0$ and $\partial z = c - \tilde{c}$.

B. Reduced cyclic homology of A -infinity-algebras

$$\begin{array}{ccccc}
c_{j-1} \in \bar{D}_{k-2j+2} & \xleftarrow{\bar{B}} & z_j \in \bar{D}_{k-2j+1} & & \\
\downarrow \text{b} & & \downarrow \text{b} & & \\
& \xleftarrow{\bar{B}} & c_j, \tilde{c}_j \in \bar{D}_{k-2j} & \xleftarrow{\bar{B}} & z_{j+1} \in \bar{D}_{k-2j-1} \\
& & \downarrow \text{b} & & \downarrow \text{b} \\
& & & \xleftarrow{\bar{B}} & c_{j+1}, \tilde{c}_{j+1} \in \bar{D}_{k-2j-2}
\end{array}$$

Figure B.2.: Positions of element in BC for j odd.

$n_0 \in \mathbb{N}$ such that $c_i \in \mathcal{F}_w^{n_0} D_{k-i}$ for all $i \in \mathbb{N}_0$. However, we have $|c_i| = -k + i - 1$ for the degrees in BV , and because the degrees of V are bounded, c_i eventually, as $i \rightarrow \infty$, reach degrees which can not be produced by n_0 vectors. Therefore, there is an $i_0 \in \mathbb{N}_0$ such that $c_i = 0$ for all $i \geq i_0$; this means that $c \in \text{Tot}_I CC$.

To show injectivity of the induced map on homology, suppose that $c \in \text{Tot}_I CC$ satisfies $c = \partial z$ for some $z \in \text{Tot}_{II} CC$. Let $i_0 \in \mathbb{N}$ be such that $c_i = 0$ for all $i \geq i_0$. We use the Subclaim to alter z and obtain a chain $\tilde{z} \in \text{Tot}_I CC$ such that $\tilde{z}_i = z_i$ for $i \leq i_0$ and $\partial \tilde{z} = c$. This shows injectivity.

(b) We will prove an analogy of the Subclaim from (a):

Given a closed $c = (c_i)_{i=0}^\infty \in \text{Tot}_{IIk} \bar{BC}$, every $c_i \in \bar{D}_{k-2i} V$ can be written as

$$c_i = \tilde{c}_i + 1\hat{c}_i$$

for unique $\tilde{c}_i \in D_{q-2i} \bar{V}$ and $\hat{c}_i \in D_{q-2i-1} \bar{V}$. Using strict unitality, we have $\text{b}(1\hat{c}_i) = (\mathbb{1} - t)\hat{c}_i - 1\text{b}'\hat{c}_i$, and hence

$$\begin{aligned}
\text{b}c_i &= \text{b}\bar{c}_i + (\mathbb{1} - t)\hat{c}_i - 1\text{b}'\hat{c}_i, \\
\bar{B}c_i &= 1N\bar{c}_i.
\end{aligned}$$

For the second equality, recall the definition (B.7) and note that the \bar{p} in front “kills” any input of \bar{B} containing at least one 1 . We see that $\partial c = 0$ is equivalent to $\text{b}c_i = -\bar{B}c_{i+1}$ which is equivalent to

$$\begin{aligned}
\text{b}\bar{c}_i + (\mathbb{1} - t)\hat{c}_i &= 0 \quad \text{and} \\
\text{b}'\hat{c}_i &= N\bar{c}_{i+1}
\end{aligned}$$

for all $i \in \mathbb{N}_0$.

Suppose that $c_{j-1} \in \mathcal{F}_w^{n_0} \bar{D}_{k-2j+2}$ for some $j \geq 1$ and $n_0 \in \mathbb{N}_0$. Then $\bar{c}_{j-1} \in$

B. Reduced cyclic homology of A-infinity-algebras

$\mathcal{F}_w^{n_0} \bar{D}_{k-2j+2}$ and $\hat{c}_{j-1} \in \mathcal{F}_w^{n_0-1} \bar{D}_{k-2j+1}$. Because $b'\hat{c}_{j-1} = N\bar{c}_j$, we can find $\bar{d}_j \in \mathcal{F}_w^{n_0-1} D_{k-2j} \bar{V}$ such that $N\bar{d}_j = b'\hat{c}_{j-1}$. Because $\bar{d}_j - \bar{c}_j \in \ker N = \text{im}(\mathbb{1} - t)$, we can find $\hat{z}_j \in \bar{D}_{k-2j} V$ such that $(\mathbb{1} - t)\hat{z}_j = \bar{d}_j - \bar{c}_j$. We compute

$$\begin{aligned} b\bar{d}_j &= b(\bar{c}_j + (\mathbb{1} - t)\hat{z}_j) \\ &= -(\mathbb{1} - t)\hat{c}_j + b(\mathbb{1} - t)\hat{z}_j \\ &= -(\mathbb{1} - t)\hat{c}_j + (\mathbb{1} - t)b'\hat{z}_j \\ &= -(\mathbb{1} - t)(\hat{c}_j - b'\hat{z}_j). \end{aligned}$$

Because $\bar{d}_j \in \mathcal{F}_w^{n_0-1} D_{k-2j} \bar{V}$ and b does not increase the filtration, we can find $\hat{d}_j \in \mathcal{F}_w^{n_0-1} D_{k-2j-1} \bar{V}$ such that $(\mathbb{1} - t)\hat{d}_j = -b\bar{d}_j$. Now, $\hat{d}_j - (\hat{c}_j - b'\hat{z}_j) \in \ker(\mathbb{1} - t) = \text{im } N$, and hence there is a $\bar{z}_{j+1} \in \bar{D}_{k-2j-1}$ such that $N\bar{z}_{j+1} = \hat{d}_j - (\hat{c}_j - b'\hat{z}_j)$. We define the following elements:

$$\begin{aligned} \tilde{c}_j &:= c_j + \bar{B}\bar{z}_{j+1} + b(\mathbb{1}\hat{z}_j) \\ &= c_j + \bar{B}\bar{z}_{j+1} + (\mathbb{1} - t)\hat{z}_j - \mathbb{1}b'\hat{z}_j \\ &= c_j + \mathbb{1}N\bar{z}_{j+1} + (\mathbb{1} - t)\hat{z}_j - \mathbb{1}b'\hat{z}_j \\ &= c_j + \mathbb{1}\hat{d}_j - \mathbb{1}\hat{c}_j + \mathbb{1}b'\hat{z}_j + \bar{d}_j - \bar{c}_j - \mathbb{1}b'\hat{z}_j \\ &= \bar{d}_j + \mathbb{1}\hat{d}_j, \\ \tilde{c}_{j+1} &:= c_{j+1} + b\bar{z}_{j+1}, \\ \tilde{z}_j &:= \mathbb{1}\hat{z}_j, \\ \tilde{z}_{j+1} &:= \bar{z}_{j+1}. \end{aligned}$$

The following relations hold:

- I. $\bar{B}\tilde{c}_j = \bar{B}c_j$,
- II. $b\tilde{c}_j = bc_j + b\bar{B}\bar{z}_{j+1} = -\bar{B}c_{j+1} - \bar{B}b\bar{z}_{j+1} = -\bar{B}\tilde{c}_{j+1}$,
- III. $b\tilde{c}_{j+1} = bc_{j+1}$,
- IV. $\bar{B}\tilde{z}_j = \bar{B}\mathbb{1}\hat{z}_j = 0$,
- V. $b\tilde{z}_j = \tilde{c}_j - c_j - \bar{B}\bar{z}_{j+1}$,
- VI. $b\tilde{z}_{j+1} = \tilde{c}_{j+1} - c_{j+1}$.

Relations I–III show that \tilde{c} is closed, and relations IV–VI show that $\partial z = \tilde{c} - c$. Here \tilde{c}

B. Reduced cyclic homology of A-infinity-algebras

is defined as in (B.8) and z has one more term:

$$z_i := \begin{cases} \tilde{z}_j & \text{for } i = j, \\ \tilde{z}_{j+1} & \text{for } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{c}_j = \bar{d}_j + 1\hat{d}_j$, $\bar{d}_j \in \mathcal{F}_w^{n_0-1}D_{k-2j}\bar{V}$ and $\hat{d}_j \in \mathcal{F}_w^{n_0-1}D_{k-2j-1}\bar{V}$, we have $\tilde{c}_j \in \mathcal{F}_w^{n_0}\bar{D}_{k-2j}$.

Having the recursive step, the rest can be done in the same way as in (a). \square

Lemma B.2.7 (Loday's and Connes' bicomplexes are quasi-isomorphic). *Let $\mathcal{A} = (V, (\mu_k), 1)$ be a strictly unital A_∞ -algebra. The map*

$$\begin{aligned} I : \text{Tot}_\Pi BC &\longrightarrow \text{Tot}_\Pi CC \\ (c_0, c_1, c_2, \dots) &\longmapsto (c_0, \iota_1 N c_1, c_1, \iota_1 N c_2, c_2, \dots) \end{aligned}$$

is a chain map inducing the isomorphisms

$$\mathrm{H}(\widehat{BC}) \simeq \mathrm{H}(\widehat{CC}) \quad \text{and} \quad \mathrm{H}(BC) \simeq \mathrm{H}(CC).$$

Analogously, the map

$$\begin{aligned} P : \text{Tot}_\Pi CC^* &\longrightarrow \text{Tot}_\Pi BC^* \\ (\psi_0, \psi_1, \psi_2, \dots) &\longmapsto (\psi_0, \psi_2 + N^* \iota_1^* \psi_1, \psi_4 + N^* \iota_1^* \psi_3, \dots) \end{aligned}$$

is a chain map inducing the isomorphisms

$$\mathrm{H}(\widehat{BC}^*) \simeq \mathrm{H}(\widehat{CC}^*) \quad \text{and} \quad \mathrm{H}(BC^*) \simeq \mathrm{H}(CC^*).$$

Proof. The following computation shows that ι is a chain map:

$$\begin{aligned} \partial_{CC} I(c_0, c_1, c_2, \dots) &= (\mathrm{bc}_0 + (\mathbb{1} - t)\iota_1 N c_1, -\mathrm{b}'\iota_1 N c_1 + N c_1, \mathrm{bc}_1 + (\mathbb{1} - t)\iota_1 N c_2, \dots) \\ &= (\mathrm{bc}_0 + \mathrm{B}c_1, -N c_1 + \iota_1 \mathrm{b}' N c_1 + N c_1, \mathrm{bc}_1 + \mathrm{B}c_2, \dots) \\ &= (\mathrm{bc}_0 + \mathrm{B}c_1, \iota_1 N \mathrm{bc}_1, \mathrm{bc}_1 + \mathrm{B}c_2, \dots) \\ &= I(\mathrm{bc}_0 + \mathrm{B}c_1, \mathrm{bc}_1 + \mathrm{B}c_2, \dots) \\ &= I\partial_{BC}(c_0, c_1, c_2, \dots). \end{aligned}$$

B. Reduced cyclic homology of A -infinity-algebras

Clearly, I is injective, and hence it induces an isomorphisms of chain complexes

$$(\mathrm{Tot}_{\mathrm{II}} BC, \partial_{BC}) \simeq (\mathrm{im}(I), \partial_{CC}|_{\mathrm{im}(I)}) \subset (\mathrm{Tot}_{\mathrm{II}} CC, \partial_{CC}).$$

Consider the subcomplex $(\mathrm{Tot}_{\mathrm{II}} CC_{\mathrm{odd}, \bullet}, -b') \subset (\mathrm{Tot}_{\mathrm{II}} CC, \partial_{CC})$ which consists of odd columns of CC . It is a direct complement of $\mathrm{im}(I)$ in $\mathrm{Tot}_{\mathrm{II}} CC$. Indeed, $(0, c_1, 0, c_2, \dots) \in \mathrm{im}(I)$ implies $c_i = 0$ for all $i \in \mathbb{N}$, which gives $\mathrm{Tot}_{\mathrm{II}}(CC_{\mathrm{odd}, \bullet}) \cap \mathrm{im}(I) = 0$; also, for any $(c_i) \in \mathrm{Tot}_{\mathrm{II}} CC$, we have

$$(c_0, c_1, c_2, c_3, c_4, \dots) = I((c_0, c_2, c_4, \dots)) - (0, \iota_1 N c_2 - c_1, 0, \iota_1 N c_4 - c_3, 0, \dots),$$

which gives $\mathrm{im}(I) + \mathrm{Tot}_{\mathrm{II}}(CC_{\mathrm{odd}, \bullet}) = \mathrm{Tot}_{\mathrm{II}} CC$. Now, $\mathrm{Tot}_{\mathrm{II}}(CC_{\mathrm{odd}, \bullet})$ is contractible by CP3, and hence $H(\mathrm{Tot}_{\mathrm{II}} BC) \simeq H(\mathrm{Tot}_{\mathrm{II}} CC)$ (using an argument with the long exact sequence in homology). Clearly, I restricts to short chains $\mathrm{Tot}_{\mathrm{I}}$, and thus $H(\mathrm{Tot}_{\mathrm{I}} BC) \simeq H(\mathrm{Tot}_{\mathrm{I}} CC)$ holds too.

A similar discussion applies in cohomology. The following computation shows that P is a chain map:

$$\begin{aligned} & \mathrm{Pd}_{CC}(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \dots) \\ &= P(b^* \psi_0, (\mathbb{1} - t^*) \psi_0 - b' \psi_1, N^* \psi_1 + b^* \psi_2, (\mathbb{1} - t^*) \psi_2 - b' \psi_3, N^* \psi_3 + b^* \psi_4, \dots) \\ &= (b^* \psi_0, N^* \psi_1 + b^* \psi_2 + N^* \iota_1^*((\mathbb{1} - t^*) \psi_0 - b' \psi_1), \\ &\quad N^* \psi_3 + b^* \psi_4 + N^* \iota_1^*((\mathbb{1} - t^*) \psi_2 - b' \psi_3), \dots) \\ &= (b^* \psi_0, B^* \psi_0 + b^* \psi_2 + N^*(\psi_1 - \iota_1^* b' \psi_1), B^* \psi_2 + b^* \psi_4 + N^*(\psi_3 - \iota_1^* b' \psi_3), \dots) \\ &= (b^* \psi_0, B^* \psi_0 + b^*(\psi_2 + N^* \iota_1^* \psi_1), B^* \psi_2 + b^*(\psi_4 + N^* \iota_1^* \psi_3), \dots) \\ &= d_{BC}(\psi_0, \psi_2 + N^* \iota_1^* \psi_1, \psi_2, \psi_4 + N^* \iota_1^* \psi_3, \dots) \\ &= d_{BC}P(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \dots) \end{aligned}$$

The fourth equality uses that $\iota_1 b' + b' \iota_1 = \mathbb{1}$ and $b' N = N b$. Because $(\psi_0, \psi_1, \dots) = P(\psi_0, 0, \psi_1, 0, \dots)$, P is surjective, and hence it induces an isomorphism of cochain complexes $\mathrm{Tot}_{\mathrm{II}} CC^* / \ker(P) \simeq \mathrm{Tot}_{\mathrm{II}} BC^*$. It is easy to see that

$$\ker(P) = \{(0, \psi_1, -N^* \iota_1^* \psi_1, \psi_3, -N^* \iota_1^* \psi_3, \dots)\}$$

B. Reduced cyclic homology of A-infinity-algebras

and that the map

$$\begin{aligned} Z : \text{Tot}_{\Pi}(CC^{\text{odd}, \bullet}) &\longrightarrow \ker(p) \\ (\psi_1, \psi_3, \dots) &\longmapsto (0, \psi_1, -N^* \iota_1^* \psi_1, \psi_3, -N^* \iota_1^* \psi_3, \dots) \end{aligned}$$

is an isomorphism of the cochain complexes

$$(\text{Tot}_{\Pi}(CC^{\text{odd}, \bullet}), -b'^*) \simeq (\ker(P), \partial_{CC}|_{\ker(P)}) \subset (\text{Tot}_{\Pi}CC, \partial_{CC}).$$

Indeed, we have

$$\begin{aligned} \partial_{CC} Z(\psi_1, \psi_3, \dots) &= (0, -b'^* \psi_1, N^* \psi_1 - b^* N^* \iota_1^* \psi_1, -(\mathbb{1} - t^*) N^* \iota_1^* \psi_1 - b'^* \psi_3, \dots) \\ &= (0, -b'^* \psi_1, -N^* \iota_1^* (-b'^* \psi_1), -b'^* \psi_3, \dots) \\ &= Z(-b'^*)(\psi_1, \psi_3, \dots). \end{aligned}$$

Therefore, $\ker P$ is contractible, and the statement is implied by an argument with the long exact sequence in homology. \square

Lemma B.2.8 (Connes' cyclic bicomplexes are quasi-iso to their normalized versions). *Let $\mathcal{A} = (V, (\mu_k), \mathbb{1})$ be a strictly unital A_{∞} -algebra. The projection \bar{p} and the inclusion \bar{i} (see Definition B.2.3) induce the isomorphisms $H(BC) \simeq H(\overline{BC})$ and $H(\widehat{BC}^*) \simeq H(\overline{BC}^*)$, respectively.*

Proof. It follows from CP4 using the spectral sequence associated to the vertical filtration. In cohomology, we use (d) and (f) of Proposition B.2.1.

In homology, we have $\tilde{E}^{sd} = H(BC_{-s, -d}, \partial_v)$ for the reversed spectral sequence (see the proof of Lemma B.2.5), and hence strong convergence is implied by Theorem B.2.1 from the proof of Proposition B.2.1. Claim (e) of Proposition B.2.1 finishes the proof. \square

The following is based on [Lod92, Proposition 2.2.14] and its proof.

Lemma B.2.9 (Reduced Connes' cyclic bicomplexes and cyclic homology are quas-iso). *Let $\mathcal{A} = (V, (\mu_k), \mathbb{1})$ be a strictly unital A_{∞} -algebra. The projection $p^{\lambda} : BC^{\text{red}} \rightarrow \bar{D}^{\lambda}$ to the first column modulo $\text{im}(\mathbb{1} - t)$ is a chain map and induces an isomorphism $H(\widehat{BC}^{\text{red}}) \simeq H(\bar{D}^{\lambda})$ ($=: H^{\lambda, \text{red}}(\mathcal{A})$). The inclusion $\iota_{\lambda} : \bar{D}_{\lambda}^* \rightarrow BC_{\text{red}}^*$ into the first column of BC_{red}^* is a chain map and induces an isomorphism $H(\bar{D}_{\lambda}^*) \simeq H(BC_{\text{red}}^*)$.*

Proof. We start with the cohomology. It is easy to see that ι_1 induces $\iota_1^* : D_{\text{red}}^* V \rightarrow D_{\text{red}}^* V$

B. Reduced cyclic homology of A -infinity-algebras

and that for this map, we have

$$Z := \ker \iota_1^* = \operatorname{im} \iota_1^* \simeq D^* \bar{V}.$$

We consider the following diagonal filtration of BC_{red}^* :

$$\mathcal{F}^s BC_{\text{red}}^* : \begin{array}{ccccccc} & & b^* \uparrow & & b^* \uparrow & & b^* \uparrow \\ & & D_{\text{red}}^{s+2} & \xrightarrow{\bar{B}^*} & D_{\text{red}}^{s+1} & \xrightarrow{\bar{B}^*} & Z^s \xrightarrow{\bar{B}^*} \\ & b^* \uparrow & & b^* \uparrow & & b^* \uparrow & \\ & D_{\text{red}}^{s+1} & \xrightarrow{\bar{B}^*} & Z^s & \xrightarrow{\bar{B}^*} & 0 & \longrightarrow \\ & b^* \uparrow & & \uparrow & & \uparrow & \\ & Z^s & \xrightarrow{\bar{B}^*} & 0 & \longrightarrow & 0 & \longrightarrow \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

The first page of the corresponding spectral sequence consists of the columns

$$E_1^s = H(\operatorname{gr}_s \operatorname{Tot} I) = H(Z^s \xrightarrow{b^*} D_{\text{red}}^{s+1}/Z^{s+1} \xrightarrow{\bar{B}^*} Z^s \xrightarrow{b^*} \dots),$$

where the cochain complex starts in degree s . Because $\iota_1^* b^* = -b'^* \iota_1^* + (\mathbb{1} - t^*)$ and $\bar{B}^* = N^* \iota_1^*$ on D_{red} , we have the commutative diagram

$$\begin{array}{ccccccc} Z^s & \xrightarrow{b^*} & D_{\text{red}}^{s+1}/Z^{s+1} & \xrightarrow{\bar{B}^*} & Z^s & \xrightarrow{b^*} & \dots \\ \downarrow \mathbb{1} & & \downarrow \iota_1^* & & \downarrow \mathbb{1} & & \\ Z^s & \xrightarrow{\mathbb{1}-t^*} & Z^s & \xrightarrow{N^*} & Z^s & \xrightarrow{\mathbb{1}-t^*} & \dots \end{array}$$

Therefore, the only non-zero terms of the first page are

$$E_1^{s,0} = Z_s / \ker(\mathbb{1} - t^*) = \bar{D}_\lambda^s$$

with the differential $d_1 = b^*$. This is precisely the first page of the canonical filtration of \bar{D}_λ^* . The map induced by ι_λ on the first page is the identity, and Proposition B.2.1 implies the rest.

The situation in homology is analogous. We consider the restriction $\iota_1 : D^{\text{red}} \rightarrow D^{\text{red}}$ and the subspace

$$B := \ker \iota_1 = \operatorname{im} \iota_1.$$

The diagonal filtration is now

B. Reduced cyclic homology of A -infinity-algebras

$$\mathcal{F}^s BC^{\text{red}} : \begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow & \\ & 0 & \longleftarrow & 0 & \longleftarrow & B_s & \longleftarrow \\ & \downarrow & & \downarrow & & \downarrow \text{b} & \\ & 0 & \longleftarrow & B_s & \longleftarrow & D_{s-1}^{\text{red}} & \longleftarrow \\ & \downarrow & & \downarrow \text{b} & & \downarrow \text{b} & \\ B_s & \longleftarrow & D_{s-1}^{\text{red}} & \longleftarrow & D_{s-2}^{\text{red}} & \longleftarrow & \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

The reversed filtration $r(\mathcal{F})_s = \mathcal{F}^{-s}$ of the degree reversed cochain complexes $r(\text{Tot}_I)$ satisfies

$$\begin{aligned} \tilde{E}_1^s &= H(r(\mathcal{F})_s r(\text{Tot}_I) / r(\mathcal{F})_{s+1} r(\text{Tot}_I)) \\ &= H(D_{-s-1}^{\text{red}} / B_{-s-1} \xleftarrow{\text{b}} B_{-s} \xleftarrow{\bar{B}} D_{-s-1}^{\text{red}} / B_{-s-1} \xleftarrow{\text{b}} \dots) \end{aligned}$$

where the first group has degree s . We have the commutative diagram

$$\begin{array}{ccccccc} D_{-s-1}^{\text{red}} / B_{-s-1} & \xleftarrow{\text{b}} & B_{-s} & \xleftarrow{\bar{B}} & D_{-s}^{\text{red}} / B_{-s} & \xleftarrow{\text{b}} & \dots \\ \downarrow \iota_1 & & \downarrow \mathbb{1} & & \downarrow \iota_1 & & \\ B^{-s} & \xleftarrow{\mathbb{1}-t} & B^{-s} & \xleftarrow{N} & B^{-s} & \xleftarrow{\mathbb{1}-t} & \dots, \end{array}$$

and thus $\tilde{E}_1^{s,0} = B^{-s} / \text{im}(\mathbb{1} - t) = D_{-s}^{\lambda, \text{red}}$. The rest is as in the case of cohomology. \square

B.3. Final argument and remarks

We are finally in position to prove Proposition 3.3.13 in Part I. Precisely as in the proof sketch, we replace the unit-augmentation sequence (3.30) in Part I, up to a quasi-isomorphism, by a short exact sequence of normalized Connes' cyclic bicomplexes.

Lemma B.3.1 (Short exact sequence of normalized Connes' cyclic bicomplexes). *Let $\mathcal{A} = (V, (\mu_j), \mathbb{1}, \varepsilon)$ be a strictly augmented strictly unital A_∞ -algebra. The short exact sequences of bicomplexes*

$$0 \longrightarrow \overline{BC}(\mathbb{R}) \xrightarrow{u} \overline{BC}(V) \xrightarrow{p^{\text{red}}} BC^{\text{red}} \longrightarrow 0$$

and

$$0 \longrightarrow BC_{\text{red}}^* \xrightarrow{\iota_{\text{red}}} \overline{BC}^*(V) \xrightarrow{u^*} \overline{BC}(\mathbb{R}) \longrightarrow 0$$

B. Reduced cyclic homology of A-infinity-algebras

split. From this, we obtain the following isomorphisms:

$$\begin{aligned} H(\overline{BC}) &\simeq H(BC^{\text{red}}) \oplus H^\lambda(\mathbb{R}), & H(\widehat{\overline{BC}}) &\simeq H(\widehat{BC}^{\text{red}}) \oplus H^\lambda(\mathbb{R}), \\ H(\overline{BC}^*) &\simeq H(BC_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}), & H(\widehat{\overline{BC}}^*) &\simeq H(\widehat{BC}_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}). \end{aligned} \quad (\text{B.9})$$

Proof. Because we have a strict unit and a strict augmentation, the maps $u : D\mathbb{R} \rightarrow DV$ and $\varepsilon : DV \rightarrow D\mathbb{R}$ satisfy $\varepsilon \circ u = \mathbb{1}$. It follows that $DV = \ker \varepsilon \oplus \text{im } u$. The same holds for the induced maps $\bar{u} : \bar{D}\mathbb{R} \rightarrow \bar{D}V$ and $\bar{\varepsilon} : \bar{D}V \rightarrow \bar{D}\mathbb{R}$. We can now define the splitting $r : \overline{BC}(V) \rightarrow \overline{BC}(\mathbb{R})$ of the homological short exact sequence by projecting to $\ker \bar{\varepsilon}$ along $\text{im } \bar{u}$. It is easy to check that it is a morphism of bicomplexes. This dualizes “pointwisely” to cohomology.

Because

$$(\bar{D}\mathbb{R})_i = \begin{cases} 0 & \text{for } i \neq 0, \\ \mathbb{R} & \text{for } i = 0, \end{cases}$$

the bicomplex $\overline{BC}(\mathbb{R})$ is diagonal, and hence Tot_I and Tot_{II} are the same; both compute $H^\lambda(\mathbb{R})$ (using results from the previous sections). The same is true in cohomology. This shows (B.9). \square

We summarize our results in Figure B.3. Having $H^\lambda(\mathcal{A}) \simeq H^{\lambda, \text{red}}(\mathcal{A}) \oplus H^\lambda(\mathbb{R})$, Proposition 3.3.13 in Section 3.3 in Part I follows by dualization.

Remark B.3.2 (Some questions). (i) Suppose that V has bounded degrees and look at Figure B.3. Does $H(\widehat{\overline{CC}}^*) \simeq H(CC^*)$ hold? Does $H(BC^*) \simeq H(\overline{BC}^*)$ hold?

(ii) Does Proposition 3.3.13 in Part I hold for homological unital and homological augmented A_∞ -algebras? A strategy would be to construct a quasi-isomorphic strictly unital and strictly augmented A_∞ -algebra. Does it exist?

(iii) How is it with the conditional and strong convergence of spectral sequences associated to different filtrations of bicomplexes from Definition B.2.3? Because of the simple internal data, lots of them collapse. \triangleleft

B. Reduced cyclic homology of A-infinity-algebras

$$\begin{array}{ccccc}
H^\lambda(\mathcal{A}) & \xrightarrow{\text{Lem. B.2.5}} & H(\widehat{CC}) & \xrightarrow{\text{Lem. B.2.6}} & H(CC) \\
& & \downarrow \text{Lem. B.2.7} & & \downarrow \text{Lem. B.2.7} \\
& & H(\widehat{BC}) & & H(BC) \\
& & & & \downarrow \text{Lem. B.2.8} \\
& & H(\widehat{\overline{BC}}) & \xrightarrow{\text{Lem. B.2.6}} & H(\overline{BC}) \\
& & \downarrow \text{Lem. B.3.1} & & \downarrow \text{Lem. B.3.1} \\
& & H(\widehat{BC}^{\text{red}}) \oplus H^\lambda(\mathbb{R}) & & H(BC^{\text{red}}) \oplus H^\lambda(\mathbb{R}) \\
& & \downarrow \text{Lem. B.2.9} & & \\
& & H^{\lambda, \text{red}}(\mathcal{A}) \oplus H^\lambda(\mathbb{R}) & &
\end{array}$$

$$\begin{array}{ccc}
H(\widehat{CC}^*) & & H(CC^*) \xrightarrow{\text{Lem. B.2.5}} H_\lambda^* \\
\downarrow \text{Lem. B.2.7} & & \downarrow \text{Lem. B.2.7} \\
H(\widehat{BC}^*) & & H(BC^*) \\
\downarrow \text{Lem. B.2.8} & & \\
H(\widehat{\overline{BC}}^*) & & H(\overline{BC}^*) \\
\downarrow \text{Lem. B.3.1} & & \downarrow \text{Lem. B.3.1} \\
H(\widehat{BC}_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}) & & H(BC_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}) \\
& & \downarrow \text{Lem. B.2.9} \\
& & H_{\lambda, \text{red}}^*(\mathcal{A}) \oplus H_\lambda^*(\mathbb{R})
\end{array}$$

Figure B.3.: Isomorphisms of (co)homologies for a strictly unital strictly augmented A_∞ -algebra \mathcal{A} on a graded vector space V . A solid line denotes an isomorphism which is always valid and a dashed line an isomorphism which is valid provided that the degrees of V are bounded. The dotted line denotes the isomorphism obtained by dualizing the corresponding isomorphism in homology under the assumptions that the degrees of V are bounded.

C. Towards an invariant definition of canonical IBL-operations

The canonical IBL-operations q_{210} and q_{120} on cyclic cochains of an odd symplectic vector space V have been defined in coordinates in [CFL15]. In Part I, we used Definition 3.4.1; it takes and gives invariant objects but requires a choice of basis to describe the inner “trace” mechanism. Example A.1.5 in Part I shows that an invariant definition can be obtained from the invariant formalism for evaluation of ribbon graphs developed in Appendix A by plugging in the algebraic Schwarz kernel of the identity as a propagator. In this appendix, we ask the following question.

Question C.0.1. *Do q_{210} and q_{120} arise as a combination of some natural operations coming from the structure of Hochschild cochains on V or from odd symplectic geometry?*

In Section C.1, we define a canonical Lie bracket $[\cdot, \cdot]$ on Hochschild cochains on V with values in V (Definition C.1.2 and Proposition C.1.3), which comes from a natural pre-Lie algebra structure which is visualized as grafting of trees (Equation (C.2), Figure C.1 and Lemma C.1.1). This is, in fact, the Gerstenhaber bracket from [Ger63]. Next, we use the symplectic form to define the operator $+$ which “lowers indices” (Equation C.3) and show that it takes $[\cdot, \cdot]$ to a Lie bracket on cyclic cochains on V with values in \mathbb{R} (Definition C.1.5 and Proposition C.1.6). By writing down everything in coordinates, we show that this Lie bracket is a degree shift of q_{210} (Proposition C.1.8). This answers Question C.0.1 for q_{210} .

In Section C.2, we relate the cobracket q_{120} to the canonical BV-operator Δ_{sym} on “functions” on an odd symplectic vector space (Proposition C.2.4). We do it by rewriting q_{120} in terms of certain double derivative operators on the tensor algebra, which are associated to a given basis (Definition C.2.2). We prove that both definitions are equivalent (Proposition C.2.5). The BV-operator Δ_{sym} can be defined geometrically (see [DJP18]), and the cobracket q_{120} is a factorization of an extension of Δ_{sym} to cyclic invariants with respect to the cyclic shuffle product. We do not know whether this characterization determines q_{120} uniquely, and hence Question C.0.1 for q_{120} remains

open (see Remark C.2.6 for a list of related questions). This section was stimulated by a discussion with J. Pullman and L. Peksov at a winter school in Srn, 2019.

C.1. Bracket and grafting of trees

Let V be a \mathbb{Z} -graded vector space. We define the weight-graded vector spaces

$$DV := \bigoplus_{k=0}^{\infty} \text{Hom}(V^{\otimes k}, V) \quad \text{and} \quad \bar{D}V := \bigoplus_{k=1}^{\infty} \text{Hom}(V^{\otimes k}, V), \quad (\text{C.1})$$

where Hom denotes the graded vector space generated by homogenous morphisms. The latter space is the weight-reduced version of the former (see Definition 3.1.1 in Part I).

For homogenous $\psi_1 \in \text{Hom}(V^{\otimes k_1}, V)$, $\psi_2 \in \text{Hom}(V^{\otimes k_2}, V)$ with $k_1, k_2 \in \mathbb{N}$ and vectors $v_1, \dots, v_{k_1+k_2-1} \in V$, we define¹

$$\begin{aligned} & (\psi_1 * \psi_2)(v_1 \cdots v_{k_1+k_2-1}) \\ &:= \sum_{i=1}^{k_1} (-1)^{\psi_2(v_1+\cdots+v_{i-1})} \psi_1(v_1 \cdots v_{i-1} \psi_2(v_i \cdots v_{k_2+i-1}) v_{k_2+i} \cdots v_{k_1+k_2-1}). \end{aligned} \quad (\text{C.2})$$

Here and almost everywhere in this section, we suppress writing the tensor product. We recall that we use the same symbol v to denote both a homogenous vector and its degree in the exponent. From (C.2), we get a bilinear operation

$$* : DV \otimes DV \longrightarrow DV,$$

which preserves the degree and decreases the weight by 1. It can be visualized as grafting of trees (see Figure C.1).

Lemma C.1.1 (Pre-Lie algebra). *Let V be a graded vector space. The pair $(DV, *)$ is a graded pre-Lie algebra with respect to the degree, i.e., it holds*

$$(\psi_1 * \psi_2) * \psi_3 - \psi_1 * (\psi_2 * \psi_3) = (-1)^{\psi_2 \psi_3} [(\psi_1 * \psi_3) * \psi_2 - \psi_1 * (\psi_3 * \psi_2)]$$

for all homogenous $\psi_1, \psi_2, \psi_3 \in DV$.

Proof. In the following computations, U denotes a tensor product of homogenous vectors from the vector space V and U_i denotes a part of a decomposition of V , i.e., $U = U_1 \cdots U_k$.

¹If $k_1 = 0$ or $k_2 = 0$, we define $*$ to be 0.

C. Towards an invariant definition of canonical IBL-operations

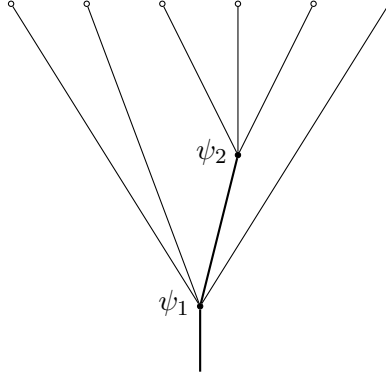


Figure C.1.: The Gerstenhaber bracket as grafting of trees.

We compute schematically

$$\begin{aligned} ((\psi_1 * \psi_2) * \psi_3)(U) = & \sum (-1)^{\psi_3(U_1+U_2+U_3)+\psi_2 U_1} \psi_1(U_1 \psi_2(U_2) U_3 \psi_3(U_4) U_5) \\ & + \sum (-1)^{\psi_3 U_1 + \psi_2(U_1+U_2+U_3+\psi_3)} \psi_1(U_1 \psi_3(U_2) U_3 \psi_2(U_4) U_5) \\ & + \sum (-1)^{\psi_3(U_1+U_2)+\psi_2 U_1} \psi_1(U_1 \psi_2(U_2 \psi_3(U_3) U_4) U_5) \end{aligned}$$

and

$$(\psi_1 * (\psi_2 * \psi_3))(U) = \sum (-1)^{(\psi_2 - \psi_3)U_1 + \psi_3 U_2} \psi_1(U_1 \psi_2(U_2 \psi_3(U_3) U_4) U_5).$$

Therefore, we have

$$\begin{aligned} [((\psi_1 * \psi_2) * \psi_3) - (\psi_1 * (\psi_2 * \psi_3))](U) &= \sum (-1)^{\psi_3(U_1+U_2+U_3)+\psi_2 U_1} \psi_1(U_1 \psi_2(U_2) U_3 \psi_3(U_4) U_5) \\ &+ \sum (-1)^{\psi_3 U_1 + \psi_2(U_1+U_2+U_3+\psi_3)} \psi_1(U_1 \psi_3(U_2) U_3 \psi_2(U_4) U_5) \\ &= (-1)^{\psi_2 \psi_3} \left[\sum (-1)^{\psi_2(U_1+U_2+U_3)+\psi_3 U_1} \psi_1(U_1 \psi_3(U_2) U_3 \psi_2(U_4) U_5) \right. \\ &\quad \left. + \sum (-1)^{\psi_2 U_1 + \psi_3(U_1+U_2+U_3+\psi_2)} \psi_1(U_1 \psi_2(U_2) U_3 \psi_3(U_4) U_5) \right] \\ &= (-1)^{\psi_2 \psi_3} [((\psi_1 * \psi_3) * \psi_2) - (\psi_1 * (\psi_3 * \psi_2))](U). \end{aligned}$$

This finishes the proof. □

Definition C.1.2 (Canonical Lie algebra on Hochschild cochains). *Let V be a graded*

C. Towards an invariant definition of canonical IBL-operations

vector space. For all $\psi_1, \psi_2 \in DV$, we define the bracket by

$$[\psi_1, \psi_2] := \psi_1 * \psi_2 - (-1)^{\psi_1 \psi_2} \psi_2 * \psi_1.$$

Proposition C.1.3 (Canonical Lie algebra on Hochschild cochains). *In the situation of Definition C.1.2, the pair $(DV, [\cdot, \cdot])$ is a graded Lie algebra with bracket $[\cdot, \cdot]$ of degree 0 and weight -1 .*

Proof. The proof is standard. □

Let $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ be a homogenous bilinear form of degree $-n$. This means that

$$\langle v_1, v_2 \rangle \neq 0 \implies v_1 + v_2 = n.$$

To every $\psi \in \text{Hom}(V^{\otimes k}, V)$, we associate $\psi^+ \in \text{Hom}(V^{\otimes k+1}, \mathbb{R})$ defined for all $v_1, \dots, v_{k+1} \in V$ by the formula

$$\psi^+(v_1 \cdots v_{k+1}) := \langle \psi(v_1 \cdots v_k), v_{k+1} \rangle. \quad (\text{C.3})$$

We define the weight-graded vector spaces

$$D^+V := \bigoplus_{k=0}^{\infty} \text{Hom}(V^{\otimes k}, \mathbb{R}) \quad \text{and} \quad \bar{D}^+V := \bigoplus_{k=1}^{\infty} \text{Hom}(V^{\otimes k}, \mathbb{R}). \quad (\text{C.4})$$

We use here the cohomological grading convention; i.e., $\psi \in D^+V$ has degree d and weight k if and only if for all homogenous $v_1, \dots, v_k \in V$, the following implication holds:

$$\psi(v_1 \cdots v_k) \neq 0 \implies v_1 + \cdots + v_k = d.$$

We say that $\varphi \in DV$ is cyclically symmetric if

$$\varphi(v_1 \cdots v_k) = \underbrace{(-1)^{v_k(v_1 + \cdots + v_{k-1})}}_{=:\varepsilon(v_1 \cdots v_k, v_k v_1 \cdots v_{k-1})} \varphi(v_k v_1 \cdots v_{k-1})$$

for all homogenous $v_1, \dots, v_k \in V$. We define the weight-graded vector spaces

$$\begin{aligned} D_{\text{cyc}}^+V &:= \{\varphi \in D^+V \mid \varphi \text{ is cyclically symmetric}\} \quad \text{and} \\ D_{\text{cyc}}V &:= \{\psi \in DV \mid \psi^+ \in D_{\text{cyc}}^+V\}, \end{aligned}$$

and similarly for the reduced versions. Formula (C.3) defines the following linear maps

C. Towards an invariant definition of canonical IBL-operations

of weight $+1$:

$$\begin{array}{ccc} + : & DV & \longrightarrow \bar{D}^+V \\ & \uparrow & \uparrow \\ + := +|_{D_{\text{cyc}}V} : & D_{\text{cyc}}V & \longrightarrow \bar{D}_{\text{cyc}}^+V \end{array} \quad (\text{C.5})$$

This might be related to Remark 3.3.2 in Part I about weight-reduced bar complexes.

Because D^+V has the cohomological grading and DV the standard grading for homogenous maps, $+$ is not homogenous; instead, it satisfies

$$|\psi^+| = n - |\psi|.$$

It is also useful to note that if $\psi^+(v_1 \cdots v_{k+1}) \neq 0$, then

$$\varepsilon(v_1 \cdots v_{k+1}, v_{k+1}v_1 \cdots v_k) = (-1)^{v_{k+1}(\psi^+-1)} = (-1)^{v_{k+1}(\psi-n-1)}.$$

Lemma C.1.4 (Restriction of Lie bracket to cyclic invariants). *Let V be a graded vector space and $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ a homogenous bilinear form of degree $-n$ which is graded antisymmetric. This means that for all homogenous $v_1, v_2 \in V$, we have*

$$\langle v_1, v_2 \rangle = (-1)^{1+v_1v_2} \langle v_2, v_1 \rangle.$$

Then the Lie bracket $[\cdot, \cdot]$ on DV satisfies $[D_{\text{cyc}}V, D_{\text{cyc}}V] \subset D_{\text{cyc}}V$, and thus it restricts to the Lie bracket (which we denote by the same symbol)

$$[\cdot, \cdot] : D_{\text{cyc}}V \otimes D_{\text{cyc}}V \longrightarrow D_{\text{cyc}}V.$$

Proof. For any homogenous $v_1, \dots, v_{k_1+k_2} \in V$, we compute

$$\begin{aligned} & \langle [\psi_1, \psi_2](v_1, \dots, v_{k_1+k_2-1}), v_{k_1+k_2} \rangle \\ &= \sum_{i=1}^{k_1} (-1)^{\psi_2(v_1+\dots+v_{i-1})} \langle \psi_1(v_1 \cdots v_{i-1} \psi_2(v_i \cdots v_{i+k_2-1}) v_{i+k_2} \cdots \\ & \quad v_{k_1+k_2-1}), v_{k_1+k_2} \rangle - (-1)^{\psi_1\psi_2} \sum_{j=1}^{k_2} (-1)^{\psi_1(v_1+\dots+v_{j-1})} \langle \psi_2(v_1 \cdots \\ & \quad v_{j-1} \psi_1(v_j \cdots v_{j+k_1-1}) v_{j+k_1} \cdots v_{k_1+k_2-1}), v_{k_1+k_2} \rangle \end{aligned}$$

C. Towards an invariant definition of canonical IBL-operations

$$\begin{aligned}
&= (-1)^{v_{k_1+k_2}(v_1+\dots+v_{k_1+k_2-1})} \left[\sum_{i=1}^{k_1-1} (-1)^{\psi_2(v_{k_1+k_2}+v_1+\dots+v_{i-1})} \right. \\
&\quad \langle \psi_1(v_{k_1+k_2}v_1 \dots \psi_2(v_i \dots v_{i+k_2-1})v_{i+k_2} \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle \\
&\quad + (-1)^{\psi_2(v_1+\dots+v_{k_1-1})+v_{k_1+k_2}\psi_2} \langle \psi_1(v_{k_1+k_2}v_1 \dots v_{k_1-1}), \psi_2(v_{k_1} \dots \\
&\quad v_{k_1+k_2-1}) \rangle - (-1)^{\psi_1\psi_2} \left(\sum_{j=1}^{k_2-1} (-1)^{\psi_1(v_{k_1+k_2}+v_1+\dots+v_{j-1})} \langle \psi_2(v_1 \dots \psi_1(v_j \dots \right. \\
&\quad v_{j+k_1-1})v_{j+k_1} \dots v_{k_1+k_2-1}), v_{k_1+k_2} \rangle + (-1)^{\psi_1(v_1+\dots+v_{k_2-1})+v_{k_1+k_2}\psi_1} \\
&\quad \left. \left. \langle \psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1}), \psi_1(v_{k_2}, \dots, v_{k_1+k_2-1}) \rangle \right) \right].
\end{aligned}$$

Using the graded antisymmetry, we have

$$\begin{aligned}
&(-1)^{\psi_1(v_1+\dots+v_{k_2-1}+v_{k_1+k_2})} \langle \psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1}), \psi_1(v_{k_2}, \dots, v_{k_1+k_2-1}) \rangle \\
&= (-1)^{1+\psi_1\psi_2+(\psi_2+v_{k_1+k_2}+v_1+\dots+v_{k_2-1})(v_{k_2}+\dots+v_{k_1+k_2-1})} \\
&\quad \langle \psi_1(v_{k_2} \dots v_{k_1+k_2-1}), \psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1}) \rangle \\
&= (-1)^{1+\psi_1\psi_2} \langle \psi_1(\psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1})v_{k_2} \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle \\
&=: (*),
\end{aligned}$$

and similarly

$$\begin{aligned}
&(-1)^{\psi_2(v_1+\dots+v_{k_1-1}+v_{k_1+k_2})} \langle \psi_1(v_{k_1+k_2}v_1 \dots v_{k_1-1}), \psi_2(v_{k_1} \dots v_{k_1+k_2-1}) \rangle \\
&= (-1)^{1+\psi_1\psi_2} \langle \psi_2(\psi_1(v_{k_1+k_2}v_1 \dots v_{k_1-1})v_{k_1} \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle.
\end{aligned}$$

We see that

$$(*) = (-1)^{v_{k_1+k_2}(v_1+\dots+v_{k_1+k_2-1})} \langle [\psi_1, \psi_2](v_{k_1+k_2}v_1 \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle.$$

This proves the lemma. \square

The horizontal arrows in (C.5) become isomorphisms provided that $\langle \cdot, \cdot \rangle$ is non-degenerate (injectivity) and V is of finite type (surjectivity). Therefore, the following definition is possible.

Definition C.1.5 (Candidate for IBL-product). *Let V be a graded vector space of finite type, and let $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ be a non-degenerate homogenous graded antisymmetric bilinear form of degree $-n$. Let ν be a formal symbol of degree n realizing the degree shift*

C. Towards an invariant definition of canonical IBL-operations

by $-n$. We define the operation

$$\mathfrak{q}_{210}^* : (\bar{D}_{\text{cyc}}^+ V)[-n] \otimes (\bar{D}_{\text{cyc}}^+ V)[-n] \longrightarrow (\bar{D}_{\text{cyc}}^+ V)[-n]$$

for all $\psi_1, \psi_2 \in D_{\text{cyc}} V$ by

$$\mathfrak{q}_{210}^*(\nu\psi_1^+ \otimes \nu\psi_2^+) := \nu[\psi_2, \psi_1]^+.$$

Proposition C.1.6 (Candidate for IBL-product). *In the situation of Definition C.1.5, the pair $((\bar{D}_{\text{cyc}}^+ V)[-n], \mathfrak{q}_{210}^*)$ is a graded Lie algebra with the bracket \mathfrak{q}_{210}^* of degree $-2n$ and weight -2 .*

Proof. As for the degree, we have

$$\begin{aligned} |\nu[\psi_2, \psi_1]^+| &= n + |[\psi_2, \psi_1]^+| \\ &= n - |[\psi_2, \psi_1]| + n \\ &= n - |\psi_1| - |\psi_2| + n \\ &= n + |\psi_1^+| + |\psi_2^+| - n \\ &= -n + |\nu\psi_1^+| + |\nu\psi_2^+| - n \\ &= |\nu\psi_1^+| + |\nu\psi_2^+| - 2n \end{aligned}$$

The weights are clear. As for the graded anticommutativity, we have

$$\begin{aligned} \mathfrak{q}_{210}^*(\nu\psi_2^+ \otimes \nu\psi_1^+) &= \nu[\psi_1, \psi_2]^+ \\ &= -(-1)^{|\psi_1\psi_2|} \nu[\psi_2, \psi_1]^+ \\ &= -(-1)^{|\nu\psi_1| |\nu\psi_2|} \mathfrak{q}_{210}^*(\nu\psi_1^+ \otimes \nu\psi_2^+), \end{aligned}$$

where we used that

$$|\nu\psi^+| = n + |\psi^+| = n + n - |\psi| = 2n - |\psi|.$$

From the same reason, the graded Jacobi identity for \mathfrak{q}_{210}^* is implied by the graded Jacobi identity for $[\cdot, \cdot]$. \square

In the situation of Definition C.1.5, let (e_i) be a basis of V , and let (e^i) be the dual basis of V such that $\langle e_i, e^j \rangle = \delta_i^j$. To an index i , we will assign the degree of e_i , so that, e.g., we can write $(-1)^i$ instead of $(-1)^{e_i}$. We will also use the Einstein summation convention.

C. Towards an invariant definition of canonical IBL-operations

We define the coordinates

$$\begin{aligned} g_{ij} &= \langle e_i, e_j \rangle, \\ g^{ij} &= \langle e^i, e^j \rangle, \\ \psi(e_{i_1} \cdots e_{i_{k_1}}) &= \psi_{i_1 \cdots i_{k_1}}^i e_i, \\ \psi^+(e_{i_1} \cdots e_{i_{k_1+1}}) &= \psi_{i_1 \cdots i_{k_1+1}}, \end{aligned}$$

where $\psi \in DV$ and $\psi^+ \in D^+V$. It holds

$$g_{ij} = (-1)^{ij+1} g_{ji}, \quad g^{ij} = (-1)^{ij+1} g^{ji} \quad \text{and} \quad g_{ik} g^{jk} = \delta_i^j.$$

Equation (C.3) is now equivalent to

$$\psi_{i_1 \cdots i_{k_1}}^i g_{ii_{k_1+1}} = \psi_{i_1 \cdots i_{k_1+1}}, \quad \text{or to} \quad \psi_{i_1 \cdots i_{k_1}}^i = \psi_{i_1 \cdots i_{k_1}} g^{ik}. \quad (\text{C.6})$$

We define the operation $\mu : (\bar{D}_{\text{cyc}}^+ V)^{\otimes 2} \rightarrow \bar{D}_{\text{cyc}}^+ V$ for $\varphi_1 \in \text{Hom}_{\text{cyc}}(V^{k_1+1}, \mathbb{R})$, $\varphi_2 \in \text{Hom}_{\text{cyc}}(V^{\otimes k_2+1}, \mathbb{R})$ with $k_1, k_2 \in \mathbb{N}_0$ such that $k_1 + k_2 \geq 1$ in coordinates by the formula

$$\begin{aligned} \mu(\varphi_1 \otimes \varphi_2)_{i_1 \cdots i_{k_1+k_2}} &:= \sum_{i,j} \sum_{c=1}^{k_1+k_2} \varepsilon(i_1 \cdots i_{k_1+k_2}, i_c \cdots i_{c-1}) (-1)^{(\varphi_1-i)j} \\ &\quad (-1)^i g^{ij} (\varphi_1)_{ii_c \cdots i_{c+k_1-1}} (\varphi_2)_{ji_{c+k_1} \cdots i_{c-1}}. \end{aligned} \quad (\text{C.7})$$

Remark C.1.7 (Comparison to [CFL15]). In order to compare the μ from (C.7) to the μ from [CFL15, Section 10] (and to the definition of \mathfrak{q}_{210} from Section 3.4 in Part I), we must make the replacements

$$V \mapsto V[1], \quad n \mapsto n-2 \quad \text{and} \quad \bar{D}_{\text{cyc}}^+ V \mapsto B_{\text{cyc}}^* V.$$

Then \mathfrak{q}_{210} and $\mathfrak{q}_{210}^* : (B_{\text{cyc}}^* V)[2-n]^{\otimes 2} \rightarrow (B_{\text{cyc}}^* V)[2-n]$ clearly have the same degree $2(2-n)$.² Also, $\varphi_1 j + i j$ is precisely the sign needed to move j over $i_c \cdots i_{c+k_1-1}$. We also recall the identification of $B_{\text{cyc}}^* V$ with the weight-graded dual $(B^{\text{cyc}} V)^{*_{\text{wg}}}$ from Remark 3.3.5 in Part I and remind that $B^{\text{cyc}} V$ and $B_{\text{cyc}}^* V$ were defined to be weight-reduced in order to ease the notation (see Definition 3.3.1 in Part I). \triangleleft

²The degree of \mathfrak{q}_{210} on tensor powers of $C = B_{\text{cyc}}^* V[2-n]$, is obtained from the degree on $EC = S(C[1])$ in Definition 3.4.1 in Part I by subtracting 1 (and by adding 1 for the cobracket).

C. Towards an invariant definition of canonical IBL-operations

Taking into account Remark C.1.7, the equality

$$\bar{\pi} \circ \mathfrak{q}_{210}^* \circ (\bar{\iota} \otimes \bar{\iota}) = \mathfrak{q}_{210},$$

where $\bar{\pi} : D_{\text{cyc}}^+ V \rightarrow \bar{D}_{\text{cyc}}^+ V$ and $\bar{\iota} : \bar{D}_{\text{cyc}}^+ V \rightarrow D_{\text{cyc}}^+ V$ are the canonical projection and inclusion, respectively, is obtained from the next proposition.

Proposition C.1.8 (Equivalence of definitions of IBL-product). *Let V be a graded vector space of finite type, and let $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ be a non-degenerate homogenous graded antisymmetric bilinear form of degree $-n$. Then the operation $\mathfrak{q}_{210}^* : (D_{\text{cyc}}^+ V)[-n]^{\otimes 2} \rightarrow (D_{\text{cyc}}^+ V)[-n]$ defined invariantly in Definition C.1.5 is a degree shift of the operation $\mu : (\bar{D}_{\text{cyc}}^+ V)^{\otimes 2} \rightarrow \bar{D}_{\text{cyc}}^+ V$ defined in coordinates above. More precisely, for all $\varphi_1, \varphi_2 \in \bar{D}_{\text{cyc}}^+ V$, it holds*

$$\mathfrak{q}_{210}^*(\nu^2 \varphi_1 \otimes \varphi_2) = \nu \mu(\varphi_1 \otimes \varphi_2) \quad \text{in } (\bar{D}_{\text{cyc}}^+ V)[-n].$$

Proof. Consider $\psi_1, \psi_2 \in D_{\text{cyc}} V$ with k_1, k_2 inputs, respectively. Writing the definition of $[\cdot, \cdot]$ in coordinates, we get

$$\begin{aligned} & [\psi_1, \psi_2]_{i_1 \dots i_{k_1+k_2-1}}^i \\ &= \sum_{j=1}^{k_1} (-1)^{\psi_2(i_1+\dots+i_{j-1})} (\psi_1)_{i_1 \dots i_{j-1} k i_{j+k_2} \dots i_{k_1+k_2-1}}^i (\psi_2)_{i_j \dots i_{j+k_2-1}}^k \\ &+ \sum_{j=1}^{k_2} (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1} (\psi_2)_{i_1 \dots i_{j-1} k i_{j+k_1} \dots i_{k_1+k_2-1}}^i (\psi_1)_{i_j \dots i_{j+k_1-1}}^k. \end{aligned}$$

Using (C.6) and the cyclic symmetry of ψ_1^+ and ψ_2^+ , we compute

$$\begin{aligned} & [\psi_1, \psi_2]_{i_1 \dots i_{k_1+k_2}} \\ &= [\psi_1, \psi_2]_{i_1 \dots i_{k_1+k_2-1}}^i g_{ii_{k_1+k_2}} \\ &= \sum_{j=1}^{k_1} (-1)^{\psi_2(i_1+\dots+i_{j-1})} (\psi_1)_{i_1 \dots i_{j-1} k i_{j+k_2} \dots i_{k_1+k_2}} (\psi_2)_{i_j \dots i_{j+k_2-1}}^k \\ &+ \sum_{j=1}^{k_2} (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1} (\psi_2)_{i_1 \dots i_{j-1} k i_{j+k_1} \dots i_{k_1+k_2}} (\psi_1)_{i_j \dots i_{j+k_1-1}}^k \\ &= \sum_{j=1}^{k_1} (-1)^{\psi_2(i_1+\dots+i_{j-1})} g^{ab} (\psi_1)_{i_1 \dots i_{j-1} a i_{j+k_2} \dots i_{k_1+k_2}} (\psi_2)_{i_j \dots i_{j+k_2-1}}^b \\ &+ \sum_{j=1}^{k_2} (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1} g^{ab} (\psi_2)_{i_1 \dots i_{j-1} a i_{j+k_1} \dots i_{k_1+k_2}} (\psi_1)_{i_j \dots i_{j+k_1-1}}^b \end{aligned}$$

C. Towards an invariant definition of canonical IBL-operations

$$\begin{aligned}
&= \sum_{j=1}^{k_1} \underbrace{(-1)^{\psi_2(i_1+\dots+i_{j-1})+(i_1+\dots+i_{j-1})(\psi_1-n-1)+b(\psi_2-n-1)}}_{=:\varepsilon_1} \\
&\quad g^{ab}(\psi_1) a i_{j+k_2} \dots i_{k_1+k_2} i_1 \dots i_{j-1} (\psi_2) b i_j \dots i_{j+k_2-1} \\
&+ \sum_{j=1}^{k_2} \underbrace{(-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1+(i_1+\dots+i_{j-1})(\psi_2-n-1)+a(\psi_1-n-1)+ab+1}}_{=:\varepsilon_2} \\
&\quad g^{ab}(\psi_1) a i_j \dots i_{j+k_1-1} (\psi_2) b i_{j+k_1} \dots i_{k_1+k_2} i_1 \dots i_{j-1}.
\end{aligned}$$

Notice that the lower indices $i_1, \dots, i_{k_1+k_2}$ appear in cyclic permutations. The first sum consists of cyclic permutation starting with i_{k_2+1} and going up to $i_{k_1+k_2+1}$, and the second sum consists of cyclic permutations starting with i_1 and going up to i_{k_2} ; hence, all cyclic permutations appear, and it just remains to check the signs. We see that

$$\begin{aligned}
\varepsilon_1 &= (-1)^{\psi_2(i_1+\dots+i_{j-1})+(i_1+\dots+i_{j-1})(\psi_1-n-1)+b(\psi_2-n-1)} \\
&= \underbrace{(-1)^{(i_1+\dots+i_{j-1}+\psi_2-n-b)(\psi_1-n-a+\psi_2-n-b-1)}}_{=:\varepsilon(i_1 \dots i_{k_1+k_2}, i_{j+k_2} \dots i_{k_1+k_2} i_1 \dots i_{j+k_2-1})} (-1)^{b(\psi_1-n-a)} (-1)^a \\
&\quad (-1)^{n(\psi_1-n)+\psi_2\psi_1}
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_2 &= (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+(i_1+\dots+i_{j-1})(\psi_2-n-1)+a(\psi_1-n-1)+ab} \\
&= \underbrace{(-1)^{(i_1+\dots+i_{j-1})(\psi_1-n-a+\psi_2-n-b-1)}}_{=:\varepsilon(i_1 \dots i_{k_1+k_2}, i_j \dots i_{k_1+k_2} i_1 \dots i_{j-1})} (-1)^{b(\psi_1-n-a)} (-1)^a \\
&\quad (-1)^{n(\psi_1-n)+\psi_1\psi_2}.
\end{aligned}$$

This finishes the proof. \square

C.2. Cobracket and odd symplectic geometry

We start with a remark about the naive dualization of the bracket on the dual.

Remark C.2.1 (Naive dualization). If we ignore the fact that the dualization of spaces like DV is problematic and we might not get $(DV^{*\mathfrak{g}})^{*_{\mathfrak{g}}} \simeq DV$,³ a naive way to obtain a cobracket $\bar{\delta} : DV \rightarrow DV \otimes DV$ would be to dualize the canonical bracket $[\cdot, \cdot]^* : DV^{*\mathfrak{g}} \otimes$

³It holds $V \simeq (V^{*\mathfrak{g}})^{*_{\mathfrak{g}}}$ provided that V is of finite type. The tensor product of graded vector spaces of finite type is also of finite type provided that the grading is, e.g., bounded from below. Moreover, one has to require that $W^0 = W^1 = 0$ for W for which $V = W[1]$ or take suitable completions with respect to weights in order to deal with arbitrary long tensor products in the same degree in the dual of DV .

C. Towards an invariant definition of canonical IBL-operations

$DV^{*\mathfrak{g}} \rightarrow DV^{*\mathfrak{g}}$ from the previous section (replacing V by $V^{*\mathfrak{g}}$). Because $w([\cdot, \cdot]^*) = -1$, the dual would have $w(\bar{\delta}) = 1$, where w denotes the weights. However, the IBL-cobacket has $w(\mathfrak{q}_{120}) = -2$ (see Definition 3.4.1 in Part I), and hence

$$w(((+)^{-1} \otimes (+)^{-1}) \circ \mathfrak{q}_{120} \circ +) = w(\mathfrak{q}_{120}) - 1 = -3 \neq 1 = w(\bar{\delta}),$$

where the map $^+ : DV \rightarrow D^+V$ of weight $+1$ was defined in (C.5). It might be interesting to write down the coordinate expression for such $\bar{\delta}$ which comes from \mathfrak{q}_{120} and study its origin and properties on DV . \triangleleft

In the following definition, we will use the terms “cobacket” and “BV-operator” to denote some operations without actually knowing that they satisfy the corresponding relations (see Remark C.2.6 at the end of this section).

Definition C.2.2 (Hochschild cobacket and BV-operator). *Let V be a graded vector space, let $(e_i) \in V$ be its homogenous basis, and let $(\eta^i) \subset V^{*\mathfrak{g}}$ be the dual basis, i.e., it holds $\eta^i(e_j) = \delta_j^i$. We denote by $\mathrm{TV}^{*\mathfrak{g}}$ the tensor algebra over $V^{*\mathfrak{g}}$ and consider its basis $\eta^{\otimes I} := \eta^{i_1} \otimes \cdots \otimes \eta^{i_k}$ for all multiindices $I = (i_1, \dots, i_k)$.*

For each i, j , we define the cyclic double derivative $\mathrm{Der}_{ij} : \mathrm{TV}^{\mathfrak{g}} \rightarrow \mathrm{TV}^{*\mathfrak{g}} \otimes \mathrm{TV}^{*\mathfrak{g}}$ on the basis by the following formula, which we explain below:*

$$\mathrm{Der}_{ij}(\eta^{\otimes I}) := \sum_{(i,j) \in I} \frac{1}{k_1} \frac{1}{k_2} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\mathrm{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\mathrm{cyc}}}} \varepsilon(I, ij I_1^{c_1} I_2^{c_2}) (\eta^{\otimes I_1^{c_1}} \otimes \eta^{\otimes I_2^{c_2}}) \quad (\text{C.8})$$

The first sum is over all pairs of indices (i, j) at two distinct positions in I . We consider the cyclic order on positions in I (of length k) starting from the left and define I_1 and I_2 as the substrings between i and j (of length k_1) and between j and i (of length k_2), respectively. We include neither i nor j in I_1 or I_2 , i.e., $k = k_1 + k_2 + 2$. The second sum is over all cyclic permutations c_1 and c_2 of I_1 and I_2 , respectively. As usual, ε stands for the Koszul sign and i has the degree of e_i .

On $\mathrm{TV}^{\mathfrak{g}}$, we consider the shuffle product $\mu^{\mathrm{sh}} : \mathrm{TV}^{*\mathfrak{g}} \otimes \mathrm{TV}^{*\mathfrak{g}} \rightarrow \mathrm{TV}^{*\mathfrak{g}}$ which is defined by*

$$\mu^{\mathrm{sh}}(\eta^{\otimes I_1}, \eta^{\otimes I_2}) := \frac{k_1! k_2!}{(k_1 + k_2)!} \sum_{\mu \in \mathbb{S}_{k_1, k_2}} \varepsilon(I_1 I_2, \eta(I_1 I_2)) \eta^{\otimes \eta(I_1 I_2)},$$

where k_1 and k_2 are the lengths of I_1 and I_2 , respectively, and \mathbb{S}_{k_1, k_2} denotes the shuffle permutations.

If V is of finite type and $\langle \cdot, \cdot \rangle$ is a non-degenerate homogenous bilinear form on V , we consider the coordinates $g^{ij} = \langle e^i, e^j \rangle$, where $(e^i) \subset V$ is the basis such that $\langle e^i, e_j \rangle = \delta_j^i$.

C. Towards an invariant definition of canonical IBL-operations

We define the Hochschild cobracket $\zeta_H : TV^{*\mathfrak{g}} \rightarrow TV^{*\mathfrak{g}} \otimes TV^{*\mathfrak{g}}$ by

$$\zeta_H = \frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} \text{Der}_{ij} \quad (\text{C.9})$$

and the Hochschild BV-operator $\Delta_H : TV^{*\mathfrak{g}} \rightarrow TV^{*\mathfrak{g}}$ by

$$\Delta_H := \mu^{\text{sh}} \circ \zeta_H. \quad (\text{C.10})$$

It is easy to check that the definitions of μ^{sh} , ζ_H and Δ_H do not depend on the choice of the basis (e_i) of V .

We denote by $(TV^{*\mathfrak{g}})_{\text{cyc}}$ the subspace of cyclic invariants of $TV^{*\mathfrak{g}}$ and by $(TV^{*\mathfrak{g}})_{\text{sym}}$ the subspace of symmetric invariants. We consider the cyclization map $\pi_{\text{cyc}} : TV^{*\mathfrak{g}} \rightarrow (TV^{*\mathfrak{g}})_{\text{cyc}}$ and the symmetrization map $\pi_{\text{sym}} : TV^{*\mathfrak{g}} \rightarrow (TV^{*\mathfrak{g}})_{\text{sym}}$; they are defined by

$$\begin{aligned} \pi_{\text{cyc}}(\eta^{\otimes I}) &= \frac{1}{k} \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \varepsilon(I, I^c) \eta^{\otimes I^c} \quad \text{and} \\ \pi_{\text{sym}}(\eta^{\otimes I}) &= \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \varepsilon(I, I^\sigma) \eta^{\otimes I^\sigma}, \end{aligned}$$

respectively, where k is the length of I . We denote by p_{cyc} the projection to cyclic coinvariants and by p_{sym} the projection to symmetric coinvariants.

Definition C.2.3 (Cyclic and symmetric versions). *In the situation of Definition C.2.2, we define the cyclic and symmetric versions of the operations Der_{ij} , ζ_H , μ^{sh} and Δ_H by precomposing with the inclusions $\iota_{\text{cyc}} : (TV^{*\mathfrak{g}})_{\text{cyc}} \rightarrow TV^{*\mathfrak{g}}$ and $\iota_{\text{sym}} : (TV^{*\mathfrak{g}})_{\text{sym}} \rightarrow TV^{*\mathfrak{g}}$ and postcomposing with the projections $\pi_{\text{cyc}} : TV^{*\mathfrak{g}} \rightarrow (TV^{*\mathfrak{g}})_{\text{cyc}}$ and $\pi_{\text{sym}} : TV^{*\mathfrak{g}} \rightarrow (TV^{*\mathfrak{g}})_{\text{sym}}$, respectively. In formulas, we have*

$$\begin{aligned} D_{ij}^{\text{cyc}} &:= (\pi_{\text{cyc}} \otimes \pi_{\text{cyc}}) \circ \text{Der}_{ij} \circ \iota_{\text{cyc}}, & D_{ij}^{\text{sym}} &:= (\pi_{\text{sym}} \otimes \pi_{\text{sym}}) \circ \text{Der}_{ij} \circ \iota_{\text{sym}}, \\ \zeta_{\text{cyc}} &:= (\pi_{\text{cyc}} \otimes \pi_{\text{cyc}}) \circ \zeta_H \circ \iota_{\text{cyc}}, & \zeta_{\text{sym}} &:= (\pi_{\text{sym}} \otimes \pi_{\text{sym}}) \circ \zeta_H \circ \iota_{\text{sym}}, \\ \mu_{\text{cyc}} &:= \pi_{\text{cyc}} \circ \mu^{\text{sh}} \circ (\iota_{\text{cyc}} \otimes \iota_{\text{cyc}}), & \mu_{\text{sym}} &:= \pi_{\text{sym}} \circ \mu^{\text{sh}} \circ (\iota_{\text{sym}} \otimes \iota_{\text{sym}}), \\ \Delta_{\text{cyc}} &:= \pi_{\text{cyc}} \circ \Delta_H \circ \iota_{\text{cyc}}, & \Delta_{\text{sym}} &:= \pi_{\text{sym}} \circ \Delta_H \circ \iota_{\text{sym}}. \end{aligned} \quad (\text{C.11})$$

Proposition C.2.4 (Hochschild, cyclic and symmetric operations). *In the situation of Definition C.2.3, we consider for every multiindex I the linear functional $\alpha^I : TV \rightarrow \mathbb{R}$ given with respect to the basis $e_{\otimes J}$ by*

$$\alpha^I(e_{\otimes J}) := \varepsilon(I, J).$$

C. Towards an invariant definition of canonical IBL-operations

We set

$$\alpha_{\text{sym}}^I = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \varepsilon(I, I^\sigma) \alpha^{I^\sigma} \quad \text{and} \quad \alpha_{\text{cyc}}^I = \frac{1}{k} \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \varepsilon(I, I^c) \alpha^{I^c}.$$

We define the monomorphisms I , I_{cyc} and I_{sym} by

$$\begin{array}{ccc} TV^{*\mathfrak{g}} & \xhookrightarrow{I} & D^+V : & \eta^{\otimes I} & \hookrightarrow & \alpha^I, \\ \uparrow & & \uparrow & & & \\ (TV^{*\mathfrak{g}})_{\text{cyc}} & \xhookrightarrow{I_{\text{cyc}}} & D_{\text{cyc}}^+V : & \pi_{\text{cyc}}(\eta^{\otimes I}) & \hookrightarrow & \alpha_{\text{cyc}}^I, \\ \uparrow & & \uparrow & & & \\ (TV^{*\mathfrak{g}})_{\text{sym}} & \xhookrightarrow{I_{\text{sym}}} & D_{\text{sym}}^+V : & \pi_{\text{sym}}(\eta^{\otimes I}) & \hookrightarrow & \alpha_{\text{sym}}^I. \end{array} \quad (\text{C.12})$$

The vertical arrows are inclusions and the diagram commutes. Moreover, I , I_{cyc} and I_{sym} become isomorphisms provided that V is of finite type with degrees bounded from below.⁴ In this case, we transfer the operations and obtain the following commutative diagram:

$$\begin{array}{ccccc} \Delta_{\text{H}} : & D^+V & \xrightarrow{\zeta_{\text{H}}} & D^+V \otimes D^+V & \xrightarrow{\mu^{\text{sh}}} & D^+V \\ & \uparrow & & \uparrow & & \uparrow \\ \Delta_{\text{cyc}} : & D_{\text{cyc}}^+V & \xrightarrow{\zeta_{\text{cyc}}} & D_{\text{cyc}}^+V \otimes D_{\text{cyc}}^+V & \xrightarrow{\mu_{\text{cyc}}} & D_{\text{cyc}}^+V \\ & \uparrow & & \uparrow & & \uparrow \\ \Delta_{\text{sym}} : & D_{\text{sym}}^+V & \xrightarrow{\zeta_{\text{sym}}} & D_{\text{sym}}^+V \otimes D_{\text{sym}}^+V & \xrightarrow{\mu_{\text{sym}}} & D_{\text{sym}}^+V. \end{array} \quad (\text{C.13})$$

The following formulas hold:

$$\begin{aligned} \zeta_{\text{cyc}}(\alpha_{\text{cyc}}^I) &= \frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} \sum_{(i,j) \in I} \varepsilon(I, ijI_1I_2) (\alpha_{\text{cyc}}^{I_1} \otimes \alpha_{\text{cyc}}^{I_2}), \\ \mu_{\text{sym}}(\alpha_{\text{sym}}^{I_1}, \alpha_{\text{sym}}^{I_2}) &= \alpha_{\text{sym}}^{I_1I_2}, \\ \Delta_{\text{sym}}(\alpha_{\text{sym}}^I) &= \frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} \sum_{(i,j) \in I} \varepsilon(I, ijI_1I_2) \alpha_{\text{sym}}^{I_1I_2}. \end{aligned} \quad (\text{C.14})$$

Proof. The commutativity of (C.12) is clear from the definitions. The commutativity of (C.13) is equivalent to the commutativity of the same diagram with D^+V , D_{cyc}^+V , D_{sym}^+V replaced by $TV^{*\mathfrak{g}}$, $(TV^{*\mathfrak{g}})_{\text{cyc}}$, $(TV^{*\mathfrak{g}})_{\text{sym}}$, respectively. This commutativity then follows from the defining relations (C.11) if we show the following inclusions:

$$(1) \quad \text{Der}_{ij}((TV^{*\mathfrak{g}})_{\text{cyc}}) \subset (TV^{*\mathfrak{g}})_{\text{cyc}} \otimes (TV^{*\mathfrak{g}})_{\text{cyc}},$$

⁴We need just $(V^{\otimes k})^{*\mathfrak{g}} \simeq V^{*\mathfrak{g}} \otimes^k$.

C. Towards an invariant definition of canonical IBL-operations

$$(2) \text{ Der}_{ij}((TV^{*\mathfrak{g}})_{\text{sym}}) \subset (TV^{*\mathfrak{g}})_{\text{sym}} \otimes (TV^{*\mathfrak{g}})_{\text{sym}},$$

$$(3) \mu^{\text{sh}}((TV^{*\mathfrak{g}})_{\text{cyc}} \otimes (TV^{*\mathfrak{g}})_{\text{cyc}}) \subset (TV^{*\mathfrak{g}})_{\text{cyc}},$$

$$(4) \mu^{\text{sh}}((TV^{*\mathfrak{g}})_{\text{sym}} \otimes (TV^{*\mathfrak{g}})_{\text{sym}}) \subset (TV^{*\mathfrak{g}})_{\text{sym}}.$$

Inclusion (1) is clear from (C.8). In fact, it holds even $\text{Der}_{ij}(TV^{*\mathfrak{g}}) \subset (TV^{*\mathfrak{g}})_{\text{cyc}} \otimes (TV^{*\mathfrak{g}})_{\text{cyc}}$. As for inclusion (2), we have

$$\text{Der}_{ij}(\pi_{\text{sym}}(\eta^{\otimes I})) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \sum_{(i,j) \in I^\sigma} \frac{1}{k_1} \frac{1}{k_2} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \varepsilon(I, ij I_1^{\sigma c_1} I_2^{\sigma c_2}) \eta^{\otimes I_1^{\sigma c_1}} \otimes \eta^{\otimes I_2^{\sigma c_2}}.$$

A pair $(\sigma_1, \sigma_2) \in \mathbb{S}_{k_1} \times \mathbb{S}_{k_2}$ induces a bijection of the domain of summation, which consists of the elements $\sigma, (i, j), c_1, c_2$, by keeping $(i, j), c_1, c_2$ fixed and defining a new $\tilde{\sigma}$ by altering σ to compensate for the effect of (σ_1, σ_2) . The signs fit since we only consider the Koszul sign. Therefore, we have

$$(\sigma_1 \otimes \sigma_2) \text{Der}_{ij}(\pi_{\text{sym}}(\eta^{\otimes I})) = \text{Der}_{ij}(\pi_{\text{sym}}(\eta^{\otimes I})),$$

and (2) follows. As for (3), we have

$$\begin{aligned} & \mu^{\text{sh}}(\pi_{\text{cyc}}(\eta^{\otimes I_1}), \pi_{\text{cyc}}(\eta^{\otimes I_2})) \\ &= \frac{k_1! k_2!}{(k_1 + k_2)!} \sum_{\mu \in \mathbb{S}_{k_1, k_2}} \frac{1}{k_1} \frac{1}{k_2} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \varepsilon(I_1 I_2, \mu(I_1^{c_1}, I_2^{c_2})) \eta^{\otimes \mu(I_1^{c_1}, I_2^{c_2})}. \end{aligned}$$

A permutation $c \in \mathbb{S}_k$ again induces a bijection of the domain of summation, which consists of the elements μ, c_1, c_2 . A similar argument holds for (4).

We now derive (C.14). Because

$$\text{Der}_{ij}(\alpha_{\text{cyc}}^I) = \sum_{(i,j) \in I} \varepsilon(I, ij I_1 I_2) \alpha_{\text{cyc}}^{I_1} \otimes \alpha_{\text{cyc}}^{I_2},$$

the formula for ζ_{cyc} is clear. As for Δ_{sym} , because $\mu_{\text{sym}} = \pi_{\text{sym}} \circ \mu_{\text{con}}$ for the concatenation product $\mu_{\text{con}}(\eta^{\otimes I_1}, \eta^{\otimes I_2}) = \eta^{\otimes I_1 I_2}$, we have

$$(\mu_{\text{sym}} \circ \text{Der}_{ij})(\pi_{\text{sym}}(\eta^{\otimes I})) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \sum_{(i,j) \in I^\sigma} \frac{1}{k_1} \frac{1}{k_2} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \pi_{\text{sym}}(\varepsilon(I, ij I_1^{\sigma c_1} I_2^{\sigma c_2}) \eta^{\otimes I_1^{\sigma c_1}} I_2^{\sigma c_2})$$

C. Towards an invariant definition of canonical IBL-operations

$$= \sum_{(i,j) \in I} \varepsilon(I, ijI_1I_2) \pi_{\text{sym}}(\eta^{\otimes I_1I_2}),$$

where we used that $\pi_{\text{sym}} \circ \sigma = \pi_{\text{sym}}$ for every permutation σ . The formula for Δ_{sym} follows. The formula for μ_{sym} is also clear. \square

In the formula (C.14) for Δ_{sym} , we recognize the canonical BV-operator on $SV^{*\text{g}}$ from [DJP18, Definition 4].⁵

We will now relate ζ_{cyc} to \mathbf{q}_{120} . The steps are similar to Section C.1. We define the coproduct $\mathbf{q}_{120}^* : (D_{\text{cyc}}^+ V)[-n] \rightarrow (D_{\text{cyc}}^+ V)[-n]^{\otimes 2}$, where $-n$ is the degree of $\langle \cdot, \cdot \rangle$, by

$$\mathbf{q}_{120}^*(\nu\varphi) = \nu^2 \zeta_{\text{cyc}}(\varphi) \quad \text{for all } \varphi \in D_{\text{cyc}}^+ V, \quad (\text{C.15})$$

where ν is a formal symbol of degree n .

For $\varphi \in \text{Hom}_{\text{cyc}}(V^{\otimes k}, \mathbb{R})$ with $k \geq 4$, we define $(\delta\varphi) \in (\bar{D}_{\text{cyc}}^+ V)^{\otimes 2}$ using the coordinates

$$\varphi_I := \varphi(e_{\otimes I}) \quad \text{and} \quad (\delta\varphi)_{I_1;I_2} = (\delta\varphi)(e_{\otimes I_1} \otimes e_{\otimes I_2}) \quad (\text{C.16})$$

by

$$(\delta\varphi)_{I_1;I_2} := \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \sum_{i,j} (-1)^{e_i} g^{ij} \varepsilon(I_1, I_1^c) \varepsilon(I_2, I_2^c) (-1)^{e_j I_1} \varphi_{iI_1^{c_1} jI_2^{c_2}}. \quad (\text{C.17})$$

We set $\delta\varphi = 0$ for $k \leq 3$. In order to obtain $(\delta\varphi) \in (\bar{D}_{\text{cyc}}^+ V)^{\otimes 2}$, we use the embedding $(D_{\text{cyc}}^+ V)^{\otimes k} \subset (\text{TV})^{\otimes k*}$ from Remark 3.3.5 in Part I. Note that we can also use the canonical identification $(\text{TV}/\text{cyc})^{*\text{wg}} \simeq D_{\text{cyc}}^+ V$ and work with the cyclic words $e_I := \text{p}_{\text{cyc}}(e_{\otimes I})$ instead. We see that (C.17) is precisely the defining equation [CFL15, Equation (10.6)] for the IBL-cobracket $\delta : \bar{D}_{\text{cyc}}^+ V \rightarrow (\bar{D}_{\text{cyc}}^+ V)^{\otimes 2}$, and $\mathbf{q}_{120} : (\bar{D}_{\text{cyc}}^+ V)[-n] \rightarrow (\bar{D}_{\text{cyc}}^+ V)[-n]^{\otimes 2}$ is the degree shift of δ .

Taking into account Remark C.1.7, the equality

$$(\bar{\pi} \otimes \bar{\pi}) \circ \mathbf{q}_{120}^* \circ \bar{\iota} = \mathbf{q}_{120}$$

is obtained from the next proposition.

Proposition C.2.5 (Equivalence of definitions of IBL-cobracket). *Let V be a graded vector space of finite type, and let $\langle \cdot, \cdot \rangle$ be a non-degenerate homogenous bilinear form of*

⁵In fact, they work on a bigger “algebra of functions” on V , which is a completion of $SV^{*\text{g}}$ with respect to a suitable filtration. Their statement is that the algebra of functions on an odd symplectic vector space carries a canonical BV-operator. It is given by the divergence of the Hamiltonian vector field.

C. Towards an invariant definition of canonical IBL-operations

degree $-n$. For every $\varphi \in \bar{D}_{\text{cyc}}^+ V$, it holds

$$\mathfrak{q}_{120}^*(\nu\varphi) = \nu^2\delta\varphi \quad \text{in } (\bar{D}_{\text{cyc}}^+ V)[-n].$$

Proof. We consider the basis (α_{cyc}^I) of $D_{\text{cyc}}^+ V$ from Proposition C.2.4. Given $\varphi \in D_{\text{cyc}}^+ V$, we can write $\varphi = \sum_I \frac{1}{|I|} \varphi_I \alpha_{\text{cyc}}^I$, where we sum over all multiindices I and $\varphi_I \in \mathbb{R}$ are coefficients cyclically symmetric in I . For every I of length k , we have

$$\varphi(e_{\otimes I}) = \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \frac{1}{k} \varphi_{I^c} \alpha_{\text{cyc}}^{I^c}(e_{\otimes I}) = \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \frac{1}{k} \varphi_{I^c} \varepsilon(I^c, I) = \varphi_I,$$

and hence φ_I correspond to the coefficients (C.16). Using (C.14), we compute

$$\begin{aligned} \zeta_{\text{cyc}}(\varphi) &= \frac{1}{2} \sum_I \sum_{(i,j) \in I} \frac{1}{|I|} \varphi_I (-1)^{e_i} g^{ij} \varepsilon(I, ijI_1I_2) \alpha_{\text{cyc}}^{I_1} \otimes \alpha_{\text{cyc}}^{I_2} \\ &= \frac{1}{2} \sum_J \sum_{c \in \mathbb{S}_{|J|}^{\text{cyc}}} \sum_{j \in J \setminus \{j_1\}} \frac{1}{|J|} \varphi_{J^c} (-1)^{e_{j_1}} g^{j_1j} (-1)^{jJ_1} \varepsilon(J, J^c) \alpha_{\text{cyc}}^{J_1} \otimes \alpha_{\text{cyc}}^{J_2} \\ &= \frac{1}{2} \sum_J \sum_{j \in J \setminus \{j_1\}} \varphi_J (-1)^{e_{j_1}} g^{j_1j} (-1)^{jJ_1} \alpha_{\text{cyc}}^{J_1} \otimes \alpha_{\text{cyc}}^{J_2} \\ &= \sum_{J_1, J_2} \underbrace{\left(\frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} (-1)^{jJ_1} \varphi_{iJ_1jJ_2} \right)}_{=:(*)_{J_1;J_2}} \alpha_{\text{cyc}}^{J_1} \otimes \alpha_{\text{cyc}}^{J_2}. \end{aligned}$$

On the second line, we used the bijection of the summation domains consisting of I , (i, j) , resp. J , c , b , which assigns to I its cyclic permutation J such that the position of i corresponds to the first position of J , i.e., it holds $j_1 = i$. On the third line, we used that φ_I are cyclic symmetric, and on the fourth line, we just rewrote the summation using $J = iJ_1jJ_2$. We now relate $(*)_{J_1;J_2}$ to $(\delta\varphi)_{J_1;J_2}$. We compute (notice that there is no Koszul sign from the application of the tensor product, c.f., Remark 3.3.5 in Part I)

$$\begin{aligned} \zeta_{\text{cyc}}(\varphi)(e_{I_1} \otimes e_{I_2}) &= \left(\sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} (*)_{I_1^{c_1}; I_2^{c_2}} \alpha_{\text{cyc}}^{I_1^{c_1}} \otimes \alpha_{\text{cyc}}^{I_2^{c_2}} \right) (e_{I_1} \otimes e_{I_2}) \\ &= \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \varepsilon(I_1^{c_1}, I_1) \varepsilon(I_2^{c_2}, I_2) (*)_{I_1^{c_1}; I_2^{c_2}} \\ &= (\delta\varphi)_{J_1;J_2}. \end{aligned}$$

C. Towards an invariant definition of canonical IBL-operations

This shows the proposition. \square

Remark C.2.6 (Some questions). (i) Are $(D_{\text{cyc}}^+ V, \Delta_{\text{cyc}}, \mu_{\text{cyc}})$ and $(D^+ V, \Delta_{\text{H}}, \mu^{\text{sh}})$ also BV-algebras like $(D_{\text{sym}}^+ V, \Delta_{\text{sym}}, \mu_{\text{sym}})$? Recall the 2nd-derivative identity (or the "seven term identity") which is used to define a BV-algebra:

$$\begin{aligned} \Delta(abc) = & \Delta(ab)c + (-1)^{a(b+c)} \Delta(bc)a + (-1)^{bc} \Delta(ac)b \\ & - \Delta(a)bc - (-1)^{ab} \Delta(b)ac - (-1)^{c(a+b)} \Delta(c)ab. \end{aligned} \quad (\text{C.18})$$

It clearly uses associativity (and in this form also commutativity). It is a known fact that μ^{sh} is both associative and commutative (see [Lod92]). Therefore, the question for Δ_{H} makes sense. However, μ_{cyc} might not be associative (is it?), and hence Δ_{cyc} might not be a BV-operator in the classical sense. Is it possible to define a second order derivative, and hence the notion of a BV-operator, on a non-associative algebra?

(ii) Do the factorizations $\Delta_{\text{cyc}} = \mu_{\text{cyc}} \circ \zeta_{\text{cyc}}$, resp. $\Delta_{\text{sym}} = \mu_{\text{sym}} \circ \zeta_{\text{sym}}$ define ζ_{cyc} , resp. ζ_{sym} uniquely? Do other BV-operators factorize like this? Is ζ_{sym} a cobracket and is there a bracket on "symmetric Hochschild cochains" inducing a bialgebra structure on $D_{\text{sym}}^+ V$?

(iii) Are the new operations chain maps with respect to b (Hochschild differential) and $b+d$? How are our constructions related to the work of other authors, e.g., [Iri14]? \triangleleft

D. Filtered IBL-infinity-algebras as filtered MV-algebras

In this appendix, we want to understand the BV-formalism for IBL_∞ -algebras and the equivalence

$$\text{IBL}_\infty : \begin{array}{c} \text{Surface calculus} \\ (q_{klg}), (f_{klg}), \text{gluing relations} \end{array} \iff \begin{array}{c} \text{BV-formalismus} \\ \Delta, \mathfrak{f}, \text{relations } \Delta^2 = 0, e^{\mathfrak{f}} \Delta^+ = \Delta^- e^{\mathfrak{f}}. \end{array}$$

This was originally done in [CFL15] using filtrations and formal power series in \hbar as a “bookkeeping”. Nevertheless, the author thinks that some details were not fully addressed and was also curious about extending the MV-formalism from [MV15] to the filtered setting.

As a matter of fact, the main motivation for better understanding of these details was a brief discussion with Kyler Siegel at a workshop about BV-quantization in Stony Brook, 2019. It was pointed out that a twisted ∞ -algebra is, in some sense, “isomorphic” to the untwisted one because the twisting is just conjugation with the exponential of the Maurer-Cartan element. The case of A_∞ -algebras from [Fuk+10] was mentioned. The author of this text was confused because, by Part I, the twisting with the Chern-Simons Maurer-Cartan element “adds” the information about string topology to the canonical structure and one expects to obtain a non-isomorphic structure. The precise notions of isomorphisms needed to be clarified.

In Section D.1, we start by dealing with filtrations and completions in terms of series in more details. We formulate the Resummation Lemma (Lemma D.1.1), define (complete) filtered algebras (Definition D.1.2) and comment on units and augmentations (Remark D.1.4). We then consider combinations of two filtrations (Equation (D.1)) and show that the symmetric bialgebra with the filtration which is the union of the induced filtration and the filtration by weights might not be a filtered bialgebra (Example D.1.5); it is, however, under a boundedness assumption on the filtration (Lemma D.1.6). We then consider the exponential and the logarithm on a complete filtered algebra (Lemma D.1.7) and generalize it for the convolution product of morphisms with codomain a complete

filtered algebra and domain a complete filtered coalgebra satisfying the limit conilpotency property (Proposition D.1.8). This is the case of completions of conilpotent coalgebras, e.g., the symmetric bialgebra (Lemma D.1.9). We remark that the exponential of a morphism might not be invertible as a map (Remark D.1.10).

In Section D.2, we recall MV-algebras from [MV15] and introduce their complete filtered version (Definition D.2.1). Based on this, in analogy to [MV15], we define the notion of a complete filtered IBL_∞ -algebra in MV-formalism and its morphisms (Definition D.2.3). We study the equivalent formulation in components via the surface calculus (Proposition D.2.6). We show that the notion of a filtered IBL_∞ -algebra in MV-formalism is equivalent to the notion of a filtered IBL_∞ -algebra from [CFL15] in some cases, e.g., for the dual cyclic bar complex from Part I (Proposition D.2.7). The formalism of [CFL15] is symmetric on exchanging inputs and outputs whereas the MV-formalism is not; we illustrate how bubblings at inputs and outputs are handled differently (Example D.2.9). Finally, we consider twisting with a Maurer-Cartan element (Proposition D.2.10).

In Section D.3, we define the BV-chain complexes associated to a complete filtered IBL_∞ -algebra in MV-formalism (Definition D.3.1) and observe that morphisms and the twisting with the Maurer-Cartan element induce chain maps of these chain complexes (Proposition D.3.3). We formulate some open questions (Remark D.3.4).

In Section D.4, we study the composition of maps which are convolutions of other maps (Lemma D.4.1). This leads to the definition of compositions controlling the number of “veins” between the individual maps (Definition D.4.2). This can be used to formulate the surface calculus algebraically (Proposition D.4.3). We give proofs of some formulas for partial compositions from Section 3.2 in Part I (Proposition D.4.4).

D.1. Filtered bialgebras and exponential in convolution product

We use the definitions of a filtration, completion and the induced filtration on the completion from Section 3.1 in Part I. We formulate the statements in the category of vector spaces, but similar statements hold in the category of graded vector spaces (writing “homogenous” and “graded” everywhere). All filtrations are decreasing, i.e., $\mathcal{F}^{\lambda_1} \supseteq \mathcal{F}^{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$. We work over a field \mathbb{K} of characteristics zero.

We start with the following technical lemma, which shows that convergence in completion is very much like absolute convergence in analysis.¹

¹In fact, for complex numbers $v_k \in \mathbb{C}$ for $k \in \mathbb{N}_0$, property (b) of Lemma D.1.1 is equivalent to the absolute convergence of $\sum_{k=0}^{\infty} v_k$ in \mathbb{C} . Thanks to Jiří Zeman for noting that.

Lemma D.1.1 (Resummation Lemma). *Let V be a vector space filtered by a decreasing filtration \mathcal{F} , and let \hat{V} be its completion.*

- (a) *Let $v_{ij} \in V$ for all $i, j \in \mathbb{N}_0$ be vectors such that $\sum_{j=0}^{\infty} v_{ij}$ converges for every $i \geq 0$ (i.e., $\|v_{ij}\| \rightarrow \infty$ as $j \rightarrow \infty$, so that $\sum_{j=0}^{\infty} v_{ij} \in \hat{V}$) and $\|\sum_{j=0}^{\infty} v_{ij}\| \rightarrow \infty$ as $i \rightarrow \infty$. Then for any bijection $r : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$, the sum $\sum_{i=0}^{\infty} v_{r(i)}$ converges and the limit does not depend on r ; we denote it by $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_{ij} \in \hat{V}$.²*
- (b) *Suppose that $\sum_{k=0}^{\infty} v_k$ for $v_k \in V$ converges, and let $s : Z \subset \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$ be a bijection. If we define $w_{ij} := v_{s(i,j)}$ for $(i, j) \in Z$ and $w_{ij} := 0$ otherwise, then $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij}$ converges (in the sense of (a)) and equals $\sum_{k=0}^{\infty} v_k$.*
- (c) *Let V and V' be vector spaces filtered by exhaustive filtrations, and let $\sum_{i=0}^{\infty} v_i$ for $v_i \in V$ and $\sum_{i=0}^{\infty} v'_i$ for $v'_i \in V'$ converge. Then for any bijection $r : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$, $i \rightarrow (r_1(i), r_2(i))$, the sum $\sum_{i=0}^{\infty} v_{r_1(i)} \otimes v'_{r_2(i)}$ converges and the limit does not depend on r ; we denote it by $\sum_{i=0}^{\infty} v_i \otimes \sum_{j=0}^{\infty} v'_j \in V \hat{\otimes} V$.*

Moreover, we have the following statements about maps of filtered vector spaces:

- (d) *If $f : \hat{V} \rightarrow \hat{V}$ is a linear map of finite filtration degree, i.e., $\|f\| > -\infty$, and $\sum_{i=0}^{\infty} v_i$ with $v_i \in V$ a convergent series, then $\sum_{i=0}^{\infty} f(v_i)$ converges to $f(\sum_{i=0}^{\infty} v_i)$.*
- (e) *Given linear maps $f_i : \hat{V} \rightarrow \hat{V}$ for $i \in \mathbb{N}_0$, we say that the sum $\sum_{i=0}^{\infty} f_i$ converges if $\sum_{i=0}^{\infty} f_i(v)$ converges for all $v \in \hat{V}$. In this case, $f(v) := \sum_{i=0}^{\infty} f_i(v)$ defines a linear map $\hat{V} \rightarrow \hat{V}$ which satisfies $\|f\| \geq \inf_{i \in \mathbb{N}_0} \|f_i\|$.*

Proof. (a) Let $r : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ be a bijection. We first prove the convergence of $\sum_{i=0}^{\infty} v_{r(i)}$. Given $K > 0$, let $i_0 \in \mathbb{N}_0$ be such that $\|v_{ij}\| \geq K$ for $(i, j) \in \{i \geq i_0, j \geq 0\}$. Such i_0 exists by the assumption that $\|\sum_{j=0}^{\infty} v_{ij}\| \rightarrow \infty$ as $i \rightarrow \infty$. Let $j_0 \in \mathbb{N}_0$ be such that $\|v_{ij}\| \geq K$ for $(i, j) \in \{i \leq i_0, j \geq j_0\}$. Such j_0 exists from the convergence of $\sum_{j=0}^{\infty} v_{0j}, \dots, \sum_{j=0}^{\infty} v_{i_0j}$. We can now pick $k_0 \in \mathbb{N}$ such that $r(\{k < k_0\}) \supset \{i \leq i_0, j \leq j_0\}$, and hence $\|v_{r(k)}\| \geq K$ for all $k \geq k_0$. This implies the convergence of $\sum_{i=0}^{\infty} v_{r(i)}$.

Let now $r' : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ be another bijection. Given $K > 0$, we find $i_0, j_0 \in \mathbb{N}_0$ so that $\|v_{ij}\| \geq K$ for all $(i, j) \notin \{i \leq i_0, j \leq j_0\}$ (like in the previous paragraph). There exists $k_0 \in \mathbb{N}$ such that $r(\{k < k_0\}), r'(\{k < k_0\}) \supset \{i \leq i_0, j \leq j_0\}$. Hence, for any $k \geq k_0$, the contribution of v_{ij} with $(i, j) \in \{i \leq i_0, j \leq j_0\}$ cancels in $\Delta_k := \sum_{i=0}^k v_{r(i)} - \sum_{i=0}^k v_{r'(i)}$, and we get $\|\Delta_k\| \geq K$. It follows that the sums are equal in the limit.

²It is an exercise to prove a similar statement for nested infinite sums $\sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} v_{i_1, \dots, i_n}$.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

(b) It is easy to see that w_{ij} satisfy the assumptions of (a). The claim follows from (a) by defining $r := s^{-1} \sqcup \mathbb{1} : \mathbb{N}_0 = \mathbb{N}_0 \sqcup (\mathbb{N}_0 \times \mathbb{N}_0 \setminus Z) \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$, where the notation $\mathbb{N}_0 = \mathbb{N}_0 \sqcup (\mathbb{N}_0 \times \mathbb{N}_0 \setminus Z)$ means an infinite shuffle permutation. Clearly, $\sum_{i=0}^{\infty} w_{r(i)}$ is precisely $\sum_{i=0}^{\infty} v_i$ after crossing out possibly infinitely many zeros.

(c) We start with convergence. Let $K > 0$. Pick indices $i_0 \in \mathbb{N}_0$ and $i'_0 \in \mathbb{N}_0$ such that $\|v_i\| \geq K - \inf_j \|v'_j\|$ and $\|v'_i\| \geq K - \inf_j \|v_j\|$ for all $i \geq i_0$ and $i \geq i'_0$, respectively. Note that the right-hand sides of the inequalities are finite because the filtrations are exhaustive. Pick $k_0 \in \mathbb{N}$ such that $r(\{k < k_0\}) \supset \{i \leq i_0, j \leq j_0\}$. Then, for $k \geq k_0$, we have

$$\begin{aligned} \|v_{r_1(k)} \otimes v'_{r_2(k)}\| &\geq \|v_{r_1(k)}\| + \|v'_{r_2(k)}\| \\ &\geq \begin{cases} (K - \inf_j \|v'_j\|) + \inf_j \|v'_j\| = K & \text{if } r_1(k) \geq i_0, \\ \inf_j \|v_j\| + (K - \inf_j \|v_j\|) = K & \text{if } r_2(k) \geq i'_0. \end{cases} \end{aligned}$$

It follows that $\sum_{i=0}^{\infty} v_{r_1(i)} \otimes v'_{r_2(i)}$ converges.

If $r' : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ is another bijection, we write $r' = r \circ s$ for a bijection $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ and apply (b) to $V \otimes V'$ with the bijection $s : Z := \mathbb{N}_0 \times \{0\} \simeq \mathbb{N}_0 \rightarrow \mathbb{N}_0$. It follows that $\sum_{i=0}^{\infty} v_{r(i)} = \sum_{i=0}^{\infty} v_{r'(i)}$.

(d) This is clear because $f(\sum_{i=0}^{\infty} v_i) = \sum_{i=0}^n f(v_i) + f(\sum_{i=n+1}^{\infty} v_i)$ for any $n \in \mathbb{N}_0$ and $\|f(\sum_{i=n+1}^{\infty} v_i)\| \geq \|f\| + \inf_{i>n} \|v_i\| \rightarrow \infty$ as $n \rightarrow \infty$.

(e) Linearity is clear. The inequality $\|f\| \geq \inf_i \|f_i\|$ follows immediately from $\|f(v)\| \geq \inf_i \|f_i(v)\| \geq \inf_i \|f_i\| + \|v\|$ for all $v \in \hat{V}$. \square

Note that (a) and (c) of the Resummation Lemma are, in fact, reformulations of the facts that the canonical inclusions induce the isomorphisms $\hat{V} \simeq \hat{\hat{V}}$ and $V \hat{\otimes} V \simeq \hat{V} \hat{\otimes} \hat{V}$, respectively.

We will work with (complete) filtered algebras, coalgebras and bialgebras, which we now define schematically. They are basically algebras over the corresponding (pr)operad in the category of (complete) filtered vector spaces.

Definition D.1.2 ((Complete) filtered algebras, coalgebras, bialgebras). *A filtered algebra, coalgebra or bialgebra is a filtered vector space V with linear operations $\mu_{klg} : V^{\otimes k} \rightarrow V^{\otimes l}$ of non-negative filtration degree for $k, l, g \in \mathbb{N}_0$ such that $(V, (\mu_{klg}))$ is an algebra, coalgebra or bialgebra, respectively.*

D. Filtered IBL-infinity-algebras as filtered MV-algebras

A complete filtered algebra, coalgebra or bialgebra is a complete filtered vector space V with linear operations $\mu_{klg} : V^{\hat{\otimes} k} \rightarrow V^{\hat{\otimes} l}$ of non-negative filtration degree for $k, l, g \in \mathbb{N}_0$ such that μ_{klg} satisfy the relations of an algebra, coalgebra or bialgebra with \otimes replaced by $\hat{\otimes}$, respectively.

A morphism of (complete) filtered algebras, coalgebras or bialgebras is a linear map $f : V_1 \rightarrow V_2$ with non-negative filtration degree intertwining μ_{klg} .

Remark D.1.3 (On (complete) filtered algebras). (i) Clearly, completion is a functor from filtered algebras to complete filtered algebras. In deformation theory, we may deal with complete filtered algebras which are not in the image of this functor.

(ii) The IBL_∞ -algebra on C from Definition 3.2.4 in Part I for $\gamma = 0$ is, in fact, a “complete filtered IBL_∞ -algebra on \hat{C} ” in the sense of Definition D.1.2 plus the data of the filtered vector space C . It might not be the completion of a “filtered IBL_∞ -algebra on C ”; in fact, we called an IBL_∞ -algebra “completion-free” if the operations on \hat{C} arose as continuous extensions of operations on C . Definition D.1.2 seemed natural when we were trying to compute examples of the twisted structure, hoping that we do not encounter infinite sums. After dealing with completions, however, Definition D.1.2 is more logical as an abstract definition.

Notice that for $\gamma > 0$, the norm $\|\mathbf{q}_{klg}\| \geq \gamma(2 - 2g - k - l)$ is allowed to be negative; in particular, $\|\mathbf{q}_{210}\| \geq -\gamma$ and $\|\mathbf{q}_{120}\| \geq -\gamma$. Therefore, one has to allow finite filtration degree in Definition D.1.2 in order to accommodate it to IBL_∞ -algebras with $\gamma \geq 0$. \triangleleft

Remark D.1.4 (Units and augmentations). The ground field \mathbb{K} is filtered by the trivial complete filtration $\mathcal{F}^{\lambda \leq 0} \mathbb{K} = \mathbb{K}$ and $\mathcal{F}^{\lambda > 0} \mathbb{K} = 0$ (see also (3.9) in Part I). Suppose that $\eta : \mathbb{K} \rightarrow V$ is a unit and $\varepsilon : V \rightarrow \mathbb{K}$ an augmentation of a complete filtered algebra V ; in particular, we have $\varepsilon \circ \eta = \mathbb{1}$ and $\|\eta\|, \|\varepsilon\| \geq 0$. From this, we deduce the following implications for $v \in V$:

$$\begin{aligned} \|v\| > 0 &\implies v \in \ker \varepsilon =: \bar{V}, \\ v \in \text{im } \eta &\implies \|v\| = 0. \end{aligned}$$

Recall that $V = \text{im } \eta \oplus \bar{V}$. \triangleleft

Given a vector space U over \mathbb{K} , we will work with the *symmetric bialgebra* $SU = \bigoplus_{k=0}^{\infty} S_k U$ from Definition 3.1.7 in Part I. It has the concatenation product $\mu : SU \otimes SU \rightarrow SU$ and the shuffle coproduct $\delta : SU \rightarrow SU \otimes SU$ which are given for all $u_{ij}, u_i \in U$ and

D. Filtered IBL-infinity-algebras as filtered MV-algebras

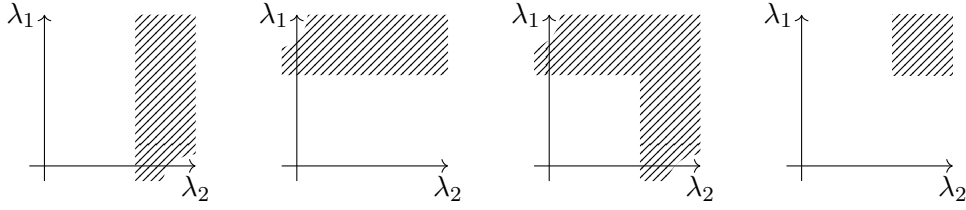


Figure D.1.: For a bigraded vector space $V = \bigoplus_i \bigoplus_j V_{ij}$, let $\mathcal{F}_v^{\lambda_1} := \bigoplus_{i \geq \lambda_1} \bigoplus_j V_{ij}$ be the vertical and $\mathcal{F}_h^{\lambda_2} := \bigoplus_i \bigoplus_{j \geq \lambda_2} V_{ij}$ the horizontal filtration. The illustrations above depict \mathcal{F}_v , \mathcal{F}_h , $\mathcal{F}_v \cup \mathcal{F}_h$ and $\mathcal{F}_v \cap \mathcal{F}_h$, respectively. We imagine the component V_{ij} sitting at the position (i, j) in the plane. Given $v_{ij} \in V_{ij}$, the sum $\sum_{i,j} v_{ij}$ converges with respect to the given filtration if and only if for every λ , only finitely many v_{ij} 's in the white region are non-zero.

$k, k' \in \mathbb{N}_0$ by

$$\begin{aligned} \mu(u_{11} \cdots u_{1k} \otimes u_{21} \cdots u_{2k'}) &= u_{11} \cdots u_{1k} u_{21} \cdots u_{2k'} \quad \text{and} \\ \delta(u_1 \cdots u_k) &= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = k}} \sum_{\sigma \in \mathbb{S}_{k_1, k_2}} \varepsilon(\sigma, u) u_{\sigma_1^{-1}}^{-1} \cdots u_{\sigma_{k_1}^{-1}}^{-1} \otimes u_{\sigma_{k_1+1}^{-1}}^{-1} \cdots u_{\sigma_{k_1+k_2}^{-1}}^{-1}, \end{aligned}$$

respectively. Here \mathbb{S}_{k_1, k_2} denotes the shuffle permutations and $\varepsilon(\sigma, u)$ the Koszul sign. The unit $\eta : \mathbb{K} \rightarrow SU$ is given by $\eta(1) = 1 \in S_0 U = \mathbb{K}$, and the augmentation $\varepsilon : SU \rightarrow \mathbb{K}$ is determined by its reduced algebra $\bar{S}U = \bigoplus_{k=1}^{\infty} S_k U$. If U is filtered, then μ , δ , η and ε preserve the induced filtration on SU (see Definition 3.1.8 in Part I), and hence SU is a filtered bialgebra. We will also use the canonical projection $\pi_k : SU \rightarrow S_k U$ and the canonical inclusion $\iota_l : S_l U \rightarrow SU$ for $k \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$. They too preserve the filtration.

The symmetric bialgebra of a filtered vector space has naturally two filtrations — the induced filtration and the filtration by weights $\mathcal{F}_w^\lambda SU := \bigoplus_{k \geq \lambda} S_k U$.³ In general, having two filtration \mathcal{F}_1 and \mathcal{F}_2 on a vector space V , we define the filtrations

$$\mathcal{F}_\cup^\lambda := \mathcal{F}_1^\lambda + \mathcal{F}_2^\lambda \quad \text{and} \quad \mathcal{F}_\cap^\lambda := \mathcal{F}_1^\lambda \cap \mathcal{F}_2^\lambda \quad (\text{D.1})$$

and call them the *combined filtrations* — the union and the intersection. It is easy to see that

$$\|\cdot\|_\cup \geq \max(\|\cdot\|_1, \|\cdot\|_2) \quad \text{and} \quad \|\cdot\|_\cap \leq \min(\|\cdot\|_1, \|\cdot\|_2)$$

hold for the corresponding filtration degrees.

³Note that the filtration by weights can be also viewed as the induced filtration from the filtration on U defined by $\mathcal{F}^{\leq 1} U := U$, $\mathcal{F}^{> 1} U := 0$.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

The following example shows that SU might not be a filtered bialgebra with respect to combined filtrations.

Example D.1.5 (Combined filtrations of symmetric bialgebra SU). Let U be a vector space filtered by a filtration \mathcal{F} . Consider the symmetric bialgebra SU with the induced filtration \mathcal{F} and the filtration by weights \mathcal{F}_w^λ . It is easy to see that the operations μ, δ, η and ε preserve both \mathcal{F} and \mathcal{F}_w . Consider the combined filtrations \mathcal{F}_\cup and \mathcal{F}_\cap . Because

$$\begin{aligned} \min(\lambda_1 + \lambda_2, k_1 + k_2) &\geq \min(\lambda_1, k_1) + \min(\lambda_2, k_2) \quad \text{and} \\ \max(\lambda_1 + \lambda_2, k_1 + k_2) &\leq \max(\lambda_1, k_1) + \max(\lambda_2, k_2), \end{aligned}$$

it holds $\|\mu\|_\cap \geq 0$ and $\|\delta\|_\cup \geq 0$, respectively.

We now demonstrate that it might happen that $\|\mu\|_\cup = -\infty$ and $\|\delta\|_\cap = -\infty$. Let $U = TW$ be the tensor algebra over a non-zero vector space W . On U , consider the filtration by weights $\mathcal{F}^\lambda U = \bigoplus_{k \geq \lambda} W^{\otimes k}$. Let $0 \neq w \in W$, and define

$$v_k := \underbrace{w \otimes \cdots \otimes w}_{k\text{-times}} \in \mathcal{F}^k S_1 U \quad \text{for all } k \in \mathbb{N},$$

where \otimes denotes the tensor product on TW . For $0 \neq u \in T_0 W = \mathbb{K}$, define

$$v'_k := \underbrace{u \cdots u}_{k\text{-times}} \in \mathcal{F}^0 S_k U \quad \text{for all } k \in \mathbb{N},$$

where \cdot denotes the concatenation product on SU . It follows that

$$\|v_k\|_\cup = k, \quad \|v'_k\|_\cup = k \quad \text{and} \quad \|v_k v'_k\|_\cup = k + 1.$$

Therefore, for all $k \in \mathbb{N}$, it holds

$$\begin{aligned} \|\mu\|_\cup &\leq \|v_k v'_k\|_\cup - \|v_k \otimes v'_k\|_\cup \\ &\leq k + 1 - \|v_k\|_\cup - \|v'_k\|_\cup \\ &= 1 - k. \end{aligned}$$

Consequently $\|\mu\|_\cup = -\infty$. Next, it holds

$$\|v_k v'_k\|_\cap = k \quad \text{and} \quad \|v_k \otimes v'_k\|_\cap = 1,$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

and we compute

$$\begin{aligned}\|\delta\|_\cap &\leq \|\delta(v_k v'_k)\|_\cap - \|v_k v'_k\|_\cap \\ &= 1 - k.\end{aligned}$$

We used here that the summands of $\delta(v_k v'_k)$ are tensor products of $v''_1 = u^i v_k \in \mathcal{F}^1 S_{i+1} U$ and $v''_2 = u^{k-i} \in \mathcal{F}^0 S_{k-i} U$ for $i = 0, \dots, k$, so that $\|v''_1 \otimes v''_2\|_\cap = 1$. It follows that $\|\delta\|_\cap = -\infty$. A heuristic explanation of these phenomenons is that the tensor product of combined filtrations is not the combined filtration of the tensor product.

In the next paragraph, we will argue that it might still be possible to extend μ to $SU \hat{\otimes}_U SU$ — the completion of $SU \otimes SU$ with respect to the tensor product of the union filtrations — even though $\|\mu\| = -\infty$. A similar discussion applies for δ .

Suppose that the filtration on U is bounded from above. Let $v_i \in \mathcal{F}^{\lambda_i} S_{k_i} U$ and $v'_i \in \mathcal{F}^{\lambda'_i} S_{k'_i} U$ for all $i \in \mathbb{N}_0$ be such that $\max(\lambda_i, k_i) + \max(\lambda'_i, k'_i) \rightarrow \infty$ as $i \rightarrow \infty$, so that $\sum_{i=0}^\infty v_i \otimes v'_i$ converges in $SU \hat{\otimes}_U SU$ (in fact, any element of $SU \hat{\otimes}_U SU$ can be written in this way). We have $\mu(v_i, v'_i) \in \mathcal{F}^{\lambda_i + \lambda'_i} S_{k_i + k'_i} U$ for all $i \in \mathbb{N}_0$. Suppose that $\sum_{i=0}^\infty v_i v'_i$ does not converge with respect to \mathcal{F}_U . Thus, $\max(\lambda_i + \lambda'_i, k_i + k'_i)$ is bounded, hence $\lambda_i + \lambda'_i$ and $k_i + k'_i$ are bounded, and since $k_i, k'_i \geq 0$, also k_i and k'_i are bounded. Nevertheless, in order to comply with the assumption on the convergence of $\sum_{i=0}^\infty v_i \otimes v'_i$, we need one of λ_i or λ'_i , let's say λ_i , to diverge to ∞ . Because $\lambda_i + \lambda'_i$ is bounded, λ'_i has to diverge to $-\infty$. But this is not allowed by the boundedness assumption. \triangleleft

It turns out that the union filtration is often identical with the induced filtration.

Lemma D.1.6 (Combined filtration and boundedness condition). *Let U be a vector space filtered by a decreasing filtration \mathcal{F} . Suppose that there is $\gamma > 0$ such that*

$$\mathcal{F}^\gamma U = U. \tag{D.2}$$

Then the following holds:

$$\begin{aligned}\gamma \geq 1 &\implies \forall \lambda \in \mathbb{R} : \mathcal{F}_U^\lambda SU = \mathcal{F}^\lambda SU, \\ 0 < \gamma < 1 &\implies \forall v \in SU : \|v\|_U \geq \|v\| \geq \gamma \|v\|_U.\end{aligned}$$

Proof. Under the assumption (D.2), we have for any $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ the following:

$$\lambda \leq k\gamma \implies \mathcal{F}^\lambda S_k U = \sum_{\lambda_1 + \dots + \lambda_k = \lambda} \mathcal{F}^{\lambda_1} U \otimes \dots \otimes \mathcal{F}^{\lambda_k} U / S_k = S_k U. \tag{D.3}$$

We compute

$$\begin{aligned} \mathcal{F}_U^\lambda SU &= \bigoplus_{k < \lambda} \mathcal{F}^\lambda S_k U \oplus \bigoplus_{k \geq \lambda} S_k U \\ &= \begin{cases} \bigoplus_{k=0}^{\infty} \mathcal{F}^\lambda S_k U = \mathcal{F}^\lambda SU & \text{for } \gamma \geq 1, \\ \bigoplus_{k < \lambda \text{ or } k \geq \frac{\lambda}{\gamma}} \mathcal{F}^\lambda S_k U \oplus \bigoplus_{\lambda \leq k < \frac{\lambda}{\gamma}} S_k U \subset \mathcal{F}^{\lambda\gamma} SU & \text{for } 0 < \gamma < 1. \end{cases} \end{aligned}$$

The first case holds because if $\gamma \geq 1$, then $k \geq \lambda$ implies $\gamma k \geq \lambda$, and thus $S_k U = \mathcal{F}^\lambda S_k U$ by (D.3). The second case holds from the following reasons. If $0 < \gamma < 1$ and $k \geq \frac{\lambda}{\gamma}$, so that $S_k U = \mathcal{F}^\lambda S_k U$ by (D.3), then $k > \lambda$. Now, $\mathcal{F}^\lambda S_k U \subset \mathcal{F}^{\lambda\gamma} S_k U$ because the filtration is decreasing and because $\lambda \geq \lambda\gamma$, and if $v \in S_k U$ with $\lambda \leq k < \frac{\lambda}{\gamma}$, then $k\gamma \geq \lambda\gamma$, and hence $S_k U = \mathcal{F}^{\lambda\gamma} S_k U$ by (D.3). The claim follows. \square

We consider the exponential and the logarithm on filtered algebras.

Lemma D.1.7 (Exponential and logarithm on filtered algebras). *Let V be a complete filtered associative algebra with unit 1. Given $v \in V$ with $\|v\| > 0$, we define the exponential*

$$e^v := 1 + v + \frac{1}{2!}v^2 + \frac{1}{3!}v^3 + \cdots \in V.$$

It holds $e^v e^{-v} = 1$. Given $v \in V$ with $\|v - 1\| > 0$, we define the logarithm

$$\log(v) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} (v - 1)^r \in V.$$

It holds $\log(e^v) = v$ for $v \in V$ with $\|v\| > 0$ and $e^{\log v} = v$ for $v \in V$ with $\|v - 1\| > 0$.

Proof. For any $v \in V$, we have

$$\begin{aligned} e^{-v} e^v &= e^{-v} \left(\sum_{i=0}^{\infty} \frac{1}{i!} v^i \right) \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} e^{-v} v^i \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i! j!} (-1)^j v^{i+j} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^n (-1)^k \binom{n}{k}}_{=0 \text{ for } n > 0} v^n. \end{aligned}$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

On the second line, we used that the multiplication with e^{-v} from the left has finite filtration degree, and hence it commutes with infinite sums; on the third line, we used that the multiplication with v^i from the right has finite filtration degree; on the fourth line, we used the Resummation Lemma (Lemma D.1.1). The facts about \log are proven similarly, the equations reducing to known combinatorial identities. \square

Let (V, δ) be a coalgebra and (V', μ') an algebra. The *convolution product* $*$ on \mathbb{K} -linear maps $f_1, f_2 : V \rightarrow V'$ is defined by

$$f_1 * f_2 := \mu' \circ (f_1 \otimes f_2) \circ \delta.$$

See also [LV12, Section 1.6]. If $\varepsilon : V \rightarrow \mathbb{K}$ is a counit for V and $\eta' : \mathbb{K} \rightarrow V'$ a unit for V' , then

$$e := \eta' \circ \varepsilon : V \rightarrow V' \tag{D.4}$$

is the unit for $*$. The convolution product is associative and commutative provided that the algebra and coalgebra are.

Let us recall the conilpotency property of a counitented counital coassociative coalgebra $(V, \delta, \varepsilon, \eta)$, where $\varepsilon : V \rightarrow \mathbb{K}$ is the counit and $\eta : \mathbb{K} \rightarrow V$ the counitented. We write

$$\delta = \bar{\delta} + \delta_0,$$

where $\delta_0 : V \rightarrow V \otimes V$ is defined by

$$\delta_0(v) := \begin{cases} 1 \otimes 1 & \text{if } v = 1, \\ 1 \otimes v + v \otimes 1 & \text{if } v \in \bar{V}, \end{cases} \tag{D.5}$$

with respect to the decomposition $V = \langle 1 \rangle \oplus \bar{V}$. The other map is then defined simply as the difference

$$\bar{\delta} := \delta - \delta_0 : V \longrightarrow V \otimes V.$$

We call δ_0 the *trivial coproduct* and $\bar{\delta}$ the *reduced diagonal*. They are both coassociative, and δ_0 is even cocommutative. The *conilpotency property* reads: for every $v \in V$, there exists an $n \in \mathbb{N}$ such that $\bar{\delta}^{(n)}(v) = 0$, where $\bar{\delta}^{(n)} := (\bar{\delta} \otimes \mathbb{1}^{\otimes n-2}) \circ \dots \circ \bar{\delta}$ is the iterated reduced diagonal with n outputs.

Using the conilpotency property, it is possible to weaken the condition $\|f\| > 0$ on the convergence of a power series in $f \in \text{Hom}(V, V')$ in the convolution algebra $(\text{Hom}(V, V'), *, e)$ to $\|f\| \geq 0$ and $\|f(1)\| > 0$. The latter condition can not be weakened

D. Filtered IBL-infinity-algebras as filtered MV-algebras

because $e^f(1) = e^{f(1)}$ is the exponential in (V', μ', η') .

Proposition D.1.8 (Power series in $*$ with coefficients in \mathbb{K}). *Let $(V, \delta, \varepsilon, \eta)$ be a complete filtered coaugmented counital cocommutative coassociative coalgebra such that the following limit conilpotency property holds:*

$$\bar{\delta}^{(n)}(v) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } v \in V. \quad (\text{D.6})$$

Let $(V', \mu', \eta', \varepsilon')$ be a complete filtered augmented unital commutative associative algebra. Let R be a complete filtered \mathbb{K} -algebra. Then the following holds:

- (a) *The convolution product $*$ extends naturally to R -linear maps $\mathfrak{f}_i : V \hat{\otimes} R \rightarrow V' \hat{\otimes} R$ ($\hat{\otimes} = \hat{\otimes}_{\mathbb{K}}$) of finite filtration degrees, and for its iterations, we have*

$$\mathfrak{f}_1 * \cdots * \mathfrak{f}_n = \mu'^{(n)} \circ (\mathfrak{f}_1 \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n) \circ \delta^{(n)}, \quad (\text{D.7})$$

where $\mu'^{(n)}$ and $\delta^{(n)}$ are the continuous R -linear extensions of the iterated product and coproduct, respectively. (Recall that the continuous extension of a linear map f of finite filtration degree on V to the completion \hat{V} is defined by $f(\sum_{i=0}^{\infty} v_i) := \sum_{i=0}^{\infty} f(v_i)$ for all $v_i \in V$ with $\|v_i\| \rightarrow \infty$.)

- (b) *Let $\mathfrak{f} : V \hat{\otimes} R \rightarrow V' \hat{\otimes} R$ be an R -linear map of finite filtration degree such that*

$$\|\mathfrak{f}\| \geq 0 \quad \text{and} \quad \|\mathfrak{f}(1)\| > 0. \quad (\text{D.8})$$

*Then any power series $\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k}$ with coefficients $\alpha_k \in \mathbb{K}$, where $\mathfrak{f}^{*0} := \mathfrak{e}$ is the unit of the convolution product, converges to an R -linear map $V \hat{\otimes} R \rightarrow V' \hat{\otimes} R$ of non-negative filtration degree, and it holds*

$$\left(\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k} \right) (1) = \sum_{k=0}^{\infty} \alpha_k \mathfrak{f}(1)^k.$$

Proof. (a) Given a \mathbb{K} -linear map $u : V^{\hat{\otimes} k} \rightarrow \hat{V}^{\otimes l}$ of finite filtration degree for some $k, l \geq 0$, we consider its R -linear extension $u \otimes \mathbb{1} : V^{\hat{\otimes} k} \otimes R \rightarrow V^{\hat{\otimes} l} \otimes R$ and extend it continuously to $V^{\hat{\otimes} k} \hat{\otimes} R \rightarrow V^{\hat{\otimes} l} \hat{\otimes} R$. Because $V^{\hat{\otimes} n} \hat{\otimes} R \simeq (V \hat{\otimes} R)^{\hat{\otimes}_{R^n}}$ via canonical maps (this can be proven explicitly using Proposition 3.1.11 in Part I), we get the desired continuous R -linear extension $(V \hat{\otimes} R)^{\hat{\otimes}_{R^k}} \rightarrow (V \hat{\otimes} R)^{\hat{\otimes}_{R^l}}$. We apply this construction to $u = \mu'^{(n)}, \delta^{(n)}$ and define $\mathfrak{f}_1 * \cdots * \mathfrak{f}_n$ using (D.7) for all $n \geq 2$. It is then easy to see that

$$(\cdots ((\mathfrak{f}_1 * \mathfrak{f}_2) * \mathfrak{f}_3) * \cdots) * \mathfrak{f}_n$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

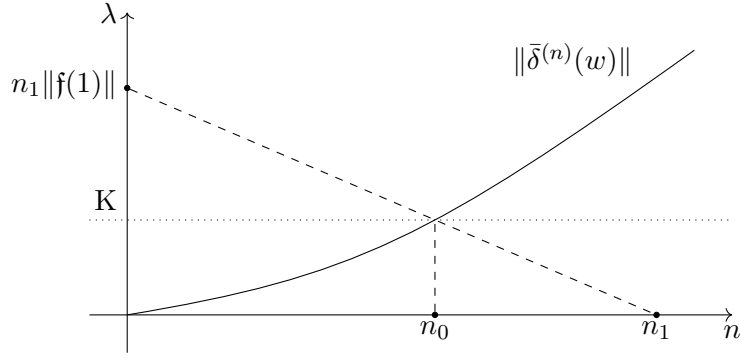


Figure D.2.: Given $K > 0$, pick $n_0 \in \mathbb{N}$ such that $\|\bar{\delta}^{(n)}(w)\| \geq K$ for $n \geq n_0$. Find $n_1 \geq n_0$ such that $n_1 \|f(1)\| \geq K + n_0 \|f(1)\|$. Then $(n - i) \|f(1)\| + \|\bar{\delta}^{(i)}(w)\| \geq K$ for $i = 0, \dots, n$ for any $n \geq n_1$.

$$\begin{aligned}
&= \mu'(\mu'(\cdots \mu'(\mathfrak{f}_1 \hat{\otimes}_R \mathfrak{f}_2) \delta \hat{\otimes}_R \mathfrak{f}_3) \delta \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n) \delta \\
&= \underbrace{\mu'(\mu' \hat{\otimes}_R \mathbb{1}) \cdots (\mu' \hat{\otimes}_R \mathbb{1}^{n-2})}_{=\mu'^{(n)}} (\mathfrak{f}_1 \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n) \underbrace{(\delta \hat{\otimes}_R \mathbb{1}^{n-2}) \cdots (\delta \hat{\otimes}_R \mathbb{1}) \delta}_{\delta^{(n)}} \\
&= \mathfrak{f}_1 * \cdots * \mathfrak{f}_n.
\end{aligned}$$

(b) Let $w \in V \hat{\otimes} R$ and $n \geq 2$. For the map $\delta^{(n)} : V \hat{\otimes} R \rightarrow (V \hat{\otimes} R)^{\hat{\otimes}_{R^2} n}$, we write

$$\delta^{(n)}(w) = \sum_{i=0}^n \sum_{(n-i) \times 1} \sum_{\delta^{(i)}} w_{(1)} \hat{\otimes}_R 1 \hat{\otimes}_R 1 \hat{\otimes}_R w_{(2)} \hat{\otimes}_R 1 \hat{\otimes}_R \cdots \hat{\otimes}_R w_{(i)} \hat{\otimes}_R 1, \quad (\text{D.9})$$

where the second sum is over all insertions of $n - i$ units 1 into the tensor product and the third sum denotes the sum from the Sweedler's notation

$$\bar{\delta}^{(i)}(w) = \sum w_{(1)} \hat{\otimes}_R \cdots \hat{\otimes}_R w_{(i)}.$$

Formula (D.9) is easy to see from $\delta = \delta_0 + \bar{\delta}$.

Before we continue, let us check that $\bar{\delta} : V \hat{\otimes} R \rightarrow (V \hat{\otimes} R)^{\hat{\otimes}_{R^2} 2}$ also satisfies the limit conilpotency property (the proof is similar to the proof of Lemma D.1.9 below). Every element of $V \hat{\otimes} R$ can be written as $\sum_{i=0}^{\infty} r_i v_i$ for $r_i \in R$ and $v_i \in V$ such that $\|r_i v_i\| \rightarrow \infty$ as $i \rightarrow \infty$; this follows from the Resummation Lemma (Lemma D.1.1). Given $K > 0$, we pick $i_0 \in \mathbb{N}$ such that $\|r_i v_i\| \geq K$ for all $i \geq i_0$. We pick $n_0 \in \mathbb{N}$ such that $\|\bar{\delta}^{(n)}(v_0)\| + \|r_0\|, \dots, \|\bar{\delta}^{(n)}(v_{i_0-1})\| + \|r_{i_0-1}\| \geq K$ for all $n \geq n_0$. According to the

construction in (a), we have

$$\bar{\delta}^{(n)}\left(\sum_{i=0}^{\infty} r_i v_i\right) = \sum_{i=0}^{\infty} r_i \bar{\delta}^{(n)}(v_i).$$

For $n \geq n_0$, it holds $\|r_i \bar{\delta}^{(n)}(v_i)\| \geq \|r_i\| + \|\bar{\delta}^{(n)}(v_i)\| \geq K$ for $i < i_0$ and $\|r_i \bar{\delta}^{(n)}(v_i)\| \geq \|r_i v_i\| \geq K$ for $i \geq i_0$. It follows that $\bar{\delta}^{(n)}\left(\sum_{i=0}^{\infty} r_i v_i\right) \rightarrow 0$ as $n \rightarrow \infty$.

Going on with the main proof, using (D.7) and (D.9), we have

$$\begin{aligned} \|\mathfrak{f}^{*n}(w)\| &= \|\mu^{(n)}(\mathfrak{f} \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}) \bar{\delta}^{(n)}(w)\| \\ &= \left\| \sum_{i=0}^n \sum_{(n-i) \times 1} \sum_{\bar{\delta}^{(i)}} \mathfrak{f}(w_{n,(1)}) \hat{\otimes}_R \mathfrak{f}(1) \hat{\otimes}_R \mathfrak{f}(1) \hat{\otimes}_R \mathfrak{f}(w_{n,(2)}) \hat{\otimes}_R \mathfrak{f}(1) \hat{\otimes}_R \cdots \right. \\ &\quad \left. \hat{\otimes}_R \mathfrak{f}(w_{n,(n-i)}) \hat{\otimes}_R \mathfrak{f}(1) \right\| \\ &\geq \min_{i=0, \dots, n} ((n-i)\|\mathfrak{f}(1)\| + i\|\mathfrak{f}\| + \|\bar{\delta}^{(i)}(w)\|) \\ &\geq \underbrace{\min_{i=0, \dots, n} ((n-i)\|\mathfrak{f}(1)\| + \|\bar{\delta}^{(i)}(w)\|)}_{=:(*)_n}. \end{aligned}$$

Let $K > 0$ be arbitrary. Because $\|\mathfrak{f}\| \geq 0$, $\|\mathfrak{f}(1)\| > 0$ and $\bar{\delta}^i(w) \rightarrow 0$ as $i \rightarrow \infty$ by the assumptions, we obtain an $n_1 \in \mathbb{N}$ such that $(*)_n \geq K$ for all $n \geq n_1$ (see Figure D.2). This implies the convergence of $\sum_{n=0}^{\infty} \alpha_n \mathfrak{f}^{*n}(w)$ in $V \hat{\otimes} R$. Using (e) of the Resummation Lemma, we get a well-defined \mathbb{K} -linear map $\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k} : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$ with $\|\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k}\| \geq \inf_{k=0, \dots, \infty} \|\mathfrak{f}^{*k}\| \geq 0$. It is easy to see that it is R -linear as well. \square

Lemma D.1.9 (Completion of conilpotent coalgebra satisfies limit conilpotency). *Let $(V, \delta, \varepsilon, \eta)$ be a filtered coaugmented counital coassociative coalgebra which is conilpotent. Then the completion $(\hat{V}, \delta, \varepsilon, \eta)$ is a complete filtered coaugmented counital coassociative coalgebra which satisfies the limit conilpotency property (D.6).*

Proof. Let $v \in \hat{V}$, and write $v = \sum_{i=0}^{\infty} v_i$ for $v_i \in V$ with $\|v_i\| \rightarrow \infty$. Given $K > 0$, find $i_0 \in \mathbb{N}$ such that $\|v_i\| \geq K$ for all $i \geq i_0$. From the conilpotency, we can find $n_0 \in \mathbb{N}$ such that $\bar{\delta}^{(n)}(v_0) = \dots = \bar{\delta}^{(n)}(v_{i_0-1}) = 0$ for all $n \geq n_0$. Because $\bar{\delta}$ has finite filtration degree, $\bar{\delta}^{(n)}$ has finite filtration degree too, and so we can permute it with the infinite sum and write

$$\bar{\delta}^{(n)}(v) = \sum_{i=0}^{\infty} \bar{\delta}^{(n)}(v_i) = \sum_{i=i_0}^{\infty} \bar{\delta}^{(n)}(v_i).$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

Because $\|\bar{\delta}\| \geq 0$, we have $\|\bar{\delta}^{(n)}\| \geq 0$, and hence

$$\|\bar{\delta}^{(n)}(v)\| \geq \inf_{i=i_0, \dots, \infty} \|\bar{\delta}^{(n)}(v_i)\| \geq \inf_{i=i_0, \dots, \infty} \|v_i\| \geq K$$

for all $n \geq n_0$. This shows the limit conilpotency property. \square

Remark D.1.10 (Exponential and logarithm in $*$). (i) It is obvious that Lemma D.1.7 holds for $*$ with the weakened conditions (D.8). To sum up, the conditions for the existence of $\exp(\mathfrak{f})$ are $\|\mathfrak{f}\| \geq 0$ and $\|\mathfrak{f}(1)\| > 0$, and the conditions for the existence of $\log(\mathfrak{f})$ are $\|\mathfrak{f}\| \geq 0$ and $\|\mathfrak{f}(1) - 1\| > 0$.

(ii) Given $\mathfrak{f} : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$ satisfying (D.8), it holds $e^{-\mathfrak{f}} * e^{\mathfrak{f}} = e$ by Lemma D.1.7 because e is the unit for the convolution product. However, there might not be any $\mathfrak{g} : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$ such that $\mathfrak{g} \circ e^{\mathfrak{f}} = \mathbb{1}$ (or $e^{\mathfrak{f}} \circ \mathfrak{g} = \mathbb{1}$). To see this, let $V = \hat{S}U$, $R = \mathbb{K}$, and let $\mathfrak{f}_{110} : U \rightarrow U$ be a linear map with $\|\mathfrak{f}_{110}\| \geq 0$ such that there is $0 \neq v \in U$ with $\mathfrak{f}_{110}(v) = 0$. Consider the trivial extension $\mathfrak{f} := \mathfrak{f}_{110} : \hat{S}U \rightarrow \hat{S}U$ (it equals \mathfrak{f}_{110} on $\hat{S}_1U \rightarrow \hat{S}_1U$ and 0 otherwise). Because

$$\mathfrak{f}^{*n}(\underbrace{v \cdots v}_{k\text{-times}}) = \begin{cases} n! \mathfrak{f}_{110}(v)^n & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

for all $k, n \in \mathbb{N}_0$, it holds $e^{\mathfrak{f}}(v) = \mathfrak{f}_{110}(v) = 0$, and so $e^{\mathfrak{f}} : \hat{S}U \rightarrow \hat{S}U$ is not invertible.

(iii) In the case of $V = \hat{S}U$, one can check that $\log(\mathbb{1})$ is the trivial extension of $\mathbb{1}_{110} : \hat{U} \rightarrow \hat{U}$ (see also [MV15, Example 21] in the non-filtered setting). \triangleleft

D.2. Filtered IBL-infinity-algebras in filtered MV-formalism

In [MV15], MV-algebras were introduced; they are precisely the algebras governed by the BV-relations $\Delta^2 = 0$ and $e^{\mathfrak{f}} \Delta^+ = \Delta^- e^{\mathfrak{f}}$ (following [CFL15], we will denote by $^+$ the source and by $^-$ the target). Schematically, there is the following inclusion of categories

(explanations will be given below):

$$\begin{array}{lll}
 \text{MV-algebras} & \supset & \text{BV}_\infty\text{-algebras}^* & \supset & \text{IBL}_\infty\text{-algebras}^{**} \\
 \text{Obj.: } (V, \mu, \delta, \eta, \varepsilon), R, & & R = \mathbb{K}[[\hbar]], & & V = SU, \\
 \Delta : V \hat{\otimes} R \rightarrow V \hat{\otimes} R, & & \Delta = \Delta_1 + \hbar \Delta_2 + \hbar^2 \Delta_3 + \cdots, & & S_{k>i}(U) \subset \ker \mathfrak{f}_i. \\
 \Delta^2 = 0, |\Delta| = 1, & & \Delta_i : V \rightarrow V \text{ differential} & & \\
 \Delta(1) = 0. & & \text{operator of order } \leq i, & & \\
 \text{Mor.: } \mathfrak{f} : V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R, & & \mathfrak{f} = \mathfrak{f}_1 + \hbar \mathfrak{f}_2 + \hbar^2 \mathfrak{f}_3 + \cdots. & & \\
 e^{\mathfrak{f}} \circ \Delta^+ = \Delta^- \circ e^{\mathfrak{f}}, & & & & \\
 \mathfrak{f}(1) \subset V^- \hat{\otimes} \mathfrak{m}^-. & & & &
 \end{array} \tag{D.10}$$

Here, R is a complete local Noetherian ring with residue field \mathbb{K} of characteristics 0 and maximal ideal \mathfrak{m} , $(V, \mu, \eta, \varepsilon)$ is a graded commutative associative algebra (V, μ) over \mathbb{K} with unit $\eta : \mathbb{K} \rightarrow V$ and augmentation $\varepsilon : V \rightarrow \mathbb{K}$, $(V, \delta, \varepsilon, \eta)$ is a *conilpotent* graded cocommutative coassociative coalgebra with counit ε and coaugmentation η (the same maps as ε and η for (V, μ)), $V \hat{\otimes} R := \varprojlim_n (V \otimes R / V \otimes \mathfrak{m}^n)$, where $\mathfrak{m}^n = \mathfrak{m} \cdots \mathfrak{m}$, is the completed tensor product, the operators Δ and \mathfrak{f} are R -linear and have non-negative filtration degrees with respect to the filtration by $V \hat{\otimes} \mathfrak{m}^n$.

If R is a \mathbb{K} -algebra,⁴ then this is precisely the same setting as that of Proposition D.1.8 for the filtrations $\mathcal{F}^{\leq 0}V = V$, $\mathcal{F}^{> 0}V = 0$ and $\mathcal{F}^{\lambda \leq 0}R = R$, $\mathcal{F}^{\lambda > 0}R = \mathfrak{m}^{[\lambda]}$. Clearly, $\mathfrak{f}(1) \subset V^- \hat{\otimes} \mathfrak{m}^-$ is equivalent to $\|\mathfrak{f}(1)\| > 0$. Therefore, $e^{\mathfrak{f}}$ exists by Proposition D.1.8.

Recall from [MV15, p. 5] that a linear operator $D : V \rightarrow V$ on a commutative associative algebra V with unit 1 is called a *differential operator* of order $\leq k$ for $k \in \mathbb{N}_0$ if it holds

$$\psi_{k+1}^D(v_1, \dots, v_{k+1}) := [[\cdots [D, L_{v_1}], L_{v_2}], \dots], L_{v_{k+1}}] = 0 \quad \text{for all } v_1, \dots, v_{k+1} \in V,$$

where

$$L_v(w) := vw \quad \text{for all } w \in V$$

is the left-multiplication with $v \in V$ and $[\cdot, \cdot]$ is the graded commutator.

The categories of BV_∞ - and IBL_∞ -algebras contained in the MV-formalism are not the most general ones; this is what $*$ and $**$ indicate. Before we explain this, let us agree on calling an IBL_∞ -algebra (\mathfrak{q}_{klg}) *strict* if $\mathfrak{q}_{0lg} = \mathfrak{q}_{k0g} = 0$ for all $k, l, g \in \mathbb{N}_0$, *input-strict*

⁴The ring \mathbb{Z}_4 is a local ring which is not a \mathbb{K} -algebra over its residue field $\mathbb{K} = \mathbb{Z}_2$. The ring of polynomials $\mathbb{R}[x]$ in a single variable x is a \mathbb{K} -algebra over its residue field $\mathbb{K} = \mathbb{R}$, but it is not a local ring because both (x) and $(x+1)$ are maximal ideals. In contrast to this, the ring of power series $\mathbb{K}((\hbar))$ is both a local ring and a \mathbb{K} -algebra. Thanks to Thorsten Hertl for pointing this out.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

if $q_{0lg} = 0$ for all $l, g \in \mathbb{N}_0$, and *weak* if we want to emphasize that operations with no input and output are allowed. The same terminology is used for morphisms.

* It is the category of *augmented strictly commutative* BV_∞ -algebras (see [CL07, Section 5] for the definition of a commutative BV_∞ -algebra). The augmentation ε is an additional data to get an MV-algebra. The coproduct is defined by $\delta := \delta_0$ (see (D.5)).

** It is the category of *(non-filtered) input-strict* IBL_∞ -algebras $(q_{klg})_{k \geq 1, l, g \geq 0}$ and input-strict morphisms $(f_{klg})_{k \geq 1, l, g \geq 0}$ which satisfy the following *finiteness condition* for all $k \geq 1$ and $g \geq 0$: for any $v \in SU$, we have

$$q_{klg}(v) = 0 \quad \text{and} \quad f_{klg}(v) = 0 \quad \text{for all but finitely many } l \in \mathbb{N}_0. \quad (D.11)$$

See [CFL15, Remark (6), p. 14] for the same finiteness condition.

(The transformation formulas between q_{klg} and Δ_i and f_{klg} and f_i were written down in Remark 3.2.9 in Part I. We will repeat them in Proposition D.2.6 below.)

The following filtered version of MV-algebras will allow us to describe more general weak IBL_∞ -algebras. In particular, we will be able to remove the restriction (D.11).

Definition D.2.1 (Complete filtered MV-algebra). *A complete filtered MV-algebra over a complete filtered graded algebra R over a field \mathbb{K} is a complete filtered graded vector space V over \mathbb{K} , a homogenous R -linear map $\Delta : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$ of finite filtration degree satisfying*

$$|\Delta| = -1, \quad \|\Delta\| \geq 0, \quad \|\Delta(1)\| > 0 \quad \text{and} \quad \Delta \circ \Delta = 0,$$

and operations $\mu : V \hat{\otimes} V \rightarrow V$, $\delta : V \rightarrow V \hat{\otimes} V$, $\eta : \mathbb{K} \rightarrow V$ and $\varepsilon : V \rightarrow \mathbb{K}$ such that $(V, \mu, \eta, \varepsilon)$ is a complete filtered augmented unital commutative associative algebra and $(V, \delta, \varepsilon, \eta)$ is a complete filtered coaugmented counital cocommutative coassociative coalgebra satisfying the limit conilpotency property (D.6). We often denote a complete filtered MV-algebra simply by (V, Δ) .

A morphism of complete filtered MV-algebras (V^+, Δ^+) and (V^-, Δ^-) over R is a homogenous R -linear map $f : V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R$ of finite filtration degree such that

$$|f| = 0, \quad \|f\| \geq 0, \quad \|f(1)\| > 0 \quad \text{and} \quad e^f \circ \Delta^+ = \Delta^- \circ e^f.$$

The composition \diamond of two composable morphisms f_1 and f_2 (i.e., the target of f_2 is the

D. Filtered IBL-infinity-algebras as filtered MV-algebras

source of \mathfrak{f}_1) is defined by

$$\mathfrak{f}_1 \diamond \mathfrak{f}_2 := \log(e^{\mathfrak{f}_1} \circ e^{\mathfrak{f}_2}). \quad (\text{D.12})$$

(The exponential and logarithm are well-defined by Proposition D.1.8.)

Remark D.2.2 (On complete filtered MV-algebras). (i) The condition $\|\mathfrak{f}(1)\| > 0$ is required for the exponential to converge; i.e., it might be seen as a technical condition. On the other hand, the condition $\|\Delta(1)\| > 0$ is optional, and we impose it in the manner of [MV15] and [CFL15].

(ii) We do not define “non-complete filtered MV-algebras” because it is not clear whether $\mathfrak{f}_1 \diamond \mathfrak{f}_2 : V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R$ for $V^+ \otimes R \xrightarrow{\mathfrak{f}_2} V^{+-} \otimes R \xrightarrow{\mathfrak{f}_1} V^- \otimes R$ restricts to a map $V^+ \otimes R \rightarrow V^- \otimes R$. This forces us to define morphisms as maps of completions $V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R$. In such category, two filtered MV-algebras would be isomorphic if and only if their completions were. This is not desired, and thus we define only “complete filtered MV-algebras”. \triangleleft

Inspired by (D.10), we define the following version of weak IBL_∞ -algebras.

Definition D.2.3 (Complete filtered IBL_∞ -algebra in MV-formalism). *Let $d \in \mathbb{Z}$ and $\gamma > 0$. A complete filtered IBL_∞ -algebra of bidegree (d, γ) in MV-formalism is a complete filtered MV-algebra $(V, \Delta, R, \mu, \delta, \varepsilon, \eta)$ satisfying the following conditions:*

- (1) $R = \mathbb{K}((\hbar))$ is the ring of Laurent series⁵ in a formal variable \hbar of degree $|\hbar| = 2d$ equipped with the complete, Hausdorff and exhaustive filtration

$$\mathcal{F}^\lambda \mathbb{K}((\hbar)) := \left\{ \sum_{i=-\infty}^{\infty} a_i \hbar^i \in \mathbb{K}((\hbar)) \mid a_i = 0 \text{ for } i < \frac{\lambda}{2\gamma} \right\} \quad \text{for } \lambda \in \mathbb{R}. \quad (\text{D.13})$$

- (2) There is a complete filtered graded vector space W such that if we define

$$U := W[1] \quad \text{and} \quad \mathcal{F}^\lambda U := (\mathcal{F}^{\lambda-\gamma} W)[1] \quad \text{for all } \lambda \in \mathbb{R}, \quad (\text{D.14})$$

then $(V = \hat{S}U, \mu, \delta, \eta, \varepsilon)$ is the completion of the symmetric bialgebra on U filtered by the induced filtration.

- (3) The BV-operator $\Delta : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$, where $\hat{S}U((\hbar)) := SU \hat{\otimes} \mathbb{K}((\hbar))$,⁶ decom-

⁵Another notation for $\mathbb{K}((\hbar))$ is $\mathbb{K}[[\hbar]][\hbar^{-1}]$ to emphasize that it is the localization of the ring of power series $\mathbb{K}[[\hbar]]$ at the powers of \hbar . An element of $\mathbb{K}((\hbar))$ is a formal power series $\sum_{i=-\infty}^{\infty} a_i \hbar^i$ with only finitely many non-zero $a_i \in \mathbb{K}$ for $i \leq 0$.

⁶In contrast to $\mathbb{K}((\hbar))$, an element of $\hat{S}U((\hbar))$, when seen as a power series, can have a non-zero coefficient at every power \hbar^i .

D. Filtered IBL-infinity-algebras as filtered MV-algebras

poses as

$$\Delta = \hbar^{-1} \Delta_0 + \Delta_1 + \hbar \Delta_2 + \hbar^2 \Delta_3 + \cdots, \quad (\text{D.15})$$

where for all $i \geq 0$ the operator $\Delta_i : \hat{S}U \rightarrow \hat{S}U$ is a linear differential operator of order $\leq i$.⁷

We often denote the data of a complete filtered IBL_∞ -algebra in MV-formalism simply by (W, Δ) .

A morphism of complete filtered IBL_∞ -algebras (W^+, Δ^+) and (W^-, Δ^-) of bidegree (d, γ) in MV-formalism is a morphism of complete filtered MV-algebras $\mathfrak{f} : \hat{S}U^+((\hbar)) \rightarrow \hat{S}U^-((\hbar))$ which decomposes as

$$\mathfrak{f} = \hbar^{-1} \mathfrak{f}_0 + \mathfrak{f}_1 + \hbar \mathfrak{f}_2 + \hbar^2 \mathfrak{f}_3 + \cdots, \quad (\text{D.16})$$

where for all $i \geq 0$ the map $\mathfrak{f}_i : \hat{S}U^+ \rightarrow \hat{S}U^-$ is \mathbb{K} -linear and satisfies

$$\hat{S}_j U^+ \subset \ker \mathfrak{f}_i \quad \text{for all } j > i. \quad (\text{D.17})$$

Notation D.2.4 (Replacing $*$ with \odot for $V = SU$). From now on, we will denote the convolution product $*$ on morphisms of SU by \odot . It is namely the same operation on morphisms which is denoted by \odot in Section 3.2 in Part I and in [CFL15].

For a map $\mathfrak{f} = \sum_{i=-\infty}^{\infty} \mathfrak{f}_{i+1} \hbar^i : \hat{S}U^+((\hbar)) \rightarrow \hat{S}U^-((\hbar))$, where $\mathfrak{f}_i : \hat{S}U^+ \rightarrow \hat{S}U^-$ is \mathbb{K} -linear, we denote

$$\langle \mathfrak{f} \rangle_{klg} := \pi_l \circ \mathfrak{f}_{k+g} \circ \iota_k : \hat{S}_k U^+ \rightarrow \hat{S}_l U^- \quad \text{for all } k, l \geq 0, g \in \mathbb{Z}.$$

This is the notation introduced in [CFL15, Equation (2.14)].

Based on [CFL15], we will now give an equivalent characterization of complete filtered IBL_∞ -algebras in MV-formalism and their morphisms in terms of their components (\mathfrak{q}_{klg}) and (\mathfrak{f}_{klg}) within the surface calculus. We will not repeat the interpretation of the algebraic relations in terms of gluing of surface; for this, see [CFL15]. We need the following lemma.

Lemma D.2.5 (Differential operators on filtered symmetric bialgebras). *Let U be a complete filtered graded vector space. Then the following holds for the complete filtered symmetric bialgebra $\hat{S}U$ and all $k \in \mathbb{N}_0$:*

⁷The definition of a differential operator on p. 286 generalizes in a straightforward way to complete filtered algebras.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

- (a) A linear homogenous map $D : \hat{S}U \rightarrow \hat{S}U$ of finite filtration degree is a differential operator of order $\leq k$ if and only if it can be written as

$$D = \sum_{i=0}^k D_i \odot \mathbb{1} \quad (\text{D.18})$$

for linear homogenous maps $D_i : \hat{S}_i U \subset \hat{S}U \rightarrow \hat{S}U$ of finite filtration degrees (here $D_i = 0$ on $\hat{S}_j U$ for $j \neq i$ is the trivial extension). The maps D_i are uniquely determined by D .

- (b) A linear homogenous map $\mathfrak{f} : \hat{S}U \rightarrow \hat{S}U$ of finite filtration degree satisfies $\hat{S}_{i>k} U \subset \ker \mathfrak{f}$ if and only if it can be written as $\mathfrak{f} = \mathfrak{f}_0 + \cdots + \mathfrak{f}_k$ for linear homogenous maps $\mathfrak{f}_i : \hat{S}_i U \rightarrow \hat{S}U$ of finite filtration degrees. The maps \mathfrak{f}_i are uniquely determined by \mathfrak{f} .

Proof. Before we begin, we recall that an operator $D : SU \rightarrow SU$ is called a *derivative* of order $\leq k$ if

$$D(1) = 0 \quad \text{and} \quad \psi_i^D(v_1, \dots, v_i)(1) = 0 \quad \text{for all } i \geq k+1 \text{ and } v_1, \dots, v_i \in V.$$

This definition can be extended to complete filtered algebras in a straightforward way.

- (a) The claim is implied by the following subclaim and proposition, which generalize in a straightforward way to the filtered case:

Subclaim (Derivatives and differential operators). A homogenous linear operator $D : SU \rightarrow SU$ is a differential operator of order $\leq k$ if and only if the linear operator $D' := D - D_0 \odot \mathbb{1}$, where $D_0 = D \circ \iota_0$, is a derivative of order $\leq k$ (recall that $\iota_k : S_k U \rightarrow SU$ is the inclusion).

Proof. Given a derivative D' of order $\leq k$ and a linear map $D_0 : S_0 U \subset SU \rightarrow SU$, we will check that $D := D' + D_0 \odot \mathbb{1}$ is a differential operator of order $\leq k$. We have

$$\begin{aligned} \underbrace{\psi_{k+2}^{D'}(v_1, \dots, v_{k+2})(1)}_{=0} &= [\psi_{k+1}^{D'}(v_1, \dots, v_{k+1}), L_{v_{k+2}}](1) \\ &= \psi_{k+1}^{D'}(v_1, \dots, v_{k+1})(v_{k+2}) - \underbrace{\psi_{k+1}^{D'}(v_1, \dots, v_{k+1})(1)}_{=0} v_{k+2} \end{aligned}$$

for all $v_1, \dots, v_{k+2} \in V$, and hence D' is also a differential operator of order $\leq k$. For all $v \in SU$, it holds

$$(D_0 \odot \mathbb{1})(v) = D_0(1)v,$$

and hence for all $v, v_1 \in SU$, we have

$$\begin{aligned}\psi_1^{D_0 \odot \mathbb{1}}(v_1)(v) &= (D_0 \odot \mathbb{1})(v_1 v) - (-1)^{D_0 v_1} v_1 (D_0 \odot \mathbb{1})(v) \\ &= D_0(1) v_1 v - (-1)^{D_0 v_1} v_1 D_0(1) v \\ &= 0.\end{aligned}$$

It follows that $D_0 \odot \mathbb{1}$ is a differential operator of order 0. Clearly, a sum of differential operators of orders $\leq k_1$ and $\leq k_2$ is a differential operator of order $\leq \max(k_1, k_2)$. Therefore, $D = D' + D_0 \odot \mathbb{1}$ is a differential operator of order $\leq k$. Moreover, since $D'(1) = 0$ by definition, we have $D(1) = D_0(1)$, i.e., $D_0 = D \circ \iota_0$.

Given a differential operator D of order $\leq k$, we define $D_0 := D \circ \iota_0$ and $D' := D - D_0 \odot \mathbb{1}$. Firstly, it holds

$$D'(1) = D(1) - (D_0 \odot \mathbb{1})(1) = 0.$$

Secondly, we have

$$\psi_i^D(v_1, \dots, v_i) = \psi^{D'}(v_1, \dots, v_i) = 0,$$

and hence $\psi_i^D(v_1, \dots, v_i)(1) = 0$ for all $i \geq k + 1$ and $v_1, \dots, v_i \in V$. Therefore, D' is a derivative of order $\leq k$. (Subclaim) \square

Proposition D.2.1 ([Mar97, Proposition 3.2]). A linear operator $D : SU \rightarrow SU$ is a derivative of order $\leq k$ if and only if it can be written as $D = \sum_{i=1}^k D_i \odot \mathbb{1}$ for unique $D_i : S_i U \rightarrow SU$.

(b) This is clear. We have $\mathfrak{f}_i = \mathfrak{f} \circ \iota_i$ for all $i = 0, \dots, k$. \square

Proposition D.2.6 (Filtered IBL_∞ -algebras in MV-formalism in components). *We have the following equivalences of structures:*

(a) *The data of a complete filtered IBL_∞ -algebra of bidegree (d, γ) in MV-formalism (W, Δ) is equivalent to the data of a complete filtered graded vector space W and a collection of \mathbb{K} -linear maps $\mathfrak{q}_{klg} : \hat{S}_k U \rightarrow \hat{S}_l U$ for all $k, l, g \in \mathbb{N}_0$ which are homogenous, have finite filtration degrees and satisfy the following conditions for all $k, l, g \geq 0$:*

$$(1) \quad |\mathfrak{q}_{klg}| = -2d(k + g - 1) - 1.$$

$$(2) \quad \|\mathfrak{q}_{klg}\| \geq -2\gamma(k + g - 1).$$

(3) *The inequality in (2) is strict whenever $k = 0$.*

(4) *The sum of maps $Q_{kg} := \sum_{l=0}^{\infty} \mathfrak{q}_{klg} : \hat{S}_k U \rightarrow \hat{S} U$ converges (in the sense of Lemma D.1.1).*

D. Filtered IBL-infinity-algebras as filtered MV-algebras

(5) The following equation holds:⁸

$$\sum_{\substack{h \geq 1 \\ k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k \\ l_1 + l_2 - h = l \\ g_1 + g_2 + h - 1 = g}} \mathfrak{q}_{k_1 l_1 g_1} \circ_h \mathfrak{q}_{k_2 l_2 g_2} = 0. \quad (\text{D.19})$$

(b) The data of a strict morphism $\mathfrak{f} : (W^+, \Delta^+) \rightarrow (W^-, \Delta^-)$ of complete filtered IBL_∞ -algebras of bidegree (d, γ) in MV-formalism, strict meaning that $\varepsilon^- \circ \mathfrak{f} = \mathfrak{f} \circ \eta^+ = 0$, is equivalent to a collection of \mathbb{K} -linear maps $\mathfrak{f}_{klg} : \hat{S}_k U^+ \rightarrow \hat{S}_l U^-$ for all $k, l, g \in \mathbb{N}_0$ which are homogenous, have finite filtration degrees and satisfy the following conditions for all $k, l, g \geq 0$:

- (1) $|\mathfrak{f}_{klg}| = -2d(k + g - 1)$.
- (2) $\|\mathfrak{f}_{klg}\| \geq -2\gamma(k + g - 1)$.
- (3) The inequality in (2) is strict whenever $k = 0$.
- (4) The sum $F_{kg} := \sum_{l=0}^{\infty} \mathfrak{f}_{klg} : \hat{S}_k U^+ \rightarrow \hat{S} U^-$ converges.
- (5) It holds $\mathfrak{f}_{k0g} = \mathfrak{f}_{0lg} = 0$.
- (6) The following equation holds:

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{h_1, \dots, h_r \geq 1 \\ k_1, \dots, k_r, k^-, l_1, \dots, l_r, l^-, g_1, \dots, g_r, g^- \geq 0 \\ k_1 + \dots + k_r = k \\ l_1 + \dots + l_r - k^- + l^- = l \\ g_1 + \dots + g_r + g^- + k^- - r = g}} \mathfrak{q}_{k^- l^- g^-} \circ_{h_1, \dots, h_r} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{h_1, \dots, h_r \geq 1 \\ k_1, \dots, k_r, k^+, l_1, \dots, l_r, l^+, g_1, \dots, g_r, g^+ \geq 0 \\ k_1 + \dots + k_r + k^+ - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1, \dots, h_r} \mathfrak{q}_{k^+ l^+ g^+}. \end{aligned} \quad (\text{D.20})$$

The term $r = 0$ on the left- or right-hand side is possible only for $k = 0$ or $l = 0$ and equals $\mathfrak{q}_{0lg}^- \circ \mathfrak{e} : \hat{S}_0 U^+ \rightarrow \hat{S}_l U^-$ or $\mathfrak{e} \circ \mathfrak{q}_{k0g}^+ : \hat{S}_k U^+ \rightarrow \hat{S}_0 U^-$, respectively, where \mathfrak{e} is the $\hat{S}_0 U^+ \rightarrow \hat{S}_0 U^-$ component of the unit for the

⁸Notice that (D.19) does not contain \mathfrak{q}_{00g} with $g \geq 0$ for any $k, l, g \geq 0$ (otherwise all \mathfrak{q}_{klg} appear). In fact, it will be clear from the proof that there is no condition on \mathfrak{q}_{00g} coming from $\Delta^2 = 0$; if we work over a general ring, then $\mathfrak{q}_{00g}(1)$ for $g \geq 0$ can be arbitrary odd elements in it. Note that this contradicts [CFL15, p. 47, bottom-most paragraph].

D. Filtered IBL-infinity-algebras as filtered MV-algebras

convolution product (D.4) (the identity under $\hat{S}_0 U^\pm \simeq \mathbb{K}$ via η^\pm).⁹

The equivalences (a) and (b) are given by the formulas

$$\Delta_i = \sum_{\substack{k,g \geq 0 \\ k+g=i}} Q_{kg} \odot \mathbb{1} \quad \text{and} \quad \mathfrak{f}_i = \sum_{\substack{k,g \geq 0 \\ k+g=i}} F_{kg} \quad \text{for all } i \geq 0, \text{ respectively.} \quad (\text{D.21})$$

We remark that the operations \circ_{h_1, \dots, h_r} are defined in Definition D.4.2 in the next section or in Definition 3.2.2 in Part I.

Suppose that \mathfrak{f}^+ and \mathfrak{f}^- are strict composable morphisms of complete filtered IBL_∞ -algebras in MV-formalism. Then we have the following:

- (c) The composition $\mathfrak{f} := \mathfrak{f}^- \diamond \mathfrak{f}^+$ is a strict morphism of complete filtered IBL_∞ -algebras in MV-formalism and its components \mathfrak{f}_{klg} for $k, l \geq 1, g \geq 0$ are given by

$$\begin{aligned} \mathfrak{f}_{klg} = \sum_{\substack{r^-, r^+ \geq 0 \\ k_1^-, l_1^-, \dots, k_{r^-}^-, l_{r^-}^- \geq 1 \\ k_1^+, l_1^+, \dots, k_{r^+}^+, l_{r^+}^+ \geq 1 \\ g_1^+, \dots, g_{r^+}^+, g_1^-, \dots, g_{r^-}^- \geq 0 \\ k_1^- + \dots + k_{r^-}^- = l_1^+ + \dots + l_{r^+}^+ \\ k_1^+ + \dots + k_{r^+}^+ = k \\ l_1^- + \dots + l_{r^-}^- = l \\ g_1^+ + \dots + g_{r^+}^+ + g_1^- + \dots + g_{r^-}^- - r^+ - r^- \\ + k_1^- + \dots + k_{r^-}^- + 1 = g}} \frac{1}{r^+! r^-!} (\mathfrak{f}_{k_1^- l_1^- g_1^-}^-, \dots, \mathfrak{f}_{k_{r^-}^- l_{r^-}^- g_{r^-}^-}^-) \circ_{\text{con}} (\mathfrak{f}_{k_1^+ l_1^+ g_1^+}^+, \dots, \mathfrak{f}_{k_{r^+}^+ l_{r^+}^+ g_{r^+}^+}^+), \quad (\text{D.22}) \end{aligned}$$

where \circ_{con} denotes the connected composition (see Definition D.4.2 in the next section).

Proof. (a) Suppose first that (W, Δ) is a complete filtered IBL_∞ -algebra of bidegree (d, γ) in MV-formalism; i.e., we have $\Delta = \Delta_0 \hbar^{-1} + \Delta_1 + \Delta_2 \hbar + \dots$ for $\Delta_i : \hat{S}U \rightarrow \hat{S}U$ differential operators of order $\leq i$, $|\Delta| = -1$, $\|\Delta\| \geq 0$ and $\|\Delta(1)\| > 0$. Using Lemma D.2.5, we write $\Delta_i = \sum_{j=0}^i D_{ij} \odot \mathbb{1}$ for \mathbb{K} -linear maps $D_{ij} : \hat{S}_j U \subset \hat{S}U \rightarrow \hat{S}U$ of finite filtration degrees which are uniquely determined by Δ_i , and we define $Q_{kg} := D_{k+g, k} : \hat{S}_k U \rightarrow \hat{S}U$ for all $k, g \in \mathbb{N}_0$. Next, we define $\mathfrak{q}_{klg} := \pi_l \circ Q_{kg} : \hat{S}_k U \rightarrow \hat{S}_l U$ for all $k, l, g \in \mathbb{N}_0$.

⁹If \mathfrak{f} is not strict, it would be interesting to know whether $e^\mathfrak{f} \Delta^+ = \Delta^- e^\mathfrak{f}$ is still equivalent to the connected calculus (D.20) or whether there are some other equations possibly involving disconnected gluing. Note that trying to incorporate \mathfrak{f}_{0lg} to the right-hand side and \mathfrak{f}_{k0g} to the left-hand side of (D.20) always leads to the disconnected calculus. Also note that \mathfrak{f}_{00g} for $g \geq 0$ do not appear in (D.20) for any $k, l, g \geq 0$; otherwise all \mathfrak{f}_{klg} appear. Finally, note that \mathfrak{q}_{klg}^+ and \mathfrak{q}_{klg}^- appear for all $k, l, g \geq 0$ and that it follows from (D.20) that $e \circ \mathfrak{q}_{00g}^+$ and $\mathfrak{q}_{00g}^- \circ e$ have to be equal for all $g \geq 0$.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

In the following computations, we will keep in mind that $|\hbar| = 2d$ and $\|\hbar\| = 2\gamma$. Using the formula $|f \circ g| = |f| + |g|$, which is valid for homogenous linear maps f and g , we have

$$\begin{aligned} |\Delta| = -1 &\iff \forall i \geq -1 : |\Delta_{i+1}| = -1 - di \\ &\iff \forall k, g \geq 0, k + g = i + 1 : |Q_{kg}| = -1 - di \\ &\iff \forall k, l, g \geq 0, k + g = i + 1 : |\mathbf{q}_{klg}| = -1 - d(k + g - 1). \end{aligned}$$

Thus (1) follows. Using that $\|f \circ g\| \geq \|f\| + \|g\|$ and $\|\sum_i v_i\| \geq \inf_i \|v_i\|$, we have

$$\begin{aligned} \|\Delta\| \geq 0 &\iff \forall i \geq -1 : \|\Delta_{i+1}\| \geq -2\gamma i \\ &\iff \forall k, g \geq 0, k + g = i + 1 : \|Q_{kg}\| \geq -2\gamma i \\ &\iff \forall k, l, g \geq 0, k + g = i + 1 : \|\mathbf{q}_{klg}\| \geq -2\gamma(k + g - 1). \end{aligned}$$

Thus (2) follows. The argument for “ \implies ” on the second line is inductive using that

$$Q_{kg} = \left(\Delta_{k+g} - \sum_{j=0}^{k-1} Q_{j, k+g-j} \odot \mathbb{1} \right) \circ \iota_k$$

for all $k, g \geq 0$. Evaluation of $\Delta(1)$ gives

$$\begin{aligned} \Delta(1) &= \Delta_0(1)\hbar^{-1} + \Delta_1(1) + \Delta_2(1)\hbar + \dots \\ &= Q_{00}(1)\hbar^{-1} + Q_{01}(1) + Q_{02}(1)\hbar + \dots \\ &= \left(\sum_{l=0}^{\infty} \mathbf{q}_{0l0}(1) \right) \hbar^{-1} + \left(\sum_{l=0}^{\infty} \mathbf{q}_{0l1}(1) \right) + \hbar \left(\sum_{l=0}^{\infty} \mathbf{q}_{0l2}(1) \right) + \dots, \end{aligned}$$

and we see that $\|\Delta(1)\| > 0$ is equivalent to (3). As for (4), it is implied by the Resummation Lemma (Lemma D.1.1). Namely, we have

$$\sum_{l=0}^{\infty} \mathbf{q}_{klg}(v) = \sum_{l=0}^{\infty} \pi_l(Q_{kg}(v)) \stackrel{D.1.1}{=} Q_{kg}(v) \quad \text{for all } v \in \hat{S}_k U.$$

Thus $\sum_{l=0}^{\infty} \mathbf{q}_{klg}$ converges to Q_{kg} .

On the other hand, given \mathbf{q}_{klg} satisfying (1), (2), (3) and (4), we clearly get an operator $\Delta : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$ which has the decomposition (D.15) and which satisfies $|\Delta| = 0$, $\|\Delta\| \geq 0$ and $\|\Delta(1)\| > 0$.

Assuming (1), (2), (3) and (4), it remains to check that $\Delta^2 = 0$ is equivalent to (5).

D. Filtered IBL-infinity-algebras as filtered MV-algebras

This is the same computation as in [CFL15, Section 2], just in the filtered setting and allowing $k = 0$ or $l = 0$ (it will turn out that it does not change anything). Recall the notation from Section 3.2 in Part I that $\hat{D} := D \odot \mathbb{1} = \mu(D \hat{\otimes} \mathbb{1})\delta$. We will be using formulas from Remark 3.2.3 in that section, which are proven in Proposition D.4.4 in the next section. Note that by the Resummation Lemma, elements of the completion can be summed up in any order, even as nested infinite sums, provided one (and hence all) of these sums converges. Therefore, we could, in principle, write just one sum \sum and think of Δ as of the sum $\sum \hat{\mathbf{q}}_{klg} \hbar^{k+g-1}$ over all $k, l, g \geq 0$.

We have

$$\begin{aligned} \Delta \circ \Delta &= \sum_{i=-2}^{\infty} \left(\sum_{\substack{k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ k_1 + k_2 + g_1 + g_2 = i+2}} \hat{\mathbf{q}}_{k_1 l_1 g_1} \circ \hat{\mathbf{q}}_{k_2 l_2 g_2} \right) \hbar^i \\ &= \sum_{i=-2}^{\infty} \left(\sum_{\substack{k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ k_1 + k_2 + g_1 + g_2 = i+2}} \sum_{h=0}^{\min(k_1, l_2)} \overline{\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}} \right) \hbar^i, \end{aligned}$$

and we see that

$$\langle \Delta^2 \rangle_{klg} = \sum_{\substack{k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ k_1 + k_2 + g_1 + g_2 = k+g+1}} \sum_{h=0}^{\min(k_1, l_2)} \pi_l \circ \overline{\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}} \circ \iota_k \quad \text{for all } k, l \geq 0, g \geq -1.$$

Because $\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2} : \hat{S}_{k_1+k_2-h}U \rightarrow \hat{S}_{l_1+l_2-h}U$, it follows from the definition $\hat{f} := \mu(f \otimes \mathbb{1})\delta$ that $\pi_l \circ \overline{\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}} \circ \iota_k = 0$ unless $0 \leq k_1 + k_2 - h \leq k$, $0 \leq l_1 + l_2 - h \leq l$ and $l_1 + l_2 + h - (k_1 + k_2 + h) = l - k$. For fixed $k, l, h \geq 0, g \geq -1$, it holds

$$\left\{ \begin{array}{l} k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ 0 \leq k_1 + k_2 - h \leq k \\ 0 \leq l_1 + l_2 - h \leq l \\ l_1 + l_2 - k_1 - k_2 = l - k \\ k_1 + k_2 + g_1 + g_2 = k + g + 1 \end{array} \right\} = \bigsqcup_{s=0, \dots, k} \left\{ \begin{array}{l} k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k - s \\ l_1 + l_2 - h = l - s \\ g_1 + g_2 + h - 1 = g + s \end{array} \right\},$$

and hence

$$\begin{aligned}
 \langle \Delta^2 \rangle_{klg} &= \sum_{s=0}^k \sum_{h=0}^{g+s+1} \sum_{\substack{k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k - s \\ l_1 + l_2 - h = l - s \\ g_1 + g_2 + h - 1 = g + s}} \pi_l \circ \overline{\mathfrak{q}_{k_1 l_1 g_1} \circ_h \mathfrak{q}_{k_2 l_2 g_2}} \circ \iota_k \\
 &= \underbrace{\sum_{h=0}^{g+1} \sum_{\substack{k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k \\ l_1 + l_2 - h = l \\ g_1 + g_2 + h - 1 = g}} \mathfrak{q}_{k_1 l_1 g_1} \circ_h \mathfrak{q}_{k_2 l_2 g_2}}_{=: (\square)_{klg}} + \sum_{s=1}^k \pi_l \circ \overline{(\square)_{k-s, l-s, g+s}} \circ \iota_k. \tag{D.23}
 \end{aligned}$$

We see immediately that

$$\langle \Delta^2 \rangle_{klg} = 0 \quad \text{for all } k, l \geq 0, g \geq -1, \tag{D.24}$$

which is equivalent to $\Delta^2 = 0$, follows from $(\square)_{klg} = 0$ for all $k, l \geq 0, g \geq -1$. The reverse implication is proven by induction on the linear order \prec on signatures (k, l, g) defined in [CFL15, Definition 2.4]. It holds $(k', l', g') \prec (k, l, g)$, by definition, if one of the following conditions is satisfied:

- (i) $k' + l' + 2g' < k + l + 2g$,
- (ii) $k' + l' + 2g' = k + l + 2g$ and $g' > g$, or
- (iii) $k' + l' + 2g' = k + l + 2g$ and $g' = g$ and $k' < k$.

For the last sum in (D.23), we denote $k_s := k - s$, $l_s := l - s$ and $g_s := g + s$ and compute

$$(k - k_s) + (l - l_s) + 2(g - g_s) = s + s + 2(-s) = 0.$$

Therefore, case (ii) applies, and so $(k_s, l_s, g_s) \prec (k, l, g)$ for all $s = 1, \dots, k$. This shows the equivalence of $\Delta^2 = 0$ and $(\square)_{klg} = 0$ for all $k, l \geq 0, g \geq -1$. Finally, $(\square)_{k, l, -1} = 0$ holds automatically because $g_1 + g_2 + h = 0$ implies $h = 0$, and all terms $\mathfrak{q}_{k_1 l_1 g_1} \circ_h \mathfrak{q}_{k_2 l_2 g_2}$ with $h = 0$ vanish. This is because \mathfrak{q}_{klg} are odd and $f_1 \circ_0 f_2 = (-1)^{|f_1||f_2|} f_2 \circ_0 f_1$.

(b) Exactly as in (a), we first prove the equivalence of $|\mathfrak{f}| = 0$, $\|\mathfrak{f}\| \geq 0$, $\|\mathfrak{f}(1)\| > 0$ and the conditions (D.16) and (D.17) to (1), (2), (3) and (4) under the correspondence (D.21). Clearly, $\varepsilon^- \circ \mathfrak{f} = \mathfrak{f} \circ \eta^+ = 0$ is equivalent to (5).

Assuming (1), (2), (3) and (4), it remains to check that the equation $e^\mathfrak{f} \Delta^+ = \Delta^- e^\mathfrak{f}$ is equivalent to (6). This is again the same computation as in [CFL15, Section 2]. We will

do it in the weak case and then restrict to the strict case for the induction. We have

$$\begin{aligned}
 e^{\mathfrak{f}} \circ \Delta^+ &= \left(\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i_1, \dots, i_r \geq -1} \sum_{\substack{k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k_1 + g_1 = i_1 + 1, \dots, k_r + g_r = i_r + 1}} \mathfrak{f}_{k_1 l_1 g_1} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \hbar^{i_1 + \dots + i_r} \right) \\
 &\quad \circ \left(\sum_{i^+ = -1}^{\infty} \sum_{\substack{k^+, l^+, g^+ \geq 0 \\ k^+ + g^+ = i^+ + 1}} \hat{\mathfrak{q}}_{k^+ l^+ g^+}^+ \hbar^{i^+} \right) \\
 &= \sum_{i=-\infty}^{\infty} \left(\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{k^+, l^+, g^+, k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^+ + k_1 + \dots + k_r + g^+ + g_1 + \dots + g_r = i + r + 1}} (\mathfrak{f}_{k_1 l_1 g_1} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r}) \circ \hat{\mathfrak{q}}_{k^+ l^+ g^+}^+ \right) \hbar^i \\
 &= \sum_{i=-\infty}^{\infty} \left(\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^+ + k_1 + \dots + k_r + g^+ + g_1 + \dots + g_r = i + r + 1}} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = l^+}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1, \dots, h_r} \mathfrak{q}_{k^+ l^+ g^+}^+ \right) \hbar^i \\
 &\quad \underbrace{\hspace{15em}}_{=: (*)^+_{i,r} : \hat{S}U^+ \rightarrow \hat{S}U^-}
 \end{aligned}$$

and similarly

$$\begin{aligned}
 \Delta^- \circ e^{\mathfrak{f}} &= \sum_{i=-\infty}^{\infty} \left(\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{k^-, l^-, g^- \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^- + k_1 + \dots + k_r + g^- + g_1 + \dots + g_r = i + r + 1}} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k^-}} \mathfrak{q}_{k^- l^- g^-}^- \circ_{h_1, \dots, h_r} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \right) \hbar^i \\
 &\quad \underbrace{\hspace{15em}}_{=: (*)^-_{i,r} : \hat{S}U^+ \rightarrow \hat{S}U^-}
 \end{aligned}$$

We will consider $\pi_l \circ (*)^+_{i,r} \circ \iota_k$ for fixed $i \in \mathbb{Z}$ and $k, l \geq 0$. Because of the definition of \circ_{h_1, \dots, h_r} , only the terms with $l_1 + \dots + l_r = l$ and $k^+ + k_1 + \dots + k_r - h_1 - \dots - h_r = k^+ + k_1 + \dots + k_r - l^+ = k$ survive in $\pi_l \circ (*)^+_{i,r} \circ \iota_k$. Denoting $g := i - k + 1$, we have

$$\begin{aligned}
 k^+ + k_1 + \dots + k_r + g^+ + g_1 + \dots + g_r &= i + r + 1 & g_1 + \dots + g_r + g^+ + l^+ - r &= g \\
 k^+ + k_1 + \dots + k_r - l^+ &= k & \iff & k^+ + k_1 + \dots + k_r - l^+ = k \\
 l_1 + \dots + l_r &= l & & l_1 + \dots + l_r = l.
 \end{aligned}$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

Therefore, it holds

$$\pi_l \circ (*)_{i,r}^+ \circ \iota_k = \sum_{\substack{k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^+ + k_1 + \dots + k_r - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g}} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = l^+}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1, \dots, h_r} \mathfrak{q}_{k^+ + l^+ + g^+}^+$$

Collecting the terms with $h_i = 0$ and using the graded commutativity of \odot , we obtain

$$\begin{aligned} \pi_l \circ (*)_{i,r}^+ \circ \iota_k &= \underbrace{\sum_{\substack{h_1, \dots, h_r \geq 1 \\ k^+, l^+, g^+, k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ h_1 + \dots + h_r = l^+ \\ k_1 + \dots + k_r + k^+ - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g}} (\square^+)^r_{klg} \\ &\quad + \sum_{r'=0}^{r-1} \binom{r}{r'} \sum_{\substack{h_1, \dots, h_{r'} \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_{r'}, l_{r'}, g_{r'} \geq 0 \\ h_1 + \dots + h_{r'} = l^+ \\ k_1 + \dots + k_{r'} + k^+ - l^+ = k \\ l_1 + \dots + l_{r'} = l \\ g_1 + \dots + g_{r'} + g^+ + l^+ - r' = g}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_{r'} l_{r'} g_{r'}}) \circ_{h_1, \dots, h_{r'}} \mathfrak{q}_{k^+ + l^+ + g^+}^+ \\ &\quad \odot \mathfrak{f}_{k_{r'+1} l_{r'+1} g_{r'+1}} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \\ &= (\square^+)^r_{klg} + \sum_{r'=0}^{r-1} \binom{r}{r'} \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ 0 \leq g' \leq g + r - r'}} \left((\square^+)^{r'}_{k' l' g'} \right. \\ &\quad \left. \odot \sum_{\substack{k_{r'+1}, l_{r'+1}, g_{r'+1}, \dots, k_r, l_r, g_r \geq 0 \\ k_{r'+1} + \dots + k_r = k - k' \\ l_{r'+1} + \dots + l_r = l - l' \\ g_{r'+1} + \dots + g_r - (r - r') = g - g'}} \mathfrak{f}_{k_{r'+1} l_{r'+1} g_{r'+1}} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \right). \\ &=: (\triangle)^{r-r'}_{k-k', l-l', g-g'} \end{aligned}$$

We used here that for fixed $r \geq 0$, $0 \leq r' \leq r - 1$, $k, l \geq 0$ and $g \in \mathbb{Z}$, we have

$$\left\{ \begin{array}{l} h_1, \dots, h_{r'} \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ \frac{k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0}{h_1 + \dots + h_{r'} = l^+} \\ k_1 + \dots + k_r + k^+ - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g \end{array} \right\} = \bigsqcup_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ -r' \leq g' \leq g+r-r'}} \left\{ \begin{array}{l} h_1, \dots, h_{r'} \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ \frac{k_1, l_1, g_1, \dots, k_{r'}, l_{r'}, g_{r'} \geq 0}{h_1 + \dots + h_{r'} = l^+} \\ k_1 + \dots + k_{r'} + k^+ - l^+ = k' \\ l_1 + \dots + l_{r'} = l' \\ g_1 + \dots + g_{r'} + g^+ + l^+ - r' = g' \end{array} \right\} \\ \times \left\{ \begin{array}{l} \frac{k_{r'+1}, l_{r'+1}, g_{r'+1}, \dots, k_r, l_r, g_r \geq 0}{k_{r'+1} + \dots + k_r = k - k'} \\ l_{r'+1} + \dots + l_r = l - l' \\ g_{r'+1} + \dots + g_r - (r - r') = g - g' \end{array} \right\},$$

where the notation is the vertical version of $\{\cdot \mid \cdot\}$. In fact, the summation starts from $g' = 0$ because if $g' < 0$, then $(\square^+)^{r'}_{k'l'g'} = 0$. Summing over $r \in \mathbb{N}_0$, we get

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)_{i,r}^+ \circ \iota_k \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} (\square^+)^r_{klg} + \sum_{r=0}^{\infty} \sum_{r'=0}^{r-1} \frac{1}{r'!} \frac{1}{(r-r')!} \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ 0 \leq g' \leq g+r-r'}} (\square^+)^{r'}_{k'l'g'} \odot (\Delta)^{r-r'}_{k-k', l-l', g-g'} \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} (\square^+)^r_{klg} + \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ g' \geq 0}} \left(\sum_{r'=0}^{\infty} \frac{1}{r'!} (\square^+)^{r'}_{k'l'g'} \right) \odot \left(\sum_{t=1}^{\infty} \frac{1}{t!} (\Delta)^t_{k-k', l-l', g-g'} \right), \end{aligned} \quad (\text{D.25})$$

where we used the substitution $t = r - r'$ and the fact that if $g - g' < -t$, then $(\Delta)^t_{k-k', l-l', g-g'} = 0$. Similarly, we obtain

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)_{i,r}^- \circ \iota_k \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} (\square^-)^r_{klg} + \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ g' \geq 0}} \left(\sum_{t=1}^{\infty} \frac{1}{t!} (\Delta)^t_{k-k', l-l', g-g'} \right) \odot \left(\sum_{r'=0}^{\infty} \frac{1}{r'!} (\square^-)^{r'}_{k'l'g'} \right). \end{aligned} \quad (\text{D.26})$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

Now, $e^{\mathfrak{f}} \Delta^+ = \Delta^- e^{\mathfrak{f}}$ is equivalent to

$$\sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)_{i,r}^+ \circ \iota_k = \sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)_{i,r}^- \circ \iota_k \quad \text{for all } k, l \geq 0 \text{ and } i \in \mathbb{Z}. \quad (\text{D.27})$$

By our previous computations, this follows from

$$\sum_{r=0}^{\infty} \frac{1}{r!} (\square^+)^r_{klg} = \sum_{r=0}^{\infty} \frac{1}{r!} (\square^-)^r_{klg} \quad \text{for all } k, l, g \geq 0,$$

which are precisely equations (D.20). In order to prove the reverse implication, one needs to do induction on signatures (k, l, g) as in (a). It is possible in the strict case by the following lemma.

Subclaim (Induction in strict case I). Suppose that $\mathfrak{f}_{0lg} = \mathfrak{f}_{k0g} = 0$ for all $k, l, g \geq 0$. Then for any $k, l \geq 0$ and $i \in \mathbb{Z}$, $g = i - k + 1$, only the terms with $(k', l', g') \prec (k, l, g)$ contribute to the sums on the right-hand side of (D.25) and (D.26). These sums vanish if $(k, l, g) = (0, 0, 0)$.

Proof. Recall the definition of \prec on p. 296. First of all, $(\Delta)_{k-k', l-l', g-g'}^t \neq 0$ implies

$$D := (k + l + 2g) - (k' + l' + 2g') = (k - k') + (l - l') + 2(g - g') \geq t + t + 2(-t) = 0,$$

where $k - k', l - l' \geq t$ holds due to strictness. Suppose that $D = 0$. Then $g' \geq g$ must hold because $k - k' \geq 0$ and $l - l' \geq 0$. If $g' = g$, then $l = l'$ and $k = k'$ must hold; this is a contradiction with $k - k' \geq t \geq 1$. Therefore, it holds $g' > g$, and case (ii) of the definition of \prec applies.

If $(k, l, g) = (0, 0, 0)$, then $(\Delta)_{0,0,-g'}^t = 0$ because $t \geq 1$ and $k - k' \geq t$. (*Subclaim*) \square

(c) We have

$$e^{\mathfrak{f}} = \sum_{k,l,g \geq 0} \langle e^{\mathfrak{f}} \rangle_{klg} h^{k+g-1},$$

where

$$\langle e^{\mathfrak{f}} \rangle_{klg} = \sum_{\substack{r \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k_1 + \dots + k_r = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r - r + 1 = g}} \frac{1}{r!} \mathfrak{f}_{k_1 l_1 g_1} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r}$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

$$= \mathfrak{f}_{klg} + \sum_{\substack{r \geq 2 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k_1 + \dots + k_r = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r - r + 1 = g}} \frac{1}{r!} \mathfrak{f}_{k_1 l_1 g_1} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \quad (\text{D.28})$$

for all $(k, l, g) \neq (0, 0, 1)$ with $k, l \geq 0, g \in \mathbb{Z}$. For $(k, l, g) = (0, 0, 1)$, one has to add the unit e coming from the summand with $r = 0$. We find that

$$\begin{aligned} \langle e^{\mathfrak{f}^-} e^{\mathfrak{f}^+} \rangle_{klg} = & \sum_{\substack{r^-, r^+ \geq 0 \\ k_1^-, l_1^-, g_1^-, \dots, k_{r^-}^-, l_{r^-}^-, g_{r^-}^- \geq 0 \\ k_1^+, l_1^+, g_1^+, \dots, k_{r^+}^+, l_{r^+}^+, g_{r^+}^+ \geq 0 \\ k_1^- + \dots + k_{r^-}^- = l_1^+ + \dots + l_{r^+}^+ \\ k_1^+ + \dots + k_{r^+}^+ = k \\ l_1^- + \dots + l_{r^-}^- = l \\ g_1^+ + \dots + g_{r^+}^+ + g_1^- + \dots + g_{r^-}^- - r^+ - r^- + k_1^- + \dots + k_{r^-}^- + 1 = g}} \frac{1}{r^+! r^-!} (\mathfrak{f}_{k_1^- l_1^- g_1^-}^- \odot \dots \odot \mathfrak{f}_{k_{r^-}^- l_{r^-}^- g_{r^-}^-}^-) \circ (\mathfrak{f}_{k_1^+ l_1^+ g_1^+}^+ \odot \dots \odot \mathfrak{f}_{k_{r^+}^+ l_{r^+}^+ g_{r^+}^+}^+) \end{aligned}$$

for all $k, l \geq 0, g \in \mathbb{Z}$. The composition $\mathfrak{f}^- \diamond \mathfrak{f}^+ = \log(e^{\mathfrak{f}^-} \circ e^{\mathfrak{f}^+})$ is the unique \mathfrak{f} such that $e^{\mathfrak{f}} = e^{\mathfrak{f}^-} \circ e^{\mathfrak{f}^+}$; this is equivalent to

$$\langle e^{\mathfrak{f}} \rangle_{klg} = \langle e^{\mathfrak{f}^-} e^{\mathfrak{f}^+} \rangle_{klg} \quad \text{for all } k, l \geq 0, g \in \mathbb{Z}. \quad (\text{D.29})$$

In the strict case, this can be solved for (\mathfrak{f}_{klg}) by induction on signatures.

Subclaim (Induction in strict case II). Suppose that \mathfrak{f}^+ and \mathfrak{f}^- are strict. Then it holds $(k_i, l_i, g_i) \prec (k, l, g)$ for all $i = 1, \dots, r$ and $r \geq 2$ in (D.28) for all $k, l \geq 0, g \in \mathbb{Z}$.

Proof. We compute

$$k + l + 2g - 2 = (k_1 + l_1 + 2g_1 - 2) + \dots + (k_r + l_r + 2g_r - 2),$$

where $k_i + l_i + 2g_i - 2 \geq 0$ by strictness. It follows that $k + l + 2g \geq k_i + l_i + 2g_i$ holds for all $i = 1, \dots, r$. If, e.g., $k + l + 2g = k_1 + l_1 + 2g_1$, then $k_2 = l_2 = \dots = k_r = l_r = 1, g_2 = \dots = g_r = 0$, and hence $g = 1 - r$. However, this can not happen as $r \geq 2$ and $g \geq 0$. Therefore, case (i) of \prec always occurs. *(Subclaim)* \square

Using this, we can set $\mathfrak{f}_{110} = \mathfrak{f}_{110}^- \circ \mathfrak{f}_{110}^+$ and $\mathfrak{f}_{0lg} = \mathfrak{f}_{k0g} := 0$ for all $k, l, g \geq 0$ and solve (D.29) for (\mathfrak{f}_{klg}) for all $k, l \geq 1, g \geq 0$ by induction over (k, l, g) . The solution (D.22) will solve (D.29) for all $k, l \geq 0, g \in \mathbb{Z}$ (the equations for $g < 0$ consist of disconnected gluings and can be checked by splitting into connected components). Conditions on the filtration degree are easy to check as in [CFL15, Lemma 8.5]. \square

D. Filtered IBL-infinity-algebras as filtered MV-algebras

We expect that weak complete filtered IBL_∞ -algebras in MV-formalism with weak morphisms form a subcategory of the category of complete filtered MV-algebras (to see this, it remains to prove a version of (c) of Proposition D.2.6 for weak morphisms). The identity morphism in this category is the continuous $\mathbb{K}((\hbar))$ -linear extension of the trivial extension of the identity $\mathbb{1} : \hat{U} \rightarrow \hat{U}$ to $\hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$ (see (iii) of Remark D.1.10).

In the following, we will compare Definition D.2.3 to the definition of filtered IBL_∞ -algebras from [CFL15, Section 8].

A *filtered IBL_∞ -algebra of bidegree (d, γ) over \mathbb{K} on a complete filtered graded vector space W according to [CFL15]* is a collection of homogenous \mathbb{K} -linear maps $\mathbf{q}_{klg} : \hat{S}_k U \rightarrow \hat{S}_l U$ of finite filtration degrees for all $k, l, g \in \mathbb{N}_0$, where U was defined in (D.14), which satisfy, firstly, (1) and (2) of (a) of Proposition D.2.6 with the strict inequality in (2) for all (k, l, g) from the set

$$\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (2, 0, 0), (0, 2, 0), (0, 0, 1)\} \quad (\text{D.30})$$

and, secondly, the equations (D.24). A *morphism of filtered IBL_∞ -algebras of bidegree (d, γ) over \mathbb{K} on complete filtered graded vector spaces W^+ and W^- according to [CFL15]* is a collection of homogenous \mathbb{K} -linear maps $\mathbf{f}_{klg} : \hat{S}_k U^+ \rightarrow \hat{S}_l U^-$ of finite filtration degrees for all $k, l, g \in \mathbb{N}_0$ which satisfy, firstly, (1) and (2) of (b) of Proposition D.2.6 with the strict inequality in (2) for $(k, l, g) \in (\text{D.30})$ and, secondly, the equations (D.27).

Recall (D.14) and notice that if we define the filtration $\mathcal{F}_W^\lambda U := (\mathcal{F}^\lambda W)[1]$ for all $\lambda \in \mathbb{R}$ and denote the corresponding filtration degree by $\|\cdot\|_W$, then it holds $\|\cdot\| = \|\cdot\|_W + \gamma$ on U , and for a map $\mathbf{q}_{klg} : \hat{S}_k U \rightarrow \hat{S}_l U$, we have

$$\|\mathbf{q}_{klg}\| \geq -2\gamma(k + g - 1) \iff \|\mathbf{q}_{klg}\|_W \geq \gamma(2 - 2g - k - l).$$

Signatures (D.30) together with $(1, 1, 0)$ correspond to unstable surfaces, i.e., those (k, l, g) for which $\chi_{klg} = 2 - 2g - k - l \geq 0$. Allowing morphisms \mathbf{f} which have non-zero components of these signatures leads to the appearance of an infinite number of summands in $\langle \Delta^- e^{\mathbf{f}} \rangle_{klg}$, $\langle e^{\mathbf{f}} \Delta^+ \rangle_{klg}$ and $\langle e^{\mathbf{f}^-} e^{\mathbf{f}^+} \rangle_{klg}$ (and in the Maurer-Cartan equation (D.38) later); the strict inequalities for filtration degrees of these components seem to be the minimal condition to algebraically handle this situation. Again, the author does not see any technical reason for imposing strict filtration degree conditions for \mathbf{q}_{klg} . See Figure D.3 for the graphical explanation of the *bubbling*.

We observe the following differences between our approach to weak IBL_∞ -algebras and the original approach from [CFL15]:

D. Filtered IBL-infinity-algebras as filtered MV-algebras

- (i) *Bubbling.* The definition of [CFL15] is symmetric in inputs and outputs, whereas our definition is not (compare (D.30) and (3) of Proposition D.2.6). We will illustrate this in Example D.2.9 below.
- (ii) *General ring.* The theory of [CFL15] is formulated over a filtered graded commutative ring R , i.e., W is a filtered R -module and the maps are R -linear. We can tweak our formalism to handle this situation if R is a \mathbb{K} -algebra (e.g., the Novikov ring) by replacing $\mathbb{K}((\hbar))$ with $R((\hbar))$ in Definition D.2.3.
- (iii) *Completions.* The BV-formalism of [CFL15] uses the completion $\hat{S}U$ with respect to the union filtration \mathcal{F}_\cup of the induced filtration and the filtration by weights (see Section D.1); however, this is not necessarily a filtered bialgebra (see Example D.1.5), Proposition D.1.8 might not apply, and it is not clear whether e^\flat (or the multiplication with the exponential of the Maurer-Cartan element later) are well-defined operators on $\hat{S}U((\hbar))$.

In the following case, our theory and the theory of [CFL15] agree.

Proposition D.2.7 (Equivalence of definitions in bounded case over \mathbb{K}). *Let $d \in \mathbb{Z}$ and $\gamma > 0$. For $\zeta \in \mathbb{R}$, we denote by \mathcal{W}_ζ the class of complete filtered graded vector spaces W for which there is an $\alpha > \zeta$ such that*

$$\mathcal{F}^\alpha W = W. \tag{D.31}$$

For such W , we will consider weak IBL_∞ -algebras of bidegree (d, γ) over \mathbb{K} and their weak IBL_∞ -morphisms. We have the following:

- (a) *Filtered IBL_∞ -algebras and morphisms from [CFL15] over $\mathcal{W}_{-\gamma}$ are also complete filtered MV-algebras and morphisms from Definition D.2.3, respectively.*
- (b) *The two definitions agree over \mathcal{W}_0 . If in addition $\gamma \geq 1$, then also the BV-formalisms are identical.*
- (c) *The canonical dIBL -structure on the (reduced) dual cyclic bar complex*

$$W = \hat{\mathrm{B}}_{\mathrm{cyc}}^* V[2 - n]$$

for a Poincaré duality algebra V of degree n from Section 3.4 in Part I satisfies (b).

Proof. For $W \in \mathcal{W}_\zeta$, there is an $\alpha > \zeta$ such that

$$\|\cdot\| = \|\cdot\|_W + \gamma \geq \alpha + \gamma \quad \text{on } U,$$

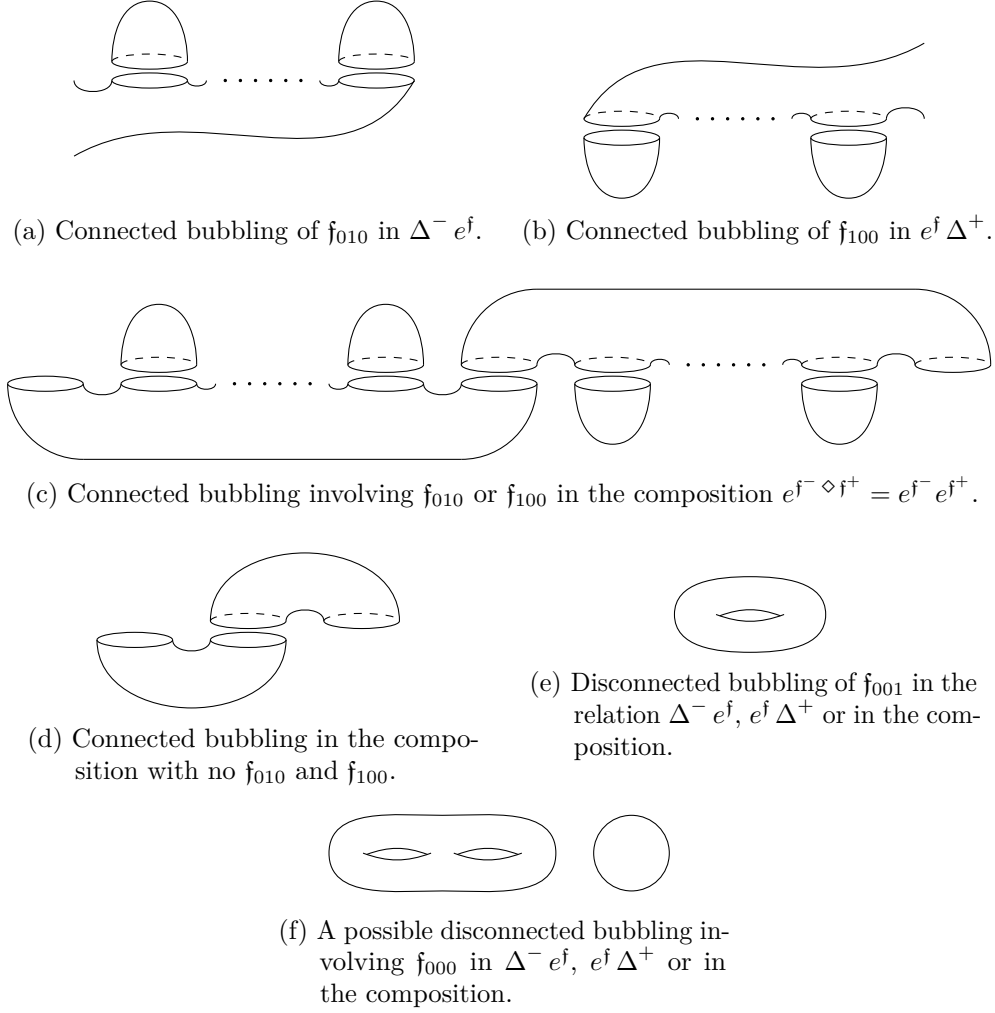


Figure D.3.: Bubbling in IBL_∞ -relations. If we glue any of the components above to a surface of signature (k, l, g) , the signature remains the same. Note that since g is defined via the Euler characteristics, adding a disconnected component without inputs and outputs decreases g by one.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

and hence

$$\|\mathbf{q}_{klg}(v)\|_U \geq l(\alpha + \gamma) \quad \text{for all } v \in \hat{S}_k U \quad (\text{D.32})$$

and any $k \geq 0, l \geq 1, g \geq 0$. It holds also for $l = 0$ because the filtration on $\hat{S}_0 U \simeq \mathbb{K}$ is non-negative. We see that if $\zeta \geq -\gamma$, which is the case of both (a) and (b), then the sums $\sum_{l=0}^{\infty} \mathbf{q}_{klg}$ converge automatically; i.e., (4) of (a) of Proposition D.2.6 holds. It remains to study the relation of the strict filtration degree conditions

$$\forall l, g \in \mathbb{N}_0 : \quad \|\mathbf{q}_{0lg}\| = \|\mathbf{q}_{0lg}(1)\| > -2\gamma(g-1) \quad (\text{D.33})$$

and

$$\forall (k, l, g) \in (\text{D.30}) : \quad \|\mathbf{q}_{klg}\| > -2\gamma(k+g-1). \quad (\text{D.34})$$

Here, $\|\mathbf{q}_{0lg}\| = \|\mathbf{q}_{0lg}(1)\|$ holds because the trivial filtration on \mathbb{K} is non-negative and it holds $\|1\| = 0$. For morphisms, the story is the same, and the rest is implied by Proposition D.2.6.

(a) Assuming (D.34), we have to prove (D.33). It is easy to see that if $\zeta \geq -\gamma$, then (D.33) follows from (D.32) except for $(k, l, g) = (0, 0, 0), (0, 1, 0), (0, 2, 0)$ and $(0, 0, 1)$. These are precisely the signatures from (D.30) with no input; in particular, they are implied by (D.34).

(b) Assuming (D.33), we have to prove (D.34). Clearly, (D.33) implies the strict inequality for all $(k, l, g) \in (\text{D.30})$ with no input. It remains to check it for $(1, 0, 0)$ and $(2, 0, 0)$. Using $\zeta \geq -\gamma$, we obtain

$$\|\mathbf{f}_{100}(v)\| \geq \|\mathbf{f}_{100}\| + \|v\| \geq 0 + (\gamma + \alpha) > 0 \quad \text{for all } v \in \hat{S}_1 V,$$

and using $\zeta \geq 0$, we obtain

$$\|\mathbf{f}_{200}(v)\| \geq \|\mathbf{f}_{200}\| + \|v\| \geq -2\gamma + 2(\gamma + \alpha) > 0 \quad \text{for all } v \in \hat{S}_2 V.$$

Because $\text{im } \mathbf{f}_{100}, \text{im } \mathbf{f}_{200} \subset \mathbb{K}$, it must hold $\mathbf{f}_{100} = \mathbf{f}_{200} = 0$, and hence the strict filtration degree condition is automatically satisfied.

The equivalence of BV-formalisms for $\gamma \geq 1$ follows from Lemma D.1.6 because $\mathcal{F}^1 U = (\mathcal{F}^{1-\gamma} W)[1] = W[1] = U$, where $\alpha > 0 \geq 1 - \gamma$.

(c) For any $k \in \mathbb{N}_0$ and $k \leq \lambda < k+1$, we have

$$\mathcal{F}_w^\lambda \mathbf{B}^{\text{cyc}} V = \mathbf{B}_1^{\text{cyc}} V \oplus \cdots \oplus \mathbf{B}_k^{\text{cyc}} V,$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

$$\begin{aligned}\mathcal{F}^\lambda B_{\text{cyc}}^* V &= \{\psi \in B_{\text{cyc}}^* V \mid \psi|_{\mathcal{F}^\lambda B_{\text{cyc}} V} = 0\} \\ &\simeq (B_{\text{cyc}}^* V)_{k+1} \oplus (B_{\text{cyc}}^* V)_{k+2} \oplus \cdots,\end{aligned}$$

where \mathcal{F}_w is the increasing filtration by weights, \mathcal{F} the dual filtration and $(B_{\text{cyc}}^* V)_i$ the graded dual to $B_i^{\text{cyc}} V$. We also see that $\mathcal{F}^\lambda B_{\text{cyc}}^* V = B_{\text{cyc}}^* V$ for all $\lambda < 1$. Since $\hat{B}_{\text{cyc}}^* V$ is the completion of $B_{\text{cyc}}^* V$ in the graded category, we have $\hat{B}_{\text{cyc}}^* V \in W_0$. Finally, we know that $\gamma = 2$ for the canonical dIBL-algebra. \square

Remark D.2.8 (Generalization over algebra). It is easy to see that (a) of Proposition D.2.7 also holds when we work over a non-negatively filtered augmented unital graded \mathbb{K} -algebra R . However, (b) should not generalize as there are no conditions in the filtered MV-formalism implying the strict filtration degree condition for $(1, 0, 0)$ and $(2, 0, 0)$. \triangleleft

Example D.2.9 (Asymmetry of MV-formalism). How is it possible that there are no strict filtration degree conditions on $(1, 0, 0)$ and $(2, 0, 0)$ in the filtered MV-formalism even though these unstable surfaces can obviously bubble off (see Figure D.3)?¹⁰

Heuristically, increasing the number of $(1, 0, 0)$'s glued to the outputs increases the number of times $\bar{\delta}$ has to be applied to the input to split it and feed it into the new $(1, 0, 0)$'s. The limit conilpotency property (D.6) steps in, increases the filtration degree and provides convergence of the infinite sum. As for the bubbling of $(2, 0, 0)$, we look at (c) and (d) of Figure D.3 and see that there is always an increasing number of $(0, 2, 0)$'s or $(1, 0, 0)$'s in the adjacent components. Now, $(0, 2, 0)$ increases the filtration degree due to the strict filtration degree condition, and $(1, 0, 0)$ increases the filtration degree due to the limit conilpotency property as explained above. The convergence is again established.

We will now illustrate the bubbling of f_{100} and f_{010} in the relations for morphisms with a concrete computation. Let $f_{100} : \hat{S}_1 U \rightarrow \hat{S}_0 U$, $f_{010} : \hat{S}_0 U \rightarrow \hat{S}_1 U$, $q_{1l0} : \hat{S}_1 U \rightarrow \hat{S}_l U$ and $q_{kl0} : \hat{S}_k U \rightarrow \hat{S}_l U$ for $k, l \geq 0$ be \mathbb{K} -linear maps such that the following sums converge to $\mathbb{K}((\hbar))$ -linear operators $\hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$:

$$\begin{aligned}\Delta &:= \sum_{l=1}^{\infty} \hat{q}_{1l0}, & f &:= f_{100}, \\ \Delta' &:= \sum_{k=1}^{\infty} \hat{q}_{k10} \hbar^{k-1}, & f' &:= f_{010} \hbar^{-1}.\end{aligned}$$

¹⁰Note that this question is relevant only when we work over a general \mathbb{K} -algebra R , so that f_{100} and f_{200} do not necessarily vanish.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

Assume that $\|\mathfrak{f}\| = \|\mathfrak{f}_{100}\| \geq 0$. Because $\mathfrak{f}_{100}(1) = 0$, convergence of the exponential

$$e^{\mathfrak{f}} = \sum_{r=0}^{\infty} \frac{1}{r!} \mathfrak{f}_{100}^{\odot r} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$$

relies purely on the limit conilpotency property (see the proof of Proposition D.1.8). Assume that $\|\mathfrak{f}_{010}\| = \|\mathfrak{f}_{010}(1)\| > 2\gamma$, i.e., $\|\mathfrak{f}'\| = \|\mathfrak{f}'(1)\| = \|\mathfrak{f}_{010}(1)\| - 2\gamma > 0$. Clearly, convergence of the exponential

$$e^{\mathfrak{f}'} = \sum_{r=0}^{\infty} \frac{1}{r!} \mathfrak{f}_{010}^{\odot r} \hbar^{-r} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar)) \quad (\text{D.35})$$

relies purely on the strict filtration degree condition.

For all $v \in S_1U$, we have

$$\begin{aligned} (e^{\mathfrak{f}} \Delta)(v) &= \sum_{l=1}^{\infty} e^{\mathfrak{f}} \hat{\mathfrak{q}}_{1l0}(v) \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \mu^{(l)} \mathfrak{f}_{100}^{\otimes l} \bar{\delta}^{(l)} \mathfrak{q}_{1l0}(v) \\ &= \sum_{l=1}^{\infty} \mathfrak{f}_{100}^{\otimes l} (\mathfrak{q}_{1l0}(v)). \end{aligned} \quad (\text{D.36})$$

In this simplest case, the limit conilpotency property takes the form $\bar{\delta}^{(k)}(S_l U) = 0$ for $k > l$, which bounds the number of \mathfrak{f}_{100} 's applied to individual summands. Next, we have

$$\begin{aligned} (\Delta' e^{\mathfrak{f}'})(1) &= \sum_{r=0}^{\infty} \frac{1}{r!} \Delta' \mathfrak{f}_{010}(1)^r \hbar^{-r} \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i=1}^{\infty} \hat{\mathfrak{q}}_{i10} \mathfrak{f}_{010}(1)^r \\ &= \sum_{r=1}^{\infty} \sum_{i=1}^r \frac{1}{i!(r-i)!} \mathfrak{q}_{i10} (\mathfrak{f}_{010}(1)^i) \mathfrak{f}_{010}(1)^{r-i} \hbar^{-r+i-1} \\ &= \left(\sum_{i=1}^{\infty} \frac{1}{i!} \mathfrak{q}_{i10} (\mathfrak{f}_{010}(1)^i) \right) \underbrace{\left(\sum_{t=0}^{\infty} \frac{1}{t!} \mathfrak{f}_{010}(1)^t \hbar^{-t-1} \right)}_{=e^{\mathfrak{f}'}(1) \hbar^{-1}}. \end{aligned} \quad (\text{D.37})$$

Clearly, the condition $\|\mathfrak{f}_{010}(1)\| > 0$ is required for the convergence of the sum in the first bracket. \triangleleft

Let us now consider the twisting of a complete filtered IBL_{∞} -algebra in MV-formalism (\mathfrak{q}_{klg}) on W with a Maurer-Cartan element \mathfrak{n} . The *Maurer-Cartan element* is, by definition,

D. Filtered IBL-infinity-algebras as filtered MV-algebras

a morphism from the trivial IBL_∞ -algebra 0 to W . Because $\hat{S}0 = \hat{S}_0 0 = \mathbb{K}$, only the $(0, l, g)$ -components denote by $\mathbf{n}_{lg} : \mathbb{K} \rightarrow \hat{S}_l U$ for $l, g \geq 0$ might be non-zero, and thus

$$\mathbf{n} = \sum_{l, g \geq 0} \mathbf{n}_{lg} \hbar^{g-1} : \hat{S}0((\hbar)) \longrightarrow \hat{S}U((\hbar)).$$

Clearly, $e^n(1) = e^{n(1)} \in \hat{S}U((\hbar))$. Consider the left multiplication $L_{e^{n(1)}} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$, and let Δ be the BV-operator for (\mathbf{q}_{klg}) . The BV-operator for the twisted IBL_∞ -algebra (\mathbf{q}_{klg}^n) is defined by

$$\Delta^n := L_{e^{-n(1)}} \circ \Delta \circ L_{e^{n(1)}} : \hat{S}U((\hbar)) \longrightarrow \hat{S}U((\hbar)). \quad (\text{D.38})$$

The twisting always produces an input-strict IBL_∞ -algebra. Indeed, we have

$$\Delta \circ e^n = 0 \iff \Delta(e^{n(1)}) = 0 \iff \Delta^n(1) = 0.$$

This is called the *Maurer-Cartan equation*.

Proposition D.2.10 (Twisted IBL_∞ -algebra). *Let (W, Δ) be a complete filtered IBL_∞ -algebra in MV-formalism, and let \mathbf{n} be a Maurer-Cartan element. Then Δ^n defined by (D.38) is an input-strict complete filtered IBL_∞ -algebra in MV-formalism of the same bidegree as (W, Δ) , and for its components (\mathbf{q}_{klg}^n) , it holds*

$$\mathbf{q}_{klg}^n = \sum_{\substack{r \geq 0 \\ k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ k' - h_1 - \dots - h_r = k \\ l' + l_1 + \dots + l_r - h_1 - \dots - h_r = l \\ g' + h_1 + \dots + h_r - r + g_1 + \dots + g_r = g}} \mathbf{q}_{k'l'g'} \circ_{h_1, \dots, h_r} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_r g_r}) \quad \text{for all } k \geq 1, l, g \geq 0,$$

where (\mathbf{q}_{klg}) and (\mathbf{n}_{lg}) are components of Δ and \mathbf{n} , respectively.

Proof. Using $L_{e^{n(1)}} = e^n \odot \mathbb{1}$, we compute

$$\begin{aligned} & \Delta \circ L_{e^{n(1)}} \\ &= \sum_{\substack{r \geq 0 \\ k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0}} \frac{1}{r!} \hat{\mathbf{q}}_{k'l'g'} \circ (\mathbf{n}_{l_1 g_1} \odot \dots \odot \mathbf{n}_{l_r g_r} \odot \mathbb{1}) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \\ &= \sum_{\substack{r \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r + k = k'}} \frac{1}{r!} \mathbf{q}_{k'l'g'} \circ_{h_1, \dots, h_r, k} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_r g_r}, \mathbb{1}) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \end{aligned}$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

$$\begin{aligned}
&= \sum_{\substack{r \geq 0 \\ 0 \leq r' \leq r \\ k, k', l', g', l_1, g_1, \dots, l_{r'}, g_{r'} \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ h_1 + \dots + h_{r'} + k = k'}} \frac{1}{(r - r')!} \frac{1}{r'!} \mathbf{n}_{l_{r'} + 1, g_{r'} + 1} \odot \dots \odot \mathbf{n}_{l_r, g_r} \\
&\quad \odot \left(\mathbf{q}_{k' l' g'} \circ_{h_1, \dots, h_{r'}, k} (\mathbf{n}_{l_1, g_1}, \dots, \mathbf{n}_{l_{r'}, g_{r'}}) \right) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \\
&= \left(\sum_{t \geq 0} \frac{1}{t!} \mathbf{n}_{l'_1, g'_1} \odot \dots \odot \mathbf{n}_{l'_t, g'_t} \hbar^{g'_1 + \dots + g'_t - t} \right) \\
&\quad \odot \left(\sum_{k, l, g \geq 0} \sum_{\substack{r' \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ k' - h_1 - \dots - h_{r'} = k \\ l' + l_1 + \dots + l_{r'} - h_1 - \dots - h_{r'} = l \\ g' + h_1 + \dots + h_{r'} - r' + g_1 + \dots + g_{r'} = g}} \frac{1}{r'!} \mathbf{q}_{k' l' g'} \circ_{h_1, \dots, h_{r'}, k} (\mathbf{n}_{l_1, g_1}, \dots, \mathbf{n}_{l_{r'}, g_{r'}}) \hbar^{k + g - 1} \right) \\
&= \left(\sum_{t \geq 0} \frac{1}{t!} (\mathbf{n}_{l'_1, g'_1} \odot \dots \odot \mathbf{n}_{l'_t, g'_t} \odot \mathbb{1}) \hbar^{g'_1 + \dots + g'_t - t} \right) \\
&\quad \odot \left(\sum_{k, l, g \geq 0} \sum_{\substack{r' \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ k' - h_1 - \dots - h_{r'} = k \\ l' + l_1 + \dots + l_{r'} - h_1 - \dots - h_{r'} = l \\ g' + h_1 + \dots + h_{r'} - r' + g_1 + \dots + g_{r'} = g}} \frac{1}{r'!} \mathbf{q}_{k' l' g'} \circ_{h_1, \dots, h_{r'}, k} (\mathbf{n}_{l_1, g_1}, \dots, \mathbf{n}_{l_{r'}, g_{r'}}) \odot \mathbb{1} \hbar^{k + g - 1} \right) \\
&= L_{e^{n(1)}} \odot \left(\sum_{k, l, g \geq 0} \sum_{\substack{r' \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ k' - h_1 - \dots - h_{r'} = k \\ l' + l_1 + \dots + l_{r'} - h_1 - \dots - h_{r'} = l \\ g' + h_1 + \dots + h_{r'} - r' + g_1 + \dots + g_{r'} = g}} \frac{1}{r'!} \mathbf{q}_{k' l' g'} \circ_{h_1, \dots, h_{r'}, k} (\mathbf{n}_{l_1, g_1}, \dots, \mathbf{n}_{l_{r'}, g_{r'}}) \hbar^{k + g - 1} \right).
\end{aligned}$$

The claim follows. \square

D.3. BV-complexes for IBL-infinity-algebras

We plan to use the filtered MV-formalism from the previous section to study chain-maps between the following chain complexes.

Definition D.3.1 (Chain complexes for IBL_∞ -algebras). *Given a complete filtered IBL_∞ -algebra in MV-formalism (U, Δ) , we call the chain complex $(\hat{\text{S}}U((\hbar)), \Delta)$ the BV-complex. If (U, Δ) is input-strict, i.e., if $\Delta(1) = 0$, then $(\hat{\text{S}}U[[\hbar]], \Delta)$ is a chain complex as well, and we call it the strict BV-complex. If (U, Δ) is strict, we also have the chain complex $(\hat{\text{S}}_1 U, \mathbf{q}_{110})$, and we call it the IBL_∞ -chain complex (c.f., Definition 3.2.8 in Part I).*

Clearly, the definition works also for filtered IBL_∞ -algebras from [CFL15] (we are just

not sure about Proposition D.3.3 below, and thus we rather stick to our formalism).

Remark D.3.2 (BV-bicomplex for surfaces). Writing $\Delta = \Delta_1 + \Delta_2 \hbar + \Delta_3 \hbar^2 + \dots$ in the input-strict case, we have

$$0 \stackrel{!}{=} \Delta^2 = \Delta_1^2 + \hbar(\Delta_1 \Delta_2 + \Delta_2 \Delta_1) + \hbar^2(\Delta_2^2 + \Delta_1 \Delta_3 + \Delta_3 \Delta_1) + \hbar^3(\dots) + \dots.$$

We see that $\Delta_1^2 = 0$ and $[\Delta_1, \Delta_2] = 0$, where $[\cdot, \cdot]$ is the graded commutator, always hold. Moreover, if $[\Delta_1, \Delta_3] = 0$, then also $\Delta_2^2 = 0$. Now, $|\Delta_1| = -1$ always holds, and if $d = -1$, then $|\hbar| = -2$ and $|\Delta_2| = -1 - 2d = 1$. We see that if $[\Delta_1, \Delta_3] = 0$ and $d = -1$, then $(\hat{S}U, \Delta_1, \Delta_2)$ is a (homological) mixed complex. If $\Delta_k = 0$ for $k \geq 3$, then the $[\hbar^{-1}, \hbar]$ - and $[[\hbar]]$ -versions of the total complex (see Remark B.2.4) compute the BV- and strict BV-homology, respectively. Note that by [CVonb, Proposition 2.2], these versions are quasi-isomorphism invariants of mixed complexes.¹¹

In the case of the canonical dIBL-algebra on cyclic cochains of a Poincaré duality algebra of degree n , we have $d = n - 3$, and so $d = -1$ corresponds to $n = 2$. \triangleleft

Proposition D.3.3 (Morphisms of BV-complexes). *In the category of complete filtered IBL_∞-algebras in MV-formalism, the following holds:*

- (a) *A weak morphism of weak IBL_∞-algebras induces a chain map e^\natural of BV-complexes.*
- (b) *An input-strict morphism of input-strict IBL_∞-algebras induces a chain map of both the BV-complex and its strict version.*
- (c) *The $(1, 1, 0)$ -part of a strict IBL_∞-morphism of strict IBL_∞-algebras induces a chain map of IBL_∞-chain complexes.*
- (d) *For a Maurer-Cartan element \mathfrak{n} , the left multiplication $L_{e^{\mathfrak{n}}(1)} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$ is an isomorphism of the BV-complexes $(\hat{S}U((\hbar)), \Delta) \simeq (\hat{S}U((\hbar)), \Delta^n)$.*

Proof. Clear. \square

Notice that whereas $L_{e^{\mathfrak{n}}(1)}$ is an isomorphism of chain complexes and it even has non-negative filtration degree, it might not be, and it should not be, an IBL_∞-morphism (see question (iii) of Remark D.3.4). This would clarify the author's confusion explained in the preamble.

¹¹A morphism of mixed complexes commutes with both the boundary operator and the differential. A quasi-isomorphism of homological mixed complexes is a morphism of mixed complexes which is a quasi-isomorphism with respect to the boundary operator.

Remark D.3.4 (Some questions). (i) What is the geometric meaning of

$$H(\hat{S}U((\hbar)), \Delta) \simeq H(\hat{S}U((\hbar)), \Delta^n), \quad H(\hat{S}U[[\hbar]], \Delta) \quad \text{and} \quad H(\hat{S}U[[\hbar]], \Delta^n), \quad (\text{D.39})$$

where $U = B_{\text{cyc}}^* H_{\text{dR}}(M)[3 - n]$ is the (reduced) dual cyclic bar complex of the de Rham cohomology of a closed oriented n -manifold M with the filtration dual to the (homological, i.e., increasing) filtration by weights, Δ represents the canonical dIBL-structure for the intersection pairing on $H_{\text{dR}}(M)$ and \mathfrak{n} is the Chern-Simons Maurer-Cartan element? Are some of the homologies (D.39) isomorphic? How about the bicomplex from Remark D.3.2 for surfaces?

(ii) Are there any implications between the following statements (a)–(c)? A strict IBL_∞ -morphism of strict IBL_∞ -algebras induces

- (a) a quasi-isomorphism of IBL_∞ -complexes,
- (b) a quasi-isomorphism of strict BV-complexes and
- (c) a quasi-isomorphism of BV-complexes?

(iii) For a Maurer-Cartan element \mathfrak{n} , what is

$$\log L_{e^{\mathfrak{n}}(1)} : \hat{S}U((\hbar)) \longrightarrow \hat{S}U((\hbar)),$$

and can one see that this map is, in general, not in the form of an IBL_∞ -morphism? \triangleleft

D.4. Composition of polynomials in convolution product

We will work on a (non-filtered) \mathbb{N}_0 -graded bialgebra V over \mathbb{K} and consider \mathbb{K} -linear maps $V \rightarrow V$. Nevertheless, the results extend in a straightforward way to weight-graded complete filtered bialgebras \hat{V} and \mathbb{K} -linear maps of finite filtration degrees, and ultimately to the formal series $\hat{V}((\hbar))$ and $\mathbb{K}((\hbar))$ -linear maps of finite filtration degrees.

The theory is based on the following observation.

Lemma D.4.1 (Iterated compatibility condition). *Let $(V, \mu, \delta, \eta, \varepsilon)$ be an \mathbb{N}_0 -graded bialgebra. Write $V = \bigoplus_{i=0}^\infty V_i$,¹² and for every $i \in \mathbb{N}_0$, let $\iota_i : V_i \rightarrow V$ and $\pi_i : V \rightarrow V_i$ be the canonical inclusion and projection, respectively. Let $m, n \in \mathbb{N}$, and let $\sigma_{m,n} \in \mathbb{S}_{mn}$ be*

¹²We use the lower index because it should correspond to the weight in the weight-graded setting of $V = SU$.

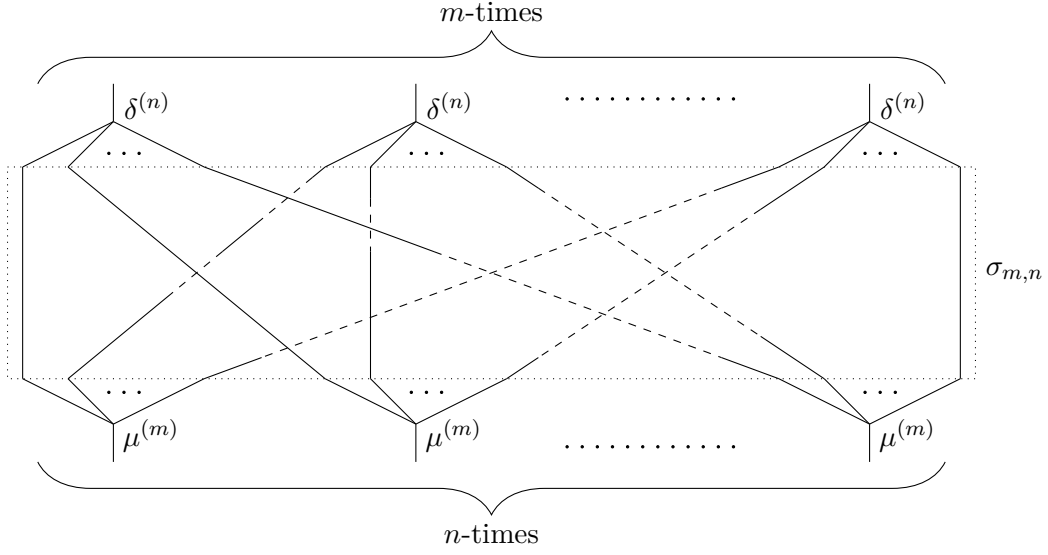


Figure D.4.: The “heart and veins” of the iterated bialgebra compatibility condition $\delta^{(n)}\mu^{(m)}$, and the definition of $\sigma_{m,n}$.

the permutation of mn elements depicted in Figure D.4. For $A \in \mathbb{N}_0^m \times \mathbb{N}_0^n$, let

$$F_A := \bigotimes_{i=1}^m \bigotimes_{j=1}^n \pi_{A_{ij}} \circ \iota_{A_{ij}} : V^{\otimes mn} \longrightarrow V^{\otimes mn}.$$

Then for the iterated product $\mu^{(m)} : V^{\otimes m} \rightarrow V$ and the iterated coproduct $\delta^{(n)} : V \rightarrow V^{\otimes n}$, the following holds:

$$\delta^{(n)}\mu^{(m)} = \mu^{(m)\otimes n}\sigma_{m,n}\delta^{(n)\otimes m} = \sum_{A \in \mathbb{N}_0^m \times \mathbb{N}_0^n} \mu^{(m)\otimes n}\sigma_{m,n}F_A\delta^{(n)\otimes m}. \quad (\text{D.40})$$

For $n = m = 2$, this reduces to the well-known compatibility relation

$$\delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{1} \otimes \tau \otimes \mathbb{1}) \circ (\delta \otimes \delta). \quad (\text{D.41})$$

Proof. Clearly, the second equality of (D.40) holds because $\sum_{A \in \mathbb{N}_0^m \times \mathbb{N}_0^n} F_A = \mathbb{1}^{mn}$. As for the first equality, it holds for $m = n = 2$ from the definition of a bialgebra; the rest is obtained by induction on m, n , distinguishing the two cases $\delta^{(n)}\mu^{(m)} = (\delta \otimes \mathbb{1}^{\otimes n-2})\delta^{(n-1)}\mu^{(m)}$ and $\delta^{(n)}\mu^{(m)} = \delta^{(n)}\mu^{(m-1)}(\mu \otimes \mathbb{1}^{\otimes n-2})$, respectively. In the first case, the induction step reads

$$\delta^{(n)}\mu^{(m)} = (\delta \otimes \mathbb{1}^{\otimes n-2})\delta^{(n-1)}\mu^{(m)}$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

$$\begin{aligned}
&= (\delta \otimes \mathbb{1}^{\otimes n-2}) \mu^{(m) \otimes n-1} \sigma_{m,n-1} \delta^{(n-1) \otimes m} \\
&= \mu^{(m) \otimes n} (\sigma_{m,2} \delta^{\otimes m} \otimes \mathbb{1}^{\otimes m(n-2)}) \sigma_{m,n-1} \delta^{(n-1) \otimes m} \\
&= \mu^{(m) \otimes n} \sigma_{m,n} \delta^{(n) \otimes m}.
\end{aligned}$$

On the third line, we used that $\delta \mu^{(m)} = (\mu^{(m)} \otimes \mu^{(m)}) \sigma_{m,2} \delta^{\otimes m}$, which equals (D.41) for $m = 2$, and for $m \geq 3$, it is proven by induction with the induction step

$$\begin{aligned}
\delta \mu^{(m)} &= \delta \mu^{(m-1)} (\mu \otimes \mathbb{1}^{\otimes m-2}) \\
&= (\mu^{(m-1)} \otimes \mu^{(m-1)}) \sigma_{m-1,2} \delta^{\otimes m-1} (\mu \otimes \mathbb{1}^{\otimes m-2}) \\
&= (\mu^{(m-1)} \otimes \mu^{(m-1)}) \sigma_{m-1,2} (\mu \otimes \mu \otimes \mathbb{1}^{\otimes m-2}) (\mathbb{1} \otimes \tau \otimes \mathbb{1} \otimes \mathbb{1}^{\otimes m-2}) \delta^{\otimes m} \\
&= (\mu^{(m)} \otimes \mu^{(m)}) \sigma_{m,2} \delta^{\otimes m}.
\end{aligned}$$

The second case is analogous. □

Definition D.4.2 (Compositions of monomials in convolution product). *Consider an \mathbb{N}_0 -graded bialgebra $(V, \mu, \delta, \eta, \varepsilon)$. Given linear maps $\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'} : V \rightarrow V$ and a matrix $A \in \mathbb{N}_0^{r'} \times \mathbb{N}_0^r$ for $r, r' \in \mathbb{N}$, we define the A -composition $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}) : V \rightarrow V$ by the formula*

$$\begin{aligned}
&(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}) \\
&:= \mu^{(r)} \circ (\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r) \circ \mu^{(r') \otimes r} \circ \sigma_{r',r} \circ F_A \circ \delta^{(r) \otimes r'} \circ (\mathfrak{f}'_1 \otimes \dots \otimes \mathfrak{f}'_{r'}) \circ \delta^{(r')},
\end{aligned}$$

where $\sigma_{r',r}$ and F_A were defined in Lemma D.4.1.

We call a matrix $A \in \mathbb{N}_0^{r'} \times \mathbb{N}_0^r$ connected if for the matrix

$$B := \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

the matrix product $B^{r+r'}$ has at least one row with all entries non-zero.¹³ In this case, we say that the A -composition $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'})$ is connected.

Suppose that V is connected, i.e., $V_0 = \langle 1 \rangle$. Given linear maps $\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'} : V \rightarrow V$ for $r, r' \in \mathbb{N}$, we define the connected composition $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{\text{con}} (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}) :$

¹³Such A represents a connected weighted bipartite graph.

D. Filtered IBL-infinity-algebras as filtered MV-algebras

$V \rightarrow V$ by

$$(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{\text{con}} (\mathfrak{f}_1, \dots, \mathfrak{f}_{r'}) = \sum_{\substack{A \in \mathbb{N}_0^{r'+1} \times \mathbb{N}_0^{r+1} \\ A \text{ connected} \\ A_{r'+1, r+1} = 0}} (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}, \mathbb{1}).$$

Given linear maps $\mathfrak{f}, \mathfrak{f}_1, \dots, \mathfrak{f}_r : V \rightarrow V$ and parameters $h_1, \dots, h_r \in \mathbb{N}_0$ for $r \in \mathbb{N}$, we define the partial compositions $\mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r)$, $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f} : V \rightarrow V$ by

$$\begin{aligned} & \mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \\ &:= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([(\mu^{(r)} \otimes \mu^{(r)})(F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r})\sigma_r \delta^{\otimes r}] \otimes \mathbb{1}) \\ & \quad (\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1})\delta^{(r+1)}, \\ & (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f} \\ &:= \mu^{(r+1)}(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1})([\mu^{\otimes r} \sigma_r^{-1}(F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r})(\delta^{(r)} \otimes \delta^{(r)})] \otimes \mathbb{1}) \\ & \quad (\mathbb{1} \otimes \tau)(\delta \otimes \mathbb{1})(\mathfrak{f} \otimes \mathbb{1})\delta, \end{aligned}$$

where we set $F_{h_1, \dots, h_r} := F_{(h_1, \dots, h_r)} = F_{(h_1, \dots, h_r)}^T = \iota_{h_1} \pi_{h_1} \otimes \dots \otimes \iota_{h_r} \pi_{h_r}$ and $\sigma_r := \sigma_{r,2}$, and we omit writing the composition \circ . For $r = 1$, we have $\mu^{(1)} = \delta^{(1)} := \mathbb{1}$ by definition, and both equations above reduce to

$$\begin{aligned} \mathfrak{f} \circ_h \mathfrak{f}_1 &= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)(F_h \otimes \mathbb{1}^{\otimes 2})(\delta \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \mathbb{1})\delta \\ & \quad (= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(F_h \otimes \mathbb{1}^{\otimes 2})(\mathbb{1} \otimes \tau)(\delta \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \mathbb{1})\delta). \end{aligned}$$

(c.f., Definition 3.2.2 in Section 3.2 in Part I for the case of $V = \mathbf{S}(C[1])$)

Proposition D.4.3 (Partial compositions). *Let $\mathfrak{f}, \mathfrak{f}_1, \dots, \mathfrak{f}_r : V \rightarrow V$ be linear maps on a connected \mathbb{N}_0 -graded bialgebra V , and let $h_1, \dots, h_r \in \mathbb{N}_0$ for $r \in \mathbb{N}$. Then the following formulas hold:*

$$\begin{aligned} \mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) &= \sum_{\substack{A \in \mathbb{N}_0^{r+1} \times \mathbb{N}_0^2 \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} (\mathfrak{f}, \mathbb{1}) \square_A (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}), \\ (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f} &= \sum_{\substack{A \in \mathbb{N}_0^2 \times \mathbb{N}_0^{r+1} \\ A = \begin{pmatrix} h_1 & \dots & h_r & \bullet \\ \bullet & \dots & \bullet & 0 \end{pmatrix}}} (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}) \square_A (\mathfrak{f}, \mathbb{1}). \end{aligned}$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

Proof. We compute

$$\begin{aligned}
& \sum_{\substack{A \in \mathbb{N}_0^{r+1} \times \mathbb{N}_0^2 \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} (\mathfrak{f} * \mathbb{1}) \square_A (\mathfrak{f}_1 * \cdots * \mathfrak{f}_r * \mathbb{1}) \\
&= \sum_{\substack{A \in \mathbb{N}_0^{r+1} \times \mathbb{N}_0^2 \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} \mu(\mathfrak{f} \otimes \mathbb{1}) \mu^{(r+1) \otimes 2} \sigma_{r+1,2} F_A \delta^{\otimes r+1} (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1}) \mu^{(r+1) \otimes 2} \sigma_{r+1} (\iota_{h_1} \pi_{h_1} \otimes \mathbb{1} \otimes \cdots \otimes \iota_{h_r} \pi_{h_r} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \iota_0 \pi_0) \delta^{\otimes r+1} \\
&\quad (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1}) \mu^{(r+1) \otimes 2} (F_{h_1, \dots, h_r} \otimes \mathbb{1} \otimes \mathbb{1}^{\otimes r} \otimes \iota_0 \pi_0) \sigma_{r+1} \delta^{\otimes r+1} \\
&\quad (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1}) (\mu \otimes \mathbb{1}) (\mathbb{1} \otimes \tau) ([\mu^{(r) \otimes 2} (F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r}) \sigma_r \delta^{\otimes r}] \otimes \mathbb{1}) \\
&\quad (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r).
\end{aligned}$$

On the line before the last line we used that for a connected bialgebra V , it holds

$$(\mathbb{1} \otimes \iota_0 \pi_0) \delta(v) = v \otimes \mathbb{1} \quad \text{and} \quad (\iota_0 \pi_0 \otimes \mathbb{1}) \delta(v) = \mathbb{1} \otimes v \quad \text{for all } v \in V.$$

The case of $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f}$ is treated analogously, using that $\sigma_{m,n} = \sigma_{n,m}^{-1}$ (this is visible by looking at the highlighted square in Figure D.4, which is symmetric under the rotation by 180°). \square

We see that the operations \circ_{h_1, \dots, h_r} and \circ_{con} , which are the cornerstone of the surface calculus for SU in Section D.2, originate naturally from \square_A . Clearly, we can now replace SU by any connected weight-graded bialgebra V and develop a surface calculus for BV_∞ -algebras over V .

The following formulas were stated in Remark 3.2.3 in Section 3.2 in Part I and come from [CFL15], where they were used in a slightly different form (and without a proof).

Proposition D.4.4 (Formulas involving partial compositions). *For $r \in \mathbb{N}$, let $\mathfrak{f}, \mathfrak{f}_1, \dots, \mathfrak{f}_r : V \rightarrow V$ be linear maps on a connected \mathbb{N}_0 -graded bialgebra V . Then the following*

formulas hold:

$$\hat{\mathfrak{f}} \circ \hat{\mathfrak{f}}_1 = \sum_{h \geq 0} \widehat{\mathfrak{f} \circ_h \mathfrak{f}_1}, \quad (\text{D.42})$$

$$\hat{\mathfrak{f}} \circ (\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) = \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k}} (\mathfrak{f} \iota_k \pi_k) \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r), \quad (\text{D.43})$$

$$(\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) \circ \hat{\mathfrak{f}} = \sum_{l \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = l}} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} (\iota_l \pi_l \mathfrak{f}), \quad (\text{D.44})$$

$$\mathfrak{f} \circ_{h_1, \dots, h_{r-1}, 0} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) = \mathfrak{f} \circ_{h_1, \dots, h_{r-1}} (\mathfrak{f}_1, \dots, \mathfrak{f}_{r-1}) * \mathfrak{f}_r, \quad (\text{D.45})$$

$$(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{0, h_2, \dots, h_r} \mathfrak{f} = \mathfrak{f}_1 * (\mathfrak{f}_2, \dots, \mathfrak{f}_r) \circ_{h_2, \dots, h_r} \mathfrak{f}. \quad (\text{D.46})$$

(Recall that $\hat{\mathfrak{f}} := \mathfrak{f} * \mathbb{1} : V \rightarrow V$.)

Proof. As for (D.42), we compute

$$\begin{aligned} \hat{\mathfrak{f}} \circ \hat{\mathfrak{f}}_1 &= \mu(\mathfrak{f} \otimes \mathbb{1}) \delta \mu(\mathfrak{f}_1 \otimes \mathbb{1}) \delta \\ &= \mu(\mathfrak{f} \otimes \mathbb{1}) (\mu \otimes \mu) (\mathbb{1} \otimes \tau \otimes \mathbb{1}) (\delta \otimes \delta) (\mathfrak{f}_1 \otimes \mathbb{1}) \delta \\ &= \mu(\mu \otimes \mathbb{1}) (\mathfrak{f} \otimes \mathbb{1}^{\otimes 2}) (\mu \otimes \mathbb{1}^{\otimes 2}) (\mathbb{1} \otimes \tau \otimes \mathbb{1}) (\delta \otimes \mathbb{1}^{\otimes 2}) (\mathfrak{f}_1 \otimes \mathbb{1}^{\otimes 2}) (\delta \otimes \mathbb{1}) \delta \\ &= \mu \left(\underbrace{[\mu(\mathfrak{f} \otimes \mathbb{1}) (\mu \otimes \mathbb{1}) (\mathbb{1} \otimes \tau) (\delta \otimes \mathbb{1}) (\mathfrak{f}_1 \otimes \mathbb{1}) \delta] \otimes \mathbb{1}}_{= \sum_{h \geq 0} \mathfrak{f} \circ_h \mathfrak{f}_1} \right) \delta \\ &= \sum_{h \geq 0} \widehat{\mathfrak{f} \circ_h \mathfrak{f}_1}. \end{aligned}$$

On the second line, we used the compatibility condition; on the third line, we used associativity and coassociativity.

In order to see (D.43), the easiest is to use Proposition D.4.3:

$$\begin{aligned} \hat{\mathfrak{f}}(\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) &= \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k}} \sum_{\substack{A \in \mathbb{N}_0^r \times \mathbb{N}_0^2 \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \end{pmatrix}}} (\mathfrak{f} \iota_k \pi_k, \mathbb{1}) \square_A (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \\ &= \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k}} \sum_{\substack{A' \in \mathbb{N}_0^{r+1} \times \mathbb{N}_0^2 \\ A' = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} (\mathfrak{f} \iota_k \pi_k, \mathbb{1}) \square_{A'} (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}) \end{aligned}$$

D. Filtered IBL-infinity-algebras as filtered MV-algebras

$$= \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k}} (\mathfrak{f}_{\ell_k \pi_k}) \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r).$$

On the second line, the bottom-left \bullet , i.e., how many “veins” of $\mathbb{1}$ go into $\mathfrak{f}_{\ell_k \pi_k}$, is forced to be 0 because $\mathfrak{f}_{\ell_k \pi_k}$ has k inputs and $h_1 + \dots + h_r = k$. Equation (D.44) is proven analogously.

As for (D.45), we prefer to manipulate the expression with bialgebra operations:

$$\begin{aligned} & \mathfrak{f} \circ_{h_1, \dots, h_{r-1}, 0} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \\ &= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([\mu^{(r)} \otimes 2(F_{h_1, \dots, h_{r-1}, 0} \otimes \mathbb{1}^{\otimes r}) \sigma_r \delta^{\otimes r}] \otimes \mathbb{1}) \\ & \quad (\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\ &= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([(\mu^{(r-1)} \otimes \mu^{(r)})(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r})(\sigma_{r-1} \otimes \mathbb{1}) \\ & \quad (\delta^{\otimes r-1} \otimes \mathbb{1})] \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\ &= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)(\mathbb{1} \otimes \mu \otimes \mathbb{1})([\mu^{(r-1)} \otimes 2(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r-1}) \sigma_{r-1} \\ & \quad \delta^{\otimes r-1}] \otimes \mathbb{1}^{\otimes 2})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\ &= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \mu)(\mathbb{1} \otimes \tau \otimes \mathbb{1})([\mu^{(r-1)} \otimes 2(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r-1}) \sigma_{r-1} \\ & \quad \delta^{\otimes r-1}] \otimes \mathbb{1}^{\otimes 2})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_{r-1} \otimes \mathbb{1} \otimes \mathfrak{f}_r) \delta^{(r+1)} \\ &= \mu([\mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([\mu^{(r-1)} \otimes 2(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r-1}) \sigma_{r-1} \\ & \quad \delta^{\otimes r-1}] \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_{r-1} \otimes \mathbb{1}) \delta^{(r)}] \otimes \mathfrak{f}_r) \delta \\ &= \mu(\mathfrak{f} \circ_{h_1, \dots, h_{r-1}} (\mathfrak{f}_1, \dots, \mathfrak{f}_{r-1}) \otimes \mathfrak{f}_r) \delta \\ &= \mathfrak{f} \circ_{h_1, \dots, h_{r-1}} (\mathfrak{f}_1, \dots, \mathfrak{f}_{r-1}) * \mathfrak{f}_r \end{aligned}$$

Equation (D.46) is proven analogously. □

Bibliography

- [Aka07] M. Akaho. “Quasi-isomorphisms of A_∞ -algebras and oriented planar trees”. In: *Illinois J. Math.* 51.2 (2007).
- [Alb12] P. Albin. *Math 524 – Linear Analysis on Manifolds*. 2012. URL: <https://faculty.math.illinois.edu/~palbin/Math524.Spring2012/LectureNotesMay1.pdf>.
- [AS91] S. Axelrod and I. M. Singer. *Chern-Simons Perturbation Theory*. Oct. 1991. arXiv: [hep-th/9110056](https://arxiv.org/abs/hep-th/9110056).
- [AS93] S. Axelrod and I. M. Singer. *Chern-Simons Perturbation Theory II*. Apr. 1993. arXiv: [hep-th/9304087](https://arxiv.org/abs/hep-th/9304087).
- [Bas11] S. Basu. “Transversal String Topology & Invariants of Manifolds”. PhD thesis. Stony Brook University, 2011. URL: <http://www.math.stonybrook.edu/alumni/2011-Somnath-Basu.pdf>.
- [BGV04] N. Berline, E. Getzler and M. Vergne. *Heat Kernels and Dirac Operators*. Springer, 2004.
- [Boa98] J. M. Boardman. “Conditionally convergent spectral sequences”. In: *Homotopy invariant algebraic structures*. American Mathematical Society, 1998. DOI: [10.1090/conm/239/03597](https://doi.org/10.1090/conm/239/03597).
- [Bry18] R. Bryant. *A primitive to $\text{Vol}(x) \pm \text{Vol}(y)$ on $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$* . Jan. 30, 2018. URL: <https://mathoverflow.net/q/291535>.
- [BT82] R. Bott and L. W. Tu. *Differential Forms in Algebraic Topology*. New York: Springer, 1982. DOI: [10.1007/978-1-4757-3951-0](https://doi.org/10.1007/978-1-4757-3951-0).
- [Cen98] M. Cencelj. “Spectral sequences of unbounded bicomplexes”. In: *Glasnik Matematicki* 33.53 (1998), pp. 161–165.
- [CFL15] K. Cieliebak, K. Fukaya and J. Latschev. *Homological algebra related to surfaces with boundary*. Aug. 2015. arXiv: [1508.02741](https://arxiv.org/abs/1508.02741).
- [Che12] X. Chen. *An algebraic chain model of string topology*. Tech. rep. 5. 2012.

BIBLIOGRAPHY

- [CHV06] R. L. Cohen, K. Hess and A. A. Voronov. *String Topology and Cyclic Homology*. Birkhäuser, 2006, p. 163. DOI: 10.1007/3-7643-7388-1.
- [Cie13] K. Cieliebak. *Lectures on String Topology*. Jan. 2013. URL: <https://www.math.uni-augsburg.de/prof/geo/Dokumente/string.pdf>.
- [CJ02] R. L. Cohen and J. D. Jones. “A homotopy theoretic realization of string topology”. In: *Mathematische Annalen* 324.4 (July 2002), pp. 773–798. DOI: 10.1007/s00208-002-0362-0.
- [CL07] K. Cieliebak and J. Latschev. *The role of string topology in symplectic field theory*. June 2007. arXiv: 0706.3284.
- [CM08] A. S. Cattaneo and P. Mnev. *Remarks on Chern-Simons invariants*. Nov. 2008. arXiv: 0811.2045.
- [CMR15] A. S. Cattaneo, P. Mnev and N. Reshetikhin. *Perturbative quantum gauge theories on manifolds with boundary*. July 2015. arXiv: 1507.01221.
- [CR19] J. Cirici and A. Roig. “Sullivan minimal models of operad algebras”. In: *Publicacions Matemàtiques* 63.1 (2019), pp. 125–154. DOI: 10.5565/PUBLMAT6311904.
- [Cra04] M. Crainic. *On the perturbation lemma, and deformations*. Mar. 2004. arXiv: math/0403266.
- [CS02] M. Chas and D. Sullivan. *Closed string operators in topology leading to Lie bialgebras and higher string algebra*. Dec. 2002. arXiv: math/0212358.
- [CS04] M. Chas and D. Sullivan. “Closed String Operators in Topology Leading to Lie Bialgebras and Higher String Algebra”. In: *The Legacy of Niels Henrik Abel*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2004, pp. 771–784.
- [CS99] M. Chas and D. Sullivan. *String Topology*. Nov. 1999. arXiv: math/9911159.
- [CVona] K. Cieliebak and E. Volkov. *Chern-Simons theory and string topology*. In preparation.
- [CVonb] K. Cieliebak and E. Volkov. *Cyclic homology and Chen’s iterated integrals*. In preparation.
- [DJP17] M. Doubek, B. Jurčo and L. Peksová. *Properads and Homotopy Algebras Related to Surfaces*. Aug. 2017. arXiv: 1708.01195.
- [DJP18] M. Doubek, B. Jurčo and J. Pulmann. *Quantum L_∞ Algebras and the Homological Perturbation Lemma*. Feb. 2018. arXiv: 1712.02696v2.
- [DPR15] G. C. Drummond-Cole, K. Poirier and N. Rounds. “Chain-Level String Topology Operations”. In: (June 2015). arXiv: 1506.02596.

BIBLIOGRAPHY

- [Fio+19] D. Fiorenza et al. *Almost formality of manifolds of low dimension*. 2019. arXiv: 1902.08406.
- [FOT08] Y. Félix, J. Oprea and D. Tanré. *Algebraic Models in Geometry*. Oxford University Press, 2008.
- [Fre17] B. Fresse. *Homotopy of Operads and Grothendieck–Teichmüller Groups: Parts 1 and 2*. USA: AMS, 2017.
- [Fuk+10] K. Fukaya et al. *Lagrangian intersection Floer theory–anomaly and obstruction*. U.S.A.: AMS/IP, 2010. ISBN: 978-0-8218-4831-9.
- [Ger63] M. Gerstenhaber. “The cohomology structure of an associative ring”. In: *Ann. Math.* 78.2 (1963), pp. 267–288. DOI: 10.2307/1970343.
- [GJP91] E. Getzler, J. D. S. Jones and S. Petrack. “Differential forms on loop spaces and the cyclic bar complex”. In: (1991).
- [Gol86] W. M. Goldman. “Invariant functions on Lie groups and Hamiltonian flows of surface group representations”. In: *Inventiones Mathematicae* 85.2 (June 1986), pp. 263–302. DOI: 10.1007/BF01389091.
- [Háj14] P. Hájek. “On Manifolds with Corners”. PhD thesis. LMU Munich, 2014.
- [Háj18] P. Hájek. *Program in Wolfram Mathematica 10.4 to compute the integrals for \mathbb{S}^n* . 2018. URL: https://github.com/p135246/Integrals_for_Sn.
- [Har04] B. Harris. *Iterated integrals and cycles on algebraic manifolds*. Singapore: World Scientific Publishing, 2004. ISBN: 981-238-720-X.
- [Hör07] L. Hörmander. *The Analysis of Linear Partial Differential Operators III — Pseudo-Differential Operators*. Springer, 2007.
- [Huy04] D. Huybrechts. *Complex Geometry: An Introduction*. New York: Springer, 2004. ISBN: 3-540-21290-6.
- [Idr18] N. Idrissi. *DGA-formality via A_∞* . Apr. 3, 2018. URL: <https://math.stackexchange.com/q/2719961>.
- [Iri14] K. Irie. *A chain level Batalin-Vilkovisky structure in string topology via de Rham chains*. 2014. arXiv: 1404.0153.
- [Kel99] B. Keller. *Introduction to A -infinity algebras and modules*. Nov. 1999. arXiv: math/9910179.
- [Koh02] T. Kohno. *Conformal Field Theory and Topology*. Oxford University Press, 2002. ISBN: 978-0821821305.

BIBLIOGRAPHY

- [Kot00] D. Kotschick. *On products of harmonic forms*. Apr. 2000. arXiv: [math/0004009](#).
- [Laz03] A. Lazarev. *Hochschild cohomology and moduli spaces of strongly homotopy associative algebras*. Apr. 2003. arXiv: [math/0204062](#).
- [Lip14] M. Lipyanskiy. “Geometric Homology”. In: (Sept. 2014). eprint: [1409.1121](#).
- [LO15] “Rational homotopy — Sullivan models”. In: *Free Loop Spaces in Geometry and Topology*. Ed. by J. Latschev and A. Oancea. European Mathematical Society, 2015. Chap. 3, pp. 111–136. ISBN: 978-3-03719-153-8.
- [Lod92] J.-L. Loday. *Cyclic Homology*. New York: Springer, 1992. ISBN: 3-540-53339-7.
- [LS07] P. Lambrechts and D. Stanley. *Poincare duality and commutative differential graded algebras*. Jan. 2007. arXiv: [math/0701309](#).
- [LV12] J.-L. Loday and B. Vallette. *Algebraic Operads*. Heidelberg: Springer, 2012. DOI: [10.1007/978-3-642-30362-3](#).
- [Mar97] M. Markl. *Loop Homotopy Algebras in Closed String Field Theory*. Nov. 1997. arXiv: [hep-th/9711045](#).
- [Mes16] S. Mescher. *A primer on A-infinity-algebras and their Hochschild homology*. Jan. 2016. arXiv: [1601.03963](#).
- [Mil79] T. J. Miller. “On the formality of $k - 1$ connected compact manifolds of dimension less than or equal to $4k - 2$ ”. In: *Illinois J. Math.* 23.2 (1979).
- [Mne17] P. Mnev. *Lectures on Batalin-Vilkovisky formalism and its applications in topological quantum field theory*. July 2017. arXiv: [1707.08096](#).
- [MS11] K. Münster and I. Sachs. *Quantum Open-Closed Homotopy Algebra and String Field Theory*. Oct. 2011. arXiv: [1109.4101](#).
- [MV15] M. Markl and A. A. Voronov. *The MV formalism for IBL_∞ - and BV_∞ -algebras*. Nov. 2015. arXiv: [1511.01591](#).
- [nLa19] nLab. *Model Structure on DG-Algebras*. 2019. URL: <https://ncatlab.org/nlab/show/model+structure+on+dg-algebras>.
- [Pek18] L. Peksová. “Properads and homological differential operators related to surfaces”. In: *Archivum Mathematicum* 54 (2018), pp. 299–312. DOI: [10.5817/AM2018-5-299](#).
- [Rog19] J. Rognes. *Conditionally convergent spectral sequences with exiting and entering differentials*. July 23, 2019. URL: <https://mathoverflow.net/q/336781>.

BIBLIOGRAPHY

- [Sch92] A. Schwarz. *Geometry of Batalin-Vilkovisky quantization*. July 1992. arXiv: [hep-th/9205088](https://arxiv.org/abs/hep-th/9205088).
- [Sul05] D. Sullivan. “String topology: background and present state”. In: *Current Developments in Mathematics* 2005.1 (2005), pp. 41–88.
- [Tam10] H. Tamanoi. “Loop coproducts in string topology and triviality of higher genus TQFT operations”. In: *Journal of Pure and Applied Algebra* 214.5 (June 2010), pp. 605–615. URL: <http://arxiv.org/abs/0706.1276>.
- [Tur91] V. G. Turaev. “Skein quantization of Poisson algebras of loops on surfaces”. In: *Annales scientifiques de l’École normale supérieure* 24.6 (1991), pp. 635–704. DOI: [10.24033/asens.1639](https://doi.org/10.24033/asens.1639).
- [Val12] B. Vallette. *Algebra+Homotopy=Operad*. Feb. 2012. arXiv: [1202.3245](https://arxiv.org/abs/1202.3245).
- [War83] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Graduate Texts in Mathematics. New York: Springer, 1983. DOI: [10.1007/978-1-4757-1799-0](https://doi.org/10.1007/978-1-4757-1799-0).
- [Wei94] C. A. Weibel. *An introduction to homological algebra*. USA: Cambridge University Press, 1994.
- [Zwi92] B. Zwiebach. *Closed String Field Theory: Quantum Action and the BV Master Equation*. June 1992. arXiv: [hep-th/9206084](https://arxiv.org/abs/hep-th/9206084).