IBL-Infinity Model of String Topology from Perturbative Chern-Simons Theory

Dissertation

zur Erlangung des akademischen Grades

Dr. rer. nat.

eingereicht an der

Mathematisch-Naturwissenschaftlich-Technischen Fakultät

der Universität Augsburg

von

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Augsburg, Oktober 2019

Universität
Augsburg
University

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Tag der mündlichen Prüfung: Freitag, 20. Dezember, 2019

Abstract

Pavel Hájek: *IBL-Infinity Model of String Topology from Perturbative Chern-Simons Theory*

Following Cieliebak, Fukaya, Latschev and Volkov, we construct an IBL-infinity chain model for equivariant string topology on cyclic Hochschild cochains of de Rham cohomology. We study its properties and perform explicit computations.

Keywords: chain model for equivariant string topology, IBL-infinity algebra, Maurer-Cartan element, Hodge propagator, perturbative Chern-Simons theory, Poincaré duality models, BV-formalism, cyclic homology

The thesis was typeset in LaTeX using the scrbook class of KOMA-Script.
The bibliography was created using BibLaTeX and checked against MathSciNet.
Figures were drawn in Ti*k*Z.
Font: Linux Libertine, 11 pt.
Computations were done in Wolfram *Mathematica*® 10.4.

Acknowledgements

First and foremost, I thank my parents, *Dana Hájková* and *Libor Hájek,* for supporting me throughout my studies, both emotionally and materially, for discussing various decisions I had to make, for listening when I needed to talk and for wishing me all the best in whatever I wanted to do. Díky moc mami a tati!

I thank my grandparents, Anděla Hájková, Anna Uhlíková and Jaroslav Uhlík, for their love and support, although they would rather see me being happy and having a family than pursuing a doctorate. Díky moc babi a dědo!

Next, I thank my supervisor, Prof. Dr. Kai Cieliebak, for giving me time and support to get through the graduate studies, to develop academically and to finish the thesis. I enjoyed discussions with him very much and learned from his insights and methods. I appreciate and admire his skill to explain everything anytime starting from elementary facts, his passion for mathematics and physics, scientific productivity and lecturing skills. Danke schön, Kai!

I thank Dr. Evgeny Volkov for discussing his work on Chern-Simons theory, string topology, cyclic homology and Chen's iterated integrals, for providing me his preprints in progress and for supportive chats at the university. Thank you, Evgeny, good luck fighting with lions in Siberia!

I thank Prof. Robert Bryant for proposing an alternative notation for the Hodge propagator for spheres in an online discussion. I thank Prof. John Rognes for answering my question about spectral sequences in an online discussion. I thank Dr. Najib Idrissi for answering my question about formality in an online discussion. I also thank the online discussion — Mathoverflow and Math Stackexchange — for its existence.

I thank Dr. Alexandru Doicu for checking a tedious sign computation and for friendly talks. I thank Dr. Andreas Hermann for discussing the standard Hodge propagator, for sending me his notes on the Green kernel for spheres and for his interest in my work. I thank Prof. Dr. Hông Vân Lê for discussing her work on formality and Poincaré DGA's and for her interest in my work. I thank Ján Pulmann and Lada Peksová for discussing BV-formalism and properads. I thank Ondřej Hulík for a Skype discussion about physics and for friendly talks. I thank Dr. Oliver Lindblad Petersen for bringing up the name

Melrose blow-up for our blow-up construction. I thank Jiří Zeman for pointing out that absolute convergence is equivalent to invariance on resummations and for friendly talks. I thank Dr. Kyler Siegel for a discussion about A-infinity in Stony Brook. I thank Dr. Raymond Puzio for his interest in my work, for thinking about an action for our propagator and for friendly talks. I thank Prof. Alberto Cattaneo and Dr. Pavel Mnev for discussing Feynman integrals and an action for our propagator in Stony Brook and for sending me a list of relevant literature later. I thank Prof. Dr. Braňo Jurčo for discussing BV-formalism and for offering me a position in Prague.

I thank Prof. Dr. Christian Bär, Prof. Dr. Kaoru Ono and Prof. Dr. Janko Latschev for inviting me to their institutes to discuss and give a talk, for their hospitality and for valuable feedback and insights. I am grateful to Prof. Dr. Janko Latschev for giving me a position in Hamburg and for waiting until I finish the thesis.

Although everything mentioned in the last paragraphs occurred in the final fifth year of my Ph.D., it inspired me and motivated me to complete the work.

I thank members of the department for mathematics at the University of Augsburg for creating and maintaining a rich and inspiring academic environment. I thank Prof. Dr. Urs Frauenfelder for various discussions and for always coming up with a highly interesting topic for the symplectic seminar, which, at the beginning, I often preferred to my own work. I thank Prof. Dr. Edward Belbruno, a visiting professor from Princeton, for his series of talks about chaotic dynamics in celestial mechanics, for the explanation of the weak stability boundary, for organizing exciting events related to space travel and for stopping by our office to chat about various topics.

I thank my colleagues at the university; those who I liked for being friendly and encouraging, and those who I did not for putting up with me and for not fueling conflicts. Specifically, I thank Alexei Kudryashov and Thorsten Hertl, who hanged out with me in the final two years when I was spending the most of my time at the university, and who had to listen to my laments about lost years, starting to work on concrete problems too late, terrible topic which is not geometric at all, internet overuse and screwed up personal life. I can not forget to highlight their top-level expertise in model categories and algebraic topology, respectively.

Unrelated to academia, I thank my tennis trainer, Thorsten Moser, alias Mr. T, for his university tennis courses in a cheerful atmosphere, him being dressed in one of his crazy costumes and constantly making fun of my tennis skills. There were times when the regular tennis training was the only thing I was looking forward to and the only order I could stick to. I thank my sparring partners and friends, Lukas Moosbrugger, Thomas Gumpinger and Tobias Watzka, for playing tennis and drinking beer with me.

I thank Surf Club Augsburg, Rügen Piraten and Windsurfing F4 for the great time I had windsurfing on Mandichosee, Ostsee and Bol, respectively, during my holiday. Memories of planing along the coast when a storm was coming helped me to overcome times when I felt really down. I thank Cuban Salsa Power in Augsburg, where I learned to dance Salsa properly, and where I spent almost every other evening in the beginnings of my studies. I thank my dance partner, Beatrix Fertl, for visiting me in Augsburg and for going out with me. I thank Ushi Haller for her interest in my well-being and for baking cookies and cakes for me. Finally, I want to mention an old friend, Milan Vaňkát, and thank him for the occasional obscene phone calls we had and for his few visits when we made fun of every possible social norm. He also left me the two African snails (die Schnecken) who accompanied me from the beginning to the end; I never felt too slow with them by my side.

Structure of the thesis

The thesis consists of the *Introduction* and the following three parts:

- *Part I—IBL-infinity chain model:* the goal is to compute the twisted IBL_∞-algebra on de Rham cohomology of $\mathbb{S}^n$ explicitly and compare it to string topology (the original task). A version of Part I was previously made available online and can be found under the following reference:

  P. Hájek. *Twisted IBL-infinity-algebra and string topology: First look and examples.* Nov. 2018. arXiv: `1811.05281 [math-ph]`

- *Part II—Follow-up topics:* additional topics discovered while working on Part I.
- *Part III—Appendices:* algebraic details of Part I and related questions.

The three parts were written chronologically and reflect how author's view of the theory developed. The mathematical notation is consistent but the terminology and understanding might vary slightly from part to part. All in all, Part I is more about "getting things done" while Parts II and III are more about "optimizing".

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1. Introduction

1.1. String topology and Chen's iterated integrals

String topology of a manifold M is the study of the *free loop space*

$$\mathrm{L}M = \{\gamma : \mathbb{S}^1 \to M \text{ continuous}\},$$

which is equipped with the compact-open topology, and of natural structures on it. Each loop γ is parametrized, with base-point 1, and there is a natural $\mathbb{S}^1$-action changing the base-point. Therefore, we can distinguish the following two homology theories:

$\mathrm{H}(\mathrm{L}M)$	…	the *singular homology* and
$\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)$	…	the *equivariant homology* — "the singular homology of the space of parametrized loops with the base-point forgotten."

In this thesis, we consider $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)$ with coefficients in $\mathbb{R}$ only.

If $M = \Sigma$ is an oriented surface, we consider immersed loops with transverse double points and define a bracket and cobracket by Figure 1.1. In words:

$\mathbb{m}_2$**:** Imagine putting one base-point b_1 on the first loop γ_1 and another base-point b_2 on the second loop γ_2 in all possible positions. Whenever $\gamma_1(b_1) = \gamma_2(b_2) = p$, construct a new loop $\gamma = \gamma_1 *_p \gamma_2$ by running first along γ_1 with double speed starting and ending at b_1 and continuing along γ_2 starting and ending at b_2. Forget the base-points and multiply γ with the sign of the intersection $\varepsilon(p; \gamma_1, \gamma_2)$. Should more intersections occur, take the sum.

$\mathbb{c}_2$**:** Imagine putting the basepoints b_1 and b_2 on γ in all possible positions such that $b_1 \neq b_2$. Whenever $\gamma(b_1) = \gamma(b_2) = p$, split γ into γ_1 and γ_2 as follows. The first loop γ_1 is the portion of γ from b_1 to b_2 and the second loop γ_2 is the portion from b_2 to b_1 ran along with the correspondingly scaled speed. Forget the base-points, form the tensor product $\gamma_1 \otimes \gamma_2$ and multiply it with $\varepsilon(p; \gamma)$. Should more self-intersections occur, take the sum of the tensors.

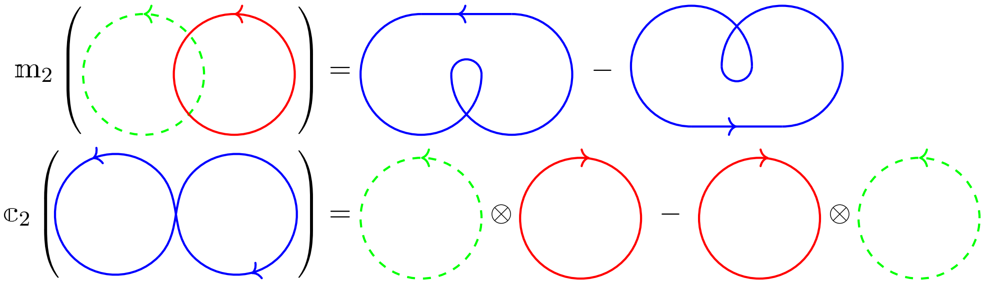

Figure 1.1.: The bracket and cobracket in equivariant string topology. Imagine some genus so that the operations are homotopically non-trivial.

The operations $\mathbb{m}_2$ and $\mathbb{c}_2$ are known as the *Goldman bracket* and the *Turaev cobracket* and were defined and studied in [Gol86] and [Tur91], respectively.

In [CS99], it was demonstrated that the construction of $\mathbb{m}_2$ extends to families of loops and to an arbitrary dimension n of an oriented manifold M; it produces a Lie bracket on the equivariant homology. The construction of $\mathbb{c}_2$ generalizes too and gives a Lie cobracket; this is explained for instance in [CL09]. The picture is always Figure 1.1, just the intersection points come from transverse intersections of smooth parameter spaces of points on loops in M; consequently, both $\mathbb{m}_2$ and $\mathbb{c}_2$ have degrees $2-n$. Another definition of the coproduct is used in [Bas11], where loops in M are viewed as open strings in $M \times M$ with endpoints at the diagonal.

In order to make these geometric constructions rigorous, the most straightforward way (which would work over $\mathbb{Z}$) is to use a *geometric homology theory* of M based on smooth chains such that the transverse intersection of two smooth chains is again a smooth chain. Such theory for smooth manifolds was constructed in [Lip14]. A version for general topological spaces is proposed in [Cie13], and some details regarding triangulations are addressed in [Háj14] (see also the discussion at [Par14]). Note that $\mathbb{m}_2$ and $\mathbb{c}_2$ are only "transversally defined" on the chain level, and in order to define them on homology, it is important that we can homotop to a generic situation within the homology class.

The main theorem of [CS04] asserts that $\mathbb{m}_2$ and $\mathbb{c}_2$ induce the structure of an *involutive bi-Lie algebra*, abbreviated IBL, of degree $2-n$ on the equivariant homology $H^{\mathbb{S}^1}(\mathrm{L}M, M)$ relative to constant loops $M \hookrightarrow \mathrm{L}M$. Modding out constant loops is necessary for $\mathbb{c}_2$ to be well defined because of the phenomenon of *"vanishing of small loops"* illustrated for instance in [CL09]: Let $\sigma \in C_1(\mathrm{L}M)$ be a 1-chain supported on $[0,1]$ which for $t=0$ agrees with the loop in the argument of $\mathbb{c}_2$ in Figure 1.1, next, for $t \in (0,1)$, the left knot L contracts to the mid-point p, and for $t=1$, only the right knot R remains (thus the name “vanishing of small loops”). It is then easy to see that

$$0 \neq \partial \mathbb{c}_2(\sigma) - \mathbb{c}_2(\partial\sigma) = p \otimes R - L \otimes p \in C(M) \otimes C(\mathrm{L}M) + C(\mathrm{L}M) \otimes C(M).$$

Since the string bracket $\mathbb{m}_2$ applied to a chain of constant loops gives a degenerate chain, it restricts to a Lie bracket on the relative homology.[1] In work in progress [CHOon], Poincaré duality on the Rabinowitz-Floer homology of the unit cotangent bundle of M is introduced and related to the non-equivariant string topology via a long exact sequence. This shall give a canonical extension of $\mathbb{c}_2$ to $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M, \mathrm{pt})$, the relative homology modulo one point, which together with $\mathbb{m}_2$ would make $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M, \mathrm{pt})$ into a bi-Lie algebra. In fact, this is what our chain model is supposed to compute. Being aware of this context, we will use the symbol $\overline{\mathrm{H}}^{\mathbb{S}^1}(\mathrm{L}M)$ as an avatar for either $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M, M)$ or $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M, \mathrm{pt})$.

It is expected that the IBL-structure on $\overline{\mathrm{H}}^{\mathbb{S}^1}(\mathrm{L}M)$ is induced from a much richer and in some sense natural algebraic structure on the chain level, whose homotopy type is an invariant of M. In fact, there is a notion of *strong homotopy involutive bi-Lie algebra*, abbreviated IBL_∞, which was developed in [CFL15]. An IBL_∞-algebra consists of operations $(\mathfrak{q}_{klg})$ for k, $l \geq 1$ and $g \geq 0$, where $\mathfrak{q}_{110}$ is a boundary operator, $\mathfrak{q}_{210}$ a bracket and $\mathfrak{q}_{120}$ a cobracket which satisfy the IBL-relations up to a coherent system of higher homotopies $(\mathfrak{q}_{klg})$.

Consider the string space

$$\mathrm{L}_{\mathbb{S}^1} M \coloneqq (\mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M)/\mathbb{S}^1,$$

i.e., the homotopically correct version of the quotient $\mathrm{L}M/\mathbb{S}^1$, and let $(C(\mathrm{L}_{\mathbb{S}^1}M), \partial)$ be the singular chain complex of $\mathrm{L}_{\mathbb{S}^1}M$. Recall that $\mathbb{m}_2$ and $\mathbb{c}_2$ are partially defined on transverse smooth chains therein. An IBL_∞*-chain model for equivariant string topology* is an IBL_∞-algebra $(\mathcal{M}, (\mathfrak{q}_{klg}))$ ($\mathcal{M}$ stands for “model”) together with a weak homotopy equivalence ($\coloneqq$ zig-zag of quasi-isomorphisms) of $(C(\mathrm{L}_{\mathbb{S}^1}M), \partial)$ and $(\mathcal{M}, \mathfrak{q}_{110})$ which induces an isomorphism of IBL-algebras

$$(\overline{\mathrm{H}}^{\mathbb{S}^1}(\mathrm{L}M), \mathbb{m}_2, \mathbb{c}_2) \simeq (\mathrm{H}(\mathcal{M}, \mathfrak{q}_{110}), \mathfrak{q}_{210}, \mathfrak{q}_{120}).$$

Note that there can be various non-homotopically equivalent IBL_∞-chain models. On

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[1] This contrasts with the situation on the non-equivariant homology $\mathrm{H}(\mathrm{L}M)$, where constant loops do not always form an ideal for the associative loop product; this is easy to see in the case of torus $\mathbb{T}^2$. They do form an ideal, however, provided that the Euler characteristics $\chi(M)$ is non-zero, see [Tam10]. On the other hand, if $\chi(M) = 0$, then the loop coproduct admits an extension to $\mathrm{H}(\mathrm{L}M)$; this is possibly dependent on the choice of a non-vanishing vector field, see [Bas11].

the other hand, the properad IBL_∞ is a quasi-free resolution of the properad IBL and as such has convenient homotopy theoretical properties; for example, homotopy inverses of quasi-isomorphisms exist. These properties imply that any two weakly homotopy equivalent IBL_∞-algebras are homotopy equivalent.

In this thesis, we use a version of *perturbative Chern-Simons theory* for an oriented compact Riemannian manifold M to construct an IBL_∞-chain model for the equivariant string topology of M. The chains are cyclic Hochschild cochains of the de Rham cohomology $\mathrm{H}_{\mathrm{dR}} \coloneqq \mathrm{H}_{\mathrm{dR}}(M)$, and the homotopy type of the model is supposed to be an invariant of M (perhaps topological). The construction involves a version of Feynman integrals, and the proof that it is well-defined relies on the theory of integration on certain compactifications of configuration spaces, which is currently being developed in [CVon]. The concrete form of this chain model was sketched in [CFL15]. For the sake of the big picture we remark that it is expected that evaluations at boundaries of pseudo-holomorphic curves in the symplectization of the unit cotangent bundle of M induce an IBL_∞-quasi-isomorphism of the corresponding symplectic field theory and the IBL_∞-chain model of string topology; see [CL09].

We now describe the underlying chain complex of our IBL_∞-chain model and the quasi-isomorphism to string topology in more details. Let $\Omega \coloneqq \Omega(M)$ be the space of smooth de Rham forms on M, and let $\mathrm{B}^{\mathrm{cyc}}\Omega$ be the graded vector space generated by cyclic words

$$\omega_1 \cdots \omega_k = (-1)^{|\omega_k|(|\omega_1|+\cdots+|\omega_k|)}\omega_k\omega_1\cdots\omega_{k-1}$$

with homogenous components $\omega_1, \ldots, \omega_k \in \Omega$ for $k \geq 1$. The grading satisfies $|\omega_i| = \deg(\omega_i) - 1$, where $\deg(\omega_i)$ denotes the form-degree of ω_i. We call $\mathrm{B}^{\mathrm{cyc}}\Omega$ the *cyclic bar complex of* Ω (it might be described as "reduced" because we omit $k = 0$). On $\mathrm{B}^{\mathrm{cyc}}\Omega$, we consider the *Hochschild differential*

$$\begin{aligned} \mathrm{b}(\omega_1\cdots\omega_k) = {} & \sum_{i=1}^{k}(-1)^{|\omega_1|+\cdots+|\omega_{i-1}|}\omega_1\cdots \mathrm{d}\omega_i\cdots\omega_k \\ & + \sum_{i=1}^{k-1}(-1)^{|\omega_1|+\cdots+|\omega_{i-1}|+|\omega_i|+1}\omega_1\cdots\omega_i\wedge\omega_{i+1}\cdots\omega_k \\ & + (-1)^{|\omega_k|(|\omega_1|+\cdots+|\omega_{k-1}|)+|\omega_k|+1}\omega_k\wedge\omega_1\cdots\omega_{k-1}. \end{aligned}$$

It descends from the Hochschild differential on the bar construction $\mathrm{B}\Omega$, which is defined as the sum of the unique extensions of degree shifts of d and $\wedge$ to coderivatives of $\mathrm{B}\Omega$ plus the wrap-around term.

Chen's iterated integral is the map

$$\begin{aligned} I: \quad \mathrm{B}^{\mathrm{cyc}}\Omega &\longrightarrow C^*(\mathrm{L}_{\mathbb{S}^1}M) \\ \omega_1\cdots\omega_k &\longmapsto \Big(\sigma \mapsto \varepsilon(\omega)\int_{K_\sigma\times\Delta^k}\omega_1(\tilde{\sigma}(x,t_1))\cdots\omega_k(\tilde{\sigma}(x,t_k))\Big), \end{aligned}$$

where $\varepsilon(\omega)$ is the sign

$$\varepsilon(\omega) = (-1)^{(k-1)(|\omega_1|+1)+(k-2)(|\omega_2|+1)+\cdots+|\omega_{k-1}|+1},$$

K_σ is a smooth chain in M, i.e., a manifold with corners, and $\tilde{\sigma}: K_\sigma \times \mathbb{S}^1 \to M$ is the projection of a lift $K_\sigma \xrightarrow{\sigma} \mathrm{L}_{\mathbb{S}^1}M \dashrightarrow \mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M$ to the second factor. We also identify $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The map I is a chain map with respect to the grading of $\mathrm{B}^{\mathrm{cyc}}\Omega$ by $|\omega_1\dots\omega_k| = |\omega_1| + \cdots + |\omega_k|$.

If $\pi_1(M) = \{1\}$, then I induces an isomorphism

$$\mathrm{H}(\mathrm{B}^{\mathrm{cyc}}\Omega, \mathrm{b})/\mathrm{span}\{[1^{2k-1}] \mid k \in \mathbb{N}\} \simeq \mathrm{H}^*(\mathrm{L}_{\mathbb{S}^1}M)/\mathbb{R}[u], \tag{1.1}$$

where $\mathrm{span}\{[1^{2k-1}] \mid k\in\mathbb{N}\} = \mathrm{H}(\mathrm{B}^{\mathrm{cyc}}\mathbb{R}, \mathrm{b})$ and the polynomial ring $\mathbb{R}[u]$ with $|u| = 2$ comes from the module structure on $\mathrm{H}^*(\mathrm{L}_{\mathbb{S}^1}M)$ induced from $\mathrm{L}_{\mathbb{S}^1}M = (\mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M)/\mathbb{S}^1 \to \mathrm{E}\,\mathbb{S}^1/\mathbb{S}^1 = \mathbb{C}\mathrm{P}^\infty$. Note that the quotient on the left hand side agrees with the homology of $\mathrm{coker}(\mathrm{B}^{\mathrm{cyc}}\mathbb{R} \hookrightarrow \mathrm{B}^{\mathrm{cyc}}\Omega)$.

Proving (1.1) is the goal of [CV20]. They study different totalizations of the Connes' cyclic bicomplex of Ω and identify the one which $(\mathrm{B}^{\mathrm{cyc}}\Omega, \mathrm{b})$ is weakly equivalent to. Then they use the isomorphism from [GJP91].

1.2. IBL-infinity chain model and Chern-Simons theory

We consider the *dual cyclic bar complex*

$$\hat{\mathrm{B}}^*_{\mathrm{cyc}}\Omega = \bigoplus_{d\in\mathbb{Z}}\prod_{k=1}^{\infty}(\mathrm{B}^{\mathrm{cyc}}_k\Omega)^{d*},$$

where $(\mathrm{B}^{\mathrm{cyc}}_k\Omega)^{d*}$ denotes the linear dual to the degree d component of the weight k component $\mathrm{B}^{\mathrm{cyc}}_k\Omega$ of $\mathrm{B}^{\mathrm{cyc}}\Omega$ (k is the number of letters in the generating word). We equip $\hat{\mathrm{B}}^*_{\mathrm{cyc}}\Omega$ with the dual Hochschild differential b^*. By taking $\prod_k$, i.e., the completion of $\bigoplus_k$ with respect to the filtration by weights, we allow "bubbling off" of forms ω_i of form-degree 1 with $|\omega_i| = 0$ and constants 1 with $|1| = -1$. Notice that if we take H_{dR}

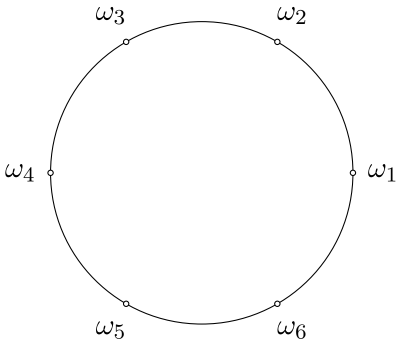


Figure 1.2.: Inserting fields on strings.

instead of Ω, then the completion is relevant only if $\mathrm{H}^1_{\mathrm{dR}} \neq 0$. The dual of Chen's iterated integral map provides a quasi-isomorphism

$$I^* : (C(\mathrm{L}_{\mathbb{S}^1} M), \partial) \to (\hat{\mathrm{B}}^*_{\mathrm{cyc}}\Omega, \mathrm{b}^*)$$

for simply-connected M.

In the following discussion, which does not aim to be rigorous at all, we introduce a *physical interpretation.* We think of

- elements of Ω as *fields,*
- elements of $\mathrm{B}^{\mathrm{cyc}}\Omega$ as *field strings* (not to confuse with string fields :-)) and
- elements of the space

$$\mathcal{F}(\mathrm{B}^{\mathrm{cyc}}\Omega[1]) \coloneqq \hat{\mathrm{S}}(\mathrm{B}^*_{\mathrm{cyc}}\Omega[1]) \mathbin{\hat{\otimes}} \mathbb{R}((\hbar))$$

as *observables on field strings.* Here, $\mathbb{R}((\hbar))$ is the ring of Laurent series in Planck's constant $\hbar$, $\hat{\mathrm{S}}$ the completed symmetric algebra and $\hat{\otimes}$ the completed tensor product.

If $\sigma : \mathbb{S}^1 \to M$ is a string, the observable $I^*(\sigma)$ "localizes" on field strings which approximate σ; for instance, it holds $I^*(\omega) = \int_\sigma \omega$, and thus $I^*(\omega)$ "localizes" at fundamental forms of $\sigma(\mathbb{S}^1)$. We imagine that we decorate σ with a field string $\omega_1 \ldots \omega_k$ as in Figure 1.2 and get a number $I(\sigma)(\omega_1 \ldots \omega_k)$. The isomorphism (1.1) guarantees that the observable $I(\sigma)$ determines σ up to a boundary term.

We will be dealing with *two "dynamical" theories:* one is the theory of fields $\omega \in \Omega$ and one of field strings $\omega_1 \cdots \omega_k \in \mathrm{B}^{\mathrm{cyc}}\Omega$.

The field theory is at hand. We know that $(\Omega, \mathrm{d}, \wedge, \langle\cdot,\cdot\rangle)$, where $\langle\omega_1, \omega_2\rangle = \int_M \omega_1 \wedge \omega_2$ for ω_1, $\omega_2 \in \Omega$, is a *symmetric dg-Frobenius algebra.* It is well-known that finite-

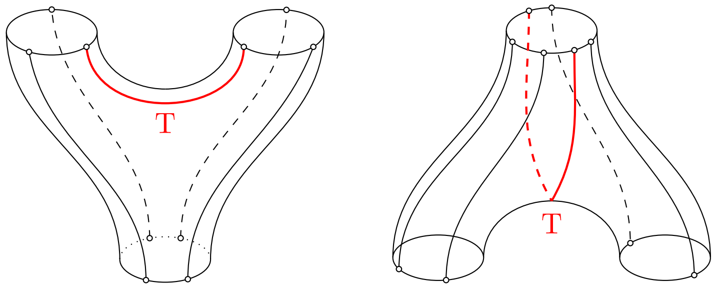


Figure 1.3.: Operations $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$. Fields propagate from the bottom to the top. An additional dualization is required when two outputs/inputs are connected; hence the emergence of the identity propagator.

dimensional symmetric Frobenius algebras V are equivalent to $2d$ topological quantum field theories (TQFT). A finite dimension is necessary so that one can write the identity as $\mathbb{1} = \sum \langle \cdot, e^i \rangle e_i$, or, in other words, that the identity propagator

$$T = \sum \pm e^i \otimes e_i$$

is well defined. We will ignore this issue and substitute $V = \Omega$ for now, although we will soon transfer to H_{dR}, where everything works just fine. We will represent interactions of fields via Feynman graphs drawn on surfaces — the trivial cylinder for fields, i.e., the free propagation, will be a line, and the pair of pants, i.e., the interaction via the intersection $\wedge$, will be a point with 3 segments emanating from it (we do not have to distinguish inputs and outputs by cyclic symmetry).

Let us now consider field strings. Figure 1.3 defines the operations

$$\mathfrak{q}_{210} : \mathrm{E}_2\mathrm{B}^*_{\mathrm{cyc}}V \longrightarrow \mathrm{B}^*_{\mathrm{cyc}}V \quad \text{and} \quad \mathfrak{q}_{120} : \mathrm{B}^*_{\mathrm{cyc}}V \longrightarrow \mathrm{E}_2\mathrm{B}^*_{\mathrm{cyc}}V,$$

where $\mathrm{E}_k\mathrm{B}^*_{\mathrm{cyc}}V$ denotes the k-th exterior power of $\mathrm{B}^*_{\mathrm{cyc}}V$ seen as the k-th symmetric power of the degree shift $(\mathrm{B}^*_{\mathrm{cyc}}V)[1]$. We read the diagram from the top to the bottom but imagine fields $\omega \in \mathrm{B}^{\mathrm{cyc}}V$ being fed into $\psi \in \mathrm{B}^*_{\mathrm{cyc}}V$ from the bottom to the top. We might think of these digrams as of *string interaction diagrams* for strings freely moving in a topological space M, connecting and disconnecting. This suggests that $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ are related to $\mathbb{m}_2$ and $\mathbb{c}_2$. Formulas for $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ were written down in [CFL15]; they are also clear from the figure by decorating the world-lines with the identity (or T) and evaluating in a straightforward way. It was proven that $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ indeed constitute an IBL-algebra on $\mathrm{B}^*_{\mathrm{cyc}}V$ (note that this is not a TQFT for strings!).

As a mathematical remark, we will show that $\mathfrak{q}_{210}$ is obtained from the Gerstenhaber bracket on Hochschild cochains via cyclization by $\langle\cdot,\cdot\rangle$ and that $\mathfrak{q}_{120}$ is a factorization of an extension of the canonical Schwarz's BV-operator on $\mathcal{F}(V[1])$ to cyclic invariants with respect to the cyclic shuffle product. This makes sense because odd degree shift of a finite-dimensional symmetric dg-Frobenius algebra is an odd symplectic vector space.

Since $\mathrm{B}^*_{\mathrm{cyc}}V[1]$ is not naturally an odd symplectic vector space, there is no Schwarz's BV-operator on $\mathcal{F}(\mathrm{B}^*_{\mathrm{cyc}}V[1])$. However, the following canonical operator

$$\Delta_{\mathrm{s}} \coloneqq \hat{\mathfrak{q}}_{120} + \hbar\hat{\mathfrak{q}}_{210} : \mathcal{F}(\mathrm{B}^*_{\mathrm{cyc}}V[1]) \to \mathcal{F}(\mathrm{B}^*_{\mathrm{cyc}}V[1]),$$

where $\hat{}$ denotes the canonical extension to (co)derivatives, is a BV-operator with respect to the function multiplication; we call it the *string* BV*-operator.*

In physics, a BV-operator Δ on $\mathcal{F}(U)$, where U is the space of fields (typically an odd cotangent bundle with classical fields in the base and ghost fields in the fibers), is related to the measure in the path integral $\int \mu$. An action $S \in \mathcal{F}(U)$ satisfying the *quantum master equation* (QME)

$$\Delta\, S + \frac{1}{2}\{S, S\} = 0$$

defines a new measure $e^{-S}\mu$, and the corresponding twisted BV-operator (or rather BV_∞-operator) satisfies $\Delta^S = e^{-S}\,\Delta\, e^S$.

For field strings, we define the following *actions* S_{free}, $S_{\mathrm{int}} \in \mathcal{F}(\mathrm{B}^{\mathrm{cyc}}V[1])$, which remind us of the *Chern-Simons functional:*

$$S_{\mathrm{free}}(\omega_1\omega_2) \coloneqq \pm\hbar^{-1}\int_M \omega_1 \wedge \mathrm{d}\omega_2 \quad \text{and} \quad S_{\mathrm{int}}(\omega_1\omega_2\omega_3) \coloneqq \pm\hbar^{-1}\int_M \omega_1 \wedge \omega_2 \wedge \omega_3.$$

More precisely, S_{free} and S_{int} are linear functions on $\mathrm{B}^{\mathrm{cyc}}V[1]$ which vanish everywhere but on field strings of lengths 2 and 3, respectively. It turns out that S_{free} and $S \coloneqq S_{\mathrm{free}} + S_{\mathrm{int}}$ satisfy the QME for Δ_s. The twisted BV-operators look like

$$\Delta_s^{S_{\mathrm{free}}} = \hat{\mathfrak{q}}_{110} + \Delta_{\mathrm{s}} \quad \text{and} \quad \Delta_s^{S} = \hat{\mathfrak{q}}_{110} + \widehat{\mathfrak{q}_{210}(S_{\mathrm{int}},\cdot)} + \Delta_{\mathrm{s}}\,.$$

Figure 1.4 depicts the new terms $\mathfrak{q}_{110}$ and $\mathfrak{q}_{210}(S_{\mathrm{int}},\cdot)$ in Δ_s^S. The corresponding dIBL-algebra reads

$$\Big(\mathrm{B}^*_{\mathrm{cyc}}V, \mathfrak{q}^{\mathfrak{m}}_{110} \coloneqq \mathfrak{q}_{110} + \mathfrak{q}_{210}(S_{\mathrm{int}},\cdot), \mathfrak{q}_{210}, \mathfrak{q}_{120}\Big).$$

One can show that $\mathfrak{q}^{\mathfrak{m}}_{110}$ is the Hochschild differential. If this was well-defined for $V = \Omega$, then it would surely be a model of string topology.

As in quantum field theories, we are going to "formally" *integrate out redundant degrees of freedom in the path integral* of our ill-defined theory and obtain a well-defined theory

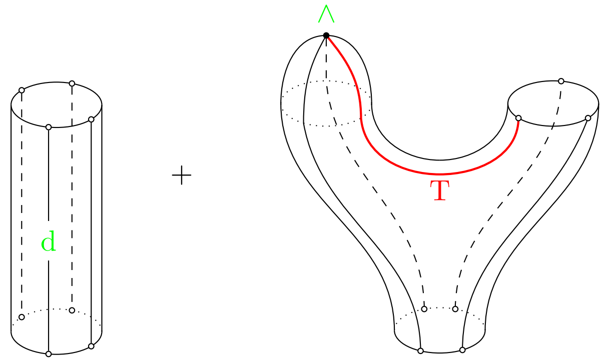


Figure 1.4.: Adding d and $\wedge$ via S.

on $\mathrm{B}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}$, which is "formally" homotopy equivalent to the original one. We pick a Riemannian metric on M and consider the Hodge decomposition

$$\Omega = \mathcal{H} \oplus \mathrm{d}\Omega \oplus \mathrm{d}^*\Omega,$$

where $\mathcal{H} \simeq \mathrm{H}_{\mathrm{dR}}$ is the space of harmonic forms defined by $\mathrm{d}\omega = \mathrm{d}^*\omega = 0$. One may interpret $\mathrm{d}\omega = 0$ as the Euler-Lagrange equation and $\mathrm{d}^*\omega = 0$ as the Lorentz gauge. The inverse of $\mathrm{d} : \mathrm{d}^*\Omega \to \mathrm{d}\Omega$ extended by 0 to $\mathcal{H}$ and $\mathrm{d}^*\Omega$ is called the *standard Hodge homotopy* $\mathcal{P}_{\mathrm{std}}$*;* equivalently, it is the unique coexact solution of

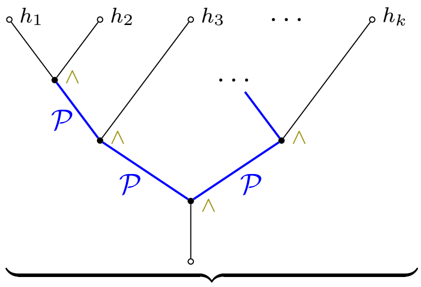
$$\mathrm{d}\mathcal{P} + \mathcal{P}\mathrm{d} = \pi_{\mathcal{H}} - \mathbb{1},$$

where $\pi_{\mathcal{H}} : \Omega \to \mathcal{H}$ is the orthogonal projection. The Schwartz kernel of $\mathcal{P}_{\mathrm{std}}$ is the *standard Hodge propagator* $\mathrm{P}_{\mathrm{std}}$.

A formula for the *effective action* $W \in \mathcal{F}(\mathrm{B}^{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}[1])$ was given in [CFL15]; in their terminology, W is equivalent to the *(formal) pushforward Maurer-Cartan element.* We have

$$W = \hbar^{-1} \sum_{l\geq 1, g\geq 0} \mathfrak{n}_{lg}\hbar^g,$$

where $\mathfrak{n}_{lg} \in \hat{\mathrm{E}}_l\mathrm{B}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}$ is computed by summing over $(l+g-1)$-loop Feynman diagrams with interaction vertices $\wedge$ and propagator $\mathrm{P}_{\mathrm{std}}$. We remark that the Feynman diagrams in W have at least one external vertex. One might try to construct a refinement $(W^0_{lg})_{l\geq 1, g\geq 0}$ of the Chern-Simons invariant by summing over diagrams with no external vertex, but it seems to be unrelated to the IBL_∞-theory so far.



$$\wedge \circ (\mathcal{P} \otimes \mathcal{P}) \circ (\wedge \otimes \wedge) \circ (\mathcal{P} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) \\ \circ (\wedge \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})(h_1, h_2, h_3, \dots, h_k).$$

Figure 1.5.: Kontsevich-Soibelman evaluation of a decorated tree.

The twisted string BV-operator on $\mathcal{F}(\mathrm{B}^{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}[1])$ reads

$$\Delta_s^W = \hat{\mathfrak{q}}^{\mathfrak{n}}_{110} + \hbar \mathfrak{q}_{210} + \sum_{l \geq 2, g \geq 0} \hat{\mathfrak{q}}^{\mathfrak{n}}_{1lg} \hbar^g,$$

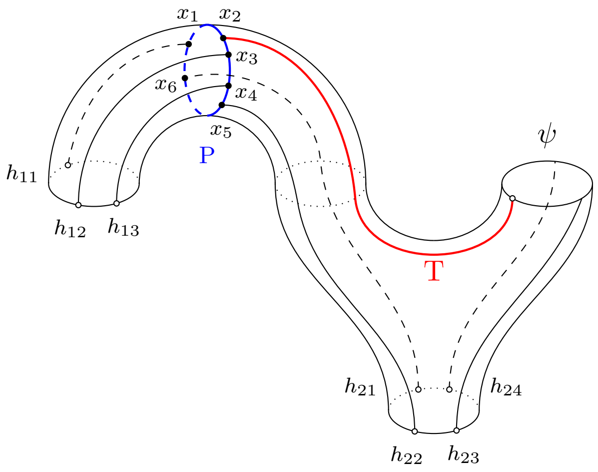
where $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}_{210}(\mathfrak{n}_{10}, \cdot) = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$ and $\mathfrak{q}^{\mathfrak{n}}_{1lg} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg}$, where $\circ_1$ means that precisely one output of the first operation is connected to precisely one input of the following operation. The resulting IBL_∞-structure on $\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}$ has lots of vanishing operations. It is in fact a *quantum* coL_∞*-algebra* $(\mathfrak{q}^{\mathfrak{n}}_{1lg})$ *with Drinfeld-compatible Lie bracket* $\mathfrak{q}_{210}$. The boundary operator $\mathfrak{q}^{\mathfrak{n}}_{110}$ is precisely the Hochschild differential of the homotopy transfered A_∞-structure (m_k) on $\mathcal{H}$. The quasi-isomorphism of $(\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}, \mathfrak{q}^{\mathfrak{n}}_{110})$ and $(C(\mathrm{L}_{\mathbb{S}^1}M), \partial)$ inducing an isomorphism of the IBL-structure on homology is given by the composition $F \circ I^* : C(\mathrm{L}_{\mathbb{S}^1}M) \to \hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}$ for

$$F = \mathfrak{f}_{110} + \mathfrak{f}_{210} \circ_1 \mathfrak{m}_{10} + \frac{1}{2!}\mathfrak{f}_{310} \circ_{1,1} (\mathfrak{m}_{10}, \mathfrak{m}_{10}) + \cdots,$$

where $\mathfrak{f}_{110} = \iota^*$ for $\iota : \mathrm{H}_{\mathrm{dR}} \simeq \mathcal{H} \hookrightarrow \Omega$ and $\mathfrak{f}_{k10} \circ_{1,\dots,1} (\mathfrak{m}_{10}, \dots, \mathfrak{m}_{10})$ is obtained by summing over trivalent trees as in Figure 1.5.

Note that in order to evaluate trees, we do not need the Schwartz kernel P and hence any pairing $\langle \cdot, \cdot \rangle$. The homotopy $\mathcal{P}$ is enough because the graph is directed and we can distinguish inputs and outputs. On the other hand, an evaluation of the 1-loop Feynman graph in Figure 1.6 contributing to $\mathfrak{q}^{\mathfrak{n}}_{120}$ requires P, and hence also $\langle \cdot, \cdot \rangle$.

It is well-known from Sullivan's minimal model theory of a simply-connected manifold M that the homotopy type of the homotopy transferred A_∞-structure (m_k) on $\mathcal{H}$ is a topological invariant which encodes the rational homotopy theory of M. The IBL_∞-



$$= \sum_{a,b} \sum_{c=1}^{4} \pm \mathrm{T}^{ab} \psi(e_a h_{2,c+2} h_{2,c+3}) \Big(\int_{x_1x_2x_3x_4x_5x_6} \mathrm{P}(x_1,x_2)\mathrm{P}(x_2,x_3)\mathrm{P}(x_3,x_4)$$
$$\mathrm{P}(x_4,x_5)\mathrm{P}(x_5,x_6)\mathrm{P}(x_6,x_1)\big(h_{11}(x_1)h_{12}(x_3)h_{13}(x_4)\big)\big(e_b(x_2)h_{2,c}(x_6)h_{2,c+1}(x_5)\big)\Big)$$

Figure 1.6.: A 1-loop diagram and its contribution to the twisted cobracket.

construction is associated to the Poincaré DGA $(\Omega, \mathrm{d}, \wedge, \langle\cdot,\cdot\rangle)$, i.e, a DGA whose homology is a Poincaré duality algebra. It is not clear yet to which extent it depends on the pairing and what kind of invariant of M it is. However, if M is formal in the sense of DGA's, then M is formal in the sense of Poincaré DGA's, and we conjecture that it is formal also in the sense of IBL_∞-algebras; by this we mean that the twisted IBL_∞-algebra on $\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}$ is homotopy equivalent to the canonical IBL-algebra on $\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}$.

As a final remark, it is well-known from the theory of Koszul (pr)operads that IBL is Koszul dual to Frob, i.e., $\mathrm{IBL}^! = \mathrm{Frob}$ and $\mathrm{Frob}^! = \mathrm{IBL}$. Here, Frob is the properad of Frobenius bialgebras. It follows that $\mathrm{IBL}_\infty = \Omega(\mathrm{Frob}^*)$, where Ω denotes the cobar construction and * the linear dual coproperad. This precisely reflects our situation of having a Frobenius bialgebra structure on H_{dR}, where the coproduct is obtained from $\wedge$ via dualization, and an IBL_∞-structure on $\mathrm{B}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}$ if $\mathrm{B}^{\mathrm{cyc}}$ is understood as a cyclic version of the cobar construction.

1.3. Other relevant work

In [CJ02], a homotopy theoretical realization of the Chas-Sullivan loop product on $\mathrm{H}(\mathrm{L}M)$ by constructing the "wrong way map" using the Thom-Pontryagin construction was described. Note that having the loop product, one constructs $\mathbb{m}_2$ on $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)$ via the Gysin sequence for the Borel construction.

In [Che12], the Chas-Sullivan BV-algebra on $\mathrm{H}(\mathrm{L}M)$ and the gravity algebra on $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)$ were constructed using an algebraic model based on Whitney polynomial forms with coefficients in $\mathbb{Q}$. The advantage of Whitney forms A over de Rham forms Ω is that the dualization of the product gives a complete coproduct with values in the currents $C = A^*$, which together with the product on A constitutes a dg Frobenius-like algebra. It is shown that $A\hat{\otimes}\hat{\Omega}(C)$, where $\hat{\Omega}$ denotes the complete cobar construction, carries a natural dg-algebra structure which corresponds to the loop product on $\mathrm{H}(\mathrm{L}M)$ under the Jones, et al., quasi-isomorphism from the singular chain complex of $\mathrm{L}M$. The equivariant case is handled with methods of Connes' cyclic homology.

In [Iri18], de Rham chains of marked Moore loops and their fiber products and concatenations are used to construct a non-symmetric dg operad $\mathcal{O}$ with a cyclic structure, multiplication and unit together with a morphism $\mathcal{O} \to \mathrm{End}_{\Omega}$. The cyclic Deligne conjecture is applied to obtain an algebra $\widetilde{\mathcal{O}}$ over a chain model of the framed little disk operad (whose homology is the BV-operad) such that the induced quasi-isomorphism from $\widetilde{\mathcal{O}}$ to Hochschild cochains induces an isomorphism of the BV-structures on homology. The latter is known to be isomorphic to the BV-structure on $\mathrm{H}(\mathrm{L}M)$ via iterated integrals. In addition, they use the homotopy transfer from $\widetilde{\mathcal{O}}$ to obtain A_∞- and L_∞-structures on $\mathrm{H}(LM)$ whose operations with 2 inputs are the loop product and the Gerstenhaber bracket, respectively. Interestingly, their chain model works for non-simply connected M.

In [DPR15], they use diffuse intersection and short geodesic segments to associate to metric chord diagrams operations on the singular chain complex of $\mathrm{L}M$. They should recover the full positive boundary TQFT on $\mathrm{H}(\mathrm{L}M)$ described in [CG04].

In [Sul07], a rich structure of operations on equivariant chains of $\mathrm{L}M$ parametrized by chains in a certain compactification of the moduli space of Riemann surfaces is discussed. A part of this structure is an IBL_∞-chain model on the reduced chains.

In [Abo15], it is proven that the BV-algebra of symplectic homology of the cotangent bundle of an oriented manifold M is isomorphic to the BV-algebra $\mathrm{H}(\mathrm{L}M)$.

In [CL09], they sketch a proof that symplectic field theory of the unit cotangent bundle of M and chain level equivariant string topology of M are IBL_∞-quasi-isomorphic via a map induced from evaluations at boundaries of holomorphic curves in symplectizations. From this point of view, string topology operations arise naturally from the structure of codimension 1 boundary stratas of the moduli space of holomorphic curves. In fact, a precise formulation of this correspondence was perhaps the main reason for Cieliebak & Latschev to think about an IBL_∞-chain model of string topology.

In [Fuk06], it is argued that the compactified moduli space $\widehat{\mathcal{M}}$ of holomorphic discs with boundaries on a Lagrangian submanifold L of a symplectic manifold M gives rise to a filtered A_∞-structure on $\mathrm{H}^*(L)$. Evaluation at the boundary allows to interpret $\widehat{\mathcal{M}}$ as a chain in the free loop space. The structure of codimension 1 boundary strata of $\widehat{\mathcal{M}}$ implies the relation $\partial\widehat{\mathcal{M}} + \frac{1}{2}\{\widehat{\mathcal{M}}, \widehat{\mathcal{M}}\} = 0$, where $\{\cdot,\cdot\}$ is the chain level string bracket. Under iterated integrals, this translates to the Maurer-Cartan equation on the cyclic bar construction of $\Omega(L)$ with the Gerstenhaber bracket. The twisted coderivation gives the A_∞-structure on $\Omega(L)$ which is then homotopy transferred to $\mathrm{H}^*(L)$.

1.4. Summary of results

1) The starting point was setting up a formalism and deducing signs for a definition of the formal-pushforward Maurer-Cartan element, aka Chern-Simons Maurer-Cartan element, and the IBL_∞-chain model in the de Rham setting. A big part of the work was about trying to understand what is happening and discovering and formulating the structure and possible claims.

2) We compute the IBL_∞-chain model for $\mathbb{S}^n$ with $n \neq 2$ by finding an explicit Hodge propagator and computing Feynman integrals. In fact, for $n \geq 3$, all integrals which are relevant for the IBL_∞-theory vanish.

   A trick from [CM10] is based on modifying an abstract Hodge propagator to obtain special properties implying vanishing of the integrals. The author of this thesis was not aware of this trick and tried to compute integrals with an explicit propagator in spherical coordinates for around 3 years until he rediscovered a part of this trick himself. The interesting thing is that the discovery was made via explicit computations, and it was a coincidence that the constructed Hodge propagator satisfied the special properties.

3) Using the trick from [CM10], we generalize the previous computation to geometrically formal manifolds and show that the Feynman integrals vanish provided that $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$. For a general manifold, all higher coproducts vanish unless M is a surface or a 3-manifold with $\mathrm{H}^1_{\mathrm{dR}}(M) \neq 0$. In fact, the homotopy type of the IBL_∞-chain model for a manifold with $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$ is determined by the tree-level perturbative Chern-Simons theory for a special Hodge propagator.

4) We conjecture that the IBL_∞-chain model for formal manifolds with $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$ is homotopy equivalent to the canonical dIBL-structure.

5) There are two approaches of associating an IBL_∞-homotopy type to a Poincaré DGA like $\Omega(M)$. One is using the homotopy transfer and integrals as explained in the previous section (geometric approach) and one is by taking a Poincaré duality model $\mathcal{M}$ and constructing the canonical dIBL structure on cyclic cochains of $\mathcal{M}$ (algebraic approach). We study both and conjecture that they are equivalent.

6) We study DGA's of Hodge type and give an alternative proof of the existence of a Poincaré duality model in the category of PDGA's. In the DGA category, this is originally due to Lambrechts & Stanley. The new method is based on adding exact partners to non-degenerates rather than adding killers of orphans.

7) We prove a proposition that the cyclic homology of a strictly unital A_∞-algebra can be computed from its reduced version. We do it by extending Loday's cyclic homology theory for DGA's to A_∞-algebras.

8) We relate $\mathfrak{q}_{210}$ to Gerstenhaber bracket and its cyclization and $\mathfrak{q}_{120}$ to the Schwarz's BV-operator and cyclic shuffle product.

9) We extend the MV-formalism to filtered MV-formalism and use it to construct a BV-formulation of the weak IBL_∞-theory. This has the advantage that the exponentials are honest exponentials and honest maps. This will be useful for studying BV-chain complexes.

10) We formulate the composition at k-common channels $\circ_k$ using "heart with veins" which appears in the iterated bialgebra compatibility condition.

11) We propose a BV-formulation of the IBL_∞-theory with an action, effective action and quantum master equation.

12) We find the standard Hodge propagator for $\mathbb{S}^2$ up to a constant and prove that it smoothly extends to spherical blow-up.

Part I.

IBL-infinity chain model

2. Overview

An IBL_∞-*algebra* is essentially a collection of multilinear operations $\mathfrak{q}_{klg}$ with k inputs, l outputs and "genus" g satisfying certain relations; in particular, $\mathfrak{q}_{110}$ is a boundary operator, and the pair $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ induces the structure of an involutive Lie bialgebra on the homology of $\mathfrak{q}_{110}$. It was introduced in [CFL15] and applications to string topology, symplectic field theory and higher genus Lagrangian Floer theory were proposed.

This part of the thesis is an attempt to understand its application to *string topology*. The idea was to carry out some explicit computations according to the plan sketched in [CFL15, Section 13] and test the string topology conjecture (see below).

The following results from [CFL15, Corollary 11.9] are our starting point. Precise definitions of all the notions will be given in the next chapters. IBL_∞-algebras in Part I will be strict and filtered in the terminology of [CFL15]. We denote by $[d]$ the degree shift by d; we handle the additional signs in [CFL15] by working with degree shifted objects.

(A) For a finite-dimensional cyclic cochain complex (V, Π, m_1) of degree $2-n$, where $\Pi : V[1] \otimes V[1] \to \mathbb{R}$ is a pairing of degree $2-n$ and $m_1 : V[1] \to V[1]$ a differential, there is a canonical dIBL-structure $\mathfrak{p}_{110}$, $\mathfrak{p}_{210}$, $\mathfrak{p}_{120}$ of bidegree $(n-3, 2)$ on the degree shifted dual cyclic bar complex

$$C(V) \coloneqq \mathrm{B}^*_{\mathrm{cyc}} V[2-n] \simeq \Big(\bigoplus_{k\geq 1} (V[1]^{\otimes k}/\mathrm{cyc})^{*_{\mathrm{g}}}\Big)[2-n],$$

where cyc stands for cyclic permutations with the Koszul sign and $*_{\mathrm{g}}$ denotes the graded dual. This structure is denoted by $\mathrm{dIBL}(C(V))$.

(B) Let $(\mathcal{H}, \Pi, m_1) \subset (V, \Pi, m_1)$ be a subcomplex such that the restriction $\Pi : \mathcal{H}[1] \otimes \mathcal{H}[1] \to \mathbb{R}$ is non-degenerate. We apply (A) to $(\mathcal{H}, \Pi, m_1)$ to get the canonical dIBL-algebra $\mathrm{dIBL}(C(\mathcal{H})) = (C(\mathcal{H}), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$. Suppose that $\pi : V[1] \to V[1]$ is a projection to $\mathcal{H}[1]$ which satisfies

$$\pi \circ m_1 = m_1 \circ \pi \quad \text{and}$$
$$\Pi(\pi(v_1), v_2) = \Pi(v_1, \pi(v_2))$$

for all v_1, $v_2 \in V[1]$, and let $\iota : \mathcal{H}[1] \to V[1]$ be the inclusion. A linear map $\mathcal{P} : V[1] \to V[1]$ of degree -1 such that

$$\begin{aligned} m_1 \circ \mathcal{P} + \mathcal{P} \circ m_1 = \iota \circ \pi - \mathbb{1}_{V[1]} \quad \text{and} \\ \Pi(\mathcal{P}(v_1), v_2) = (-1)^{|v_1|} \Pi(v_1, \mathcal{P}(v_2)) \end{aligned} \tag{2.1}$$

for all v_1, $v_2 \in V[1]$ induces an IBL_∞-homotopy equivalence

$$\mathfrak{f} = (\mathfrak{f}_{klg}) : \mathrm{dIBL}(C(V)) \longrightarrow \mathrm{dIBL}(C(\mathcal{H}))$$

such that $\mathfrak{f}_{110} : C(V)[1] \to C(\mathcal{H})[1]$ is the map given by the precomposition with ι in every component. We recall from [CFL15] that $\mathfrak{f}_{klg} : \mathrm{E}_k C(V) \to \mathrm{E}_l C(\mathcal{H})$ is a linear map between exterior powers of C which are realized as symmetric powers of the degree shift $C[1]$.

The map $\mathfrak{f}_{klg}$ is constructed as a sum of contributions coming from isomorphism classes of *ribbon graphs* (=: multigraphs with a cyclic ordering of half-edges at every internal vertex) with k internal vertices, l boundary components and genus g. To compute the contribution of a labeled ribbon graph Γ to the value

$$\mathfrak{f}_{klg}(\Psi_1 \otimes \cdots \otimes \Psi_k)(\mathrm{w}_1 \otimes \cdots \otimes \mathrm{w}_l)$$

for $\Psi_1, \ldots, \Psi_k \in \mathrm{B}^*_{\mathrm{cyc}} V[3-n]$ and $\mathrm{w}_1, \ldots, \mathrm{w}_l \in \mathrm{B}^{\mathrm{cyc}} \mathcal{H}[3-n]$, we decorate the i-th internal vertex of Γ with Ψ_i, external vertices lying on the i-th boundary component with components $v_{i1}, \ldots, v_{is_i} \in V[1]$ of $\mathrm{w}_i = \mathrm{s}(v_{i1} \otimes \cdots \otimes v_{is_i}/\mathrm{cyc})$, where s is a formal symbol of degree $n-3$, and internal edges with the Schwartz kernel P of $\mathcal{P}$ with respect to Π. Decorated ribbon graphs are then evaluated in a consistent way to obtain real numbers (see Appendix A for an invariant formalism or [CFL15, Section 10] for the original definition in coordinates).

We will also use the following results from [CFL15, Proposition 12.5 and Theorem 12.9] about deformations of IBL_∞-algebras:

(C) If in addition to (A) there is a product $m_2 : V[1] \otimes V[1] \to V[1]$ making (V, m_1, m_2) into a cyclic dga, then $(-1)^{n-2} m_2^+ \in C(V)$, where $m_2^+ = \Pi(m_2(\cdot,\cdot),\cdot)$, defines a canonical Maurer-Cartan element $\mathfrak{m} := (\mathfrak{m}_{10})$ for $\mathrm{dIBL}(C(V))$. The twisted IBL_∞-algebra is again a dIBL-algebra of bidegree $(n-3, 2)$; it is denoted by $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$

and satisfies

$$\begin{aligned}&\mathrm{dIBL}^{\mathfrak{m}}(C(V))\\&\quad=\big(C(V),\mathfrak{p}^{\mathfrak{m}}_{110}=\mathfrak{p}_{110}+\mathfrak{p}_{210}\circ_1\mathfrak{m}_{10},\ \mathfrak{p}^{\mathfrak{m}}_{210}=\mathfrak{p}_{210},\ \mathfrak{p}^{\mathfrak{m}}_{120}=\mathfrak{p}_{120}\big).\end{aligned}\tag{2.2}$$

(D) The IBL_∞-morphism $\mathfrak{f}$ from (B) can be used to pushforward $\mathfrak{m}$ and obtain a Maurer-Cartan element $\mathfrak{n}=(\mathfrak{n}_{lg})$ for $\mathrm{dIBL}(C(\mathcal{H}))$. The twist by $\mathfrak{n}$ is an IBL_∞-algebra of bidegree $(n-3,2)$; it is denoted by $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}))$ and satisfies

$$\begin{aligned}&\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}))\\&\quad=\big(C(\mathcal{H}),\mathfrak{q}^{\mathfrak{n}}_{110}=\mathfrak{q}_{110}+\mathfrak{q}_{210}\circ_1\mathfrak{n}_{10},\ \mathfrak{q}^{\mathfrak{n}}_{210}=\mathfrak{q}_{210},\ \mathfrak{q}^{\mathfrak{n}}_{120}=\mathfrak{q}_{120}\\&\qquad\quad+\mathfrak{q}_{210}\circ_1\mathfrak{n}_{20},\ \text{plus higher operations } \mathfrak{q}^{\mathfrak{n}}_{1lg}=\mathfrak{q}_{210}\circ_1\mathfrak{n}_{lg}\big).\end{aligned}$$

This IBL_∞-algebra is IBL_∞-homotopy equivalent to $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$ via a twisted IBL_∞-morphism

$$\mathfrak{f}^{\mathfrak{m}}=(\mathfrak{f}^{\mathfrak{m}}_{klg}):\mathrm{dIBL}^{\mathfrak{m}}(C(V))\longrightarrow\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H})).$$

The pushforward Maurer-Cartan element $\mathfrak{n}=\mathfrak{f}_*\mathfrak{m}$ can be expressed as a sum of contributions of isomorphism classes of *trivalent ribbon graphs* (m_2^+ has namely three inputs), where a labeled ribbon graph Γ is decorated with m_2^+ at internal vertices, with the components of the i-th argument of $\mathfrak{n}_{lg}$, i.e., elements of $\mathcal{H}[1]$, at the i-th boundary component and with P at internal edges.

The application to string topology of an oriented closed manifold M of dimension n comes from studying generalizations of (A) – (D) to the infinite-dimensional oriented dga (Ω,Π,m_1,m_2). Here, $\Omega=\Omega(M)$ is the de Rham complex of M and the maps $\Pi:\Omega[1]^{\otimes 2}\to\mathbb{R}$, $m_1:\Omega[1]\to\Omega[1]$ and $m_2:\Omega[1]^{\otimes 2}\to\Omega[1]$ are defined for all η, η_1, $\eta_2\in\Omega$ as follows:

$$\text{de Rham cyc. dga}\left\{\begin{aligned}\Pi(\theta\eta_1,\theta\eta_2)&:=(-1)^{\deg\eta_1}\int_M\eta_1\wedge\eta_2,\\ m_1(\theta\eta)&:=\theta\mathrm{d}\eta,\\ m_2(\theta\eta_1,\theta\eta_2)&:=(-1)^{\deg\eta_1}\theta(\eta_1\wedge\eta_2),\end{aligned}\right.\tag{2.3}$$

where d is the de Rham differential, $\wedge$ the wedge product, θ a formal symbol of degree -1 and $\deg\eta_1$ is the form-degree of η_1.

By picking a Riemannian metric on M, we obtain the subcomplex of harmonic forms $(\mathcal{H},\Pi,m_1\equiv 0)$ with the projection $\pi_{\mathcal{H}}:\Omega\to\mathcal{H}$ coming from the Hodge decomposition. This cyclic cochain complex shall be taken as the subcomplex in (B).

Because the intersection pairing on $\Omega[1]$ is not perfect, one has to restrict the construction in (A) to the subspace $\mathrm{B}^*_{\mathrm{cyc}}\Omega_\infty$ of elements with smooth Schwartz kernel. Then (A) and (B) work in the setting of the so called Fréchet IBL_∞-algebras introduced in [CFL15, Section 13]. However, the element $\mathfrak{m}_{10} \in \mathrm{B}^*_{\mathrm{cyc}}\Omega[3-n]$, which translates into the *Chern-Simons term*

$$m_2^+(\theta\eta_1, \theta\eta_2, \theta\eta_3) \coloneqq (-1)^{\deg \eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \eta_1, \eta_2, \eta_3 \in \Omega(M),$$

does not define the canonical Maurer-Cartan element $\mathfrak{m}$ in (C) directly because $m_2^+ \notin \hat{\mathrm{B}}^*_{\mathrm{cyc}}\Omega_\infty$. This also means that one cannot use (D) to conclude the existence of the pushforward Maurer-Cartan element $\mathfrak{n}$.

Nevertheless, it was proposed to define $\mathfrak{n}$ formally using a summation over trivalent ribbon graphs as in the finite-dimensional case. We call such $\mathfrak{n}$ the *Chern-Simons* or *formal pushforward Maurer-Cartan element.* In order to compute the contribution of a labeled trivalent ribbon graph Γ with k internal vertices, l boundary components and genus g to the value

$$\mathfrak{n}_{lg}(\Omega_1 \otimes \dots \otimes \Omega_l),$$

where $\Omega_i = \mathrm{s}\omega_i$ for $\omega_1, \dots, \omega_l \in \mathrm{B}^{\mathrm{cyc}}\mathcal{H}$, one starts by decorating internal vertices with integration variables $x_1, \dots, x_k$ on the k-fold product $M \times \dots \times M$, external vertices on the i-th boundary component with the components $\alpha_{i1}, \dots, \alpha_{is_i} \in \mathcal{H}[1]$ of ω_i and internal edges with the Hodge propagator P. In the setting of Ω and $\mathcal{H}$, P shall be the Schwartz kernel of a homotopy $\mathcal{P}$ in the sense of pseudo-differential operators (it is necessarily singular at the diagonal and smooth outside of it, i.e., $\mathrm{P} \in \Omega^{n-1}(M \times M \backslash \Delta)$). One then takes the wedge product of all forms in the decorated graph in the order and with the sign deduced from the labeling of Γ and computes the integral over $x_1, \dots, x_k$. Similar integrals appear in *perturbative Chern-Simons quantum field theory.*

Because of the singularity of P at Δ, the integrand described above is smooth only on the k-th configuration space of M. It is not clear that all the integrals converge and that the resulting $(\mathfrak{n}_{lg})$ are well-defined and satisfy the Maurer-Cartan equation. The idea of work in progress [CVon] of K. Cieliebak and E. Volkov is to use iterated spherical blow-ups of multiple diagonals to resolve the singularities and obtain integrals of smooth forms on compact manifolds with corners; this guarantees integrability. The Maurer-Cartan equation for $\mathfrak{n} = (\mathfrak{n}_{lg})$ is then proven with the help of a Stokes' formula on stratified spaces; the key part is to show that contributions of hidden codimension-1 faces cancel. This method is similar to the method from [AS92] and [AS94], where Feynman integrals of perturbative Chern-Simons theory were considered.

Having $\mathfrak{n}$, the twisted IBL_∞-algebra $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}))$, which can be equivalently written as $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}))$ using the Hodge isomorphism $\mathcal{H} \simeq \mathrm{H}_{\mathrm{dR}}$, should satisfy the following conjecture:

String topology conjecture (Quotation of [CFL15, Conjecture 1.12]). *Let M be a closed oriented manifold of dimension n and H_{dR} its de Rham cohomology. Then there exists an IBL_∞-structure on (a suitable version of) $\mathrm{B}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}[2-n]$ whose homology equals the cyclic cohomology of the de Rham complex of M.*

The idea is as follows. The $\mathbb{S}^1$*-equivariant homology of the free loop space* $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)$ is for simply-connected M isomorphic to a version of *Connes' cyclic cohomology of the de Rham algebra* $\mathrm{H}^*_\lambda(\Omega)$. The precise relation will be established in yet another work in progress [CVon] of K. Cieliebak and E. Volkov using a chain-map coming from a cyclic version of Chen's iterated integrals. Now, a suitable degree shift of $\mathrm{H}^*_\lambda(\Omega)$ is isomorphic to the homology of the boundary operator $\mathfrak{q}^{\mathfrak{m}}_{110}$ of the hypothetical dIBL-algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(\Omega))$. This hypothetical dIBL-algebra is then by (D) hypothetically quasi-isomorphic to the IBL_∞-algebra $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}))$ via the twisted IBL_∞-morphism $\mathfrak{f}^{\mathfrak{m}}$.

The space $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M, M)$ carries an IBL-structure consisting of the Chas-Sullivan string bracket $\mathbb{m}_2$ and string cobracket $\mathbb{c}_2$; these operations were defined geometrically on suitably transverse smooth chains in [CS99] and [CS04], respectively. A natural question is: How is the IBL-structure $\mathbb{m}_2$, $\mathbb{c}_2$ related to the IBL-structure $\mathfrak{q}^{\mathfrak{n}}_{210}$, $\mathfrak{q}^{\mathfrak{n}}_{120}$ induced on $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)$ via the isomorphism from the string topology conjecture? An extended string topology conjecture asserts that these structures agree, and hence the operations $\mathfrak{q}^{\mathfrak{n}}_{210}$, $\mathfrak{q}^{\mathfrak{n}}_{120}$ defined on cyclic cochains provide a *chain model* for $\mathbb{m}_2$, $\mathbb{c}_2$. Based on our observations and explicit computations, we formulate an up-to-date version of the string topology conjecture for simply-connected manifolds (see Conjecture 4.5.1).

Our first result is an explicit computation of $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(\mathbb{S}^n)))$ by finding a particular Hodge propagator and showing that all integrals which contribute to $\mathfrak{n}$ vanish for $n \geq 3$; for $n = 1$, there is a non-vanishing integral whose value we compute (see Section 5.1); for $n = 2$, the existence of a non-vanishing integral remains open.

Theorem A (Explicit computation for $\mathbb{S}^n$). *Consider the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Define $\mathrm{1} \coloneqq \theta 1$, $\mathrm{v} \coloneqq \theta \mathrm{Vol} \in \mathrm{H}_{\mathrm{dR}}(\mathbb{S}^n)[1]$, where Vol is the volume form, 1 the constant one and θ a formal symbol of degree -1. There exists a Hodge propagator P such that the* *following holds. For the homology of the twisted boundary operator* $\mathfrak{q}^{\mathfrak{n}}_{110}$*, we have:*

$$\begin{aligned}\mathbb{H}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(\mathbb{S}^n)))[1] &\coloneqq \mathrm{H}(\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}(\mathbb{S}^n)[3-n], \mathfrak{q}^{\mathfrak{n}}_{110}) \\ &= \begin{cases} \langle \mathrm{s}\mathrm{v}^{i*}, \mathrm{s}1^{2j-1*} \mid i, j \geq 1\rangle & \textit{for } n \geq 3 \textit{ odd,} \\ \langle \mathrm{s}\mathrm{v}^{2i-1*}, \mathrm{s}1^{2j-1*} \mid i, j \geq 1\rangle & \textit{for } n \textit{ even,} \\ \langle \mathrm{s}\sum_{k=1}^{\infty} c_k \mathrm{v}^{k*}, \mathrm{s}1^{2j-1*} \mid c_k \in \mathbb{R}, j \geq 1\rangle & \textit{for } n = 1. \end{cases}\end{aligned}$$

Here $\langle\cdot\rangle$ *denotes the linear span over* $\mathbb{R}$*,* $*$ *the dual and* s *is a formal symbol of degree* $n-3$*. The product* $\mathfrak{q}^{\mathfrak{n}}_{210}$ *vanishes on* $\mathbb{H}^{\mathfrak{n}}$ *except for the following relations for* $n \geq 3$ *odd*

$$\mathfrak{q}^{\mathfrak{n}}_{210}(\mathrm{s}1^* \otimes \mathrm{s}\mathrm{v}^{k*}) = \mathfrak{q}^{\mathfrak{n}}_{210}(\mathrm{s}\mathrm{v}^{k*} \otimes \mathrm{s}1^*) = -(k-1)\mathrm{v}^{k-1*}$$

and the following relations for $n = 1$*:*

$$\mathfrak{q}^{\mathfrak{n}}_{210}\Big(\mathrm{s}1^* \otimes \mathrm{s}\sum_{k=1}^{\infty} c_k \mathrm{v}^{k*}\Big) = -\mathrm{s}\sum_{k=1}^{\infty} k c_{k+1} \mathrm{v}^{k*}.$$

The coproduct $\mathfrak{q}^{\mathfrak{n}}_{120}$ *as well as all higher operations* $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ *vanish on* $\mathbb{H}^{\mathfrak{n}}$ *in every dimension* n*. For* $\mathbb{S}^1$*, we have* $\mathfrak{q}^{\mathfrak{n}}_{120} \neq \mathfrak{q}_{120}$ *on the chain level; i.e., the twisting is non-trivial. For* $n \neq 2$*, all higher operations vanish on the chain level.*

If we mod out $\mathrm{s}1^{2j-1*}$, i.e., if we consider the point-reduced version, then, after dropping s, the results agree with the string topology of M relative to one constant loop and with Chas-Sullivan operations. The only exception is $M = \mathbb{S}^1$. This supports the string topology conjecture for simply-connected manifolds and provides a counterexample for non-simply connected manifolds.

Our second result generalizes the previous explicit computation and shows that in many cases, the twists with $\mathfrak{n}$ and $\mathfrak{m}$ coincide. Its proof is a combination of facts from Section 4.4.

Theorem B (Triviality of the twist with $\mathfrak{n}$ on the chain level)**.** *Let* M *be a closed oriented* n*-manifold. There exists a Hodge propagator* P*, the so called special Hodge propagator, such that the following holds for the twisted* IBL_∞*-structure* $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$*:*

(1) For the basic operations $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$*,* $\mathfrak{q}^{\mathfrak{n}}_{210} = \mathfrak{q}_{210}$*,* $\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}_{120} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$*, we have:*

  a) If $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$*, then* $\mathfrak{n}_{20} = 0$*, and hence* $\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}_{120}$*.*

  b) If M *is geometrically formal, then* $\mathfrak{n}_{10} = \mathfrak{m}_{10}$*, and hence* $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}^{\mathfrak{m}}_{110}$*. (In fact, if in addition* $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$*, then* $\mathfrak{n} = \mathfrak{m}$*, at least for* $n \neq 2$*.)*

(2) *For the higher operations* $\mathfrak{q}^{\mathfrak{n}}_{1lg} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$ *with* $(l,g) \neq (1,0)$, $(2,0)$, *we have* $\mathfrak{n}_{lg} = 0$, *and hence* $\mathfrak{q}^{\mathfrak{n}}_{1lg} = 0$ *with the possible exception of surfaces and* 3*-manifolds with* $\mathrm{H}^1_{\mathrm{dR}}(M) \neq 0$.

The following are some interesting directions to work on sorted from the most concrete to the most unclear:

(1) Improve Theorem B by showing that the higher operations for $\mathbb{S}^2$ vanish. If this is the case, then the statement that for every manifold M with $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$ there is a propagator such that all higher operations vanish is true.

(2) Let M be a formal and simply-connected manifold. Are $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ and $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ homotopy equivalent as IBL_∞-algebras? If not, we would like to understand the obstruction.

(3) Does the Schwartz kernel $\mathrm{P}_{\mathrm{std}}$ of $\mathcal{P}_{\mathrm{std}} = -\mathrm{d}^*\Delta^{-1}$, the so called standard Hodge propagator, where d^* is the codifferential and Δ the Hodge-de Rham Laplacian, extends smoothly to a blow-up? If yes, then it is a canonical special Hodge propagator for which the statement of Theorem B holds.

(4) Compute $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ for surfaces Σ_g with $g \geq 1$ and formulate and proof a version of string topology conjecture for non-simply connected manifolds. One can consider differential forms with values in a Lie algebra and obtain an IBL_∞-theory with gauge group. How is this structure related to [Gol86] and [AMR96]?

(5) The IBL_∞-theory uses only graphs with at least one external vertex to construct $\mathfrak{n}_{lg}$. Let W_{lg} be the real number obtained by summing over graphs with l boundary components, genus g and with no external vertex. How is this related to the partition function and Chern-Simons invariants and does it fit into a "weak non-reduced IBL_∞-formalism" and possibly satisfy some relations?

(6) What physical field theory has P as a propagator and trivalent ribbon graphs with harmonic forms as Feynman graphs? Due to the polarization $\mathrm{P} = \sum_{k=0}^{n-1} \mathrm{P}^{(k)}(x,y)$, such theory must have a field in every degree (the standard $3d$ Chern-Simons theory before gauge fixing has just one 1-form).

(7) Can the IBL_∞-theory be interpreted from the point of view of string field theory or topological strings? Together with symplectic field theory, what is the global picture?

3. Algebraic structures

In Section 3.1, we recall weight-grading (Definition 3.1.1), Koszul sign (Definition 3.1.2), degree shift (Definition 3.1.3), filtrations (Definition 3.1.8) and completions (Definition 3.1.9). We prove the Künneth formula for completed symmetric (co)homology (Proposition 3.1.13).

In Section 3.2, we review basics of IBL_∞-algebras from [CFL15]. We define the exterior algebra $\mathrm{E}C$ over a graded vector space C as the symmetric algebra S over $C[1]$ (Definition 3.2.1) and use the operations μ and Δ from the structure of an associative bialgebra on $\mathrm{S}(C[1])$ to give explicit formulas for the partial compositions $\circ_{h_1,\ldots,h_k}$ (Definition 3.2.2). We use the compositions to define the notion of an IBL_∞-algebra $(\mathfrak{q}_{klg})$ on C (Definition 3.2.4), a Maurer-Cartan element $(\mathfrak{n}_{lg})$ (Definition 3.2.6) and twisted operations $(\mathfrak{q}^{\mathfrak{n}}_{klg})$ (Definition 3.2.7). We mention that an IBL-algebra according to our definition is an odd degree shift of a classical IBL-algebra (Proposition 3.2.5). We define the induced IBL-structure on homology (Definition 3.2.8) and briefly discuss the BV-formalism and mention weak IBL_∞-algebras (Remark 3.2.9). Finally, we summarize the situation for twisted dIBL-algebras (Proposition 3.2.10) and briefly discuss higher operations (Remark 3.2.11).

In Section 3.3, we define the (weight-reduced) dual cyclic bar-complex $\mathrm{B}^*_{\mathrm{cyc}}V$ of a graded vector space V (Definition 3.3.1) and introduce some notation (Notation 3.3.3). We then summarize some facts about the completions $\hat{\mathrm{B}}^*_{\mathrm{cyc}}V$ and $\hat{\mathrm{E}}_k\mathrm{B}^*_{\mathrm{cyc}}V$ (Proposition 3.3.6). We define the notion of a cyclic A_∞-structure on V (Definition 3.3.7) and its Hochschild and cyclic homology (Definition 3.3.9). We recall strict units and strict augmentations (Definition 3.3.10), define the reduced dual cyclic bar complex $\mathrm{B}^*_{\mathrm{cyc,red}}V$ (Definition 3.3.12) and sketch a proof of the fact that the cyclic homology is a direct sum of the reduced cyclic homology and the cyclic homology of the ground field (Proposition 3.3.13). Our version of cyclic homology of a dga V is based on the degree shift $V[1]$. We undo the degree shift and obtain a version based on V, which can be compared to the standard version from [Lod92] (Proposition 3.3.14). We also show that the reduced spaces for a simply connected and connected V are complete (Proposition 3.3.15).

In Section 3.4, we review the construction of the canonical dIBL-structure $\mathrm{dIBL}(C(V))$

(Definition 3.4.1) and the canonical Maurer-Cartan element $\mathfrak{m}$ (Definition 3.4.2) starting from a cyclic dga (V, Π, m_1, m_2). We give formulas for the operations $(\mathfrak{q}^{\mathfrak{n}}_{1lg})$ of the IBL$_\infty$-algebra dIBL$^{\mathfrak{n}}(C(V))$ twisted by a Maurer-Cartan element $\mathfrak{n}$ (Proposition 3.4.4). We consider the A$_\infty$-structure induced on V by $\mathfrak{n}_{10}$ (Definition 3.4.5) and relate its cyclic homology to the homology of $\mathfrak{q}^{\mathfrak{n}}_{110}$ (Proposition 3.4.6). We define the reduced canonical dIBL-algebra dIBL$(C_{\mathrm{red}}(V))$ (Definition 3.4.7) and the notion of a strictly reduced Maurer-Cartan element (Definition 3.4.8). The twisted IBL$_\infty$-structure then splits into the reduced part and the part generated by $\mathbb{1}^{i*}$, which we can explicitly compute (Proposition 3.4.10).

3.1. Gradings, degree shifts and completions

We will work with vector spaces over $\mathbb{R}$, possibly infinite-dimensional, graded by the *degree* $d \in \mathbb{Z}$ and the *weight* $k \in \mathbb{N}_0$.

Definition 3.1.1 (Weight-graded vector spaces)**.** *A* graded vector space *is a vector space W together with a collection of subspaces $W^d \subset W$ for all $d \in \mathbb{Z}$ such that*

$$W = \bigoplus_{d\in\mathbb{Z}} W^d.$$

Elements of W^d are called homogenous *of degree d; given $w \in W^d$, we denote the degree of w by $|w| := d$.*

A linear map of graded vector spaces $f : W_1 \to W_2$ is called homogenous of degree $|f| \in \mathbb{Z}$ if it holds

$$f(W_1^d) \subset W_2^{d+|f|} \quad \text{for all } d \in \mathbb{Z}. \tag{3.1}$$

A weight-graded vector space *is a graded vector space W together with a collection of subspaces $W^d_k \subset W^d$ for all $k \in \mathbb{N}_0$ and $d \in \mathbb{Z}$ such that*

$$W^d = \bigoplus_{k\in\mathbb{N}_0} W^d_k \quad \text{for all } d \in \mathbb{Z}.$$

We define the weight-k component *of W by*

$$W_k := \bigoplus_{d\in\mathbb{Z}} W^d_k \quad \text{for all } k \in \mathbb{N}_0.$$

If $W^d_0 = 0$ for all $d \in \mathbb{Z}$, we say that W is weight-reduced*. We define the* weight-reduced subspace $\overline{W}$ *of a weight-graded vector space* W *by*

$$\overline{W} \coloneqq \bigoplus_{d\in\mathbb{Z}}\bigoplus_{k\in\mathbb{N}} W_k^d.$$

The superscript * *usually denotes the dual in the category we work in, e.g., chain complexes. For a (weight-)graded vector space* W*, we also introduce the following notation to avoid confusion:*

$$\begin{aligned} W^{*\mathrm{lin}} &\coloneqq \{\psi : W \to \mathbb{R} \textit{ linear}\} \;\ldots\; && \text{linear dual,} \\ W^{*\mathrm{g}} &\coloneqq \bigoplus_{d\in\mathbb{Z}}\prod_{k\in\mathbb{N}_0} W_k^{d*} \;\ldots\; && \text{graded dual,} \\ W^{*\mathrm{wg}} &\coloneqq \bigoplus_{d\in\mathbb{Z}}\bigoplus_{k\in\mathbb{N}_0} W_k^{d*} \;\ldots\; && \text{weight-graded dual.} \end{aligned} \tag{3.2}$$

Here, $W^{*\mathrm{lin}}$ *is a vector space,* $W^{*\mathrm{g}}$ *a graded vector space and* $W^{*\mathrm{wg}}$ *a weight-graded vector space. The grading convention from* (3.2) *is the* cohomological grading convention*, which differs from the convention* (3.1) *for maps* $f : W \to \mathbb{R}$ *by degree reversal (for this, see Definition 3.1.3).*

We identify $W^{*\mathrm{g}}$ *with the subspace of* $W^{*\mathrm{lin}}$ *generated by homogenous maps and* $W^{*\mathrm{wg}}$ *with the subspace of* $W^{*\mathrm{lin}}$ *generated by maps which are non-zero only on finitely many* W_k^d*; hence, under this identification, we have the tower of inclusions*

$$W^{*\mathrm{wg}} \subset W^{*\mathrm{g}} \subset W^{*\mathrm{lin}}.$$

Definition 3.1.2 (Koszul sign)**.** *Let* $k \geq 1$*, and let* $\sigma \in \mathbb{S}_k$ *be a permutation on* k *elements, i.e., a bijection of the set* $\{1, \ldots, k\}$*. For* $i = 1, \ldots, k$*, let* a_i *and* b_i *be graded symbols of degrees* $|a_i|$ *and* $|b_i|$*, respectively. We denote by*

$$\varepsilon(\sigma, a) \quad \textit{and} \quad \varepsilon(a, b)$$

the Koszul signs *of the transformations*

$$a_1 \ldots a_k \longmapsto a_{\sigma_1^{-1}} \ldots a_{\sigma_k^{-1}} \quad \textit{and} \quad a_1 \ldots a_k b_1 \ldots b_k \longmapsto a_1 b_1 \ldots a_k b_k,$$

respectively. Here $\sigma_i^{-1} \coloneqq \sigma^{-1}(i)$*. The Koszul sign is computed by permuting the left-hand side to the right-hand side via transpositions of two adjacent elements such that whenever we transpose two graded symbols, e.g.,* $a_i \longleftrightarrow a_j$*, we multiply with* $(-1)^{|a_i||a_j|}$*.*

We emphasize that the Koszul sign depends only on the initial and the final order of the graded symbols and not on the sequence of transpositions; this is easy to show.

Definition 3.1.3 (Degree shift and grading reversal)**.** *Let $A \in \mathbb{Z}$. The* degree shift by A *is a functor which associates to a graded vector space W the graded vector space $W[A]$ with degree-d components*

$$W[A]^d := W^{d+A} \quad \text{for all } d \in \mathbb{Z}.$$

There is the canonical degree shift morphism

$$W \longrightarrow W[A] \tag{3.3}$$

of degree $-A$ mapping W^d identically to $W[A]^{d-A}$. We view this morphism as multiplication from the left with a formal symbol s_A of degree $|\mathrm{s}_A| = -A$, so that (3.3) *can be written as $w \in W \longmapsto \mathrm{s}_A w \in W[A]$.*

Given graded vector spaces W_1, W_2 and constants A_1, $A_2 \in \mathbb{Z}$, we associate to a morphism $f : W_1 \to W_2$ its degree shift $f : W_1[A_1] \to W_2[A_2]$ *by defining*[1]

$$f(\mathrm{s}_{A_1} w) := \mathrm{s}_{A_2} f(w) \quad \text{for all } w \in W_1. \tag{3.4}$$

Notice that if $f : W_1 \to W_2$ has degree $|f|$, then $f : W_1[A_1] \to W_2[A_2]$ has degree $|f| + A_1 - A_2$.

The grading reversal r *is a functor which associates to a graded vector space W the graded vector space $r(W)$ with*

$$r(W)^d := W^{-d} \quad \text{for all } d \in \mathbb{Z}.$$

There is the canonical morphism $W \to r(W)$ mapping W^d identically to W^{-d} for every $d \in \mathbb{Z}$. The degree reversal of a morphism $f : W_1 \to W_2$ is the morphism $f : r(W_1) \to r(W_2)$ defined by conjugating f with the canonical morphism. If $|f|$ is the degree of $f : W_1 \to W_2$, then $-|f|$ is the degree of $f : r(W_1) \to r(W_2)$.

In our main reference [CFL15], they view W and $W[A]$ as one vector space with two different gradings $\deg(\cdot)$ and $|\cdot|$, respectively; these are related by

$$|w| = \deg(w) - A \quad \text{for all homogenous } w \in W.$$

On the other hand, we think of W and $W[A]$ as of two different graded vector spaces

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[1] This convention is not optimal; see Remark 3.1.4.

and never use the same symbol for an element $w \in W$ and its degree shift $\mathrm{s}_A w \in W[A]$. It allows us to use just one notation $|\cdot|$ for the gradings on both W and $W[A]$. However, in order to preserve compatibility with [CFL15], we will also sometimes use the notation $\deg(w)$ (in the exponent just $(-1)^w$) for the degrees on W.

For graded vector spaces $W_1, \ldots, W_k$ and constants $A_1, \ldots, A_k \in \mathbb{Z}$, we identify

$$W_1[A_1] \otimes \cdots \otimes W_k[A_k] \simeq (W_1 \otimes \cdots \otimes W_k)[A_1 + \cdots + A_k]$$

using the *Koszul convention for the tensor product*; for homogenous elements $w_1 \in W_1, \ldots, w_k \in W_k$, it reads

$$\mathrm{s}_{A_1} w_1 \otimes \cdots \otimes \mathrm{s}_{A_k} w_k = \varepsilon(\mathrm{s}_A, w) \underbrace{\mathrm{s}_{A_1} \ldots \mathrm{s}_{A_k}}_{=:\, \mathrm{s}_{A_1 + \cdots + A_k}} w_1 \otimes \cdots \otimes w_k. \tag{3.5}$$

If $A_1 = \cdots = A_k =: A$ is fixed in the context, which is our usual case, we omit the subscript A and write just s.

In the case of the multilinear map $f : W_1 \otimes \cdots \otimes W_k \to V_1 \otimes \cdots \otimes V_l$, the combination of (3.4) and (3.5) gives for $f : W_1[A_1] \otimes \cdots \otimes W_k[A_k] \to V_1[B_1] \otimes \cdots \otimes V_l[B_l]$ the following:

$$f(\mathrm{s}_{A_1 + \cdots + A_k} w_1 \otimes \cdots \otimes w_k) = \mathrm{s}_{B_1 + \cdots + B_l} f(w_1 \otimes \cdots \otimes w_k). \tag{3.6}$$

Remark 3.1.4 (Why is this sign convention bad?). Let us illustrate that (3.6) is not compatible with the following standard Koszul rule:

$$(K): \qquad (f_1 \otimes f_2)(w_1 \otimes w_2) = (-1)^{|f_2||w_1|} f_1(w_1) \otimes f_2(w_2).$$

On one hand, we get

$$\begin{aligned} (f_1 \otimes f_2)(\mathrm{s}^2 w_1 \otimes w_2) &\overset{(3.6)}{=} \mathrm{s}^2 (f_1 \otimes f_2)(w_1 \otimes w_2) \\ &\overset{(K)}{=} (-1)^{|f_2||w_1|} \mathrm{s}^2 f_1(w_1) \otimes f_2(w_2) \\ &\overset{(3.5)}{=} (-1)^{|f_2||w_1| + A(|f_1| + |w_1|)} \mathrm{s} f_1(w_1) \otimes \mathrm{s} f_2(w_2). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} (f_1 \otimes f_2)(\mathrm{s}^2 w_1 \otimes w_2) &\overset{(3.5)}{=} (-1)^{A|w_1|} (f_1 \otimes f_2)(\mathrm{s} w_1 \otimes \mathrm{s} w_2) \\ &\overset{(K)}{=} (-1)^{A|w_1| + |f_2|(A + |w_1|)} f_1(\mathrm{s} w_1) \otimes f_2(\mathrm{s} w_2) \\ &\overset{(3.6)}{=} (-1)^{A|w_1| + |f_2|(A + |w_1|)} \mathrm{s} f_1(w_1) \otimes \mathrm{s} f_2(w_2). \end{aligned}$$

The results differ by $(-1)^{A(|f_1|+|f_2|)}$. Therefore, we can not use (K) to identify the tensor product $\mathrm{Hom}(W_1,V_1)\otimes\mathrm{Hom}(W_2,V_2)$ with a subspace of $\mathrm{Hom}(W_1\otimes W_2,V_1\otimes V_2)$ in general. We will rather define an ad-hoc pairing in the case where we need it (see Definition 3.3.4).

Another caveat is that in the case of the tensor product, the degree shift by A_1 followed by the degree shift by A_2 is not the same as the degree shift by A_1+A_2. Indeed, we compute

$$\begin{aligned}(\mathrm{s}_{A_1+A_2}w_1)\otimes(\mathrm{s}_{A_1+A_2}w_2) &= (\mathrm{s}_{A_2}\mathrm{s}_{A_1}w_1)\otimes(\mathrm{s}_{A_2}\mathrm{s}_{A_1}w_2)\\ &= (-1)^{A_2(A_1+|w_1|)}\mathrm{s}_{A_2}^2(\mathrm{s}_{A_1}w_1)\otimes(\mathrm{s}_{A_1}w_2)\\ &= (-1)^{A_2A_1+(A_1+A_2)|w_1|}\mathrm{s}_{A_2}^2\mathrm{s}_{A_1}^2(w_1\otimes w_2)\\ &= (-1)^{A_2A_1+(A_1+A_2)|w_1|}\mathrm{s}_{2(A_1+A_2)}(w_1\otimes w_2),\end{aligned}$$

which differs by $(-1)^{A_1A_2}$ from the direct degree shift by A_1+A_2. Therefore, we have to always remember the vector spaces which we started with and the sequence of degree shifts.

Note that we also have the unnatural "$\mathrm{s}_{A_1}\mathrm{s}_{A_2}=\mathrm{s}_{A_1+A_2}=\mathrm{s}_{A_2}\mathrm{s}_{A_1}$" due to (3.5). $\triangleleft$

Remark 3.1.5 (Is there a better sign convention?). The author originally respected the Koszul rule for the algebra with formal symbols and considered the following map $\mathrm{s}_*^l\bar{\mathrm{s}}^{k*}f : W[A]^{\otimes k}\to V[A]^{\otimes l}$ as the degree shift of $f: W^{\otimes k}\to V^{\otimes l}$:

$$(\mathrm{s}_*^l\bar{\mathrm{s}}^{k*}f)(\mathrm{s}^k w_1\otimes\cdots\otimes w_k)=(-1)^{k|f|A+\frac{1}{2}k(k-1)A}\mathrm{s}^l f(w_1\otimes\cdots\otimes w_k). \tag{3.7}$$

Here $\bar{\mathrm{s}}$ denotes the "inverse" of s with $|\bar{\mathrm{s}}|=-|\mathrm{s}|$, $\mathrm{s}_*^l f=\mathrm{s}^l\circ f$ is the post-composition, $\bar{\mathrm{s}}^{k*}f=(-1)^{kA|f|}f\circ\bar{\mathrm{s}}^k$ the pre-composition, and the sign $\varepsilon(\mathrm{s},\bar{\mathrm{s}})=(-1)^{\frac{1}{2}k(k-1)A}$ comes from the "collision" $\bar{\mathrm{s}}_1\ldots\bar{\mathrm{s}}_k\mathrm{s}_1\ldots\mathrm{s}_k\mapsto\bar{\mathrm{s}}_1\mathrm{s}_1\ldots\bar{\mathrm{s}}_k\mathrm{s}_k$.

However, the author did not manage to reprove the theory in [CFL15] using (3.7) (because of too many "external" signs appearing and a problem with disconnected graphs). A motivation to try a different sign convention was to explain some artificial signs in [CFL15] and formulate their coordinate constructions invariantly in order to generalize them to the "continuous" de Rham case.

It might be possible to deduce a "universal" sign convention "respecting" the Koszul rules by considering the category of chain complexes and graded morphisms $\mathcal{C}$ as the category enriched in the closed monoidal category of chain complexes and chain maps of degree 0. One can then define the enriched degree shift functor $\mathrm{s}_A:\mathcal{C}\to\mathcal{C}$, embed $\mathcal{C}^{\otimes k}\subset\mathcal{C}$ using (K) and study enriched natural transformations in the algebra of functors consisting of tensor products and compositions of s_A, $\mathrm{Hom}(\cdot,\cdot)$ and the dual $*$. The question is whether the sign rules for degree shifts, in particular (3.7), are uniquely determined by the requirements on enriched functors and enriched natural transformations.

Another idea for how to specify the degree shift convention (3.7) is the observation that maps $f : W^{\otimes k} \to W^{\otimes l}$ under consideration usually form an algebra $\mathcal{P} \to \mathrm{End}_W$ of a certain PROP $\mathcal{P}$ over W; here End_W denotes the endomorphism PROP. Suppose that there is a natural notion of the degree shift $\mathcal{P}[A]$ of a PROP $\mathcal{P}$ and a canonical degree shift morphism $\mathcal{P} \to \mathcal{P}[A]$. The PROPs $(\mathrm{End}_W)[A]$ and $\mathrm{End}_{W[A]}$ are better to be isomorphic, and any intelligent degree shift convention $\psi_{k,l} : \mathrm{Hom}(W^{\otimes k}, W^{\otimes l})[A(k-l)] \to \mathrm{Hom}(W[A]^{\otimes k}, W[A]^{\otimes l})$ should induce an isomorphism of the PROPs $(\mathrm{End}_W)[A]$ and $\mathrm{End}_{W[A]}$. $\lhd$

Definition 3.1.6 (Standard action of permutations)**.** *For $k \geq 1$ and $\sigma \in \mathbb{S}_k$ (:= the group of permutations on k elements), we define the* standard action of σ *on $W^{\otimes k}$ by*

$$\sigma(w_1 \otimes \cdots \otimes w_k) \coloneqq \varepsilon(\sigma, w) w_{\sigma_1^{-1}} \otimes \cdots \otimes w_{\sigma_k^{-1}} \tag{3.8}$$

for all homogenous $w_1, \ldots, w_k \in W$.

Notice that the i-th vector is permuted to the σ_i-th place — this is the "active" convention for permutations.

Definition 3.1.7 (Symmetric algebra)**.** *Let $\mathrm{T}(V) \coloneqq \bigoplus_{k\geq 0} V^{\otimes k}$ be the tensor algebra over a graded vector space V. The* symmetric algebra *over V is defined by $\mathrm{S}(V) \coloneqq \bigoplus_{k\geq 0} \mathrm{S}_k(V)$, where*

$$\mathrm{S}_k(V) \coloneqq V^{\otimes k} / \sum_{\sigma\in\mathbb{S}_k} \mathrm{im}(\mathbb{1} - \sigma) \quad (=: \mathbb{S}_k\text{-}\textit{coinvariants}).$$

It is a weight-graded vector space with components denoted by $(\mathrm{S}_k V)^d$ for all $d \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. Note that $\mathrm{S}_0 V = \mathbb{R}$ has degree 0 by definition. Consider the canonical projection

$$\begin{aligned} \pi : \mathrm{T}(V) &\longrightarrow \mathrm{S}(V) \\ v_1 \otimes \cdots \otimes v_k &\longmapsto v_1 \cdots v_k. \end{aligned}$$

If $v_i \in V$ are homogenous, we call $v_1 \cdots v_k$ a generating word*; we have*

$$v_1 \cdots v_k = \varepsilon(\sigma, v) v_{\sigma_1^{-1}} \cdots v_{\sigma_k^{-1}} \quad \textit{for every } \sigma \in \mathbb{S}_k.$$

Let $\iota : \mathrm{S}(V) \to \mathrm{T}(V)$ *be the section of* π *defined by*

$$\iota(v_1 \cdots v_k) := \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \varepsilon(\sigma, v) v_{\sigma_1^{-1}} \otimes \ldots \otimes v_{\sigma_k^{-1}}.$$

We use it to identify $\mathrm{S}(V)$ *with the subspace of symmetric tensors*

$$\iota(\mathrm{S}_k(V)) = \bigcap_{\sigma \in \mathbb{S}_k} \ker(\mathbb{1} - \sigma) \subset \mathrm{T}_k(V) \quad (=: \mathbb{S}_k\text{-}invariants).$$

Definition 3.1.8 (Filtrations)**.** *Let* W *be a graded vector space. A filtration of* W *is a collection of linear subspaces* $\mathcal{F}^\lambda W \subset W$ *for* $\lambda \in \mathbb{R}$ *such that we have either*

- $\mathcal{F}^{\lambda_1} W \subset \mathcal{F}^{\lambda_2} W$ *for all* $\lambda_1 \leq \lambda_2 \quad :\Longleftrightarrow:$ increasing filtration, *or*
- $\mathcal{F}^{\lambda_1} W \supset \mathcal{F}^{\lambda_2} W$ *for all* $\lambda_1 \leq \lambda_2 \quad :\Longleftrightarrow:$ decreasing filtration.

We will assume that our filtrations are graded *in the following sense:*

$$\forall \lambda \in \mathbb{R} : \quad \mathcal{F}^\lambda W = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^\lambda W^d, \quad where \quad \mathcal{F}^\lambda W^d := \mathcal{F}^\lambda W \cap W^d.$$

A filtration $\mathcal{F}^\lambda W$ *is called:*

$$\begin{aligned}
\text{exhaustive} \quad &:\Longleftrightarrow \quad \bigcup_{\lambda \in \mathbb{R}} \mathcal{F}^\lambda W = W; \\
\text{Hausdorff} \quad &:\Longleftrightarrow \quad \bigcap_{\lambda \in \mathbb{R}} \mathcal{F}^\lambda W = 0; \\
\mathbb{Z}\text{-gapped} \quad &:\Longleftrightarrow \quad \mathcal{F}^\lambda W = \mathcal{F}^{\lfloor \lambda \rfloor} W \textit{ for all } \lambda \in \mathbb{R}; \\
\text{bounded from below} \quad &:\Longleftrightarrow \quad \exists \lambda \in \mathbb{R} : \mathcal{F}^\lambda W = 0; \\
\text{bounded from above} \quad &:\Longleftrightarrow \quad \exists \lambda \in \mathbb{R} : \mathcal{F}^\lambda W = W.
\end{aligned}$$

Given a graded vector space W *filtered by a* $\mathbb{Z}$*-gapped filtration* $\mathcal{F}^\lambda W$*, we associate to it the bi-graded vector space*

$$\mathrm{gr}(W) = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{\lambda \in \mathbb{Z}} \mathrm{gr}(W)^d_\lambda$$

called the graded module *whose components are given as follows:*[2]

$$\forall d,\lambda\in\mathbb{Z}:\quad \mathrm{gr}(W)^d_\lambda := \begin{cases}\mathcal{F}^\lambda W^d/\mathcal{F}^{\lambda-1}W^d & \text{for increasing } \mathcal{F}^\lambda W,\\ \mathcal{F}^\lambda W^d/\mathcal{F}^{\lambda+1}W^d & \text{for decreasing } \mathcal{F}^\lambda W.\end{cases}$$

We naturally extend a filtration over degree shifts, graded duals, direct sums, tensor products and symmetric products as follows:

$$\begin{aligned}\mathcal{F}^\lambda W[A]^d &:= \mathcal{F}^\lambda W^{d+A},\\ \mathcal{F}^\lambda(W^{*_{\mathrm{g}}})^d &:= \{\psi\in W^{d*}\mid \psi|_{\mathcal{F}^\lambda W}=0\},\\ \mathcal{F}^\lambda\Big(\bigoplus_{i\in I}W_i\Big)^d &:= \bigoplus_{i\in I}\mathcal{F}^\lambda W_i^d,\\ \mathcal{F}^\lambda(W_1\otimes\cdots\otimes W_k)^d &:= \bigoplus_{\substack{d_1,\dots,d_k\in\mathbb{Z}\\ d_1+\dots+d_k=d}}\ \sum_{\substack{\lambda_1,\dots,\lambda_k\in\mathbb{R}\\ \lambda_1+\dots+\lambda_k=\lambda}}\mathcal{F}^{\lambda_1}W_1^{d_1}\otimes\cdots\otimes\mathcal{F}^{\lambda_k}W_k^{d_k},\\ \mathcal{F}^\lambda(\mathrm{S}_kV)^d &:= \pi(\mathcal{F}^\lambda(V^{\otimes k})^d),\end{aligned}$$

where $\pi:\mathrm{T}(V)\to\mathrm{S}(V)$ *is the canonical projection. The ground field* $\mathbb{R}$ *is filtered by the trivial filtration:*

$$\mathcal{F}^\lambda\mathbb{R} := \begin{cases}\mathbb{R} & \lambda\le 0,\\ 0 & \lambda>0.\end{cases} \tag{3.9}$$

If (W,∂) *is a filtered chain complex, we filter the homology as follows:*

$$\forall\lambda\in\mathbb{R}, d\in\mathbb{Z}:\quad \mathcal{F}^\lambda\mathrm{H}_d(W,\partial) := \{\alpha\in\mathrm{H}_d(C,\partial)\mid \exists w\in\alpha: w\in\mathcal{F}^\lambda W^d\}.$$

Definition 3.1.9 (Completions)**.** *Let* W *be a graded vector space filtered by a decreasing filtration* $\mathcal{F}^\lambda W$*. The* filtration degree *of* $w\in W$ *is defined by*

$$\|w\| := \sup\{\lambda\in\mathbb{R}\mid w\in\mathcal{F}^\lambda W\}.$$

The filtration degree of a linear map $f:W_1\to W_2$ *is defined by*

$$\|f\| := \sup\{\lambda\in\mathbb{R}\mid \|f(w)\|\ge\|w\|+\lambda\ \forall w\in W_1\}.$$

---

[2] The definition is made in such a way that if r is the functor which reverses λ, i.e., $r(\mathcal{F})^\lambda=\mathcal{F}^{-\lambda}$, then it holds $r(\mathrm{gr}(\mathcal{F}))=\mathrm{gr}(r(\mathcal{F}))$.

We say that the filtration degree is finite *if* $\|f\| > -\infty$. *Note that* $\|0\| = \infty$.

The completion *of* W *is the graded vector space*

$$\hat{W} := \bigoplus_{d\in\mathbb{Z}} \hat{W}^d,$$

where for all $d \in \mathbb{Z}$ *we define*

$$\hat{W}^d := \Bigl\{\sum_{i=0}^{\infty} w_i \Bigm| \forall i \in \mathbb{N}_0 : w_i \in W^d; \|w_i\| \to \infty \text{ as } i \to \infty\Bigr\}\Big/\sim .$$

Here $\sum_{i=0}^{\infty} w_i \sim \sum_{i=0}^{\infty} w'_i$ *if and only if* $\|\sum_{i=0}^{n}(w_i - w'_i)\| \to \infty$ *as* $n \to \infty$.[3] *The completion* $\hat{W}$ *is canonically filtered by the filtration* $\mathcal{F}^\lambda\hat{W}$ *defined as follows:*

$$\forall\lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda\hat{W}^d := \Bigl\{\sum_{i=0}^{\infty} w_i \in \hat{W}^d \Bigm| \forall i \in \mathbb{N}_0 : w_i \in \mathcal{F}^\lambda W^d\Bigr\}.$$

We denote the completion of $W_1 \otimes \cdots \otimes W_k$ *by* $W_1 \hat{\otimes} \cdots \hat{\otimes} W_k$ *and the completion of* $\mathrm{S}_k V$ *by* $\hat{\mathrm{S}}_k V$.

A map $f : W_1 \to W_2$ *of finite filtration degree* extends continuously *to a linear map* $f : \hat{W}_1 \to \hat{W}_2$*; this continuous extension is defined by*

$$f\Bigl(\sum_{i=0}^{\infty} w_i\Bigr) := \sum_{i=0}^{\infty} f(w_i) \quad \text{for all } \sum_{i=0}^{\infty} w_i \in \hat{W}.$$

Remark 3.1.10 (Completed tensor product). Using Proposition 3.1.11 below, one can show that the *completed tensor product* $\hat{\otimes}$ is associative and that $W_1 \hat{\otimes} W_2 \simeq \hat{W}_1 \hat{\otimes} \hat{W}_2$. By refining this argument, one can show that $\hat{\mathrm{S}}_k V \simeq \hat{\mathrm{S}}_k \hat{V}$ for any $k \in \mathbb{N}$. ◁

A weight-graded vector space W is canonically *filtered by weights:*

$$\forall\lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda W^d := \bigoplus_{k\leq\lambda} W^d_k. \tag{3.10}$$

This filtration is $\mathbb{Z}$-gapped, exhaustive, Hausdorff, increasing and bounded from below. The induced filtration on the graded dual $W^{*_{\mathrm{g}}}$ is $\mathbb{Z}$-gapped, Hausdorff, decreasing and bounded from above (and thus automatically exhaustive). It holds $\mathrm{gr}(W) \simeq W$, and it is

---

[3] In fact, $\hat{W}$ is the inverse limit $\varprojlim^{\mathrm{gr}}_\lambda(W/\mathcal{F}^\lambda W)$ in the category of graded vector spaces and $\hat{W}^d$ the inverse limit $\varprojlim_\lambda(W^d/\mathcal{F}^\lambda W^d)$ in the category of vector spaces. As a side-remark, if we forget the grading on W, we might also consider $\varprojlim_\lambda(W/\mathcal{F}^\lambda W)$, which would be a vector space containing $\hat{W}$ as a subspace.

easy to see from (3.2) that the canonical map $W^{*_{\mathrm{wg}}} \to W^{*_{\mathrm{g}}}$ induces the isomorphism

$$\widehat{W^{*_{\mathrm{wg}}}} \simeq W^{*_{\mathrm{g}}}.$$

We also see that the condition

$$(WG0): \quad \forall d \in \mathbb{Z}\ \exists J \subset \mathbb{N}_0, |J| < \infty\ \forall k \in \mathbb{N}_0 \backslash J: \quad W_k^d = 0$$

is equivalent to $W^{*_{\mathrm{wg}}} = W^{*_{\mathrm{g}}}$.

A useful tool to compare completions is the following proposition:

Proposition 3.1.11 ([Fre17, Proposition 7.3.7], Isomorphism criterion)**.** *Let W_1 and W_2 be graded vector spaces filtered by $\mathbb{Z}$-gapped filtrations which are decreasing and bounded from above. Suppose that $f: W_1 \to W_2$ is a filtration preserving homogenous linear map. Then the continuous extension $f: \hat{W}_1 \to \hat{W}_2$ is an isomorphism if and only if the induced map $f: \mathrm{gr}(W_1) \to \mathrm{gr}(W_2)$ is an isomorphism.*

Proof. The implication from the right to the left is obtained from the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}(W_1)_\lambda \arrow[r, hook] \arrow[d, "f"] & W_1/\mathcal{F}^{\lambda+1}W_1 \arrow[r, two heads] \arrow[d, "f"] & W_1/\mathcal{F}^{\lambda}W_1 \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathrm{gr}(W_2)_\lambda \arrow[r, hook] & W_2/\mathcal{F}^{\lambda+1}W_2 \arrow[r, two heads] & W_2/\mathcal{F}^{\lambda}W_2 \arrow[r] & 0
\end{tikzcd}$$

by induction using the definition of $\hat{W}$ as the inverse limit of $W/\mathcal{F}^\lambda W$ (see Footnote 3 on page 32). The other implication follows from $\mathcal{F}^\lambda \hat{W}/\mathcal{F}^{\lambda-1}\hat{W} \simeq \mathcal{F}^\lambda W/\mathcal{F}^{\lambda-1}W$. ☐

For a graded vector space W filtered by a $\mathbb{Z}$-gapped filtration, consider the following conditions:

$$\begin{aligned}
&(WG1): \quad \forall \lambda \in \mathbb{Z}\ \exists I \subset \mathbb{Z}, |I| < \infty\ \forall d \in \mathbb{Z}\backslash I: && \mathrm{gr}(W)_\lambda^d = 0, \\
&(WG2): \quad \forall d, \lambda \in \mathbb{Z}: && \dim(\mathrm{gr}(W)_\lambda^d) < \infty.
\end{aligned} \tag{3.11}$$

Lemma 3.1.12 (Completion of symmetric powers of the graded dual)**.** *Let W be a graded vector space filtered by an exhaustive $\mathbb{Z}$-gapped filtration $\mathcal{F}^\lambda W$ which is increasing and bounded from below. If (WG1) & (WG2) are satisfied, then the natural map $\mathrm{S}_k(W^{*_{\mathrm{g}}}) \to (\mathrm{S}_k W)^{*_{\mathrm{g}}}$ induces the isomorphism*

$$\hat{\mathrm{S}}_k(W^{*_{\mathrm{g}}}) \simeq (\mathrm{S}_k W)^{*_{\mathrm{g}}} \quad \textit{for every } k \in \mathbb{N}.$$

Note that we filter graded duals by the induced filtration from Definition 3.1.8.

Proof. The natural map $\mathrm{S}_k(W^{*_{\mathrm{g}}}) \to (\mathrm{S}_k W)^{*_{\mathrm{g}}}$ is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space $(\mathrm{S}_k W)^{*_{\mathrm{g}}}$ is already complete (the dual space $W^{*_{\mathrm{g}}}$ is complete, provided that the filtration of W is exhaustive), and thus we obtain the map $\hat{\mathrm{S}}_k(W^{*_{\mathrm{g}}}) \to (\mathrm{S}_k W)^{*_{\mathrm{g}}}$. According to Proposition 3.1.11, this map is an isomorphism if and only if the induced map $\mathrm{gr}(\mathrm{S}_k(W^{*_{\mathrm{g}}})) \to \mathrm{gr}((\mathrm{S}_k W)^{*_{\mathrm{g}}})$ is. This is shown by the following computation (the maps involved are natural in at least one direction):

$$\begin{aligned}
& \frac{\mathcal{F}^{\lambda}(W^{\otimes k *_{\mathrm{g}}})^d}{\mathcal{F}^{\lambda+1}(W^{\otimes k *_{\mathrm{g}}})^d} \simeq \frac{\mathcal{F}^{\lambda}(W^{\otimes k})^{d*}}{\mathcal{F}^{\lambda+1}(W^{\otimes k})^{d*}} \simeq \Big(\frac{\mathcal{F}^{\lambda+1}(W^{\otimes k})^d}{\mathcal{F}^{\lambda}(W^{\otimes k})^d}\Big)^* \\
& \simeq \left(\frac{\bigoplus_{|\vec{d}|=d}\sum_{|\vec{\lambda}|=\lambda+1}\mathcal{F}^{\lambda_1}W^{d_1}\otimes\cdots\otimes\mathcal{F}^{\lambda_k}W^{d_k}}{\bigoplus_{|\vec{d}|=d}\sum_{|\vec{\lambda}|=\lambda}\mathcal{F}^{\lambda_1}W^{d_1}\otimes\cdots\otimes\mathcal{F}^{\lambda_k}W^{d_k}}\right)^* \\
& \simeq \left(\bigoplus_{|\vec{d}|=d}\frac{\sum_{|\vec{\lambda}|=\lambda+1}\mathcal{F}^{\lambda_1}W^{d_1}\otimes\cdots\otimes\mathcal{F}^{\lambda_k}W^{d_k}}{\sum_{|\vec{\lambda}|=\lambda}\mathcal{F}^{\lambda_1}W^{d_1}\otimes\cdots\otimes\mathcal{F}^{\lambda_k}W^{d_k}}\right)^* \\
& \simeq \Big(\bigoplus_{|\vec{d}|=d}\bigoplus_{|\vec{\lambda}|=\lambda}\frac{\mathcal{F}^{\lambda_1+1}W^{d_1}}{\mathcal{F}^{\lambda_1}W^{d_1}}\otimes\cdots\otimes\frac{\mathcal{F}^{\lambda_k+1}W^{d_k}}{\mathcal{F}^{\lambda_k}W^{d_k}}\Big)^* \\
& \simeq \Big(\bigoplus_{|\vec{\lambda}|=\lambda}\bigoplus_{|\vec{d}|=d}\frac{\mathcal{F}^{\lambda_1+1}W^{d_1}}{\mathcal{F}^{\lambda_1}W^{d_1}}\otimes\cdots\otimes\frac{\mathcal{F}^{\lambda_k+1}W^{d_k}}{\mathcal{F}^{\lambda_k}W^{d_k}}\Big)^* \\
\begin{matrix}\mathbb{Z}\text{−gapped}\\ \text{\& bounded below} \to\\ \text{\& } (WG1)\end{matrix}\quad & \simeq \bigoplus_{|\vec{\lambda}|=\lambda}\bigoplus_{|\vec{d}|=d}\Big(\frac{\mathcal{F}^{\lambda_1+1}W^{d_1}}{\mathcal{F}^{\lambda_1}W^{d_1}}\otimes\cdots\otimes\frac{\mathcal{F}^{\lambda_k+1}W^{d_k}}{\mathcal{F}^{\lambda_k}W^{d_k}}\Big)^* \\
(WG2) \to\quad & \simeq \bigoplus_{|\vec{\lambda}|=\lambda}\bigoplus_{|\vec{d}|=d}\Big(\frac{\mathcal{F}^{\lambda_1+1}W^{d_1}}{\mathcal{F}^{\lambda_1}W^{d_1}}\Big)^*\otimes\cdots\otimes\Big(\frac{\mathcal{F}^{\lambda_k+1}W^{d_k}}{\mathcal{F}^{\lambda_k}W^{d_k}}\Big)^* \\
& \simeq \bigoplus_{|\vec{d}|=d}\bigoplus_{|\vec{\lambda}|=\lambda}\frac{\mathcal{F}^{\lambda_1}(W^{*_{\mathrm{g}}})^{d_1}}{\mathcal{F}^{\lambda_1+1}(W^{*_{\mathrm{g}}})^{d_1}}\otimes\cdots\otimes\frac{\mathcal{F}^{\lambda_k}(W^{*_{\mathrm{g}}})^{d_k}}{\mathcal{F}^{\lambda_k+1}(W^{*_{\mathrm{g}}})^{d_k}} \\
& \simeq \frac{\mathcal{F}^{\lambda}(W^{*_{\mathrm{g}}\otimes k})^d}{\mathcal{F}^{\lambda+1}(W^{*_{\mathrm{g}}\otimes k})^d}.
\end{aligned}$$

In fact, this computation shows that $\hat{\mathrm{T}}_k(W^{*_{\mathrm{g}}}) \simeq (\mathrm{T}_k W)^{*_{\mathrm{g}}}$. The conclusion for S_k follows by checking that the maps above are $\mathbb{S}_k$-equivariant. □

Given a chain complex (W, ∂), the boundary operator ∂ induces the boundary operator $\partial_k : W^{\otimes k} \to W^{\otimes k}$ for all $k \in \mathbb{N}$; for all $w_1, \ldots, w_k \in W$, it is defined by

$$\partial_k(w_1 \otimes \cdots \otimes w_k) \coloneqq \sum_{i=1}^{k}(-1)^{|w_1|+\cdots+|w_{i-1}|} w_1 \otimes \cdots \otimes \partial w_i \otimes \cdots \otimes w_k. \tag{3.12}$$

The map ∂_k is clearly $\mathbb{S}_k$-equivariant, and thus induces the boundary operator $\partial_k : \mathrm{S}_k W \to \mathrm{S}_k W$.

Proposition 3.1.13 (Künneth formula for completed symmetric cohomology)**.** *Let (W, ∂) be a $\mathbb{Z}$-graded chain complex over $\mathbb{R}$ filtered by an exhaustive $\mathbb{Z}$-gapped filtration $\mathcal{F}^\lambda W$ which is increasing and bounded from below. Consider the dual cochain complex $(W^{*_\mathrm{g}}, \mathrm{d} := \partial^*)$. Suppose that d has finite filtration degree, so that $\mathrm{d}_k : \mathrm{S}_k(W^{*_\mathrm{g}}) \to \mathrm{S}_k(W^{*_\mathrm{g}})$ extends continuously to $\mathrm{d}_k : \hat{\mathrm{S}}_k(W^{*_\mathrm{g}}) \to \hat{\mathrm{S}}_k(W^{*_\mathrm{g}})$ for every $k \in \mathbb{N}$. If (WG1) & (WG2) are satisfied, then the natural map $\mathrm{S}_k\mathrm{H}(W^{*_\mathrm{g}}, \mathrm{d}) \to \mathrm{H}(\hat{\mathrm{S}}_k(W^{*_\mathrm{g}}), \mathrm{d}_k)$ induces the isomorphism*

$$\hat{\mathrm{S}}_k\mathrm{H}(W^{*_\mathrm{g}}, \mathrm{d}) \simeq \mathrm{H}(\hat{\mathrm{S}}_k(W^{*_\mathrm{g}}), \mathrm{d}_k) \quad \textit{for all } k \in \mathbb{N}.$$

Proof. The natural map $\mathrm{S}_k\mathrm{H}(W^{*_\mathrm{g}}, \mathrm{d}) \to \mathrm{H}(\hat{\mathrm{S}}_k W^{*_\mathrm{g}}, \mathrm{d}_k)$ is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space $\mathrm{H}(\hat{\mathrm{S}}_k W^{*_\mathrm{g}}, \mathrm{d}_k)$ is already complete (the homology of a complete space is complete), and hence we obtain the map $\hat{\mathrm{S}}_k\mathrm{H}(W^{*_\mathrm{g}}, \mathrm{d}) \to \mathrm{H}(\hat{\mathrm{S}}_k W^{*_\mathrm{g}}, \mathrm{d}_k)$. The following facts are easy to verify:

(1) The isomorphism from Lemma 3.1.12 is an isomorphism of cochain complexes

$$(\hat{\mathrm{S}}_k W^{*_\mathrm{g}}, \mathrm{d}_k) \simeq ((\mathrm{S}_k W)^{*_\mathrm{g}}, \partial_k^*).$$

(2) If the filtration on W satisfies (WG1) and (WG2), then the filtration on $\mathrm{H}(W)$ also satisfies (WG1) and (WG2), respectively. Consequently, Lemma 3.1.12 holds for symmetric powers of $\mathrm{H}(W, \partial)^{*_\mathrm{g}}$ as well.

(3) The Künneth formula $\mathrm{H}(W^{\otimes k}) \simeq \mathrm{H}(W)^{\otimes k}$ implies $\mathrm{H}(\mathrm{S}_k W) \simeq \mathrm{S}_k\mathrm{H}(W)$ for any $\mathbb{Z}$-graded chain complex W over $\mathbb{R}$.

(4) We have $(\mathrm{H}(W))^{*_\mathrm{g}} \simeq \mathrm{H}(W^{*_\mathrm{g}})$ over $\mathbb{R}$ by the universal coefficient theorem.

Now, we compute

$$\begin{aligned}\mathrm{H}(\hat{\mathrm{S}}_k W^{*_\mathrm{g}}, \mathrm{d}_k) &\underset{\substack{\uparrow\\(1)}}{\simeq} \mathrm{H}((\mathrm{S}_k W)^{*_\mathrm{g}}, \partial_k^*) \underset{\substack{\uparrow\\(4)}}{\simeq} \mathrm{H}(\mathrm{S}_k W, \partial_k)^{*_\mathrm{g}} \underset{\substack{\uparrow\\(3)}}{\simeq} (\mathrm{S}_k\mathrm{H}(W, \partial))^{*_\mathrm{g}}\\ &\underset{\substack{\uparrow\\(2)}}{\simeq} \hat{\mathrm{S}}_k(\mathrm{H}(W, \partial)^{*_\mathrm{g}}) \underset{\substack{\uparrow\\(4)}}{\simeq} \hat{\mathrm{S}}_k\mathrm{H}(W^{*_\mathrm{g}}, \mathrm{d}).\end{aligned}$$

This proves the proposition. □

3.2. Basics of IBL-infinity-algebras

Definition 3.2.1 (Exterior algebra)**.** *Given a graded vector space C over $\mathbb{R}$, we define the* exterior algebra *over C by*

$$\mathrm{E}C \coloneqq \mathrm{S}(C[1]).$$

The weight-k component is denoted by E_kC and the weight-reduced part by $\bar{\mathrm{E}}C$. If C is in addition filtered, then E_kC is filtered by the induced filtration and its completion is denoted by $\hat{\mathrm{E}}_kC$.

We have the concatenation product $\mu : \mathrm{E}C \otimes \mathrm{E}C \to \mathrm{E}C$ and the shuffle coproduct $\delta : \mathrm{E}C \to \mathrm{E}C \otimes \mathrm{E}C$ defined by

$$\begin{aligned} &\mu(c_{11} \dots c_{1k} \otimes c_{21} \dots c_{2k'}) \coloneqq c_{11} \dots c_{1k} c_{21} \dots c_{2k'} \quad \text{and} \\ &\delta(c_1 \dots c_k) \coloneqq \sum_{\substack{k_1,\, k_2 \geq 0 \\ k_1+k_2=k}} \sum_{\sigma \in \mathbb{S}_{k_1,k_2}} \varepsilon(\sigma, c) c_{\sigma_1^{-1}} \dots c_{\sigma_{k_1}^{-1}} \otimes c_{\sigma_{k_1+1}^{-1}} \dots c_{\sigma_{k_1+k_2}^{-1}} \end{aligned}$$

for all homogenous c_{ij}, $c_i \in C[1]$ and k, $k' \geq 0$, respectively, where $\mathbb{S}_{k_1,k_2} \subset \mathbb{S}_{k_1+k_2}$ denotes the set of shuffle permutations with blocks of lengths k_1 and k_2. These operations satisfy the relations of an *associative bialgebra* (see [LV12]):

$$\text{Assoc. bialg.} \quad \begin{cases} \mu \circ (\mathbb{1} \otimes \mu) = \mu \circ (\mu \otimes \mathbb{1}), \\ (\mathbb{1} \otimes \delta) \circ \delta = (\delta \otimes \mathbb{1}) \circ \delta, \\ \qquad \delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{1} \otimes \tau \otimes \mathbb{1}) \circ (\delta \otimes \delta). \end{cases} \tag{3.13}$$

Here $\tau : C_1 \otimes C_2 \to C_2 \otimes C_1$, $c_1 \otimes c_2 \mapsto (-1)^{|c_1||c_2|} c_2 \otimes c_1$ denotes the *twist map*.

We will use the bialgebra calculus ($\coloneqq$ relations (3.13)) to write down explicit formulas for the operations $\circ_{h_1,\dots,h_r}$ which were briefly introduced in [CFL15]; these operations take symmetric maps f_1, …, f_r and connect h_1, …, h_r of their outputs to the inputs of a symmetric map f in all possible ways, so that the result, which we denote by $f \circ_{h_1,\dots,h_r} (f_1, \dots, f_r)$, becomes a symmetric map again.

Definition 3.2.2 (Partial compositions)**.** *Let C be a graded vector space. For i, $j \geq 0$, we denote by*

$$\begin{aligned} &\pi_i : \mathrm{E}C \longrightarrow \mathrm{E}_iC, \quad \iota_i : \mathrm{E}_iC \longrightarrow \mathrm{E}C, \\ &\mathbb{1}_i : \mathrm{E}_iC \longrightarrow \mathrm{E}_iC, \quad \tau_{i,j} : \mathrm{E}_iC \otimes \mathrm{E}_jC \longrightarrow \mathrm{E}_jC \otimes \mathrm{E}_iC \end{aligned}$$

the components of the canonical projection π, the canonical inclusion ι, the identity $\mathbb{1}$

and the twist map τ*, respectively. We also set*

$$\delta_{i,j} \coloneqq (\pi_i \otimes \pi_j) \circ \delta \circ \iota_{i+j} \quad \text{and} \quad \mu_{i,j} \coloneqq \pi_{i+j} \circ \mu \circ (\iota_i \otimes \iota_j).$$

For k'*,* k_1*,* l'*,* $l_1 \geq 0$*, let* $f : \mathrm{E}_{k'}C \to \mathrm{E}_{l'}C$ *and* $f_1 : \mathrm{E}_{k_1}C \to \mathrm{E}_{l_1}C$ *be linear maps, and let* $0 \leq h \leq \min(k', l_1)$*. We set*

$$k \coloneqq k' + k_1 - h \quad \text{and} \quad l \coloneqq l' + l_1 - h$$

and define the composition of f and f_1 at h common outputs *to be the linear map* $f \circ_h f_1 : \mathrm{E}_k C \to \mathrm{E}_l C$ *given by*

$$\begin{aligned} f \circ_h f_1 \coloneqq\ & \mu_{l',l_1-h} \circ (f \otimes \mathbb{1}_{l_1-h}) \circ (\mu_{h,k'-h} \otimes \mathbb{1}_{l_1-h}) \circ (\mathbb{1}_h \otimes \tau_{l_1-h,k'-h}) \\ & \circ (\delta_{h,l_1-h} \otimes \mathbb{1}_{k'-h}) \circ (f_1 \otimes \mathbb{1}_{k'-h}) \circ \delta_{k_1,k'-h} . \end{aligned} \tag{3.14}$$

More generally, we define the composition of $f : E_{k'} \to E_{l'}$ *with* $r \geq 1$ *linear maps* $f_i : E_{k_i} \to E_{l_i}$ *with* k_i*,* $l_i \geq 0$ *for* $i = 1, \ldots, r$ *at* $0 \leq h_i \leq l_i$ *common outputs such that* $h \coloneqq h_1 + \cdots + h_r \leq k'$ *as follows. We set*

$$k \coloneqq k' + k_1 + \cdots + k_r - h \quad \text{and} \quad l \coloneqq l' + l_1 + \cdots + l_r - h$$

and define $f \circ_{h_1,\ldots,h_r} (f_1, \ldots, f_r) : \mathrm{E}_k C \to \mathrm{E}_l C$ *by*

$$\begin{aligned} & f \circ_{h_1,\ldots,h_r} (f_1, \ldots, f_r) \\ & \quad \coloneqq \mu \circ (f \otimes \mathbb{1}) \circ (\mu \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \tau) \\ & \qquad \circ ([(\mu^{(r)} \otimes \mu^{(r)}) \circ (F_{h_1,\ldots,h_r} \otimes \mathbb{1}^{\otimes r}) \circ \sigma_r \circ \delta^{\otimes r}] \otimes \mathbb{1}) \\ & \qquad\qquad \circ (f_1 \otimes \cdots \otimes f_r \otimes \mathbb{1}) \circ \delta^{(r+1)}, \end{aligned} \tag{3.15}$$

where we have:

- *The operation* $\mu^{(r)}$ *is the "product with r inputs" and the operation* $\delta^{(r)}$ *the "coproduct with r outputs"; they are defined by*

$$\begin{aligned} \mu^{(r)} &\coloneqq \mu(\mathbb{1} \otimes \mu) \cdots (\mathbb{1}^{\otimes r-2} \otimes \mu), & \mu^{(1)} &\coloneqq \mathbb{1}, \\ \delta^{(r)} &\coloneqq (\mathbb{1}^{\otimes r-2} \otimes \delta) \cdots (\mathbb{1} \otimes \delta)\, \delta, & \delta^{(1)} &\coloneqq \mathbb{1}. \end{aligned}$$

- $F_{h_1,\ldots,h_r} \coloneqq (\iota_{h_1}\pi_{h_1}) \otimes \cdots \otimes (\iota_{h_r}\pi_{h_r})$.

- *The permutation* $\sigma_r \in \mathbb{S}_{2r}$ *is given by*

$$\sigma_r : (1,2,3,4\dots,2r-1,2r) \longmapsto (1,r+1,2,r+2,\dots,r,2r).$$

- *The symbols* f *and* f_i *inside the formula denote the* trivial extensions *of* f *and* f_i, *respectively; we extend a linear map* $f : E_{k'}C \to E_{l'}C$ *trivially to* $f : \mathrm{E}C \to \mathrm{E}C$ *by defining* $f(\mathrm{E}_i C) = 0$ *for* $i \neq k'$.

Remark 3.2.3 (On partial compositions). (i) Defining $f \circ_{h_1,\dots,h_r} (f_1,\dots,f_r) : \mathrm{E}_k C \to \mathrm{E}_l C$ using (3.15) makes sense because the right hand side is a trivial extension of its component $\mathrm{E}_k C \to \mathrm{E}_l C$. In fact, all μ, δ, π, ι in (3.15) can be replaced with $\mu_{i,j}$, $\delta_{i,j}$, π_i, ι_i for unique i, j, so that trivial extensions do not have to be used at all. In this way, it can be seen that (3.14) is indeed a special case of (3.15).

(ii) If $h = k' = l_1$, then $f \circ_h f_1 = f \circ f_1$.

(iii) It holds $f \circ_0 f_1 = (-1)^{|f||f_1|} f_1 \circ_0 f$ and

$$f \circ_{h_1,\dots,h_r} (f_1,\dots,f_r) = \varepsilon(\sigma,f) f \circ_{h_{\sigma_1^{-1}},\dots,h_{\sigma_r^{-1}}} (f_{\sigma_1^{-1}},\dots,f_{\sigma_r^{-1}}).$$

(iv) Consider the ("non-trivial") extension $\hat{f} \coloneqq \mu(f \otimes \mathbb{1})\,\delta : \mathrm{E}C \to \mathrm{E}C$ and the symmetric product $f_1 \odot \dots \odot f_r \coloneqq \mu^{(r)}(f_1 \otimes \dots \otimes f_r)\,\delta^{(r)} : \mathrm{E}C \to \mathrm{E}C$. The following formulas from [CFL15] hold:

$$\begin{aligned} f \circ_{h_1,\dots,h_{r-1},0} (f_1,\dots,f_r) &= f \circ_{h_1,\dots,h_{r-1}} (f_1,\dots,f_{r-1}) \odot f_r, \\ \hat{f} \circ \hat{f}_1 &= \sum_{h=0}^{\min(k',l_1)} \widehat{f \circ_h f_1}, \\ \hat{f} \circ (f_1 \odot \dots \odot f_r) &= \sum_{\substack{h_1,\dots,h_r \geq 0 \\ h_1+\dots+h_r = k'}} f \circ_{h_1,\dots,h_r} (f_1,\dots,f_r). \end{aligned} \tag{3.16}$$

We also have the "weak associativity"

$$\sum_{\substack{0 \leq h_2 \leq \min(f_3^-, f_2^+) \\ 0 \leq h_1 \leq \min(f_1^+, f_2^- + f_3^- - h_2) \\ h_1 + h_2 = h}} f_1 \circ_{h_1} (f_2 \circ_{h_2} f_3) = \sum_{\substack{0 \leq h_1 \leq \min(f_1^+, f_2^-) \\ 0 \leq h_2 \leq \min(f_1^+ + f_2^+ - h_1, f_3^-) \\ h_1 + h_2 = h}} (f_1 \circ_{h_1} f_2) \circ_{h_2} f_3 \tag{3.17}$$

for every $0 \leq h \leq \min(k_1 + k_2 + k_3, l_1 + l_2 + l_3)$, where f^+ denotes the number of inputs and f^- the number of outputs of f. The weak associativity of $\circ_h$ can be proven using the associativity of $\hat{\cdot}$ and the second relation of (3.16).

(v) We refer to Section D.4 of Appendix D for a thorough treatment of partial compositions. We show there that $\circ_{h_1,\dots,h_r}$ can be defined on maps on any connected weight-graded bialgebra using natural bilinear operations $\square_A$ on polynomials in the convolution product. In Proposition D.4.4 in the appendix, we prove the relations above. ◁

If C is filtered by a decreasing filtration, then the bialgebra operations extend continuously to

$$\begin{aligned} \mu : \hat{\mathrm{E}}_{k_1}C \mathbin{\hat{\otimes}} \hat{\mathrm{E}}_{k_2}C &\longrightarrow \hat{\mathrm{E}}_{k_1+k_2}C \quad \text{and} \\ \delta : \hat{\mathrm{E}}_{k}C &\longrightarrow \bigoplus_{\substack{l_1,l_2\geq 0\\ l_1+l_2=k}} \hat{\mathrm{E}}_{l_1}C \mathbin{\hat{\otimes}} \hat{\mathrm{E}}_{l_2}C \end{aligned}$$

for all k_1, k_2, $k \in \mathbb{N}_0$ because they preserve the filtration degree (see [Fre17] for a similar construction). Next, if $f_1 : \hat{\mathrm{E}}_{k_1}C \to \hat{\mathrm{E}}_{l_1}C$ and $f_2 : \hat{\mathrm{E}}_{k_2}C \to \hat{\mathrm{E}}_{l_2}C$ have finite filtration degrees, then $f_1 \otimes f_2 : \hat{\mathrm{E}}_{k_1}C \otimes \hat{\mathrm{E}}_{k_2}C \to \hat{\mathrm{E}}_{l_1}C \otimes \hat{\mathrm{E}}_{l_2}C$ has finite filtration degree too, and hence it extends continuously to $f_1 \hat{\otimes} f_2 : \hat{\mathrm{E}}_{k_1}C \mathbin{\hat{\otimes}} \hat{\mathrm{E}}_{k_2}C \to \hat{\mathrm{E}}_{l_1}C \mathbin{\hat{\otimes}} \hat{\mathrm{E}}_{l_2}C$. Using these facts, we can canonically extend Definition 3.2.2 to maps $f : \hat{\mathrm{E}}_{k'}C \to \hat{\mathrm{E}}_{l'}C$ and $f_i : \hat{\mathrm{E}}_{k_i}C \to \hat{\mathrm{E}}_{l_i}C$ of finite filtration degrees. The resulting map $f \circ_{h_1,\dots,h_r} (f_1,\dots,f_r) : \hat{\mathrm{E}}_{k}C \to \hat{\mathrm{E}}_{l}C$ will have finite filtration degree too. Moreover, the formulas in Remark 3.2.3 will still hold.

We will now rephrase the definitions of an IBL_∞-algebra, a Maurer-Cartan element and twisted operations from [CFL15] in terms of $\circ_{h_1,\dots,h_r}$.

Definition 3.2.4 (IBL_∞-algebra). *Let C be a graded vector space equipped with a decreasing filtration, and let $d \in \mathbb{Z}$ and $\gamma \geq 0$ be fixed constants. An* IBL_∞-algebra of bidegree (d,γ) *on C is a collection of linear maps $\mathfrak{q}_{klg} : \hat{\mathrm{E}}_{k}C \to \hat{\mathrm{E}}_{l}C$ for all $k, l \geq 1$, $g \geq 0$ which are homogenous, of finite filtration degree and satisfy the following conditions:*

1) $|\mathfrak{q}_{klg}| = -2d(k+g-1)-1$.

2) $\|\mathfrak{q}_{klg}\| \geq \gamma\chi_{klg}$, *where* $\chi_{klg} \coloneqq 2-2g-k-l$.

3) The IBL_∞-relations *hold: for all $k, l \geq 1$, $g \geq 0$, we have*

$$\sum_{h=1}^{g+1} \sum_{\substack{k_1,k_2,l_1,l_2\geq 1\\ g_1,g_2\geq 0\\ k_1+k_2=k+h\\ l_1+l_2=l+h\\ g_1+g_2=g+1-h}} \mathfrak{q}_{k_2l_2g_2} \circ_h \mathfrak{q}_{k_1l_1g_1} = 0. \tag{3.18}$$

We denote a given IBL_∞*-algebra structure on* C *by* $\mathrm{IBL}_\infty(C)$*; i.e., we write* $\mathrm{IBL}_\infty(C) = (C, (\mathfrak{q}_{klg}))$*.*

If $\mathfrak{q}_{klg} \equiv 0$ *for all* $(k,l,g) \neq (1,1,0)$*,* $(2,1,0)$*,* $(1,2,0)$*, then we call* $\mathrm{IBL}_\infty(C)$ *a* dIBL-algebra *and denote it by* $\mathrm{dIBL}(C)$*. If in addition* $\mathfrak{q}_{110} \equiv 0$*, then we have an* IBL-algebra $\mathrm{IBL}(C)$*.*

If the operations on the completed exterior powers $\hat{\mathrm{E}}_k C$ *arise as continuous extensions of operations* $\mathfrak{q}_{klg} : \mathrm{E}_k C \to \mathrm{E}_l C$*, then we call the* IBL_∞*-algebra* completion-free *and denote* C *together with the operations* $\mathfrak{q}_{klg} : \mathrm{E}_k C \to \mathrm{E}_l C$ *by* $\mathrm{IBL}^0_\infty(C)$*.*

The acronym IBL stands for an *involutive Lie bialgebra.* It follows namely from the IBL_∞-relations (3.18) that for $\mathrm{IBL}(C) = (C, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ the following holds:

$$\begin{array}{rll} \text{Lie bialg.} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. \!\!\!\!\!\!\!\!& \begin{array}{l} 0 = \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{210} \\ 0 = \mathfrak{q}_{120} \circ_1 \mathfrak{q}_{120} \\ 0 = \mathfrak{q}_{120} \circ_1 \mathfrak{q}_{210} + \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{120} \end{array} & \begin{array}{l} \leftarrow \text{Jacobi id.} \\ \leftarrow \text{co-Jacobi id.} \\ \leftarrow \text{Drinfeld id.} \end{array} \\ & 0 = \mathfrak{q}_{210} \circ_2 \mathfrak{q}_{120} & \leftarrow \text{Involutivity} \end{array}$$

The acronym dIBL stands for a *differential involutive Lie bialgebra* — an involutive Lie bialgebra together with a differential (a boundary operator in our case) such that the bracket and cobracket are chain maps.

Proposition 3.2.5 (Odd degree shift of an IBL-algebra)**.** *Let* $(C, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ *be an* IBL-*algebra of degree* d *from Definition 3.2.4, and let* $\tilde{\mathfrak{q}}_{210} : C^{\otimes 2} \to C$ *and* $\tilde{\mathfrak{q}}_{120} : C \to C^{\otimes 2}$ *be the linear maps defined by*

$$\begin{aligned} \theta\tilde{\mathfrak{q}}_{210}(x_1 \otimes x_2) &:= \mathfrak{q}_{210}(\pi(\theta^2 x_1 \otimes x_2)) \quad \textit{and} \\ \theta^2\tilde{\mathfrak{q}}_{120}(x) &:= \iota(\mathfrak{q}_{120}(\theta x)) \end{aligned} \tag{3.19}$$

for all x_1*,* x_2*,* $x \in C$*, where* $\iota : \mathrm{S}_2(C[1]) \to C[1]^{\otimes 2}$ *is the section of* $\pi : C[1]^{\otimes 2} \to \mathrm{S}_2(C[1])$ *from Definition 3.1.7 and* θ *is a formal symbol of degree* $|\theta| = -1$*. Then the degrees satisfy*

$$|\tilde{\mathfrak{q}}_{210}| = |\mathfrak{q}_{210}| - 1 = -2d - 2 \quad \textit{and} \quad |\tilde{\mathfrak{q}}_{120}| = |\mathfrak{q}_{120}| + 1 = 0,$$

the operations $\tilde{\mathfrak{q}}_{210}$ *and* $\tilde{\mathfrak{q}}_{120}$ *are graded antisymmetric, i.e., we have*

$$\tilde{\mathfrak{q}}_{210} \circ \tau = -\tilde{\mathfrak{q}}_{210} \quad \textit{and} \quad \tau \circ \tilde{\mathfrak{q}}_{120} = -\tilde{\mathfrak{q}}_{120}$$

for the twist map τ, and the relations

$$\begin{aligned}
0 &= \tilde{\mathfrak{q}}_{210} \circ (\tilde{\mathfrak{q}}_{210} \otimes \mathbb{1}) \circ (\mathbb{1}^{\otimes 3} + t_3 + t_3^2), \\
0 &= (\mathbb{1}^{\otimes 3} + t_3 + t_3^2) \circ (\tilde{\mathfrak{q}}_{120} \otimes \mathbb{1}) \circ \tilde{\mathfrak{q}}_{120}, \\
0 &= x_1 \cdot \tilde{\mathfrak{q}}_{120}(x_2) - (-1)^{x_1 x_2} x_2 \cdot \tilde{\mathfrak{q}}_{120}(x_1) - \tilde{\mathfrak{q}}_{120}(\tilde{\mathfrak{q}}_{210}(x_1, x_2)), \\
0 &= \tilde{\mathfrak{q}}_{210} \circ \tilde{\mathfrak{q}}_{120},
\end{aligned}$$

hold for all x_1, $x_2 \in C$. Here, $t_3 \in \mathbb{S}_3$ denotes the cyclic permutation with $t_3(1) = 2$ acting on $C^{\otimes 3}$, and we define

$$x \cdot (y_1 \otimes y_2) \coloneqq \tilde{\mathfrak{q}}_{210}(x, y_1) \otimes y_2 + (-1)^{x y_1} y_1 \otimes \tilde{\mathfrak{q}}_{210}(x, y_2)$$

for all x, y_1, $y_2 \in C$.

Proof. The proof is a lengthy but straightforward computation. ☐

Consider the *sign-action of* $\mathbb{S}_k$ on $C^{\otimes k}$ given by $\sigma \mapsto \bar{\sigma}$, where

$$\bar{\sigma}(c_1 \otimes \cdots \otimes c_k) \coloneqq (-1)^{\sigma} \varepsilon(\sigma, c) c_{\sigma_1^{-1}} \otimes \cdots \otimes c_{\sigma_k^{-1}}$$

for all $c_1, \ldots, c_k \in C$ and $\sigma \in \mathbb{S}_k$. We define

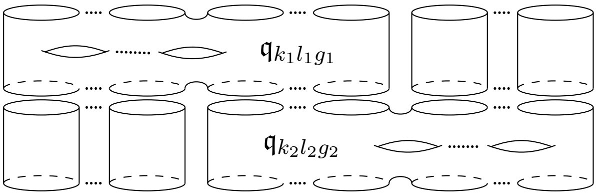
$$\Lambda C \coloneqq \bigoplus_{k=0}^{\infty} \Lambda_k C \quad \text{with} \quad \Lambda_k C \coloneqq C^{\otimes k} / \sim,$$

where $c \sim \bar{\sigma}(c)$ for all $c \in C^{\otimes k}$ and $\sigma \in \mathbb{S}_k$. It is easy to see that the degree-shift map $\theta^{\otimes k} : c_1 \otimes \cdots \otimes c_k \in C^{\otimes k} \mapsto \varepsilon(c, \theta)(\theta c_1) \otimes \cdots \otimes (\theta c_k) \in C[1]^{\otimes k}$ is equivariant with respect to the sign-action of $\mathbb{S}_k$ on $C^{\otimes k}$ and the standard action of $\mathbb{S}_k$ on $C[1]^{\otimes k}$ for all k, and thus it induces an isomorphism of vector spaces ΛC and $\mathrm{E}C$. We use the following notation:

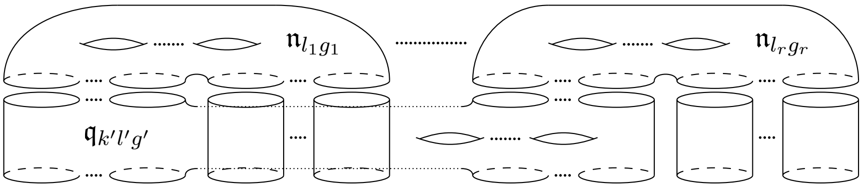
$$\begin{tikzcd}
\mathfrak{q}_{klg} : \hat{\mathrm{E}}_k C \arrow[r] \arrow[d, "\theta^{\otimes k}"] & \hat{\mathrm{E}}_l C \arrow[d, "\theta^{\otimes l}"] \\
\tilde{\mathfrak{q}}_{klg} : \hat{\Lambda}_k C \arrow[r] & \hat{\Lambda}_l C.
\end{tikzcd}$$

In fact, $\mathfrak{q}_{klg}$ and $\tilde{\mathfrak{q}}_{klg}$ are related precisely by the degree shift (3.6).

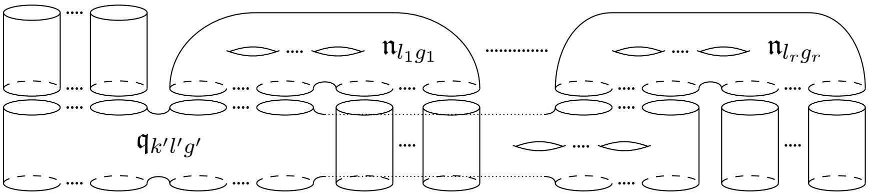
Definition 3.2.6 (Maurer-Cartan element)**.** *A* Maurer-Cartan element *for an* IBL_∞-*algebra* $\mathrm{IBL}_\infty(C)$ *from Definition 3.2.4 is a collection* $\mathfrak{n} \coloneqq (\mathfrak{n}_{lg})_{l \geq 1, g \geq 0}$ *of elements* $\mathfrak{n}_{lg} \in \hat{\mathrm{E}}_l C$ *which are homogenous, of finite filtration degree and satisfy the following*



(a) The term $\mathfrak{q}_{k_2 l_2 g_2} \circ_h \mathfrak{q}_{k_1 l_1 g_1}$ in the IBL$_\infty$-equation (3.18).



(b) The term $\mathfrak{q}_{k' l' g'} \circ_{h_1,\dots,h_r} (\mathfrak{n}_{l_1 g_1}, \dots, \mathfrak{n}_{l_r g_r})$ in the Maurer-Cartan equation (3.20). We remark that the contour of the surface corresponding to $\mathfrak{q}_{k' l' g'}$ starts on the left and continues to the right along the dotted line behind the two trivial cylinders.



(c) The term $\mathfrak{q}_{k' l' g'} \circ_{l_1,\dots,l_r} (\mathfrak{n}_{l_1 g_1}, \dots, \mathfrak{n}_{l_r g_r})$ in the twisted operation (3.21). The remark to Figure (b) applies too.

Figure 3.1.: Graphical representation of compositions appearing in Definitions 3.2.4, 3.2.6 and 3.2.7 as gluing of connected Riemannian surfaces. The figure is to be read from the top to the bottom, the empty cylinder represents the identity, and the resulting surface must be connected. We emphasize that the gluing is not associative (c.f., weak associativity (3.17)).

conditions:

1) $|\mathfrak{n}_{lg}| = -2d(g-1)$.

2) $\|\mathfrak{n}_{lg}\| \geq \gamma \chi_{0lg}$ *with* $>$ *for* $(l,g) = (1,0)$, $(2,0)$ *(see Definition 3.2.4 for* χ_{klg}*).*

3) *The* Maurer-Cartan equation *holds: for all* $l \geq 1$, $g \geq 0$, *we have*

$$\sum_{r \geq 1} \frac{1}{r!} \sum_{\substack{l', k', l_1, \dots, l_r \geq 1 \\ g', g_1, \dots, g_r \geq 0 \\ h_1, \dots, h_r \geq 1 \\ l_1 + \dots + l_r + l' - k' = l \\ g_1 + \dots + g_r + g' + k' = g + r \\ h_1 + \dots + h_r - k' = 0}} \mathfrak{q}_{k' l' g'} \circ_{h_1, \dots, h_r} (\mathfrak{n}_{l_1 g_1}, \dots, \mathfrak{n}_{l_r g_r}) = 0, \tag{3.20}$$

where we view $\mathfrak{n}_{lg}$ *as a linear map* $\mathfrak{n}_{lg} : \hat{\mathrm{E}}_0 C = \mathbb{R} \to \hat{\mathrm{E}}_l C$ *with* $\mathfrak{n}_{lg}(1) = \mathfrak{n}_{lg}$*.*

Definition 3.2.7 (Twisted operations)**.** *In the setting of Definition 3.2.6, the* twisted operations $\mathfrak{q}^{\mathfrak{n}}_{klg} : \hat{\mathrm{E}}_k C \to \hat{\mathrm{E}}_l C$ *for* $k, l \geq 1$, $g \geq 0$ *are defined by*

$$\mathfrak{q}^{\mathfrak{n}}_{klg} = \sum_{r\geq 0} \frac{1}{r!} \sum_{\substack{k',l',l_1,\ldots,l_r\geq 1\\ g',g_1,\ldots,g_r\geq 0\\ h_1,\ldots,h_r\geq 1\\ l_1+\cdots+l_r+l'-k'=l-k\\ g_1+\cdots+g_r+g'+k'=g+r+k\\ h_1+\cdots+h_r-k'=-k}} \mathfrak{q}_{k'l'g'} \circ_{h_1,\ldots,h_r} (\mathfrak{n}_{l_1 g_1}, \ldots, \mathfrak{n}_{l_r g_r}). \tag{3.21}$$

In [CFL15, Proposition 9.3], they prove that $(\mathfrak{q}^{\mathfrak{n}}_{klg})_{k,l\geq 1, g\geq 0}$ *is again an* IBL_∞*-algebra of bidegree* (d,γ) *on* C — the twisted IBL_∞-algebra*. We denote it by* $\mathrm{IBL}^{\mathfrak{n}}_\infty(C)$*.*

Let $(\mathfrak{q}_{klg})$ be an IBL_∞-algebra on C. The boundary operator $\mathfrak{q}_{110} : C[1] \to C[1]$ induces the boundary operator $\partial_k : \mathrm{E}_k C \to \mathrm{E}_k C$ for every $k \in \mathbb{N}$ (see (3.12)). Because of the finite filtration degree, ∂_k continuously extends to $\partial_k : \hat{\mathrm{E}}_k C \to \hat{\mathrm{E}}_k C$. The following is easy to see using (3.14):

$$\begin{aligned} \mathfrak{q}_{klg} \circ_1 \mathfrak{q}_{110} &= \mathfrak{q}_{klg} \circ \partial_k, \\ \mathfrak{q}_{110} \circ_1 \mathfrak{q}_{klg} &= \partial_l \circ \mathfrak{q}_{klg}. \end{aligned}$$

Because $\mathfrak{q}_{klg}$ are odd ($:=$ have odd degree), we have

$$\begin{aligned} [\partial, \mathfrak{q}_{klg}] &:= \partial_l \circ \mathfrak{q}_{klg} - (-1)^{|\partial||\mathfrak{q}_{klg}|} \mathfrak{q}_{klg} \circ \partial_k \\ &= \partial_l \circ \mathfrak{q}_{klg} + \mathfrak{q}_{klg} \circ \partial_k \\ &= \mathfrak{q}_{110} \circ_1 \mathfrak{q}_{klg} + \mathfrak{q}_{klg} \circ_1 \mathfrak{q}_{110}. \end{aligned}$$

With this notation, the IBL_∞-relations (3.18) for $\mathfrak{q}_{210} : \hat{\mathrm{E}}_2 C \to \hat{\mathrm{E}}_1 C$ and $\mathfrak{q}_{120} : \hat{\mathrm{E}}_1 C \to \hat{\mathrm{E}}_2 C$ become $[\partial, \mathfrak{q}_{210}] = 0$ and $[\partial, \mathfrak{q}_{120}] = 0$, respectively. If moreover the canonical maps $\mathrm{E}_k \mathrm{H}(\hat{C}, \tilde{\mathfrak{q}}_{110}) \to \mathrm{H}(\hat{\mathrm{E}}_k C, \partial_k)$ induce the isomorphisms $\hat{\mathrm{E}}_k \mathrm{H}(\hat{C}, \tilde{\mathfrak{q}}_{110}) \simeq \mathrm{H}(\hat{\mathrm{E}}_k C, \partial_k)$ for $k = 1$, 2, e.g., when Proposition 3.1.13 holds, then we obtain the maps

$$\mathfrak{q}_{210} : \hat{\mathrm{E}}_2 \mathrm{H}(\hat{C}, \tilde{\mathfrak{q}}_{110}) \to \hat{\mathrm{E}}_1 \mathrm{H}(\hat{C}, \tilde{\mathfrak{q}}_{110}) \quad \text{and} \quad \mathfrak{q}_{120} : \hat{\mathrm{E}}_1 \mathrm{H}(\hat{C}, \tilde{\mathfrak{q}}_{110}) \to \hat{\mathrm{E}}_2 \mathrm{H}(\hat{C}, \tilde{\mathfrak{q}}_{110}),$$

and $(\mathrm{H}(\hat{C}, \tilde{\mathfrak{q}}_{110}), \mathfrak{q}_{210}, \mathfrak{q}_{120})$ becomes an IBL-algebra according to Definition 3.2.4 — the *induced* IBL*-algebra on homology.*

Definition 3.2.8 (Homology)**.** *We define the homology of an* IBL_∞*-algebra* $\mathrm{IBL}_\infty(C)$ *by*

$$\mathbb{H}(C)[1] := \mathrm{H}(\hat{C}[1], \mathfrak{q}_{110}).$$

It is a graded vector space with the induced filtration. If $\mathfrak{n}$ *is a Maurer-Cartan element for* $\mathrm{IBL}_\infty(C)$*, we denote by* $\mathbb{H}^{\mathfrak{n}}(C)$ *the homology of* $\mathrm{IBL}^{\mathfrak{n}}_\infty(C)$*.*

Remark 3.2.9 (Weak IBL_∞-algebras and BV-formalism). (i) A possible generalization of the IBL_∞-theory is to allow $k=0$ and $l=0$. Such structures would be called *weak* IBL_∞*-algebras* while the structures from this section *strict* IBL_∞*-algebras.* In fact, one does not need filtrations and completions to deal with the category of strict IBL_∞-algebras unless deformations (twisting) are considered. On the other hand, one needs filtrations and completions for the definition of a morphism of weak IBL_∞-algebras already. We refer to Appendix D for more details.

(ii) Let $\hat{\mathrm{E}}C[[\hbar]]$, resp. $\hat{\mathrm{E}}C((\hbar))$ be the spaces of formal power, resp. Laurent series in the variable $\hbar$ of degree $|\hbar|=2d$ with coefficients in $\mathrm{E}C$ completed with respect to a suitable completion. Operations of an IBL_∞-algebra on C can be encoded in a degree -1 operator $\Delta : \hat{\mathrm{E}}C[[\hbar]] \to \hat{\mathrm{E}}C[[\hbar]]$ called the BV*-operator,* while the data of a Maurer-Cartan element $(\mathfrak{n}_{lg})$ give rise to an element $e^{\mathfrak{n}} \in \hat{\mathrm{E}}C((\hbar))$. The prescriptions are

$$\Delta \coloneqq \sum_{i\geq 0} \Delta_{i+1}\,\hbar^i \quad \text{and} \quad e^{\mathfrak{n}} \coloneqq \sum_{j\in\mathbb{Z}} (e^{\mathfrak{n}})_j \hbar^j,$$

where Δ_i and $(e^{\mathfrak{n}})_j$ for $i\geq 1$, $j\in\mathbb{Z}$ are defined by

$$\Delta_i \coloneqq \sum_{\substack{k\geq 1, g\geq 0\\ k+g=i}} \sum_{l\geq 1} \hat{\mathfrak{q}}_{klg} \quad \text{and}$$

$$(e^{\mathfrak{n}})_j \coloneqq \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{g_1,\dots,g_r\geq 0\\ g_1+\dots+g_r-r=j}} \sum_{l_1,\dots,l_r\geq 1} \mathfrak{n}_{l_1 g_1}\odot\dots\odot\mathfrak{n}_{l_r g_r}.$$

It can be shown that the IBL_∞-relations (3.18) and the Maurer-Cartan equation (3.20) are equivalent to

$$\Delta\circ\Delta = 0 \quad \text{and} \quad \Delta(e^{\mathfrak{n}}) = 0, \tag{3.22}$$

respectively, and that the BV_∞-operator $\Delta^{\mathfrak{n}}$ for the twisted IBL_∞-structure $(\mathfrak{q}^{\mathfrak{n}}_{klg})$ satisfies

$$\Delta^{\mathfrak{n}}(\bullet) = e^{-\mathfrak{n}}\,\Delta(e^{\mathfrak{n}}\bullet), \tag{3.23}$$

where we multiply with $e^{-\mathfrak{n}}$ and $e^{\mathfrak{n}}$, respectively. These facts were shown in [CFL15] using (3.16). We refer to Appendix D to the precise formulation of the BV-formalism using a filtered version of the MV-formalism from [MV17]. ◁

In our applications in string topology, a canonical dIBL-algebra $\mathrm{dIBL}(C)$ with a natural

Maurer-Cartan element $\mathfrak{n}$ coming from the Chern-Simons theory is given, and we want to study $\mathrm{dIBL}^{\mathfrak{n}}(C)$, which will be a chain model of string topology. We are also interested in the homology $\mathbb{H}^{\mathfrak{n}}(C)$, the IBL-structure $\mathrm{IBL}(\mathbb{H}^{\mathfrak{n}}(C))$ and possible higher operations on $\mathbb{H}^{\mathfrak{n}}(C)$ induced by $\mathfrak{q}^{\mathfrak{n}}_{klg}$; however, these higher maps are not chain maps in general. The following proposition summarizes some observations in this situation:

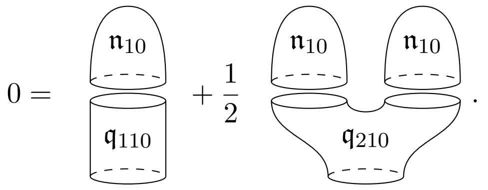
Proposition 3.2.10 (Twist of a dIBL-algebra)**.** *Let* $\mathrm{dIBL}(C) = (C, \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ *be a* dIBL-*algebra, and let* $\mathfrak{n} = (\mathfrak{n}_{lg})$ *be a Maurer-Cartan element. The Maurer-Cartan equation* (3.20) *reduces to the following:*

$$\begin{aligned} 0 = \mathfrak{q}_{110} \circ_1 \mathfrak{n}_{lg} + \mathfrak{q}_{120} \circ_1 \mathfrak{n}_{l-1,g} + \mathfrak{q}_{210} \circ_2 \mathfrak{n}_{l+1,g-1} \\ + \frac{1}{2} \sum_{\substack{l_1,l_2\geq 1\\ g_1,g_2\geq 0\\ l_1+l_2=l+1\\ g_1+g_2=g}} \mathfrak{q}_{210} \circ_{1,1} (\mathfrak{n}_{l_1 g_1}, \mathfrak{n}_{l_2 g_2}) \quad \forall l \geq 1, g \geq 0. \end{aligned}$$

In particular, the "lowest" equation is given by[4]

$$(l,g) = (1,0): \qquad \mathfrak{q}_{110}(\mathfrak{n}_{10}) + \frac{1}{2}\mathfrak{q}_{210}(\mathfrak{n}_{10}, \mathfrak{n}_{10}) = 0. \tag{3.24}$$

This can be visualized as



The twisted IBL_∞-*algebra* $\mathrm{dIBL}^{\mathfrak{n}}(C)$ *consists of the operations* $\mathfrak{q}^{\mathfrak{n}}_{110}$, $\mathfrak{q}^{\mathfrak{n}}_{210}$ *and* $\mathfrak{q}^{\mathfrak{n}}_{120}$, *which we call the* basic operations, *and of the operations* $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ *for the pairs* $(l,g) \in \mathbb{N} \times \mathbb{N}_0 \backslash \{(1,0),(2,0)\}$, *which we call the* higher operations*. These operations are given by*

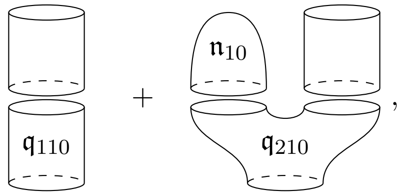
$$\begin{aligned} \mathfrak{q}^{\mathfrak{n}}_{110} &= \mathfrak{q}_{110} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}, \\ \mathfrak{q}^{\mathfrak{n}}_{210} &= \mathfrak{q}_{210}, \\ \mathfrak{q}^{\mathfrak{n}}_{120} &= \mathfrak{q}_{120} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}, \\ \mathfrak{q}^{\mathfrak{n}}_{1lg} &= \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg}. \end{aligned}$$

---

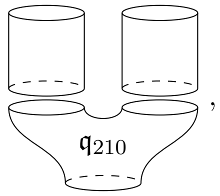
[4] In [CFL15, Definition 2.4.], they define a partial ordering on the signatures (k,l,g).

This can be visualized as

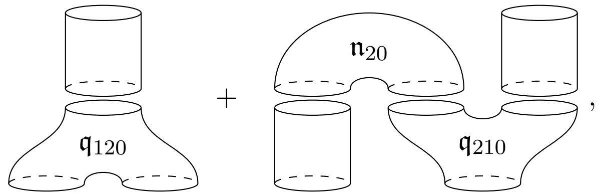
$\mathfrak{q}^{\mathfrak{n}}_{110} =$



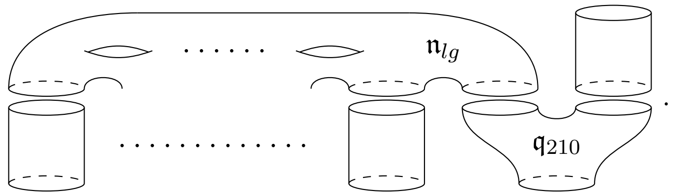
$\mathfrak{q}^{\mathfrak{n}}_{210} =$



$\mathfrak{q}^{\mathfrak{n}}_{120} =$



$\mathfrak{q}^{\mathfrak{n}}_{1lg} =$



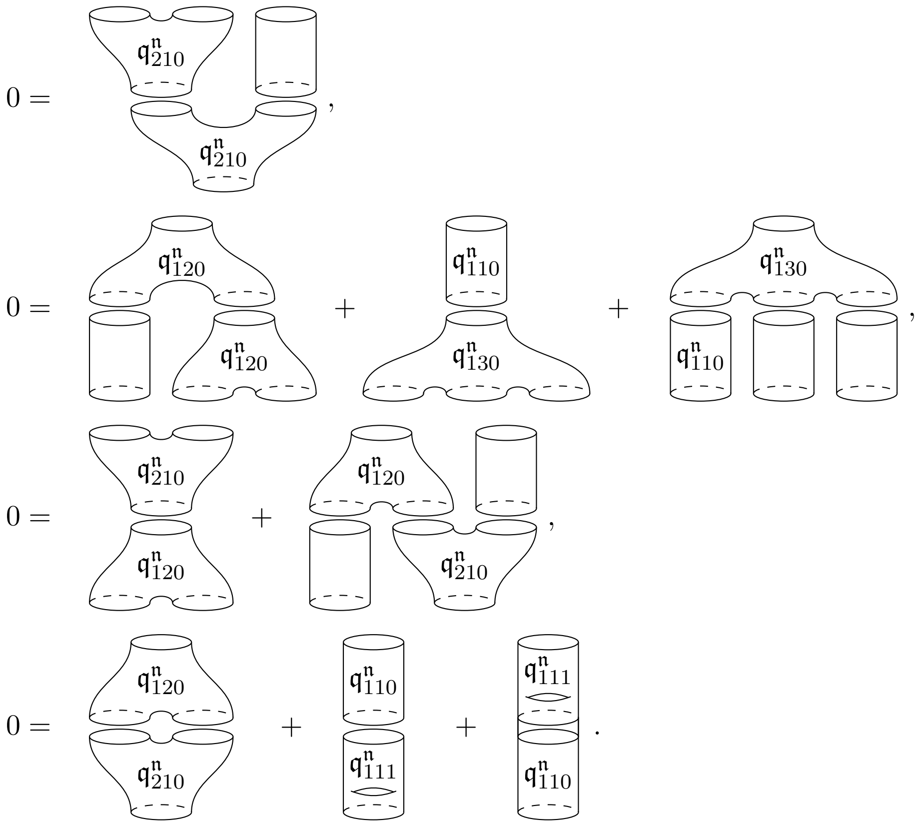
The IBL_∞*-relations satisfied by* $(\mathfrak{q}^{\mathfrak{n}}_{klg})$ *read for all* $l \geq 1$, $g \geq 0$ *as follows:*

$$\begin{aligned}
(3,1,0): \quad & 0 = \mathfrak{q}^{\mathfrak{n}}_{210} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{210}, \\
(2,l,g): \quad & 0 = \mathfrak{q}^{\mathfrak{n}}_{1lg} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{210} + \mathfrak{q}^{\mathfrak{n}}_{210} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{1lg}, \\
(1,l,g): \quad & 0 = \sum_{\substack{l_1,l_2\geq 1 \\ g_1,g_2 \geq 0 \\ l_1+l_2=l+1 \\ g_1+g_2=g}} \mathfrak{q}^{\mathfrak{n}}_{1l_1g_1} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{1l_2g_2} + \mathfrak{q}^{\mathfrak{n}}_{210} \circ_2 \mathfrak{q}^{\mathfrak{n}}_{1,l+1,g-1}.
\end{aligned} \tag{3.25}$$

We call the relations for $(k,l,g) = (1,1,0)$, $(2,1,0)$, $(1,2,0)$, $(3,1,0)$, $(1,3,0)$, $(2,2,0)$, $(1,1,1)$ basic relations *because they contain all compositions of basic operations. In the* *order above, they read:*

$$
\begin{aligned}
0 &= \mathfrak{q}^{\mathfrak{n}}_{110} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{110}, \\
0 &= \mathfrak{q}^{\mathfrak{n}}_{110} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{210} + \mathfrak{q}^{\mathfrak{n}}_{210} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{110}, \\
0 &= \mathfrak{q}^{\mathfrak{n}}_{110} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{120} + \mathfrak{q}^{\mathfrak{n}}_{120} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{110}, \\
0 &= \mathfrak{q}^{\mathfrak{n}}_{210} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{210}, && \leftarrow \textit{Jacobi identity} \\
0 &= \mathfrak{q}^{\mathfrak{n}}_{120} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{120} + \mathfrak{q}^{\mathfrak{n}}_{110} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{130} + \mathfrak{q}^{\mathfrak{n}}_{130} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{110}, && \leftarrow \textit{co-Jacobi id. up to htpy.} \\
0 &= \mathfrak{q}^{\mathfrak{n}}_{120} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{210} + \mathfrak{q}^{\mathfrak{n}}_{210} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{120}, && \leftarrow \textit{Drinfeld identity} \\
0 &= \mathfrak{q}^{\mathfrak{n}}_{210} \circ_2 \mathfrak{q}^{\mathfrak{n}}_{120} + \mathfrak{q}^{\mathfrak{n}}_{111} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{110} + \mathfrak{q}^{\mathfrak{n}}_{110} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{111}. && \leftarrow \textit{Involutivity up to htpy.}
\end{aligned}
$$

The last four equations can be visualized as



Proof. The proof is clear by specializing (3.18), (3.20) and (3.21). □

Remark 3.2.11 (Higher operations). (i) We see from Proposition 3.2.10 that if $\mathfrak{q}^{\mathfrak{n}}_{120} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{120} = 0$ and $\mathfrak{q}^{\mathfrak{n}}_{210} \circ_2 \mathfrak{q}^{\mathfrak{n}}_{120} = 0$, then $[\partial^{\mathfrak{n}}, \mathfrak{q}^{\mathfrak{n}}_{130}] = 0$ and $[\partial^{\mathfrak{n}}, \mathfrak{q}^{\mathfrak{n}}_{111}] = 0$, respectively, and hence the operations $\mathfrak{q}^{\mathfrak{n}}_{130} : \hat{\mathrm{E}}_1\mathbb{H}^{\mathfrak{n}} \to \hat{\mathrm{E}}_3\mathbb{H}^{\mathfrak{n}}$ and $\mathfrak{q}^{\mathfrak{n}}_{111} : \hat{\mathrm{E}}_1\mathbb{H}^{\mathfrak{n}} \to \hat{\mathrm{E}}_1\mathbb{H}^{\mathfrak{n}}$ are well-defined (provided that the assumption of Definition 3.2.8 holds). Likewise, the higher operation $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ defines a map $\hat{\mathrm{E}}_1\mathbb{H}^{\mathfrak{n}} \to \hat{\mathrm{E}}_l\mathbb{H}^{\mathfrak{n}}$, provided that the following equation holds:

$$\mathfrak{q}^{\mathfrak{n}}_{210} \circ_2 \mathfrak{q}^{\mathfrak{n}}_{1,l+1,g-1} + \sum_{\substack{l_1,l_2\geq 1\\ g_1,g_2\geq 0\\ l_1+l_2=l+1\\ g_1+g_2=g\\ (l_i,g_i)\neq(1,0)}} \mathfrak{q}^{\mathfrak{n}}_{1l_1g_1} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{1l_2g_2} = 0.$$

This expression is just the left-over after subtracting the commutator $[\mathfrak{q}^{\mathfrak{n}}_{1lg}, \mathfrak{q}^{\mathfrak{n}}_{110}] = \mathfrak{q}^{\mathfrak{n}}_{110} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{1lg} + \mathfrak{q}^{\mathfrak{n}}_{1lg} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{110}$ from (3.25).

(ii) In the genus-0 case, i.e., $\mathfrak{q}^{\mathfrak{n}}_{1lg} = 0$ whenever $g \geq 1$, relations (3.25) reduce to

$$\begin{aligned} 0 &= \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{210}, \\ 0 &= \mathfrak{q}^{\mathfrak{n}}_{1lg} \circ_1 \mathfrak{q}_{210} + \mathfrak{q}_{210} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{1lg}, \\ 0 &= \sum_{\substack{l_1,l_2\geq 1\\ g_1,g_2\geq 0\\ l_1+l_2=l+1\\ g_1+g_2=g}} \mathfrak{q}^{\mathfrak{n}}_{1l_1g_1} \circ_1 \mathfrak{q}^{\mathfrak{n}}_{1l_2g_2}. \end{aligned}$$

The first relation is the Jacobi identity for $\mathfrak{q}_{210}$, the second relation is a generalization of the Drinfeld identity to higher coproducts $\mathfrak{q}^{\mathfrak{n}}_{1l0}$, and the third relations are coL$_\infty$-*relations* for $\mathfrak{q}^{\mathfrak{n}}_{1lg}$. Therefore, allowing $g \geq 0$, we see that the twisted dIBL-algebra $\mathfrak{q}^{\mathfrak{n}}_{110}$, $\mathfrak{q}_{210}$, $(\mathfrak{q}^{\mathfrak{n}}_{1lg})$ is, in fact, a *quantum* coL$_\infty$-*algebra.* ◁

3.3. Dual cyclic bar complex and cyclic (co)homology

Definition 3.3.1 (Bar complexes)**.** *Let V be a graded vector space. The* bar- and dual bar-complex *of V are the weight-graded vector spaces defined by*

$$\mathrm{B}V := \bar{\mathrm{T}}(V[1]) \quad \textit{and} \quad \mathrm{B}^*V := (\mathrm{B}V)^{*_{\mathrm{wg}}},$$

respectively, where $\bar{T}V := \bigoplus_{k=1}^{\infty} V^{\otimes k}$ is the weight-reduced tensor algebra. For every $k \in \mathbb{N}$, let $t_k \in \mathbb{S}_k$ be the cyclic permutation $t_k : (1, \ldots, k) \mapsto (2, \ldots, k, 1)$, so that for all v_1, …, $v_k \in V[1]$ we have

$$t_k(v_1 \otimes \cdots \otimes v_k) = (-1)^{|v_k|(|v_1|+\cdots+|v_{k-1}|)} v_k \otimes v_1 \otimes \cdots \otimes v_{k-1}.$$

We set

$$t := \sum_{k=1}^{\infty} t_k : \mathrm{B}V \longrightarrow \mathrm{B}V.$$

The cyclic bar-complex *is defined by*

$$\mathrm{B}^{\mathrm{cyc}}V := \mathrm{B}V/\operatorname{im}(1-t).$$

We denote the image of $v_1\otimes\cdots\otimes v_k \in \mathrm{B}V$ *under the canonical projection* $\pi : \mathrm{B}V \to \mathrm{B}^{\mathrm{cyc}}V$ *by* $v_1 \dots v_k$. *If* $v_i \in V[1]$ *are homogenous, then* $v_1 \dots v_k$ *is called a* generating word*; we have*

$$v_1 \dots v_k = (-1)^{|v_k|(|v_1|+\cdots+|v_{k-1}|)} v_k v_1 \dots v_{k-1}.$$

We define the section $\iota : \mathrm{B}^{\mathrm{cyc}}V \to \mathrm{B}V$ *of* π *by*

$$\iota(v_1 \dots v_k) := \frac{1}{k}\sum_{i=0}^{k-1} \underbrace{t_k^i}_{=:\, t_k \circ \cdots \circ t_k \ i\text{-times}} (v_1 \otimes \cdots \otimes v_k)$$

and use it to identify $\mathrm{B}^{\mathrm{cyc}}V$ *with the subspace* $\operatorname{im}\iota = \ker(1-t) \subset \mathrm{B}V$ *consisting of cyclic symmetric tensors.*

We define the dual cyclic bar-complex *by*

$$\mathrm{B}^*_{\mathrm{cyc}}V := \{\psi \in \mathrm{B}^*V \mid \psi \circ t = \psi\}.$$

Remark 3.3.2 (Non-weight-reduced bar complex)*.* In fact, our $\mathrm{B}^*_{\mathrm{cyc}}V$ is weight-reduced. The non-weight-reduced version would be $\mathrm{B}^*_{\mathrm{cyc}}V \oplus \mathbb{R}$ with $\mathbb{R}$ of degree 0. This might play a role in the theory of weak A_∞-algebras (:= operation μ_0 added; c.f., Definition 3.3.7), and it might also be possible to consider IBL_∞-algebras on non-weight-reduced cyclic cochains (c.f., Section 3.4). ◁

Notice that $\psi \in \mathrm{B}^*V$ is homogenous of degree $|\psi| \in \mathbb{Z}$ if and only if for all homogenous $v_1, \dots, v_k \in V[1]$ the following implication holds:

$$|v_1| + \cdots + |v_k| \neq |\psi| \quad \Longrightarrow \quad \psi(v_1 \otimes \cdots \otimes v_k) = 0.$$

This is the cohomological grading convention.

Notation 3.3.3 (Degree shifts of bar complexes)**.** *Let* $A \in \mathbb{Z}$. *In the following, we write* $\mathrm{B}^*_{\mathrm{cyc}}V$, *but the convention applies to all complexes from Definition 3.3.1. We denote by*

s_A *and* θ *the formal symbols of degrees*

$$|\mathrm{s}_A| = -A \quad \text{and} \quad |\theta| = -1,$$

respectively. The degree shift $V \mapsto V[1]$ *will be realized as multiplication with* θ *and the degree shift* $\mathrm{B}^*_{\mathrm{cyc}}V \mapsto \mathrm{B}^*_{\mathrm{cyc}}V[A]$ *as multiplication with* s_A*. In addition, the following notation will be used consistently:*

- $\tilde{v} \in V \longleftrightarrow v = \theta\tilde{v} \in V[1]$

  To clarify this, given $\tilde{v} \in V$*, then* v *automatically means* $v = \theta\tilde{v} \in V[1]$*, and the other way round. Recall that the degree of* $\tilde{v} \in V$ *is denoted by* $\deg(\tilde{v})$ *or simply by* $\tilde{v}$ *in the exponent, e.g.,* $(-1)^{\tilde{v}}$*.*
- $\psi \in \mathrm{B}^*_{\mathrm{cyc}}V \longleftrightarrow \Psi = \mathrm{s}_A\psi \in \mathrm{B}^*_{\mathrm{cyc}}V[A]$.
- *A generating word of* $\mathrm{B}^{\mathrm{cyc}}V$ *of weight* k *will be denoted by the symbol* w *and written as* $w = v_1 \dots v_k$*, where* $v_i = \theta\tilde{v}_i \in V[1]$*. A generating word of* $\mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V$ *is an element* $w_1 \cdots w_k \in \mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V$ *such that each* w_i *is a generating word of* $\mathrm{B}^{\mathrm{cyc}}V$*.*
- $w \in \mathrm{B}^{\mathrm{cyc}}V \longleftrightarrow W = \mathrm{s}_A w \in \mathrm{B}^{\mathrm{cyc}}V[A]$.

We abbreviate

$$\mathrm{B}^*_{\mathrm{cyc}}V[A] \coloneqq (\mathrm{B}^*_{\mathrm{cyc}}V)[A].$$

In contrast to this, we would write $\mathrm{B}^*_{\mathrm{cyc}}(V[A])$ *for the dual cyclic bar-complex of* $V[A]$*. We also identify* $(\mathrm{B}^*_{\mathrm{cyc}}V[A])[1] = \mathrm{B}^*_{\mathrm{cyc}}V[A+1]$ *in* $\mathrm{EB}^*_{\mathrm{cyc}}V[A]$*.*

Definition 3.3.4 (Pairing of tensor powers of bar complexes)**.** *For every* $A \in \mathbb{Z}$ *and* $k \in \mathbb{N}$*, we define the pairing as follows:*

$$\begin{aligned} (\mathrm{B}^*V[A])^{\otimes k} \otimes (\mathrm{B}V[A])^{\otimes k} &\longrightarrow \mathbb{R} \\ (\Psi_1 \otimes \cdots \otimes \Psi_k, W_1 \otimes \cdots \otimes W_k) &\longmapsto \underbrace{\psi_1(w_1) \dots \psi(w_k)}_{(\Psi_1 \otimes \cdots \otimes \Psi_k)(W_1 \otimes \cdots \otimes W_k) \coloneqq}. \end{aligned} \tag{3.26}$$

This means that we evaluate elements from the left-hand side on the elements from the right-hand side in this way without any signs (see the discussion in Remark 3.1.4). We extend the pairing by 0 *if the number of* Ψ_i*'s and the number of* W_i*'s differ.*

Remark 3.3.5 (Dual bar complex and dual of the bar complex). Because the pairing (3.26) is non-degenerate, we can embed the space on the left into the the linear dual of the space on the right. From Definition 3.3.1 we have $\mathrm{B}^*_{\mathrm{cyc}}V \subset \mathrm{B}^*V$, and $\mathrm{B}^{\mathrm{cyc}}V$ is identified with

$\operatorname{im}\iota \subset \mathrm{B}V$. Therefore, we can restrict (3.26) to obtain the pairing of $\mathrm{B}^*_{\mathrm{cyc}}V$ and $\mathrm{B}^{\mathrm{cyc}}V$. It is easy to see that for any $\psi \in \mathrm{B}^*_{\mathrm{cyc}}V$ and any generating word $v_1 \dots v_k \in \mathrm{B}^{\mathrm{cyc}}V$, we have

$$\psi(v_1 \dots v_k) = \psi(v_1 \otimes \cdots \otimes v_k).$$

The subspace of $(\mathrm{B}^{\mathrm{cyc}}V)^{*\mathrm{lin}}$ corresponding to $\mathrm{B}^*_{\mathrm{cyc}}V$ is then precisely $(\mathrm{B}^{\mathrm{cyc}}V)^{*\mathrm{wg}}$.

More generally, for every $k \in \mathbb{N}$, the spaces $\mathrm{E}_k\mathrm{B}^*_{\mathrm{cyc}}V$ and $\mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V$ are embedded into $(\mathrm{B}^*_{\mathrm{cyc}}V[1])^{\otimes k}$ and $(\mathrm{B}^{\mathrm{cyc}}V[1])^{\otimes k}$, respectively, using ι and π from Definition 3.1.7. Therefore, the restriction of (3.26) gives the pairing of $\mathrm{E}_k\mathrm{B}^*_{\mathrm{cyc}}V$ and $\mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V$. It is easy to see that for any generating word $w_1 \cdots w_k \in \mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V$ and any $\psi_1 \cdots \psi_k \in \mathrm{E}_k\mathrm{B}^*_{\mathrm{cyc}}V$, we have

$$(\psi_1 \cdots \psi_k)(w_1 \cdots w_k) = \frac{1}{k!}\sum_{\sigma \in \mathbb{S}_k} \varepsilon(\sigma, w)\psi_1(w_{\sigma_1^{-1}}) \dots \psi_k(w_{\sigma_k^{-1}}).$$

The subspace of $(\mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V)^{*\mathrm{lin}}$ corresponding to $\mathrm{E}_k\mathrm{B}^*_{\mathrm{cyc}}V$ lies in $(\mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V)^{*\mathrm{wg}}$; it is equal to $(\mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V)^{*\mathrm{wg}}$, provided that V is finite-dimensional.[5] $\triangleleft$

The weight-graded vector spaces $\mathrm{B}V$ and $\mathrm{B}^{\mathrm{cyc}}V$ are canonically filtered by the filtration by weights (3.10). Their weight-graded duals B^*V and $\mathrm{B}^*_{\mathrm{cyc}}V$ are filtered by the dual filtrations and the exterior powers $\mathrm{E}_k\mathrm{B}^*V$ and $\mathrm{E}_k\mathrm{B}^*_{\mathrm{cyc}}V$ by the induced filtration from Definition 3.1.8.

Proposition 3.3.6 (Completed dual cyclic bar complex)**.** *Let V be a graded vector space and $A \in \mathbb{Z}$. The filtration of $\mathrm{B}^*_{\mathrm{cyc}}V$ dual to the weight-filtration of $\mathrm{B}^{\mathrm{cyc}}V$ is $\mathbb{Z}$-gapped, Hausdorff, decreasing and bounded from above. Moreover, the following holds:*

$$\dim(V) < \infty \quad \Longrightarrow \quad (WG1)\ \&\ (WG2)\ \textit{are satisfied.}$$

The same holds for the induced filtration of $\mathrm{E}_k\mathrm{B}^*_{\mathrm{cyc}}V[A]$.

In the sense of Remark 3.3.5, we have

$$\hat{\mathrm{B}}^*_{\mathrm{cyc}}V \simeq (\mathrm{B}^{\mathrm{cyc}}V)^{*\mathrm{g}} \quad \textit{and} \quad \hat{\mathrm{E}}_k\mathrm{B}^*_{\mathrm{cyc}}V[A] \subset (\mathrm{E}_k\mathrm{B}^{\mathrm{cyc}}V[A+1])^{*\mathrm{g}},$$

where "=" holds if V is finite-dimensional.

The filtration degree *of $\Psi \in \hat{\mathrm{E}}_m\mathrm{B}^*_{\mathrm{cyc}}V[A]$ satisfies*

$$\|\Psi\| = \min\{k \in \mathbb{N}_0 \mid \exists W \in (\mathrm{E}_m\mathrm{B}^{\mathrm{cyc}}V[A])_k : \Psi(W) \neq 0\}.$$

Proof. The proof is clear. $\square$

---

[5] The problem is that if $\dim(V) = \infty$, then $(V \otimes V)^* \neq V^* \otimes V^*$.

Definition 3.3.7 (Cyclic A_∞-algebra)**.** *A graded vector space V together with a pairing*

$$\Pi : V[1] \otimes V[1] \to \mathbb{R}$$

of degree $d \in \mathbb{Z}$ and a collection of homogenous linear maps

$$\mu_k : V[1]^{\otimes k} \to V[1] \quad \text{for } k \geq 1$$

is called a cyclic A_∞-algebra of degree d *if the following conditions are satisfied:*

(1) The pairing Π is non-degenerate and graded antisymmetric; i.e., we have

$$\Pi(v_1, v_2) = (-1)^{1+|v_1||v_2|}\Pi(v_2, v_1) \quad \text{for all } v_1, v_2 \in V[1].$$

(2) The degrees satisfy $|\mu_k| = 1$ for all $k \geq 1$.

(3) The A_∞-relations *are satisfied: for all $k \geq 1$, we have*

$$\sum_{\substack{k_1,k_2 \geq 1 \\ k_1+k_2=k+1}} \sum_{p=1}^{k_1} \mu_{k_1} \circ_1^p \mu_{k_2} = 0, \tag{3.27}$$

where for all $p = 1, \ldots, k$ and $v_1, \ldots, v_k \in V[1]$ we define

$$\begin{aligned}(\mu_{k_1} \circ_1^p \mu_{k_2})(v_1, \ldots, v_k) \coloneqq (-1)^{|v_1|+\cdots+|v_{p-1}|}\mu_{k_1}(v_1, \ldots, v_{p-1}, \\ \mu_{k_2}(v_p, \ldots, v_{p+k_2-1}), v_{p+k_2} \ldots, v_k).\end{aligned}$$

(4) The operations $\mu_k^+ : V[1]^{\otimes k+1} \to \mathbb{R}$ defined by

$$\mu_k^+ \coloneqq \Pi \circ (\mu_k \otimes \mathbb{1})$$

for all $k \geq 1$ are cyclic symmetric; i.e., we have

$$\mu_k^+ \circ t_{k+1} = \mu_k^+.$$

We denote by $\tilde{\Pi} : V \otimes V \to \mathbb{R}$ and $\tilde{\mu}_k : V^{\otimes k} \to \mathbb{R}$ the operations before the degree shift; i.e., for all $k \geq 1$ and $\tilde{v}_1, \ldots, \tilde{v}_k \in V$ with $v_i = \theta\tilde{v}_i$, we have

$$\tilde{\Pi}(\tilde{v}_1, \tilde{v}_2) \coloneqq (-1)^{\tilde{v}_1}\Pi(v_1, v_2) \quad \text{and}$$

$$\tilde{\mu}_k(\tilde{v}_1, \ldots, \tilde{v}_k) \coloneqq \varepsilon(\theta, \tilde{v})\mu_k(v_1, \ldots, v_k).$$

We define $\tilde{\mu}_k^+ : V^{\otimes k+1} \to \mathbb{R}$ *similarly.*

If $\mu_k \equiv 0$ *for all* $k \geq 2$*, then* (V, Π, μ_1) *is called a* cyclic cochain complex*. If* $\mu_k \equiv 0$ *for all* $k \geq 3$*, then* (V, Π, μ_1, μ_2) *is called a* cyclic dga*. We use the same terminology but omit "cyclic" if there is no pairing* Π *and 1) and 4) are thus irrelevant.*

Remark 3.3.8 (A difference in sign conventions)*.* Our definition of μ_k^+ differs from the definition of m_k^+ in [CFL15, Definition 12.1] by a sign. To compensate this, we have to add this artificial sign in the definitions of Maurer-Cartan elements later; e.g., in Definition 3.4.2 or in the formula (A.12). $\lhd$

Definition 3.3.9 (Cyclic (co)homology of an A_∞-algebra)**.** *Let* $\mathcal{A} = (V, (\mu_k))$ *be an* A_∞*-algebra. For every* $k \geq 1$*, we consider the maps* b'_k*,* $R_k : V[1]^{\otimes k} \to \mathrm{B}V$ *given by*

$$\begin{aligned} \mathrm{b}'_k &\coloneqq \sum_{j=1}^{k} \sum_{i=0}^{k-j} t_{k-j+1}^{i} \circ (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^{-i} \quad \text{and} \\ R_k &\coloneqq \sum_{j=2}^{k} \sum_{i=1}^{j-1} (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^{i}, \end{aligned} \tag{3.28}$$

respectively, and define the following maps $\mathrm{B}V \to \mathrm{B}V$*:*

$$\mathrm{b}' \coloneqq \sum_{k=1}^{\infty} \mathrm{b}'_k, \quad R \coloneqq \sum_{k=2}^{\infty} R_k \quad \text{and} \quad \mathrm{b} \coloneqq \mathrm{b}' + R.$$

We denote by $\mathrm{b}^* : \hat{\mathrm{B}}^* V = (\mathrm{B}V)^{*_\mathrm{g}} \to \hat{\mathrm{B}}^* V$ *the dual map to* $\mathrm{b} : \mathrm{B}V \to \mathrm{B}V$*. The following holds:*[6]

$$|\mathrm{b}| = 1 \;(|\mathrm{b}^*| = -1), \quad \mathrm{b} \circ \mathrm{b} = 0 \quad \text{and} \quad \mathrm{b}(1-t) = (1-t)\mathrm{b}'. \tag{3.29}$$

From the last equation we see that b *restricts to* $\mathrm{B}^{\mathrm{cyc}} V = \mathrm{B}V / \operatorname{im}(1-t)$*. We define the following graded vector spaces:*

$$\begin{aligned} D(V) &\coloneqq r(\mathrm{B}V)[1], & D^*(V) &\coloneqq r(\hat{\mathrm{B}}^* V)[1], \\ D^\lambda(V) &\coloneqq r(\mathrm{B}^{\mathrm{cyc}} V)[1], & D^*_\lambda(V) &\coloneqq r(\hat{\mathrm{B}}^*_{\mathrm{cyc}} V)[1], \end{aligned}$$

where r *denotes the grading reversal. For instance, we have*

$$D^q_\lambda(V) = r(\hat{\mathrm{B}}^*_{\mathrm{cyc}} V)^{q+1} = (\hat{\mathrm{B}}^*_{\mathrm{cyc}} V)^{-q-1} \quad \text{for all } q \in \mathbb{Z}.$$

Then $(D(V), \mathrm{b})$ *and* $(D^\lambda(V), \mathrm{b})$ *are chain complexes and* $(D^*(V), \mathrm{b}^*)$ *and* $(D^*_\lambda(V), \mathrm{b}^*)$

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[6] The facts (3.29), in some form, are generally known; see [Mes16] or [Laz03]. We prove them in our setting in Appendix B.

the dual cochain complexes, respectively. We define the following (co)homologies:

$$\begin{aligned} \mathrm{HH}(\mathcal{A};\mathbb{R}) &\coloneqq \mathrm{H}(D(V),\mathrm{b}), & \mathrm{HH}^*(\mathcal{A};\mathbb{R}) &\coloneqq \mathrm{H}(D^*(V),\mathrm{b}^*), \\ \mathrm{H}^\lambda(\mathcal{A};\mathbb{R}) &\coloneqq \mathrm{H}(D^\lambda(V),\mathrm{b}), & \mathrm{H}^*_\lambda(\mathcal{A};\mathbb{R}), &\coloneqq \mathrm{H}(D^*_\lambda(V),\mathrm{b}^*). \end{aligned}$$

We call HH *the* Hochschild homology *and* H^λ *the* cyclic homology *of the* A_∞*-algebra* $\mathcal{A}$*. We call* HH^* *the* Hochschild cohomology *and* H^*_λ *the* cyclic cohomology *of* $\mathcal{A}$*.*

For a dga $\mathcal{A} = (V, \mu_1, \mu_2)$, we have for all $v_1, \ldots, v_k \in V[1]$ the formula

$$\begin{aligned} \mathrm{b}(v_1 \ldots v_k) = &\sum_{i=1}^{k} (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \ldots \mu_1(v_i) \ldots v_k \\ &+ \sum_{i=1}^{k-1} (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \ldots \mu_2(v_i, v_{i+1}) \ldots v_k \\ &+ (-1)^{|v_k|(|v_1|+\cdots+|v_{k-1}|)} \mu_2(v_k, v_1) v_2 \ldots v_{k-1}. \end{aligned}$$

Definition 3.3.10 (Strict units and strict augmentations)**.** *Let* $\mathcal{A} = (V, (\mu_k))$ *be an* A_∞*-algebra. A non-zero homogenous element* $\mathbb{1} \in V[1]$ *with* $|\mathbb{1}| = -1$ *is called a* strict unit *for* $\mathcal{A}$ *if the following holds:*

$$\begin{aligned} \mu_2(\mathbb{1}, v) = (-1)^{|v|+1}\mu_2(v, \mathbb{1}) &= v & &\forall v \in V[1], \\ \mu_k(v_1, \ldots, v_{i-1}, \mathbb{1}, v_{i+1}, \ldots, v_k) &= 0 & &\forall\ k \neq 2,\ 1 \leq i \leq k,\ v_j \in V[1]. \end{aligned}$$

The pair $(\mathcal{A}, \mathbb{1})$ *is called a* strictly unital A_∞-algebra.

A strictly unital A_∞*-algebra* $(\mathcal{A}, \mathbb{1})$ *is called* strictly augmented *if it is equipped with a linear map* $\varepsilon : V[1] \to \mathbb{R}[1]$ *which satisfies*

$$\varepsilon(\mathbb{1}_V) = \mathbb{1}_\mathbb{R}, \quad \varepsilon \circ \mu_1 = 0 \quad \textit{and} \quad \varepsilon \circ \mu_2 = \mu_2 \circ (\varepsilon \otimes \varepsilon),$$

where $\mathbb{1}_\mathbb{R}$ *is the strict unit for* $\mathbb{R}$ *endowed with the standard multiplication. The map* ε *is called a* strict augmentation..

If the homological dga $\mathrm{H}(\mathcal{A}) \coloneqq (\mathrm{H}(V, \tilde{\mu}_1), \mu_1 \equiv 0, \mu_2)$ *of* $\mathcal{A}$ *is strictly unital and strictly augmented, then* $\mathcal{A}$ *is called* homologically unital *and* homologically augmented*, respectively. A strictly unital and strictly augmented cochain complex* $(V, \mu_1, \mathbb{1}, \varepsilon)$ *is called just augmented.*

We denote by $u : \mathbb{R}[1] \to V[1]$ the injective linear map defined by $u(\mathbb{1}_\mathbb{R}) \coloneqq \mathbb{1}_V$, and by $u^* : \mathrm{B}^*_{\mathrm{cyc}}V \to \mathrm{B}^*_{\mathrm{cyc}}\mathbb{R}$ and $\varepsilon^* : \mathrm{B}^*_{\mathrm{cyc}}\mathbb{R} \to \mathrm{B}^*_{\mathrm{cyc}}V$ the precompositions with $u^{\otimes k}$ and $\varepsilon^{\otimes k}$ in every weight-k component, respectively.

Remark 3.3.11 (On units and augmentations). (i) A strict unit 1_V for $\mathcal{A}$ induces an A_∞-morphism $(u_k) : \mathbb{R} \to V$ given by $u_1(1_\mathbb{R}) \coloneqq 1_V$ and $u_k \equiv 0$ for all $k \geq 2$. A (general) augmentation of $(\mathcal{A}, 1_V)$ is by definition any A_∞-morphism $(\varepsilon_k) : V \to \mathbb{R}$ such that $(\varepsilon_k) \circ (u_k) = \mathbb{1}$ as A_∞-morphisms (see [Kel01]). Strict augmentations are precisely the maps ε_1 coming from augmentations (ε_k) with $\varepsilon_k \equiv 0$ for all $k \geq 2$.

(ii) As for $(V, \mu_1, 1, \varepsilon)$, we need the chain map ε to provide the splitting of the short exact sequence of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}[1] \arrow[r, hook, "u"] & V[1] \arrow[r, two heads] \arrow[l, bend left, "\varepsilon"] & \operatorname{coker}(u) \arrow[r] & 0, \end{tikzcd}$$

so that we get $\mathrm{H}(V) \simeq \mathrm{H}_{\mathrm{red}}(V) \oplus \mathbb{R}$, where $\mathrm{H}_{\mathrm{red}}(V) \coloneqq \mathrm{H}(\operatorname{coker}(u))$. If (V, μ_1) is non-negatively graded and we are given an injective chain map $u : \mathbb{R}[1] \to V[1]$ ($\eqqcolon$ the classical augmentation), then one can show that such ε always exists. ◁

Definition 3.3.12 (Reduced dual cyclic bar complex). *Let* $(\mathcal{A}, 1)$ *be a strictly unital* A_∞*-algebra. Consider the injection* $\iota_1 : \mathrm{B}V \to \mathrm{B}V$, $v_1 \otimes \cdots \otimes v_k \mapsto 1 \otimes v_1 \otimes \cdots \otimes v_k$. *We define the* reduced dual cyclic bar-complex *by*

$$\mathrm{B}^*_{\mathrm{cyc,red}}V \coloneqq \{\psi \in \mathrm{B}^*_{\mathrm{cyc}}V \mid \psi \circ \iota_1 = 0\}.$$

Under the assumption of strict unitality, b^* *preserves* $\mathrm{B}^*_{\mathrm{cyc,red}}V$*, and hence we can consider the reduced cyclic cochain complex*

$$D^*_{\lambda,\mathrm{red}}(V) \coloneqq r(\hat{\mathrm{B}}^*_{\mathrm{cyc,red}}V)[1]$$

and define the reduced cyclic cohomology of $\mathcal{A}$ *by*

$$\mathrm{H}^*_{\lambda,\mathrm{red}}(\mathcal{A}; \mathbb{R}) \coloneqq \mathrm{H}(D^*_{\lambda,\mathrm{red}}(V), \mathrm{b}^*).$$

Proposition 3.3.13 (Reduction to the reduced cyclic cohomology). *Let* $\mathcal{A} = (V, (\mu_k))$ *be an* A_∞*-algebra with a strict unit* 1 *and a strict augmentation* ε*. Then the inclusions* $\mathrm{B}^*_{\mathrm{cyc,red}}V$, $\varepsilon^*(\mathrm{B}^*_{\mathrm{cyc}}\mathbb{R}) \subset \mathrm{B}^*_{\mathrm{cyc}}V$ *induce the decomposition*

$$\mathrm{H}^*_\lambda(\mathcal{A}; \mathbb{R}) \simeq \mathrm{H}^*_{\lambda,\mathrm{red}}(\mathcal{A}; \mathbb{R}) \oplus \mathrm{H}^*_\lambda(\mathbb{R}; \mathbb{R}).$$

Here we have

$$\mathrm{H}^q_\lambda(\mathbb{R};\mathbb{R}) = \begin{cases} \langle 1^{q+1*} \rangle & \textit{for } q \geq 0 \textit{ even,} \\ 0 & \textit{for } q > 0 \textit{ odd and } q < 0, \end{cases}$$

where $1^{i*} : \mathbb{R}[1]^{\otimes i} \to \mathbb{R}$ *is defined by* $1^{i*}(1^i) \coloneqq 1$.

Sketch of the proof. The maps $\varepsilon^* : D_\lambda(\mathbb{R}) \to D_\lambda(V)$ and $u^* : D_\lambda(V) \to D_\lambda(\mathbb{R})$ are chain maps with $u^* \circ \varepsilon^* = \mathbb{1}$. Therefore, we have the sequence of cochain complexes

$$\begin{tikzcd} 0 \arrow[r] & D_{\lambda,\mathrm{red}}(V) \arrow[r, hook] & D_\lambda(V) \arrow[r, two heads, "u^*"] & D_\lambda(\mathbb{R}) \arrow[l, bend left, "\varepsilon^*"] \arrow[r] & 0, \end{tikzcd} \tag{3.30}$$

which is exact everywhere except for the middle, and where ε^* is a splitting map. The idea of [Lod92] is to replace these cochain complexes with quasi-isomorphic bicomplexes consisting of normalized Hochschild cochains $\bar{D}(V)$ such that the sequence becomes exact. The work then reduces to proving that $\bar{D}(V)$ computes $\mathrm{HH}(\mathcal{A};\mathbb{R})$; a variant of this result for A_∞-algebras was proven in [Laz03]. See Appendix B for the full proof. $\square$

We will now compare our version of the cyclic cohomology of a dga (V, μ_1, μ_2) to the version from [Lod92, Section 5]. In order to do this, we have to undo the degree shift $V[1]$ first since it is not considered in [Lod92].

Let $\tilde{\mathrm{b}}$, $\tilde{\delta} : \bar{T}V \to \bar{T}V$ be the linear maps defined for all $\tilde{v}_1, \ldots, \tilde{v}_k \in V$ by

$$\begin{aligned} \tilde{\mathrm{b}}(\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_k) &\coloneqq \sum_{i=1}^{k-1} (-1)^{i-1} \tilde{v}_1 \otimes \cdots \otimes \tilde{\mu}_2(\tilde{v}_i, \tilde{v}_{i+1}) \otimes \cdots \otimes \tilde{v}_k \\ &\qquad + (-1)^{k-1+\tilde{v}_k(\tilde{v}_1+\cdots+\tilde{v}_{k-1})} \tilde{\mu}_2(\tilde{v}_k, \tilde{v}_1) \otimes \tilde{v}_2 \otimes \cdots \otimes \tilde{v}_{k-1}, \\ \tilde{\delta}(\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_k) &\coloneqq \sum_{i=1}^{k} (-1)^{\tilde{v}_1+\cdots+\tilde{v}_{i-1}} \tilde{v}_1 \otimes \cdots \otimes \tilde{\mu}_1(\tilde{v}_i) \otimes \cdots \otimes \tilde{v}_k. \end{aligned}$$

For all $q \geq 0$, we define

$$\tilde{D}_q(V) \coloneqq \bigoplus_{\substack{k \geq 1 \\ d \in \mathbb{Z} \\ k-d=q+1}} (V^{\otimes k})^d \tag{3.31}$$

and $\tilde{\partial} : \tilde{D}_{q+1}(V) \to \tilde{D}_q(V)$ by

$$\tilde{\partial}(\tilde{v}_1 \cdots \tilde{v}_k) = \tilde{\mathrm{b}}(\tilde{v}_1 \cdots \tilde{v}_k) + (-1)^{k+1} \tilde{\delta}(\tilde{v}_1 \cdots \tilde{v}_k).$$

It can be checked that $\tilde{\partial} \circ \tilde{\partial} = 0$ and $\tilde{\partial}(\mathrm{im}(1 - \tilde{t})) \subset \mathrm{im}(1 - \tilde{t})$, so that $\tilde{\partial}$ induces a boundary operator on the chain complexes

$$\tilde{D}(V) := \bigoplus_{q\in\mathbb{Z}} \tilde{D}_q(V) \quad \text{and} \quad \tilde{D}^\lambda(V) := \tilde{D}(V)/\operatorname{im}(1-\tilde{t}).$$

Here, we have $\tilde{t}(\tilde{v}_1\cdots\tilde{v}_k) := (-1)^{k+|\tilde{v}_k|(|\tilde{v}_1|+\cdots+|\tilde{v}_{k-1}|)}\tilde{v}_k\tilde{v}_1\cdots\tilde{v}_{k-1}$.

We call $(\tilde{D}(V),\tilde{\partial})$ the *non-degree-shifted Hochschild complex* and $(\tilde{D}^\lambda(V),\tilde{\partial})$ the *non-degree-shifted cyclic complex* of the dga (V,μ_1,μ_2). We denote their homologies by $\mathrm{HH}^{\mathrm{nds}}(V)$ and $\mathrm{H}^{\lambda,\mathrm{nds}}(V)$, respectively.

Looking at (3.31), the chain complex $(\tilde{D}(V),\tilde{\partial})$ is the total complex of the bicomplex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
{} & (V^{\otimes 3})^2 \arrow[l] \arrow[d, "\tilde{\mathrm{b}}"] & (V^{\otimes 3})^1 \arrow[l, "\tilde{\delta}"'] \arrow[d, "\tilde{\mathrm{b}}"] & (V^{\otimes 3})^0 \arrow[l, "\tilde{\delta}"'] \arrow[d, "\tilde{\mathrm{b}}"] \\
{} & (V^{\otimes 2})^2 \arrow[l] \arrow[d, "\tilde{\mathrm{b}}"] & (V^{\otimes 2})^1 \arrow[l, "-\tilde{\delta}"'] \arrow[d, "\tilde{\mathrm{b}}"] & (V^{\otimes 2})^0 \arrow[l, "-\tilde{\delta}"'] \arrow[d, "\tilde{\mathrm{b}}"] \\
{} & V^2 \arrow[l] & V^1 \arrow[l, "\tilde{\delta}"'] & V^0 \arrow[l, "\tilde{\delta}"']
\end{tikzcd} \tag{3.32}$$

with chain groups being the direct sums of the top-left/right-bottom diagonals. This differs from the bicomplex [Lod92, Equation (5.3.2.1)], whose total complex is used to define the Hochschild homology of V in [Lod92], by the reversed grading in degree. The convention of [Lod92] is namely $|\tilde{\mu}_1| = -1$, whereas ours is $|\tilde{\mu}_1| = 1$. The total complex of [Lod92] corresponds to the bottom-left/right-top diagonal in (3.32). Therefore, the homologies might differ!

We also warn the careful reader that the degree is called “weight” in [Lod92].

The next proposition shows that our $\mathrm{H}^\lambda(V)$ indeed computes $\mathrm{H}^{\lambda,\mathrm{nds}}(V)$.

Proposition 3.3.14 (Non-degree-shifted case). *Let $\mathcal{A} = (V,\mu_1,\mu_2)$ be a dga. Then the degree shift map*

$$\begin{aligned} U : \tilde{D}_q(V) &\longrightarrow D_q(V), \\ \tilde{v}_1\otimes\cdots\otimes\tilde{v}_k &\longmapsto \varepsilon(\theta,\tilde{v})v_1\otimes\cdots\otimes v_k, \end{aligned}$$

where we denote $v_i = \theta\tilde{v}_i$ for a formal symbol θ with $|\theta| = -1$, is an isomorphism of the chain complexes $(\tilde{D}(V),\tilde{\partial})$ and $(D(V),\mathrm{b})$, resp. $(\tilde{D}^\lambda(V),\tilde{\partial})$ and $(D^\lambda(V),\mathrm{b})$.

Proof. First of all, it holds $|\tilde{\mu}_j| = 2-j$ for every $j\geq 1$. For every j, k, $l \geq 1$ such that

$j+l\le k+1$ and for every $\tilde{v}_1,\,\dots,\,\tilde{v}_k\in V$, we compute

$$\begin{aligned}&[U^{-1}(\mathbb{1}^{l-1}\otimes\mu_j\otimes\mathbb{1}^{k-j-l+1})U](\tilde{v}_1\cdots\tilde{v}_k)\\&\quad=(-1)^{l-1+(j-2)(\tilde{v}_1+\dots+\tilde{v}_{l-1}+k-l-j+1)}\tilde{v}_1\cdots\tilde{v}_{l-1}\tilde{\mu}_j(\tilde{v}_l\cdots\tilde{v}_{l+j-1})\tilde{v}_{l+j}\cdots\tilde{v}_k,\\&[U^{-1}t_kU](\tilde{v}_1\cdots\tilde{v}_k)=(-1)^{k-1}\tilde{v}_1\cdots\tilde{v}_k,\end{aligned}$$

where we use the Koszul convention $(f_1\otimes f_2)(v_1\otimes v_2)=(-1)^{|f_2||v_1|}f_1(v_1)\otimes f_2(v_2)$. Using this, we obtain

$$\begin{aligned}U^{-1}\mathrm{b}'_kU&=\sum_{j=1}^{k}\sum_{i=0}^{k-1}(-1)^{i+j(i+k+1)}t^i_{k-j+1}(\tilde{\mu}_j\otimes\mathbb{1}^{k-j})t_k^{-i}\quad\text{and}\\U^{-1}R_kU&=\sum_{j=1}^{k}\sum_{i=1}^{j-1}(-1)^{(i+j)(k+1)}(\tilde{\mu}_j\otimes\mathbb{1}^{k-j})t^i_k.\end{aligned}$$

It is now easy to check that $U^{-1}\circ\mathrm{b}\circ U=\tilde{\partial}$.

If $k\in\mathbb{N}$ is a weight and $d\in\mathbb{Z}$ a degree such that $k-d-1=q$ for some $q\in\mathbb{Z}$, we have schematically $U:(k,d)\mapsto(k,d-k)=(k,-q-1)$. Therefore, U preserves the grading of chain complexes. This finishes the proof. □

Proposition 3.3.15 (Reduced cochains are complete in $0,1$-connected case)**.** *Suppose that $V=\bigoplus_{d\ge0}V^d$ is a non-negatively graded vector space with $V^0=\langle 1\rangle$ for some $1\in V$ (=: V is* connected*) and $V^1=0$ (=: V is* simply-connected*). Then for all $m\ge1$, we have*

$$\hat{\mathrm{E}}_m\mathrm{B}^*_{\mathrm{cyc,red}}V=\mathrm{E}_m\mathrm{B}^*_{\mathrm{cyc,red}}V.$$

Proof. Let $\bar{V}\coloneqq\bigoplus_{d\ge2}V^d$. We clearly have $\mathrm{B}^*_{\mathrm{cyc,red}}V\simeq\mathrm{B}^*_{\mathrm{cyc}}\bar{V}$. Since $\bar{V}[1]$ is positively graded, we have $(\mathrm{B}\bar{V})^d_k=0$ whenever $k>d$. Therefore, a map $\Psi\in\hat{\mathrm{E}}_m\bar{V}$, which is non-zero only on finitely many homogenous components of $\mathrm{B}^{\mathrm{cyc}}V[1]^{\otimes m}$, will be non-zero only on finitely many weights. This implies that $\Psi\in\mathrm{E}_m\bar{V}$. □

Remark 3.3.16 (Universal coefficient theorem)*.* We have

$$\mathrm{B}V=\bigoplus_{d\in\mathbb{Z}}\bigoplus_{k=1}^{\infty}(V[1]^{\otimes k})^d\quad\text{and}\quad\mathrm{B}^*V=\bigoplus_{d\in\mathbb{Z}}\bigoplus_{k=1}^{\infty}(V[1]^{\otimes k})^{d*},$$

and hence

$$(\mathrm{B}V)^{*_{\mathrm{g}}}=\bigoplus_{d\in\mathbb{Z}}\prod_{k=1}^{\infty}(V[1]^{\otimes k})^{d*}=\bigoplus_{d\in\mathbb{Z}}\widehat{(\mathrm{B}^*V)^d}=\hat{\mathrm{B}}^*V.$$

Therefore, $(D^*_\lambda(V), \mathrm{b}^*)$ is dual to $(D^\lambda(V), \mathrm{b})$ as a chain complex. Now, because we work over $\mathbb{R}$, the universal coefficient theorem gives

$$\mathrm{H}^q_\lambda(\mathcal{A}, \mathrm{b}^*) \simeq [\mathrm{H}^\lambda_q(\mathcal{A}, \mathrm{b})]^* \quad \text{for all } q \in \mathbb{Z}.$$

Suppose that we have found closed homogenous elements $(w_i)_{i\in I} \subset D^\lambda(V)$ for some index set I which induce a basis of $\mathrm{H}^\lambda(\mathcal{A};\mathbb{R})$. For every $i \in I$, we define the linear map $w_i^* : D^\lambda(V) \to \mathbb{R}$ by prescribing

$$w_i^*(w_j) = \delta_{ij} \qquad \text{for all } j \in I$$

and $w_i^* \equiv 0$ on $\operatorname{im}\mathrm{b}$ and on a complement of $\ker(\mathrm{b})$ in $D^\lambda(V)$. Then $(w_i^*)_{i\in I} \subset D^*_\lambda(V)$ are closed homogenous elements which generate linearly independent cohomology classes in $\mathrm{H}^*_\lambda(\mathcal{A};\mathbb{R})$; if we denote $I_q \coloneqq \{i \in I \mid w_i \in C^\lambda_q(V)\}$, then we can write

$$\mathrm{H}^q_\lambda(\mathcal{A};\mathbb{R}) = \Big\{\sum_{i\in I_q} \alpha_i w_i^* \mid \alpha_i \in \mathbb{R}\Big\} \quad \text{for all } q \in \mathbb{Z}. \qquad \lhd$$

3.4. Canonical dIBL-structure on cyclic cochains

In this section, we will consider a cyclic dga (V, Π, m_1, m_2) of degree $2-n$ for some $n \in \mathbb{N}$ which is finite-dimensional.

For all v_1, v_2, $v_3 \in V[1]$, the following relations holds:

$$\text{cyc. dga}\begin{cases} \left.\begin{aligned} \Pi(v_1,v_2) &= (-1)^{1+|v_1||v_2|}\Pi(v_2,v_1),\\ m_1(m_1(v_1)) &= 0,\\ m_1^+(v_1,v_2) &= (-1)^{|v_1||v_2|}m_1^+(v_2,v_1),\end{aligned}\right\}\begin{array}{l}\text{cyc.}\\ \text{cochain}\\ \text{complex}\end{array}\\ \begin{aligned} m_1(m_2(v_1,v_2)) &= -m_2(m_1(v_1),v_2)\\ &\quad -(-1)^{|v_1|}m_2(v_1,m_1(v_2)),\\ m_2(m_2(v_1,v_2),v_3) &= (-1)^{|v_1|+1}m_2(v_1,m_2(v_2,v_3)),\\ m_2^+(v_1,v_2,v_3) &= (-1)^{|v_3|(|v_1|+|v_2|)}m_2^+(v_3,v_1,v_2).\end{aligned}\end{cases}\tag{3.33}$$

The facts (A) and (C) from the Overview apply, and we get the canonical dIBL-algebra $\mathrm{dIBL}(\mathrm{B}^*_{\mathrm{cyc}}V[2-n])$ of bidegree $(n-3, 2)$ and the canonical Maurer-Cartan element $\mathfrak{m} = (\mathfrak{m}_{10})$. We will denote

$$C(V) \coloneqq \mathrm{B}^*_{\mathrm{cyc}}V[2-n]$$

and call it the space of *cyclic cochains on V*. If V is fixed, we will write just C.

Definition 3.4.1 (Canonical dIBL-algebra)**.** *Let (V, Π, m_1) be a cyclic cochain complex of degree $2-n$ which is finite-dimensional. Let $(e_0, \dots, e_m) \subset V[1]$ be a basis of $V[1]$, and let $(e^0, \dots, e^m)$ be the dual basis with respect to Π; this means that*

$$\Pi(e_i, e^j) = \delta_{ij} \quad \textit{for all } i, j = 0, \dots, m.$$

We define the tensor $T = \sum_{i,j=0}^{m} T^{ij} e_i \otimes e_j \in V[1]^{\otimes 2}$ by[7]

$$T^{ij} = (-1)^{|e_i|}\Pi(e^i, e^j) \quad \textit{for all } i, j = 0, \dots, m. \tag{3.34}$$

The canonical dIBL-algebra *on $C(V)$ is the quadruple*

$$\mathrm{dIBL}(C(V)) \coloneqq (C(V), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}),$$

where the operations $\mathfrak{q}_{110}$, $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ are defined for all ψ, ψ_1, $\psi_2 \in \hat{\mathrm{B}}^*_{\mathrm{cyc}} V$ and generating words $w = v_1 \dots v_k$, $w_1 = v_{11} \dots v_{1k_1}$, $w_2 = v_{21} \dots v_{2k_2} \in \mathrm{B}^{\mathrm{cyc}} V$ with k, k_1, $k_2 \geq 1$ as follows:

- *The* dIBL-boundary operator $\mathfrak{q}_{110} : \hat{\mathrm{E}}_1 C \to \hat{\mathrm{E}}_1 C$ *of degree $|\mathfrak{q}_{110}| = -1$ is defined by*

$$\mathfrak{q}_{110}(\mathrm{s}\psi)(\mathrm{s}w) \coloneqq \mathrm{s}\sum_{i=1}^{k}(-1)^{|v_1|+\dots+|v_{i-1}|}\psi(v_1 \dots v_{i-1} m_1(v_i) v_{i+1} \dots v_k). \tag{3.35}$$

- *The* product $\mathfrak{q}_{210} : \hat{\mathrm{E}}_2 C \longrightarrow \hat{\mathrm{E}}_1 C$ *of degree $|\mathfrak{q}_{210}| = -2(n-3)-1$ is written schematically as*

  $$\mathfrak{q}_{210}(\mathrm{s}^2\psi_1 \otimes \psi_2)(\mathrm{s}w) \coloneqq \sum \varepsilon(w \mapsto w^1 w^2)(-1)^{|e_j||w^1|} T^{ij} \psi_1(e_i w^1)\psi_2(e_j w^2)$$

  and defined "algorithmically" as follows:

  For every cyclic permutation $\sigma \in \mathbb{S}_k$, consider the tensor

  $$\sigma(w) \coloneqq \varepsilon(\sigma, w) v_{\sigma_1^{-1}} \otimes \dots \otimes v_{\sigma_k^{-1}},$$

  and split it into two parts w^1 and w^2 of possibly zero length such that $v_{\sigma_1^{-1}} \otimes \dots \otimes v_{\sigma_k^{-1}} = w^1 \otimes w^2$. Feed w^1 and w^2 into ψ_1 and ψ_2 preceded by e_i and e_j, respectively, and multiply the result with the sign $(-1)^{|e_j||w^1|}$, which is the Koszul sign to order

  $$e_i e_j w^1 w^2 \longmapsto e_i w^1 e_j w^2.$$

---

[7] See Appendix A for the invariant meaning of T as the Schwartz kernel of $\pm\mathbb{1}$.

Finally, sum over all $\sigma \in \mathbb{S}_k$, all splittings of $\sigma(w)$ and all indices $i, j = 0, \ldots, m$. The sign $\varepsilon(\sigma, w)$ is denoted by $\varepsilon(w \mapsto w^1 w^2)$ to indicate the splitting.

- *The* coproduct $\mathfrak{q}_{120} : \hat{\mathrm{E}}_1 C \longrightarrow \hat{\mathrm{E}}_2 C$ *of degree* $|\mathfrak{q}_{120}| = -1$ *is written schematically as*

  $$\begin{aligned}&\mathfrak{q}_{120}(\mathrm{s}\psi)(\mathrm{s}^2 w_1 \otimes w_2)\\ &\qquad = \frac{1}{2} \sum \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1)(-1)^{|e_j||w_1^1|} T^{ij} \psi(e_i w_1^1 e_j w_2^1)\end{aligned}$$

  and defined "algorithmically" as follows:

  For all cyclic permutations $\sigma \in \mathbb{S}_{k_1}$ and $\mu \in \mathbb{S}_{k_2}$, denote $w_1^1 := \sigma(w_1)$ and $w_2^1 := \mu(w_2)$ and let $\varepsilon(w_1 \mapsto w_1^1)$ and $\varepsilon(w_2 \mapsto w_2^1)$ be the corresponding Koszul signs, respectively. Feed w_1^1 and w_2^1 into ψ in the indicated order interleaved by e_i and e_j and multiply the result with the sign $(-1)^{|e_j||w_1^1|}$, which is the Koszul sign to order

  $$e_i e_j w_1^1 w_2^1 \mapsto e_i w_1^1 e_j w_2^1.$$

  Finally, sum over all $\sigma \in \mathbb{S}_{k_1}$, $\mu \in \mathbb{S}_{k_2}$ and all indices $i, j = 0, \ldots, m$.

The operations are extended continuously to the completion.

Definition 3.4.2 (Canonical Maurer-Cartan element)**.** *Let (V, Π, m_1, m_2) be a finite-dimensional cyclic dga of degree $2 - n$. The* canonical Maurer-Cartan element $\mathfrak{m}$ *for* dIBL$(C(V))$ *consists of only one element $\mathfrak{m}_{10} \in \hat{\mathrm{E}}_1 C$ of degree $|\mathfrak{m}_{10}| = 2(n-3)$ which is defined by*

$$\mathfrak{m}_{10}(\mathrm{s} v_1 v_2 v_3) := (-1)^{n-2} \mu_2^+(v_1, v_2, v_3) \quad \textit{for all } v_1, v_2, v_3 \in V[1]$$

on the weight-three component of $\mathrm{B}^{\mathrm{cyc}} V[3-n]$ *and extended by* 0 *to other weight-k components.*

Remark 3.4.3 (On canonical dIBL-structure). (i) Elements of the completion $\hat{C}(V)$ which are not in $C(V)$ will be called *long cyclic cochains.* Because there are no infinite sums in Definition 3.4.1, dIBL(C) is completion-free. Clearly, the twist dIBL$^{\mathfrak{n}}(C)$ remains completion-free as long as $\mathfrak{n}_{lg} \in \mathrm{E}_l C$ for all l, g.

(ii) The constructions of $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ do not depend on the choice of a basis and can be rephrased in terms of summation over ribbon graphs (see Example A.1.5).

(iii) According to Proposition 3.3.6, the filtration on $C(V)$ satisfies (WG1) & (WG2). Thus, the IBL-structures IBL$(\mathbb{H}(C))$ and IBL$(\mathbb{H}^{\mathfrak{m}}(C))$ are well-defined (see Definition 3.2.8). ◁

Proposition 3.4.4 (Formulas for twisted operations)**.** *Let* $\mathrm{dIBL}(C(V))$ *be the canonical* dIBL*-algebra for a finite-dimensional cyclic cochain complex* (V,Π,m_1) *of degree* $2-n$*, and let* $\mathfrak{n}=(\mathfrak{n}_{lg})$ *be a Maurer-Cartan element. Then for all* $l\geq 1$*,* $g\geq 0$*,* $\Psi\in\hat{\mathrm{B}}^*_{\mathrm{cyc}}V[3-n]$ *and generating words* $W_1,\,\dots,\,W_l\in\mathrm{B}^{\mathrm{cyc}}V[3-n]$*, we have*

$$\begin{aligned}&[(\mathfrak{q}_{210}\circ_1\mathfrak{n}_{lg})(\Psi)](W_1\otimes\dots\otimes W_l)\\&\quad=\sum_{j=1}^{l}\sum\varepsilon'\varepsilon(w_j\mapsto w_j^1w_j^2)T^{ab}\Psi(\mathrm{s}e_aw_j^1)\mathfrak{n}_{lg}(W_1\otimes\cdots W_{j-1}\otimes(\mathrm{s}e_bw_j^2)\\&\qquad\qquad\otimes W_{j+1}\otimes\dots\otimes W_l),\end{aligned}\tag{3.36}$$

where the sum without limits is the sum in Definition 3.4.1 for $\mathfrak{q}_{210}$ *and* ε' *is the Koszul sign of the following operation:*

$$\begin{aligned}&(\mathrm{s}e_ae_b)\,W_1\dots W_{j-1}(\mathrm{s}w_j^1w_j^2)\,W_{j+1}\dots W_l\\&\quad\longmapsto(\mathrm{s}e_aw_j^1)\,W_1\dots W_{j-1}(\mathrm{s}e_bw_j^2)\,W_{j+1}\dots W_l.\end{aligned}$$

In particular, for $l=1$*,* $g\geq 0$ *and* $W\in\mathrm{B}^{\mathrm{cyc}}V[3-n]$*, we have*

$$(\mathfrak{q}_{210}\circ_1\mathfrak{n}_{1g})(W)=(-1)^{n-3}\sum T^{ab}\varepsilon(w\mapsto w^1w^2)\mathfrak{n}_{1g}(\mathrm{s}e_aw^1)\psi(e_bw^2),\tag{3.37}$$

and for $l=2$*,* $g\geq 0$ *and* $W_1,\ W_2\in\mathrm{B}^{\mathrm{cyc}}V[3-n]$*, we have*

$$\begin{aligned}&[(\mathfrak{q}_{210}\circ_1\mathfrak{n}_{2g})(\Psi)](W_1\otimes W_2)\\&\quad=(-1)^{(n-3)(|\Psi|+1)}\Big[\sum T^{ab}\varepsilon(w_1\mapsto w_1^1w_1^2)(-1)^{|e_b||w_1^1|}\Psi(\mathrm{s}e_aw_1^1)\\&\qquad\mathfrak{n}_{20}(\mathrm{s}e_bw_1^2\otimes W_2)+(-1)^{|W_1||W_2|}\sum T^{ab}\varepsilon(w_2\mapsto w_2^1w_2^2)\\&\qquad\qquad(-1)^{|e_b||w_2^1|}\Psi(\mathrm{s}e_aw_2^1)\mathfrak{n}_{20}(\mathrm{s}e_bw_2^2\otimes W_1)\Big].\end{aligned}\tag{3.38}$$

Proof. Let us first discuss the completions. Given $\mathfrak{n}_{lg}\in\hat{\mathrm{E}}_lC$, we can write it as $\mathfrak{n}_{lg}=\sum_{i=1}^{\infty}\Phi_1^i\cdots\Phi_l^i$ with generating words $\Phi_1^i\cdots\Phi_l^i\in\mathrm{E}_lC$ of weights approaching ∞. The canonical extension of $\circ_h$ to maps with finite filtration degree commutes with convergent infinite sums, and hence we have $\mathfrak{q}_{klg}\circ_h\mathfrak{n}_{lg}=\sum_{i=1}^{\infty}\mathfrak{q}_{klg}\circ_h(\Phi_1^i\cdots\Phi_l^i)$. Therefore, it suffices to prove the formulas for generating words $\Phi_1^i\cdots\Phi_l^i\in\mathrm{E}_lC$.

From (3.14), we get for every Ψ, Φ_1, …, $\Phi_l\in C$ the equation

$$[\mathfrak{q}_{210}\circ_1(\Phi_1\cdots\Phi_l)](\Psi)=\sum_{i=1}^{l}(-1)^{|\Phi_i|(|\Phi_1|+\dots+|\Phi_{i-1}|)}\mathfrak{q}_{210}(\Psi,\Phi_i)\Phi_1\cdots\hat{\Phi}_i\cdots\Phi_l,$$

where $\Phi_1 \cdots \Phi_l$ on the left-hand-side is considered as a map $\mathrm{E}_0 C = \mathbb{R} \to \mathrm{E}_l C$ mapping 1 to $\Phi_1 \cdots \Phi_l$. For $\mathrm{W}_1, \ldots, \mathrm{W}_l \in \mathrm{B}^{\mathrm{cyc}} V[3-n]$ and $\sigma \in \mathbb{S}_l$, we use

$$[\sigma(\Phi_1 \otimes \cdots \otimes \Phi_l)](\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_l) = (\Phi_1 \otimes \cdots \otimes \Phi_l)[\sigma^{-1}(\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_l)]$$

and Definition 3.4.1 to get

$$\begin{aligned}
&\big([\mathfrak{q}_{210} \circ_1 (\Phi_1 \cdots \Phi_l)](\Psi)\big)(\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_l) = \\
&\quad = \sum_{i=1}^{l} (-1)^{|\Phi_i|(|\Phi_1| + \cdots + |\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in \mathbb{S}_l} \varepsilon(\sigma^{-1}, \mathrm{W}) [\mathfrak{q}_{210}(\Psi, \Phi_i)](\mathrm{W}_{\sigma_1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \Phi_1(\mathrm{W}_{\sigma_2}) \cdots \hat{\Phi}_i(\emptyset) \cdots \Phi_l(\mathrm{W}_{\sigma_l}) \\
&\quad = \sum_{i=1}^{l} (-1)^{|\Phi_i|(|\Phi_1| + \cdots + |\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in \mathbb{S}_l} \varepsilon(\sigma^{-1}, \mathrm{W}) (-1)^{|\mathrm{s}|\Psi} \\
&\qquad \sum \varepsilon(w_{\sigma_1} \mapsto w^1_{\sigma_1} w^2_{\sigma_1}) (-1)^{|e_b||w^1_{\sigma_1}|} T^{ab} \Psi(e_a w^1_{\sigma_1}) \Phi_i(e_b w^2_{\sigma_1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \Phi_1(\mathrm{W}_{\sigma_2}) \ldots \hat{\Phi}_i(\emptyset) \ldots \Phi_l(\mathrm{W}_{\sigma_l}) \\
&\quad =: (*),
\end{aligned}$$

where $\hat{\Phi}_i(\emptyset)$ means omission of the corresponding term. Consider the bijection

$$\begin{aligned}
I : \{1, \ldots, l\} \times \mathbb{S}_l &\longrightarrow \{1, \ldots, l\} \times \mathbb{S}_l \\
(i, \sigma) &\longmapsto \left(j := \sigma_1, \mu := \begin{pmatrix} 1 & \ldots & i-1 & i & i+1 & \ldots & l \\ \sigma_2 & \ldots & \sigma_i & \sigma_1 & \sigma_{i+1} & \ldots & \sigma_l \end{pmatrix} \right).
\end{aligned}$$

Given $(i, \sigma) \in \{1, \ldots, l\} \times \mathbb{S}_l$ and $b \in \{1, \ldots, m\}$, let $(j, \mu) := I(i, \sigma)$ and

$$\mathrm{W}' := \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_{j-1} \otimes (\mathrm{s} e_b w^2_j) \otimes \mathrm{W}_{j+1} \otimes \cdots \otimes \mathrm{W}_l.$$

Suppose that $(\Phi_1 \otimes \cdots \otimes \Phi_l)(\mathrm{W}') \neq 0$. We compute the Koszul sign $\varepsilon(\mu^{-1}, \mathrm{W}')$ in the following way:

$$\begin{aligned}
\mathrm{W}' &\mapsto (-1)^{(|w^1_j| + |e_b| + |\mathrm{W}_j|)(|\mathrm{W}_1| + \cdots + |\mathrm{W}_{j-1}|)} (\mathrm{s} e_b w^2_j) \mathrm{W}_1 \ldots \hat{\mathrm{W}}_j \ldots \mathrm{W}_l \\
&\mapsto \underbrace{(-1)^{(|w^1_j| + |e_b|)(|\mathrm{W}_1| + \cdots + |\mathrm{W}_{j-1}|)}}_{=:\varepsilon_1} \varepsilon(\sigma^{-1}, \mathrm{W}) (\mathrm{s} e_b w^2_j) \mathrm{W}_{\sigma_2} \ldots \mathrm{W}_{\sigma_l} \\
&\mapsto \underbrace{\varepsilon_1 \varepsilon(\sigma^{-1}, \mathrm{W}) (-1)^{|\Phi_i|(|\Phi_1| + \cdots + |\Phi_{i-1}|)}}_{=\varepsilon(\mu^{-1}, \mathrm{W}')} \underbrace{\mathrm{W}_{\sigma_2} \ldots \mathrm{W}_{\sigma_i} (\mathrm{s} e_b w^2_j) \mathrm{W}_{\sigma_{i+1}} \ldots \mathrm{W}_{\sigma_l}}_{=\mathrm{W}'_{\mu_1} \ldots \mathrm{W}'_{\mu_l}}.
\end{aligned}$$

Using this, we can rewrite $(*)$ as

$$\begin{aligned}(*) &= (-1)^{|s||\Psi|} \sum_{j=1}^{l} \sum \varepsilon(w_j \mapsto w_j^1 w_j^2)(-1)^{|e_b||w_j^1|} T^{ab} \Psi(e_a w_j^1) \\ &\qquad\qquad \varepsilon_1 \frac{1}{l!} \sum_{\mu \in \mathbb{S}_l} \varepsilon(\mu^{-1}, \mathrm{W}') \Phi_1(\mathrm{W}'_{\mu_1}) \dots \Phi_l(\mathrm{W}'_{\mu_l}) \\ &= \sum_{j=1}^{l} \sum \varepsilon(w_j \mapsto w_j^1 w_j^2)(-1)^{|s||\Psi| + |e_b||w_j^1| + (|w_j^1| + |e_b|)(|\mathrm{W}_1| + \dots + |\mathrm{W}_{j-1}|)} T^{ab} \\ &\qquad \Psi(\mathrm{s}e_a w_j^1)(\Phi_1 \cdots \Phi_l)(\mathrm{W}_1 \otimes \dots \otimes \mathrm{W}_{j-1} \otimes (\mathrm{s}e_b w_j^2) \otimes \mathrm{W}_{j+1} \otimes \dots \otimes \mathrm{W}_l).\end{aligned}$$

Finally, we use

$$T^{ab} \neq 0 \implies |e_a| + |e_b| = n - 2$$

to write

$$\begin{aligned}|s||\Psi| &= |s|(|w_j^1| + |e_a|) = (n-3)(|w_j^1| + n - 2 - |e_b|) \\ &= |s|(|w_j^1| + |e_b|) \mod 2,\end{aligned}$$

and the formula (3.36) follows.

As for (3.37), we first compute ε' for $l = 1$ as follows:

$$\begin{aligned}\ln_{-1}(\varepsilon') &= |w_1||e_b| + (|e_b| + |w_1|)|s| \underset{\substack{\uparrow \\ 2(n-3) = |\mathfrak{n}_{10}| = |s| + |e_b| + |w^2|}}{=} |w^1||w^2| + |s||e_b| \\ &\underset{\substack{\uparrow \\ |e_a| + |e_b| = |s| + 1}}{=} |w^1||w^2| + |e_a||e_b| \mod 2.\end{aligned}$$

Using this, we obtain

$$\begin{aligned}[(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{1g})(\Psi)](W) &= \sum \varepsilon' \varepsilon(w \mapsto w^1 w^2) T^{ab} \Psi(\mathrm{s}e_a w^1) \mathfrak{n}_{1g}(\mathrm{s}e_b w^2) \\ &\underset{\substack{\uparrow \\ T^{ab} = (-1)^{|s| + |e_a||e_b|} T^{ba} \\ \varepsilon(w \mapsto w^1 w^2) = (-1)^{|w^1||w^2|} \varepsilon(w \mapsto w^2 w^1)}}{=} (-1)^{|s|} \sum \varepsilon(w \mapsto w^2 w^1) T^{ba} \mathfrak{n}_{1g}(\mathrm{s}e_b w^2) \Psi(\mathrm{s}e_a w^1),\end{aligned}$$

which implies (3.37).

The proof of (3.38) is a combination of the same arguments. $\square$

We will now relate homology of the twisted boundary operator $\mathfrak{q}_{110}^{\mathfrak{n}}$ to cohomology of an A_∞-algebra on V induced by $\mathfrak{n}_{10}$.

Definition 3.4.5 (A_∞-operations and compatible Maurer-Cartan element)**.** *Suppose that (V, Π, m_1) is a finite-dimensional cyclic cochain complex of degree $2 - n$, and let*

$\mathfrak{n} = (\mathfrak{n}_{lg})$ *be a Maurer-Cartan element for* $\mathrm{dIBL}(C(V))$. *We define the operations* $\mu_k : V[1]^{\otimes k} \to V[1]$ *for all* $k \geq 1$ *by*

$$\mu_k(v_1, \ldots, v_k) := (-1)^{n-3} \sum_{i,j} T^{ij} \mathfrak{n}_{10}(\mathrm{s}e_i v_1 \ldots v_k) e_j$$

for all $v_1, \ldots, v_k \in V[1]$, *where* T^{ij} *is the matrix from Definition 3.4.1.*

If (V, Π, m_1, m_2) *is in addition a cyclic dga and* $\mathfrak{m}$ *the canonical Maurer-Cartan element for* $\mathrm{dIBL}(C(V))$, *then we say that* $\mathfrak{n}$ *is* compatible with $\mathfrak{m}$ *if*

$$\mathfrak{n}_{10}(\mathrm{s}v_1v_2v_3) = \mathfrak{m}_{10}(\mathrm{s}v_1v_2v_3) \quad \textit{for all } v_1, v_2, v_3 \in V[1].$$

Proposition 3.4.6 (Twisted boundary operator $\mathfrak{q}_{110}^{\mathfrak{n}}$ and A_∞-cyclic cohomology)**.** *In the setting of Definition 3.4.5, the triple* $\mathcal{A}_{\mathfrak{n}}(V) := (V, \Pi, (\mu_k))$ *is a cyclic* A_∞*-algebra. We always have* $\mu_1 = m_1$, *and if* $\mathfrak{n}$ *is compatible with* $\mathfrak{m}$ *for a cyclic dga* (V, Π, m_1, m_2), *then also* $\mu_2 = m_2$.

The following holds for the homologies:

$$\mathbb{H}^{\mathfrak{n}}(C(V)) = r(\mathrm{H}^*_\lambda(\mathcal{A}_{\mathfrak{n}}(V); \mathbb{R}))[3-n].$$

Proof. First of all, according to Definition 3.2.6 we must have $\|\mathfrak{n}_{10}\| > 2$, and hence

$$\mathfrak{n}_{10}(\mathrm{s}v_1v_2) = \mathfrak{n}_{10}(\mathrm{s}v_1) = 0 \quad \text{for all } v_1, v_2 \in V[1].$$

This implies $\mu_1 = m_1$.

Now, let $e_0, \ldots, e_m$ be a basis of $V[1]$ and let $e^0, \ldots, e^m$ be the dual basis with respect to Π. For all $k \geq 2$ and $v_1, \ldots, v_k \in V[1]$, we compute the following:

$$\begin{aligned}
&\Pi(\mu_k(v_1, \ldots, v_k), v_{k+1}) \\
&\quad = (-1)^{n-3} \sum_{ij} (-1)^{|e_i|} \Pi(e^i, e^j) \mathfrak{n}_{10}(\mathrm{s}e_i v_1 \ldots v_k) \Pi(e_j, v_{k+1}) \\
&\quad \underset{\substack{\uparrow \\ \forall v \in V[1]:\ \sum_j \Pi(v, e^j) e_j = v}}{=} (-1)^{n-3} \sum_{i} (-1)^{|e_i|} \mathfrak{n}_{10}(\mathrm{s}e_i v_1 \ldots v_k) \Pi(e^i, v_{k+1}) \\
&\quad \underset{\substack{\uparrow \\ (|v|_1+\cdots+|v_k|)|e_i| = \\ (|\mathfrak{n}_{10}|+|s|+|e_i|)|e_i| = (|s|+1)|e_i|}}{=} (-1)^{n-3} \sum_{i} (-1)^{|e_i|(n-3)} \mathfrak{n}_{10}(\mathrm{s}v_1 \ldots v_k e_i) \Pi(e^i, v_{k+1})
\end{aligned}$$

$$\begin{aligned}
&\underset{\substack{\uparrow \\ 1+|v_{k+1}||e^i|=1+(|e|^i+2-n)|e^i| \\ =1+(3-n)|e^i|=1+(3-n)|e_i|}}{=} (-1)^{n-2}\sum_i \mathfrak{n}_{10}(\mathrm{s}v_1\dots v_k e_i)\Pi(v_{k+1},e^i) \\
&= (-1)^{n-2}\mathfrak{n}_{10}(\mathrm{s}v_1\dots v_{k+1}).
\end{aligned}$$

Therefore, we have

$$\mathfrak{n}_{10} = (-1)^{n-2}\sum_{k\geq 2}\mu_k^+.$$

In this case, [CFL15, Proposition 12.3] asserts that the A_∞-relations (3.27) for $(\mu_k)_{k\geq 1}$ are equivalent to the "lowest" Maurer-Cartan equation (3.24) for $\mathfrak{n}_{10}$. The degree condition $|\mu_k| = 1$ and the cyclic symmetry of μ_k^+ are easy to check. Therefore, $\mathcal{A}_\mathfrak{n}(V)$ is a cyclic A_∞-algebra.

As for the compatibility with $\mathfrak{m}$, we have for all v_1, $v_2 \in V[1]$ the following:

$$\begin{aligned}
m_2(v_1,v_2) &= \sum_i \Pi(e_i, m_2(v_1,v_2))e^i \\
&\underset{\substack{\uparrow \\ T^{ij}=(-1)^{|e_i|}\Pi(e^i,e^j)}}{=} \sum_{i,j}(-1)^{|e_i|}T^{ij}\Pi(e_i,m_2(v_1,v_2))e_j \\
&\underset{\substack{\uparrow \\ \Pi(v_1,v_2)=(-1)^{1+(n-3)|v_1|}\Pi(v_2,v_1)}}{=} \sum_{i,j}(-1)^{1+(n-2)|e_i|}T^{ij}\underbrace{\Pi(m_2(v_1,v_2),e_i)}_{=(-1)^{n-2}\mathfrak{m}_{10}(\mathrm{s}v_1v_2e_i)}e_j \\
&\underset{\substack{\uparrow \\ (|v_1|+|v_2|)|e_i|=(|\mathfrak{m}_{10}|-|s|-|e_i|)|e_i| \\ =(n-2)|e_i|}}{=} (-1)^{n-3}\sum_{i,j}T^{ij}\mathfrak{m}_{10}(\mathrm{s}e_iv_1v_2)e_j \\
&= \mu_2(v_1,v_2).
\end{aligned}$$

We will now clarify the relation to the cyclic cohomology of $\mathcal{A}_\mathfrak{n}(V)$. Recall from Proposition 3.2.10 that $\mathfrak{q}^\mathfrak{n}_{110}(\Psi) = \mathfrak{q}_{110}(\Psi) + \mathfrak{q}_{210}(\mathfrak{n}_{10},\Psi)$ for $\Psi \in \hat{\mathrm{B}}^*_{\mathrm{cyc}}V[3-n]$, where the first term is given by (3.35) and the second by (3.37). Consider now b'^k and R^k from (3.28), whose sum gives the Hochschild boundary operator b. Using the cyclic symmetry, we can rewrite a summand of b'^k for $j = 1, \dots, k$ and $i = 0, \dots, k-j$ applied to a generating word $v_1 \dots v_k \in \mathrm{B}^{\mathrm{cyc}} V$ as follows:

$$\begin{aligned}
&[t_{k-j+1}^{i} \circ (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^{-i}](\underbrace{v_1 \dots v_k}_{=:w}) = \\
&\quad = (-1)^{|v_1|+\dots+|v_i|} v_1 \dots v_i \mu_j(v_{i+1} \dots v_{i+j}) v_{i+j+1} \dots v_k \qquad (3.39)\\
&\quad = \varepsilon(w \mapsto w^1 w^2) \mu_j (\underbrace{v_{i+1} \dots v_{i+j}}_{=:w^1}) \underbrace{v_{i+j+1} \dots v_k v_1 \dots v_i}_{=:w^2}
\end{aligned}$$

Clearly, summing (3.39) over $j = 1$ and $i = 0, \dots, k-1$ gives the dual to $\mathfrak{q}_{110}$. For $j = 2$, $\dots$, k, we can write (3.39) as

$$(-1)^{n-3} \sum_{i,j} \varepsilon(w \mapsto w^1 w^2) T^{ij} \mathfrak{n}_{10}(\mathrm{s}e_i w^1) e_j w^2.$$

Therefore, the sum over $j = 2, \dots, k$ and $i = 0, \dots, k-j$ gives the part of the dual to $\mathfrak{q}_{210}(\mathfrak{n}_{10}, \Psi)$ corresponding to the cyclic permutations $\sigma \in \mathbb{S}_k$ with $\sigma_1 = 1, j+1, \dots, k$. The rest, i.e., the cyclic permutations with $\sigma_1 = 2, \dots, j$, is obtained analogously from the summands $(\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^i$ of R^k for $j = 2, \dots, k$ and $i = 1, \dots, j-1$. We conclude that $\mathfrak{q}_{110}^{\mathfrak{n}} : \hat{\mathrm{B}}_{\mathrm{cyc}}^* V[3-n] \to \hat{\mathrm{B}}_{\mathrm{cyc}}^* V[3-n]$ is a degree shift of $\mathrm{b}^* : \hat{\mathrm{B}}_{\mathrm{cyc}}^* V \to \hat{\mathrm{B}}_{\mathrm{cyc}}^* V$. As for the gradings, we have:

$$\begin{aligned}
r(D_\lambda(V))[3-n]^i = r(D_\lambda(V))^{i+3-n} = (D_\lambda(V))^{-i-3+n} = (\hat{\mathrm{B}}_{\mathrm{cyc}}^* V)^{i+3-n-1} \\
= \hat{\mathrm{B}}_{\mathrm{cyc}}^* V[2-n]^i.
\end{aligned}$$

This finishes the proof. ☐

We will now turn to units and augmentations.

Definition 3.4.7 (Reduced canonical dIBL-algebra). *Let $(V, \Pi, m_1, \mathbb{1}, \varepsilon)$ be an augmented cyclic cochain complex of degree $2-n$ from Definition 3.3.10. We define the space of* reduced cyclic cochains *on V by*

$$C_{\mathrm{red}}(V) \coloneqq \mathrm{B}_{\mathrm{cyc,red}}^* V[2-n].$$

We define the reduced canonical dIBL-algebra *by*

$$\mathrm{dIBL}(C_{\mathrm{red}}(V)) \coloneqq (C_{\mathrm{red}}(V), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}),$$

where $\mathfrak{q}_{110}$, $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ are restrictions of the operations of $\mathrm{dIBL}(C(V))$.

Definition 3.4.8 (Strictly reduced Maurer-Cartan element)**.** *In the setting of Definition 3.4.7, we call a Maurer-Cartan element* $\mathfrak{n} = (\mathfrak{n}_{lg})$ *for* $\mathrm{dIBL}(C(V))$ strictly reduced *if* $\mathfrak{n}_{lg} \in \hat{\mathrm{E}}_l C_{\mathrm{red}}(V)$ *for all* $(l, g) \neq (1, 0)$ *and if the* A_∞*-algebra* $(\mathcal{A}_{\mathfrak{n}}(V), 1, \varepsilon)$ *induced by* $\mathfrak{n}_{10}$ *is strictly unital and strictly augmented. Given a strictly reduced Maurer-Cartan element* $\mathfrak{n}$*, we can define the twisted* IBL_∞*-algebra*

$$\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}}(V)) \coloneqq (C_{\mathrm{red}}(V), (\mathfrak{q}^{\mathfrak{n}}_{klg})),$$

where $\mathfrak{q}^{\mathfrak{n}}_{klg}$ *are the restrictions of the operations of* $\mathrm{dIBL}^{\mathfrak{n}}(C(V))$*. We denote the homology of* $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}})$ *by* $\mathbb{H}^{\mathfrak{n}}(C_{\mathrm{red}})$ *or* $\mathbb{H}^{\mathfrak{n},\mathrm{red}}(C)$*.*[8]

Remark 3.4.9 (On strictly reduced Maurer-Cartan element). (i) We can say that the IBL_∞-algebra $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}})$ is a *subalgebra* of $\mathrm{dIBL}^{\mathfrak{n}}(C)$, which means that the inclusion $C_{\mathrm{red}} \hookrightarrow C$ induces the following commutative diagram for all $k, l \geq 1$, $g \geq 0$:

$$\begin{tikzcd}
\hat{\mathrm{E}}_k C \arrow[r, "\mathfrak{q}^{\mathfrak{n}}_{klg}"] & \hat{\mathrm{E}}_l C \\
\hat{\mathrm{E}}_k C_{\mathrm{red}} \arrow[r, "\mathfrak{q}^{\mathfrak{n}}_{klg}"] \arrow[u, hook] & \hat{\mathrm{E}}_l C_{\mathrm{red}}. \arrow[u, hook]
\end{tikzcd}$$

We denote this fact by $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}}) \subset \mathrm{dIBL}^{\mathfrak{n}}(C)$.

(ii) The canonical Maurer-Cartan element $\mathfrak{m}$ of a strictly augmented strictly unital dga $(V, m_1, m_2, 1, \varepsilon)$ is strictly reduced (this follows from Proposition 3.4.6).

(iii) In the situation of Definition 3.4.8, we denote

$$\bar{V}[1] \coloneqq \ker(\varepsilon),$$

so that $V = \bar{V} \oplus \langle 1 \rangle$. We use the canonical projection $\pi : V \to \bar{V}$ to identify $\hat{\mathrm{B}}^*_{\mathrm{cyc}} \bar{V} \xrightarrow{\simeq} \hat{\mathrm{B}}^*_{\mathrm{cyc,red}} V$ via the componentwise pullback π^*. In this way, we obtain the IBL_∞-algebras $\mathrm{dIBL}(C(\bar{V}))$ and $\mathrm{dIBL}^{\mathfrak{n}}(C(\bar{V}))$, which are isomorphic to $\mathrm{dIBL}(C_{\mathrm{red}}(V))$ and $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}}(V))$, respectively. $\lhd$

In the following list, we sum up our main reasons for considering units, augmentations and reduced Maurer-Cartan elements. Suppose that we are in the situation of Definition 3.4.8, then:

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[8] The latter option suggests that it might be possible to define the reduced homology with the induced IBL-algebra even if $\mathfrak{n}$ is not strictly reducible, e.g., if $(\mathcal{A}_{\mathfrak{n}}(V), 1, \varepsilon)$ is only homologically unital and augmented.

- Proposition 3.3.13 implies the splitting

$$\mathbb{H}^{\mathfrak{n}}(C)[1] = \mathbb{H}^{\mathfrak{n}}(C_{\mathrm{red}})[1] \oplus \langle \mathrm{s}\mathbb{1}^{2q-1*} \mid q \in \mathbb{N} \rangle. \tag{3.40}$$

  Here $\mathbb{1}^{i*} \in \mathrm{B}^*_{\mathrm{cyc}}V$ is the componentwise pullback ε^* of $\mathbb{1}^{i*} \in \mathrm{B}^*_{\mathrm{cyc}}(\mathbb{R})$. To get this, we used

$$\mathbb{H}^{\mathfrak{n}}(C_{\mathrm{red}}) = r(\mathrm{H}^*_{\lambda,\mathrm{red}}(\mathcal{A}_{\mathfrak{n}}))[3-n],$$

  which can be seen by redoing the proof of Proposition 3.4.6 with reduced cochains.

- The subalgebra $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}}) \subset \mathrm{dIBL}^{\mathfrak{n}}(C)$ induces the subalgebra

$$\mathrm{IBL}(\mathbb{H}^{\mathfrak{n}}(C_{\mathrm{red}})) \subset \mathrm{IBL}(\mathbb{H}^{\mathfrak{n}}(C)),$$

  and any higher operation $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ which induces a map $\hat{\mathrm{E}}_1\mathbb{H}(C) \to \hat{\mathrm{E}}_l\mathbb{H}(C)$ induces a map $\hat{\mathrm{E}}_1\mathbb{H}(C_{\mathrm{red}}) \to \hat{\mathrm{E}}_l\mathbb{H}(C_{\mathrm{red}})$ as well.

- If V is non-negatively graded, connected and simply-connected, then we have $\hat{\mathrm{E}}_k C_{\mathrm{red}} \simeq \mathrm{E}_k C_{\mathrm{red}}$ for all $k \in \mathbb{N}_0$ by Proposition 3.3.15, and hence $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}})$ is completion-free.

Proposition 3.4.10 (Operations on units)**.** *Suppose that* $(V, \Pi, m_1, \mathbb{1}, \varepsilon)$ *is a finite-dimensional augmented cyclic cochain complex of degree* $2-n$ *such that* $n \geq 1$*, and let* $\mathfrak{n}$ *be a strictly reduced Maurer-Cartan element for* $\mathrm{dIBL}(C(V))$*. The following relations are the only relations containing* $\mathbb{1}^{i*}$ *which may be non-zero on the homology* $\mathbb{H}^{\mathfrak{n}}(C)$*:*

For all $\Psi \in C_{\mathrm{red}}(V)$ *and* $l \geq 1$*,* $g \geq 0$*, we have*

$$\begin{aligned} &\mathfrak{q}_{210}(\mathrm{s}\mathbb{1}^* \otimes \Psi) = (-1)^{(n-2)|\Psi|}\mathfrak{q}_{210}(\Psi \otimes \mathrm{s}\mathbb{1}^*) = (-1)^{n-2}\Psi \circ \iota_{\mathrm{v}} \quad \textit{and} \\ &\mathfrak{q}^{\mathfrak{n}}_{1lg}(\mathrm{s}\mathbb{1}^*) = -\mathfrak{n}_{lg} \circ \iota_{\mathrm{v}}, \end{aligned}$$

where ι_{v} *is defined as follows:*

- *The element* $\mathrm{v} \in V[1]$ *is the unique vector such that* $\Pi(\mathbb{1}, \mathrm{v}) = 1$ *and* $\mathrm{v} \perp \bar{V}[1]$ *with respect to* Π*. Note that* $|\mathrm{v}| = n-1$ *and that such* v *always exists due to non-degeneracy.*

- *We start by defining* $\iota^0_{\mathrm{v}} : \mathrm{B}^{\mathrm{cyc}}V \to \mathrm{B}^{\mathrm{cyc}}V$ *by*

$$\iota^0_{\mathrm{v}}(v_1 \dots v_k) \coloneqq \sum_{i=1}^{k} (-1)^{|\mathrm{v}|(|v_1|+\dots+|v_{i-1}|)} v_1 \dots v_{i-1}\mathrm{v}v_i \dots v_k$$

for all generating words $v_1 \dots v_k \in \mathrm{B}^{\mathrm{cyc}} V$. *Next, for all* $k \geq 1$, *we define* $\iota_{\mathrm{v}} : (\mathrm{B}^{\mathrm{cyc}} V)^{\otimes k} \to (\mathrm{B}^{\mathrm{cyc}} V)^{\otimes k}$ *by*

$$\begin{aligned}\iota_{\mathrm{v}}(w_1 \otimes \dots \otimes w_k) \coloneqq (-1)^{|\mathrm{v}|k} \sum_{j=1}^{k} (-1)^{|\mathrm{v}|(|w_1|+\dots+|w_{j-1}|)} w_1 \otimes \dots \otimes w_{j-1} \\ \otimes \iota^0_{\mathrm{v}}(w_j) \otimes w_{j+1} \otimes \dots \otimes w_k\end{aligned}$$

for all generating words $w_1, \dots, w_k \in \mathrm{B}^{\mathrm{cyc}} V$. *Finally, we take the degree shift* $\iota_{\mathrm{v}} : (\mathrm{B}^{\mathrm{cyc}} V[3-n])^{\otimes k} \to (\mathrm{B}^{\mathrm{cyc}} V[3-n])^{\otimes k}$ *according to the degree shift convention* (3.4).

Proof. Pick a basis $(e_0, \dots, e_m)$ of $V[1]$ such that $e_0 = \mathbb{1}$ and $\bar{V}[1] = \langle e_1, \dots, e_m \rangle$. If $(e^0, \dots, e^m)$ is the dual basis, then we have $\mathrm{v} = e^0$. We will often use the following relation:

$$\sum_{j=0}^{m} T^{1j} e_j = \sum_{j=0}^{m} (-1)^{|\mathbb{1}|} \Pi(\mathrm{v}, e^j) e_j = -\mathrm{v}. \tag{3.41}$$

We consider only those generating words $w = v_1 \dots v_k$ of $\mathrm{B}^{\mathrm{cyc}} V$ with either $v_i \in \bar{V}$ for each i (shortly $w \in \mathrm{B}^{\mathrm{cyc}} \bar{V}$) or $v_i = \mathbb{1}$ for each i with k odd (i.e., $w = \mathbb{1}^{2j-1}$ for some j). Let $w_1, \dots, w_l$ with $w_j = v_{j1} \dots v_{jk_j}$ denote such generating words. Clearly, if $\Phi \in \hat{\mathrm{E}}_l C(V)$ is a $\mathfrak{q}^{\mathfrak{n}}_{110}$-closed element which vanishes on all $w_1 \otimes \dots \otimes w_l$, then (3.40) implies that $[\Phi] = 0$ in $\hat{\mathrm{E}}_l \mathbb{H}(C)$.

For $\Psi \in C_{\mathrm{red}}(V)$ and $q \geq 1$ odd, we compute using (3.41) the following:

$$\begin{aligned}\mathfrak{q}_{210}(\mathrm{s}^2 \mathbb{1}^{q*} \otimes \psi)(\mathrm{s}w) &= \sum \varepsilon(w \mapsto w^1 w^2)(-1)^{(n-1)|w^1|} T^{1j} \mathbb{1}^{q*}(\mathbb{1} w^1) \psi(e_j w^2) \\ &= -\sum \varepsilon(w \mapsto w^1 w^2)(-1)^{(n-1)|w^1|} \mathbb{1}^{q*}(\mathbb{1} w^1) \psi(\mathrm{v} w^2) \\ &=: (*).\end{aligned}$$

Now, in order to get $(*) \neq 0$, we need either $q = 1$ and $w \in \mathrm{B}^{\mathrm{cyc}} \bar{V}$, in which case

$$\begin{aligned}(*) &= -\sum \varepsilon(w \mapsto \underbrace{w^1}_{=\emptyset} w^2) \psi(\mathrm{v} w^2) \\ &= -\sum_{j=1}^{k} (-1)^{|v|(|v_1|+\dots+|v_{j-1}|)} \psi(v_1 \dots v_{j-1} \mathrm{v} v_{j+1} \dots v_k) \\ &= -(\psi \circ \iota^0_{\mathrm{v}})(w) \\ &= (-1)^{n-2} (\Psi \circ \iota_{\mathrm{v}})(\mathrm{W}),\end{aligned}$$

or $q > 1$ odd and $w = \mathbb{1}^{q-1}$, in which case

$$\begin{aligned}(*) &= \sum \varepsilon(w \mapsto w^1 \underbrace{w^2}_{=\emptyset})\mathbb{1}^{q*}(\mathbb{1}^q)\psi(\mathrm{v}) \\ &= \psi(\mathrm{v})\sum_{j=1}^{q-1}(-1)^j \\ &= 0.\end{aligned}$$

Next, because $n \geq 1$, we get $T^{11} = 0$, and hence

$$\mathfrak{q}_{120}(\mathbb{1}^{q*}) = 0 \quad \text{for all } q \in \mathbb{N}$$

on the chain level. Therefore, we have $\mathfrak{q}^{\mathfrak{n}}_{1lg} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg}$ for all $l \geq 1$, $g \geq 0$, and using Proposition 3.4.4 and (3.41), we obtain

$$\begin{aligned}&[(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg})(\mathbb{1}^{q*})](\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_l) \\ &\quad = -\sum_{j=1}^{l}\sum \varepsilon'\varepsilon(w_j \mapsto w_j^1 w_j^2)\mathbb{1}^{q*}(\mathbb{1}w_j^1)\mathfrak{n}_{lg}(\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_{j-1} \otimes (\mathrm{s}\mathrm{v}w_j^2) \\ &\qquad\qquad \otimes \mathrm{W}_{j+1} \otimes \cdots \otimes \mathrm{W}_l) \\ &\quad =: (**).\end{aligned}$$

In order to get $(**) \neq 0$, we need either $q = 1$ and $w_j \in \mathrm{B}^{\mathrm{cyc}}\bar{V}$ for all j, in which case

$$\begin{aligned}(**) &= -\sum_{j=1}^{l}\sum_{i=1}^{k_j}(-1)^{|\mathrm{v}|(|\mathrm{W}_1|+\cdots+|\mathrm{W}_{j-1}|+|\mathrm{s}|)}(-1)^{|\mathrm{v}|(|v_1|+\cdots+|v_{i-1}|)}\mathfrak{n}_{lg}(\mathrm{W}_1 \otimes \cdots \\ &\qquad \otimes \mathrm{W}_{j-1} \otimes (\mathrm{s}v_1 \ldots v_{i-1}\mathrm{v}v_i \ldots v_{k_j}) \otimes \mathrm{W}_{j+1} \otimes \cdots \otimes \mathrm{W}_l) \\ &= -(\mathfrak{n}_{lg} \circ \iota_{\mathrm{v}})(\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_l),\end{aligned}$$

or $q > 1$ odd and $w_j = \mathbb{1}^{q-1}$ for some j, in which case

$$\begin{aligned}(**) &= -\sum_{\substack{1 \leq j \leq l \\ w_j = \mathbb{1}^{q-1}}} \varepsilon'\Big(\sum \varepsilon(w_j \mapsto w_j^1 \underbrace{w_j^2}_{=\emptyset} \mathbb{1}^{q*}(\mathbb{1}w_j^1))\Big)\mathfrak{n}_{lg}(\mathrm{W}_1 \otimes \cdots \\ &\qquad \otimes \mathrm{W}_{j-1} \otimes (\mathrm{s}\mathrm{v}) \otimes \mathrm{W}_{j+1} \otimes \cdots \otimes \mathrm{W}_l) \\ &= -\sum_{\substack{1 \leq j \leq l \\ w_j = \mathbb{1}^{q-1}}} \varepsilon'\underbrace{\Big(\sum_{i=1}^{q-1}(-1)^i\Big)}_{=0}\mathfrak{n}_{lg}(\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_{j-1} \otimes (\mathrm{s}\mathrm{v}) \otimes \mathrm{W}_{j+1} \otimes \\ &\qquad \cdots \otimes \mathrm{W}_l) \\ &= 0.\end{aligned}$$

The only relation left to check is

$$\mathfrak{q}_{210}(\mathrm{s}1^{q_1 *}, \mathrm{s}1^{q_1 *}) = 0 \quad \text{for all } q_1, q_2 \in \mathbb{N}.$$

However, this is easy to see, and the proof is done. $\square$

Remark 3.4.11 (Finite type). Everything in this section works if V is just of *finite type*, i.e., if $\dim V^d < \infty$ for all $d \in \mathbb{Z}$. The only difference is that T is not a tensor in $V[1]^{\otimes 2}$ anymore. $\triangleleft$

4. Chern-Simons Maurer-Cartan element and string topology

In Section 4.1, we consider the cyclic dga's $\Omega(M)$, $\mathrm{H}_{\mathrm{dR}}(M)$ and $\mathcal{H}(M)$ for a closed oriented n-manifold M (Proposition 4.1.2) and apply the theory from Section 3.4 to the last two, which are finite-dimensional.

In Section 4.2, we define the admissible Hodge propagator P (Definition 4.2.4). It is a primitive to the Schwartz kernel H of the harmonic projection $\pi_{\mathcal{H}}$ (see Proposition 4.2.7) outside the diagonal and extends smoothly to the spherical blow-up of the diagonal. These ideas come from an early version of [CVon]. We consider conditions (P1)–(P5) on a linear operator $\mathcal{P}$ and its Schwartz kernel P (see p. 83) and show that P satisfying all these conditions always exists (Proposition 4.2.11). We also mention the standard Hodge propagator $\mathrm{P}_{\mathrm{std}}$ (see (4.15)), which might be a canonical Hodge propagator satisfying (P1)–(P5).

In Section 4.3, we review ribbon graphs, labelings, compatibility of the order and orientation of internal edges, and the edge and vertex order from [CFL15] (Definitions 4.3.1, 4.3.3, 4.3.4 and 4.3.5). We then define $\mathfrak{n}$ as a signed sum of integrals of products of Hodge propagators and harmonic forms which are associated to labeled trivalent ribbon graphs (Definition 4.3.6). We do not show that $\mathfrak{n}$ satisfies the Maurer-Cartan equation, but we do show all other properties of a Maurer-Cartan element (Lemma 4.3.7 and Proposition 4.3.10). We define the Y-graph, trees, circular graphs, vertices of types A, B, C and their contributions A_{α_1,α_2}, B_α, C, respectively (Definitions 4.3.8 and 4.3.11).

In Section 4.4, we observe that vanishing of some subintegrals associated to special configurations of vertices in a graph implies $\mathfrak{n}_{lg} = \mathfrak{m}_{lg}$. For example, if all configurations with $\imath$ at an external vertex vanish, which holds if P satisfies (P4) and (P5) (Proposition 4.4.2), then all higher operations $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ vanish on the chain level in dimensions $n > 3$ (Proposition 4.4.1). Next, if all configurations with an A-vertex vanish, then $\mathfrak{n}_{10} = \mathfrak{m}_{10}$, and hence $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}^{\mathfrak{m}}_{110}$ (Proposition 4.4.3). We show that $\mathfrak{n} = \mathfrak{m}$ for simply-connected geometrically formal manifolds with $n \neq 2$ (Proposition 4.4.4). Using the results of [CVon], we argue that the chain complexes of $\mathfrak{q}^{\mathfrak{m}}_{110}$ and $\mathfrak{q}^{\mathfrak{n}}_{110}$ are quasi-isomorphic provided M is simply-connected and formal (Proposition 4.4.6).

In Section 4.5, we recall basic facts about the Chas-Sullivan operations $\mathbb{m}_2$ and $\mathbb{c}_2$ on the $\mathbb{S}^1$-equivariant homology of the free loop space and formulate a version of the string topology conjecture for simply-connected manifolds (Conjecture 4.5.1).

4.1. Canonical dIBL-structures on cyclic cochains of de Rham cohomology

Let M be an oriented closed Riemannian manifold of dimension n. We consider the following graded vector spaces:

$$\begin{aligned} \Omega^*(M) &\ \dots\ \text{smooth de Rham forms,}\\ \mathcal{H}^*(M) &\ \dots\ \text{harmonic forms,}\\ \mathrm{H}^*_{\mathrm{dR}}(M) &\ \dots\ \text{de Rham cohomology.}\end{aligned}$$

Since M is fixed, we often write just Ω, $\mathcal{H}$ and H_{dR}. We consider the Hodge decomposition $\Omega = \mathcal{H} \oplus \operatorname{im} \mathrm{d} \oplus \operatorname{im} \mathrm{d}^*$, where d is the de Rham differential and d^* the codifferential. We call the corresponding projection

$$\pi_{\mathcal{H}} : \Omega^*(M) \longrightarrow \mathcal{H}^*(M) \tag{4.1}$$

the *harmonic projection* and the induced isomorphism $\pi_{\mathcal{H}} : \mathrm{H}_{\mathrm{dR}} \to \mathcal{H}$ mapping a cohomology class into its unique harmonic representative the *Hodge isomorphism.*

Notation 4.1.1 (Updated notation for bar complexes)**.** *We use Notation 3.3.3 for $V = \Omega$, $\mathcal{H}$, H_{dR} and $A = n-3$ with the following changes:*

$$\tilde{v} \sim \eta \in V, \quad v \sim \alpha \in V[1], \quad w \sim \omega \in \mathrm{B}^{\mathrm{cyc}}V, \quad W \sim \Omega \in \mathrm{B}^{\mathrm{cyc}}V[n-3].$$

We use the formal symbols s *and θ with $|\mathrm{s}| = n-3$ and $|\theta| = -1$, so that $\alpha = \theta\eta$ and $\Omega = \mathrm{s}\omega$.*

Proposition 4.1.2 (De Rham cyclic dga's)**.** *Let M be an oriented closed Riemannian manifold of dimension n. The quadruple $(\Omega(M), \Pi, m_1, m_2)$ with the operations from* (2.3) *is a cyclic dga of degree $2-n$. For the operations before the degree shift, we have*

$$\begin{aligned}\tilde{m}_1(\eta_1) &= \mathrm{d}\eta_1,\\ \tilde{m}_2(\eta_1, \eta_2) &= \eta_1 \wedge \eta_2,\end{aligned}$$

$$\tilde{\Pi}(\eta_1, \eta_2) = \int_M \eta_1 \wedge \eta_2 =: (\eta_1, \eta_2),$$

where d *is the de Rham differential,* $\wedge$ *the wedge product and* $\tilde{\mathcal{P}}$ *the* intersection pairing*. The operations restrict to* $\mathrm{H}_{\mathrm{dR}}(M)$ *and make* $(\mathrm{H}_{\mathrm{dR}}(M), \Pi, m_1 \equiv 0, m_2)$ *into a cyclic dga. If we define* $\mu_1 \equiv 0$ *and*

$$\mu_2(\alpha_1, \alpha_2) \coloneqq \pi_{\mathcal{H}}(m_2(\alpha_1, \alpha_2)) \quad \textit{for all } \alpha_1, \alpha_2 \in \mathcal{H}(M)[1], \tag{4.2}$$

then $(\mathcal{H}(M), \Pi, \mu_1, \mu_2)$ *is a cyclic dga as well, and* $\pi_{\mathcal{H}} : \mathrm{H}_{\mathrm{dR}} \to \mathcal{H}$ *is an isomorphism of cyclic dga's. All three dga's* Ω*,* H_{dR} *and* $\mathcal{H}$ *are strictly unital and strictly augmented with the unit* $\mathbb{1} \coloneqq \theta 1 \in \Omega[1]$*, where* 1 *is the constant one.*

Proof. The relations (3.33) follow from the classical properties of d and $\wedge$ and from the Stokes' theorem for oriented closed manifolds. The Poincaré duality asserts that $(\cdot,\cdot)$ is non-degenerate on H_{dR} and $\mathcal{H}$, and thus they are cyclic dga's as well. The fact that $\pi_{\mathcal{H}} : \mathrm{H}_{\mathrm{dR}} \to \mathcal{H}$ is an isomorphism of vector spaces follows from the Hodge theory. As for compatibility with the product, given η_1, $\eta_2 \in \mathcal{H}$, then $\eta_1 \wedge \eta_2$ is closed, and since $\ker \mathrm{d} = \mathcal{H} \oplus \operatorname{im} \mathrm{d}$, we see that $\pi_{\mathcal{H}}(\eta_1 \wedge \eta_2) = \eta_1 \wedge \eta_2 + \mathrm{d}\eta$ for some $\eta \in \Omega$ is a harmonic representative of the cohomology class $[\eta_1 \wedge \eta_2] = [\eta_1] \wedge [\eta_2]$. Unitality is obvious, and the construction of an augmentation map clear. Note that a strict augmentation for $\Omega(M)$ is the evaluation at a point, for instance. ☐

The facts (A) and (C) from the Overview apply to the cyclic dga's $\mathcal{H}$ and H_{dR} (not to Ω because it is infinite-dimensional!), and we get the canonical dIBL-algebras $\mathrm{dIBL}(C(\mathcal{H}))$ and $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}))$ of bidegrees $(n-3,2)$ with the canonical Maurer-Cartan element $\mathfrak{m} = (\mathfrak{m}_{10})$. The Hodge isomorphism induces an isomorphism of these dIBL-algebras, and hence we can use $\mathcal{H}$ and H_{dR} interchangeably. The reduced versions (see Definition 3.4.7) also satisfy $\mathrm{dIBL}(C_{\mathrm{red}}(\mathcal{H})) \simeq \mathrm{dIBL}(C_{\mathrm{red}}(\mathrm{H}_{\mathrm{dR}}))$, $\mathrm{dIBL}^{\mathfrak{m}}(C_{\mathrm{red}}(\mathcal{H})) \simeq \mathrm{dIBL}^{\mathfrak{m}_{\mathrm{dR}}}(C_{\mathrm{red}}(\mathrm{H}_{\mathrm{dR}}))$. We have $\mathfrak{q}_{110} \equiv 0$, and hence $\mathrm{dIBL}(C(\mathcal{H}))$ is, in fact, an IBL-algebra. However, we will denote it by dIBL and call it a dIBL algebra as a reminder of the canonical dIBL-structure. The canonical Maurer-Cartan element $\mathfrak{m}$ satisfies

$$\mathfrak{m}_{10}(\mathrm{s}\alpha_1\alpha_2\alpha_3) = (-1)^{n-2+\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \alpha_1, \alpha_2, \alpha_3 \in \mathcal{H}[1]. \tag{4.3}$$

We get the canonical twisted dIBL-algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{H}))$ from (2.2) with, in general, non-trivial boundary operator $\mathfrak{q}_{110}^{\mathfrak{m}}$ whose homology is the cyclic homology of H_{dR} up to degree shifts.

4.2. Hodge propagator

We will use fiberwise integration and spherical blow-ups, which we now recall.

Definition 4.2.1 (Fiberwise integration). *Let* $\mathrm{pr} : E \to B$ *be a smooth oriented fiber bundle with an oriented fiber* F *over an oriented manifold* B *with* $\partial B = \emptyset$*. We orient* E *as* $F \times B$*. Let* $\Omega_{\mathrm{c}}(B)$ *denote the space of forms with compact support and* $\Omega_{\mathrm{cv}}(E)$ *the space of forms with compact vertical support. For any* $\kappa \in \Omega_{\mathrm{cv}}(E)$*, let* $\int^F \kappa \in \Omega(B)$ *be the unique smooth form such that*

$$\int_E \kappa \wedge \mathrm{pr}^* \eta = \int_B \Big(\int^F \kappa \Big) \wedge \eta \quad \textit{for all } \eta \in \Omega_{\mathrm{c}}(B).$$

Definition 4.2.2 (Spherical blow-up). *Let* X *be a smooth* n*-dimensional manifold and* $Y \subset X$ *a smooth* k*-dimensional submanifold. The* blow-up *of* X *at* Y *is as a set defined by*

$$\mathrm{Bl}_Y X \coloneqq X \backslash Y \sqcup P^+ NY,$$

where P^+NY *is the real oriented projectivization of the normal bundle* NY *of* Y *in* X*. This means that* P^+NY *is the quotient of* $\{v \in NY \mid v \neq 0\}$ *by the relation* $v \sim av$ *for all* $a \in (0, \infty)$*. The* blow-down map *is defined by*

$$\begin{aligned} \pi : \mathrm{Bl}_Y X &\longrightarrow X \\ p \in X \backslash Y &\longmapsto p, \\ [v]_p \in P^+NY &\longmapsto p. \end{aligned}$$

In the following, we will equip the blow-up with the structure of a smooth manifold with boundary such that its interior becomes diffeomorphic to $X\backslash Y$ via the blow-down map and the boundary becomes P^+NY. Consider an adapted chart (U, ψ) for Y in X with $\psi(U) = \mathbb{R}^n$ and $\psi(U \cap Y) = \{(0, y) \mid y \in \mathbb{R}^k\}$. It induces the bijection

$$\begin{aligned} \tilde{\psi} : \mathrm{Bl}_{U\cap Y} U &\longrightarrow [0, \infty) \times \mathbb{S}^{n-k-1} \times \mathbb{R}^k \\ p \in U \backslash Y &\longmapsto \Big(|\pi_1 \psi(p)|, \frac{\pi_1 \psi(p)}{|\pi_1 \psi(p)|}, \pi_2 \psi(p) \Big), \\ [v] \in P^+ N_p Y &\longmapsto \Big(0, \frac{\pi_1 \, \mathrm{d}\psi(v)}{|\pi_1 \, \mathrm{d}\psi(v)|}, \pi_2 \psi(p) \Big), \end{aligned}$$

where π_1 and π_2 are the canonical projections to the factors of $\mathbb{R}^{n-k} \times \mathbb{R}^k$. Notice that we have the canonical inclusion $\mathrm{Bl}_{U\cap Y} U \subset \mathrm{Bl}_Y X$. It can be checked that for any two overlapping adapted charts (U_1, ψ_1) and (U_2, ψ_2), the transition function $\tilde{\psi}_1 \circ \tilde{\psi}_2^{-1}$ is a

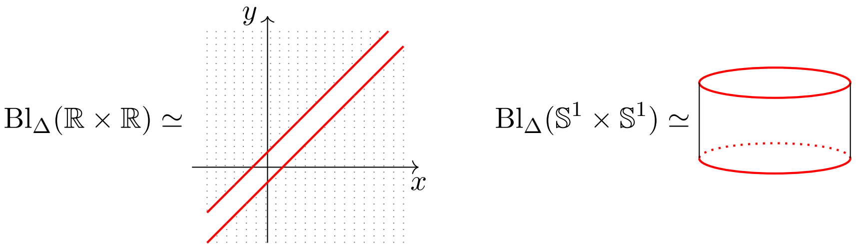

Figure 4.1.: The simplest blow-up examples.

diffeomorphism of manifolds with boundary. Therefore, we can use the charts $(\mathrm{Bl}_{U\cap Y}U, \tilde{\psi})$ to define a smooth atlas on $\mathrm{Bl}_Y X$. If X is oriented, we orient $\mathrm{Bl}_Y X$ so that π restricts to an orientation preserving diffeomorphism of the interior.

An important obvious fact is that if X is compact, then $\mathrm{Bl}_Y X$ *is compact.*

We are interested in the case when $X = M\times M$ for an oriented closed manifold M and $Y = \Delta \coloneqq \{(m,m) \mid m\in M\}$ is the *diagonal*. Given a chart $\varphi: U\to\mathbb{R}^n$ on M, the following is a smooth chart on $\mathrm{Bl}_\Delta(M\times M)$:

$$\begin{aligned}\tilde{\varphi}: \mathrm{Bl}_\Delta(U\times U) &\longrightarrow [0,\infty)\times\mathbb{S}^{n-1}\times\mathbb{R}^n\\ (x,y)\in(U\times U)\backslash\Delta &\longmapsto (r,w,u) \coloneqq \Big(\frac{1}{2}|\varphi(x)-\varphi(y)|, \frac{\varphi(x)-\varphi(y)}{|\varphi(x)-\varphi(y)|},\\ &\qquad\qquad\qquad\qquad \frac{1}{2}(\varphi(x)+\varphi(y))\Big),\\ [(v,-v)]_{(x,x)} &\longmapsto \Big(0, \frac{\mathrm{d}\varphi_x(v)}{|\mathrm{d}\varphi_x(v)|}, \varphi(x)\Big).\end{aligned} \tag{4.4}$$

The inverse relations for $r>0$ read

$$\varphi(x) = u + wr \quad\text{and}\quad \varphi(y) = u - wr.$$

We will denote by M_i the i-th factor of $M\times M$; i.e., we will write $M\times M = M_1\times M_2$. We denote the corresponding projection by pr_i. We define $\widetilde{\mathrm{pr}}_i \coloneqq \mathrm{pr}_i\circ\pi$, where $\pi: \mathrm{Bl}_\Delta(M\times M)\to M\times M$ is the blow-down map. We also identify $(M\times M)\backslash\Delta$ with the interior of $\mathrm{Bl}_\Delta(M\times M)$ via π.

Example 4.2.3 (Product minus open thickening of diagonal). Have a look at Figure 4.1 for the simplest examples of what kind of manifold $\mathrm{Bl}_\Delta(M\times M)$ is. In fact, it is always diffeomorphic to $M\times M$ minus an open thickening of the diagonal. The blow-down map is therefore an essential part of the blow-up construction. ◁

The map $\widetilde{\mathrm{pr}}_2 : \mathrm{Bl}_\Delta(M\times M)\to M_2$ is an oriented fiber bundle with fiber $\mathrm{Bl}_*(M_1)$, which is the blow-up of M_1 at a point (we shall assume that M is connected). The *fiberwise integration along* $\widetilde{\mathrm{pr}}_2$ will be denoted by $\int^{\mathrm{Bl}_* M_1}$.

Definition 4.2.4 (Hodge propagator)**.** *Let M be an oriented closed n-dimensional Riemannian manifold. Consider the harmonic projection $\pi_{\mathcal{H}}$ from (4.1), and let $\iota_{\mathcal{H}} : \mathcal{H}(M) \hookrightarrow \Omega(M)$ be the inclusion. A smooth $(n-1)$-form P on $(M \times M) \backslash \Delta$ is called an* admissible Hodge propagator *if the following conditions are satisfied:*

(1) The form P admits a smooth extension to $\mathrm{Bl}_\Delta(M\times M)$. More precisely, the pullback $(\pi|_{\mathrm{int}})^*\mathrm{P}$ along the blow-down map restricted to the interior is a restriction of a smooth form on $\mathrm{Bl}_\Delta(M \times M)$. We denote this form by P again by uniqueness.

(2) The operator $\mathcal{P} : \Omega^*(M) \to \Omega^{*-1}(M)$ defined by

$$\mathcal{P}(\eta) \coloneqq \int^{\mathrm{Bl}_* M_1} \mathrm{P} \wedge \widetilde{\mathrm{pr}}_1^* \eta \quad \textit{for all } \eta \in \Omega(M) \tag{4.5}$$

satisfies

$$\mathrm{d} \circ \mathcal{P} + \mathcal{P} \circ \mathrm{d} = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1}. \tag{4.6}$$

Any homogenous linear operator $\mathcal{P} : \Omega^*(M) \to \Omega^{*-1}(M)$ satisfying (4.6) will be called a Hodge homotopy*.*

(3) For the twist map $\tau : M \times M \to M \times M$ defined by $(x, y) \mapsto (y, x)$, the following symmetry property holds:

$$\tau^*\mathrm{P} = (-1)^n \mathrm{P}. \tag{4.7}$$

Remark 4.2.5 (On Hodge propagator). (i) Given a homogenous linear operator $\mathcal{P} : \Omega^*(M) \to \Omega^{*-1}(M)$, if there is a $\mathrm{P} \in \Omega^{n-1}(\mathrm{Bl}_\Delta(M \times M))$ such that (4.5) holds, then it is unique.

(ii) Because $\tau : M \times M \to M \times M$ preserves Δ, it extends to a diffeomorphism $\tilde{\tau}$ of $\mathrm{Bl}_\Delta(M \times M)$. The condition (4.7) is then equivalent to $\tilde{\tau}^*\tilde{\mathrm{P}} = (-1)^n\tilde{\mathrm{P}}$ for the extension $\tilde{\mathrm{P}}$ of P to $\mathrm{Bl}_\Delta(M \times M)$. We denote both extensions by τ and P, respectively.

(iii) Using the intersection pairing $(\eta_1, \eta_2) = \int_M \eta_1 \wedge \eta_2$, we have

$$\begin{aligned} (\mathcal{P}(\eta_1), \eta_2) &= \int_{\mathrm{Bl}_\Delta(M\times M)} \mathrm{P} \wedge \widetilde{\mathrm{pr}}_1^*\eta_1 \wedge \widetilde{\mathrm{pr}}_2^*\eta_2 \\ &= (-1)^n \int_{\mathrm{Bl}_\Delta(M\times M)} \tau^*\mathrm{P} \wedge \widetilde{\mathrm{pr}}_2^*\eta_1 \wedge \widetilde{\mathrm{pr}}_1^*\eta_2 \end{aligned}$$

and
$$\begin{aligned}(\eta_1, \mathcal{P}(\eta_2)) &= (-1)^{\eta_1(\eta_2-1)}(\mathcal{P}(\eta_2), \eta_1) \\ &= (-1)^{\eta_1}\int_{\mathrm{Bl}_\Delta(M\times M)} \mathrm{P}\wedge \widetilde{\mathrm{pr}}_2^*\eta_1 \wedge \widetilde{\mathrm{pr}}_1^*\eta_2\end{aligned}$$

for all η_1, $\eta_2 \in \Omega(M)$. This implies the following:

$$\tau^*\mathrm{P} = (-1)^n\mathrm{P} \quad\Longleftrightarrow\quad (\mathcal{P}(\eta_1), \eta_2) = (-1)^{\eta_1}(\eta_1, \mathcal{P}(\eta_2)) \quad \forall \eta_1, \eta_2 \in \Omega(M). \tag{4.8}$$

(iv) Because $\mathrm{Bl}_\Delta(M \times M)$ is compact, $\mathrm{P} \in \Omega(\mathrm{Bl}_\Delta(M \times M))$ induces an L^1-integrable form on $M \times M$.

(v) In the original article [CFL15], Hodge propagator was called "Green kernel". However, this name is standardly used for the Schwartz kernel of the generalized inverse of an elliptic pseudo-differential operator, e.g., of the Laplacian Δ. Because this was a source of confusion in various discussions, we decided to change the name to Hodge propagator. Doing a random search on the internet, we found that this name appeared already in [CMR18]. This is the story of how we discovered [CM10], where Hodge propagators were also treated, and where we took the trick in the proof of Proposition 4.2.11 from. ◁

We will now prove three propositions which will allow us to rewrite (4.6) equivalently as a differential equation for P on $M \times M\backslash\Delta$.

Proposition 4.2.6 (Identities for fiberwise integration)**.** *In the situation of Definition 4.2.1, assume that F has a boundary ∂F. We orient ∂F using $T_pF = N(p) \oplus T_p\partial F$ for $p \in \partial F$, where N is an outward pointing normal vector field. The following formulas hold for all $\kappa \in \Omega_{\mathrm{cv}}(E)$ and $\eta \in \Omega_{\mathrm{c}}(B)$:*

- *The* projection formula
$$\int^F (\kappa \wedge \pi^*\eta) = \Bigl(\int^F \kappa\Bigr) \wedge \eta,$$
- Stokes' formula
$$(-1)^F \mathrm{d}\int^F \kappa = \int^F \mathrm{d}\kappa - \int^{\partial F} \kappa,$$
where F in the exponent denotes the dimension of F.

Proof. The projection formula is proven by a straightforward calculation from the definition.

As for Stokes' formula, we get the oriented fiber bundle $\partial E \to B$ with fiber ∂F locally by restricting an oriented local trivialization of E. There are two orientations on ∂E — as the total space of $\partial E \to B$ and as the boundary of E. They agree due to our orientation convention. Using standard Stokes' theorem, we get

$$\begin{aligned}(-1)^F \int_B \mathrm{d}\Big(\int^F \kappa\Big) \wedge \eta &= (-1)^{\kappa+1} \int_B \Big(\int^F \kappa\Big) \wedge \mathrm{d}\eta \\ &= (-1)^{\kappa+1} \int_E \kappa \wedge \mathrm{d}\pi^* \eta \\ &= \int_E (\mathrm{d}\kappa \wedge \pi^*\eta - \mathrm{d}(\kappa \wedge \pi^*\eta)) \\ &= \int_B \Big(\int^F \mathrm{d}\kappa\Big) \wedge \eta - \int_{\partial E} \kappa \wedge \pi^* \eta \\ &= \int_B \Big(\int^F \mathrm{d}\kappa - \int^{\partial F} \kappa\Big) \wedge \eta.\end{aligned}$$

This proves the proposition. ☐

In what comes next, we will view the canonical projection $\mathrm{pr}_2 : M_1 \times M_2 \to M_2$ as an oriented fiber bundle such that the orientation of the total space agrees with the product orientation. The fiberwise integration for this bundle will be denoted by $\int^{M_1}$.

Proposition 4.2.7 (Schwartz kernel of the harmonic projection)**.** *Let M be an oriented closed n-dimensional Riemannian manifold. Let $\nu_1, \ldots, \nu_m$ be a homogenous basis of $\mathcal{H}(M)$ which is orthonormal with respect to the L^2-inner product*

$$(\eta_1, \eta_2)_{L^2} \coloneqq \int_M \eta_1 \wedge *\eta_2 \quad \text{for } \eta_1, \eta_2 \in \Omega(M),$$

where $*$ denotes the Hodge star. The smooth form $\mathrm{H} \in \Omega^n(M \times M)$ defined by

$$\mathrm{H} \coloneqq \sum_{i=1}^m (-1)^{n\nu_i} \mathrm{pr}_1^*(*\nu_i) \wedge \mathrm{pr}_2^*(\nu_i) \tag{4.9}$$

satisfies the following properties:

(a) For all $\eta \in \Omega(M)$, we have

$$\pi_{\mathcal{H}}(\eta) = \int^{M_1} \mathrm{H} \wedge \mathrm{pr}_1^* \eta.$$

(b) The form H is closed and Poincaré dual to $\Delta \subset M \times M$.

(c) The following symmetry condition is satisfied:

$$\tau^*\mathrm{H} = (-1)^n \mathrm{H}. \tag{4.10}$$

Proof. (a) For the purpose of the proof, we denote $\mathcal{H}(\eta) \coloneqq \int^{M_1} \mathrm{H} \wedge \mathrm{pr}_1^* \eta$. For every $k = 1, \ldots, m$, we use the projection formula to compute

$$\begin{aligned}\mathcal{H}(\nu_k) &= \sum_{i=1}^m (-1)^{\nu_i n + \nu_i \nu_k} \int^{M_1} \mathrm{pr}_1^*(*\nu_i \wedge \nu_k) \wedge \mathrm{pr}_2^*(\nu_i) \\ &= \sum_{i=1}^m (-1)^{\nu_i(n+\nu_k)+\nu_k(n+\nu_i)} \Big(\int_M \nu_k \wedge *\nu_i\Big) \nu_i \\ &= \nu_k.\end{aligned}$$

It is easy to see that $\mathcal{H}(\eta) \in \mathcal{H}(M)$ for all $\eta \in \Omega(M)$. Therefore, $\mathcal{H}$ is a projection to $\mathcal{H}(M)$. Relations $\mathcal{H}(\mathrm{d}\eta) = \mathcal{H}(\mathrm{d}^*\eta) = 0$ for all $\eta \in \Omega(M)$ follow from the second line of the computation above with ν_k replaced by $\mathrm{d}\eta$ and $\mathrm{d}^*\eta$ using that $\operatorname{im}\mathrm{d}^* \oplus \operatorname{im}\mathrm{d}$ is L^2-orthogonal to $\mathcal{H}(M)$. We see that $\mathcal{H} = \pi_{\mathcal{H}}$.

(b) Using $\mathrm{d} \circ \mathcal{H} = \mathcal{H} \circ \mathrm{d} = 0$ and Stokes' theorem, we get

$$\int^{M_1} \mathrm{dH} \wedge \mathrm{pr}_1^* \eta = (-1)^n \mathrm{d}\mathcal{H}(\eta) - \mathcal{H}(\mathrm{d}\eta) = 0 \quad \text{for all } \eta \in \Omega(M).$$

It follows that $\mathrm{dH} = 0$. Using the Künneth formula, we can write a given $\kappa \in \Omega(M \times M)$ with $\mathrm{d}\kappa = 0$ as $\kappa = \mathrm{pr}_1^* \eta_1 \wedge \mathrm{pr}_2^* \eta_2 + \mathrm{d}\eta$ for some $\eta_1, \eta_2 \in \mathcal{H}(M)$ and $\eta \in \Omega(M)$. Then

$$\begin{aligned}\int_{M\times M} \mathrm{H} \wedge \kappa &= \int_{M \times M} \mathrm{H} \wedge \mathrm{pr}_1^* \eta_1 \wedge \mathrm{pr}_2^* \eta_2 \\ &= \int_M \mathcal{H}(\eta_1) \wedge \eta_2 \\ &= \int_M \eta_1 \wedge \eta_2 = \int_\Delta \kappa.\end{aligned}$$

This shows that H is Poincaré dual to Δ.

(c) It follows from the Hodge decomposition that

$$(\pi_{\mathcal{H}}(\eta_1), \eta_2) = (\pi_{\mathcal{H}}(\eta_1), \pi_{\mathcal{H}}(\eta_2)) = (\eta_1, \pi_{\mathcal{H}}(\eta_2)) \quad \text{for all } \eta_1, \eta_2 \in \Omega(M), \tag{4.11}$$

where $(\cdot,\cdot)$ is the intersection pairing. As in (iii) of Remark 4.2.5, one shows that this is equivalent to (4.10). □

Proposition 4.2.8 (Differential condition)**.** *Let M be an oriented closed n-dimensional Riemannian manifold. For $\mathrm{P}\in\Omega^{n-1}(\mathrm{Bl}_\Delta(M\times M))$, the following claims are equivalent:*

(1) The operator $\mathcal{P}:\Omega^*(M)\to\Omega^{*-1}(M)$ defined by $\mathcal{P}(\eta):=\int^{\mathrm{Bl}_*M_1}\mathrm{P}\wedge\widetilde{\mathrm{pr}}_1^*\eta$ for $\eta\in\Omega(M)$ is a Hodge homotopy.

(2) It holds

$$\mathrm{dP}=(-1)^n\mathrm{H}\quad\text{on } (M\times M)\backslash\Delta. \tag{4.12}$$

Proof. Before we begin, note that (4.12) is equivalent to the equation $\mathrm{d}\tilde{\mathrm{P}}=(-1)^n\pi^*\mathrm{H}$ on $\mathrm{Bl}_\Delta(M\times M)$ for the extension $\tilde{\mathrm{P}}$ of P; we denote $\tilde{\mathrm{P}}$ by P and $\pi^*\mathrm{H}$ by H by uniqueness.

We will first prove 2) $\Longrightarrow$ 1). Using Stokes' formula, we get for every $\eta\in\Omega(M)$ the following:

$$\begin{aligned}\mathrm{d}\mathcal{P}(\eta)&=\mathrm{d}\int^{\mathrm{Bl}_*M_1}\mathrm{P}\wedge\widetilde{\mathrm{pr}}_1^*\eta\\&=(-1)^n\Big(\int^{\mathrm{Bl}_*M_1}\mathrm{d}(\mathrm{P}\wedge\widetilde{\mathrm{pr}}_1^*\eta)-\int^{\partial\mathrm{Bl}_*M_1}\mathrm{P}\wedge\widetilde{\mathrm{pr}}_1^*\eta\Big)\\&=\pi_{\mathcal{H}}(\eta)-\mathcal{P}(\mathrm{d}\eta)+\int^{\partial\mathrm{Bl}_*M_1}(-1)^{n+1}\mathrm{P}\wedge\widetilde{\mathrm{pr}}_1^*\eta.\end{aligned}$$

Since $\widetilde{\mathrm{pr}}_1=\widetilde{\mathrm{pr}}_2$ on $\partial\mathrm{Bl}_\Delta(M\times M)$, we get with the help of the projection formula the following:

$$\int^{\partial\mathrm{Bl}_*M_1}\mathrm{P}\wedge\widetilde{\mathrm{pr}}_1^*\eta=\int^{\partial\mathrm{Bl}_*M_1}\mathrm{P}\wedge\widetilde{\mathrm{pr}}_2^*\eta=\Big(\int^{\partial\mathrm{Bl}_*M_1}\mathrm{P}\Big)\wedge\eta.$$

We will show that the 0-form $\int^{\partial\mathrm{Bl}_*M_1}\mathrm{P}$ is constant $(-1)^n$. Stokes' formula implies

$$\int^{\partial\mathrm{Bl}_*M_1}\mathrm{P}=\int^{\mathrm{Bl}_*M_1}\mathrm{dP}=(-1)^n\int^{\mathrm{Bl}_*M_1}\mathrm{H}.$$

Using that H is Poincaré dual to Δ, we get for every $\eta\in\Omega^n(M)$ the following:

$$\begin{aligned}\int_M\Big(\int^{\mathrm{Bl}_*M_1}\mathrm{H}\Big)\wedge\eta&=\int_{\mathrm{Bl}_\Delta(M\times M)}\mathrm{H}\wedge\widetilde{\mathrm{pr}}_2^*\eta\\&=\int_{M\times M}H\wedge\mathrm{pr}_2^*\,\eta\\&=\int_\Delta\mathrm{pr}_2^*\,\eta=\int_M1\wedge\eta.\end{aligned}$$

The implication follows.

We will now prove 1) $\Longrightarrow$ 2). Assume that (4.6) holds and that P extends smoothly to the blow-up. Denote

$$K \coloneqq (-1)^n \mathrm{dP} - \mathrm{H} \quad \text{and} \quad L \coloneqq -1 + \int^{\partial \mathrm{Bl}_*(M_1)} (-1)^n \mathrm{P}.$$

Notice that L is a function on M. From the previous computations, we deduce that

$$\int^{\mathrm{Bl}_*(M_1)} K \wedge \widetilde{\mathrm{pr}}_1^* \eta = L\eta \quad \text{for all } \eta \in \Omega(M),$$

and hence

$$\int_{\mathrm{Bl}_\Delta(M \times M)} K \wedge \widetilde{\mathrm{pr}}_1^*(\eta_1) \wedge \widetilde{\mathrm{pr}}_2^*(\eta_2) = \int_M L\eta_1 \wedge \eta_2 \quad \text{for all } \eta_1, \eta_2 \in \Omega(M).$$

If $K(x,y) \neq 0$ for some $(x,y) \in (M \times M)\backslash\Delta$, we can choose η_1, η_2 with disjoint supports such that the left-hand side is non-zero. This is a contradiction. Consequently, we have $K \equiv 0$. ☐

In general, the *Schwartz kernel* of a linear operator $\mathcal{P} : \Omega(M) \to \Omega(M)$ is a distributional form P on $M \times M$ which satisfies[1]

$$\mathcal{P}(\eta)(x) = \int_{y \in M_1} \mathrm{P}(y,x)\eta(y) \quad \text{for all } \eta \in \Omega(M) \text{ and } x \in M_2.$$

We consider the following conditions on $\mathcal{P}$ and P:

(P1) The Schwartz kernel P of $\mathcal{P}$ is a restriction of a smooth form on $\mathrm{Bl}_\Delta(M \times M)$.

(P2) $\mathrm{d} \circ \mathcal{P} + \mathcal{P} \circ \mathrm{d} = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1}$.

(P3) $(\mathcal{P}(\eta_1), \eta_2) = (-1)^{\eta_1}(\eta_1, \mathcal{P}(\eta_2))$ for all η_1, $\eta_2 \in \Omega(M)$.

(P4) $\mathcal{P} \circ \iota_{\mathcal{H}} = 0$, $\pi_{\mathcal{H}} \circ \mathcal{P} = 0$.

(P5) $\mathcal{P} \circ \mathcal{P} = 0$.

Clearly, (P1)–(P3) are equivalent to P being a Hodge propagator from Definition 4.2.4. Conditions (P4) and (P5) play a crucial role in the vanishing results for the Chern-Simons Maurer-Cartan element $\mathfrak{n}$ in Section 4.4 — the more conditions are satisfied, the more vanishing we get.

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[1] We may consider such class of $\mathcal{P}$'s, e.g., pseudo-differential operators, such that P exists and is unique (c.f., the well-known Schwartz kernel theorem).

Definition 4.2.9 (Special Hodge propagator)**.** *We call a Hodge propagator satisfying (P1)–(P5)* special.

The following lemma will be used in the proof of the upcoming proposition.

Lemma 4.2.10. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be two linear operators $\Omega(M) \to \Omega(M)$ with Schwartz kernels* P_1, $\mathrm{P}_2 \in \Omega(\mathrm{Bl}_\Delta(M \times M))$*. Then $\mathcal{P} \coloneqq \mathcal{P}_1 \circ \mathcal{P}_2$ is a smoothing operator, i.e., its Schwartz kernel* P *is a smooth form on $M \times M$.*

Proof. It holds $\mathrm{P}(x_1, x_2) = \pm \int_x \mathrm{P}_2(x_1, x)\mathrm{P}_1(x, x_2)$. The lemma follows from properties of convolution. ☐

A version of the following proposition can be found in [CM10].

Proposition 4.2.11 (Existence of special Hodge propagator)**.** *Every oriented closed Riemannian manifold M admits a special Hodge propagator.*

Proof. Because H is Poincaré dual to Δ, we have for any closed $\kappa \in \Omega_{\mathrm{c}}((M \times M)\backslash\Delta)$ the following:

$$\int_{(M\times M)\backslash\Delta} \mathrm{H} \wedge \kappa = \int_{M\times M} \mathrm{H} \wedge \kappa = \int_{\Delta} \kappa = 0.$$

Poincaré duality for non-compact oriented manifolds (see [BT82]) implies that H is exact on $(M \times M)\backslash\Delta$. Because a manifold with boundary is homotopy equivalent to its interior, the restriction of the blow-down map induces an isomorphism $\pi^* : \mathrm{H}_{\mathrm{dR}}((M \times M)\backslash\Delta) \to \mathrm{H}_{\mathrm{dR}}(\mathrm{Bl}_\Delta(M \times M))$. Poincaré duality for non-compact oriented manifolds (see [BT82]) implies that H is exact on $(M \times M)\backslash\Delta$. Because a manifold with boundary is homotopy equivalent to its interior, the restriction of the blow-down map induces an isomorphism $\pi^* : \mathrm{H}_{\mathrm{dR}}((M \times M)\backslash\Delta) \to \mathrm{H}_{\mathrm{dR}}(\mathrm{Bl}_\Delta(M \times M))$. It follows that $(-1)^n\pi^*\mathrm{H}$ admits a primitive $\mathrm{P} \in \Omega(\mathrm{Bl}_\Delta(M \times M))$. According to Proposition 4.2.8, the corresponding $\mathcal{P}$ satisfies (P1) and (P2).

If we define

$$\tilde{\mathrm{P}} \coloneqq \frac{1}{2}(\mathrm{P} + (-1)^n\tau^*\mathrm{P}) \in \Omega^{n-1}(\mathrm{Bl}_\Delta(M \times M)),$$

then $\tilde{\mathrm{P}}$ satisfies $\tau^*\tilde{\mathrm{P}} = (-1)^n\tilde{\mathrm{P}}$ and is still a primitive to $(-1)^n\pi^*\mathrm{H}$. Proposition 4.2.8 and (4.8) imply that the corresponding $\mathcal{P}$ satisfies (P1)–(P3).

Given $\mathcal{P}$ satisfying (P1)–(P3), we will now show that we can arrange (P4). Let us define

$$\tilde{\mathcal{P}} \coloneqq (\mathbb{1} - \pi_{\mathcal{H}}) \circ \mathcal{P} \circ (\mathbb{1} - \pi_{\mathcal{H}}),$$

where we view $\pi_{\mathcal{H}}$ as a map $\pi_{\mathcal{H}} : \Omega \to \mathcal{H} \subset \Omega$. Then $\tilde{\mathcal{P}}$ is a Hodge homotopy because

$$\mathrm{d} \circ \tilde{\mathcal{P}} + \tilde{\mathcal{P}} \circ \mathrm{d} = (\mathbb{1} - \pi_{\mathcal{H}}) \circ (\mathrm{d} \circ \mathcal{P} + \mathcal{P} \circ \mathrm{d}) \circ (\mathbb{1} - \pi_{\mathcal{H}}) = \mathbb{1} - \pi_{\mathcal{H}}.$$

Using (4.11) and (4.8), we see that $\tilde{\mathcal{P}}$ satisfies (P3). Using the intersection pairing and Proposition 4.2.7, we can write

$$\pi_{\mathcal{H}}(\eta) = \sum_{i=1}^{m} (-1)^{(n+\eta)\nu_i} (*\nu_i, \eta)\nu_i \quad \text{for all } \eta \in \Omega(M),$$

and hence we have for all η_1, $\eta_2 \in \Omega(M)$ the following:

$$\begin{aligned}
&(\mathcal{P}(\pi_{\mathcal{H}}(\eta_1)), \eta_2) \\
&\quad = \sum_{i=1}^{m} (-1)^{(n+\eta_1)\nu_i} (*\nu_i, \eta_1)(\mathcal{P}(\nu_i), \eta_2) \\
&\quad = \sum_{i=1}^{m} (-1)^{(n+1)\nu_i} \int_{M\times M} \mathrm{pr}_1^*(*\nu_i) \wedge \mathrm{pr}_2^*(\mathcal{P}(\nu_i)) \wedge \mathrm{pr}_1^*(\eta_1) \wedge \mathrm{pr}_2^*(\eta_2).
\end{aligned}$$

It follows that the Schwartz kernel of $\mathcal{P} \circ \pi_{\mathcal{H}}$ is the smooth form

$$\mathcal{K}_{\mathcal{P}\circ\pi_{\mathcal{H}}} \coloneqq \sum_{i=1}^{m} (-1)^{(n+1)\nu_i} \mathrm{pr}_1^*(*\nu_i) \wedge \mathrm{pr}_2^*(\mathcal{P}(\nu_i)).$$

Moreover, if we replace $\mathcal{P}$ with $\pi_{\mathcal{H}} \circ \mathcal{P}$, we get the smooth Schwartz kernel $\mathcal{K}_{\pi_{\mathcal{H}}\circ\mathcal{P}\circ\pi_{\mathcal{H}}}$ of $(\pi_{\mathcal{H}} \circ \mathcal{P}) \circ \pi_{\mathcal{H}}$. In the same way, but now using in addition (4.8), we can write

$$\begin{aligned}
&(\pi_{\mathcal{H}}(\mathcal{P}(\eta_1)), \eta_2) \\
&\quad = (-1)^{\eta_1} (\eta_1, \mathcal{P}(\pi_{\mathcal{H}}(\eta_2))) \\
&\quad = (-1)^{\eta_1\eta_2} (\mathcal{P}(\pi_{\mathcal{H}}(\eta_2)), \eta_1) \\
&\quad = \sum_{i=1}^{m} (-1)^{\eta_1\eta_2 + (n+1)\nu_i} \int_{M\times M} \mathrm{pr}_1^*(*\nu_i) \wedge \mathrm{pr}_2^*(\mathcal{P}(\nu_i)) \wedge \mathrm{pr}_1^*(\eta_2) \wedge \mathrm{pr}_2^*(\eta_1) \\
&\quad = \sum_{i=1}^{m} (-1)^{(n+1)\nu_i + n} \int_{M\times M} \mathrm{pr}_2^*(*\nu_i) \wedge \mathrm{pr}_1^*(\mathcal{P}(\nu_i)) \wedge \mathrm{pr}_1^*(\eta_1) \wedge \mathrm{pr}_2^*(\eta_2),
\end{aligned}$$

where in the last equality we pulled back the integral along the twist map. It follows that the Schwartz kernel of $\pi_{\mathcal{H}} \circ \mathcal{P}$ is the smooth form

$$\mathcal{K}_{\pi_{\mathcal{H}}\circ\mathcal{P}} \coloneqq \sum_{i=1}^{m} (-1)^{\nu_i} \mathrm{pr}_1^*(\mathcal{P}(\nu_i)) \wedge \mathrm{pr}_2^*(*\nu_i).$$

The Schwartz kernel of $\tilde{\mathcal{P}} = \mathcal{P} - \pi_{\mathcal{H}} \circ \mathcal{P} - \mathcal{P} \circ \pi_{\mathcal{H}} + \pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}$ is then

$$\tilde{\mathrm{P}} = \mathrm{P} - \pi^* \mathcal{K}_{\mathcal{P} \circ \pi_{\mathcal{H}}} - \pi^* \mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P}} + \pi^* \mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}},$$

which is a smooth form on $\mathrm{Bl}_{\Delta}(M \times M)$. Therefore, $\tilde{\mathrm{P}}$ satisfies (P1)–(P4).

Given $\mathcal{P}$ satisfying (P1)–(P4), we will show that we can arrange (P5). The trick from [CM10] is to define

$$\tilde{\mathcal{P}} = \mathcal{P} \mathrm{d} \mathcal{P}.$$

Applying (P1) and (P2) repeatedly, we compute

$$\begin{aligned} \mathrm{d}\mathcal{P}\mathcal{P}\mathcal{P}\mathrm{d} &= \mathrm{d}\mathcal{P}\mathcal{P} - \mathrm{d}\mathcal{P}\mathcal{P}\mathrm{d}\mathcal{P} \\ &= \mathrm{d}\mathcal{P}\mathcal{P} - \mathrm{d}\mathcal{P}\mathcal{P} + \mathrm{d}\mathcal{P}\mathrm{d}\mathcal{P}\mathcal{P} \\ &= \mathrm{d}\mathcal{P}\mathcal{P} - \mathrm{d}\mathrm{d}\mathcal{P}\mathcal{P}\mathcal{P} \\ &= \mathrm{d}\mathcal{P}\mathcal{P} \\ &= \mathcal{P} - \mathcal{P}\mathrm{d}\mathcal{P}, \end{aligned} \tag{4.13}$$

and hence

$$\tilde{\mathcal{P}} = \mathcal{P} - \mathrm{d}\mathcal{P}\mathcal{P}\mathcal{P}\mathrm{d}.$$

Clearly, $\tilde{\mathcal{P}}$ satisfies (P1) and (P2). As for (P3), we compute

$$(\eta_1, \tilde{\mathcal{P}}\eta_2) = (-1)^{\eta_1} (\mathcal{P}\eta_1, \mathrm{d}\mathcal{P}\eta_2) = (\mathrm{d}\mathcal{P}\eta_1, \mathcal{P}\eta_2) = (-1)^{\eta_1} (\tilde{\mathcal{P}}\eta_1, \eta_2).$$

As for (P5), we have

$$\begin{aligned} \tilde{\mathcal{P}}\tilde{\mathcal{P}} &= \mathcal{P}\mathrm{d}\mathcal{P}(\mathcal{P}\mathrm{d})\mathcal{P} \\ &= \mathcal{P}\mathrm{d}\mathcal{P}\mathcal{P} - \mathcal{P}\mathrm{d}(\mathcal{P}\mathrm{d})\mathcal{P}\mathcal{P} \\ &= \mathcal{P}\mathrm{d}\mathcal{P}\mathcal{P} - \mathcal{P}\mathrm{d}\mathcal{P}\mathcal{P} + \mathcal{P}\mathrm{d}\mathrm{d}\mathcal{P}\mathcal{P} \\ &= 0. \end{aligned} \tag{4.14}$$

In order to show (P4), we have to compute the Schwartz kernel of $\mathrm{d}\mathcal{P}\mathcal{P}\mathcal{P}\mathrm{d}$. By Lemma 4.2.10, the Schwartz kernel T of $T \coloneqq \mathcal{P}\mathcal{P}\mathcal{P}$ is a smooth form on $M \times M$. Therefore, Stokes' formula without the boundary term applies, and we get

$$(\mathrm{d}T\mathrm{d})\eta = \mathrm{d} \int^{M_1} \mathrm{T} \wedge \mathrm{d}\pi_1^*(\eta) = \int^{M_1} \mathrm{d}\mathrm{T} \wedge \mathrm{d}\pi_1^*(\eta) = (-1)^{\mathrm{T}} \int^{M_1} \mathrm{d}_1 \mathrm{d}\mathrm{T} \wedge \pi_1^*(\eta).$$

Here $\mathrm{d}_1 : \Omega(M \times M) \to \Omega(M \times M)$ is the operator defined in local coordinates by

$$\mathrm{d}_1(f(x,y)\,\mathrm{dx}^I\,\mathrm{dy}^J) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x,y)\,\mathrm{dx}^{\mathrm{i}}\,\mathrm{dx}^I\,\mathrm{dy}^J\,.$$

It follows that the Schwartz kernel $\tilde{\mathrm{P}}$ of $\tilde{\mathcal{P}}$ satisfies

$$\tilde{\mathrm{P}} = \mathrm{P} + (-1)^n \mathrm{d}_1 \mathrm{d}\mathrm{T}$$

and is a smooth $(n-1)$-form on $\mathrm{Bl}_\Delta(M \times M)$. Conditions (P1)–(P5) are satisfied. □

Remark 4.2.12 (Property (P5) in dimensions 1 and 2). In dimension 1, every operator of degree -1 satisfies (P5) from degree reasons. In dimension 2, every operator satisfying (P1) and (P2) satisfies (P5) as well, which follows from (4.13) and (4.14). ◁

Remark 4.2.13 (The standard Hodge propagator). Consider the Hodge-de Rham Laplacian $\Delta = \mathrm{d} \circ \mathrm{d}^* + \mathrm{d}^* \circ \mathrm{d} : \Omega(M) \to \Omega(M)$ and its "Green operator" $\mathcal{G}$ of degree 0 (see (v) of Remark 4.2.5 for the collision of terminology) which was defined in [War83, Definition 6.9] by

$$\mathcal{G} \coloneqq \left(\Delta|_{\mathcal{H}(M)^\perp}\right)^{-1} \circ \pi_{\mathcal{H}(M)^\perp},$$

where $\perp$ denotes the L^2-orthogonal complement. We introduce the *standard Hodge homotopy* by

$$\mathcal{P}_{\mathrm{std}} \coloneqq -\mathrm{d}^* \circ \mathcal{P}_\Delta. \tag{4.15}$$

Using the properties of $\mathcal{G}$, d and d^*, one can show that $\mathcal{P}_{\mathrm{std}}$ satisfies (P2)–(P5).

As for (P1), there is the following formula inspired by [Har04]:

$$\mathrm{P}_{\mathrm{std}} = (-1)^{n+1} \lim_{t \to 0} \int_t^\infty \frac{1}{2} \mathrm{d}^* K_\tau \,\mathrm{d}\tau, \tag{4.16}$$

where $\mathrm{K}_t(x,y) = \sum_i (-1)^{ne_i} e^{-\lambda_i t} (*e_i)(x) \wedge e_i(y)$ is the heat kernel of Δ and e_i the L^2-orthonormal eigenbasis of Δ with eigenvalues λ_i (the signs come from our convention for fiberwise integration, c.f., (4.9)). In order to show (P1), a plan is to consider (4.16) in local coordinates and transform it to blow-up coordinates. For $\mathbb{R}^n$ with the standard Euclidean metric g_0, the integral and limit can be computed explicitly, see Section 7.4, and $\mathrm{P}_{\mathrm{std}}$ indeed extends smoothly to the blow-up. See Chapter 7 for some more theory on Hodge propagator. There is a "rather condensed proof" of a version of (P1) in [AS94, Section 4.3] using the Hadamard parametrix construction. ◁

4.3. Chern-Simons Maurer-Cartan element

We first recall ribbon graphs and their labelings based on [CFL15].

Definition 4.3.1 (Ribbon graph). *A* graph Γ *is a quadruple* $(V, H, \mathcal{V}, \mathcal{E})$*, where* V *is a finite set of vertices,* H *a finite set of half-edges,* $\mathcal{V} : H \to V$ *the "vertex map" and* $\mathcal{E} : H \to H$ *with* $\mathcal{E} \circ \mathcal{E} = \mathbb{1}$ *and without fixed points the "edge map". The preimage* $\mathcal{E}^{-1}(h_1) = \{h_1, h_2\}$ *for some* h_1*,* $h_2 \in H$ *is called an* edge*; the set of edges is denoted by* E*. We assume that the graphs are* connected*, i.e., that for any* v_1*,* $\mathrm{v}_2 \in V$ *there exists a path in* E *connecting* v_1 *to* v_2*.*

A ribbon graph *is a graph* Γ *together with a free transitive action* $\mathbb{Z}_{d(\mathrm{v})} \circlearrowright \mathcal{V}^{-1}(\mathrm{v})$ *for every* $\mathrm{v} \in V$*, where*

$$d(\mathrm{v}) \coloneqq |\mathcal{V}^{-1}(\mathrm{v})|$$

is the valency of v*. We denote by* $\mathcal{N} : H \to H$ *the bijection induced by* $1 \in \mathbb{Z}_{d(\mathrm{v})}$ *for every* $\mathrm{v} \in V$*.*

For a ribbon graph Γ*, consider the set of sequences* $(h_n)_{n\in\mathbb{Z}} \subset H$ *such that the following conditions holds:*

$$\forall n \in \mathbb{Z} : \quad h_{n+1} = \begin{cases} \mathcal{E}(h_n) & n \text{ even}, \\ \mathcal{N}(h_n) & n \text{ odd}. \end{cases}$$

Two such sequences $(h_n)_{n\in\mathbb{Z}}$ *and* $(h'_n)_{n\in\mathbb{Z}}$ *are equivalent if and only if there exist* n_0*,* $n'_0 \in \mathbb{Z}$ *both even or both odd such that* $h_{n_0} = h'_{n'_0}$*. An equivalence class* $[(h_n)_{n\in\mathbb{Z}}]$ *is called a* boundary (or a boundary component) of Γ*. The set of boundaries of* Γ *is denoted by* $\partial\Gamma$*.*

An IE ribbon graph *is a ribbon graph* Γ *together with the decomposition* $V = V_{\mathrm{int}} \sqcup V_{\mathrm{ext}}$ *into* internal *and* external vertices V_{int} *and* V_{ext}*, respectively, such that* $d(\mathrm{v}) = 1$ *for all* $\mathrm{v} \in V_{\mathrm{ext}}$*. This decomposition induces the decomposition* $E = E_{\mathrm{int}} \sqcup E_{\mathrm{ext}}$*, where an edge* e *is internal if it connects two internal vertices and is external otherwise. We allow only graphs with* at least one internal vertex*. We often identify an external vertex with its unique adjacent half-edge or the unique adjacent external edge; we call either of these an* external leg*. For any* $\mathrm{b} \in \partial\Gamma$*, we define the* valency of b *by*

$$s(\mathrm{b}) \coloneqq |\mathcal{V}(\mathrm{b}) \cap V_{\mathrm{ext}}|,$$

where $\mathcal{V}(\mathrm{b}) = \{\mathcal{V}(h_n) \mid n \in \mathbb{Z}\}$*. We also have the free transitive* $\mathbb{Z}_{s(\mathrm{b})}$*-action on* $\mathcal{V}(\mathrm{b}) \cap V_{\mathrm{ext}}$ *mapping* $\mathrm{v} \in \mathcal{V}(\mathrm{b}) \cap V_{\mathrm{ext}}$ *to the next external vertex in the sequence* $(\mathcal{V}(h_n))_{n\in\mathbb{Z}}$*. We will denote this action by* $\mathcal{N}$ *again.*

We say that an IE ribbon graph Γ *is* reduced *if* $s(\mathrm{b}) \geq 1$ *for all* $\mathrm{b} \in \partial\Gamma$. *The following letters will be used to denote the numerical invariants of a graph:*

k ... *the number of internal vertices,*
s ... —″— *external vertices.*
l ... —″— *boundary components,*
e ... —″— *internal edges.*

Moreover, we define the genus $g \in \mathbb{N}_0$ *so that the following* Euler formula *holds:*

$$k - e + l = 2 - 2g. \tag{4.17}$$

We denote by RG_{klg} *the set of isomorphism classes of connected IE ribbon graphs with fixed k, l, g. We let* $\overline{\mathrm{RG}}_{klg} \subset \mathrm{RG}_{klg}$ *be the subset of reduced graphs. For* $m \in \mathbb{N}_0$, *we denote by* $\mathrm{RG}^{(m)}_{klg} \subset \mathrm{RG}_{klg}$ *the set of isomorphism classes of connected IE ribbon graphs with all internal vertices* m-valent, *i.e., with*

$$d(\mathrm{v}) = m \quad \textit{for all } \mathrm{v} \in V_{\mathrm{int}}.$$

The notation $\Gamma \in \mathrm{RG}_{klg}$ *means that* Γ *is a representative of an equivalence class* $[\Gamma] \in \mathrm{RG}_{klg}$.

Remark 4.3.2 (On ribbon graphs). (i) An m-valent ribbon graph with $m \geq 2$ has a unique decomposition $V = V_{\mathrm{int}} \sqcup V_{\mathrm{ext}}$, and hence we can omit writing IE.

(ii) In this text, we will use only reduced ribbon graphs. Non-reduced ribbon graphs may play a role in the extension of the theory of $\mathrm{dIBL}^{\mathrm{n}}(C(\mathcal{H}))$ to non-reduced cyclic cochains or in the weak IBL_∞-theory (see Remarks 3.2.9 and 3.3.2). ◁

Definition 4.3.3 (Labeling). *A* labeling *of an IE ribbon graph* Γ *is the triple* $L = (L_1, L_2, L_3)$, *where* L_i *have the following meanings:*

- *The symbol* L_1 *represents an ordering of internal vertices* $(=: L_1^v)$, *and of boundary components* $(=: L_1^b)$. *Given* L_1, *we write* $V_{\mathrm{ext}} = \{\mathrm{v}_1, \ldots, \mathrm{v}_k\}$, $\partial\Gamma = \{\mathrm{b}_1, \ldots, \mathrm{b}_l\}$ *and denote*
  $$d_i \coloneqq d(\mathrm{v}_i) \quad \textit{and} \quad s_j \coloneqq s(\mathrm{b}_j).$$
- *The symbol* L_2 *represents an ordering and orientation of internal edges. Given* L_2, *we write* $E_{\mathrm{int}} = \{\mathrm{e}_1, \ldots, \mathrm{e}_e\}$ *and* $\mathrm{e}_i = \{h_{i,1}, h_{i,2}\}$ *for* $h_{i,1}$, $h_{i,2} \in H$.

- *The symbol L_3 represents an ordering of half-edges at every internal vertex (=: L_3^v) and of external vertices at every boundary component (=: L_3^b), both compatible with the* ribbon structure *(:= the $\mathbb{Z}_m$-actions). Given L_3, we write $\mathcal{V}^{-1}(\mathrm{v}) = \{h_{\mathrm{v},1}, \ldots, h_{\mathrm{v},d(\mathrm{v})}\}$ and $\mathcal{V}(\mathrm{b}) \cap V_{\mathrm{ext}} = \{\mathrm{v}_{\mathrm{b},1}, \ldots, \mathrm{v}_{\mathrm{b},s(\mathrm{b})}\}$ with $\mathcal{N}(h_{\mathrm{v},i}) = h_{\mathrm{v},i+1}$ and $\mathcal{N}(\mathrm{v}_{\mathrm{b},j}) = \mathrm{v}_{\mathrm{b},j+1}$ for all i, j, respectively.*

We sometimes call L_i partial labelings *and L a* full labeling*. A ribbon graph Γ together with a labeling L is called a* labeled ribbon graph*.*

Given a ribbon graph Γ, one can construct an oriented surface with boundary Σ_Γ — the thickening of Γ — in the obvious way and a closed oriented surface $\hat{\Sigma}_\Gamma$ by gluing oriented disks to the oriented boundaries of Σ_Γ. If partial labelings L_1 and L_2 are given, we obtain the following chain complex with oriented chain groups (vector spaces over $\mathbb{R}$):

$$C_2 \coloneqq \langle \mathrm{b}_1, \ldots, \mathrm{b}_l \rangle \xrightarrow{\partial_2} C_1 \coloneqq \langle \mathrm{e}_1, \ldots, \mathrm{e}_e \rangle \xrightarrow{\partial_1} C_0 \coloneqq \langle \mathrm{v}_k, \ldots, \mathrm{v}_1 \rangle. \tag{4.18}$$

Here b_i stands for the oriented disc glued to the i-th boundary component of Σ_Γ and now being mapped into $\hat{\Sigma}_\Gamma$, e_i stands for the 1-simplex in $\hat{\Sigma}_\Gamma$ corresponding to the i-th internal edge, v_i stands for the 0-simplex in $\hat{\Sigma}_\Gamma$ corresponding to the i-th internal vertex, and the boundary map ∂ is the "geometric" boundary operator. The homology of this chain complex is isomorphic to the singular homology $\mathrm{H}(\hat{\Sigma}) \coloneqq \mathrm{H}(\hat{\Sigma}_\Gamma; \mathbb{R})$.

The orientation of C_i (:= the order of generators in (4.18)) induces naturally an orientation of $\mathrm{H}(\hat{\Sigma}_\Gamma)$. The construction from [CFL15, Appendix A] is as follows. We pick complements H_i of $\mathrm{im}(\partial_{i+1})$ in $\ker(\partial_i)$ and complements V_i of $\ker(\partial_i)$ in C_i and write

$$C_2 = V_2 \oplus H_2 \xrightarrow{\partial_2} C_1 = V_1 \oplus H_1 \oplus \mathrm{im}(\partial_2) \xrightarrow{\partial_1} C_0 = \mathrm{im}(\partial_1) \oplus H_0.$$

We orient V_i arbitrarily and transfer the orientation to $\mathrm{im}(\partial_i)$ via $\partial_i : V_i \xrightarrow{\simeq} \mathrm{im}(\partial_i)$. Then, assuming the direct sum orientation, orienting H_i is equivalent to orienting C_i, and we obtain the orientation of $\mathrm{H}_i(\hat{\Sigma}_\Gamma)$ via the canonical projection $\pi : H_i \xrightarrow{\simeq} \mathrm{H}_i(\hat{\Sigma}_\Gamma) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$. This construction does not depend on the choices of complements and orientations of V_i.

Definition 4.3.4 (Compatibility of L_1 and L_2)**.** *Given a ribbon graph Γ with partial labelings L_1 and L_2, we say that L_2* is compatible with *L_1 if the orientation on* $\mathrm{H}(\widehat{\Sigma}_\Gamma)$ *induced by* (4.18) *agrees with the canonical orientation*

$$\mathrm{H}(\widehat{\Sigma}_\Gamma) = \langle \mathrm{v}_1 + \cdots + \mathrm{v}_k \rangle \oplus \mathrm{H}_1(\widehat{\Sigma}_\Gamma) \oplus \langle \mathrm{b}_1 + \cdots + \mathrm{b}_l \rangle,$$

where $\mathrm{H}_1(\widehat{\Sigma}_\Gamma)$ *is oriented using the canonical symplectic intersection form.*

Given a labeled IE ribbon graph Γ, the set of half-edges adjacent to internal vertices $\mathcal{V}^{-1}(V_{\mathrm{int}})$ can be ordered in two ways corresponding to writing

$$2e + (s_1 + \cdots + s_l) = d_1 + \cdots + d_k.$$

This leads to the following definition.

Definition 4.3.5 (Edge order and vertex order)**.** *For a labeled IE ribbon graph* Γ*, we define the following two orders on the set of half-edges* H*:*

- Edge order: *The first* $2e$ *half-edges* $h^{\mathrm{e}}_{i,j}$ *are the ones from internal edges; they are ordered according to* L_2*. They are followed by blocks of* $s_1, \ldots, s_l$ *half-edges* $h^{\mathrm{b}}_{i,j}$ *which come from the boundary components* $i = 1, \ldots, l$*, respectively, and which are ordered according to* L^b_3 *inside the blocks. Schematically, we have*

$$(h^{\mathrm{e}}_{1,1}h^{\mathrm{e}}_{1,2})\ldots(h^{\mathrm{e}}_{e,1}h^{\mathrm{e}}_{e,2})(h^{\mathrm{b}}_{1,1}\ldots h^{\mathrm{b}}_{1,s_1})\ldots(h^{\mathrm{b}}_{l,1}\ldots h^{\mathrm{b}}_{l,s_l}).$$

- Vertex order: *It consists of blocks of* $d_1, \ldots, d_k$ *half-edges* $h^{\mathrm{v}}_{i,j}$ *which come from internal vertices* $1, \ldots, k$*, and which are ordered according to* L^{v}_3 *inside the blocks. Schematically, we have*

$$(h^{\mathrm{v}}_{1,1}\ldots h^{\mathrm{v}}_{1,d_1})\ldots(h^{\mathrm{v}}_{k,1}\ldots h^{\mathrm{v}}_{k,d_k}).$$

We denote by $\sigma_L \in \mathbb{S}_{|H|}$ *the* permutation from the edge to the vertex order *which is constructed such that the* i*-th half-edge in the edge order is the same as the* $\sigma_L(i)$*-th half-edge in the vertex order.*

From now on, we will consider only reduced trivalent ribbon graphs $\overline{\mathrm{RG}}^{(3)}_{klg}$ with k, $l \geq 1$, $g \geq 0$. We will often use the equation

$$2e + s = 3k. \tag{4.19}$$

Definition 4.3.6 (Chern-Simons Maurer-Cartan element)**.** *Let* M *be an oriented closed Riemannian manifold, and let* $\mathrm{P} \in \Omega^{n-1}(\mathrm{Bl}_\Delta(M \times M))$ *be an admissible Hodge propagator from Definition 4.2.4. The* Chern-Simons Maurer-Cartan element $\mathfrak{n}$*, or* formal pushforward Maurer-Cartan element*, is the collection of*

$$\mathfrak{n}_{lg} \in \hat{\mathrm{E}}_l C(\mathcal{H}(M)) \quad \textit{for all } l \geq 1, g \geq 0$$

defined on generating words $\omega_i = \alpha_{i1} \dots \alpha_{is_i} \in \mathrm{B}^{\mathrm{cyc}}\mathcal{H}(M)$*, where* $\alpha_{ij} = \theta\eta_{ij}$ *with* $\eta_{ij} \in \mathcal{H}(M)$ *for* $s_i \geq 1$ *and* $i = 1, \dots, l$*, by the formula*

$$\begin{aligned}&\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \dots \otimes \omega_l)\\ &\quad\coloneqq \frac{1}{l!}\sum_{[\Gamma]\in\overline{\mathrm{RG}}^{(3)}_{klg}} \frac{1}{|\operatorname{Aut}(\Gamma)|}(-1)^{s(k,l)+P(\omega)} \sum_{L_1,\, L_3^b} (-1)^{\sigma_L} I(\sigma_L),\end{aligned} \tag{4.20}$$

which we explain as follows:

- *The second sum is over all partial labelings* L_1 *and* L_3^b *of a representative* Γ *of* $[\Gamma]$*. In every summand, we complete* L_1 *and* L_3^b *to a full labeling* $L = (L_1, L_2, L_3)$ *by picking an arbitrary* L_3^v *and an arbitrary* L_2 *compatible with* L_1*.*
- *Suppose that* Γ *and* L_1 *are* admissible *with respect to the input* $\omega_1, \dots, \omega_l$*; this means that* Γ *has* l *boundary components and that the* i*-th boundary component has valency* s_i *for every* $i = 1, \dots, l$*. In this case, denoting* $\sigma = \sigma_L$*, we define*

  $$\begin{aligned}I(\sigma_L) \coloneqq \int_{x_1,\dots,x_k} &\mathrm{P}(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots \mathrm{P}(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})})\\ &\eta_{11}(x_{\xi(\sigma_{2e+1})}) \cdots \eta_{ls_l}(x_{\xi(\sigma_{2e+s})}),\end{aligned} \tag{4.21}$$

  where $\xi : \{1, \dots, 3k\} \to \{1, \dots, k\}$ *is the function defined by*

  $$\xi(3j-2) = \xi(3j-1) = \xi(3j) \coloneqq j$$

  for all $j = 1, \dots, k$*,* $s = s_1 + \dots + s_l$*,* $\eta(x_i)$ *denotes the pullback of* η *along the canonical projection* $\pi_i : M^{\times k} \to M$ *to the* i*-th component* M_i*,* $\mathrm{P}(x_i, x_j)$ *denotes the pullback of* P *along* $\pi_i \times \pi_j : M^{\times k} \to M_i \times M_j$*, and* $\int_{x_1,\dots,x_k}$ *denotes the integral of an* nk*-form over* k *copies of* M*.*

  If Γ *and* L_1 *are not admissible, then we set* $I(\sigma_L) \coloneqq 0$*.*
- $s(k,l) \coloneqq k + kl(n-1) + \frac{1}{2}k(k-1)n \mod 2$*.*
- $P(\omega) \coloneqq \sum_{i=1}^{l}\sum_{j=1}^{s_i}(s - s_1 - \dots - s_{i-1} - j)\eta_{ij} \mod 2$*.*

In order to show that $\mathfrak{n}_{lg}$ is well-defined and that the collection $(\mathfrak{n}_{lg})$ satisfies Definition 3.2.6 for $\mathrm{dIBL}(C(\mathcal{H}(M)))$, there are several things to check:

1. The integral $I(\sigma_L)$ converges.
2. The sums are finite.

(3) The product $(-1)^{\sigma_L} I(\sigma_L)$ is independent of the choice of L_3^v and L_2 compatible with L_1.

(4) The sum over labelings is independent of the chosen representative Γ in an isomorphism class from $\overline{\mathrm{RG}}_{klg}^{(3)}$.

(5) The map $\mathfrak{n}_{lg} : \mathrm{B}^{\mathrm{cyc}}\mathcal{H}(M)[3-n]^{\otimes l} \to \mathbb{R}$ is graded symmetric on permutations of its inputs $\mathrm{s}\omega_i$.

(6) The map $\mathfrak{n}_{lg}$ is graded symmetric on cyclic permutations of the components α_{ij} of each ω_i.

(7) The degree condition 1) from Definition 3.2.6 holds with $d = n - 3$.

(8) The filtration-degree condition 2) from Definition 3.2.6 holds with $\gamma = 2$.

(9) The Maurer-Cartan equation (3.20) holds.

Condition 9) will be proven in [CVon] using the theory of iterated blow-ups.

Condition 1) follows from the property (P1) of a Hodge propagator P. The idea is to associate one integration variable to each internal half-edge, obtaining the space $\{x_i^{(1)}, x_i^{(2)}, x_i^{(3)} \mid i = 1, \dots, k\}$, and blowing-up pairs of variables connected by internal edges. By (P1), the integrand of (4.21) lifts to a smooth form on such a space. The integral $\int_{x_1 \dots x_k}$ is then realized as an integral over the closure of a lift of the subspace $\{x_i^{(1)} = x_i^{(2)} = x_i^{(3)}\}$. By compactness, the integral converges. A detailed proof is under preparation in [CVon].

In this text, we will take 1) and 9) for granted.

Lemma 4.3.7. *Conditions 2) – 8) hold.*

Proof. As for 2), the fixed input $\omega_1, \dots, \omega_l$ fixes the number s of external vertices of Γ by admissibility. Expressing e from (4.17) and plugging it in (4.19) gives

$$k = s + 2l + 4g - 4. \tag{4.22}$$

We see that all parameters are fixed. Now, there is only finitely many elements with fixed s in $\overline{\mathrm{RG}}_{klg}^{(3)}$, and each of them has only finitely many labelings. Therefore, the sums are finite.

As for 3), we have to consider the orientation of the complex (4.18). Clearly, if two L_2's are compatible with L_1, then they differ by an even number of the following operations: a transposition of two internal edges or a change of the orientation of an internal edge.

The former operation introduces no sign in $(-1)^{\sigma_L}$ but generates the sign $(-1)^{n-1}$ in $I(\sigma_L)$ from swapping the corresponding P's. The latter operation induces the sign -1 in $(-1)^{\sigma_L}$ and the sign $(-1)^n$ in $I(\sigma_L)$ from the symmetry $\mathrm{P}(x,y) = (-1)^n\mathrm{P}(y,x)$. Because the overall signs in $(-1)^{\sigma_L}I(\sigma_L)$ are the same, an even number of these operations preserves $(-1)^{\sigma_L}I(\sigma_L)$. This implies the independence of an L_2 compatible with L_1. A change in L_3^v produces no sign in $(-1)^{\sigma_L}$ because every internal vertex is trivalent and a cyclic permutation of an odd number of elements is even. The integral $I(\sigma_L)$ remains unchanged because the change in σ_L is compensated by the composition with ξ. Independence of the choice of L_3^v follows.

As for 4), every isomorphism of ribbon graphs $\Gamma \to \Gamma'$ induces the bijection $L \mapsto L'$ of compatible labelings such that $\sigma_L = \sigma_{L'}$ (L' is the "pushforward" labeling). The independence of the choice of a representative of $[\Gamma]$ follows.

As for 5), let $\mu \in \mathbb{S}_l$ be a permutation of the inputs $\mathrm{s}\omega_1$, …, $\mathrm{s}\omega_l$. The set of graphs which admit an admissible labeling is the same for both $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \omega_l)$ and $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_{\sigma_1^{-1}} \otimes \cdots \otimes \omega_{\sigma_l^{-1}})$; we will pick one such Γ and study the admissible labelings L and L', respectively. We write $\eta_i = \eta_{i1}\dots\eta_{is_i}$ and $\Omega_i = \mathrm{s}\omega_i$ for all i, j, and denote by $I'(\sigma_{L'})$ the integral in the definition of $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_{\mu_1^{-1}} \otimes \cdots \otimes \omega_{\mu_l^{-1}})$. Let $\tilde{\mu} \in \mathbb{S}_{3k}$ be the permutation which acts as the identity on 1, …, $2e$ and as the block permutation determined by μ on $2e+1$, …, $2e+s$ divided into l blocks of lengths s_1, …, s_l. For any $\sigma \in \mathbb{S}_{3k}$, we have

$$\begin{aligned}
I'(\sigma) = \int_{x_1,\dots,x_k} & \mathrm{P}(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots \mathrm{P}(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) \\
&\qquad \eta_{\mu_1^{-1}1}(x_{\xi(\sigma_{2e+1})}) \dots \eta_{\mu_l^{-1}s_{\mu_l^{-1}}}(x_{\xi(\sigma_{2e+s})}) \\
= \varepsilon(\mu,\eta) \int_{x_1,\dots,x_k} & \mathrm{P}(x_{\xi((\sigma\circ\tilde{\mu})_1)}, x_{\xi((\sigma\circ\tilde{\mu})_2)}) \cdots \mathrm{P}(x_{\xi((\sigma\circ\tilde{\mu})_{2e-1})}, x_{\xi((\sigma\circ\tilde{\mu})_{2e})}) \\
&\qquad\qquad \eta_{11}(x_{\xi((\sigma\circ\tilde{\mu})_{2e+1})}) \dots \eta_{ls_l}(x_{\xi((\sigma\circ\tilde{\mu})_{2e+s})}) \\
= \varepsilon(\mu,\eta) & I(\sigma\circ\tilde{\mu}).
\end{aligned}$$

The precomposition with $\tilde{\mu}$ corresponds to a bijection $(L_1, L_3^b) \mapsto (L_1', L_3^{b\prime})$ of partial labelings for $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1\dots\omega_l)$ and $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_{\mu_1^{-1}} \otimes \cdots \otimes \omega_{\mu_l^{-1}})$, respectively. However, if L_2 is compatible with L_1, then in order to get an L_2' compatible with L_1', the labeling L_2 has to be altered by as many operations of switching two internal edges or changing the orientation of an internal edge as there are transpositions in μ. We explained in the proof of 3) that this produces the sign $(-1)^{(n-1)\mu}$ in $(-1)^{\sigma_{L'}}I(\sigma_{L'})$. Therefore, after the choice of compatible L_2 and L_2', we have

$$(-1)^{\sigma_{L'}} I'(\sigma_{L'}) = (-1)^{(n-1)\mu}(-1)^{\tilde{\mu}}\varepsilon(\mu,\eta)(-1)^{\sigma_L} I(\sigma_L).$$

If we view η as $\eta_{11}\dots\eta_{ls_l}$, we can understand $(-1)^{P(\omega)}$ as the Koszul sign $\varepsilon(\theta,\eta)$. Similarly, we write $(-1)^{P(\mu(\omega))} = \varepsilon(\theta,\mu(\eta))$, where we first view η as $\eta_1 \otimes \dots \otimes \eta_l$ to apply μ and then as the list of components η_{ij} to compute the Koszul sign (this is a little ambiguity in our notation). If we denote by $\overline{\mu}$ the permutation of 1, …, s permuting the l blocks of lengths s_1, …, s_l according to μ, then $\overline{\mu}$ has the same sign as $\tilde{\mu}$, and the decomposition of $\varepsilon(\theta,\mu(\eta))$ into the moves

$$\begin{aligned}\theta_1\dots\theta_s\eta_{\mu_1^{-1}1}\dots\eta_{\mu_l^{-1}s_{\mu_l^{-1}}} &\xrightarrow{(1)} \theta_{\overline{\mu}_1}\dots\theta_{\overline{\mu}_s}\eta_{11}\dots\eta_{ls_l} \xrightarrow{(2)} \theta_{\overline{\mu}_1}\eta_{11}\dots\theta_{\overline{\mu}_s}\eta_{ls_l}\\ &\xrightarrow{(3)} \theta_1\eta_{\mu_1^{-1}1}\dots\theta_s\eta_{\mu_l^{-1}s_l}\end{aligned}$$

shows that

$$(-1)^{P(\mu(\omega))} = \underbrace{(-1)^{\tilde{\mu}}\varepsilon(\mu,\eta)}_{(1)}\underbrace{(-1)^{P(\omega)}}_{(2)}\underbrace{\varepsilon(\mu,\omega)}_{(3)}.$$

Using this, we write

$$(-1)^{P(\mu(\omega))}(-1)^{\sigma_{L'}} I'(\sigma_{L'}) = \varepsilon(\mu,\omega)(-1)^{(n-1)\mu}(-1)^{P(\omega)}(-1)^{\sigma_L} I(\sigma_L),$$

and compute

$$\begin{aligned}&\mathfrak{n}_{lg}(\Omega_{\mu_1^{-1}}\otimes\dots\otimes\Omega_{\mu_l^{-1}})\\ &\quad= \varepsilon(\mu(\mathrm{s}),\mu(\omega))\mathfrak{n}_{lg}(\mathrm{s}^l\omega_{\mu_1^{-1}}\otimes\dots\otimes\omega_{\mu_l^{-1}})\\ &\quad= \varepsilon(\mu(\mathrm{s}),\mu(\omega))(-1)^{|s|\mu}\varepsilon(\mu,\omega)\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1\otimes\dots\otimes\omega_l)\\ &\quad= \underbrace{\varepsilon(\mu(\mathrm{s}),\mu(\omega))}_{(1)}\underbrace{(-1)^{|s|\mu}\varepsilon(\mu,\omega)}_{(2)}\underbrace{\varepsilon(\mathrm{s},\omega)}_{(3)}\mathfrak{n}_{lg}(\mathrm{s}\omega_1\otimes\dots\otimes\mathrm{s}\omega_l)\\ &\quad= \varepsilon(\mu,\Omega)\mathfrak{n}_{lg}(\Omega_1\otimes\dots\otimes\Omega_l).\end{aligned}$$

We used $|s| = n-1 \bmod 2$, and the last equality follows from the decomposition of $\varepsilon(\mu,\Omega)$ into the moves

$$\begin{aligned}\mathrm{s}_1\omega_1\dots\mathrm{s}_l\omega_l \xrightarrow{(3)} \mathrm{s}_1\dots\mathrm{s}_l\omega_1\dots\omega_l &\xrightarrow{(2)} \mathrm{s}_{\mu_1^{-1}}\dots\mathrm{s}_{\mu_l^{-1}}\omega_{\mu_1^{-1}}\dots\omega_{\mu_l^{-1}}\\ &\xrightarrow{(1)} \mathrm{s}_{\mu_1^{-1}}\omega_{\mu_1^{-1}}\dots\mathrm{s}_{\mu_l^{-1}}\omega_{\mu_l^{-1}}.\end{aligned}$$

This proves the symmetry of $\mathfrak{n}_{lg}$.

As for 6), fix an $i = 1, \ldots, l$ and let $\mu \in \mathbb{S}_{s_i}$ be a cyclic permutation permuting the components of $\omega_i = \alpha_{i1} \ldots \alpha_{is_i}$. Similarly to the previous case, we denote by $\tilde{\mu}$ the corresponding permutation of $1, \ldots, 3k$ and get a bijection $(L_1, L_3^b) \mapsto (L_1' = L_1, L_3^{b\prime})$ of admissible labelings of a given graph Γ for $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \alpha_{i1} \ldots \alpha_{is_i} \otimes \cdots \otimes \omega_l)$ and $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \alpha_{i\mu_1^{-1}} \ldots \alpha_{i\mu_{s_i}^{-1}} \otimes \cdots \otimes \omega_l)$, respectively. This time, there is no change in L_1, and thus we can take $L_2' = L_2$, producing no sign. Therefore, we have

$$(-1)^{\sigma_{L'}} I'(\sigma_{L'}) = (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta_i)(-1)^{\sigma_L} I(\sigma_L),$$

where $\varepsilon(\mu, \eta_i)$ comes from permuting the forms in $I'(\sigma_{L'})$. Further, we deduce

$$(-1)^{P(\mu(\omega))} = (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta_i)(-1)^{P(\omega)} \varepsilon(\mu, \omega_i),$$

and hence

$$\begin{aligned}&\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \alpha_{i\mu_1^{-1}} \ldots \alpha_{i\mu_{s_i}^{-1}} \otimes \cdots \otimes \omega_l) \\ &\qquad = \varepsilon(\mu, \omega_i)\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \alpha_{i1} \ldots \alpha_{is_i} \otimes \cdots \otimes \omega_l).\end{aligned}$$

This shows the symmetry of $\mathfrak{n}_{lg}$ on cyclic permutations of the components of ω_i.

As for 7), suppose that $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \omega_l) \neq 0$, and let D denote the total form-degree of the input $\eta_{11}, \ldots, \eta_{ls_l} \in \mathcal{H}(M)$; i.e., we define

$$D \coloneqq \deg(\eta_{11}) + \cdots + \deg(\eta_{1s_1}) + \cdots + \deg(\eta_{l1}) + \cdots + \deg(\eta_{ls_l}).$$

Clearly, we must have

$$nk = (n-1)e + D, \tag{4.23}$$

where the left-hand side is the dimension of $M^{\times k}$ and the right-hand side the form-degree of the integrand of $I(\sigma_L)$. If we plug in e from (4.17) and k from (4.22), we get

$$\begin{aligned}D &= nk - (n-1)e \\ &= nk - (n-1)(k + l + 2g - 2) \\ &= k - (n-1)(l + 2g - 2) \\ &= s + 2l + 4g - 4 - (n-1)(l + 2g - 2) \\ &= s - (n-3)(l + 2g - 2).\end{aligned}$$

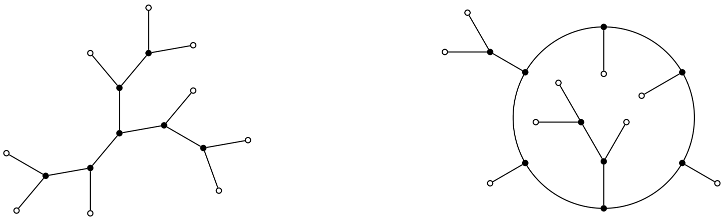


Figure 4.2.: A tree and a circular graph. Internal vertices are denoted with a full dot and external vertices with an empty dot.

It follows that

$$|\mathfrak{n}_{lg}| = |\mathrm{s}^l| + |\omega_1| + \dots + |\omega_l| = l(n-3) + D - s = -2(n-3)(g-1).$$

This is exactly the degree from Definition 3.2.6.

As for 8), if $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \dots \otimes \omega_l) \neq 0$, then

$$s = k - 2l - 4g - 4 \geq 1 + 2(2 - 2g - l) = 1 + 2\chi_{0lg},$$

and hence $\mathfrak{n}_{lg} \in \mathcal{F}^{1+2\chi_{0lg}}\hat{\mathrm{E}}_l C$ for the filtration induced from the dual of the filtration of $\mathrm{B}^{\mathrm{cyc}}\mathcal{H}$ by weights. Therefore, we get

$$\|\mathfrak{n}_{lg}\| \geq 1 + 2\chi_{0lg} > 2\chi_{0lg} \quad \text{for all } l \geq 1, g \geq 0.$$

This finishes the proof. □

Definition 4.3.8 (Vertices of types A, B, C and some special graphs). *Let* $\Gamma \in \mathrm{RG}^{(3)}_{klg}$ *be a trivalent ribbon graph and* v *its internal vertex. We say that* v *is of* type *A*, *B* or *C* *if it is connected to precisely* 1, 2 *or* 3 *internal vertices, respectively (see Figure 4.4). The graph* Γ *is called (see Figures 4.2 and 4.3):*

- *a* tree *if* $[\Gamma] \in \mathrm{RG}_{k10}$ *for some* $k \geq 1$*;*
- circular *if* $[\Gamma] \in \mathrm{RG}_{k20}$ *for some* $k \geq 1$*;*
- *the* Y-graph *is the unique tree with* $k = 1$*;*
- *an* O_k-graph *if* Γ *is circular with* k *internal vertices and no A-vertex.*

We denote the Y*-graph simply by* Y*.*

Remark 4.3.9 (On A, B, C vertices and special graphs). We observe the following:

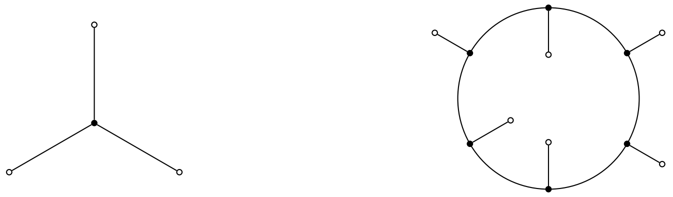


Figure 4.3.: The Y-graph and an O_6-graph.

(i) A trivalent graph $\Gamma \neq Y$ has each internal vertex of type A, B or C.

(ii) The term $\mathfrak{n}_{10}$ is a sum over trees, and the term $\mathfrak{m}_{10}$ is the contribution of the Y-graph to $\mathfrak{n}_{10}$ (see Proposition 4.3.10 below). The term $\mathfrak{n}_{20}$ is a sum over circular graphs. $\lhd$

Wee will also denote by A, B, C the numbers of internal vertices of the corresponding type. Under the change of variables

$$\begin{aligned} s &= 2A + B, \\ e &= B + \frac{1}{2}A + \frac{3}{2}C, \\ k &= A + B + C, \end{aligned} \tag{4.24}$$

the trivalent formula (4.19) becomes trivial and the Euler formula (4.17) becomes

$$C - A = 2l - 4 + 4g. \tag{4.25}$$

Proposition 4.3.10 (Chern-Simons Maurer-Cartan element). *The collection* $\mathfrak{n} = (\mathfrak{n}_{lg})$ *from Definition 4.3.6 is a Maurer-Cartan element for* $\mathrm{dIBL}(\mathcal{H}(M))$ *which is compatible with* $\mathfrak{m}$*. In particular, the* A_∞*-algebra* $\mathcal{H}(M)_{\mathfrak{n}}$ *is homologically unital and augmented.*

Proof. The fact that $\mathfrak{n}$ is a Maurer-Cartan element follows from Lemma 4.3.7 assuming 1) and 9) from [CVon].

As for the compatibility with $\mathfrak{m}$, the only graph contributing to $\mathfrak{n}_{10}(\mathrm{s}\alpha_1\alpha_2\alpha_3)$ is the Y-graph with $k = 1$. The group $\mathrm{Aut}(Y)$ consists of three rotations, and there is only one possible L_1, no L_2 and three L_3^b. In Definition 4.3.6, we get $s(1,1) = n - 2$, $(-1)^{\sigma_L} = 1$ because a cyclic permutation of an odd number of elements is even, and also

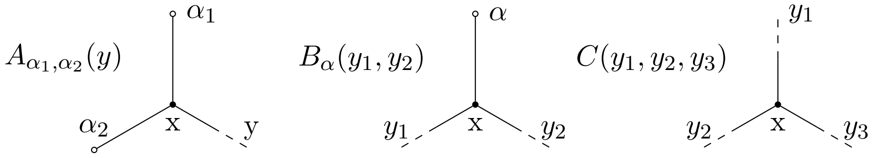


Figure 4.4.: Trivalent vertices of types A, B and C with the corresponding forms A_{α_1,α_2}, B_α and C, respectively.

$P(\alpha_1\alpha_2\alpha_3) = \eta_2$. Finally, we compute

$$\begin{aligned}\mathfrak{n}_{10}(\mathrm{s}\alpha_1\alpha_2\alpha_3) &= \frac{1}{3}(-1)^{n-2+\eta_2}\sum_{L_3^b}\int_x \alpha_1(x_{\xi(\sigma_1)})\alpha_2(x_{\xi(\sigma_2)})\alpha_3(x_{\xi(\sigma_3)})\\ &= (-1)^{n-2+\eta_2}\int_M \eta_1\wedge\eta_2\wedge\eta_3\\ &= \mathfrak{m}_{10}(\mathrm{s}\alpha_1\alpha_2\alpha_3).\end{aligned}$$

□

Definition 4.3.11 (Contributions of A, B, C vertices)**.** *Consider an internal vertex of type A, B or C as in Figure 4.4. We define the following smooth forms on M, $M^{\times 2}$ and $M^{\times 3}$, respectively:*[2]

$$\begin{aligned}A_{\alpha_1,\alpha_2}(y) &:= \int_x \mathrm{P}(y,x)\eta_1(x)\eta_2(x),\\ B_\alpha(y_1,y_2) &:= \int_x \mathrm{P}(y_1,x)\mathrm{P}(x,y_2)\eta(x),\\ C(y_1,y_2,y_3) &:= \int_x \mathrm{P}(x,y_1)\mathrm{P}(x,y_2)\mathrm{P}(x,y_3).\end{aligned}$$

4.4. Results about vanishing of Chern-Simons Maurer-Cartan element

In the situation of Definition 4.3.6, let $\Gamma \in \overline{\mathrm{RG}}^{(3)}_{klg}$ be a reduced trivalent ribbon graph, $L = (L_1, L_2, L_3)$ its labeling, x_i the integration variable associated to the i-th internal vertex, $\mathrm{P}(x_i, x_j)$ an admissible Hodge propagator on the oriented internal edge between x_i and x_j, and $\alpha_{ij} \in \mathcal{H}(M)[1]$ the harmonic form on the j-th external vertex on the i-th boundary component. Recall that we denote by $\omega_i = \mathrm{s}\alpha_{i1}\dots\alpha_{is_i}$ the i-th input of $\mathfrak{n}_{lg}$ and by D the total form-degree of all inputs.

By saying "*a graph vanishes*" we mean that $I(\sigma_L) = 0$ in the given context.

---

[2] The definitions can be made precise in local coordinates. Smoothness of A_{α_1,α_2} is clear, smoothness of B_α follows from Lemma 4.2.10, and smoothness of C can be shown by a similar argument.

Proposition 4.4.1 (Vanishing of graphs with $\mathbb{1}$)**.** *In the setting of Definition 4.3.6, suppose that the following condition is satisfied:*

(V_1) *Every graph $\Gamma \in \overline{\mathrm{RG}}^{(3)}_{klg}$, $\Gamma \neq Y$ which has $\mathbb{1} = \theta 1 \in \mathcal{H}(M)[1]$ at an external vertex vanishes.*

Then $\mathfrak{n}$ is strictly reduced, and the following holds depending on the dimension n:

- *(a) For $n > 3$: All graphs which are not trees or circular vanish. Therefore, $\mathfrak{n}_{lg} = 0$ for all $(l, g) \neq (1, 0)$, $(2, 0)$, and it follows that all higher operations $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ vanish on the chain level.*
- *(b) For $n = 3$: A tree vanishes unless all $\eta_1, \ldots, \eta_s$ are one-forms. Therefore, $\mathfrak{n}_{10}(\mathrm{s}\alpha_1 \ldots \alpha_s) \neq 0$ implies $\deg(\eta_i) = 1$ for all i.*
- *(c) For $n < 3$: All trees except for Y vanish. Therefore, we have $\mathfrak{n}_{10} = \mathfrak{m}_{10}$, and consequently $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}^{\mathfrak{m}}_{110}$.*

Moreover, we have

- *(d) A circular graph vanishes unless all $\eta_{11}, \ldots, \eta_{2s_2}$ are one-forms. Therefore, $\mathfrak{n}_{20}(\mathrm{s}^2\alpha_{11} \ldots \alpha_{1s_1} \otimes \alpha_{21} \ldots \alpha_{2s_2}) \neq 0$ implies $\deg(\eta_{ij}) = 1$ for all i, j.*

In addition to (V_1), suppose that $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$. Then:

- *(e) All circular graphs vanish. Therefore, we have $\mathfrak{n}_{20} = 0$, and consequently $\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}_{120}$.*
- *(f) For $n \leq 6$: All trees except for Y vanish. Therefore, we have $\mathfrak{n}_{10} = \mathfrak{m}_{10}$, and consequently $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}^{\mathfrak{m}}_{110}$.*

Proof. The proof is just combinatorics with D. Suppose that a trivalent ribbon graph $\Gamma \neq Y$ does not vanish on the input $\omega_1, \ldots, \omega_l$. Because all external vertices of Γ are adjacent to an A-vertex or a B-vertex, the assumption (V_1) implies $D \geq s$, where s is the total number of external vertices. A combination of (4.23) and (4.19) yields

$$nk - (n-1)e = D \geq s = 3k - 2e \quad \Longleftrightarrow \quad (n-3)k \geq (n-3)e.$$

- (a) For $n > 3$, we get $k \geq e$, which implies that Γ is either a tree or a circular graph.
- (b) If Γ is a tree, then $s = k + 2$ and $e = k - 1$. From (4.23) we get

$$D = nk - (n-1)(k-1) = k + n - 1. \tag{4.26}$$

Now D is the sum of $s = k + 2$ form-degrees $\deg(\eta_{ij}) > 0$, and hence (4.26) for $n = 3$ implies that $\deg(\eta_{ij}) = 1$ for all i, j.

(c) For $n < 3$, we get $e \geq k$, which implies that Γ is not a tree.

(d) If Γ is a circular graph, then $e = k = s$, and we get using (4.23) that

$$D = nk - (n-1)k = k.$$

Here D is the sum of $s = k$ form-degrees $\deg(\eta_{ij}) > 0$, and hence $\deg(\eta_{ij}) = 1$ for all i, j.

We will now assume, in addition, that $\mathcal{H}^1(M) \simeq \mathrm{H}^1_{\mathrm{dR}}(M) = 0$.

(e) We must have $D \geq 2s$, which is in contradiction with $D = s$ for a circular graph. Therefore, $\mathfrak{n}_{20} = 0$.

(f) Finally, for a tree $\Gamma \neq Y$, we have

$$k + n - 1 = D \geq 2s = 2(k+2) \quad \Longleftrightarrow \quad n - 5 \geq k.$$

This finishes the proof of the proposition. ☐

Proposition 4.4.2 (Special Hodge propagator)**.** *In the setting of Definition 4.3.6, suppose that the Hodge propagator* P *is special. Then the condition (*V_1*), and hence Proposition 4.4.1 holds.*

Proof. It is easy to see that $A_{\alpha_1,\alpha_2} = \mathcal{P}(\eta_1 \wedge \eta_2)$ for all α_1, $\alpha_2 \in \mathcal{H}(M)[1]$, and that $-B_1$ is the Schwartz kernel of $\mathcal{P} \circ \mathcal{P}$. Therefore, (P4) and (P5) imply $A_{\alpha_1,1} = 0$ and $B_1 = 0$, respectively.

As for the integral $I(\sigma_L)$, one has to apply the Fubini theorem in order to integrate out single vertices $A_{\alpha_1,1}$ and B_1. This step relies on L^1-integrability of the integrand which follows from [CVon] (the integrand comes from a smooth form on a compact manifold with corners). ☐

Proposition 4.4.3 (Vanishing of A-vertices)**.** *In the setting of Definition 4.3.6, suppose that the following condition is satisfied:*

(V_A**)** *Every graph with an A-vertex vanishes.*

Then we have $\mathfrak{n}_{10} = \mathfrak{m}_{10}$*, and the only contribution to* $\mathfrak{n}_{20}(\mathrm{s}^2\alpha_{11}\dots\alpha_{1s_1} \otimes \alpha_{21}\dots\alpha_{2s_2})$ *comes from* O_k*-graphs with* $k = s_1 + s_2 = D$.

Proof. The only trees and circular graphs which are not excluded by the assumption are the Y-graph and O_k-graphs, respectively (the external branches contract). The condition on form-degrees is obtained as in the proof of Proposition 4.4.1.

To argue that $I(\sigma_L) = 0$, we again need L^1-integrability as in the proof of Proposition 4.4.2. ☐

Proposition 4.4.4 (1-connected geometrically formal manifolds)**.** *Let M be a geometrically formal n-manifold and* P *a special Hodge propagator (it exists by Proposition 4.2.11). If $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$, then the following holds:*

($n \neq 2$) *All $Y \neq \Gamma \in \overline{\mathrm{RG}}_{klg}$ with k, $l \geq 1$, $g \geq 0$ vanish, and hence $\mathfrak{n} = \mathfrak{m}$.*

($n = 2$) *All $Y \neq \Gamma \in \overline{\mathrm{RG}}_{kl0}$ with k, $l \geq 1$ vanish, and hence $\mathfrak{n}_{l0} = \mathfrak{m}_{l0}$ for all $l \geq 1$.*

Proof. Given η_1, $\eta_2 \in \mathcal{H}$, geometric formality implies $\eta_1 \wedge \eta_2 \in \mathcal{H}$, and hence $A_{\alpha_1,\alpha_2} = \mathcal{P}(\eta_1 \wedge \eta_2) = 0$. We see that (V_1) and (V_A) are satisfied, and hence the implications of Propositions 4.4.1 and 4.4.3 hold. The claim for $n > 3$ follows.

As for $n = 3$, Poincaré duality implies $\mathrm{H}^2_{\mathrm{dR}}(M;\mathbb{R}) = 0$. Therefore, the total form-degree D satisfies $D = nB$, where B is the number of B-vertices. We see using (4.24) that (4.23) is equivalent to

$$B + \frac{1}{2}(3-n)C = D = nB \quad \Longleftrightarrow \quad (n-1)B = \frac{1}{2}(3-n)C. \tag{4.27}$$

It follows that $B = 0$, and hence all reduced graphs vanish.

As for $n = 2$, we get from (4.27) and (4.25) that $B \geq l$ is equivalent to $g \geq 1$. ☐

Remark 4.4.5 (A_∞-homotopy transfer). In [CVon], it will be shown that the A_∞-algebra $\mathcal{H}(M)_{\mathfrak{n}} = (\mathcal{H}(M), (\mu_k))$ induced by $\mathfrak{n}_{10}$ agrees with the A_∞-algebra obtained by the A_∞-homotopy transfer

$$\begin{pmatrix} \Omega(M) \\ m_1,\ m_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \mathcal{H}(M) \\ \mu_1 \equiv 0,\ \mu_2 = \pi_{\mathcal{H}} m_2(\iota_{\mathcal{H}}, \iota_{\mathcal{H}}),\ \mu_3,\ \dots \end{pmatrix}$$

using the homotopy retract (see [Val12])

$$\begin{tikzcd} (\Omega(M), m_1) \arrow[loop left, "\mathcal{P}"] \arrow[r, shift left, "\pi_{\mathcal{H}}"] & (\mathcal{H}(M), m_1 \equiv 0). \arrow[l, shift left, "\iota_{\mathcal{H}}"] \end{tikzcd}$$

The operation μ_k of the transferred A_∞-structure is computed as a sum over planar trees with a root and k leaves decorated by $\iota_{\mathcal{H}}$ at the leaves, $\pi_{\mathcal{H}}$ at the root and $\mathcal{P}$ at the internal edges (see [Aka07]). The result of [CVon] is plausible because the part of $\mathfrak{n}_{10}$ contributing to μ_k is a sum over trivalent ribbon trees with $k+1$ leaves.

In [CVon], they will also show that $\iota_1 \coloneqq \iota_{\mathcal{H}} : \mathcal{H} \to \Omega$ extends to an A_∞-quasi-isomorphism $(\iota_k)_{k\geq 1}$ from $(\mathcal{H}, (\mu_k))$ to (Ω, m_1, m_2). The induced chain map on the dual cyclic bar complexes is then the map $\mathfrak{f}^{\mathfrak{m}}_{110}$ coming from the IBL_∞-theory in the Overview. ◁

Proposition 4.4.6 (Twisted boundary operator for formal manifolds)**.** *In the setting of Definition 4.3.6, suppose that M is formal in the sense of rational homotopy theory. Then there is a quasi-isomorphism*

$$\mathfrak{h}_{110} : (\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}(M)[3-n], \mathfrak{q}^{\mathfrak{m}}_{110}) \longrightarrow (\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathcal{H}(M)[3-n], \mathfrak{q}^{\mathfrak{n}}_{110}).$$

Proof. Formality of M is equivalent to the existence of a zig-zag of quasi-isomorphisms of dga's (see [Val12])

$$(\mathrm{H}_{\mathrm{dR}}(M), m_1 \equiv 0, m_2) \rightsquigarrow \bullet \quad \cdots \quad \bullet \leftsquigarrow (\Omega(M), m_1, m_2).$$

Because a dga-quasi-isomorphism has a homotopy inverse in the category of A_∞-algebras, we get a direct A_∞-quasi-isomorphism

$$(g_k): \quad (\Omega(M), m_1, m_2) \rightsquigarrow (\mathrm{H}_{\mathrm{dR}}(M), m_1 \equiv 0, m_2).$$

Precomposing with (ι_k) from Remark 4.4.5, we get the A_∞-isomorphism

$$(h_k): \quad (\mathcal{H}(M), (\mu_k)) \rightsquigarrow (\mathrm{H}_{\mathrm{dR}}(M), m_1 \equiv 0, m_2).$$

This induces the quasi-isomorphism $\mathfrak{h}_{110}$ of the cyclic cochain complexes. □

Remark 4.4.7 (On formality). Geometrically formal manifolds include $\mathbb{S}^n$, $\mathbb{C}P^n$ and Lie groups (see [Kot01]). Any geometrically formal manifold is formal. Every simply-connected manifold of dimension at most 6 is formal (see [Mil79]). ◁

4.5. Conjectured relation to string topology

Given a smooth connected oriented n-dimensional manifold M, we consider the equivariant homology of the free loop space $\mathrm{L}M \coloneqq \{\gamma : \mathbb{S}^1 \to M \text{ continuous}\}$ with respect to the reparametrization action of $\mathbb{S}^1$. It is defined as the singular homology of the Borel construction

$$\mathrm{L}_{\mathbb{S}^1}M \coloneqq \mathrm{E}\,\mathbb{S}^1 \times_{\mathbb{S}^1} \mathrm{L}M \coloneqq (\mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M)/\mathbb{S}^1,$$

where $\mathrm{E}\,\mathbb{S}^1 = \mathbb{S}^\infty \to \mathrm{B}\,\mathbb{S}^1 = \mathbb{C}\mathrm{P}^\infty$ is a model for the universal bundle for $\mathbb{S}^1$, and we quotient out the diagonal action. We denote this homology by

$$\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M) \coloneqq \mathrm{H}(\mathrm{L}_{\mathbb{S}^1}M).$$

Recall that $(\mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M)/\mathbb{S}^1$ is the homotopically correct quotient replacing $\mathrm{L}M/\mathbb{S}^1$, which we prefer to use because the diagonal action of $\mathbb{S}^1$ on $\mathrm{E}\,\mathbb{S}^1$ is free, and hence $\mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M \to (\mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M)/\mathbb{S}^1$ is a principal $\mathbb{S}^1$-bundle (in contrast to the pathological map $\mathrm{L}M \to \mathrm{L}M/\mathbb{S}^1$). Recall also that $\mathrm{E}\,\mathbb{S}^1$ is contractible.

The "geometric versions" of the homologies were defined in [CS99] as the degree shifts

$$\mathbb{H}(\mathrm{L}M) \coloneqq \mathrm{H}(\mathrm{L}M)[n] \quad \text{and} \quad \mathcal{H}(\mathrm{L}M) \coloneqq \mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)[n].$$

There is the *loop product* $\bullet : \mathbb{H}(\mathrm{L}M)^{\otimes 2} \to \mathbb{H}(\mathrm{L}M)$ of degree 0 which makes $\mathbb{H}(\mathrm{L}M)$ into a graded commutative associative algebra. There is also the *loop coproduct* $\tau : \bar{\mathbb{H}}(\mathrm{L}M) \to \bar{\mathbb{H}}(\mathrm{L}M)^{\otimes 2}$ of degree $1-2n$ which is graded cocommutative and coassociative and is a derivation of $\bullet$. The geometric construction of $\bullet$ and τ on transverse smooth chains in $\mathrm{L}M$ was described in [CS99] and [Bas11], respectively. Here, the symbol $\bar{\mathbb{H}}(\mathrm{L}M)$ stands for the degree shifted relative homology

$$\bar{\mathbb{H}}(\mathrm{L}M) \coloneqq \mathrm{H}(\mathrm{L}M, M)[n]$$

with respect to constant loops $M \hookrightarrow \mathrm{L}M$. The geometric construction of τ does not work on the whole $\mathbb{H}(\mathrm{L}M)$ because of the phenomenon of "vanishing of small loops" depicted in [CL09, Figure 4, p. 13].

The projection $\mathrm{E}\,\mathbb{S}^1 \times \mathrm{L}M \to \mathrm{L}_{\mathbb{S}^1}M$ is an $\mathbb{S}^1$-principal bundle and thus induces a Gysin sequence. This sequence written using the geometric versions reads

$$\dots \longrightarrow \mathbb{H}_i \xrightarrow{\mathcal{E}} \mathcal{H}_i \xrightarrow{\cap c} \mathcal{H}_{i-2} \xrightarrow{\mathcal{M}} \mathbb{H}_{i-1} \longrightarrow \dots, \tag{4.28}$$

where the map $\mathcal{M}$ adds a marked point in each string in a family in all possible positions, the map $\mathcal{E}$ erases the marked point of each string in a family, $c \in \mathrm{H}^2_{\mathbb{S}^1}(\mathrm{L}M)$ is the Euler class of the circle bundle and $\cap$ the cap product.

The *string bracket* $\tilde{\mathfrak{m}}_2 : \mathcal{H}(\mathrm{L}M)^{\otimes 2} \to \mathcal{H}(\mathrm{L}M)$ and the *string cobracket* $\tilde{\mathfrak{c}}_2 : \bar{\mathcal{H}}(\mathrm{L}M) \to$

$\bar{\mathcal{H}}(\mathrm{L}M)^{\otimes 2}$ are defined by

$$\tilde{\mathbbm{m}}_2 \coloneqq \mathcal{E} \circ \bullet \circ \mathcal{M}^{\otimes 2} \quad \text{and} \quad \tilde{\mathbbm{c}}_2 \coloneqq \mathcal{E}^{\otimes 2} \circ \nu \circ \mathcal{M}.$$

Here, the symbol $\bar{\mathcal{H}}(\mathrm{L}M)$ stands for the degree shifted relative $\mathbb{S}^1$-equivariant homology

$$\bar{\mathcal{H}}(\mathrm{L}M) \coloneqq \underbrace{\mathrm{H}^{\mathbb{S}^1}(\mathrm{E}\,\mathbb{S}^1 \times_{\mathbb{S}^1} \mathrm{L}M, \mathrm{E}\,\mathbb{S}^1 \times_{\mathbb{S}^1} M)}_{=:\,\bar{\mathrm{H}}^{\mathbb{S}^1}(\mathrm{L}M)}[n].$$

Because $|\mathcal{M}| = 1$ and $|\mathcal{E}| = 0$, we have for all $\xi \in \bar{\mathcal{H}}(\mathrm{L}M)$ and ξ_1, $\xi_2 \in \mathcal{H}$ the relations

$$\begin{aligned} \tilde{\mathbbm{m}}_2(\xi_1, \xi_2) &= (-1)^{|\xi_1|}\mathcal{E}(\mathcal{M}(\xi_1) \bullet \mathcal{M}(\xi_2)), \\ \tilde{\mathbbm{c}}_2(\xi) &= \sum \mathcal{E}(\nu^1) \otimes \mathcal{E}(\nu^2), \end{aligned} \tag{4.29}$$

where we write $\nu(\mathcal{M}(\xi)) = \sum \nu^1 \otimes \nu^2$. The operations $\tilde{\mathbbm{m}}_2$ and $\tilde{\mathbbm{c}}_2$ have degrees 2 and $2-2n$ with respect to the grading on $\mathcal{H}(\mathrm{L}M)$, respectively. In fact, we will consider $\tilde{\mathbbm{m}}_2$ and $\tilde{\mathbbm{c}}_2$ given by (4.29) as operations on the even degree shift $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)[2-n] = \mathcal{H}(\mathrm{L}M)[2-2n]$, which have degrees $2(2-n)$ and 0, respectively. The symbols $\mathbbm{m}_2$ and $\mathbbm{c}_2$ will denote their degree shifts to $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M)[3-n]$, which have degrees of an IBL-algebra from Definition 3.2.4.

In work in progress [CV20], they consider the map

$$I_{\lambda,*} : \mathrm{H}^{\lambda}_{-\bullet-1}(\Omega(M)) \longrightarrow \mathrm{H}^{\bullet}_{\mathbb{S}^1}(\mathrm{L}M; \mathbb{R})$$

defined on the chain level as a cyclic version of Chen's iterated integrals I_λ. Recall that $\mathrm{H}^{\lambda}_{-\bullet-1}(\Omega) = \mathrm{H}_\bullet(\mathrm{B}^{\mathrm{cyc}}\Omega, \mathrm{b})$, where $\mathrm{b} : \mathrm{B}\Omega = \bigoplus_{k\geq 1} \Omega[1]^{\otimes k} \to \mathrm{B}\Omega$ is the Hochschild differential of the de Rham dga (Ω, m_1, m_2), see Section 3.3. They prove in [CV20] that if M is simply-connected, then the map $I_{\lambda,*}$ induces an isomorphism $\mathrm{H}^{\lambda,\mathrm{red}}_{-\bullet-1}(\Omega(M))) \simeq \mathrm{H}^{\bullet}_{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}M)$, where

$$\mathrm{H}_{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}M) \coloneqq \mathrm{H}_{\mathbb{S}^1}(\mathrm{E}\,\mathbb{S}^1 \times_{\mathbb{S}^1} \mathrm{L}M, \mathrm{E}\,\mathbb{S}^1 \times_{\mathbb{S}^1} \{x_0\})$$

is the *reduced* $\mathbb{S}^1$*-equivariant cohomology* with respect to a base point $x_0 \in M$ (the constant loop at x_0). Dualizing their map, we obtain the isomorphism

$$\mathrm{H}^{-\bullet-1}_{\lambda,\mathrm{red}}(\Omega(M)) \simeq \mathrm{H}^{\mathbb{S}^1,\mathrm{red}}_{\bullet}(\mathrm{L}M; \mathbb{R}). \tag{4.30}$$

Suppose from now on that M is closed. Pick a Riemannian metric and an admissible

Hodge propagator $\mathrm{P} \in \Omega^{n-1}(\mathrm{Bl}_\Delta(M \times M))$. We will assume that P is special, i.e., that it satisfes (P1)–(P5) from Section 4.2, so that the Chern-Simons Maurer-Cartan element $\mathfrak{n}$ is strictly reduced, and hence the twisted reduced IBL_∞-algebra $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}}(\mathcal{H}))$ and the induced IBL-algebra $\mathrm{IBL}(\mathbb{H}^{\mathfrak{n},\mathrm{red}}(C(\mathcal{H})))$ are well-defined. Recall that $\mathbb{H}^{\mathfrak{n}}_\bullet(C(\mathcal{H})) = \mathrm{H}^\lambda_{n-3-\bullet}(\mathcal{H}_\mathfrak{n})$, where $\mathcal{H}_\mathfrak{n}$ is the A_∞-algebra on $\mathcal{H}$ twisted by $\mathfrak{n}_{10}$. From [CVon], we have

$$\mathrm{H}_\lambda(\mathcal{H}(M)_\mathfrak{n}) \simeq \mathrm{H}_\lambda(\Omega(M)). \tag{4.31}$$

Now, (4.30) and (4.31) give the following version of the string topology conjecture.

Conjecture 4.5.1 (String topology conjecture for simply-connected manifold)**.** *Let M be an oriented closed manifold of dimension n. There is a chain map*

$$(C^{\mathrm{sing}}(\mathrm{L}_{\mathbb{S}^1}M;\mathbb{R}),\partial) \longrightarrow (\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathcal{H}(M),\mathfrak{q}^{\mathfrak{n}}_{110}),$$

where C^{sing} denotes the (smooth) singular chain complex and ∂ the standard boundary operator, which, if M is simply-connected, satisfies the following:

- *It induces an isomorphism* $\mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}M;\mathbb{R})[2-n] \simeq \mathbb{H}^{\mathfrak{n},\mathrm{red}}(C(\mathcal{H}(M)))$.
- *It intertwines* $\mathbb{m}_2$ *on* $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M;\mathbb{R})$ *and* $\mathfrak{q}_{210}$.
- *The pullback of* $\mathfrak{q}^{\mathfrak{n}}_{120}$ *to* $\mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}M;\mathbb{R})$ *is compatible with* $\mathbb{c}_2$ *on* $\bar{\mathrm{H}}^{\mathbb{S}^1}(\mathrm{L}M;\mathbb{R})$ *under the morphism induced by the inclusion* $(\mathrm{L}M,x_0) \to (\mathrm{L}M,M)$.

Remark 4.5.2 (On string topology conjecture). (i) The conjecture can be interpreted as follows. There is an IBL-structure on $\mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}M;\mathbb{R})$ compatible with Chas-Sullivan operations, and the IBL_∞-algebra $\mathrm{dIBL}^{\mathfrak{n}}(C_{\mathrm{red}}(\mathcal{H}(M)))$ is its chain model.

(ii) The loop coproduct τ is geometrically defined only on $\bar{\mathrm{H}}^{\mathbb{S}^1}(\mathrm{L}M)$; the conjecture thus provides an extension of $\mathbb{c}_2$ to $\mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}M)$. In [Bas11], it is shown that the geometric definition of τ can be extended to $\mathrm{H}(\mathrm{L}M)$ for manifolds with zero Euler characteristic, i.e., $\chi(M) = 0$. This extension depends on the choice of a non-vanishing vector field on M. By homotopy invariance (see (v) below), our extension of $\mathbb{c}_2$ should not depend on the admissible Hodge propagator P.

(iii) The loop product $\bullet$ is geometrically defined on $\mathrm{H}(\mathrm{L}M)$; however, it does not always induce an associative product on $\mathrm{H}^{\mathrm{red}}(\mathrm{L}M) = \mathrm{H}(\mathrm{L}M,x_0)$. Indeed, the examples of $\mathbb{T}^2$ (see [Bas11]) and $\mathbb{S}^3$ (see [CS99]) show that $\mathrm{H}(x_0;\mathbb{R}) \subset \mathrm{H}(\mathrm{L}M;\mathbb{R})$ is not an ideal with respect to $\bullet$. By [Tam10], this does not happen when $\chi(M) \neq 0$, and hence, in this case, $\bullet$ restricts to $\mathrm{H}(\mathrm{L}M,x_0;\mathbb{R})$.

(iv) The computation for $\mathbb{S}^n$ with $n \geq 2$ and the computation for $\mathbb{C}\mathrm{P}^n$ in Section 5 support the conjecture. The computation for $\mathbb{S}^1$ in Section 5.3 provides a counterexample for non-simply-connected M. Surfaces of genus $g \geq 1$ should be considered.

(v) We expect that if M_1 and M_2 are homotopy equivalent, then the IBL_∞-algebras $\mathrm{dIBL}^{\mathfrak{n}_1}(C(\mathrm{H}_{\mathrm{dR}}(M_1)))$ and $\mathrm{dIBL}^{\mathfrak{n}_2}(C(\mathrm{H}_{\mathrm{dR}}(M_2)))$ are IBL_∞-homotopy equivalent. $\triangleleft$

5. Explicit computations

In Section 5.1, we solve the differential equation for the Hodge propagator P for $\mathbb{S}^n$ (Proposition 5.1.2) using the Relative Poincaré Lemma (Lemma 5.1.1). In the rest of the section, we will be showing that P is admissible (Proposition 5.1.10); the most work is to show that P extends smoothly to the blow-up (Proposition 5.1.9). Another Hodge propagator for $\mathbb{S}^1$ can be obtained in an alternative simple way by writing $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and there are nice geometric formulas for P for $\mathbb{S}^2$ (Example 5.1.3).

In Section 5.2, we use P from Section 5.1 to compute the Chern-Simons Maurer-Cartan element $\mathfrak{n}$ for $\mathbb{S}^n$ (Proposition 5.2.10). We first prove that the condition (V_1) from Proposition 4.4.1 is satisfied (Lemma 5.2.1) and then perform combinatorics with degrees to show vanishing of some more integrals (Proposition 5.2.2). In fact, all the integrals vanish for $\mathbb{S}^n$ with $n \geq 3$, and the only non-vanishing integrals for $\mathbb{S}^1$ are the O_k-graphs with even k. We compute these integrals explicitly together with all signs and combinatorial coefficients required to obtain $\mathfrak{n}_{20}$ (Lemmas 5.2.5, 5.2.6, 5.2.7 and 5.2.8). There might be some non-vanishing integrals associated to reduced graphs for $\mathbb{S}^2$ as well as some non-vanishing integrals associated to graphs without external vertices for $\mathbb{S}^3$; however, the simplest examples vanish (Remarks 5.2.3 and 5.2.4).

In the remaining Sections 5.3 and 5.4, we compute $\mathrm{IBL}(\mathbb{H}^{\mathfrak{n}}(C(\mathcal{H}(M))))$ and the higher operations $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ on $\mathbb{H}^{\mathfrak{n}}$ for $M = \mathbb{S}^n$, $\mathbb{C}\mathrm{P}^n$. As soon as we argue that $\mathfrak{n}_{10} = \mathfrak{m}_{10}$ due to geometric formality, the computation of $\mathbb{H}^{\mathfrak{m}}(C(\mathcal{H}(\mathbb{S}^n)))$ and $\mathbb{H}^{\mathfrak{m}}(C(\mathcal{H}(\mathbb{C}\mathrm{P}^n)))$ is an easy exercise in cyclic homology. The operations for $\mathbb{S}^{2m}$ and $\mathbb{C}\mathrm{P}^n$ vanish for degree reasons (Remark 5.3.1). Therefore, the integrals from Section 5.2 help only in the case of $\mathbb{S}^{2m-1}$. We compare our results to Chas-Sullivan string topology from [Bas11] and confirm Conjecture 4.5.1 for $\mathbb{S}^n$ with $n \geq 2$ and for $\mathbb{C}\mathrm{P}^n$.

5.1. Construction of a Hodge propagator for spheres

The standard Riemannian volume form on the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the restriction of the following closed form on $\mathbb{R}^{n+1}\backslash\{0\}$:

$$\mathrm{Vol}(x) := \frac{1}{|x|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i+1} x^i \, \mathrm{dx}_1 \cdots \widehat{\mathrm{dx}_i} \cdots \mathrm{dx}_{n+1} \, .$$

Here $\widehat{\mathrm{dx}_i}$ means that $\mathrm{dx_i}$ is omitted. We denote the Riemannian volume of $\mathbb{S}^n$ by

$$V := \int_{\mathbb{S}^n} \mathrm{Vol}.$$

The n-form H from Proposition 4.2.7 reads

$$\mathrm{H} = \frac{1}{V}\big(\mathrm{pr}_1^* \mathrm{Vol} + (-1)^n \,\mathrm{pr}_2^* \mathrm{Vol}\big).$$

According to Proposition 4.2.8, the equation which we want to solve reads

$$\mathrm{dP} = \frac{1}{V}\big((-1)^n \,\mathrm{pr}_1^* \mathrm{Vol} + \mathrm{pr}_2^* \mathrm{Vol}\big). \tag{5.1}$$

We denote

$$\tilde{\mathrm{P}} := V\mathrm{P} \quad \text{and} \quad \tilde{\mathrm{H}} := V\mathrm{H}.$$

The following lemma will be used to construct a solution to (5.1).

Lemma 5.1.1 (Relative Poincaré Lemma)**.** *Let M be a smooth oriented manifold and $\psi : [0,1] \times M \to M$ a smooth map. Consider the operator $T : \Omega^*(M) \to \Omega^{*-1}(M)$ defined by*

$$T(\eta) := \int^{[0,1]} \psi^* \eta \quad \textit{for all } \eta \in \Omega(M),$$

where we integrate along the fiber of the oriented fiber bundle $\mathrm{pr}_2 : [0,1] \times M \to M$. Then we have

$$\mathrm{d} \circ T + T \circ \mathrm{d} = \psi_1^* - \psi_0^*.$$

Proof. Stokes' formula from Proposition 4.2.6 gives

$$\mathrm{d} \int^{[0,1]} \psi^* \eta = -\Big(\int^{[0,1]} \mathrm{d}\psi^* \eta - \int^{\partial[0,1]} \psi^* \eta \Big) = -\int^{[0,1]} \psi^* \mathrm{d}\eta + \psi_1^* \eta - \psi_0^* \eta$$

for all $\eta \in \Omega(M)$. □

Proposition 5.1.2 (Solution to (5.1))**.** *For all* $(x,y)\in(\mathbb{S}^n\times\mathbb{S}^n)\backslash\Delta$*, let*

$$\mathrm{P}(x,y)\coloneqq(-1)^n\sum_{k=0}^{n-1}g_k(x,y)\omega_k(x,y),\tag{5.2}$$

where

$$g_k(x,y)\coloneqq\int_0^1\frac{t^k(t-1)^{n-1-k}}{(2t(t-1)(1+x\cdot y)+1)^{\frac{n+1}{2}}}\,\mathrm{dt}\tag{5.3}$$

and

$$\omega_k(x,y)\coloneqq\frac{1}{k!}\frac{1}{(n-1-k)!}\sum_{\sigma\in\mathbb{S}_{n+1}}(-1)^\sigma x^{\sigma_1}y^{\sigma_2}\,\mathrm{dx}^{\sigma_3}\cdots\mathrm{dx}^{\sigma_{2+\mathrm{k}}}\tag{5.4}$$
$$\mathrm{dy}^{\sigma_{3+\mathrm{k}}}\cdots\mathrm{dy}^{\sigma_{\mathrm{n}+1}}\,.$$

The form (5.2) *is a smoooth solution to* (5.1) *on* $(\mathbb{S}^n\times\mathbb{S}^n)\backslash\Delta$.

Proof. Define the set

$$N\coloneqq(\mathbb{R}^{n+1}_{\neq0}\times\mathbb{R}^{n+1}_{\neq0})\backslash\{(x,ax)\mid x\in\mathbb{R}^{n+1},a>0\}.$$

It is an open thickening of $(\mathbb{S}^n\times\mathbb{S}^n)\backslash\Delta$ in $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\backslash\Delta$. Consider the smooth deformation retraction

$$\begin{aligned}\psi:[0,1]\times N&\longrightarrow N\\(t,x,y)&\longmapsto\psi_t(x,y)\coloneqq(x,(1-t)y-tx)\end{aligned}$$

with

$$\psi_0(x,y)=(x,y)\quad\text{and}\quad\psi_1(x,y)=(x,-x)\quad\text{for all }(x,y)\in N.$$

The retraction is depicted in Figure 5.1. Denote by $A:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$, $x\mapsto-x$ the antipodal map. It is easy to see that

$$A^*\mathrm{Vol}=(-1)^{n+1}\mathrm{Vol},$$

and hence

$$\psi_1^*\tilde{\mathrm{H}}=\psi_1^*\,\mathrm{pr}_1^*\,\mathrm{Vol}+(-1)^n\psi_1^*\,\mathrm{pr}_2^*\,\mathrm{Vol}=\mathrm{pr}_1^*\,\mathrm{Vol}+(-1)^n\,\mathrm{pr}_1^*\,A^*\mathrm{Vol}=0.$$

Define

$$\mathrm{P}\coloneqq(-1)^{n+1}\int^{[0,1]}\psi^*\mathrm{H}.\tag{5.5}$$

Let $T:\Omega^*(N)\to\Omega^{*-1}(N)$ be the cochain homotopy from Lemma 5.1.1 associated to ψ.

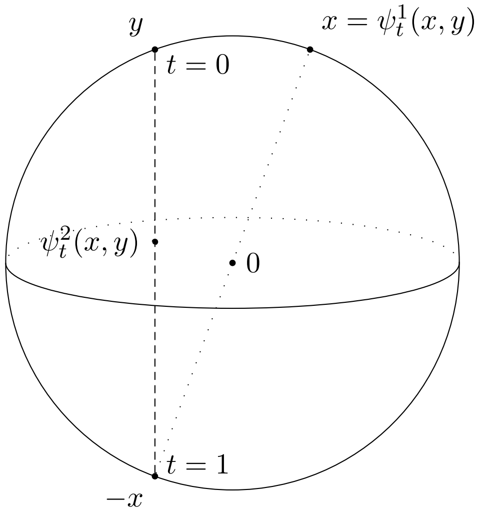


Figure 5.1.: Retraction $\psi_t = (\psi_t^1, \psi_t^2)$. A point of $\mathbb{S}^n \times \mathbb{S}^n$ is visualized as a pair of points on $\mathbb{S}^n$.

Because $\mathrm{dH} = 0$, we get

$$\mathrm{dP} = (-1)^{n+1}\mathrm{d}T(\mathrm{H}) = (-1)^{n+1}(\mathrm{d}T + T\mathrm{d})\mathrm{H} = (-1)^{n+1}(\psi_1^* - \psi_0^*)\mathrm{H} = (-1)^n\mathrm{H}.$$

For every $i = 1, \ldots, n+1$, we have

$$\psi^*(\mathrm{dx^i}) = \mathrm{dx^i} \quad \text{and} \quad \psi^*(\mathrm{dy^i}) = (1-t)\,\mathrm{dy^i} - t\,\mathrm{dx^i} - (y^i + x^i)\,\mathrm{dt}\,.$$

We compute

$$\begin{aligned}
&\int^{[0,1]} \psi^*\tilde{\mathrm{H}} \\
&\quad = (-1)^n \int^{[0,1]} \psi^*\,\mathrm{pr}_2^*\,\mathrm{Vol} \\
&\quad = (-1)^{n+1} \int^{[0,1]} \sum_{i=1}^{n+1} (-1)^i \frac{((1-t)y^i - tx^i)}{|(1-t)y - tx|^{n+1}} \psi^*(\mathrm{dy^1} \cdots \widehat{\mathrm{dy^i}} \cdots \mathrm{dy^{n+1}}) \\
&\quad = (-1)^{n+1} \sum_{1 \le i < j \le n+1} (-1)^{i+j}(x^i y^j - y^i x^j) \int^{[0,1]} \frac{\mathrm{dt}\,\psi^*(\mathrm{dy^1} \cdots \widehat{\mathrm{dy^i}} \cdots \widehat{\mathrm{dy^j}} \cdots \mathrm{dy^{n+1}})}{|(1-t)y - tx|^{n+1}}
\end{aligned}$$

$$= -\sum_{k=0}^{n-1}\Big(\int_0^1 \frac{t^k(t-1)^{n-1-k}}{|(1-t)y-tx|^{n+1}}\,\mathrm{dt}\Big)\sum_{1\le i<j\le n+1}(-1)^{i+j+1}(x^iy^j-y^ix^j)$$
$$\sum_{\substack{\sigma:\{1,\ldots,n-1\}\to\{1,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,n+1\}\\ \sigma_1<\cdots<\sigma_k\\ \sigma_{k+1}<\cdots<\sigma_{n-1}}}(-1)^\sigma\,\mathrm{dx}^{\sigma_1}\cdots\mathrm{dx}^{\sigma_{\mathrm{k}}}\,\mathrm{dy}^{\sigma_{\mathrm{k+1}}}\cdots\mathrm{dy}^{\sigma_{\mathrm{n-1}}}\,.$$

The formulas (5.3) and (5.4) are obtained from this by writing

$$|(1-t)y-tx|^2 = 2t(t-1)(1+x\cdot y)+1$$

in the denominator of the integrand and by simple combinatorics in the form part, respectively. Smoothness of P on $(\mathbb{S}^n\times\mathbb{S}^n)\backslash\Delta$ follows from the expression (5.5). ☐

Note that g_k are smooth functions on $(\mathbb{S}^n\times\mathbb{S}^n)\backslash\Delta$.

Example 5.1.3 (P for $\mathbb{S}^1$ and $\mathbb{S}^2$). (a) Let

$$\alpha : (\mathbb{S}^1\times\mathbb{S}^1)\backslash\Delta \to (0,2\pi)$$

be the smooth function assigning to a pair $(x,y)\in(\mathbb{S}^1\times\mathbb{S}^1)\backslash\Delta$ the counterclockwise angle from x to y. Let α_1, $\alpha_2\in[0,2\pi)$ be such that $x=\cos(\alpha_1)\mathrm{e}_1+\sin(\alpha_1)\mathrm{e}_2$ and $y=\cos(\alpha_2)\mathrm{e}_1+\sin(\alpha_2)\mathrm{e}_2$ for the standard Euclidean basis e_1, e_2 of $\mathbb{R}^2$. It is easy to see that

$$\alpha(x,y)=\begin{cases}\alpha_2-\alpha_1 & \text{if } \alpha_1<\alpha_2,\\ \alpha_2-\alpha_1+2\pi & \text{if } \alpha_1>\alpha_2.\end{cases}$$

Therefore, we get

$$\mathrm{d}\alpha = \mathrm{d}\alpha_2-\mathrm{d}\alpha_1 = -2\pi H \quad \text{on } (\mathbb{S}^1\times\mathbb{S}^1)\backslash\Delta.$$

On the other hand, we can compute P from (5.2) as follows. Using the substitution $u=2t-1$, we get for all x, $y\in\mathbb{S}^1$ with $x\neq\pm y$ the following:

$$\begin{aligned} g_0(x,y) &= \int_0^1 \frac{\mathrm{dt}}{2t(t-1)(1+x\cdot y)+1}\\ &= \frac{1}{1-x\cdot y}\int_{-1}^1 \frac{\mathrm{du}}{\frac{1+x\cdot y}{1-x\cdot y}u^2+1}\\ &= \frac{2}{\sqrt{1-(x\cdot y)^2}}\arctan\Big(\sqrt{\frac{1+x\cdot y}{1-x\cdot y}}\Big)\\ &= \frac{\pi-\arccos(x\cdot y)}{\sqrt{1-(x\cdot y)^2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi - \arccos(x \cdot y)}{|x^1 y^2 - x^2 y^1|} \\
&= \frac{\pi - \alpha(x, y)}{x^1 y^2 - x^2 y^1}.
\end{aligned}
$$

The third from last equality can be obtained by trigonometric considerations and the second from last equality by an algebraic manipulation with the denominator. We will explain the last equality. Consider the matrix

$$
R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

representing the counterclockwise rotation by $\frac{\pi}{2}$. The function $\arccos : (-1, 1) \to (0, \pi)$ satisfies

$$
\arccos(x \cdot y) = \begin{cases} \alpha(x, y) & \text{if } y \cdot Rx > 0, \\ 2\pi - \alpha(x, y) & \text{if } y \cdot Rx < 0. \end{cases}
$$

The last equality becomes clear when we notice that $x^1 y^2 - x^2 y^1 = y \cdot Rx$.

Finally, we have $\omega_0(x, y) = x^1 y^2 - x^2 y^1$, and hence

$$
\begin{aligned}
2\pi \mathrm{P}(x, y) &= -g_0(x, y)\omega_0(x, y) \\
&= \alpha(x, y) - \pi \\
&= \pi - \alpha(y, x).
\end{aligned}
$$

(b) For $n = 2$, we get the formulas

$$
\begin{aligned}
g_0(x, y) = -g_1(x, y) &= \frac{1}{x \cdot y - 1} \quad \text{and} \\
\omega_0(x, y) &= (x^2 y^3 - x^3 y^2)\,\mathrm{dy}^1 + (x^3 y^1 - x^1 y^3)\,\mathrm{dy}^2 + (x^1 y^2 - x^2 y^1)\,\mathrm{dy}^3 \\
&= \sum_{i=1}^{3} (x \times y)^i\,\mathrm{dy}^{\mathrm{i}}.
\end{aligned}
$$

The formula for $\omega_1(x, y)$ is obtained from the formula for $\omega_0(x, y)$ by replacing dy with dx. ◁

Consider the diagonal action of the orthogonal group $O(n+1)$ on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ by matrix multiplication.

Proposition 5.1.4 (Symmetries of P). *Consider* P *from Proposition 5.1.2. For all*

$R \in O(n+1)$, we have

$$R^* \mathrm{P} = (-1)^R \mathrm{P},$$

where $(-1)^R = \det(R)$. Moreover, if τ denotes the twist map, then

$$\tau^* \mathrm{P} = (-1)^n \mathrm{P}.$$

Proof. We will use the thickening N, the antipodal map A and the expression (5.5) for P from the proof of Proposition 5.1.2.

It is easy to check that both τ and R preserve N. Let $\tilde{\tau}$ and $\tilde{R}$ be the isomorphisms of the fiber bundle $\mathrm{pr}_2 : [0,1] \times N \to N$ given by

$$\tilde{\tau}(t,x,y) \coloneqq (1-t, y, x) \quad \text{and} \quad \tilde{R}(t,x,y) \coloneqq (t, Rx, Ry)$$

for all $(t,x,y) \in [0,1] \times N$. Then $\tilde{\tau}$ covers τ and $\tilde{R}$ covers R. A simple computation directly from Definition 4.2.1 shows that the fiberwise integration commutes with the pullback along a bundle morphism if the bundle map and the base map are both either orientation preserving or reversing. In our case, we have

$$(-1)^{\tau + \tilde{\tau}} = -1 \quad \text{and} \quad (-1)^{R + \tilde{R}} = 1.$$

Using this and the equation

$$\mathrm{pr}_2 \circ \psi \circ \tilde{\tau} = A \circ \mathrm{pr}_2 \circ \psi,$$

we get firstly

$$\begin{aligned} \tau^* \int^{[0,1]} \psi^* \tilde{\mathrm{H}} &= - \int^{[0,1]} \tilde{\tau}^* \psi^* \mathrm{pr}_2^* \mathrm{Vol} \\ &= - \int^{[0,1]} \psi^* \mathrm{pr}_2^* A^* \mathrm{Vol} \\ &= (-1)^n \int^{[0,1]} \psi^* \mathrm{pr}_2^* \mathrm{Vol} \\ &= (-1)^n \int^{[0,1]} \psi^* \tilde{\mathrm{H}} \end{aligned}$$

and secondly

$$R^* \int^{[0,1]} \psi^* \mathrm{H} = \int^{[0,1]} \tilde{R}^* \psi^* \mathrm{H} = \int^{[0,1]} \psi^* R^* \mathrm{H} = (-1)^{n+1} \int^{[0,1]} \psi^* \mathrm{H}.$$

This proves the proposition. ☐

Both diffeomorphisms R and τ preserve Δ, and hence they extend to diffeomorphisms of $\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$. If also P extends, then the statement of Proposition 5.1.4 holds for P on $\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$.

In the rest of the section, we will be proving that P extends smoothly to $\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$. This is a local problem at the boundary, where we introduce the following radial coordinates. Define the set

$$X \coloneqq \{(r, \eta, x) \in [0, \infty) \times \mathbb{S}^n \times \mathbb{S}^n \mid \eta \cdot x = 0\},$$

and let $\kappa : X \longrightarrow \mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ be the map defined by

$$\kappa(r, \eta, x) \coloneqq \begin{cases} \Big(x, \dfrac{x + r\eta}{|x + r\eta|}\Big) \in (\mathbb{S}^n \times \mathbb{S}^n)\backslash\Delta & \text{for } r > 0, \\[2ex] [(-\eta, \eta)] \in P^+ N_{(x,x)}\Delta & \text{for } r = 0. \end{cases}$$

Recall that the oriented projectivization P^+ was defined in Definition 4.2.2. For the upcoming computations, it is convention to define the map $\gamma : \mathbb{R} \to (-1, 1)$ by

$$\gamma(r) \coloneqq \frac{r}{\sqrt{1 + r^2} + 1} \quad \text{for all } r \in \mathbb{R}.$$

It is a diffeomorphism with inverse $r = \frac{2\gamma}{1-\gamma^2}$.

Lemma 5.1.5 (Parametrization of collar neighborhood)**.** *The subset $X \subset \mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ is a submanifold with boundary, and the map $\kappa : X \longrightarrow \mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ is an embedding onto a neighborhood of $\partial\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$.*

Proof. The set X is a Cartesian product of $[0, \infty)$ and a regular level set; therefore, it is a submanifold with boundary. The inclusion $\mathbb{S}^n \times \mathbb{S}^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ induces an embedding of manifolds with boundary $\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n) \subset \mathrm{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$. Consider the global chart $\tilde{\mathbb{1}} : \mathrm{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \to [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^{n+1}$ from (4.4) induced by the identity. We have

$$\begin{aligned} Y &\coloneqq \tilde{\mathbb{1}}(\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)) \\ &= \{(\tilde{r}, w, u) \in [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^{n+1} \mid |u|^2 + \tilde{r}^2 = 1,\ w \cdot u = 0\}, \end{aligned}$$

where we denote r on Y by $\tilde{r}$ in order to distinguish it from r on X. It suffices to prove the claim for the map $\mu \coloneqq \tilde{\mathbb{1}} \circ \kappa : X \to Y$. For $(r, \eta, x) \in X$, we compute

$$\mu(r, \eta, x) = \Big(\frac{\gamma}{\sqrt{1+\gamma^2}}, \frac{1}{\sqrt{1+\gamma^2}}(\gamma x - \eta), \frac{1}{1+\gamma^2}(x + \gamma\eta) \Big).$$

This formula defines a smooth map of $\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. It is a local diffeomorphism because its Jacobian is non-vanishing:

$$|\,\mathrm{D}\mu\,| = \frac{\partial \tilde{r}}{\partial r} \Big(\frac{\partial w}{\partial \eta} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} \frac{\partial u}{\partial \eta} \Big)^{n+1} = (-1)^{n+1}(1+\gamma^2)^{-\frac{n+4}{2}} \frac{\partial \gamma}{\partial r}.$$

Moreover, the map μ is injective, maps X into Y and ∂X onto ∂Y. The claim follows. □

Consider the action of $O(n+1)$ on X defined by

$$R \cdot (r, \eta, x) \coloneqq (r, R\eta, Rx) \quad \text{for all } (r, \eta, x) \in X \text{ and } R \in O(n+1).$$

Via κ, this agrees with the diagonal action of $O(n+1)$ on $\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$. Denote

$$\mathrm{P}' \coloneqq \kappa^* \mathrm{P} \in \Omega^{n-1}(\mathrm{Int}(X)).$$

From Proposition 5.1.4 we get

$$R^* \mathrm{P}' = (-1)^R \mathrm{P}' \quad \text{for all } R \in O(n+1). \tag{5.6}$$

Consider the smooth curve (see Figure 5.2)

$$\begin{aligned} \zeta' : [0, \infty) &\longrightarrow X \\ r &\longmapsto (r, e_n, e_{n+1}). \end{aligned}$$

We have the following lemma.

Lemma 5.1.6 (Smooth extension along curve). *The form* P' *extends smoothly to* X *if and only if the map* $\mathrm{P}' \circ \zeta' : (0, \infty) \to \Lambda^{n-1} T^* X$ *extends smoothly to the interval* $[0, \infty)$.

Proof. As for the non-trivial implication, let $(0, \eta_0, x_0) \in X$ be a boundary point. Pick vectors $v_1, \ldots, v_{n-1} \in \mathbb{R}^{n+1}$ so that the vectors $v_1, \ldots, v_{n-1}, \eta_0, x_0$ are linearly independent, and define the set

$$U \coloneqq \{(r, \eta, x) \in X \mid v_1, \ldots, v_{n-1}, \eta, x \text{ are linearly independent}\}.$$

It is an open neighborhood of $(0, \eta_0, x_0)$ in X. Applying the Gram-Schmidt orthog-

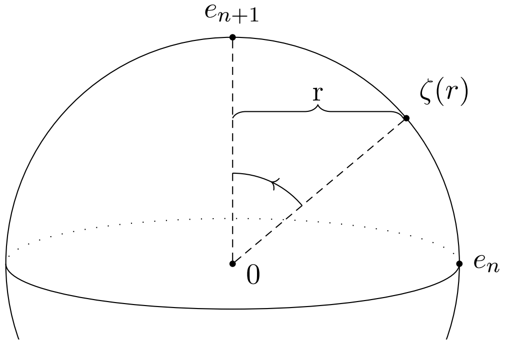


Figure 5.2.: The curve $\zeta := \kappa \circ \zeta'$ is given by $\zeta(r) = \big(e_{n+1}, \frac{e_{n+1}+re_n}{|e_{n+1}+re_n|}\big)$ for $r > 0$.

onalization to $v_1, \ldots, v_{n-1}, \eta, x$, we find a smooth map $R : U \to O(n+1)$ such that

$$R(r,\eta,x)\cdot(r,\eta,x) = (r, e_n, e_{n+1}) \quad \text{for all } (r,\eta,x) \in U.$$

The equation (5.6) implies

$$\mathrm{P}'(r,\eta,x) = (-1)^R R(r,\eta,x)^*(\mathrm{P}'(r,e_n,e_{n+1})) \quad \text{for all } (r,\eta,x) \in \mathrm{Int}(U),$$

where $R(r,\eta,x)^* : \Lambda^* T^* X \to \Lambda^* T^* X$ is the smooth cotangential map which is induced by the diffeomorphism $R(r,\eta,x) : X \to X$, and which maps the fiber over $z \in X$ to the fiber over $R(r,\eta,x)^{-1}z$. By the assumption, all maps in the composition are smooth in their arguments. The lemma follows. □

Lemma 5.1.7 (Local expression at boundary)**.** *On the interval* $(0,\infty)$*, we have*

$$\tilde{\mathrm{P}}' \circ \zeta' = (-1)^{n+1}(1+\gamma^2)^{-\frac{n-1}{2}} \sum_{k=0}^{n-1} \gamma^{n-k}(h_k \circ \gamma)(\nu_k \circ \zeta'),$$

where the functions $h_k : (0,1) \to \mathbb{R}$ *are defined by*

$$h_k(\gamma) := \int_{-1}^{1} \frac{(u+\gamma^2)^k (u-1)^{n-1-k}}{(u^2+\gamma^2)^{\frac{n+1}{2}}}\,\mathrm{du} \quad \textit{for all } \gamma \in (0,1)$$

and the forms $\nu_k \in \Omega(X)$ *are defined by*

$$\nu_k(r,x,\eta) := \frac{1}{k!(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n-1}} (-1)^\sigma \, \mathrm{dx}^{\sigma_1} \cdots \mathrm{dx}^{\sigma_\mathrm{k}} \, \mathrm{d}\eta^{\sigma_{\mathrm{k+1}}} \cdots \mathrm{d}\eta^{\sigma_{\mathrm{n-1}}} \,.$$

Proof. We start with the following formula from the proof of Proposition 5.1.2:

$$\tilde{\mathrm{P}} = \sum_{1\le i<j\le n+1} (-1)^{i+j}(x^i y^j - y^i x^j) \int^{[0,1]} \frac{\mathrm{dt}\, \psi^*(\mathrm{dy}^1 \cdots \widehat{\mathrm{dy}^{\mathrm{i}}} \cdots \widehat{\mathrm{dy}^{\mathrm{j}}} \cdots \mathrm{dy}^{\mathrm{n+1}})}{|(1-t)y - tx|^{n+1}}.$$

We restrict to the points $(x, y) = \kappa(r, e_n, e_{n+1})$ with $r > 0$. There, we have

$$\begin{aligned} &x^1 = \cdots = x^n = 0,\ x^{n+1} = 1, \\ &y^1 = \cdots = y^{n-1} = 0,\ y^n = \frac{2\gamma}{1+\gamma^2},\ y^{n+1} = \frac{1-\gamma^2}{1+\gamma^2}. \end{aligned}$$

Under the substitution $u = 2t - 1$, we get

$$|(1-t)y - tx|^2 = \frac{4t(t-1)}{1+\gamma^2} + 1 = \frac{u^2+\gamma^2}{1+\gamma^2}.$$

We make the following preliminary computations:

$$\begin{aligned} x^i y^j - y^i x^j &= 0 \quad \text{for } 1 \le i \le n-1 \text{ and } i < j \le n+1, \\ x^n y^{n+1} - y^n x^{n+1} &= -\frac{2\gamma}{1+\gamma^2}, \\ \kappa^*(\mathrm{dy}^{\mathrm{i}}) &= \frac{1}{1+\gamma^2}\big((1-\gamma^2)\, \mathrm{dx}^{\mathrm{i}} + 2\gamma\, \mathrm{d}\eta^{\mathrm{i}}\big) \quad \text{for } 1 \le i \le n-1. \end{aligned}$$

We plug these in the formula for $\tilde{\mathrm{P}}$ and get

$$\begin{aligned} \tilde{\mathrm{P}}'(\zeta'(r)) &= 2\gamma(1+\gamma^2)^{\frac{n-1}{2}} \int^{[0,1]} \mathrm{dt}\, \frac{\prod_{i=1}^{n-1}\big((1-t)\kappa^*(\mathrm{dy}^{\mathrm{i}}) - t\, \mathrm{dx}^{\mathrm{i}}\big)}{(u^2+\gamma^2)^{\frac{n+1}{2}}} \\ &= (-1)^{n+1}\gamma(1+\gamma^2)^{-\frac{n-1}{2}} \int^{[-1,1]} \mathrm{du}\, \frac{\prod_{i=1}^{n-1}\big((u+\gamma^2)\, \mathrm{dx}^{\mathrm{i}} + \gamma(u-1)\, \mathrm{d}\eta^{\mathrm{i}}\big)}{(u^2+\gamma^2)^{\frac{n+1}{2}}} \\ &= (-1)^{n+1}(1+\gamma^2)^{-\frac{n-1}{2}} \sum_{k=0}^{n-1} \gamma^{n-k} \Big(\int_{-1}^{1} \frac{(u+\gamma^2)^k (u-1)^{n-1-k}}{(u^2+\gamma^2)^{\frac{n+1}{2}}}\, \mathrm{du}\Big)\nu_k. \end{aligned}$$

The lemma follows. ☐

Lemma 5.1.8 (Integrals depending on parameter)**.** *Let $n \in \mathbb{N}$, and let $l \in \{0, 1, \ldots, n-1\}$. The function $F_{n,l} : (0,\infty) \to \mathbb{R}$ defined by*

$$F_{n,l}(t) := \int_{-1}^{1} \frac{t^{n-l} u^l}{(u^2+t^2)^{\frac{n+1}{2}}}\, \mathrm{du} \quad \textit{for all } t \in (0,\infty) \tag{5.7}$$

extends smoothly to $[0, \infty)$*.*

Proof. We have

$$F_{1,0}(t) = 2\arctan\Big(\frac{1}{t}\Big) = \pi - 2\arctan(t) \quad \text{for all } t \in (0, \infty).$$

The right-hand side is a smooth function on $\mathbb{R}$.

For $n \geq 2$, we deduce the recursive formula

$$F_{n,0}(t) = \frac{1}{n-1}\Big((n-2)F_{n-2,0}(t) + \frac{2t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}}\Big).$$

If l is odd, then $F_{n,l} \equiv 0$ for all n because the integrand of (5.7) is odd as a function of u.

For $n \geq 3$ and even $2 \leq l \leq n-1$, we deduce yet another recursive formula

$$F_{n,l}(t) = \frac{1}{n-l}\Big((l-1)F_{n,l-2}(t) - \frac{2t^{n-l}}{(1+t^2)^{\frac{n-1}{2}}}\Big).$$

The claim for all $F_{n,l}$ follows by induction. □

Proposition 5.1.9 (Smooth extension to boundary)**.** *The form* P *from* (5.2) *extends smoothly to* $\mathrm{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$*.*

Proof. According to Lemmas 5.1.5 and 5.1.6, it suffices to show that the curve $\mathrm{P}' \circ \zeta' : (0, \infty) \to \Lambda^{n-1}T^*X$ extends smoothly to $[0, \infty)$. Lemma 5.1.7 gives an expression for $\mathrm{P}' \circ \zeta'$ as a linear combination of smooth forms $\nu_k \in \Omega^{n-1}(X)$ with coefficients $\gamma^{n-k}(h_k \circ \gamma)$ for $k = 0, \ldots, n-1$ multiplied by the overall coefficient $(-1)^n(1+\gamma^2)^{-\frac{n-1}{2}}$. We expand

$$\gamma^{n-k}(h_k \circ \gamma) = \sum_{a=0}^{k} \sum_{b=0}^{n-1-k} (-1)^{n-1-k-b} \binom{k}{a} \binom{n-1-k}{b} \int_{-1}^{1} \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2+\gamma^2)^{\frac{n+1}{2}}} \,\mathrm{du}$$

and notice that we can write

$$\int_{-1}^{1} \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2+\gamma^2)^{\frac{n+1}{2}}} \,\mathrm{du} = \gamma^{k-a+b}(F_{n,a+b} \circ \gamma)$$

for the function $F_{n,l}$ from (5.7) with $l \coloneqq a+b$. Because $0 \leq l \leq n-1$, Lemma 5.1.8 asserts that $F_{n,l}$ extends smoothly to $[0, \infty)$. Because $k - a + b \geq 0$, the entire coefficient at ν_k extends smoothly to $[0, \infty)$ for every $k = 0, \ldots, n-1$. The lemma follows. □

We summarize our results in the following proposition:

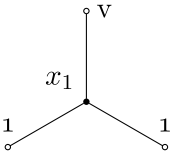


Figure 5.3.: The Y-graph for $\mathbb{S}^n$.

Proposition 5.1.10 (Hodge propagator for $\mathbb{S}^n$)**.** *The form* P *from* (5.2) *defines a Hodge propagator for* $\mathbb{S}^n$ *satisfying Definition 4.2.4. Moreover, we have the symmetries*

$$\begin{aligned} R^*\mathrm{P} &= (-1)^R\mathrm{P} \quad \textit{for all } R \in O(n+1) \textit{ and} \\ \tau^*\mathrm{P} &= (-1)^n\mathrm{P}. \end{aligned}$$

Proof. The proposition is a summary of Propositions 5.1.2, 5.1.4 and 5.1.9. □

Remark 5.1.11 (Better notation due to R. Bryant, see [Bry18]). Pick an oriented basis $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$ as generators of the exterior algebra $\Lambda^*(\mathbb{R}^{n+1})$, and view x, y, dx, dy as $\Lambda^*(\mathbb{R}^{n+1})$-valued forms on $\mathbb{R}^{n+1}$. For example, we view x as the map $x \in \mathbb{R}^{n+1} \mapsto \sum_{i=1}^{n+1} x^i e_i \in \Lambda^1(\mathbb{R}^{n+1})$ and dx as the map $x \in \mathbb{R}^{n+1} \mapsto \sum_{i=1}^{n+1} (\mathrm{dx}_i)_x e_i \in \Lambda^1(\mathbb{R}^{n+1})$. There is a natural wedge product on the space of $\Lambda^*(\mathbb{R}^{n+1})$-valued forms. If ω is a top-form, we denote by $[\omega]$ the coefficient of ω at $e_1 \wedge \dots \wedge e_{n+1}$. Then it holds

$$\omega_k(x,y) = \frac{1}{k!}\frac{1}{(n-1-k)!}[x \wedge y \wedge (\mathrm{dx})^k \wedge (\mathrm{dy})^{n-1-k}].$$

Note that if we view e_i as odd variables, then $[\cdot]$ corresponds to the odd integration $\int \mathrm{D}e(\cdot)$. It would be interesting to know whether this notation simplifies some proofs, especially if Lemma 5.2.1 can be deduced from abstract algebraic facts or rules valid for odd integration. ◁

5.2. Computation of Chern-Simons Maurer-Cartan element for spheres

We recall from Definition 4.3.6 that the Chern-Simons Maurer-Cartan element $\mathfrak{n}$ is computed as a sum over trivalent ribbon graphs decorated with the admissible Hodge Propagator P at internal edges, integration variables x_i at internal vertices and, in the case of $\mathbb{S}^n$, with $\mathrm{1}$ or v at external vertices.

The canonical Maurer-Cartan element $\mathfrak{m}$ is the contribution of the Y-graph (see

Figure 5.3), and it is easy to see that

$$\mathfrak{m}_{10}(\mathrm{sv11}) = (-1)^n \mathfrak{m}_{10}(\mathrm{s1v1}) = \mathfrak{m}_{10}(\mathrm{s11v}) = (-1)^{n-2}.$$

Throughout this section, we will be in the setting of Definition 4.3.6. In particular, $\Gamma \in \mathrm{RG}^{(3)}_{klg}$ is a ribbon graph, L its compatible labeling admissible with respect to an input $\omega_1, \ldots, \omega_l$ and $I(\sigma_L)$ the corresponding integral.

Lemma 5.2.1 (Condition (V_1) holds)**.** *For the Hodge Propagator* P *from* (5.2) *for* $\mathbb{S}^n$ *for* $n \geq 1$*, every graph* $\Gamma \neq Y$ *with* $\mathrm{1}$ *at an external vertex vanishes.*

Proof. The only contribution of an A-vertex which does not vanish for degree reasons is

$$A_{\mathrm{v},1}(y) = \int_x \mathrm{P}(x,y)\mathrm{Vol}(x).$$

From the symmetry of P and Vol under the action of $O(n+1)$, we get

$$R^* A_{\mathrm{v},1} = (-1)^R A_{\mathrm{v},1} \quad \text{for all } R \in O(n+1).$$

Therefore, it suffices to check that $A_{\mathrm{v},1}(\mathrm{e}_1) = 0$, where $\mathrm{e}_1, \ldots, \mathrm{e}_{n+1}$ denotes the standard basis of $\mathbb{R}^{n+1}$. Evaluation of (5.4) at (x, e_1) gives

$$\omega_0(x, \mathrm{e}_1) = \frac{1}{(n-1)!} \sum_{\substack{\sigma \in \mathbb{S}_{n+1} \\ \sigma_2 = 1}} (-1)^\sigma x^{\sigma_1} \, \mathrm{dy}^{\sigma_3} \cdots \mathrm{dy}^{\sigma_{\mathrm{n+1}}} \, .$$

Therefore, we get

$$\begin{aligned} A_{\mathrm{v},1}(\mathrm{e}_1) &= (-1)^n \int_x g_0(x \cdot \mathrm{e}_1)\omega_0(x, \mathrm{e}_1)\mathrm{Vol}(x) \\ &= (-1)^{n+1} \int_x g_0(x^1) \Big(\sum_{j=2}^{n+1} (-1)^j x^j \Big) \mathrm{Vol}(x) \, \mathrm{dy}^2 \cdots \widehat{\mathrm{dy}^{\mathrm{j}}} \cdots \mathrm{dy}^{\mathrm{n}+1}, \end{aligned}$$

where we view g_0 as a function of $x \cdot y$. Consider the cyclic permutation of $x^2, \ldots, x^{n+1}$:

$$\begin{aligned} I : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ (x^1, x^2, \ldots, x^n, x^{n+1}) &\longmapsto (x^1, x^{n+1}, x^2 \ldots, x^n). \end{aligned}$$

Then we have

$$\begin{aligned}\int_x g_0(x^1)\sum_{j=2}^{n+1}(-1)^j x^j \mathrm{Vol}(x) &= (-1)^{n-1}\int_x I^*\Big(g_0(x^1)\sum_{j=2}^{n+1}(-1)^j x^j \mathrm{Vol}(x)\Big)\\ &= (-1)^{n-1}\int_x g_0(x^1)\Big(-\sum_{j=2}^{n+1}(-1)^j x^j\Big)((-1)^{n-1}\mathrm{Vol}(x))\\ &= -\int_x g_0(x^1)\sum_{j=2}^{n+1}(-1)^j x^j \mathrm{Vol}(x).\end{aligned}$$

It follows that $A_{\mathrm{v},\imath}(\mathrm{e}_1) = 0$.[1]

Let us now consider the contribution of a B-vertex with $\imath$:

$$B_\imath(y,z) = \int_x \mathrm{P}(y,x)\mathrm{P}(x,z) = (-1)^n\int_x \mathrm{P}(y,x)\mathrm{P}(x,z).$$

For $n = 1$, the degree of $\mathrm{P}(y,x)\mathrm{P}(x,z)$ is 0, and hence $B_\imath = 0$ trivially. Suppose that $n \geq 2$. As in the case of $A_{\mathrm{v},\imath}$, we get that

$$R^*B_\imath = (-1)^R B_\imath \quad \text{for all } R \in O(n+1).$$

Therefore, it suffices to check that $B_\imath(\mathrm{e}_1, c_1\mathrm{e}_1 + c_2\mathrm{e}_2) = 0$ for all $(c_1, c_2) \in \mathbb{S}^1$. We have

$$\begin{aligned}B_\imath(\mathrm{e}_1, c_1\mathrm{e}_1 + c_2\mathrm{e}_2) = (-1)^n\int_x \sum_{a=1}^{n-1} g_a(x^1)g_{n-a}(c_1x^1 + c_2x^2)\omega_a(x,\mathrm{e}_1)&\\ \omega_{n-a}(x, c_1\mathrm{e}_1 + c_2\mathrm{e}_2).&\end{aligned}$$

We will show that for every $a = 1, \ldots, n-1$ we can write

$$\mu_a(x) \coloneqq \omega_a(x,\mathrm{e}_1)\omega_{n-a}(x, c_1\mathrm{e}_1 + c_1\mathrm{e}_2) = \Big(\sum_{i=3}^{n+1} \pm x^i \mathrm{Vol}(x)\Big)\eta_a(y,z) \tag{5.8}$$

with alternating signs for some form $\eta_a(y,z)$. Then we can use the same argument as for $A_{\mathrm{v},\imath}$ with the cyclic permutation of $x^3, \ldots, x^{n+1}$ given by

$$\begin{aligned}I' : \mathbb{S}^n &\longrightarrow \mathbb{S}^n\\ (x^1, x^2, x^3, \ldots, x^n, x^{n+1}) &\longmapsto (x^1, x^2, x^{n+1}, x^3 \ldots, x^n)\end{aligned}$$

---

[1] Notice that we avoided using $\sum\int = \int\sum$ because the convergence of single summands is not guaranteed.

to conclude that $B_1(\mathrm{e}_1, c_1\mathrm{e}_1 + c_2\mathrm{e}_2) = 0$.

In order to show (5.8), we have to study the product of ω_i's. From (5.4) we get

$$\begin{aligned}&\omega_a(x,y)\omega_{n-a}(x,z)\\ &\quad= \frac{1}{a!(n-1-a)!(n-a)!(a-1)!} \sum_{\sigma,\,\mu\in\mathbb{S}_{n+1}} (-1)^{\sigma+\mu} x^{\sigma_1} x^{\mu_1} y^{\sigma_2} z^{\mu_2}\\ &\qquad\qquad \mathrm{dx}^{\sigma_3}\cdots\mathrm{dx}^{\sigma_{2+a}}\,\mathrm{dx}^{\mu_3}\cdots\mathrm{dx}^{\mu_{2+n-a}}\,\mathrm{dy}^{\sigma_{3+a}}\cdots\mathrm{dy}^{\sigma_{n+1}}\\ &\qquad\qquad\qquad\qquad \mathrm{dz}^{\mu_{3+n-a}}\cdots\mathrm{dz}^{\mu_{n+1}}\,.\end{aligned} \tag{5.9}$$

In order to simplify this expression, we decompose $\sigma \in \mathbb{S}_{n+1}$ as

$$\sigma = \sigma^5 \circ \sigma^4 \circ \sigma^3 \circ \sigma^2 \circ \sigma^1, \tag{5.10}$$

where $\sigma^1, \ldots, \sigma^5 \in \mathbb{S}_{n+1}$ are permutations defined as follows:

- The permutation σ^1 is a shuffle permutation $\sigma^1 \in \mathbb{S}_{2+a,n-a-1}$ such that its first block denoted by $\sigma^1(1) = (\sigma^1_1, \ldots, \sigma^1_{2+a})$ is equal to the ordered set $\{\sigma_1, \ldots, \sigma_{2+a}\}$. The second block $\sigma^1(2)$ is then the ordered set $\{\sigma_{3+a}, \ldots, \sigma_{n+1}\}$, which will be denoted by J_σ.
- The permutation σ^2 acts on the block $\sigma^1(1)$ by moving σ_2 in front. We denote the new block $\sigma^1(1)\backslash\{\sigma_2\}$ by I_σ, so that we can write $\sigma^2 : \sigma^1(1) \mapsto (\sigma_2, I_\sigma)$.
- The permutation σ^3 acts on the block I_σ by moving σ_1 in front. Together with the previous step we get $\sigma^1(1) \mapsto (\sigma_2, \sigma_1, I_\sigma\backslash\{\sigma_1\})$.
- The permutation σ^4 is a transposition of σ_1 and σ_2.
- The permutation σ^5 is determined by the pair $(\sigma^{51}, \sigma^{52}) \in \mathbb{S}_a \times \mathbb{S}_{n-1-a}$ of permutations σ^{51} and σ^{52} acting on blocks $I_\sigma\backslash\{\sigma_1\}$ and J_σ to get $(\sigma_3, \ldots, \sigma_{2+a})$ and $(\sigma_{3+a}, \ldots, \sigma_{n+1})$, respectively.

We define the decomposition $\mu^1, \ldots, \mu^5$ for $\mu \in \mathbb{S}_{n+1}$ from (5.9) analogously with a replaced by $n-a$. Using (5.10), the product (5.9) can be written as

$$\begin{aligned}&\frac{1}{a!(n-1-a)!(n-a)!(a-1)!} \sum_{\substack{\sigma^1,\ldots,\sigma^5\\ \mu^1,\ldots,\mu^5}} (-1)^{\sigma^1+\cdots+\sigma^5+\mu^1+\cdots+\mu^5} x^{\sigma_1} x^{\mu_1} y^{\sigma_2} z^{\mu_2}\\ &\qquad\qquad \mathrm{dx}^{\sigma^{51}(I_\sigma\backslash\{\sigma_1\})}\,\mathrm{dx}^{\mu^{51}(I_\mu\backslash\{\mu_1\})}\,\mathrm{dy}^{\sigma^{52}(J_\sigma)}\,\mathrm{dz}^{\mu^{52}(J_\mu)}\end{aligned}$$

$$= -\sum_{\sigma^1,\,\mu^1}(-1)^{\sigma^1+\mu^1}\Big(\sum_{\sigma^2,\,\mu^2}(-1)^{\sigma^2+\mu^2}\sum_{\sigma^3,\,\mu^3}(-1)^{\sigma^3+\mu^3}x^{\sigma_1}x^{\mu_1}y^{\sigma_2}z^{\mu_2}$$
$$\mathrm{dx}^{I_\sigma\setminus\{\sigma_1\}}\,\mathrm{dx}^{I_\mu\setminus\{\mu_1\}}\Big)\,\mathrm{dy}^{J_\sigma}\,\mathrm{dz}^{J_\mu},$$

where -1 comes from $(-1)^{\sigma^4}$ and σ^5 is compensated by permutations of forms. For fixed σ^1 and μ^1, consider the coefficient at $\mathrm{dy}^{J_\sigma}\,\mathrm{dz}^{J_\mu}$ in the brackets. If we evaluate it at $y = \mathrm{e}_1$, $z = c_1\mathrm{e}_1 + c_2\mathrm{e}_2$, we get

$$c_1\overbrace{\sum_{\substack{\sigma^3,\,\mu^3\\ \sigma_2=1\\ \mu_2=1}}(-1)^{\sigma^3+\mu^3}x^{\sigma_1}x^{\mu_1}\,\mathrm{dx}^{I_\sigma\setminus\{\sigma_1\}}\,\mathrm{dx}^{I_\mu\setminus\{\mu_1\}}}^{=:\,\mathrm{I}}$$
$$+(-1)^{\mu^2}c_2\overbrace{\sum_{\substack{\sigma^3,\,\mu^3\\ \sigma_2=1\\ \mu_2=2}}(-1)^{\sigma^3+\mu^3}x^{\sigma_1}x^{\mu_1}\,\mathrm{dx}^{I_\sigma\setminus\{\sigma_1\}}\,\mathrm{dx}^{I_\mu\setminus\{\mu_1\}}}^{=:\,\mathrm{II}},$$

where $(-1)^{\mu^2} = -1$ if and only if $1 \in I_\mu$.

More generally, for multiindices I_1, $I_2 \subset \{1, \ldots, n+1\}$ of lengths $a+1$ and $n-a+1$, respectively, consider the sum

$$S(I_1, I_2) \coloneqq \sum_{\substack{i_1\in I_1\\ i_2\in I_2}}(-1)^{(i_1,I_1)+(i_2,I_2)}x^{i_1}x^{i_2}\,\mathrm{dx}^{I_1\setminus\{i_1\}}\,\mathrm{dx}^{I_2\setminus\{i_2\}}, \tag{5.11}$$

where (i_j, I_j) is the number of transpositions required to move i_j in front of I_j. The following implication holds:

$$S(I_1, I_2) \neq 0 \implies 1 \leq |I_1 \cap I_2| \leq 2.$$

We distinguish the two cases left:

Case $I_1 \cap I_2 = \{i, j\}$ with $i < j$: We get

$$\begin{aligned} S(I_1, I_2) &= (-1)^{(i,I_1)+(j,I_2)}x^i x^j\,\mathrm{dx}^{I_1\setminus\{i\}}\,\mathrm{dx}^{I_2\setminus\{j\}} \\ &\quad + (-1)^{(j,I_1)+(i,I_2)}x^j x^i\,\mathrm{dx}^{I_1\setminus\{j\}}\,\mathrm{dx}^{I_2\setminus\{i\}} \\ &= (-1)^{(i,I_1)+(j,I_2)+(j,I_1)+1+(i,I_2)}x^i x^j\,\mathrm{dx}^j\,\mathrm{dx}^{I_1\setminus\{i,j\}} \\ &\quad \mathrm{dx}^i\,\mathrm{dx}^{I_2\setminus\{i,j\}} + (-1)^{(j,I_1)+(i,I_2)+(i,I_1)+(j,I_2)+1}x^i x^j \\ &\quad \mathrm{dx}^i\,\mathrm{dx}^{I_1\setminus\{i,j\}}\,\mathrm{dx}^j\,\mathrm{dx}^{I_2\setminus\{i,j\}} \end{aligned}$$

$$= \pm(-1+1)x^i x^j \,\mathrm{dx}^i \,\mathrm{dx}^{I_1\backslash\{i,j\}} \,\mathrm{dx}^j \,\mathrm{dx}^{I_2\backslash\{i,j\}},$$

where in the last step we switched $\mathrm{dx}^i \leftrightarrow \mathrm{dx}^j$ in the first summand. Therefore, it holds $S(I_1, I_2) = 0$.

Case $I_1 \cap I_2 = \{i\}$*:* We must have $I_1 \cup I_2 = \{1, \ldots, n+1\}$. A non-zero summand in (5.11) has either $i_1 = i$ and $i_2 \in I_2$, in which case

$$I_1\backslash\{i_1\} \cup I_2\backslash\{i_2\} = \{1, \ldots, \widehat{i_2}, \ldots, n+1\},$$

or $i_2 = i$ and $i_1 \in I_1$ with $i_1 \neq i$, in which case

$$I_1\backslash\{i_1\} \cup I_2\backslash\{i_2\} = \{1, \ldots, \widehat{i_1}, \ldots, n+1\}.$$

Indices i_2 from the first case and i_1 from the second case constitute $\{1, \ldots, n+1\}$. Therefore, for some signs ±, we can write

$$S(I_1, I_2) = x^i \sum_{j=1}^{n+1} \pm x^j \,\mathrm{dx}^1 \cdots \widehat{\mathrm{dx^j}} \cdots \mathrm{dx^{n+1}} .$$

We will prove that the signs alternate, and hence $S(I_1, I_2) = \pm x^i \mathrm{Vol}(x)$. Suppose that j, $j+1 \in I_1$ for some $j \in \{1, \ldots, n\}$. The two summands in (5.11) with $(i_1, i_2) = (j, i)$ and $(i_1, i_2) = (j+1, i)$, respectively, give

$$\begin{aligned}
&(-1)^{(j,I_1)+(i,I_2)} x^j x^i \,\mathrm{dx}^{I_1\backslash\{j\}} \,\mathrm{dx}^{I_2\backslash\{i\}} + (-1)^{(j+1,I_1)+(i,I_2)} x^{j+1} x^i \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{dx}^{I_1\backslash\{j+1\}} \,\mathrm{dx}^{I_2\backslash\{i\}} \\
&\quad = (-1)^{(i,I_2)} x^j x^i \,\mathrm{dx}^{j+1} \,\mathrm{dx}^{I_1\backslash\{j,j+1\}} \,\mathrm{dx}^{I_2\backslash\{i\}} \\
&\qquad + (-1)^{1+(i,I_2)} x^{j+1} x^i \,\mathrm{dx}^j \,\mathrm{dx}^{I_1\backslash\{j,j+1\}} \,\mathrm{dx}^{I_2\backslash\{i\}} \\
&\quad = (-1)^{(i,I_2)} x^i (x^j \,\mathrm{dx}^{j+1} - x^{j+1} \,\mathrm{dx}^j) \,\mathrm{dx}^{I_1\backslash\{j,j+1\}} \,\mathrm{dx}^{I_2\backslash\{i\}} .
\end{aligned}$$

The signs clearly alternate. A symmetric argument holds when j, $j+1 \in I_2$. Now assume that $j \in I_1$ and $j+1 \in I_2$. The two summands in (5.11) which have $(i_1, i_2) = (j, i)$ and $(i_1, i_2) = (i, j+1)$, respectively, give

$$\begin{aligned}
&(-1)^{(j,I_1)+(i,I_2)} x^j x^i \,\mathrm{dx}^{I_1\backslash\{j\}} \,\mathrm{dx}^{I_2\backslash\{i\}} + (-1)^{(i,I_1)+(j+1,I_2)} x^i x^{j+1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{dx}^{I_1\backslash\{i\}} \,\mathrm{dx}^{I_2\backslash\{j+1\}}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{(j,I_1)+(i,I_1\setminus\{j\})+(a+1)} x^j x^i \,\mathrm{dx}^{\mathrm{I_1\setminus\{i,j\}}} \,\mathrm{dx}^{I_2} \\
&\qquad + (-1)^{(i,I_1)+(j+1,I_2)+(j,I_1\setminus\{i\})} x^i x^{j+1} \,\mathrm{dx}^j \,\mathrm{dx}^{I_1\setminus\{i,j\}} \,\mathrm{dx}^{I_2\setminus\{j+1\}} \\
&= (-1)^{(j,I_1)+(i,I_1\setminus\{j\})+(j+1,I_2)} x^i x^j \,\mathrm{dx}^{j+1} \,\mathrm{dx}^{I_1\setminus\{i,j\}} \,\mathrm{dx}^{I_2\setminus\{j+1\}} \\
&\qquad + (-1)^{1+(j,I_1)+(i,I_1\setminus\{j\})+(j+1,I_2)} x^i x^{j+1} \,\mathrm{dx}^j \,\mathrm{dx}^{I_1\setminus\{i,j\}} \,\mathrm{dx}^{I_2\setminus\{j+1\}} \\
&= (-1)^{(j,I_1)+(i,I_1\setminus\{j\})+(j+1,I_2)} x^i (x^j \,\mathrm{dx}^{j+1} - x^{j+1} \,\mathrm{dx}^j) \\
&\qquad\qquad\qquad\qquad \mathrm{dx}^{I_1\setminus\{i,j\}} \,\mathrm{dx}^{I_2\setminus\{j+1\}} .
\end{aligned}$$

The signs alternate again. A symmetric argument holds for $j \in I_2$ and $j+1 \in I_1$.

Back to the original problem, we have $\mathrm{I} = S(I_\sigma, I_\mu)$ with $I_\sigma, I_\mu \subset \{2, \ldots, n+1\}$. It follows that the first case applies, and hence $\mathrm{I} = 0$. We have $\mathrm{II} = S(I_\sigma, I_\mu)$ with $I_\sigma \subset \{2, \ldots, n+1\}$ and $I_\mu \subset \{1, \widehat{2}, \ldots, n+1\}$. It follows that either the first case or the second case with $i \geq 3$ applies. This proves (5.8) up to signs. Tracing back the signs, one can convince himself that the signs in (5.8) indeed alternate.

The last paragraph of the proof of Proposition 4.4.2 finishes the proof. $\square$

We summarize the consequences in the following proposition. The main argument is the same as in the proof of Proposition (4.4.4).

Proposition 5.2.2 (Vanishing of graphs for $\mathbb{S}^n$)**.** *Consider* $\mathbb{S}^n$ *with the Hodge Propagator* (5.2)*. Only the following trivalent ribbon graphs* $\Gamma \neq Y$ *do not necessarily vanish:*

($n = 1$)*: The* O_k*-graph with* $k \in 2\mathbb{N}$ *internal vertices of type* B *with* v *at the external vertex (see Figure 5.7).*

($n = 2$)*: It must hold* $A = 0$*,* $C = 2B$ *and all* B *vertices must have* v *at the external vertex. Moreover, if* Γ *is reduced, it must have* $g \geq 1$*.*

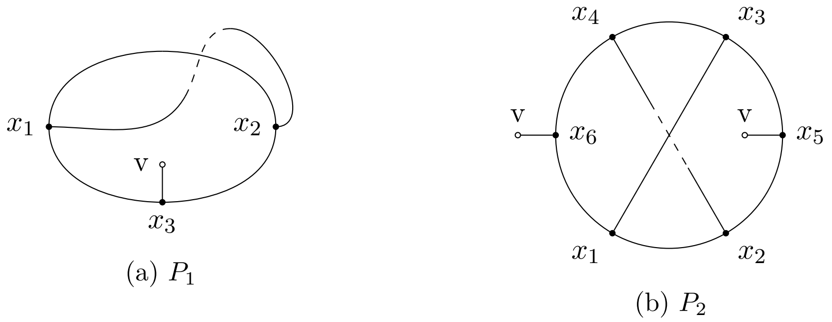
($n = 3$)*: There is no external vertex and* $4 \mid C$ *holds.*

($n > 3$)*: All graphs vanish.*

Proof. Lemma 5.2.1 implies that $A = 0$ and that the total form-degree D satisfies $D = nB$. Therefore, we get from (4.27) the following: for $n > 3$ there is neither a B-vertex nor a C-vertex; for $n = 3$, there is no B-vertex; for $n = 2$, we have $C = 2B$; and for $n = 1$, there is no C-vertex.

Consider the pullback of $I(\sigma_L)$ along the (multi)diagonal action of an $R \in O(n+1)$ with $\det(R) = -1$ on $(\mathbb{S}^n)^{\times k}$. We get schematically

$$\int_{(\mathbb{S}^n)^{\times k}} \mathrm{P}^e \mathrm{Vol}^s = (-1)^{k+e+s} \int_{(\mathbb{S}^n)^{\times k}} G^e \mathrm{Vol}^s.$$



(a) P_1 (b) P_2

Figure 5.4.: Vanishing graphs P_1 and P_2 for $\mathbb{S}^2$.

Therefore, $k+e+s$ has to be even. If we plug-in from (4.24), we get

$$k+e+s=\begin{cases}3B & \text{for } n=1,\\ 8B & \text{for } n=2,\\ \frac{5}{2}C & \text{for } n=3.\end{cases}$$

A non-vanishing reduced graph must have $B \geq l$. For $n=2$, so that $C=2B$, the formula (4.25) gives $g \geq 1$. □

Remark 5.2.3 (Graphs for $\mathbb{S}^2$). The simplest possibly non-vanishing graph for $\mathbb{S}^2$ has $A=0$, $B=1$, $C=2$. If it is reduced, we must have $l=g=1$, and hence it will contribute to $\mathfrak{n}_{11}$. Up to an isomorphism, there is only one such graph, which we denote by P_1 (see Figure 5.4). However, we see that the pair of internal vertices x_1 and x_2 is connected by two edges, which implies that $P_1=0$. Indeed, $\mathrm{P}(x,y)$ has odd degree, and hence we have[2]

$$\mathrm{P}(x,y)\mathrm{P}(y,x)=\mathrm{P}(x,y)^2=0$$

by the symmetry on the pullback along the twist map. It follows that $\mathfrak{n}_{11}=0$.

The second simplest possibly non-vanishing reduced graph is the graph P_2 from Figure 5.4. Let

$$\eta(x_1,x_2,x_3,x_4,x_5) \coloneqq \mathrm{P}(x_1,x_2)\mathrm{P}(x_1,x_3)\mathrm{P}(x_4,x_2)\mathrm{P}(x_4,x_3)\mathrm{P}(x_3,x_5)\\ \mathrm{P}(x_2,x_5)\mathrm{Vol}(x_5)$$

denote the form in the integrand coming from the part of the graph on the right-hand side of the vertical axis going through x_1, x_4. If $\tau_{1,4}$ denotes the exchange of x_1 and x_4,

---

[2] We recall from Section 4.3 that the notation $\mathrm{P}(x_i,x_j)$ means $(\pi_i \times \pi_j)^*\mathrm{P}$ and not just the evaluation at (x_i,x_j).

then clearly $\tau_{1,4}^*\eta = \eta$ because the graph is symmetric with respect to the horizontal axis going through x_5, x_6. Using this, we compute

$$\begin{aligned}
&\int_{x_1,x_2,x_3,x_4,x_5,x_6} \mathrm{v}(x_6)\mathrm{P}(x_1,x_6)\mathrm{P}(x_4,x_6)\eta(x_1,x_2,x_3,x_4,x_5) \\
&= \int_{\tau_{1,4}(x_1,x_2,x_3,x_4,x_5,x_6)} \tau_{1,4}^*(\mathrm{v}(x_6)\mathrm{P}(x_1,x_6)\mathrm{P}(x_4,x_6)\eta(x_1,x_2,x_3,x_4,x_5)) \\
&= \int_{x_4,x_2,x_3,x_1,x_5,x_6} \mathrm{v}(x_6)\mathrm{P}(x_4,x_6)\mathrm{P}(x_1,x_6)\eta(x_4,x_2,x_3,x_1,x_5) \\
&= -\int_{x_1,x_2,x_3,x_4,x_5,x_6} \mathrm{v}(x_6)\mathrm{P}(x_1,x_6)\mathrm{P}(x_4,x_6)\eta(x_1,x_2,x_3,x_4,x_5),
\end{aligned}$$

where the minus sign comes from switching the first two P's. We see that P_2 vanishes. The other variants with x_5 moved on the edge x_3, x_4 and x_2, x_4 vanish by a similar argument using the compositions $\tau_{1,3} \circ \tau_{5,6}$ and $\tau_{1,2} \circ \tau_{5,6}$, respectively. We conclude that $\mathfrak{n}_{21} = 0$, and hence $\mathfrak{q}^{\mathfrak{n}}_{121} = 0$.

We sum up some general observations about the integrals for $\mathbb{S}^2$:

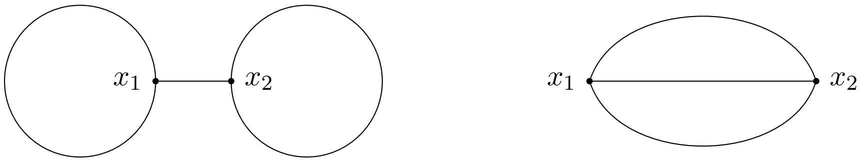
- We have $B_1 \neq 0$ and $C \neq 0$ for the corresponding forms.
- We have the multiplication formula (c.f., Example 5.1.3)

$$\omega_1(x,y)\omega_1(x,z) = x \cdot (y \times z)\mathrm{Vol}(x).$$

- The number $(-1)^{\sigma_L} I(\sigma_L)$ does not depend on the choice of L_1 provided a compatible L_2 is chosen.
- It holds $\sum_{L_3^b} (-1)^{\sigma_L} I(\sigma_L) = 0$ whenever there is a boundary component with even number of v's.
- If there is a B-vertex x such that the underlying graph (after forgetting the ribbon structure) is symmetric on the reflection along an axis going through x, then $I(\sigma_L) = 0$. $\triangleleft$

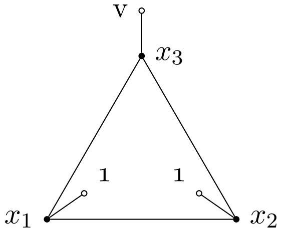
Remark 5.2.4 (Graphs for $\mathbb{S}^3$). For $\mathbb{S}^3$, we consider the non-reduced graphs K_1 and K_2 and the tadpole graph from Figure 5.5. The graphs K_1 and K_2 appear in the definition of the Chern-Simons topological invariant in [Koh02] (with a gauge group). The corresponding integrals from our theory vanish "algebraically", i.e., at the level of wedge products of ω_i. Indeed, every summand in K_1 contains

$$\omega_a(x_1, x_1) = 0 \quad \text{for some } a = 0,1,2,$$



(a) K_1

(b) K_2



(c) Tadpole

Figure 5.5.: Graphs K_1 and K_2 from the Chern-Simons theory and the tadpole graph with $(l, g) = (2, 0)$ for $n = 3$.

and, for degree reasons, the form part of K_2 can contain only

$$\omega_1(x_1, x_2)^3 = 0 \quad \text{or} \quad \omega_0(x_1, x_2)\omega_1(x_1, x_2)\omega_2(x_1, x_2) = 0.$$

The tadpole graph contains only

$$\omega_2(x_1, x_3)\omega_1(x_1, x_2)\omega_2(x_2, x_3) = 0. \qquad \lhd$$

Equations in Remark 5.2.4 were checked by the computer—this is possible as the vanishing is purely algebraic. Any author's attempts to show numerically that some integrals are non-zero or get a hint for "analytic vanishing" failed so far. The program for Wolfram Mathematica 10.4 will be made available at [Háj18a] and possibly updated.

We will now compute $\mathfrak{n}_{20}$ for $\mathbb{S}^1$, which according to Proposition 5.2.2 consists only of contributions from the O_k-graphs with k even. By analogy with the finite dimensional case (see Appendix A), we expect that the number $(-1)^{\sigma_L} I(\sigma_L)$ does not depend on L. All inputs are namely the same and the degrees even, i.e., $|m_2^+| = -2$, $|\theta^2 \mathrm{P}| = -2$ and $|\mathrm{v}| = 0$.

We fix s_1, $s_2 \geq 1$ such that $k = s_1 + s_2$ is even and make the ansatz

$$n_{20}(\mathrm{sv}^{s_1} \otimes \mathrm{sv}^{s_2}) \coloneqq \varepsilon(s_1, s_2) C(s_1, s_2) I(k),$$

where $I(k)$ is the integral

$$\frac{1}{V^k}\int_{x_1,\dots,x_k} G(x_1,x_2)\cdots G(x_{k-1},x_k)G(x_k,x_1)\mathrm{Vol}(x_1)\cdots\mathrm{Vol}(x_k), \tag{5.12}$$

$\varepsilon(s_1,s_2)$ a sign and $C(s_1,s_2)$ a combinatorial coefficient to be determined.

We fix a circle in the plane with k points (= internal vertices) and denote by $O(s_1,s_2)$ the set of ribbon graphs constructed by attaching external legs from which s_1 points in the interior and s_2 in the exterior, or the other way round, so that $O(s_1,s_2)=O(s_2,s_1)$ (see Figure 5.7). Recall that the ribbon structure is induced from the counterclockwise orientation of the plane. It is easy to see that all graphs in $O(s_1,s_2)$ admit a labeling which is admissible with respect to $\mathrm{sv}^{s_1}\otimes\mathrm{sv}^{s_2}$, and that $O(s_1,s_2)$ contains a representative of every such O_k-graph.

Lemma 5.2.5 (Integral for the O_k-graph for $\mathbb{S}^1$). *For every even $k\geq 2$, the integral $I(k)$ is equal to*

$$(-1)^{k+1}\frac{1}{2^k}\sum_{i=2,4,\dots,k}\frac{i}{(i+1)!}\sum_{\substack{i_1+\dots+i_r=k-i\\ i_1,\dots,i_r\in 2\mathbb{N},\, r\in\mathbb{N}}}(-1)^r\frac{1}{(i_1+1)!\cdots(i_r+1)!}. \tag{5.13}$$

Proof. Denote $\bar{\mathrm{P}}\coloneqq -2\pi\mathrm{P}$. For all k, $l\geq 1$, we consider the more general integral

$$I(k,l)\coloneqq\int_{x_1,\dots,x_k}\bar{\mathrm{P}}(x_1,x_2)\cdots\bar{\mathrm{P}}(x_{k-1},x_k)\bar{\mathrm{P}}(x_k,x_1)^l\mathrm{Vol}(x_1)\cdots\mathrm{Vol}(x_k).$$

Taking the pullback along $(x_1,x_2,\dots,x_{k-1},x_k)\mapsto(x_k,x_{k-1},\dots,x_2,x_1)$ and using the antisymmetry of $\bar{\mathrm{P}}(x,y)$, we get $I(k,l)=0$ whenever $k+l$ is even. We will compute $I(k,1)$ for $k\in 2\mathbb{N}$ from a recursive relation which arises from successive integration.

For the recursion step, we need to evaluate the integral

$$\int_y \bar{\mathrm{P}}(x,y)\bar{\mathrm{P}}(y,z)^l\mathrm{Vol}(y)$$

for fixed $(x,z)\in(\mathbb{S}^1\times\mathbb{S}^1)\backslash\Delta$. Pick the chart $g:\mathbb{S}^1\backslash\{z\}\to(-\pi,\pi)$ defined by

$$g(y)=\bar{\mathrm{P}}(y,z)=\pi-\alpha(y,z)\quad\text{for } y\in\mathbb{S}^1\backslash\{z\},$$

where the angle α was defined in Example 5.1.3. It holds $\mathrm{dg}(y)=\mathrm{Vol}(y)$ and

$$\bar{\mathrm{P}}(x,y)=\begin{cases}\bar{\mathrm{P}}(x,z)-g(y)-\pi & \text{for } -\pi<g(y)<\bar{\mathrm{P}}(x,z),\\ \bar{\mathrm{P}}(x,z)-g(y)+\pi & \text{for } \bar{\mathrm{P}}(x,z)<g(y)<\pi.\end{cases}$$

We compute

$$\begin{aligned}\int_y \bar{\mathrm{P}}(x,y)\bar{\mathrm{P}}(y,z)^l \mathrm{Vol}(y) &= \int_{-\pi}^{\pi} (\bar{\mathrm{P}}(x,z)-g)g^l \,\mathrm{dg} - \pi \int_{-\pi}^{\bar{\mathrm{P}}(x,z)} g^l \,\mathrm{dg} \\ &\qquad + \pi \int_{\bar{\mathrm{P}}(x,z)}^{\pi} g^l \,\mathrm{dg} \\ &= \frac{2\pi}{l+1} \begin{cases} \pi^l \bar{\mathrm{P}}(x,z) - \bar{\mathrm{P}}(x,z)^{l+1} & \text{for } l \text{ even,} \\ \dfrac{\pi^{l+1}}{l+2} - \bar{\mathrm{P}}(x,z)^{l+1} & \text{for } l \text{ odd.} \end{cases}\end{aligned}$$

From now on, $\int$ will stand for the Riemannian integral, i.e., $\int f := \int f \mathrm{Vol}$ for a function f. We compute

$$\begin{aligned} I(2,l) &= \int_{x_1,x_2} \bar{\mathrm{P}}(x_1,x_2)\bar{\mathrm{P}}(x_2,x_1)^l \\ &= -\int_{yz} \bar{\mathrm{P}}(y,z)^{l+1} \\ &= -2\pi \int_{-\pi}^{\pi} g^{l+1}\,\mathrm{dg} \\ &= \begin{cases} 0 & \text{for } l \text{ even,} \\ -\dfrac{4\pi^{l+3}}{l+2} & \text{for } l \text{ odd.} \end{cases}\end{aligned}$$

For $k \geq 4$ even and l odd, we compute

$$\begin{aligned} I(k,l) &= \frac{2\pi}{l+1} \int_{x_1,\ldots,x_{k-1}} \bar{\mathrm{P}}(x_1,x_2)\cdots\bar{\mathrm{P}}(x_{k-2},x_{k-1}) \\ &\qquad\qquad \Big(\frac{\pi^{l+1}}{l+2} - \bar{\mathrm{P}}(x_{k-1},x_1)^{l+1}\Big) \\ &= \frac{-4\pi^2}{(l+1)(l+2)} \int_{x_1,\ldots,x_{k-2}} \bar{\mathrm{P}}(x_1,x_2)\cdots\bar{\mathrm{P}}(x_{k-3},x_{k-2}) \\ &\qquad\qquad \Big(\pi^{l+1}\bar{\mathrm{P}}(x_{k-2},x_1) - \bar{\mathrm{P}}(x_{k-2},x_1)^{l+2}\Big) \\ &= \frac{4\pi^2}{(l+1)(l+2)}\big(-\pi^{l+1} I(k-2,1) + I(k-2,l+2)\big).\end{aligned}$$

For the second equality, we used $\int_{x_1} \bar{\mathrm{P}}(x_1,x_2) = 0$ to show that the term multiplied by

$\frac{\pi^{l+1}}{l+2}$ vanishes. It follows that

$$\begin{aligned} I(k,1) &= \frac{(2\pi)^{k-2}}{(k-1)!} I(2,k-1) - \sum_{l=2,4,\dots,k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l,1) \\ &= -\frac{k(2\pi^2)^k}{(k+1)!} - \sum_{l=2,4,\dots,k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l,1) \qquad \text{for all } k = 2,\, 4,\, \dots \end{aligned}$$

This is a recursive equation of the form $a_k = c_k + \sum_{l=1}^{k-1} d_{k-l} a_l$. Its solution is $a_k = \sum_{i=1}^{k} c_i D_{k-i}$ with $D_0 \coloneqq 1$ and $D_i = \sum d_{i_1} \cdots d_{i_r}$, where we sum over all $r = 1, \dots, i$ and $i_1, \dots, i_r \in \mathbb{N}$ such that $i_1 + \cdots + i_r = i$. Therefore, we get

$$I(k,1) = -(2\pi^2)^k \sum_{i=2,4,\dots,k} \frac{i}{(i+1)!} \sum_{\substack{i_1+\cdots+i_r=k-i \\ i_1,\dots,i_r\in 2\mathbb{N}, r\in\mathbb{N}}} (-1)^r \frac{1}{(i_1+1)!\cdots(i_r+1)!}.$$

The result has to be multiplied by $(-1)^k(2\pi)^{-2k}$ in order to get $I(k)$. □

Lemma 5.2.6 (Independence of labeling)**.** *The summand* $(-1)^{\sigma_L} I(\sigma_L)$ *in the definition of* $\mathfrak{n}_{20}(\mathrm{sv}^{s_1} \otimes \mathrm{sv}^{s_2})$ *for* $\mathbb{S}^1$ *is independent of the choice of* $\Gamma \in O(s_1, s_2)$ *and its labeling* L *which is compatible and admissible with respect to the input.*

Proof. Pick $\Gamma \in O(s_1, s_2)$ and its admissible labeling L. Let L' be an other admissible labeling of Γ. We distinguish the following situations:

- Suppose that L and L' differ by a permutation μ in L_3^b. A similar argument as in the proof of Lemma 4.3.7 shows that $(-1)^{\sigma_{L'}} = (-1)^\mu (-1)^{\sigma_L}$ and $I(\sigma'_L) = (-1)^\mu I(\sigma_L)$, where the sign in the integral comes from the permutation of Vol's, which have form-degree 1. Hence $(-1)^{\sigma_{L'}} I(\sigma_{L'}) = (-1)^{\sigma_L} I(\sigma_L)$.

- Suppose that the boundaries are permuted, i.e., that L and L' differ in L_1^b. Notice that $s_1 = s_2$ because otherwise one of L or L' would not be admissible. The sign from changing L_1^b cancels as in the previous case.

- Suppose that L and L' differ in L_2. It was explained in the proof of Lemma 4.3.7 that a single change of L_2 induces the sign $(-1)^{n-1} = 1$ in $(-1)^{\sigma_L} I(\sigma_L)$.

- A cyclic permutation in L_3^v induces a sign neither in $(-1)^{\sigma_L}$ nor in $I(\sigma_L)$.

- A permutation μ in L_1^v induces $(-1)^\mu$ in $(-1)^{\sigma_L}$ and a change in $I(\sigma_L)$, which can be realized by taking the pullback along $\mu : (x_1, \dots, x_k) \mapsto (x_{\mu_1}, \dots, x_{\mu_k})$. However, the sign of the Jacobian is $(-1)^\mu$, which cancels the sign from $(-1)^{\sigma_L}$.

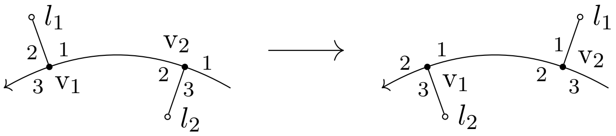


Figure 5.6.: Swapping adjacent legs.

Next, we prove the independence of $\Gamma \in O(s_1, s_2)$. Let L be an admissible and compatible labeling of Γ. Pick two adjacent internal vertices with external legs pointing to different regions, i.e., one to the interior of the circle and the other to the exterior. Suppose that the vertices are labeled by v_1 and v_2 and the legs by l_1 and l_2, respectively. Let $\Gamma' \in O(s_1, s_2)$ be the graph with the two legs turned inside out (see Figure 5.6). We can construct an admissible and compatible labeling L' of Γ' by making the following changes to L: The new leg at v_1 will be labeled by l_2 and the new leg at v_2 by l_1. The cyclic orderings at v_1 and v_2, respectively, have to be modified by a transposition in order to get compatibility with the new ribbon structure. All other labelings can be copied from L. In total, we get

$$(-1)^{\sigma_L - \sigma_{L'}} = -1.$$

This sign is compensated by swapping the one-forms in $I(\sigma_L)$:

$$\mathrm{Vol}(x_{\mathrm{v}_1}) \dots \mathrm{Vol}(x_{\mathrm{v}_2}) \longleftrightarrow \mathrm{Vol}(x_{\mathrm{v}_2}) \dots \mathrm{Vol}(x_{\mathrm{v}_1}).$$

The independence of $\Gamma \in O(s_1, s_2)$ follows from the fact that we can span the entire $O(s_1, s_2)$ by repeating the swap-of-legs operation. □

Lemma 5.2.7 (Sign). *We have*

$$\varepsilon(s_1, s_2) = (-1)^{s_1+1}.$$

Proof. By Lemma 5.2.6, in order to compute $(-1)^{\sigma_L} I(\sigma_L)$, we can pick $\Gamma^* \in O(s_1, s_2)$ and its admissible and compatible labeling L^* from Figure 5.7. We abbreviate $\sigma_0 = \sigma_{L^*}$. The corresponding integral (4.21) reads

$$I(\sigma_0) = \frac{1}{V^k} \int_{x_1, \dots, x_k} G(x_{k-1}, x_k) \cdots G(x_1, x_2) G(x_k, x_1) \mathrm{Vol}(x_{s_1}) \cdots \mathrm{Vol}(x_1)$$
$$\mathrm{Vol}(x_{s_1+1}) \cdots \mathrm{Vol}(x_k).$$

It differs from $I(k)$ from (5.12) in the order of P's and Vol's. A reordering produces the

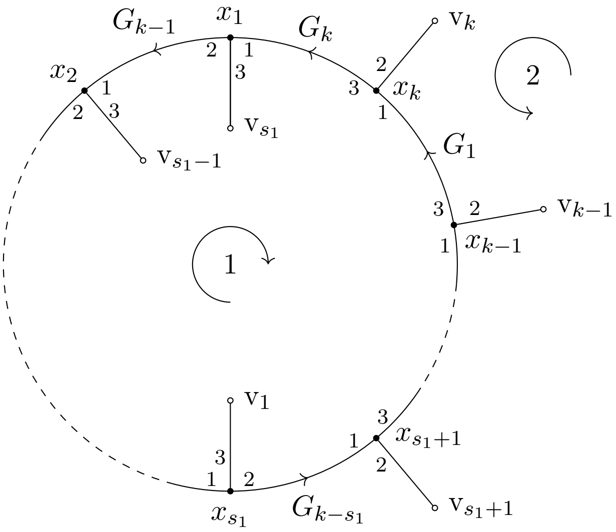


Figure 5.7.: The graph Γ^* with the labeling L^*. It can be checked that L_1 and L_2 are compatible.

sign

$$(-1)^{\frac{1}{2}s_1(s_1-1)}.$$

We will compute $(-1)^{\sigma_0}$ by ordering half-edges from the edge order back to the vertex order while looking at Figure 5.7. The steps are as follows:

- Transpose half-edges at internal vertices so that the first half-edge goes inside the vertex and the third outside with respect to the counterclockwise orientation. This gives $(-1)^a$.
- Permute external legs so that v_i is at x_i for all $i = 1, \ldots, k$. This gives
$$(-1)^{\frac{1}{2}s_1(s_1-1)}.$$
- Permute internal edges so that P_i starts at the third half-edge of x_i and ends at the first half-edge of x_{i+1} for all $i = 1, \ldots, k-1$. This does not produce any sign as swapping of two P's requires two transpositions.
- At this point, we have the permutation
$$\begin{pmatrix} 1 & 2 & \ldots & 2(e-1)-1 & 2(e-1) & 2e-1 & 2e & 2e+1 & \ldots & 3k \\ 3 & 4 & \ldots & 3k-3 & 3k-2 & 3k & 1 & 2 & \ldots & 3k-1 \end{pmatrix}.$$

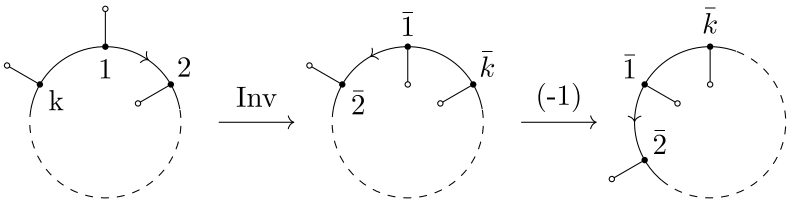


Figure 5.8.: The mirror isomorphism $M : 1 \dots k \mapsto \bar{k} \dots \bar{1}$ is a composition of the inversion and the counterclockwise rotation by one place.

We interpret the last line as $\mathrm{P}_1 \dots \mathrm{P}_k \mathrm{v}_1 \dots \mathrm{v}_k$ and permute it to the sequence $\mathrm{v}_1 \mathrm{P}_1 \mathrm{v}_2 \mathrm{P}_2 \dots \mathrm{v}_k \mathrm{P}_k$, which does not produce any sign. We end up with

$$\sigma_0' = \begin{pmatrix} 1 & 2 & 3 & \dots & 3k-1 & 3k \\ 2 & 3 & 4 & \dots & 3k & 1 \end{pmatrix}.$$

It is now easy to see that

$$(-1)^{\sigma_0'} = (-1)^{3k-1}.$$

In total, we get

$$(-1)^{\sigma_0} = (-1)^{s_1 + \frac{1}{2}s_1(s_1-1)+k+1}.$$

As for the other signs in Definition 4.3.6, we have $s(k,l) = k + \frac{1}{2}k(k-1)$ and $P(\mathrm{v}^k) = \frac{1}{2}k(k-1)$. There is no sign from $\mathrm{s}^2 \mathrm{v}^{s_1} \otimes \mathrm{v}^{s_2} = \mathrm{s}\mathrm{v}^{s_1} \otimes \mathrm{s}\mathrm{v}^{s_2}$ since $|\mathrm{s}| = -2$. Multiplying everything together, we get $\varepsilon(s_1, s_2)$. □

Lemma 5.2.8 (Combinatorial coefficient)**.** *It holds*

$$C(s_1, s_2) = \frac{1}{2} a k! \binom{k-1}{s_1}.$$

Proof. Every isomorphism of ribbon graphs Γ and Γ' from $O(s_1, s_2)$ is a composition of the clockwise rotation (r) for $r \in \mathbb{Z}$ and the mirror operation M defined as follows: If $1, \dots, k$ label internal vertices in the clockwise direction starting from the north-pole, then the result of M is $\bar{k}, \dots, \bar{1}$, where $\bar{\imath}$ means that the external leg is reversed (see Figure 5.8). These operations satisfy

$$(r+k) = (r),\ (r)(-r) = \mathbb{1},\ M^2 = \mathbb{1},\ (r)M = M(-r),$$

and hence generate a group G which is isomorphic to the dihedral group $\mathbb{Z}_k \rtimes \mathbb{Z}_2$. The orbit space $O(s_1, s_2)/\mathrm{G}$ is in $1:1$ correspondence with isomorphism classes of admissible

O_k-graphs and $\mathrm{Aut}(\Gamma)$ is in $1:1$ correspondence with $\mathrm{Stab}(\Gamma)$. From the orbit-stabilizer formula, we get

$$\begin{aligned}\sum_{\substack{[\Gamma]\text{ admiss.}\\ O_k-\text{graph}}}\frac{1}{|\operatorname{Aut}(\Gamma)|} &= \sum_{[\Gamma]\in O(s_1,s_2)/\mathrm{G}}\frac{1}{|\mathrm{Stab}(\Gamma)|}\\ &= \sum_{\Gamma\in O(s_1,s_2)}\frac{1}{|\mathrm{Orb}(\Gamma)||\mathrm{Stab}(\Gamma)|}\\ &= \frac{|O(s_1,s_2)|}{|\mathrm{G}|}\\ &= \frac{1}{2k}\binom{k}{s_1}\times\begin{cases}1 & \text{for } s_1=s_2,\\ 2 & \text{for } s_1\neq s_2.\end{cases}\end{aligned}$$

The two cases are compensated in the sum over labelings: For $s_1 = s_2$, both labelings L_1^b are admissible, and hence we get the factor 2.

Next, we multiply by $k!s_1(k-s_1)$, which is the number of L_1^v and L_3^b. There is also the factor $\frac{1}{l!}=\frac{1}{2}$. Multiplying everything together, we get $C(s_1,s_2)$. □

Before we summarize the results of our computations (see Proposition 5.2.10 below), we show directly that $\mathfrak{n}$ is a Maurer-Cartan element; i.e., in this special case, we do not need general results from [CVon] at all.

Lemma 5.2.9 (Maurer-Cartan equation for $\mathbb{S}^n$)**.** *For $n \geq 1$, consider $\mathbb{S}^n$ with the Hodge Propagator from* (5.2)*. The collection $(\mathfrak{n}_{lg})$ satisfies the Maurer-Cartan equation* (3.20) *for* $\mathrm{dIBL}(C(\mathcal{H}(\mathbb{S}^n)))$.

Proof. We will show that for every $l \geq 1$, $g \geq 0$ all summands in the relation corresponding to (l,g) vanish. The summands for $(l,g)=(1,0)$ are $\mathfrak{q}_{110}(\mathfrak{n}_{10})$ and $\frac{1}{2}\mathfrak{q}_{210}(\mathfrak{n}_{10},\mathfrak{n}_{10})$, and the summand for $(l,g)=(2,0)$ is $\mathfrak{q}_{120}(\mathfrak{n}_{10})$. The first term vanishes trivially as $\mathfrak{q}_{110}=0$, while the other two terms vanish by [CFL15, Proposition 12.5] because $\mathfrak{n}_{10}=\mathfrak{m}_{10}$ is the canonical Maurer-Cartan element. For $(l,g)\neq(1,0)$, we have the following four situations:

$\mathfrak{q}_{210}\circ_2\mathfrak{n}_{lg}$, $l \geq 2$*:* Let $\Psi=\Psi_1\cdots\Psi_l\in\mathrm{E}_l C$ be a summand of $\mathfrak{n}_{lg}$. From Proposition 5.2.2 it follows that the summands can be chosen such that $\Psi_1,\ldots,\Psi_l\in\mathrm{B}^*_{\mathrm{cyc,red}}\mathcal{H}(\mathbb{S}^n)[3-n]$, i.e., such that Ψ_i evaluates to 0 whenever $\mathbb{1}$ is a part of its argument. From

Definition 3.2.2, we compute

$$\mathfrak{q}_{210} \circ_2 (\Psi_1 \cdots \Psi_l) = \sum_{\sigma \in \mathbb{S}_{2,l-2}} \varepsilon(\sigma, \Psi)\mathfrak{q}_{210}(\Psi_{\sigma_1^{-1}} \cdot \Psi_{\sigma_2^{-1}}) \cdot \Psi_{\sigma_3^{-1}} \cdots \Psi_{\sigma_l^{-l}}.$$

We clearly have $\mathfrak{q}_{210}(\Psi_{\sigma_1^{-1}} \cdot \Psi_{\sigma_2^{-1}}) = 0$ because $\mathfrak{q}_{210}$ feeds $\mathrm{\imath}$ into one of its inputs. It follows that $\mathfrak{q}_{210} \circ_2 \mathfrak{n}_{lg} = 0$.

$\mathfrak{q}_{210} \circ_{1,1} (\mathfrak{n}_{l_1 g_1} \odot \mathfrak{n}_{l_2 g_2})$, $(l_i, g_i) \neq (1,0)$: A similar argument as above.

$\mathfrak{q}_{120} \circ_1 \mathfrak{n}_{lg}$, $(l, g) \neq (1,0)$: A similar argument as above using that $\mathfrak{q}_{120}$ also feeds $\mathrm{\imath}$ into its input.

$\mathfrak{q}_{210} \circ_{1,1} (\mathfrak{n}_{10} \odot \mathfrak{n}_{lg})$, $(l, g) \neq (1,0)$: As in the case of $\mathfrak{q}_{210} \circ_2 \mathfrak{n}_{lg}$, suppose that $\Psi_1, \dots, \Psi_l \in \mathrm{B}^*_{\mathrm{cyc,red}}\mathcal{H}(\mathbb{S}^n)[3-n]$. Recall that we write $\Omega_i = \mathrm{s}\omega_i \in \mathrm{B}^{\mathrm{cyc}}\mathcal{H}(\mathbb{S}^n)[3-n]$ and $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_l$. From Definition 3.2.2, we compute

$$\begin{aligned}
&[\mathfrak{q}_{210} \circ_{1,1} (\mathfrak{n}_{10} \odot \Psi)](\Omega_1 \otimes \cdots \otimes \Omega_l) \\
&\quad = \Big[\sum_{i=1}^{l} (-1)^{|\Psi_i|(|\Psi_1| + \cdots + |\Psi_{i-1}|)} \mathfrak{q}_{210}(\mathfrak{n}_{10}\Psi_i)\Psi_1 \cdots \widehat{\Psi_i} \cdots \Psi_l\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\Omega_1 \otimes \cdots \otimes \Omega_l) \\
&\quad = \sum_{\substack{\mu \in \mathbb{S}_l \\ i = 1, \dots, l}} \frac{1}{l!} (-1)^{|\Psi_i|(|\Psi_1| + \cdots + |\Psi_{i-1}|)} \varepsilon(\mu, \Omega)(\mathfrak{q}_{210}(\mathfrak{n}_{10} \cdot \Psi_i) \otimes \Psi_1 \otimes \cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad \widehat{\Psi_i} \cdots \otimes \Psi_l)(\Omega_{\mu_1^{-1}} \otimes \cdots \otimes \Omega_{\mu_l^{-1}}).
\end{aligned}$$

For every $i = 1, \dots, l$, we have

$$\begin{aligned}
\mathfrak{q}_{210}(\mathfrak{n}_{10} \cdot \Psi_i)(\Omega) &= \mathfrak{q}_{210}(\mathfrak{n}_{10} \otimes \Psi_i)(\Omega) \\
&= -\sum \varepsilon(\omega \mapsto \omega^1\omega^2)[(-1)^{(n-1)|\omega^1|}\mathfrak{n}_{10}(\mathrm{s}\mathrm{\imath}\omega^1)\Psi_i(\mathrm{s}\mathrm{v}\omega^2) \\
&\qquad\qquad + (-1)^{|\omega^1|}\mathfrak{n}_{10}(\mathrm{s}\mathrm{v}\omega^1)\Psi_i(\mathrm{s}\mathrm{\imath}\omega^2)] \\
&= -\sum \varepsilon(\omega \mapsto \omega^1\omega^2)(-1)^{(n-1)|\omega^1|}\mathfrak{n}_{10}(\mathrm{s}\mathrm{\imath}\omega^1)\Psi_i(\mathrm{s}\mathrm{v}\omega^2).
\end{aligned}$$

This can be non-zero only if $\omega = \mathrm{\imath}\mathrm{v}^{s-1}$ for some $s \geq 2$ (up to a cyclic permutation). For this input, we get

$$\begin{aligned}
&\mathfrak{q}_{210}(\mathfrak{n}_{10} \cdot \Psi_i)(\mathrm{s}\mathrm{\imath}\mathrm{v}^{s-1}) \\
&\quad = -[\varepsilon(\mathrm{\imath}\mathrm{v}^{s-1} \mapsto \mathrm{\imath}\mathrm{v}^{s-1})\mathfrak{n}_{10}(\mathrm{s}\mathrm{\imath}\mathrm{\imath}\mathrm{v})\Psi_i(\mathrm{s}\mathrm{v}^{s-1}) \\
&\qquad + \varepsilon(\mathrm{\imath}\mathrm{v}^{s-1} \mapsto \mathrm{v}\mathrm{\imath}\mathrm{v}^{s-2})\mathfrak{n}_{10}(\mathrm{s}\mathrm{\imath}\mathrm{v}\mathrm{\imath})\Psi_i(\mathrm{s}\mathrm{v}^{s-1})]
\end{aligned}$$

$$= (-1)^{n-3}[1 + (-1)^{ns+s-1}]\Psi_i(\mathrm{sv}^{s-1}).$$

The prefactor in brackets is 0 for n odd or s even, whereas $\mathrm{v}^{s-1} = 0$ for n even and s odd. Therefore, we have $\mathfrak{q}_{210} \circ_{1,1} (\mathfrak{n}_{10} \odot \mathfrak{n}_{lg}) = 0$. □

Proposition 5.2.10 (Chern-Simons Maurer-Cartan element for $\mathbb{S}^n$)**.** *For $n \geq 1$, consider the round sphere $\mathbb{S}^n$ with the Hodge Propagator* (5.2)*. The Chern-Simons Maurer-Cartan element $\mathfrak{n}$ is a strictly reduced Maurer-Cartan element for* $\mathrm{dIBL}(C(\mathcal{H}(\mathbb{S}^n)))$ *which satisfies*

$$\mathfrak{n}_{10} = \mathfrak{m}_{10} \quad \textit{for all } n \in \mathbb{N}$$

plus the following properties depending on the dimension:

($n = 1$): It holds $\mathfrak{n}_{lg} = 0$ for all $l \geq 1$, $g \geq 0$ such that $(l, g) \neq (1, 0)$, $(2, 0)$; the only non-trivial relation for $\mathfrak{n}_{20}$ is

$$\mathfrak{n}_{20}(\mathrm{sv}^{s_1} \otimes \mathrm{sv}^{s_2}) = (-1)^{s_1+1}\frac{1}{2}s_1 k!\binom{k-1}{s_1}I(k), \tag{5.14}$$

where s_1, $s_2 \geq 1$ are such that $k = s_1 + s_2$ is even, and $I(k)$ is given by (5.13)*.*

($n = 2$): It holds $\mathfrak{n}_{l0} = 0$ for all $l \geq 2$. We also have $\mathfrak{n}_{11} = 0$.

($n \geq 3$): It holds $\mathfrak{n}_{lg} = 0$ for all $l \geq 1$, $g \geq 0$ such that $(l, g) \neq (1, 0)$.

Notice that $\mathfrak{n}_{20} \notin \mathrm{E}_2 C(\mathcal{H}(\mathbb{S}^1))$, i.e. $\mathfrak{n}_{20}$ is a “long cochain” because it is non-zero in infinitely many weights.

5.3. Twisted IBL-infinity-structure for spheres

Let e_0, e_1 be the basis of $\mathcal{H}(\mathbb{S}^n)[1]$ defined by

$$e_0 \coloneqq \imath \coloneqq \theta 1, \quad e_1 \coloneqq \mathrm{v} \coloneqq \frac{1}{V}\theta \mathrm{Vol}.$$

The degrees satisfy

$$|\imath| = -1, \quad |\mathrm{v}| = n - 1.$$

The matrix of the pairing Π with respect to the basis e_0, e_1 reads

$$\Pi = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

The dual basis e^0, e^1 to e_0, e_1 with respect to Π is thus

$$e^0 = \mathrm{v}, \quad e^1 = (-1)^n \mathbb{1}.$$

It follows that the matrix (T^{ij}) from (3.34) satisfies

$$(T^{ij}) = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We clearly have

$$\hat{\mathrm{B}}^*_{\mathrm{cyc,red}}\mathcal{H}(\mathbb{S}^1) = \Bigl\{ \sum_{k=1}^{\infty} c_k \mathrm{v}^{k*} \Bigm| c_k \in \mathbb{R} \Bigr\},$$

where v^{k*} is the dual to the cyclic word $\mathrm{v}^k = \mathrm{v}\dots\mathrm{v}$ of length k. Observe that the cyclic symmetry gives

$$\mathrm{v}^i = (-1)^{(n-1)(i-1)}\mathrm{v}^i \quad \text{for all } i \geq 1.$$

Therefore, $\mathrm{v}^{i*} = 0$ holds if both n and i are even.

For $n \geq 2$, the vector space $\mathcal{H}(\mathbb{S}^n)$ is connected and simply-connected, and Proposition 3.3.15 implies that there are no long reduced cyclic cochains (i.e., we have only finite sums of v^{k*}'s).

The product $\mu_2 : \mathcal{H}[1]^{\otimes 2} \to \mathcal{H}[1]$ from (4.2) has the following matrix with respect to the basis $\mathbb{1}$, v:

$$\mu_2 = \begin{pmatrix} \mathbb{1} & \mathrm{v} \\ (-1)^n \mathrm{v} & 0 \end{pmatrix}.$$

Because $\mu_2(\mathrm{v},\mathrm{v}) = 0$, we get

$$\mathbb{H}^{\mathfrak{m}}(C_{\mathrm{red}}(\mathcal{H}(\mathbb{S}^n)))[1] = \begin{cases} \langle \mathrm{s}\mathrm{v}^{i*} \mid i \geq 1 \rangle & \text{for } n \geq 3 \text{ odd}, \\ \langle \mathrm{s}\mathrm{v}^{2i-1*} \mid i \geq 1 \rangle & \text{for } n \text{ even}, \\ \bigl\{ \mathrm{s} \sum_{k=1}^{\infty} c_k \mathrm{v}^{k*} \mid c_k \in \mathbb{R} \bigr\} & \text{for } n = 1. \end{cases}$$

Because we are in the strictly unital and strictly augmented case, we obtain

$$\mathbb{H}^{\mathfrak{m}}(C)[1] = \begin{cases} \langle \mathrm{s}\mathrm{v}^{i*}, \mathrm{s}\mathbb{1}^{2j-1*} \mid i,j \geq 1 \rangle & \text{for } n \geq 3 \text{ odd}, \\ \langle \mathrm{s}\mathrm{v}^{2i-1*}, \mathrm{s}\mathbb{1}^{2j-1*} \mid i,j \geq 1 \rangle & \text{for } n \text{ even}, \\ \bigl\langle \mathrm{s} \sum_{k=1}^{\infty} c_k \mathrm{v}^{k*}, \mathrm{s}\mathbb{1}^{2j-1*} \mid c_k \in \mathbb{R}, j \geq 1 \bigr\rangle & \text{for } n = 1. \end{cases} \tag{5.15}$$

The canonical IBL-operations can be written as

$$\begin{aligned}
\mathfrak{q}_{210}(\mathrm{s}^2\psi_1 \otimes \psi_2)(\mathrm{s}\omega) = &-\sum \varepsilon(\omega \mapsto \omega^1\omega^2)[(-1)^{(n-1)|\omega^1|}\psi_1(e_0\omega^1) \\
&\qquad \psi_2(e_1\omega^2) + (-1)^{|\omega_1|}\psi_1(e_1\omega^1)\psi_2(e_0\omega^2)], \\
\mathfrak{q}_{120}(\mathrm{s}\psi)(\mathrm{s}^2\omega_1 \otimes \omega_2) = &-\frac{1}{2}\sum \varepsilon(\omega_1 \mapsto \omega_1^1)\varepsilon(\omega_2 \mapsto \omega_2^1)[(-1)^{(n-1)|\omega_1^1|} \\
&\qquad \psi(e_0\omega_1^1 e_1\omega_2^1) + (-1)^{|\omega_1^1|}\psi(e_1\omega_1^1 e_0\omega_2^1)]
\end{aligned}$$

for all ψ, ψ_1, $\psi_2 \in \hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathcal{H}$ and generating words ω, ω_1, $\omega_2 \in \mathrm{B}^{\mathrm{cyc}}\mathcal{H}$. For all k, k_1, $k_2 \geq 1$, we have

$$\mathfrak{q}_{210}((\mathrm{s}\mathrm{v}^{k_1 *}) \cdot (\mathrm{s}\mathrm{v}^{k_2 *})) = 0 \quad \text{and} \quad \mathfrak{q}_{120}(\mathrm{s}\mathrm{v}^{k*}) = 0$$

because both $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ feed $\mathrm{1}$ into their inputs. For the *canonically twisted reduced* IBL-*algebra*, this implies the following:

$$\mathrm{IBL}\big(\mathbb{H}^{\mathfrak{m}}(C_{\mathrm{red}})\big) = \big(\mathbb{H}^{\mathfrak{m}}(C_{\mathrm{red}}), \mathfrak{q}_{210} \equiv 0, \mathfrak{q}_{120} \equiv 0\big) \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 3.4.10, the only possibly non-zero relation of $\mathrm{IBL}(\mathbb{H}^{\mathfrak{m}}(C))$ is

$$\begin{aligned}
&\mathfrak{q}_{210}(\mathrm{s}\mathrm{1}^* \otimes \mathrm{s}\mathrm{v}^{k*}) \\
&\quad = (-1)^{n-2}\mathrm{s}(\mathrm{v}^{k*} \circ \iota_{\mathrm{v}}) \\
&\quad = (-1)^{n-2}\Big(\sum_{i=1}^{k-1}(-1)^{i|\mathrm{v}|}\Big)\mathrm{s}\mathrm{v}^{k-1*} = \begin{cases} -(k-1)\mathrm{s}\mathrm{v}^{k-1*} & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}. \end{cases}
\end{aligned}$$

The reason for 0 for even n is that either k is odd, in which case $\sum_{i=1}^{k-1}(-1)^i = 0$, or k is even, in which case $\mathrm{v}^{k*} = 0$. Therefore, for the *canonically twisted* IBL-*algebra*, we have

$$\mathrm{IBL}\big(\mathbb{H}^{\mathfrak{m}}(C)\big) = \big(\mathbb{H}^{\mathfrak{m}}(C), \mathfrak{q}_{210}, \mathfrak{q}_{120} \equiv 0\big) \quad \text{for all } n \in \mathbb{N},$$

where $\mathbb{H}^{\mathfrak{m}}(C)$ is given by (5.15) and $\mathfrak{q}_{210}$ satisfies the following:

(n even): $\mathfrak{q}_{210} \equiv 0$.

($n \geq 3$ odd): The non-trivial relations are

$$\mathfrak{q}_{210}(\mathrm{s}\mathrm{1}^* \otimes \mathrm{s}\mathrm{v}^{k*}) = \mathfrak{q}_{210}(\mathrm{s}\mathrm{v}^{k*} \otimes \mathrm{s}\mathrm{1}^*) = -(k-1)\mathrm{v}^{k-1*} \quad \text{for } k \geq 2.$$

($n=1$): The non-trivial relations are

$$\mathfrak{q}_{210}\Big(\mathrm{s}1^* \otimes \mathrm{s}\sum_{k=1}^{\infty} c_k \mathrm{v}^{k*}\Big) = -\mathrm{s}\sum_{k=1}^{\infty} k c_{k+1}\mathrm{v}^{k*} \quad \text{for } c_k \in \mathbb{R}.$$

Recall that the twist by $\mathfrak{m}$ does not produce any higher operation $\mathfrak{q}^{\mathfrak{m}}_{1lg}$.

We will now consider $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(\mathbb{S}^n)))$. Recall that $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$, $\mathfrak{q}^{\mathfrak{n}}_{210} = \mathfrak{q}_{210}$ and $\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}_{120} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$. By Proposition 5.2.10, we have $\mathfrak{n}_{10} = \mathfrak{m}_{10}$ for all $n \in \mathbb{N}$ and $\mathfrak{n}_{20} = 0$ for all $n \geq 2$. It follows that $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathfrak{q}^{\mathfrak{m}}_{110}$ for all $n \in \mathbb{N}$ and that the only non-trivial twist may occur in $\mathfrak{q}^{\mathfrak{n}}_{120}$ for $\mathbb{S}^1$. Using (3.38), we get for all $\psi \in \hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathcal{H}(\mathbb{S}^n)$ and generating words ω_1, $\omega_2 \in \mathrm{B}^{\mathrm{cyc}}\mathcal{H}(\mathbb{S}^n)$ the following:

$$\begin{aligned}
&(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(\mathrm{s}\psi)(\mathrm{s}\omega_1 \otimes \mathrm{s}\omega_2) \\
&\quad = (-1)^{n-2}\Big[\sum \varepsilon(\omega_1 \mapsto \omega_1^1\omega_1^2)\psi(1\omega_1^1)\mathfrak{n}_{20}(\mathrm{s}\mathrm{v}\omega_1^2 \otimes \mathrm{s}\omega_2) \\
&\qquad + (-1)^{(n-3+|\omega_1|)(n-3+|\omega_2|)}\sum \varepsilon(\omega_2 \mapsto \omega_2^1\omega_2^2)\psi(1\omega_2^1) \\
&\qquad\qquad \mathfrak{n}_{20}(\mathrm{s}\mathrm{v}\omega_2^2 \otimes \mathrm{s}\omega_1)\Big].
\end{aligned} \tag{5.16}$$

In this paragraph, we suppose that $n=1$ and compute $\mathfrak{q}^{\mathfrak{n}}_{120}$. Clearly, $(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(\mathrm{s}\mathrm{v}^{k*}) = 0$ for all $k \geq 1$ since 1 is fed into v^{k*}. A non-zero evaluation of $(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(\mathrm{s}1^{k*})$ for some $k \geq 1$ odd is possible only on $\mathrm{s}1^{k-1}\mathrm{v}^{k_1} \otimes \mathrm{s}\mathrm{v}^{k_2}$ for k_1, $k_2 \geq 0$ (up to a transposition of arguments and their cyclic permutation). If $k > 1$, only the first summand of (5.16) contributes, and we get

$$\begin{aligned}
&(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(\mathrm{s}1^{k*})(\mathrm{s}1^{k-1}\mathrm{v}^{k_1} \otimes \mathrm{s}\mathrm{v}^{k_2}) \\
&\quad = (-1)^{n-2}\sum \varepsilon(1^{k-1}\mathrm{v}^{k_1} \mapsto \omega_1\omega_2)1^{k*}(1\omega_1)\mathfrak{n}_{20}(\mathrm{s}\mathrm{v}\omega_2 \otimes \mathrm{s}\mathrm{v}^{k_2}) \\
&\quad = (-1)^{n-2}1^{k*}(11^{k-1})\mathfrak{n}_{20}(\mathrm{s}\mathrm{v}\mathrm{v}^{k_1} \otimes \mathrm{s}\mathrm{v}^{k_2}) \\
&\quad = -\mathfrak{n}_{20}(\mathrm{s}\mathrm{v}^{k_1+1} \otimes \mathrm{s}\mathrm{v}^{k_2}).
\end{aligned}$$

According to Proposition 5.2.10, this is non-zero if and only if $k_1 + k_2$ is odd. It follows that

$$\mathfrak{q}^{\mathfrak{n}}_{120} \neq \mathfrak{q}^{\mathfrak{m}}_{120} = \mathfrak{q}_{120} \quad \text{on the chain level for } \mathbb{S}^1.$$

However, the chains $\mathrm{s}1^{k-1}\mathrm{v}^{k_1} \otimes \mathrm{s}\mathrm{v}^{k_2}$ for $k > 1$ do not survive to the homology (c.f., (5.15)). The only possibility is thus $k = 1$. In this case, both summands of (5.16) contribute, and using (5.14), we get for all k_1, $k_2 \geq 1$ the following:

$$(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(\mathrm{s}1^*)(\mathrm{s}\mathrm{v}^{k_1} \otimes \mathrm{s}\mathrm{v}^{k_2})$$

$$
\begin{aligned}
&= (-1)^{n-2}\Big[\sum \varepsilon(\mathrm{v}^{k_1} \mapsto \mathrm{v}^0\mathrm{v}^{k_1})\mathfrak{1}^*(\mathfrak{1})\mathfrak{n}_{20}(\mathrm{sv}^{k_1+1} \otimes \mathrm{sv}^{k_2}) \\
&\quad + (-1)^{(n-3+k_1(n-1))(n-3+k_2(n-1))} \sum \varepsilon(\mathrm{v}^{k_2} \mapsto \mathrm{v}^0\mathrm{v}^{k_2})\mathfrak{1}^*(\mathfrak{1}) \\
&\qquad\qquad \mathfrak{n}_{20}(\mathrm{sv}^{k_2+1} \otimes \mathrm{sv}^{k_1})\Big] \\
&= -k_1\mathfrak{n}_{20}(\mathrm{sv}^{k_1+1} \otimes \mathrm{sv}^{k_2}) - k_2\mathfrak{n}_{20}(\mathrm{sv}^{k_2+1} \otimes \mathrm{sv}^{k_1}) \\
&= -\frac{1}{2}(k_1+k_2+1)!I(k_1+k_2+1)\Big[(-1)^{k_1}k_1(k_1+1)\binom{k_1+k_2}{k_1+1} \\
&\qquad + (-1)^{k_2}k_2(k_2+1)\binom{k_1+k_2}{k_2+1}\Big] \\
&= -\frac{1}{2}(k_1+k_2+1)!k_1k_2\binom{k_1+k_2}{k_1}\underbrace{I(k_1+k_2+1)[(-1)^{k_1}+(-1)^{k_2}]}_{=:(*)}.
\end{aligned}
$$

Denoting $k \coloneqq k_1 + k_2 + 1$, we have that $(-1)^{k_1} + (-1)^{k_2} = 0$ for k even and $I(k) = 0$ for k odd. Therefore, $(*) = 0$ for any k_1, $k_2 \geq 1$. This implies that

$$\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}^{\mathfrak{m}}_{120} = \mathfrak{q}_{120} \quad \text{on the homology for } \mathbb{S}^1.$$

We conclude that the *twisted* IBL-*algebra* satisfies

$$\mathrm{IBL}(\mathbb{H}^{\mathfrak{n}}(C(\mathcal{H}(\mathbb{S}^n)))) = \mathrm{IBL}(\mathbb{H}^{\mathfrak{m}}(C(\mathcal{H}(\mathbb{S}^n)))) \quad \text{for all } n \in \mathbb{N}.$$

As for the *higher twisted operations*, combining Propositions 3.2.10 and 5.2.10, we see that for $\mathbb{S}^n$ with $n \in \mathbb{N}\backslash\{2\}$ all higher operations $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ vanish already on the chain level. For $n = 2$, we have that $\mathfrak{q}^{\mathfrak{n}}_{1l0} = 0$ for all $l \geq 3$ and $\mathfrak{q}^{\mathfrak{n}}_{111} = 0$ on the chain level. However, we did not prove that all higher operations vanish on the chain level. As for the operations induced on the homology, the graded vector space $\mathbb{H}^{\mathfrak{n}}(C(\mathcal{H}(\mathbb{S}^2)))$ is concentrated in even degrees and $\mathfrak{q}^{\mathfrak{n}}_{1lg}$ are odd (see Definition 3.2.4). Therefore, all higher operations vanish also on $\mathbb{H}^{\mathfrak{n}}(C(\mathcal{H}(\mathbb{S}^2)))$.

The string topology $\mathrm{H}^{\mathbb{S}^1}(\mathbb{S}^n)$ and the string operations $\mathbb{m}_2$ and $\mathbb{c}_2$ were computed in [Bas11] for all $n \in \mathbb{N}$. We review their results and basic ideas below:

We will consider *even spheres* first. The minimal model for the Borel construction $\mathrm{L}_{\mathbb{S}^1}\mathbb{S}^{2m}$ for $m \in \mathbb{N}$ is denoted by $\Lambda^{\mathbb{S}^1}(2,m)$ — it is the free graded commutative dga (=:cdga) over $\mathbb{R}$ generated by homogenous vectors x_1, y_1, x_2, y_2, u of degrees

$$|x_1| = 2m, \quad |y_1| = 2m-1, \quad |x_2| = 4m-1, \quad |y_2| = 2(2m-1), \quad |u| = 2,$$

whose differential d satisfies

$$\mathrm{d}y_1 = 0, \quad \mathrm{d}x_1 = y_1 u, \quad \mathrm{d}y_2 = -2x_1 y_1, \quad \mathrm{d}x_2 = x_1^2 + y_2 u.$$

The minimal model for the loop space $\mathrm{L}\mathbb{S}^{2m}$ is the dga $\Lambda(2,m)$ which is obtained from $\Lambda^{\mathbb{S}^1}(2,m)$ by setting $u = 0$. A computation (see [Bas11, Theorem 3.6]) gives the following for all $m \in \mathbb{N}$:

$$\begin{aligned} \mathrm{H}^*(\mathrm{L}\mathbb{S}^{2m};\mathbb{R}) &\simeq \mathrm{H}(\Lambda(2,m),\mathrm{d}) = \langle y_2^i x_1 - 2iy_1x_2y_2^{i-1}, y_1y_2^j, 1 \mid i,j \in \mathbb{N}_0\rangle, \\ \mathrm{H}^*_{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^{2m};\mathbb{R}) &\simeq \mathrm{H}(\Lambda^{\mathbb{S}^1}(2,m),\mathrm{d}) = \langle y_1y_2^i, u^j \mid i,j \in \mathbb{N}_0\rangle, \end{aligned} \tag{5.17}$$

where $y_2^0 \coloneqq u^0 \coloneqq 1$ is the unit in $\Lambda^{\mathbb{S}^1}(2,m)$ and $\langle\cdot\rangle$ denotes the linear span over $\mathbb{R}$. Clearly, the cohomology groups are degree-wise finite-dimensional, and hence, using the universal coefficient theorem, they are isomorphic to the corresponding homology groups. We can thus identify $\mathrm{H}(\mathrm{L}\mathbb{S}^{2m};\mathbb{R})$ and $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^{2m};\mathbb{R})$ with the vector spaces on the right hand side of (5.17). We have $\mathrm{H}^{\mathbb{S}^1}_{2k} = \langle u^k\rangle$ for all $k \in \mathbb{N}_0$, and hence the multiplication with u induces an isomorphism $\mathrm{H}^{\mathbb{S}^1}_{2k} \simeq \mathrm{H}^{\mathbb{S}^1}_{2k+2}$. This corresponds to the cap product with the Euler class in (4.28), and exactness of the sequence implies $\mathcal{M}(\mathrm{H}^{\mathbb{S}^1}_{2k}) = \mathcal{E}(\mathrm{H}^{\mathbb{S}^1}_{2k}) = 0$. Using this and degree considerations, we get $\mathbb{m}_2 = \mathbb{c}_2 = 0$.

We will now consider *odd spheres* with $n \geq 3$. The minimal model for $\mathrm{L}_{\mathbb{S}^1}\mathbb{S}^{2m+1}$ for $m \in \mathbb{N}$ is denoted simply by $\Lambda(x,y,u)$ — it is the free cdga on homogenous vectors x, y, u of degrees

$$|x| = 2m+1, \quad |y| = 2m, \quad |u| = 2,$$

such that

$$\mathrm{d}x = yu, \quad \mathrm{d}y = \mathrm{d}u = 0.$$

We get immediately

$$\begin{aligned} \mathrm{H}^*(\mathrm{L}\mathbb{S}^{2m+1};\mathbb{R}) &\simeq \langle x^i, y^j \mid i,j \in \mathbb{N}_0\rangle, \\ \mathrm{H}^*_{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^{2m+1};\mathbb{R}) &\simeq \langle y^i, u^j \mid i,j \in \mathbb{N}_0\rangle, \end{aligned}$$

and we can again identify H and $\mathrm{H}^{\mathbb{S}^1}$ with the vector spaces on the right hand side. Clearly, $\mathrm{H}^{\mathbb{S}^1}_{2k-1} = 0$ for all $k \in \mathbb{N}$, and hence $\mathbb{m}_2 = \mathbb{c}_2 = 0$ for degree reasons (the operations are odd).

We will now consider *the circle* $\mathbb{S}^1$. For every $i \in \mathbb{Z}$, let $\alpha_i : \mathbb{S}^1 \to \mathbb{S}^1$ and $\theta_i : \mathbb{S}^1 \to \mathrm{L}\mathbb{S}^1$ be the maps defined by

$$\alpha_i(z) \coloneqq z^i \quad \text{and} \quad \theta_i(w) \coloneqq w\alpha_i \quad \text{for all } w, z \in \mathbb{S}^1 \subset \mathbb{C}.$$

By examining the equivariant homology of connected components of $\mathrm{L}\mathbb{S}^1$ containing α_i separately as in [Bas11, Section 2.1.4], we get

$$\begin{aligned}\mathrm{H}(\mathrm{L}\mathbb{S}^1;\mathbb{R}) &= \langle \alpha_i, \theta_j \mid i, j \in \mathbb{Z}\rangle,\\ \mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^1;\mathbb{R}) &= \langle u^i, \theta_0 u^j, \alpha_k \mid i, j \in \mathbb{N}_0, k \in \mathbb{Z}\backslash\{0\}\rangle,\end{aligned}$$

where u corresponds to the Euler class and

$$|u| = 2, \quad |\theta_i| = 1, \quad |\alpha_i| = 0$$

are the degrees in the singular chain complex. On [Bas11, p. 21] they show that the string cobracket $\mathbb{c}_2$ is 0 and that all non-trivial relations for the string bracket $\mathbb{m}_2 : \mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^1)[2]^{\otimes 2} \to \mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^1)[2]$ are the following:

$$\mathbb{m}_2(\mathrm{s}\alpha_k, \mathrm{s}\alpha_{-k}) = k^2 \mathrm{s}\theta_0 \quad \forall k \in \mathbb{N}.$$

We will now compare the reduced IBL-structures motivated by Conjecture 4.5.1. The point-reduced versions $\mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}\mathbb{S}^n)$ for $n \geq 2$ are obtained from $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^n)$ by deleting u^i. We have the following isomorphisms of graded vector spaces:

$$\begin{aligned}\mathbb{H}^{\mathrm{n}}(C_{\mathrm{red}}(\mathcal{H}(\mathbb{S}^n)))[1] &\longrightarrow \mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}\mathbb{S}^n)[3-n] & \\ \mathrm{s}\mathrm{v}^i &\longmapsto \mathrm{s}y^i & \text{for } n > 1 \text{ odd},\\ \mathrm{s}\mathrm{v}^{2i+1} &\longmapsto \mathrm{s}y_1 y_2^i & \text{for } n \text{ even}.\end{aligned}$$

Because all operations are trivial, it induces the isomorphism

$$\mathrm{IBL}\big(\mathbb{H}^{\mathrm{n}}(C_{\mathrm{red}}(\mathcal{H}(\mathbb{S}^n)))\big) \simeq \mathrm{IBL}\big(\mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}\mathbb{S}^n)[2-n]\big) \quad \text{for } n \geq 2.$$

For $n = 1$, the reduced homology is seemingly different.

Remark 5.3.1 (Triviality for degree reasons). The graded vector spaces

$$\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^{2m-1})[3-n] \quad \text{and} \quad \mathbb{H}^{\mathrm{n}}(C(\mathcal{H}(\mathbb{S}^{2m})))[1]$$

are concentrated in even degrees, and so any IBL_∞-structure must be trivial for degree reasons. On the other hand, the graded vector spaces

$$\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{S}^{2m}) \quad \text{and} \quad \mathbb{H}^{\mathrm{n}}(C(\mathcal{H}(\mathbb{S}^{2m-1})))[1]$$

have both even and odd degrees, and hence an additional argument is needed to prove vanishing of the IBL-structure. This is not the case of the reduced homology, which is again concentrated in even degree. ◁

5.4. Homology of twisted IBL-infinity structure for complex projective space

Let $\mathrm{K} \in \Omega^2(\mathbb{C}\mathrm{P}^n)$ be the Fubini–Study Kähler form on $\mathbb{C}\mathrm{P}^n$ (see [Huy05, Examples 3.1.9]). The powers of K are harmonic,[3] and we get easily

$$\mathcal{H}(\mathbb{C}\mathrm{P}^n) = \langle 1, \mathrm{K}, \dots, \mathrm{K}^n \rangle.$$

We denote the Riemannian volume of $\mathbb{C}\mathrm{P}^n$ by

$$V := \int_{\mathbb{C}\mathrm{P}^n} \frac{1}{n!} \mathrm{K}^n.$$

Consider the basis $e_0, \dots, e_n$ of $\mathcal{H}(\mathbb{C}\mathrm{P}^n)[1]$ defined for all $i = 0, \dots, n$ by

$$e_i := \frac{\mathrm{k}^i}{(n!V)^{\frac{i}{n}}}, \quad \text{where} \quad \mathrm{k}^i := \theta \mathrm{K}^i.$$

The matrix of the pairing Π from (2.3) with respect to the basis $e_0, \dots, e_n$ reads:

$$(\Pi^{ij}) = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & ⋰ & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

The basis $e^0, \dots, e^n$ dual to $e_0, \dots, e_n$ with respect to Π thus satisfies

$$e^i = e_{n-i} \quad \text{for all } i = 0, \dots, n.$$

Therefore, the following holds for the matrix (T^{ij}) from (3.34):

$$(T^{ij}) = -(\Pi^{ij}).$$

---

[3] This follows by induction on the power of K using the fact that, on a general Kähler manifold M, the Lefschetz operator $\mathrm{L} : \Omega(M) \to \Omega(M)$ defined by $\mathrm{L}(\eta) := \eta \wedge \mathrm{K}$ for all $\eta \in \Omega(M)$ commutes with the Hodge–de Rham Laplacian Δ (see [Huy05, Chapter 3]).

For all $1 \leq i, j, k \leq n$, we have

$$\mu_2(e_i, e_j) = e_{i+j} \quad \text{and} \quad \mathfrak{m}_{10}(\mathrm{s}e_i e_j e_k) = \delta_{i+j+k,n}.$$

For ψ, ψ_1, $\psi_2 \in \hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathcal{H}$ and generating words ω, ω_1, $\omega_2 \in \mathrm{B}^{\mathrm{cyc}}\mathcal{H}$, we chave

$$\begin{aligned}\mathfrak{q}_{210}(\mathrm{s}^2\psi_1 \otimes \psi_2)(\mathrm{s}\omega) &= -\sum_{i=0}^{n} \sum \varepsilon(\omega \mapsto \omega^1\omega^2)(-1)^{|\omega^1|}\psi_1(e_i\omega^1)\psi_2(e_{n-i}\omega^2),\\ \mathfrak{q}_{120}(\mathrm{s}\psi)(\mathrm{s}^2\omega_1 \otimes \omega_2) &= -\sum_{i=0}^{n} \sum \varepsilon(\omega_1 \mapsto \omega_1^1)\varepsilon(\omega_2 \mapsto \omega_2^1)(-1)^{|\omega_1|}\psi(e_i\omega_1^1 e_{n-i}\omega_2^1).\end{aligned}$$

The cyclic homology of $\mathcal{H}(\mathbb{C}\mathrm{P}^n)$ is that of the truncated polynomial algebra

$$A \coloneqq \mathbb{R}[x]/(x^{n+1}) \quad \text{with } \deg(x) = 2.$$

The following lemma computes its cyclic homology.

Lemma 5.4.1 (Cyclic homology of truncated graded polynomial algebra)**.** *Consider $A \coloneqq \mathbb{R}[x]/(x^{n+1})$ with $\deg(x) = d \in \mathbb{Z}$. For all $i = 1, \ldots, n$ and $k \in \mathbb{N}_0$, there are cycles $\tilde{t}_{2k+1,i} \in \tilde{D}_q(A)$ of weights $w(\tilde{t}_{2k+1,i}) = 2k+1$ and degrees $|\tilde{t}_{2k+1,i}| = d(i + (n+1)k)$, where $q = w(\tilde{t}_{2k+1,i}) - |\tilde{t}_{2k+1,i}| - 1$, which form a basis of $\mathrm{H}^{\lambda,\mathrm{nds}}(A)$ (the non-degree shifted cyclic homology defined on page 56).*

Proof. A computation of the cyclic homology of A for $|x| = 0$ is the goal of [Lod92, Exercise 4.1.8.] or [Wei94, Exercise 9.1.1]. The hint is to compute the Hochschild homology $\mathrm{HH}_n(A) = \mathrm{Tor}_n^{A_e}(A, A)$, where A_e is the enveloping algebra of A, using a non-canonical (i.e., not the bar complex) projective resolution of the A_e-module A given by

$$\cdots \longrightarrow A_e \xrightarrow{\cdot v} A_e \xrightarrow{\cdot u} A_e \xrightarrow{\mu} A \longrightarrow 0, \tag{5.18}$$

where $u = x \otimes 1 - 1 \otimes x$ and $v = \sum_{i=0}^n x^i \otimes x^{n-i} \in A_e$. The resolution continues to the left with $\cdot u$ und $\cdot v$ periodically. Clearly, μ composed with $\cdot u$ vanishes and $u \cdot v = v \cdot u = x^{n+1} \otimes 1 - 1 \otimes x^{n+1} = 0$; one can check that (5.18) is indeed a resolution. If $\deg(x) = d$, then $\deg(u) = d$ and $\deg(v) = nd$.

We lift the resolution (5.18) to the graded category (i.e., we require that the maps are homogenous) by taking the degree shifts

$$\begin{tikzcd}
\cdots \arrow[r] & A_e[-(n+1)di] \arrow[r, "d_{2i}"] & A_e[-(n+1)di+nd] \arrow[r, "d_{2i-1}"] & A_e[-d(n+1)(i-1)] \arrow[dlll, rounded corners] \\
\cdots \arrow[r] & A_e[-(n+1)d] \arrow[r, "d_2"] & A_e[-d] \arrow[r, "d_1"] & A_e \arrow[dl, rounded corners, "\mu"] \\
 & & A \arrow[r] & 0,
\end{tikzcd}$$

where we denoted by d_i the maps from (5.18). Tensoring with A, we get the graded vector spaces $A \otimes_{A_e} A_e[\cdot] \simeq A[\cdot]$, and the maps d_j become multiplications with u, $v \in A_e$ in A as a right A_e-module. For all polynomials $p \in A$, we have

$$\begin{aligned}
d_{2i-1}(p) &= p \cdot (x \otimes 1 - 1 \otimes x) = px - (-1)^{|x||p|}xp = 0, \\
d_{2i}(p) &= p \cdot (\sum_{i=0}^{n} x^i \otimes x^{n-i}) = (-1)^{n|x||p|}(n+1)x^n p.
\end{aligned}$$

The homology of this chain complex consists of graded vector spaces $\mathrm{HH}_{(l)}$ for $l \geq 0$ which correspond to the homology of the bar complex graded by weights, i.e., $\mathrm{HH}_{(l)}$ would be represented by cycles in $A^{\otimes l+1}$ if the bar resolution was taken. We compute

$$\mathrm{HH}_{(l)} = \begin{cases} (xA)[-(n+1)di] & \text{for } l = 2i, \\ \mathbb{R}[x]/(x^n)[-(n+1)di + nd] & \text{for } l = 2i-1, \\ A & \text{for } l = 0. \end{cases} \tag{5.19}$$

Now, because the differential $\tilde{\delta}$ is zero and $\tilde{\mathrm{b}}$ is degree preserving, it holds (c.f., (3.32))

$$\mathrm{HH}^{\mathrm{nds}}(A) = \bigoplus_{l \in \mathbb{N}_0} \mathrm{HH}_{(l)}.$$

Clearly, the same will hold for $\mathrm{H}^{\lambda,\mathrm{nds}}(A)$, and hence we can ignore the gradation by degree and just use the gradation by weights in the bar complex, i.e., the non-graded theory.

In order to compute $\mathrm{H}^{\lambda}_{(l)}(A)$, we consider the Connes' exact sequence in homology, or ISB-sequence, see [Lod92, Theorem 2.2.1]. It arises from the exact sequence of bicomplexes $0 \to CC^{\{2\}} \hookrightarrow CC \twoheadrightarrow CC[2,0] \to 0$, where CC is the Loday's cyclic bicomplex from [Lod92, Paragraph 2.1.2] (bar complexes in columns), $CC^{\{2\}}$ is the subbicomplex consisting of the first two columns of CC and $CC[2,0]$ is the part of CC starting with the third column. Because A is augmented and unital, [Lod92, Theorem 4.1.13] guarantees a splitting of the ISB-sequence into

$$0 \longrightarrow \overline{\mathrm{H}}^{\lambda}_{(l-1)} \longrightarrow \overline{\mathrm{HH}}_{(l)} \longrightarrow \overline{\mathrm{H}}^{\lambda}_{(l)} \longrightarrow 0 \quad \text{for } l \geq 1, \tag{5.20}$$

where the bar $\bar{\cdot}$ denotes the reduced homology. It holds $\mathrm{H}^{\lambda}_{(0)} = \mathrm{HH}_{(0)} = A$, and hence $\overline{\mathrm{H}}^{\lambda}_{(0)} = \langle x, \dots, x^n \rangle$. Using (5.19), the first map for $l = 1$ in (5.20) reads $\langle x, \dots, x^n\rangle \hookrightarrow \langle 1, x, \dots, x^{n-1}\rangle[-d]$, and hence it is an isomorphism. It follows that $\overline{\mathrm{H}}^{\lambda}_{(1)} = 0$. For $k \geq 1$, we obtain inductively $\overline{\mathrm{H}}^{\lambda}_{(2k)} \simeq \overline{\mathrm{HH}}_{(2k)} = \langle x, \dots, x^n\rangle[-(n+1)dk] \hookrightarrow \overline{\mathrm{HH}}_{(2k+1)} = \langle 1, x, \dots, x^{n-1}\rangle[-(n+1)dk - d]$. This again has to be an isomorphism, and hence $\overline{\mathrm{H}}^{\lambda}_{(2k+1)} = 0$. □

We apply the degree shift $U : \tilde{D}(A) \to D(A)$ from Proposition 3.3.14 to get the generators

$$t_{w,i} := U(\tilde{t}_{w,i}) \in D^{\lambda}(\mathcal{H}(\mathbb{C}\mathrm{P}^n))$$

of weights w and degrees $2i + (w-1)n - 1$, so that

$$\mathrm{H}^{\lambda}(\mathcal{H}(\mathbb{C}\mathrm{P}^n)) = \langle t_{w,i}, \mathbb{1}^w \mid w \in \mathbb{N} \text{ odd}, i = 1, \dots, n\rangle.$$

By the universal coefficient theorem, we have $\mathrm{H}^*_{\lambda} = (\mathrm{H}^{\lambda})^{*_{\mathrm{g}}}$ with respect to the grading by the degree. Given $d \in \mathbb{Z}$, the equation $d = 2i + (w-1)n - 1$ has only finitely many solution $(w, i) \in \mathbb{N} \times \{1, \dots, n\}$, and hence we get

$$\mathbb{H}^{\mathfrak{m}}(C(\mathcal{H}(\mathbb{C}\mathrm{P}^n))) = \langle \mathrm{s}t^*_{w,i}, \mathrm{s}\mathbb{1}^{w*} \mid w \in \mathbb{N} \text{ odd}, i = 1, \dots, n\rangle, \tag{5.21}$$

where $t^*_{w,i}$ and $\mathbb{1}^{w*} \in \mathrm{B}^*_{\mathrm{cyc}}\mathcal{H}$ are the duals to $t_{w,i}$ and $\mathbb{1}^w$, respectively (see Remark 3.3.16). Notice that both $|\mathrm{s}t^*_{w,i}|$ and $|\mathrm{s}\mathbb{1}^{w*}|$ are even since $|\mathrm{s}| = 2n-3$.

Because $\mathbb{C}\mathrm{P}^n$ is geometrically formal, Proposition 4.4.4 implies that $\mathfrak{n}_{10} = \mathfrak{m}_{10}$. Because $\mathbb{H}^{\mathfrak{m}}(C)$ is concentrated in even degrees and because a general IBL_{∞}-operation $\mathfrak{q}_{klg}$ is odd, all operations vanish on the homology. Therefore, for the *twisted* IBL-*algebras* we have

$$\mathrm{IBL}(\mathbb{H}^{\mathfrak{n}}(C)) = \mathrm{IBL}(\mathbb{H}^{\mathfrak{m}}(C)) = (\mathbb{H}^{\mathfrak{m}}(C), \mathfrak{q}_{210} \equiv 0, \mathfrak{q}_{120} \equiv 0),$$

where $\mathbb{H}^{\mathfrak{m}}(C)$ is given by (5.21).

According to [Bas11, Section 3.1.2], the minimal model for the Borel construction $\mathrm{L}_{\mathbb{S}^1}\mathbb{C}\mathrm{P}^n$ is the cdga $\Lambda^{\mathbb{S}^1}(n+1, 1)$, which is freely generated (over $\mathbb{R}$) by the homogenous vectors x_1, x_2, y_1, y_2, u of degrees

$$|x_1| = 2, \quad |x_2| = 2n+1, \quad |y_1| = 1, \quad |y_2| = 2n, \quad |u| = 2,$$

whose differential d satisfies

$$\mathrm{d}y_1 = 0, \quad \mathrm{d}x_1 = y_1 u, \quad \mathrm{d}y_2 = -(n+1)x_1^n y_1, \quad \mathrm{d}x_2 = x_1^{n+1} + y_2 u.$$

By [Bas11, Theorem 3.6], the string cohomology $\mathrm{H}^*_{\mathbb{S}^1}(\mathrm{L}\mathbb{C}\mathrm{P}^n;\mathbb{R}) \simeq \mathrm{H}(\Lambda^{\mathbb{S}^1}(n+1,1),\mathrm{d})$ satisfies for all $m \in \mathbb{N}_0$ the following:

$$\mathrm{H}^m_{\mathbb{S}^1}(\mathrm{L}\mathbb{C}\mathrm{P}^n;\mathbb{R}) = \begin{cases} \langle u^j \rangle & \text{if } m = 2j, \\ \langle y_1 y_2^p x_1^q \mid 0 \le q \le n-1, p \ge 0; q + np = j \rangle & \text{if } m = 2j+1. \end{cases}$$

The right-hand side can be identified with $\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}\mathbb{C}\mathrm{P}^n;\mathbb{R})$ by the universal coefficient theorem. According to [Bas11, Proposition 3.7], we have $\mathbb{m}_2 = 0$ and $\mathbb{c}_2 = 0$. We conclude that the map

$$\begin{aligned} \mathbb{H}^{\mathrm{n}}(C_{\mathrm{red}}(\mathcal{H}(\mathbb{C}\mathrm{P}^n)))[1] &\longrightarrow \mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}\mathbb{C}\mathrm{P}^n;\mathbb{R})[3-n] \\ \mathrm{s}t^*_{2k+1,l} &\longmapsto \mathrm{s}y_1 y_2^k x_1^{l-1} \qquad \text{for } k \ge 0 \text{ and } l = 1,\dots,n \end{aligned}$$

induces an isomorphism of IBL-algebras

$$\mathrm{IBL}(\mathbb{H}^{\mathrm{n}}(C_{\mathrm{red}}(\mathcal{H}(\mathbb{C}\mathrm{P}^n)))) \simeq \mathrm{IBL}(\mathrm{H}^{\mathbb{S}^1,\mathrm{red}}(\mathrm{L}\mathbb{C}\mathrm{P}^n;\mathbb{R})[3-n]).$$

Part II.

Follow-up topics

6. IBL-infinity formality and Poincaré duality models

In this chapter, we study the following question:

Question 6.0.1. Let M be a connected oriented closed n-manifold which is formal in the sense of rational homotopy theory. Let $\mathfrak{m}$ be the canonical and $\mathfrak{n}$ the formal pushforward (or Chern-Simons) Maurer-Cartan element for $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}(M)))$. Are $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ and $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ homotopy equivalent as IBL_∞-algebras?

We remind that Theorem B states that for a geometrically formal manifold M with $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$, one can pick a special Hodge propagator $\mathrm{P}_{\mathrm{std}}$ such that $\mathfrak{n} = \mathfrak{m}$ at least for $n \neq 2$.[1] Question 6.0.1 asks for a generalization of Theorem B. In the upcoming sections, we will propose a strategy to answer Question 6.0.1 in the case of $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$, when the expected answer is "yes". Our main tool is the theory of Poincaré duality models from [LS08]. Nevertheless, we also study DGA's of Hodge type and their small subalgebras from [Fio+19], which, as we think, naturally fit in the picture.

In Section 6.1, we consider DGA's and define the notions of an orientation and cyclic structure (Definition 6.1.1), non-degeneracy and Poincaré duality (Definition 6.1.6), degenerate subspace and non-degenerate quotient (Definition 6.1.8), Hodge decomposition and Hodge pair (Definition 6.1.9) and Hodge and small subalgebra from [Fio+19] (Definition 6.1.14). We prove that in some cases orientations and cyclic structures are in one-to-one correspondence (Proposition 6.1.4) and that being of Hodge type is equivalent to acyclicity of the degenerate subspace (Proposition 6.1.13). We describe the small subalgebra of [Fio+19] in terms of Kontsevich-Soibelman–like evaluations of rooted binary trees with internal edges labeled with the identity or the standard Hodge homotopy (Proposition 6.1.15). We study what happens when we take small subalgebras and non-degenerate quotients iteratively (Proposition 6.1.17). We give examples when small subalgebras and their non-degenerate quotients differ for different Hodge decompositions of the same DGA (Examples 6.1.21 and 6.1.22). We prove an important lemma allowing

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[1] We did not prove that the homotopy class of $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ does not depend on $\mathrm{P}_{\mathrm{std}}$, but we expect so.

to extend an oriented DGA to a DGA of Hodge type later (Lemma 6.1.23). We also prove several little lemmas (Lemmas 6.1.12, 6.1.18, 6.1.19 and 6.1.20) and make several remarks (Remarks 6.1.2, 6.1.5, 6.1.7, 6.1.11 and 6.1.16) which might be useful for future investigations. We finish with some open questions (Questions 6.1.24)

In Section 6.2, we define differential Poincaré duality algebras[2] and Poincaré DGA's (Definition 6.2.1), i.e., DGA's whose homology is a Poincaré duality algebra. We consider Poincaré duality models (Definition 6.2.3). We argue that a Poincaré DGA is formal if and only if it is formal as a DGA (Proposition 6.2.5). We prove that an oriented DGA extends under some conditions to a DGA of Hodge type (Proposition 6.2.6) and use it to prove the existence and uniqueness of Poincaré duality models in some cases (Propositions 6.2.8 and 6.2.9). We discuss minimal models (Remark 6.2.12). We give examples of manifolds whose de Rham algebras do not admit a Poincaré duality model with just one arrow (Example 6.2.13). We finish with some open questions (Questions 6.2.14).

In Section 6.3, we prove functoriality of the canonical $\mathrm{dIBL}^{\mathfrak{m}}$ construction on differential Poincaré duality algebras up to IBL_∞-homotopy (Proposition 6.3.1) and the uniqueness up to IBL_∞-homotopy in a weak homotopy equivalence class of PDGA's in some cases (Proposition 6.3.2). We define IBL_∞-formality of a PDGA (Definition 6.3.4) and conjecture that IBL_∞-formality is implied by DGA-formality (Conjecture 6.3.5); in fact, this conjecture holds when Proposition 6.3.2 holds. We discuss another relevant notions of formality (Remark 6.3.6). For the de Rham complex of a smooth manifold, we conjecture that the IBL_∞-algebra on cyclic cochains of de Rham cohomology twisted by the Chern-Simons Maurer-Cartan element is homotopy equivalent to the dIBL-algebra on cyclic cochains of the non-degenerate quotient of the small subalgebra of the de Rham complex twisted by the canonical Maurer-Cartan element (Conjecture 6.3.7). Assuming that the conjectures hold, we obtain a positive answer to Question 6.0.1 when $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$ (Conjecture 6.3.8).

6.1. Orientation, Poincaré duality, Hodge decomposition

In this section, a DGA V is a triple $(V, \mathrm{d}, \wedge)$, where

- $V = \bigoplus_{n\in\mathbb{Z}} V^n$ is a $\mathbb{Z}$-graded vector space,
- $\mathrm{d} : V \to V$ a differential of degree 1, and

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[2] The same as cyclic DGA's for unital commutative DGA's up to a degree shift and the correspondence of orientations and cyclic structures

- $\wedge : V \otimes V \to V$ an associative product of degree 0 such that d and $\wedge$ satisfy the Leibnitz identity $\mathrm{d}(v_1 \wedge v_2) = \mathrm{d}v_1 \wedge v_2 + (-1)^{v_1} v_1 \wedge \mathrm{d}v_2$ for all homogenous v_1, $v_2 \in V$.

In other words, we consider general, possibly non-unital and non-commutative, $\mathbb{Z}$-graded DGA's. We denote by $\deg(v)$ the degree of a homogenous element $v \in V$ and write $(-1)^v$ in the exponent.

Definition 6.1.1 (Orientations and cyclic structures). *Let* (V, d) *be a* $\mathbb{Z}$*-graded cochain complex. We define the following:*

- *An* orientation in degree n *is a linear function* $\mathrm{o} : V^n \to \mathbb{R}$ *such that*

  $$\mathrm{o} \neq 0 \quad \textit{and} \quad \mathrm{o} \circ \mathrm{d} = 0.$$

  In other words, it is a surjective chain map $\mathrm{o} : (V, \mathrm{d}) \to (\mathbb{R}[-n], 0)$.

- *A* cyclic structure of degree n *is a homogenous bilinear form* $\langle \cdot, \cdot \rangle : V \otimes V \to \mathbb{R}$ *of degree* $-n$ *as a map such that for all homogenous* v_1, $v_2 \in V$*, we have*

  $$\langle \mathrm{d}v_1, v_2 \rangle = (-1)^{1+v_1 v_2} \langle \mathrm{d}v_2, v_1 \rangle. \tag{6.1}$$

A cyclic structure on a DGA $(V, \mathrm{d}, \wedge)$ *is additionally required to satisfy*

$$\langle v_1 \wedge v_2, v_3 \rangle = (-1)^{v_3(v_1+v_2)} \langle v_3 \wedge v_1, v_2 \rangle \tag{6.2}$$

for all homogenous v_1, v_2, $v_3 \in V$.

Analogously, one can define a cyclic structure on an A_∞-algebra.

Remark 6.1.2 (Cyclic DGA). A cyclic DGA (V, Π, μ_1, μ_2) of degree n from Definition 3.3.7 is the same as a DGA equipped with a cyclic structure $\langle \cdot, \cdot \rangle$ of degree n which is symmetric, i.e.,

$$\langle v_1, v_2 \rangle = (-1)^{v_1 v_2} \langle v_2, v_1 \rangle$$

for all homogenous v_1, $v_2 \in V$, and non-degenerate (see Definition 6.1.6). The correspondence is via the degree shift

$$\begin{aligned} \mu_1(\theta v) &= \theta \mathrm{d}(v), \\ \mu_2(\theta v_1, \theta v_2) &= (-1)^{v_1} \theta(v_1 \wedge v_2), \\ \Pi(\theta v_1, \theta v_2) &= (-1)^{v_1} \langle v_1, v_2 \rangle, \end{aligned}$$

where θ is a formal symbol of degree -1. From this reason, we sometimes call a DGA equipped with a non-degenerate symmetric cyclic structure a cyclic DGA, although there are degree shifts involved. ◁

The homology $\mathrm{H}(V) \coloneqq \mathrm{H}(V, \mathrm{d})$ of a DGA $(V, \mathrm{d}, \wedge)$ is also a DGA with the induced product $\wedge$ and with zero differential. Given an orientation o or a cyclic structure $\langle\cdot,\cdot\rangle$ on V, we define the maps $\mathrm{o}^{\mathrm{H}} : \mathrm{H}(V) \to \mathbb{R}$ and $\langle\cdot,\cdot\rangle^{\mathrm{H}} : \mathrm{H}(V) \otimes \mathrm{H}(V) \to \mathbb{R}$ for all closed h_1, $h_2 \in V$ by

$$\begin{aligned} \mathrm{o}^{\mathrm{H}}([h_1]) &\coloneqq \mathrm{o}(h_1) \text{ and} \\ \langle [h_1], [h_2] \rangle^{\mathrm{H}} &\coloneqq \langle h_1, h_2 \rangle, \end{aligned} \tag{6.3}$$

respectively, where $[\cdot]$ denotes the cohomology class. It is easy to see that $\langle\cdot,\cdot\rangle^{\mathrm{H}}$ is a cyclic structure on $\mathrm{H}(V)$ and that o^{H} is an orientation on $\mathrm{H}(V)$ provided that $\mathrm{o}|_{\ker \mathrm{d}} \neq 0$.

Proposition 6.1.3 (Orientation on homology)**.** *We have the following:*

(a) Let (V, d) be a $\mathbb{Z}$-graded cochain complex, and let $n \in \mathbb{Z}$. Given an orientation $\tilde{\mathrm{o}} : \mathrm{H}^n(V) \to \mathbb{R}$, there is an orientation $\mathrm{o}^V : V^n \to \mathbb{R}$ such that $\tilde{\mathrm{o}} = \mathrm{o}^{\mathrm{H}}$. If $\mathrm{d}V^n = 0$, then we have the correspondence

$$\textit{Orientations on } V \textit{ in degree } n \overset{1:1}{\simeq} \textit{Orientations on } \mathrm{H}(V) \textit{ in degree } n.$$

(b) If $(V_1, \mathrm{d}_1, \mathrm{o}_1)$ and $(V_2, \mathrm{d}_2, \mathrm{o}_2)$ are $\mathbb{Z}$-graded cochain complexes oriented in degree n with $\mathrm{d}V_1^n = 0 = \mathrm{d}V_2^n$, then a chain map $f : V_1 \to V_2$ preserves the orientation if and only if the induced map $f_* : \mathrm{H}(V_1) \to \mathrm{H}(V_2)$ preserves the induced orientation.

Proof. (a) We define $\mathrm{o}^V(v) \coloneqq \tilde{\mathrm{o}}([v])$ for all closed $v \in V$ and extend it by 0 to a complement of $\ker \mathrm{d}$ in V. It is obvious that $\mathrm{o}^{\mathrm{H}} = \tilde{\mathrm{o}}$. If $\mathrm{d}V^n = 0$, any complement is trivial and o^V is uniquely determined by o^{H}.

(b) Let $v \in V_1^n$. Because $\mathrm{d}v = 0$, we have

$$\mathrm{o}_2(f(v)) = \mathrm{o}_2^{\mathrm{H}}([f(v)]) = \mathrm{o}_2^{\mathrm{H}}(f_*[v]) = \mathrm{o}_1^{\mathrm{H}}([v]) = \mathrm{o}_1(v).$$

This finishes the proof. □

Suppose that 1 is a unit for $(V, \mathrm{d}, \wedge)$, i.e., $1 \in V^0$, $\mathrm{d}1 = 0$ and $1 \wedge v = v \wedge 1 = v$ for all

$v \in V$, and let $\langle \cdot, \cdot \rangle$ by a cyclic structure on V. For all homogenous v_1, $v_2 \in V$, we have

$$\begin{aligned}\langle v_1, v_2 \rangle &= \langle v_1 \wedge 1, v_2 \rangle \\ &= (-1)^{v_1 v_2} \langle 1 \wedge v_2, v_1 \rangle \\ &= (-1)^{v_1 v_2} \langle v_2, v_1 \rangle,\end{aligned}$$

and hence $\langle \cdot, \cdot \rangle$ is automatically symmetric.

Recall that a DGA is called commutative if $v_1 \wedge v_2 = (-1)^{v_1 v_2} v_2 \wedge v_1$ for all homogenous v_1, $v_2 \in V$. Commutativity of a DGA and symmetry of a general cyclic structure on it seem to be unrelated.

Proposition 6.1.4 (Correspondence of orientations and cyclic structures on DGA's)**.** *Let* $(V, \mathrm{d}, \wedge)$ *be a* DGA*. Then the following holds:*

(a) If $\wedge$ *is commutative, then any orientation* o *in degree* n *induces a cyclic structure* $\langle \cdot, \cdot \rangle$ *of degree* n *which is given for all homogenous* v_1, $v_2 \in V$ *by*

$$\langle v_1, v_2 \rangle \coloneqq \begin{cases} \mathrm{o}(v_1 \wedge v_2) & \text{if } \deg(v_1) + \deg(v_2) = n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

(b) If 1 *is a unit, then any non-zero cyclic structure* $\langle \cdot, \cdot \rangle$ *of degree* n *induces an orientation* o *in degree* n *by defining*

$$\mathrm{o}(v) \coloneqq \begin{cases} \langle v, 1 \rangle & \text{for } v \in V^n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

(c) For a unital commutative DGA $(V, \mathrm{d}, \wedge, 1)$*, formulas* (6.4) *and* (6.5) *define the correspondence*

$$\textit{Orientations in degree } n \overset{1:1}{\simeq} \textit{Non-zero cyclic structures of degree } n.$$

Proof. (a) Using the Leibnitz identity, properties of an orientation and commutativity, we check that

$$\begin{aligned}\langle \mathrm{d}v_1, v_2\rangle &= \mathrm{o}(\mathrm{d}v_1 \wedge v_2)\\ &= \mathrm{o}\big(\mathrm{d}(v_1 \wedge v_2) - (-1)^{v_1} v_1 \wedge \mathrm{d}v_2\big)\\ &= (-1)^{1+v_1}\mathrm{o}(v_1 \wedge \mathrm{d}v_2)\\ &= (-1)^{1+v_1 v_2}\mathrm{o}(\mathrm{d}v_2 \wedge v_1)\\ &= (-1)^{1+v_1 v_2}\langle \mathrm{d}v_2, v_1\rangle\end{aligned}$$

and

$$\begin{aligned}\langle v_1 \wedge v_2, v_3\rangle &= \mathrm{o}(v_1 \wedge v_2 \wedge v_3)\\ &= (-1)^{v_3(v_1+v_2)}\mathrm{o}(v_3 \wedge v_1 \wedge v_2)\\ &= (-1)^{v_3(v_1+v_2)}\langle v_3 \wedge v_1, v_2\rangle\end{aligned}$$

for all homogenous v_1, v_2, $v_3 \in V$.

(b) For all $v \in V$, we have

$$\mathrm{o}(\mathrm{d}v) = \langle \mathrm{d}v, 1\rangle = -\langle \mathrm{d}1, v\rangle = 0.$$

From $\langle\cdot,\cdot\rangle \neq 0$ it follows that there are homogenous v_1, $v_2 \in V$ with $\deg(v_1)+\deg(v_2) = n$ such that $\langle v_1, v_2\rangle \neq 0$. Then $v_1 \wedge v_2 \in V^n$ and

$$\begin{aligned}\mathrm{o}(v_1 \wedge v_2) &= \langle v_1 \wedge v_2, 1\rangle\\ &= \langle 1 \wedge v_1, v_2\rangle\\ &= \langle v_1, v_2\rangle \neq 0.\end{aligned}$$

Therefore, o is an orientation on V.

(c) This is a combination of (a) and (c) plus the uniqueness, which is easy to check. □

Remark 6.1.5 (Volume forms). If $\mathrm{H}^n(V) \simeq \mathbb{R}$, then orientations $\mathrm{o} : \mathrm{H}^n(V) \to \mathbb{R}$ and elements $0 \neq [\mathrm{Vol}] \in \mathrm{H}^n(V)$ called *volume forms* are in one-to-one correspondence via

$$\mathrm{o}([\mathrm{Vol}]) = 1.$$

A consequence is the following: Suppose that $(V_1, \mathrm{d}_1, \mathrm{o}_1)$ and $(V_2, \mathrm{d}_2, \mathrm{o}_2)$ are cochain complexes oriented in degree n which satisfy

$$\mathrm{H}^n(V_1) \simeq \mathrm{H}^n(V_2) \simeq \mathbb{R} \quad \text{and} \quad \mathrm{d}_1 V_1^n \simeq \mathrm{d}_2 V_2^n = 0, \tag{6.6}$$

so that Proposition 6.1.3 applies. Then a chain map $f : V_1 \to V_2$ preserves orientation if and only if the induced map $f_* : \mathrm{H}^n(V_1) \to \mathrm{H}^n(V_2)$ maps $[\mathrm{Vol}_1]$ to $[\mathrm{Vol}_2]$. In the category of unital commutative DGA's satisfying (6.6), so that also Proposition 6.1.4 holds, if the orientations come from cyclic structures, then f preserves cyclic structure if and only if f_* preserves volume form. ◁

Definition 6.1.6 (Non-degeneracy and Poincaré duality)**.** *Given a graded vector space V, a homogenous bilinear form (=: pairing) $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{R}$ of degree $-n$ as a map which is graded symmetric is called* non-degenerate *if for every $v \in V$, the following implication holds:*

$$\langle v, w\rangle = 0 \quad \textit{for all } w \in V \quad \Longrightarrow \quad v = 0.$$

We say that $\langle\cdot,\cdot\rangle$ satisfies Poincaré duality *if the map $\flat : V \to V^{*_\mathrm{g}}$ (graded dual) defined by*

$$\flat(v)(w) \coloneqq \langle v, w\rangle \quad \textit{for all } v, w \in V$$

is a graded isomorphism (it has degree $-n$ as a map) of graded vector spaces.

Remark 6.1.7 (On non-degeneracy and Poincaré duality)*.* (i) Clearly, Poincaré duality implies non-degeneracy. If the degree k component V^k of V is finite-dimensional for every $k \in \mathbb{Z}$ — we say that V is of *finite type* — then the opposite is true as well. If $n = 0$, then Poincaré duality implies that V is of finite type.

(ii) If V is non-negatively graded, then non-degeneracy of $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{R}$ implies $V = V^0 \oplus \cdots \oplus V^n$. Therefore, non-negatively graded vector spaces of finite type which admit a non-degenerate homogenous bilinear form are finite-dimensional.

(iii) The de Rham complex $(\Omega(M), \mathrm{d}, \wedge)$ of an oriented closed n-manifold M with the orientation $\int : \Omega^n(M) \to \mathbb{R}$ is an oriented DGA whose cyclic structure is non-degenerate but does not satisfy Poincaré duality. On the other hand, the induced structure on homology $\mathrm{H}(\Omega(M))$ satisfies Poincaré duality. ◁

An analog of the following definition is used in [Fio+19] and also in [LS08] (under the name "set of orphans").

Definition 6.1.8 (Degenerate subspace and non-degenerate quotient)**.** *Given a symmetric pairing $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{R}$ on a graded vector space V, we define the* degenerate subspace $V^\perp \subset V$ *by*

$$V^\perp := \{v \in V \mid \langle w, v\rangle = 0 \text{ for all } w \in V\}.$$

If $(V, \mathrm{d}, \wedge)$ *is a* DGA *and* $\langle\cdot,\cdot\rangle$ *a cyclic structure, then* (6.1) *implies that* $V^\perp$ *is a differential graded ideal in* V*, and thus we obtain the short exact sequence of* DGA*'s*

$$0 \longrightarrow V^\perp \overset{\iota}{\hookrightarrow} V \xrightarrow{\pi^{\mathcal{Q}}} \mathcal{Q}(V) := V/V^\perp \longrightarrow 0, \tag{6.7}$$

where ι *is the inclusion and* $\pi^{\mathcal{Q}}$ *the canonical projection. We call the* DGA $\mathcal{Q}(V)$ *together with the induced non-degenerate cyclic structure* $\langle\cdot,\cdot\rangle^{\mathcal{Q}}$ *such that* $\langle\pi^{\mathcal{Q}}(\cdot), \pi^{\mathcal{Q}}(\cdot)\rangle^{\mathcal{Q}} = \langle\cdot,\cdot\rangle$ *the* non-degenerate quotient.

It was observed in [Fio+19] that the question whether $(V^\perp, \mathrm{d})$ is acyclic, and hence $\pi^{\mathcal{Q}}$ is a quasi-isomorphism, turns out to be related to the existence of Hodge decomposition.

Definition 6.1.9 (Hodge decomposition)**.** *A cochain complex* (V, d) *with a symmetric cyclic structure* $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{R}$ *is of* Hodge type *if there exist subspaces* $\mathcal{H} \subset \ker \mathrm{d}$ *and* $C \subset V$ *such that*

$$V = \ker \mathrm{d} \oplus C, \quad \ker \mathrm{d} = \operatorname{im} \mathrm{d} \oplus \mathcal{H} \quad \text{and} \quad C \perp \mathcal{H} \oplus C, \tag{6.8}$$

where $\perp$ *denotes the relation of being perpendicular with respect to* $\langle\cdot,\cdot\rangle$*. Such decomposition is called a* Hodge decomposition. *We call* $\mathcal{H}$ *the* harmonic subspace *and* C *the* coexact part*.*

Given a Hodge decomposition, we define the standard Hodge homotopy $\mathcal{P}_{\mathrm{std}} : V \to V$ *by*

$$\mathcal{P}_{\mathrm{std}} := \begin{cases} -(\mathrm{d}|_C)^{-1} & \text{on } \operatorname{im} \mathrm{d}, \\ 0 & \text{on } \mathcal{H} \oplus C. \end{cases}$$

Then we have $\mathrm{d}\mathcal{P}_{\mathrm{std}} = -\pi_{\operatorname{im} \mathrm{d}}$*,* $\mathcal{P}_{\mathrm{std}}\mathrm{d} = -\pi_C$*, and hence*

$$[\mathrm{d}, \mathcal{P}_{\mathrm{std}}] = \mathrm{d}\mathcal{P}_{\mathrm{std}} + \mathcal{P}_{\mathrm{std}}\mathrm{d} = \pi_{\mathcal{H}} - \mathbb{1}.$$

We call $(\mathcal{H}, \mathcal{P}_{\mathrm{std}})$ *the* Hodge pair *associated to the Hodge decomposition* (6.8)*.*

Proposition 6.1.10 (Non-deg., fin. type implies Hodge type)**.** *Any cochain complex of finite type with a non-degenerate symmetric cyclic structure is of Hodge type.*

Proof. This is [CFL15, Lemma 11.1], and the proof uses formal Hodge theory. □

Remark 6.1.11 (Harmonic subspaces). In the situation of Proposition 6.1.10, it was shown in [Fio+19, Remark 2.6] that for any complement $\mathcal{H}$ of $\operatorname{im}\mathrm{d}$ in $\ker\mathrm{d}$ (in other words, the image of a section $\mathrm{H}(V)\to\ker\mathrm{d}$) there is a coexact part C such that $V=\mathcal{H}\oplus\operatorname{im}\mathrm{d}\oplus C$ is a Hodge decomposition. From this reason, we call any complement of $\operatorname{im}\mathrm{d}$ in $\ker\mathrm{d}$ a *harmonic subspace.*[3] ◁

The following lemma will be used in the proof of Proposition 6.1.13.

Lemma 6.1.12 (Complement of acyclic subcomplex over $\mathbb{R}$). *Let $f: V_1\to V_2$ be an injective chain map of cochain complexes (V_1,d_1) and (V_2,d_2) over $\mathbb{R}$ such that (V_1,d_1) is acyclic. Then there is a chain map $g: V_2\to V_1$ such that $g\circ f=\mathbb{1}$.*[4]

Proof. For every $i\in\mathbb{Z}$, consider the diagram

$$\begin{tikzcd}
\ker\mathrm{d}_1^i\oplus C_1^i \arrow[r, "f^i"] \arrow[d, "\mathrm{d}_1^i"] & \ker\mathrm{d}_2^i\oplus C_2^i \arrow[d, "\mathrm{d}_2^i"] \\
\ker\mathrm{d}_1^{i+1}\oplus C_1^{i+1} \arrow[r, "f^{i+1}"] & \ker\mathrm{d}_2^{i+1}\oplus C_2^{i+1},
\end{tikzcd}$$

where C_1^i is a complement of $\ker\mathrm{d}_1^i$ in V_1^i and C_2^i is a complement of $\ker\mathrm{d}_2^i$ in V_2^i. With respect to this decomposition, we write

$$f^i=\begin{pmatrix} f_{11}^i & f_{12}^i\\ f_{21}^i & f_{22}^i\end{pmatrix},\qquad g^i=\begin{pmatrix} g_{11}^i & g_{12}^i\\ g_{21}^i & g_{22}^i\end{pmatrix},$$
$$\mathrm{d}_1^i=\begin{pmatrix} 0 & d_1^i\\ 0 & 0\end{pmatrix},\qquad \mathrm{d}_2^i=\begin{pmatrix} 0 & d_2^i\\ 0 & 0\end{pmatrix}.$$

The assumption $\mathrm{H}(V_1)=0$ implies that d_1^i is an isomorphism. The fact that f is a chain map translates to

$$d_2^i f_{21}^i=0,\quad f_{21}^{i+1}d_1^i=0,\quad f_{11}^{i+1}d_1^i=d_2^i f_{22}^i. \tag{6.9}$$

From the second relation and surjectivity of d_1^i, we get that $f_{21}^{i+1}=0$. Now, f_{11}^{i+1} has to be injective because it is the only possibly non-zero part of f on $\ker d_1^{i+1}$. From the third relation of (6.9) and the fact that d_1^i is injective, we get that f_{22}^i is injective as well. Relations (6.9) hold also for g with d_1 and d_2 switched. In particular, we have $g_{21}^i=0$.

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[3] Given a Hodge decomposition $V=\mathcal{H}\oplus\operatorname{im}\mathrm{d}\oplus C$ and a harmonic subspace $\mathcal{H}'$, then it holds $\mathcal{H}'=\mathrm{gr}(\alpha:\mathcal{H}\to\mathrm{d}V)$ because $\mathcal{H}\oplus\mathrm{d}V=\mathcal{H}'\oplus\mathrm{d}V$, and one can take $C'=\mathrm{gr}(-\alpha^\dagger-\frac{1}{2}\alpha\alpha^\dagger: C\to\mathcal{H}\oplus\mathrm{d}V)$.

[4] This lemma can be used to prove that over $\mathbb{R}$, every surjective quasi-isomorphism is a deformation retraction and every injective quasi-isomorphism is a section of a deformation retraction.

The relation $g \circ f = \mathbb{1}$ translates using $f^i_{21} = g^i_{21} = 0$ to

$$g^i_{11} f^i_{11} = \mathbb{1}, \quad g^i_{11} f^i_{12} + g^i_{12} f^i_{22} = 0, \quad g^i_{22} f^i_{22} = \mathbb{1}. \tag{6.10}$$

Because d^i_1 is an isomorphism, the last equation is equivalent to

$$\mathbb{1} = d^i_1 g^i_{22} f^i_{22} (d^i_1)^{-1} = g^{i+1}_{11} d^i_2 f^i_{22} (d^i_1)^{-1} = g^{i+1}_{11} f^{i+1}_{11} d^i_1 (d^i_1)^{-1} = g^{i+1}_{11} f^{i+1}_{11}.$$

We see that g can be constructed as follows. For all $i \in \mathbb{Z}$, let g^{11}_i be an arbitrary left inverse of f^{11}_i. Set $g^i_{22} \coloneqq (d^i_1)^{-1} g^{i+1}_{11} d^i_2$ and $g^{21}_i \coloneqq 0$. Finally, g^i_{12} has to be chosen such that the second equation of (6.10) is satisfied. This is possible since we can first define g^i_{12} on $\operatorname{im} f^i_{22}$ because f^i_{22} injective and then extend it by 0 to a complement. ☐

Claim (b) of the following proposition corresponds to [Fio+19, Lemma 2.8]. Claim (c) was suggested by Prof. Hông Vân Lê via e-mail correspondence.

Proposition 6.1.13 (Hodge decomposition and acyclicity of $V^\perp$)**.** *Let (V, d) be a cochain complex with a symmetric cyclic structure $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{R}$. If V is of Hodge type, then the following implications hold:*

(a) If $\langle\cdot,\cdot\rangle$ is non-degenerate, then $\langle\cdot,\cdot\rangle^{\mathrm{H}}$ is non-degenerate.

(b) If $\langle\cdot,\cdot\rangle^{\mathrm{H}}$ is non-degenerate, then $(V^\perp, \mathrm{d})$ is acyclic.

Moreover, the following reverse implication holds:

(c) If V is of finite type and $(V^\perp, \mathrm{d})$ is acyclic, then V is of Hodge type.

Proof. (a) Let $V = \operatorname{im} \mathrm{d} \oplus \mathcal{H} \oplus C$ be a Hodge decomposition. Then $\operatorname{im} \mathrm{d} \oplus C \subset \mathcal{H}^\perp$, and hence

$$\langle \mathrm{d}\eta + b + c, b' \rangle = \langle b, b' \rangle \quad \text{for all } \eta \in V, c \in C \text{ and } b, b' \in \mathcal{H}.$$

The claim follows easily. Notice that having a Hodge decomposition, it holds

$$\langle\cdot,\cdot\rangle\big|_{\mathcal{H}\otimes\mathcal{H}} \text{ non-degenerate} \quad \iff \quad \operatorname{im} \mathrm{d} \oplus C = \mathcal{H}^\perp.$$

(b) Consider a Hodge decomposition as above, and let $v \in V^\perp \cap \ker \mathrm{d}$ be a non-zero vector. Suppose that $v \notin \operatorname{im} \mathrm{d}$. Then $[v] \neq 0$ in $\mathrm{H}(V)$, and hence there is a $b \in \mathcal{H}$ such that $\langle v, b \rangle \neq 0$ by non-degeneracy of $\langle\cdot,\cdot\rangle^{\mathrm{H}}$. This is a contradiction with $v \in V^\perp$. Therefore, it holds $V^\perp \cap \ker \mathrm{d} = V^\perp \cap \operatorname{im} \mathrm{d}$. In particular, there is an $\eta \in C$ such that

$v = \mathrm{d}\eta$. Now, for any $\eta' \in C$, $b \in \mathcal{H}$ and $c \in C$, we have using $C \perp \mathcal{H} \oplus C$ and $v \in V^\perp$ the following:

$$\begin{aligned}\langle \mathrm{d}\eta' + b + c, \eta\rangle &= \langle \mathrm{d}\eta', \eta\rangle \\ &= (-1)^{1+\eta\eta'}\langle \mathrm{d}\eta, \eta'\rangle \\ &= (-1)^{1+\eta\eta'}\langle v, \eta'\rangle \\ &= 0.\end{aligned}$$

Therefore, it holds $\eta \in V^\perp$, and we have shown that $V^\perp \cap \operatorname{im}\mathrm{d} = \mathrm{d}V^\perp$. The claim follows.

(c) Because $V^\perp \subset V$ is an acyclic subcomplex and we work over $\mathbb{R}$, there is a complementary subcomplex $Z \subset V$; i.e., $\mathrm{d}Z \subset Z$ and $V = V^\perp \oplus Z$. This follows from Lemma 6.1.12 by setting $V^\perp = \operatorname{im} f$ and $Z = \ker g$. Now, the restriction of $\langle\cdot,\cdot\rangle$ to Z is non-degenerate, and Proposition 6.1.10 provides its Hodge decomposition $Z = \mathrm{d}Z \oplus \mathcal{H} \oplus D$. Let $E \subset V^\perp$ be a graded vector space which is complementary to $\mathrm{d}V^\perp$ in $V^\perp$; i.e., $V^\perp = \mathrm{d}V^\perp \oplus E$. It is easy to check that $V = \operatorname{im}\mathrm{d} \oplus \mathcal{H} \oplus C$ with $C \coloneqq D \oplus E$ is a Hodge decomposition. ☐

The following notions were taken from [Fio+19].

Definition 6.1.14 (Hodge subalgebra and small subalgebra). *Consider a* DGA $(V, \mathrm{d}, \wedge)$ *with a symmetric cyclic structure* $\langle\cdot,\cdot\rangle$*. Suppose that it admits a Hodge decomposition with the Hodge pair* $(\mathcal{H}, \mathcal{P}_{\mathrm{std}})$*. A* Hodge subalgebra *is a differential graded subalgebra* $W \subset V$ *which satisfies*

$$\mathcal{H} \subset W \quad \text{and} \quad \mathcal{P}_{\mathrm{std}} W \subset W.$$

We denote the smallest Hodge subalgebra of V *by* $\mathcal{S}(V)$ *and call it the* small subalgebra*. We stress that the definition of* $\mathcal{S}(V)$ *depends on* $(\mathcal{H}, \mathcal{P}_{\mathrm{std}})$*!*

The following is a version of [Fio+19, Proposition 3.3] which generalizes to the non-simply-connected case (see (iii) of Remark 6.1.16 below for the comparison).

Proposition 6.1.15 (Description of small subalgebra). *Consider the situation of Definition 6.1.14. The small subalgebra* $\mathcal{S}(V)$ *is generated as a graded vector space by Kontsevich-Soibelman–like evaluations of rooted binary trees with* $k \geq 1$ *leaves labeled with homogenous elements of* $\mathcal{H}$*, interior vertices labeled with* $\wedge$ *and interior edges labeled either with* $\mathcal{P}_{\mathrm{std}}$ *or with* $\mathbb{1}$*.*

Proof. We abbreviate $\mathcal{S} \coloneqq \mathcal{S}(V)$, denote the set of labeled trees by $\mathcal{T}$ and denote the vector space generated by evaluations of elements of $\mathcal{T}$ by $\langle\mathcal{T}\rangle$.

Clearly, we have $\langle\mathcal{T}\rangle \subset \mathcal{S}$. For "=", it suffices to check that for any T, T_1, $T_2 \in \mathcal{T}$, it holds $\mathrm{d}T$, $\mathcal{P}_{\mathrm{std}}T$, $T_1 \wedge T_2 \in \langle\mathcal{T}\rangle$, i.e., that $\langle\mathcal{T}\rangle$ is a Hodge subalgebra.

As for $\mathrm{d}T$, we imagine d propagating from the root to the leaves. When it encounters $L\wedge R$, where L stands for the left and R for the right sub-branch, it duplicates the tree and continues propagating in L and R, respectively (we take the sum of the two copies in the end). This is justified by the Leibnitz identity $\mathrm{d}(L \wedge R) = \mathrm{d}L \wedge R + (-1)^L L \wedge \mathrm{d}R$. When it encouners $\mathcal{P}_{\mathrm{std}}$, it triples the tree and either goes past $\mathcal{P}_{\mathrm{std}}$ and continues propagating, or exchanges $\mathcal{P}_{\mathrm{std}}$ for $\mathbb{1}$ and stops, or exchanges $\mathcal{P}_{\mathrm{std}}$ for $\pi_{\mathcal{H}}$ and stops. This is justified by $\mathrm{d}\mathcal{P}_{\mathrm{std}} = -\mathcal{P}_{\mathrm{std}}\mathrm{d} - \mathbb{1} + \pi_{\mathcal{H}}$. If it encounters $\mathbb{1}$, nothing happens and it keeps propagating. If it reaches a leaf with $h \in \mathcal{H}$, then the corresponding tree evaluates to 0. We see that we alway obtain an element of $\langle\mathcal{T}\rangle$.

As for $\mathcal{P}_{\mathrm{std}}T$, we have either $\mathcal{P}_{\mathrm{std}}T = 0$ if the interior edge adjacent to the root (=: the root edge) is labeled with $\mathcal{P}_{\mathrm{std}}$, or $\mathcal{P}_{\mathrm{std}}T$ is a new tree in $\mathcal{T}$ which arises from T by replacing $\mathbb{1}$ by $\mathcal{P}_{\mathrm{std}}$ on the root edge.

As for $T_1 \wedge T_2$, using $\mathbb{1} = \pi_{\mathcal{H}} - \mathrm{d}\mathcal{P}_{\mathrm{std}} - \mathcal{P}_{\mathrm{std}}\mathrm{d}$ and the Leibnitz identity, we get

$$\begin{aligned}T_1 \wedge T_2 &= \pi_{\mathcal{H}}(T_1 \wedge T_2) - (\mathrm{d}\mathcal{P}_{\mathrm{std}})(T_1 \wedge T_2) - (\mathcal{P}_{\mathrm{std}}\mathrm{d})(T_1 \wedge T_2)\\ &= \pi_{\mathcal{H}}(T_1 \wedge T_2) - \mathrm{d}\big(\mathcal{P}_{\mathrm{std}}(T_1 \wedge T_2)\big) - \mathcal{P}_{\mathrm{std}}(\mathrm{d}T_1 \wedge T_2) - (-1)^{T_1}\mathcal{P}_{\mathrm{std}}(T_1 \wedge \mathrm{d}T_2).\end{aligned}$$

We see that $T_1 \wedge T_2 \in \langle\mathcal{T}\rangle$. □

Remark 6.1.16 (On Hodge subalgebra and small subalgebra). (i) Any Hodge subalgebra W inherits the Hodge decomposition

$$W = \mathcal{H} \oplus \mathrm{d}W \oplus \mathcal{P}_{\mathrm{std}}W \tag{6.11}$$

with the Hodge pair $(\mathcal{H}, \mathcal{P}_{\mathrm{std}}|_W)$.

(ii) We will be in the situation of Definition 6.1.14 and use the notation of the proof of Proposition 6.1.15. In addition, we suppose that V is non-negatively graded. Having established $\mathcal{S} = \langle\mathcal{T}\rangle$, consider the Hodge decomposition (6.11) for $W = \mathcal{S}$. Let $\bar{\mathcal{S}} = \bigoplus_{k\geq 1}\mathcal{S}^k$ denote the reduced part of $\mathcal{S}$, and let $\langle\bar{\mathcal{S}} \wedge \bar{\mathcal{S}}\rangle$ denote the graded vector space generated by products $v_1 \wedge v_2$ for v_1, $v_2 \in \bar{\mathcal{S}}$. We have

$$\begin{aligned}\mathcal{P}_{\mathrm{std}}\langle\mathcal{T}\rangle &= \big\langle\{T \in \mathcal{T} \text{ with } \mathcal{P}_{\mathrm{std}} \text{ on the root edge}\}\big\rangle\\ &= \mathcal{P}_{\mathrm{std}}\langle\bar{\mathcal{S}} \wedge \bar{\mathcal{S}}\rangle\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\langle\mathcal{T}\rangle &= \mathrm{d}(\pi_{\mathcal{H}} - \mathrm{d}\mathcal{P}_{\mathrm{std}} - \mathcal{P}_{\mathrm{std}}\mathrm{d})\langle\mathcal{T}\rangle \\ &= \mathrm{d}\mathcal{P}_{\mathrm{std}}\mathrm{d}\langle\mathcal{T}\rangle \\ &\subset \mathrm{d}\mathcal{P}_{\mathrm{std}}\langle\mathcal{T}\rangle \\ &\subset \mathrm{d}\mathcal{P}_{\mathrm{std}}\langle\bar{\mathcal{S}} \wedge \bar{\mathcal{S}}\rangle.\end{aligned}$$

It holds even $\mathrm{d}\langle\mathcal{T}\rangle = \mathrm{d}\mathcal{P}_{\mathrm{std}}\langle\bar{\mathcal{S}} \wedge \bar{\mathcal{S}}\rangle$ due to (6.11). We can now write (6.11) as

$$\mathcal{S}^k = \mathcal{H}^k \oplus \mathrm{d}\mathcal{P}_{\mathrm{std}}\langle\bar{\mathcal{S}} \wedge \bar{\mathcal{S}}\rangle^k \oplus \mathcal{P}_{\mathrm{std}}\langle\bar{\mathcal{S}} \wedge \bar{\mathcal{S}}\rangle^{k+1}. \tag{6.12}$$

If $\mathrm{H}^1 = 0$, this agrees with the formula from [Fio+19, Proposition 3.3]. In this case, it holds $\mathcal{S}^1 = 0$, and hence $\langle\bar{\mathcal{S}} \wedge \bar{\mathcal{S}}\rangle^{k+1}$ depends only on $\mathcal{S}^i$ for $i < k$. Therefore, we can compute $\mathcal{S}^k$ from (6.12) inductively starting with $\mathcal{S}^0 = \langle 1\rangle$.

(iii) The previous remark implies the following: Let $(V, \mathrm{d}, \wedge)$ be a non-negatively graded DGA with cyclic structure $\langle\cdot,\cdot\rangle$ of Hodge type. Suppose that $\mathrm{H}^1(V) = 0$. If $\mathrm{H}(V)$ is of finite type, then so is $\mathcal{S}(V)$.

(iv) The advantage of $\mathcal{S}(V)$ is that we have the diagram of pairing preserving DGA-quasi-isomorphisms

$$\begin{tikzcd} V & \mathcal{S}(V) \arrow[l, hook'] \arrow[r, two heads] & \mathcal{Q}(\mathcal{S}(V)). \end{tikzcd}$$

Moreover, in the case of (iii), the non-degenerate quotient is finite-dimensional, and hence a Poincaré duality model of V (see the next section).

(v) Notice that if there is a quasi-isomorphism $f : \mathrm{H}(V) \to V$, then $\mathcal{S}(V) = V$ for any Hodge decomposition with harmonic subspace $\operatorname{im} f$. ◁

Proposition 6.1.17 (Properties of $\mathcal{S}$ and $\mathcal{Q}$)**.** *Let* $(V, \mathrm{d}, \wedge)$ *be a* DGA *with a symmetric cyclic structure* $\langle\cdot,\cdot\rangle$*. Suppose that it admits a Hodge decomposition with Hodge pair* $(\mathcal{H}, \mathcal{P}_{\mathrm{std}})$*. Then* $\mathcal{S}(V)$ *and* $\mathcal{Q}(V)$ *admit Hodge decompositions with the Hodge pairs* $(\mathcal{H}, \mathcal{P}^{\mathcal{S}}_{\mathrm{std}} \coloneqq \mathcal{P}_{\mathrm{std}}|_{\mathcal{S}(V)})$ *and* $(\pi^{\mathcal{Q}}(\mathcal{H}), \mathcal{P}^{\mathcal{Q}}_{\mathrm{std}})$*, respectively, where* $\pi^{\mathcal{Q}} : V \to \mathcal{Q}(V)$ *is the canonical projection and* $\mathcal{P}^{\mathcal{Q}}_{\mathrm{std}}$ *the unique map on* $\mathcal{Q}(V)$ *satisfying* $\mathcal{P}^{\mathcal{Q}}_{\mathrm{std}} \circ \pi^{\mathcal{Q}} = \pi^{\mathcal{Q}} \circ \mathcal{P}_{\mathrm{std}}$*. Furthermore, with respect to the induced Hodge decompositions, we have*

$$\mathcal{S}(\mathcal{S}(V)) = \mathcal{S}(V), \quad \mathcal{Q}(\mathcal{Q}(V)) = \mathcal{Q}(V) \quad \text{and} \quad \mathcal{S}(\mathcal{Q}(\mathcal{S}(V))) = \mathcal{Q}(\mathcal{S}(V)). \tag{6.13}$$

Proof. The fact that $\mathcal{S} \coloneqq \mathcal{S}(V)$ admits a Hodge decomposition with Hodge pair $(\mathcal{H}, \mathcal{P}^{\mathcal{S}}_{\mathrm{std}})$ was stated in (i) of Remark 6.1.16 and is easy to check.

We prove that $\mathcal{Q} \coloneqq \mathcal{Q}(V)$ has a Hodge decomposition with Hodge pair $(\pi^{\mathcal{Q}}(\mathcal{H}), \mathcal{P}^{\mathcal{Q}}_{\mathrm{std}})$. Because $\pi^{\mathcal{Q}}$ is a quasi-isomorphism, $\pi^{\mathcal{Q}}(\mathcal{H})$ is a harmonic subspace of $\mathcal{Q}$, and we have $\ker \mathrm{d}^{\mathcal{Q}} = \pi^{\mathcal{Q}}(\mathcal{H}) \oplus \operatorname{im} \mathrm{d}^{\mathcal{Q}}$. Let $c \in C$ with $\mathrm{d}c \in V^{\perp}$ such that $\pi^{\mathcal{Q}}(c) \in \ker \mathrm{d}^{\mathcal{Q}} \cap \pi^{\mathcal{Q}}(C)$. From the cyclicity of $\langle \cdot, \cdot \rangle$ with respect to d and from $C \perp \mathcal{H} \oplus C$, it follows that $c \in V^{\perp}$, and thus $\pi^{\mathcal{Q}}(c) = 0$. Together with surjectivity of $\pi^{\mathcal{Q}}$ this implies that $\mathcal{Q} = \ker \mathrm{d}^{\mathcal{Q}} \oplus \pi^{\mathcal{Q}}(C)$. From $\langle \pi^{\mathcal{Q}}(\cdot), \pi^{\mathcal{Q}}(\cdot) \rangle^{\mathcal{Q}}$, we see that $\pi^{\mathcal{Q}}(C) \perp \pi^{\mathcal{Q}}(\mathcal{H}) \oplus \pi^{\mathcal{Q}}(C)$. Therefore, $\mathcal{Q} = \pi^{\mathcal{Q}}(\mathcal{H}) \oplus \operatorname{im} \mathrm{d}^{\mathcal{Q}} \oplus \pi^{\mathcal{Q}}(C)$ is a Hodge decomposition, and it is easy to see that its standard Hodge homotopy $\mathcal{P}^{\mathcal{Q}}_{\mathrm{std}}$ satisfies $\mathcal{P}^{\mathcal{Q}}_{\mathrm{std}} \circ \pi^{\mathcal{Q}} = \pi^{\mathcal{Q}} \circ \mathcal{P}_{\mathrm{std}}$. This defines $\mathcal{P}^{\mathcal{Q}}_{\mathrm{std}}$ uniquely because $\pi^{\mathcal{Q}}$ is surjective.

As for the relations (6.13), the first two are clear. The third can be seen as follows. Since $\pi^{\mathcal{Q}} : V \to \mathcal{Q}$ is a DGA-morphism mapping the harmonic subspaces isomorphically onto each other and commuting with d and $\mathcal{P}_{\mathrm{std}}$, the assignments $Y \subset V \mapsto \pi^{\mathcal{Q}}(Y) \subset \mathcal{Q}$ and $Z \subset \mathcal{Q} \mapsto (\pi^{\mathcal{Q}})^{-1}(Z) \subset V$ preserve Hodge subalgebras. Therefore, if $Z \subset \mathcal{Q}(\mathcal{S})$ is a Hodge subalgebra, then $(\pi^{\mathcal{Q}})^{-1}(Z) \subset \mathcal{S}$ is a Hodge subalgebra. It holds even $(\pi^{\mathcal{Q}})^{-1}(Z) = \mathcal{S}$ by minimality of $\mathcal{S}$, and hence $Z = \mathcal{Q}(\mathcal{S})$ by surjectivity of $\pi^{\mathcal{Q}}$. □

A natural question is, how does $\mathcal{Q}(\mathcal{S}(V))$ depend on the chosen Hodge pair and how does it behave under quasi-isomorphisms? The following lemmas might be useful.

Lemma 6.1.18 (Kernel of pairing-preserving morphism)**.** *Let V_1 and V_2 be vector spaces with symmetric bilinear forms $\langle \cdot, \cdot \rangle_1 : V_1 \otimes V_1 \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_2 : V_2 \otimes V_2 \to \mathbb{R}$, respectively. Let $f : V_1 \to V_2$ be a linear map such that*

$$\langle v_1, v_2 \rangle_1 = \langle f(v_1), f(v_2) \rangle_2 \quad \text{for all } v_1, v_2 \in V_1. \tag{6.14}$$

Then it holds

$$\ker f \subset V_1^{\perp} \quad \text{and} \quad f(V_1^{\perp}) \subset f(V_1)^{\perp}.$$

In particular, the following statements are true:

(a) If $\langle \cdot, \cdot \rangle_1$ is non-degenerate, then f is injective.

(b) If $\langle \cdot, \cdot \rangle_2$ is non-degenerate and f is surjective, then $\ker f = V_1^{\perp}$.

Proof. Clear. □

Lemma 6.1.19 (Injectivity on domain with non-degenerate orientation)**.** *Let $(V_1, \mathrm{d}_1, \wedge_1)$ be a non-negatively graded commutative* DGA *with an orientation o_1 in degree n such that the induced cyclic structure is non-degenerate. Let $(V_2, \mathrm{d}_2, \wedge_2)$ be any* DGA*, and let $f : V_1 \to V_2$ be a morphism of* DGA*'s. Then injectivity of $f_* : \mathrm{H}^n(V_1) \to \mathrm{H}^n(V_2)$ implies injectivity of $f : V_1 \to V_2$.*

Proof. Firstly, injectivity of a homogenous map is equivalent to degree-wise injectivity. Secondly, because V_1 is non-negatively graded and the cyclic structure of degree n is non-degenerate, we have $V_1 = V_1^0 \oplus \cdots \oplus V_1^n$. Now, suppose that $v \in V_1^k$ for some $k = 0, \ldots, n$ satisfies $f(v) = 0$. For any $w \in V^{n-k}$, the product $v \wedge w$ lies in V^n, and thus $\mathrm{d}(v \wedge w) = 0$. We compute

$$f_*[v \wedge w] = [f(v \wedge w)] = [f(v) \wedge f(w)] = 0,$$

and hence $v \wedge w = \mathrm{d}\eta$ for some $\eta \in V_1^{n-1}$ by injectivity of f_*. Consequently, we have

$$\mathrm{o}_1(v \wedge w) = \mathrm{o}_1(\mathrm{d}\eta) = 0,$$

and hence $v = 0$ by non-degeneracy of o_1. $\square$

Lemma 6.1.20 (Small subalgebra of cyclic DGA and quasi-iso.)**.** *Let* $(V_1, \mathrm{d}_1, \wedge_1, \langle\cdot,\cdot\rangle_1)$ *and* $(V_2, \mathrm{d}_2, \wedge_2, \langle\cdot,\cdot\rangle_2)$ *be non-negatively graded unital commutative* DGA*'s of finite type with non-degenerate cyclic structures of degree n (hence finite-dimensional). Let* $f : V_1 \to V_2$ *be a* DGA*-morphism such that* $f_* : (\mathrm{H}(V_1), \mathrm{o}_1^{\mathrm{H}}) \to (\mathrm{H}(V_2), \mathrm{o}_2^{\mathrm{H}})$ *is an isomorphism. Then a Hodge decomposition of* V_1 *with Hodge pair* $(\mathcal{H}_1, \mathcal{P}^1_{\mathrm{std}})$ *induces a Hodge decomposition of* V_2 *with Hodge pair* $(f(\mathcal{H}_1), \mathcal{P}^2_{\mathrm{std}})$*, where* $\mathcal{P}^2_{\mathrm{std}}$ *satisfies* $\mathcal{P}^2_{\mathrm{std}} \circ f = f \circ \mathcal{P}^1_{\mathrm{std}}$*. Consequently, f induces an isomorphism* $\mathcal{S}(V_1) \simeq \mathcal{S}(V_2)$*.*

Proof. Because f_* preserves orientation and it holds $V_1^{n+1} = 0 = V_2^{n+1}$, Proposition 6.1.3 implies that f preserves cyclic structure and hence is injective by Lemma 6.1.18. Let $V_1 = \mathcal{H}_1 \oplus \operatorname{im}\mathrm{d}_1 \oplus C_1$ be a Hodge decomposition. From the injectivity of f, it follows that

$$f(V_1) = f(\mathcal{H}_1) \oplus \mathrm{d}f(V_1) \oplus f(C_1)$$

and that the restriction of $\langle\cdot,\cdot\rangle_2$ to $f(V_1) \otimes f(V_1)$ is non-degenerate. Because V_1 is of finite type, $f(V_1)$ is of finite type, and so non-degeneracy implies Poincaré duality $f(V_1)^{*_{\mathrm{g}}} \simeq f(V_1)$. It follows that

$$V_2 = f(V_1) \oplus f(V_1)^{\perp}.$$

Cyclicity of d_2 with respect to $\langle\cdot,\cdot\rangle_2$ implies that $f(V_1)^{\perp} \subset V_2$ is a subcomplex. Because f is a quasi-isomorphism, we have $\mathrm{H}(V_1) \simeq \mathrm{H}(f(V_1)) \simeq \mathrm{H}(V_2)$; because the homology is additive, we have $\mathrm{H}(V_2) \simeq \mathrm{H}(f(V_1)) \oplus \mathrm{H}(f(V_1)^{\perp}) \simeq \mathrm{H}(V_2) \oplus \mathrm{H}(f(V_1)^{\perp})$; finally, because $\mathrm{H}(V_2)$ is of finite type, we have $\mathrm{H}(f(V_1)^{\perp}) = 0$. Because $V_2 = f(V_1) \oplus f(V_1)^{\perp}$ and $\langle\cdot,\cdot\rangle_2$ is non-degenerate, its restriction to $f(V_1)^{\perp}$ is non-degenerate too. As V_2 and hence $f(V_1)^{\perp}$

is of finite type, Proposition 6.1.10 gives a Hodge decomposition $f(V_1)^\perp = \mathrm{d}f(V_1)^\perp \oplus C_2'$. It is easy to check that

$$V_2 = f(\mathcal{H}_1) \oplus \underbrace{\left(\mathrm{d}f(V_1) \oplus \mathrm{d}f(V_1)^\perp\right)}_{= \operatorname{im} \mathrm{d}_2} \oplus \underbrace{\left(f(C_1) \oplus C_2'\right)}_{=: C_2}$$

is a Hodge decomposition of V_2. The corresponding standard Hodge homotopy clearly satisfies $\mathcal{P}^2_{\mathrm{std}} \circ f = f \circ \mathcal{P}^1_{\mathrm{std}}$, and it holds $\mathcal{P}^2_{\mathrm{std}}(f(V_1)) \subset f(V_1)$, so that $f(V_1)$ is a Hodge subalgebra. By injectivity, it follows that $\mathcal{S}(V_1) \simeq \mathcal{S}(V_2)$. $\square$

Since $\mathcal{S} = \mathcal{S}(V)$ is "small", a question whether it can be fit inside the image of a Sullivan's minimal model arose. This would imply, under some additional assumptions, that for any two Hodge decompositions of V, the non-degenerate quotients $\mathcal{Q}_1$, resp. $\mathcal{Q}_2$ of the corresponding small subalgebras $\mathcal{S}_1$, resp. $\mathcal{S}_2$ would be isomorphic as Poincaré duality algebras. Let us sketch the idea of this construction assuming that $f_1 : \Lambda U_1 \twoheadrightarrow \mathcal{S}_1$ and $f_2 : \Lambda U_2 \twoheadrightarrow \mathcal{S}_2$ are surjective Sullivan's minimal models. Uniqueness from [FOT08, Theorem 2.24] gives an isomorphism $\Lambda U \coloneqq \Lambda U_1 \simeq \Lambda U_2$ such that the following diagram commutes up to homotopy of DGA's:

$$\begin{tikzcd}
 & & \mathcal{S}_1 \arrow[r] \arrow[dl, hook] & \mathcal{Q}_1 \\
\Lambda U \arrow[urr, bend left, two heads, "f_1"] \arrow[drr, bend right, two heads, "f_2"'] & V & & \\
 & & \mathcal{S}_2 \arrow[r] \arrow[ul, hook] & \mathcal{Q}_2.
\end{tikzcd} \tag{6.15}$$

Now, $\Lambda U \xrightarrow{f_1} \mathcal{S}_1 \hookrightarrow V$ and $\Lambda U \xrightarrow{f_2} \mathcal{S}_2 \hookrightarrow V$ induce the same isomorphism on homology; this can be used to pullback the orientation o^{H} on $\mathrm{H}(V)$ to an orientation $\tilde{\mathrm{o}}$ on $\mathrm{H}(\Lambda U)$, so that all maps in (6.15) will preserve the orientation on homology. Under the assumptions $\mathrm{d}V^n = 0$ and $\mathrm{d}(\Lambda U)^n = 0$, Proposition 6.1.3 applies, and we obtain a cyclic structure $\langle\cdot,\cdot\rangle^{\Lambda U}$ on ΛU which is preserved by both f_1 and f_2 on the chain level. We denote $f_i^{\mathcal{Q}} \coloneqq \pi_i^{\mathcal{Q}} \circ f_i$ and write down the following diagram with pairing-preserving DGA-quasi-isomorphisms:

$$\begin{tikzcd}
 & (\Lambda U, \langle\cdot,\cdot\rangle^{\Lambda U}) \arrow[dl, two heads, "f_1^{\mathcal{Q}}"'] \arrow[dr, two heads, "f_2^{\mathcal{Q}}"] & \\
(\mathcal{Q}_1, \langle\cdot,\cdot\rangle_1^{\mathcal{Q}}) & & (\mathcal{Q}_2, \langle\cdot,\cdot\rangle_2^{\mathcal{Q}}).
\end{tikzcd} \tag{6.16}$$

Claim (b) of Lemma 6.1.18 implies that $\ker f_1^{\mathcal{Q}} = \ker f_2^{\mathcal{Q}} = (\Lambda U)^{\perp}$, and hence

$$\mathcal{Q}_1 \simeq \Lambda U/(\Lambda U)^{\perp} \simeq \mathcal{Q}_2.$$

Unfortunately, the next two examples show that one can not, in general, expect $\mathcal{S}_1$ and $\mathcal{S}_2$, or even $\mathcal{Q}_1$ and $\mathcal{Q}_2$ to be isomorphic and $\mathcal{S}$ to fit inside the image of the minimal model.[5]

Example 6.1.21 (Small algebras are, in general, not unique and not contained in images of Sullivan minimal models). Consider $M = \mathbb{C}\mathrm{P}^2$, and let K be the Fubini–Study form on M. For $\alpha \in \Omega^1(M)$, set $\mathrm{K}_\alpha \coloneqq \mathrm{K} + \mathrm{d}\alpha$. Then $K_\alpha \wedge K_\alpha = K \wedge K + \mathrm{d}(2\alpha \wedge K + \alpha \wedge \mathrm{d}\alpha)$. We can choose α such that $\mathrm{d}(2\alpha \wedge K + \alpha \wedge \mathrm{d}\alpha) \neq 0$. Consider the Riemannian Hodge decomposition of $\Omega(M)$ with $\mathcal{H} = \langle 1, \mathrm{K}, \mathrm{K} \wedge \mathrm{K}\rangle$. According to Remark 6.1.11, there is also a "twisted" Hodge decomposition with the harmonic subspace $\mathcal{H}_\alpha \coloneqq \langle 1, \mathrm{K}_\alpha, \mathrm{K} \wedge \mathrm{K}\rangle$.

The small subalgebra $\mathcal{S}$ for the Riemann Hodge decomposition is $\mathcal{S} = \langle 1, \mathrm{K}, \mathrm{K} \wedge \mathrm{K}\rangle$. The small subalgebra $\mathcal{S}_\alpha$ for the twisted Hodge decomposition must contain 1, K_α and both $\mathrm{K}_\alpha \wedge \mathrm{K}_\alpha$ and $\mathrm{K} \wedge \mathrm{K}$ (we require $\mathcal{H}_\alpha \subset \mathcal{S}_\alpha$ by definition). Therefore, it contains

$$\mathrm{K}_\alpha \wedge \mathrm{K}_\alpha - \mathrm{K} \wedge \mathrm{K} = \mathrm{d}(2\alpha \wedge K + \alpha \wedge \mathrm{d}\alpha)$$

and also

$$\mathrm{P}^{\alpha}_{\mathrm{std}} \mathrm{d}(2\alpha \wedge K + \alpha \wedge \mathrm{d}\alpha) = \pi_{C_\alpha}(2\alpha \wedge K + \alpha \wedge \mathrm{d}\alpha).$$

Proposition 6.1.15 asserts that these vectors, together with 1 and K_α, generate $\mathcal{S}_\alpha$ as a graded vector space. Clearly, $\mathcal{S}_\alpha$ is not isomorphic to $\mathcal{S}$ for generic α, but

$$\mathcal{Q}(\mathcal{S}_\alpha) = \langle 1, \mathrm{K}_\alpha, \mathrm{K}_\alpha \wedge \mathrm{K}_\alpha\rangle \simeq \langle 1, \mathrm{K}, \mathrm{K} \wedge \mathrm{K}\rangle = \mathcal{Q}(\mathcal{S}).$$

In fact, the previous argument works in general and implies that for $\mathbb{C}\mathrm{P}^2$, the non-degenerate quotients of two small subalgebras are isomorphic as Poincaré duality algebras (this does not hold for any M, see Example 6.1.22).

The Sullivan minimal model of M is the free DGA $\mathcal{M} \coloneqq \Lambda(\eta, \mu)$ with $|\eta| = 2$, $|\mu| = 5$, $\mathrm{d}\eta = 0$ and $\mathrm{d}\mu = \eta \wedge \eta$. A DGA-quasi-isomorphism $f : \mathcal{M} \to \Omega(M)$ is specified by its values on η and μ; for example, $f(\eta) \coloneqq \mathrm{K}$ and $f(\mu) \coloneqq 0$. We see that neither $\mathcal{S}_\alpha$ nor $\mathcal{H}_\alpha$ can lie in $\operatorname{im} f$ because $\dim(\operatorname{im} f)^4 = 1$ for any f. $\triangleleft$

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[5] Note that it is always possible to construct a surjective (non-minimal) Sullivan model $f : \Lambda U \twoheadrightarrow V$ by taking the minimal Sullivan model and inductively adding generators ξ_i and μ_i with $\mathrm{d}\mu_i = 0$ and $\mathrm{d}\xi_i = \mu_i$ (the minimality condition on a Sullivan's algebra would require $\mathrm{d}\xi_i$ to be decomposable), and mapping them to $v \in V$ and $\mathrm{d}v$, respectively. Nevertheless, the uniqueness property of non-minimal Sullivan models is much weaker, see [FOT08, Lemma 2.20].

Example 6.1.22 (Non-degenerate quotients of small subalgebras for different Hodge decompositions are, in general, not isomorphic)*.* Consider $M = \mathrm{SU}(6)$. It is a compact simply-connected Lie group of dimension 35 ($= n^2 - 1$ for $n = 6$) whose cohomology ring is freely generated by single elements in degrees 3, 5, …, 11; see [MT91, Corollary 3.11]. There is a biinvariant Riemannian metric and there are biinvariant differential forms x_3, x_5, …, x_{11} in the corresponding degrees such that

$$\mathcal{H} \coloneqq \Lambda(x_3, \dots, x_{11}) \subset \Omega(M)$$

is the algebra of harmonic forms, see [FOT08, Chapter 1].

For $\eta_6 \in \Omega^6(M)$ and $\eta_8 \in \Omega^8(M)$, which are going to be specified later, set

$$\tilde{x}_7 \coloneqq x_7 + \mathrm{d}\eta_6 \quad \text{and} \quad \tilde{x}_9 \coloneqq x_9 + \mathrm{d}\eta_8,$$

and let $\mathcal{H}'$ denote the graded vector space obtained from $\mathcal{H}$ by replacing the vectors x_7 and x_9 with $\tilde{x}_7$ and $\tilde{x}_9$, respectively. We emphasize that we are replacing just the vectors, not the products; e.g., $x_7 \wedge x_9$ is an element of $\mathcal{H}'$ but $\tilde{x}_7 \wedge \tilde{x}_9$ might not be. Let $\mathcal{S} = \mathcal{H}$ be the small subalgebra corresponding to the Riemannian Hodge decomposition, and let $\mathcal{S}'$ be the small subalgebra corresponding to a Hodge decomposition based on $\mathcal{H}'$ (such always exists by Remark 6.1.11). The following elements in degrees 15, resp. 20 must be contained in $\mathcal{S}'$:

$$\begin{aligned} y &\coloneqq \mathrm{P}'_{\mathrm{std}}(\tilde{x}_7 \wedge x_9 - x_7 \wedge x_9) \\ &= \mathrm{P}'_{\mathrm{std}}\mathrm{d}(\eta_6 \wedge x_9), \\ z &\coloneqq \tilde{x}_9 \wedge x_{11} - x_9 \wedge x_{11} \\ &= \mathrm{d}(\eta_8 \wedge x_{11}). \end{aligned}$$

Using Stokes theorem and $\mathrm{d} \circ \mathrm{P}'_{\mathrm{std}} = \pi_{\mathrm{im}\,\mathrm{d}}$, we get

$$\begin{aligned} \langle y, z \rangle &= \pm \int_M \mathrm{P}'_{\mathrm{std}}\mathrm{d}(\eta_6 \wedge x_9) \wedge \mathrm{d}(\eta_8 \wedge x_{11}) \\ &= \pm \int_M \mathrm{d}\eta_6 \wedge \eta_8 \wedge x_9 \wedge x_{11}. \end{aligned}$$

We claim that the integral can be made non-zero by a choice of η_6 and η_8. Indeed, because $x_9 \wedge x_{11}$ generates non-zero homology, there is an $m \in M$ such that $x_9(m) \wedge x_{11}(m) \neq 0$. Pick local coordinates (x^i) centered at m, and let $\alpha^I \,\mathrm{d}x^I$ be a non-zero coefficient in $x_9(m) \wedge x_{11}(m)$. Consider the complement $J = I^C$ and decompose $J = J_1 \cup J_2$ into two parts with $|J|_1 = 7$ and $|J|_2 = 8$. For some $0 \neq c \in \mathbb{R}$, set locally

$$\eta_6 \coloneqq ((x^{J_{11}} + c)\,\mathrm{dx}^{J_1 \backslash \{J_{11}\}}) \quad \text{and} \quad \eta_8 = \mathrm{dx}^{J_2},$$

and extend them to the whole of M by multiplying with a bump function which is constant non-zero in a neighborhood of m. We have achieved that the integrand $\omega \coloneqq \mathrm{d}\eta_6 \wedge \eta_8 \wedge x_9 \wedge x_{11}$ is non-zero around m. Because the intersection pairing is non-degenerate and because $\omega \neq 0$, there is a function $f \in C^\infty(M)$ such that $\int_M f\omega \neq 0$. We can now just rescale η_8 by f.

We have shown that z induces a non-zero element $\pi'_{\mathcal{Q}}(z) \in \mathcal{Q}(\mathcal{S}')^{20}$, where $\pi'_{\mathcal{Q}} : \mathcal{S}' \to \mathcal{S}'/\mathcal{S}'^{\perp}$ is the canonical projection. Now, $\pi'_{\mathcal{Q}}$ is a DGA-morphism, and hence $\pi'_{\mathcal{Q}}(z)$ is exact. Because $\mathcal{S}'$ is of Hodge type, $\pi'_{\mathcal{Q}}$ is also a quasi-isomorphism, and hence $\pi'_{\mathcal{Q}}(x_9 \wedge x_{11})$ generates non-trivial homology. It follows that $\pi'_{\mathcal{Q}}(z)$ is not a multiple of $\pi'_{\mathcal{Q}}(x_9 \wedge x_{11})$, and thus $\dim \mathcal{Q}(\mathcal{S}')^{20} \geq 2$. However, we have $\mathcal{Q}(\mathcal{S})^{20} = \mathcal{H}^{20} = \langle x_9 \wedge x_{11} \rangle$. This shows that $\mathcal{Q}(\mathcal{S})$ and $\mathcal{Q}(\mathcal{S}')$ can not be isomorphic as vector spaces. ◁

It is not hard to come up with artificial examples of oriented DGA's which are not of Hodge type.

The following lemma and proposition show that in some cases it is possible to extend a DGA to a DGA of Hodge type.

Lemma 6.1.23 (Giving partners to non-degenerates)**.** *Let* $(V, \mathrm{d}, \wedge, \mathrm{o})$ *be a unital commutative* DGA *which is non-negatively graded and oriented in degree* n*. For* $k = \lceil \frac{n}{2} \rceil$*, …,* n*, consider the following property* (P_k) *of a direct sum decomposition*

$$V = \mathcal{H} \oplus \mathrm{d}V \oplus C, \tag{6.17}$$

where $\mathcal{H}$ *is a harmonic subspace and* C *a complement of* $\ker \mathrm{d}$ *in* V *perpendicular to* $\mathcal{H}$ *with respect to the induced cyclic structure* $\langle \cdot, \cdot \rangle$*:*

(P_k) *There is a complement* E *of*

$$C^{\perp} \coloneqq \{c^{\perp} \in C \mid \langle c^{\perp}, c \rangle = 0 \text{ for all } c \in C\}$$

in C *and a homogenous linear map*

$$\rho : E^{\lceil n/2 \rceil} \oplus \cdots \oplus E^k \longrightarrow \mathrm{d}V$$

such that for all $e' \in E$ *and* $c^\perp \in C^\perp$*, the following holds:*

$$\langle e', \rho(e)\rangle = \langle e', e\rangle, \tag{6.18}$$

$$\langle c^\perp, \rho(e)\rangle = 0. \tag{6.19}$$

Suppose that V *is non-negatively graded, of finite type and satisfies* $V^0 = \mathrm{span}\{1\}$ *and* $V^1 = 0$*. Suppose that* $n \geq 5$ *and that* $(\mathrm{H}(V), \wedge, \mathrm{o}^{\mathrm{H}})$ *is a Poincaré duality algebra. Given* $\lceil \frac{n}{2} \rceil \leq l \leq n$*, suppose that* V *admits a decomposition of type* (6.17) *such that either* $l = \lceil \frac{n}{2} \rceil$ *or* $l > \lceil \frac{n}{2} \rceil$ *and* (P_{l-1}) *holds. Then there is an* $m \in \mathbb{N}_0$ *and a Sullivan* DGA

$$\Lambda \coloneqq \Lambda(w_1, \dots, w_m, z_1, \dots, z_m) \tag{6.20}$$

specified by $\deg w_i = l - 1$, $\deg z_i = l$, $\mathrm{d}z_i = 0$ *and* $\mathrm{d}w_i = z_i$ *for all* $i = 1, \dots, m$ *such that the tensor product* DGA

$$\hat{V} \coloneqq V \otimes \Lambda$$

admits an orientation $\hat{\mathrm{o}} : \hat{V} \to \mathbb{R}$ *which extends* $\mathrm{o} : V \to \mathbb{R}$ *on the canonical inclusion* $V \hookrightarrow \hat{V}$*, and there is a decomposition*

$$\hat{V} = \hat{\mathcal{H}} \oplus \mathrm{d}\hat{V} \oplus \hat{C} \tag{6.21}$$

of type (6.17) *for which* (P_l) *holds.*

Proof. The proof consists of a construction of $\hat{\mathrm{o}} : \hat{V} \to \mathbb{R}$, a construction of a harmonic subspace $\hat{\mathcal{H}}$ in $\hat{V}$, a construction of a complement $\hat{C}$ of $\ker \mathrm{d}$ in $\hat{V}$, a degreewise description of $\hat{C}$ and $\hat{C}^\perp$, a degreewise construction of a complement $\hat{E}$ of $\hat{C}^\perp$ in $\hat{C}$ and a proof of the property (P_l) for the constructed decomposition.

Construction of $\hat{\mathrm{o}}$*:* Consider the decomposition (6.17). Because $\mathrm{d} : C \to \mathrm{d}V$ is an isomorphism, we can write

$$V = \mathcal{H} \oplus \underbrace{\mathrm{d}E \oplus \mathrm{d}C^\perp}_{\mathrm{d}V} \oplus \underbrace{E \oplus C^\perp}_{C}. \tag{6.22}$$

The restriction of $\langle \cdot, \cdot \rangle$ to E is non-degenerate, and V is of finite type by assumption; hence, E is finite-dimensional. Set

$$m \coloneqq \dim E^l.$$

Let $\xi_1, \dots, \xi_m$ be a basis of E^l and $\xi^1, \dots, \xi^m$ its dual basis in E^{n-l}. The Sullivan algebra Λ can be written as a direct sum

$$\Lambda = \bigoplus_{k=0}^{\infty} \Lambda_k \quad \text{with} \quad \Lambda_k = \bigoplus_{\substack{r,m\geq 0\\ r+m=k}} \Lambda_r(w) \otimes \Lambda_m(z), \tag{6.23}$$

where $\Lambda_r(w)$ and $\Lambda_m(z)$ are the graded vector spaces generated by monomials $w_I = w_{i_1} \dots w_{i_r}$ and $z_J = z_{j_1} \dots z_{j_m}$ for all multiindices $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_m\}$, respectively. The direct sum decompositions (6.22) and (6.23) induce a direct sum decomposition of $\hat{V} = V \otimes \Lambda$ via the distributivity of $\otimes$ and $\oplus$. We denote

$$\hat{V}_k \coloneqq V \otimes \Lambda_k \quad \text{for } k \geq 0.$$

Let $\hat{\mathrm{o}} : \hat{V} \to \mathbb{R}$ be the linear map satisfying

$$\begin{aligned} \hat{\mathrm{o}}(v) &\coloneqq \mathrm{o}(v) && \text{for all } v \in V, && (6.24)\\ \hat{\mathrm{o}}(\xi^i \wedge z_j) &\coloneqq \mathrm{o}(\xi^i \wedge \xi_j) && \text{and} && (6.25)\\ \hat{\mathrm{o}}(\mathrm{d}\xi^i \wedge w_j) &\coloneqq (-1)^{\deg \xi^i + 1}\mathrm{o}(\xi^i \wedge \xi_j) && \text{for all } i,j = 1,\dots,m, && (6.26) \end{aligned}$$

and which is zero on $(\mathcal{H} \oplus \mathrm{d}V \oplus C^\perp \oplus \bigoplus_{i\geq 0, i\neq n-l} E^i) \otimes \Lambda_1(z)$, on $(\mathcal{H} \oplus C \oplus \mathrm{d}C^\perp \oplus \bigoplus_{i\geq 0, i\neq n-l} \mathrm{d}E^i) \otimes \Lambda_1(w)$ and on $\hat{V}_k$ for $k \geq 2$.

In order to show that $\hat{\mathrm{o}}$ is an orientation, we must check that $\hat{\mathrm{o}} \neq 0$ and $\hat{\mathrm{o}} \circ \mathrm{d} = 0$. The first condition is clear from $\hat{\mathrm{o}}\big|_V = \mathrm{o} \neq 0$. As for the second condition, $\hat{V}$ is generated by elements $v \wedge w_I \wedge z_J$ for $v \in V$ and multiindices I, J. It holds $\mathrm{d}\hat{V}_k \subset \hat{V}_k$ for all $k \geq 0$, and hence $\mathrm{d}\hat{V} = \bigoplus_{k=0}^{\infty} \mathrm{d}\hat{V}_k$. From the definition of $\hat{\mathrm{o}}$, we have immediately $\mathrm{d}\hat{V}_0 = \mathrm{d}V \subset \ker \mathrm{o} \subset \ker \hat{\mathrm{o}}$ and $\bigoplus_{k=2}^{\infty} \mathrm{d}\hat{V}_k \subset \ker \hat{\mathrm{o}}$. As for $\mathrm{d}\hat{V}_1$, we write $\hat{V}_1 = \operatorname{span}\{v \wedge w_j, v \wedge z_j \mid v \in V, j = 1,\dots,m\}$ as a graded vector space and compute

$$\begin{aligned} \mathrm{d}\hat{V}_1 &= \operatorname{span}\{\mathrm{d}(v \wedge w_j), \mathrm{d}(v \wedge z_j) \mid v \in V, j = 1,\dots,m\}\\ &= \operatorname{span}\{\mathrm{d}v \wedge w_j + (-1)^{\deg v} v \wedge z_j \mid v \in V, j = 1,\dots,m\}. \end{aligned} \tag{6.27}$$

Write $v \in V^{n-l}$ as $v = h + \mathrm{d}c + c^\perp + \sum_{i=1}^{m} \alpha_i \xi^i$ for $h \in \mathcal{H}^{n-l}$, $c \in C^{n-l-1}$, $c^\perp \in C^{\perp n-l}$

and $\alpha_i \in \mathbb{R}$, and compute for every $j = 1, \ldots, m$ the following:

$$\begin{aligned}\hat{\mathrm{o}}(\mathrm{d}v \wedge w_j) &= \hat{\mathrm{o}}\Big(\mathrm{d}c^\perp \wedge w_j + \sum_{i=1}^m \alpha_i \mathrm{d}\xi^i \wedge w_j\Big)\\ &= \sum_{i=1}^m \alpha_i \hat{\mathrm{o}}(\mathrm{d}\xi^i \wedge w_j)\\ &= \sum_{i=1}^m (-1)^{\deg \xi^i + 1} \alpha_i \hat{\mathrm{o}}(\xi^i \wedge \xi_j)\\ &= (-1)^{n-l+1} \sum_{i=1}^m \alpha_i \hat{\mathrm{o}}(\xi^i \wedge z_j)\\ &= (-1)^{n-l+1} \hat{\mathrm{o}}\Big((h + \mathrm{d}c + c^\perp) \wedge z_j + \sum_{i=1}^m \alpha_i \xi^i \wedge z_j\Big)\\ &= (-1)^{\deg v + 1} \hat{\mathrm{o}}(v \wedge z_j).\end{aligned}$$

Consequently, $\mathrm{d}\hat{V}_1 \subset \ker \hat{\mathrm{o}}$. This shows $\hat{\mathrm{o}} \circ \mathrm{d} = 0$.

The inclusion $V \hookrightarrow \hat{V}$ is clearly orientation preserving.

Construction of $\hat{\mathcal{H}}$: It holds $\bar{\mathrm{H}}(\Lambda) = 0$ for the reduced homology, and hence $\mathrm{H}(\hat{V}) \simeq \mathrm{H}(V) \otimes \mathrm{H}(\Lambda) = \mathrm{H}(V)$ by Künneth's formula. Because $\mathcal{H} \subset \ker \mathrm{d}$, $\mathcal{H} \cap \operatorname{im} \mathrm{d} = 0$ and $\dim(\mathcal{H}) = \dim \mathrm{H}(\hat{V})$, $\mathcal{H}$ is a complement of $\mathrm{d}\hat{V}$ in $\ker \mathrm{d}$. Therefore,

$$\hat{\mathcal{H}} \coloneqq \mathcal{H}$$

is a harmonic subspace of $\hat{V}$. Also note that $\mathrm{H}(\hat{V}) = \bigoplus_{k=0}^\infty \mathrm{H}(\hat{V}_k)$ and $\dim \mathrm{H}(\hat{V}) = \dim \mathrm{H}(\hat{V}_0)$, and hence $\mathrm{H}(\hat{V}_k) = 0$ for all $k \geq 1$.

Construction of $\hat{C}$: We construct $\hat{C}$ as a direct sum $\hat{C} = \bigoplus_{k=0}^\infty \hat{C}_k$. Set

$$\hat{C}_0 \coloneqq C.$$

For $k = 1$, define

$$\begin{aligned}\tilde{C}_1 &\coloneqq \operatorname{span}\{v \wedge w_i \mid v \in V, i = 1, \ldots, m\} \subset \hat{V}_1,\\ \hat{C}_1 &\coloneqq \{\tilde{c} - \pi(\tilde{c}) \mid \tilde{c} \in \tilde{C}_1\} \subset \hat{V}_0 \oplus \hat{V}_1,\end{aligned} \tag{6.28}$$

where $\pi : \hat{V} \to \hat{\mathcal{H}}$ is the orthogonal projection. For all $k \geq 2$, let $\hat{C}_k \subset \hat{V}_k$ be an arbitrary complement of $\ker \mathrm{d}$ in $\hat{V}_k$ as a graded vector space.

We show first that $\hat{C}_i$ for $i \geq 0$ are disjoint. Clearly, $\hat{C}_j \cap \sum_{i=0, i\neq j}^\infty \hat{C}_i = 0$ for all $j \geq 2$.

Because $(\mathcal{H}+C)\cap\tilde{C}_1 = 0$ and $\mathcal{H}\cap C = 0$, it holds $\hat{C}_1\cap\hat{C}_0 = 0$, and $(\hat{C}_0+\hat{C}_1)\cap\sum_{i\geq 2}\hat{C}_i = 0$ implies that $\hat{C}_j\cap\sum_{i=0,i\neq j}^{\infty}\hat{C}_i = 0$ holds also for $j = 0$, 1.

We show that $\hat{C} = \bigoplus_{k\geq 0}\hat{C}_k$ is a complement of $\ker \mathrm{d}$ in $\hat{V}$. For $k\geq 2$, $\hat{C}_k$ are complements of $\ker \mathrm{d}$ in $\hat{V}_k$ by construction. For $k = 0$, $\hat{C}_0\oplus\ker \mathrm{d}\cap\hat{V}_0 = \hat{V}_0$ follows from (6.17). For $k = 1$, we compare (6.27) and (6.28) to see that $\tilde{C}_1\oplus \mathrm{d}\hat{V}_1 = \hat{V}_1$. Because $\mathcal{H}\subset\ker \mathrm{d}$, it is easy to see that $\hat{C}_0\oplus\hat{C}_1$ is a complement of $\ker \mathrm{d}$ in $\hat{V}_0\oplus\hat{V}_1$.

Finally, $\hat{C}_0\perp\hat{\mathcal{H}}$ holds by (6.17), $\hat{C}_1\perp\mathcal{H}$ holds by the construction of $\hat{C}_1$ from $\tilde{C}_1$ and $\hat{C}_k\perp\mathcal{H}$ follows from the definition of $\hat{\mathrm{o}}$.

Degreewise description of $\hat{C}$ *and* $\hat{C}^{\perp}$*:* We are interested in $\hat{C}^i$ for $0\leq i\leq l$. For all $k\geq 2$, the graded vector space $\hat{C}_k$ is concentrated in degrees $i\geq 2(l-1)$. But $2(l-1)>l$ due to $n\geq 5$. Therefore, $\hat{C}_k^i = 0$ for $k\geq 2$ and $0\leq i\leq l$. We denote $\bar{V} := \bigoplus_{i=1}^{\infty}V^i$ and write $\tilde{C}_1 = \Lambda_1(w)\oplus(\bar{V}\otimes\Lambda_1(w))$. Because $V^1 = 0$, the graded vector space $\bar{V}\otimes\Lambda_1(w)$ is concentrated in degrees $i\geq 2+(l-1) = l+1$. We obtain

$$\hat{C}^i = (\hat{C}_0\oplus\hat{C}_1)^i = \begin{cases} C^i & \text{for } 0\leq i\leq l-1,\\ C^i+\mathrm{span}\{w_j\mid j=1,\dots,m\} & \text{for } i = l-1,\\ C^i & \text{for } i = l.\end{cases} \tag{6.29}$$

Here we used that $\hat{C}_1^{l-1} = \tilde{C}_1^{l-1} = \mathrm{span}\{w_i\mid i = 1,\dots,m\}$, which is true because $\Lambda_1(w)\perp\mathcal{H}$ from the definition of $\hat{\mathrm{o}}$.

We are now interested in $\hat{C}^{\perp i}$ for $n-l\leq i\leq l$. Note that $n-l\leq i\leq l$ is equivalent to $n-l\leq n-i\leq l$. By definition, $\hat{\mathrm{o}}$ vanishes on $C\wedge\Lambda_1(w) = C\otimes\Lambda_1(w)$ and $\Lambda_1(w)\wedge\Lambda_1(w)\subset\Lambda_2(w)$. Looking at (6.29), we see the following:

$$\hat{C}^{\perp i} = \begin{cases} C^{\perp i} & \text{for } n-l\leq i\leq l-2,\\ C^{\perp i}+\mathrm{span}\{w_i\mid i=1,\dots,m\} & \text{for } i = l-1,\\ C^{\perp i} & \text{for } i = l.\end{cases} \tag{6.30}$$

Because $\hat{V}$ is non-negatively graded, it holds $\hat{C}^{\perp i} = \hat{C}^i$ for $i>n$. Notice that $\hat{C}^{\perp i}$ might be smaller than C^i for $0\leq i\leq n-l-1$. The reason for this is a possible existence of $v_1, v_2\in V$ such that $v_1\wedge v_2$ has a non-trivial $\mathrm{d}E$-component and $\langle v_1, v_2\wedge w_i\rangle = \hat{\mathrm{o}}((v_1\wedge v_2)\wedge w_i)\neq 0$.

Construction of $\hat{E}$*:* Because $\hat{V}_i\wedge\hat{V}_j\subset\hat{V}_{i+j}$ and $\hat{V}_k\subset\ker\hat{\mathrm{o}}$ for $k\geq 2$, we have $\hat{C}_k\subset\hat{C}^{\perp}$

for all $k \geq 2$. It follows that

$$\hat{C}^{\perp} = (\hat{C}_0 \oplus \hat{C}_1) \cap \hat{C}^{\perp} \oplus \bigoplus_{k \geq 2} \hat{C}_k,$$

and hence it is enough to construct $\hat{E} \subset \hat{C}_0 \oplus \hat{C}_1$ such that $\hat{E} \oplus (\hat{C}_0 \oplus \hat{C}_1) \cap \hat{C}^{\perp} = \hat{C}_0 \oplus \hat{C}_1$. Because $E \cap \hat{C}^{\perp} \subset E \cap C^{\perp} = 0$, we can get $\hat{E}$ by extending E. For $0 \leq i \leq n$, we define

$$\hat{E}^i := \begin{cases} E^i \oplus \operatorname{span}\{c_1^i, \dots, c_{k_i}^i\} \text{ for } c_1^i, \dots, c_{k_i}^i \in C^i & \text{for } 0 \leq i \leq n - l - 1, \\ E^i & \text{for } n - l \leq i \leq l, \\ E^i \oplus \operatorname{span}\{\hat{v}_1, \dots, \hat{v}_k\} \text{ for } \hat{v}_1^i, \dots, \hat{v}_{k_i}^i \in (\hat{C}_0 \oplus \hat{C}_1)^i & \text{for } l + 1 \leq i \leq n, \end{cases} \tag{6.31}$$

where the existence of c_j^i and $\hat{v}_j^i$ and the fact that $\hat{C} = \hat{E} \oplus \hat{C}^{\perp}$ are justified by (6.29), (6.30) and (6.19).

Property (P_l)*:* We define $\hat{\rho} : \hat{E}^{\lceil \frac{n}{2} \rceil} \oplus \dots \oplus \hat{E}^l \to \mathrm{d}\hat{V}$ by $\hat{\rho} := \rho$ on $\hat{E}^{\lceil \frac{n}{2} \rceil} \oplus \dots \oplus \hat{E}^{l-1} = E^{\lceil \frac{n}{2} \rceil} \oplus \dots \oplus E^{l-1}$ and by

$$\hat{\rho}(\xi_i) := z_i \quad \text{for all } i = 1, \dots, m.$$

Conditions (6.18) and (6.19) are checked easily using (P_{l-1}), (6.29), (6.30), (6.31) and the definition of ô. ☐

Questions 6.1.24. (i) Given a cochain complex (V, d) with a symmetric pairing $\langle \cdot, \cdot \rangle$, which conditions have to be satisfied by the maps $\pi_{\mathcal{H}}$, $\mathcal{P}_{\mathrm{std}} : V \to V$ so that $V = \operatorname{im} \pi_{\mathcal{H}} \oplus \operatorname{im} \mathrm{d} \oplus \operatorname{im} \mathcal{P}_{\mathrm{std}}$ is a Hodge decomposition with Hodge pair $(\operatorname{im} \pi_{\mathcal{H}}, \mathcal{P}_{\mathrm{std}})$? This would characterize Hodge decompositions in terms of Hodge pairs.

(ii) It should be possible to prove Lemma 6.1.23 also for $n \leq 4$ by hand. Check that!

6.2. Poincaré DGA's and Poincaré duality models

In this section, we restrict to non-negatively graded unital commutative DGA's, which we denote by $\mathrm{CDGA}_1^{\geq 0}$. In this case, the notions of orientation and of cyclic structure agree by Proposition 6.1.4.

We modify and combine definitions from [Fio+19] and [LS08] as follows.

Definition 6.2.1 (Dif. Poincaré duality algebra, PDGA and formality)**.** *A* differential Poincaré duality algebra of degree n *is a* $\mathrm{CDGA}_1^{\geq 0}$ $(V, \mathrm{d}, \wedge)$ *of finite type with orientation* o *in degree n such that the induced pairing on V satisfies Poincaré duality. If* $\mathrm{d}=0$*, we call it just* Poincaré duality algebra.

A Poincaré DGA (shortly PDGA) of degree n *is a* $\mathrm{CDGA}_1^{\geq 0}$ $(V,\mathrm{d},\wedge)$ *together with an orientation* $\mathrm{o}^{\mathrm{H}}:\mathrm{H}(V)\to\mathbb{R}$ *of degree n which makes* $\mathrm{H}(V)$ *into a Poincaré duality algebra.*

A morphism of PDGA's $(V_1,\mathrm{d}_1,\wedge_1,\mathrm{o}_1^{\mathrm{H}})$ *and* $(V_2,\mathrm{d}_2,\wedge_2,\mathrm{o}_2^{\mathrm{H}})$ *is a* DGA*-morphism* $f: V_1\to V_2$ *such that the induced map* $f_*:\mathrm{H}(V_1)\to\mathrm{H}(V_2)$ *preserves orientation, i.e., it holds* $\mathrm{o}_2^{\mathrm{H}}\circ f_*=\mathrm{o}_1^{\mathrm{H}}$.

A quasi-isomorphism (or weak equivalence) *of* PDGA*'s is a morphism of* PDGA*'s* $f:V_1\to V_2$ *such that* $f_*:\mathrm{H}(V_1)\to\mathrm{H}(V_2)$ *is an isomorphism of oriented* DGA*'s.*

Two PDGA*'s are* weakly homotopy equivalent (or isomorphic in the homotopy category) *if they are connected by a zig-zag of* PDGA*-quasi-isomorphisms.*

A PDGA *is* formal *if it is weakly homotopy equivalent (as a* PDGA*) to its homology.*

It follows from Proposition 6.1.4 that a differential Poincaré duality algebra according to Definition 6.2.1 is precisely a cyclic DGA from Part I. In particular, it is finite dimensional. However, if we relax finite type, unitality or commutativity, we obtain a different notion.

Remark 6.2.2 (Frobenius algebra)*.* A differential Poincaré duality algebra, resp. a cyclic DGA, is precisely a finite-dimensional symmetric dg-Frobenius algebra from [Val12, p. 13] or [CHV06, Theorem 1.1]. ◁

Definition 6.2.3 (Poincaré duality model)**.** *A* Poincaré duality model *of a* PDGA $(V,\mathrm{d},\wedge,\mathrm{o}^{\mathrm{H}})$ *is a differential Poincaré duality algebra* $(\mathcal{M},\mathrm{d}^{\mathcal{M}},\wedge^{\mathcal{M}},\mathrm{o}^{\mathcal{M}})$ *which is weakly homotopy equivalent to V as a* PDGA.

We call a Poincaré duality model small *if* $\mathcal{Q}(\mathcal{S}(\mathcal{M}))\simeq\mathcal{M}$ *for every Hodge decomposition.*

Remark 6.2.4 (On Poincaré duality models)*.* (i) The definition of a Poincaré duality model in [LS08] requires only weak homotopy equivalence of DGA's, i.e., it does not require quasi-isomorphisms to preserve orientation on homology.

(ii) In general, a Sullivan minimal model ΛU fails easily to be a Poincaré duality model because it often has non-zero elements in degree $> n$, e.g., powers of even generators; see Example 6.2.7 for $\mathbb{S}^2$.

For a compact connected Lie group G, the subalgebra of harmonic forms $\mathcal{H}$ for any biinvariant Riemannian metric is isomorphic to a free algebra on odd generators, see [FOT08, Chapter 1]. Therefore, $\mathcal{H}$ with zero differential and the induced cyclic structure is at the same time the Sullivan minimal model and a Poincaré duality model for $\Omega(G)$.

(iii) Poincaré duality models are not “strongly unique” in the sense that two Poincaré duality models of the same algebra must not be isomorphic; see Example 6.1.22 for SU(6). However, there is a “weak uniqueness” statement in Proposition 6.2.9 below. The situation is similar to the situation with Sullivan and minimal Sullivan models; see [FOT08]. We introduced “smallness” in Definition 6.2.3 as a candidate for a minimality condition on a Poincaré duality model which might imply its “strong uniqueness”; see Question 6.2.14. ◁

Consider the functor from PDGA's to DGA's which forgets the orientation on homology. We have the following trivial yet somewhat surprising observation.

Proposition 6.2.5 (PDGA-formality is the same as DGA-formality)**.** *A Poincaré* DGA $(V, \mathrm{d}, \wedge, \mathrm{o}^{\mathrm{H}})$ *is formal (as a* PDGA*) if and only if it is formal as a* DGA*.*

Proof. The “only if” part is clear.

As for the “if” part, let

$$V \longleftarrow \bullet \cdots \bullet \longrightarrow \mathrm{H}(V) \tag{6.32}$$

be a weak homotopy equivalence of DGA's. Denote by $f : \mathrm{H}(V) \to \mathrm{H}(V)$ the isomorphism on homology induced by (6.32) from the left to the right. We adjoin $\mathrm{H}(V)$ to the right of (6.32) to obtain the homotopy

$$V \longleftarrow \bullet \cdots \bullet \longrightarrow \mathrm{H}(V) \xrightarrow{f^{-1}} \mathrm{H}(V) \tag{6.33}$$

whose induced map on homology from the left to the right is the identity. Therefore, we can orient homologies of the inner nodes of (6.33) so that all maps preserve orientation on homology. ☐

The next proposition will be used to show the existence of Poincaré duality models.

Proposition 6.2.6 (Extension of Hodge type)**.** *Let V be a* PDGA *of degree $n \geq 5$ which is of finite type and satisfies $V^0 = \mathrm{span}\{1\}$ and $V^1 = 0$. Then it is a retract of an oriented* DGA $\mathrm{E}(V)$ *of Hodge type in the category of* PDGA*'s.*

Proof. Pick an arbitrary harmonic subspace $\mathcal{H}$ and an arbitrary complement C of $\ker \mathrm{d}$ in V. If C is not perpendicular to $\mathcal{H}$, replace it with $\{c - \pi(c) \mid c \in C\}$, where $\pi : V \to \mathcal{H}$ is the orthogonal projection. We start with $l = \lceil \frac{n}{2} \rceil$ and apply Lemma 6.1.23 inductively to get an extension $\hat{V} = V \otimes \Lambda$ which admits a decomposition

$$\hat{V} = \hat{\mathcal{H}} \oplus \mathrm{d}\hat{V} \oplus \hat{C} \tag{6.34}$$

of type (6.17) such that there is a complement $\hat{E}$ of $\hat{C}^\perp$ in $\hat{C}$ and a linear map $\hat{\rho}$: $\hat{E}^{\lceil n/2 \rceil} \oplus \cdots \oplus \hat{E}^n \to \mathrm{d}\hat{V}$ such that (6.18) and (6.19) hold. We consider a linear map

$$\kappa : \hat{C} \longrightarrow \mathrm{d}\hat{V}$$

such that

$$\kappa(e) = \begin{cases} 0 & \text{for } e \in \hat{E}^i \text{ with } i < \lceil \frac{n}{2} \rceil, \\ \hat{\rho}(e) & \text{for } e \in \hat{E}^i \text{ with } i > \lceil \frac{n}{2} \rceil, \end{cases} \quad \text{and} \quad \kappa(c^\perp) = 0 \quad \text{for } c^\perp \in \hat{C}^\perp.$$

The case $n = 2k$ and $e \in \hat{E}^k$ is specified as follows. If k is even, then $\langle \cdot, \cdot \rangle : \hat{E}^k \otimes \hat{E}^k \to \mathbb{R}$ is an inner product, and there is an orthonormal basis $\eta_1, \ldots, \eta_m$ for some $m \in \mathbb{N}$. We require

$$\kappa(\eta_i) = \frac{1}{2}\hat{\rho}(\eta_i) \quad \text{for all } i = 1, \ldots, m. \tag{6.35}$$

If k is odd, then $\langle \cdot, \cdot \rangle : \hat{E}^k \otimes \hat{E}^k \to \mathbb{R}$ is a symplectic form, and there is a symplectic basis $\eta_1, \theta_1, \ldots, \eta_m, \theta_m$ for some $m \in \mathbb{N}$. We use the convention $\langle \theta_i, \eta_j \rangle = \delta_{ij}$ for $i, j = 1, \ldots, m$. We require

$$\kappa(\eta_i) = \hat{\rho}(\eta_i) \quad \text{and} \quad \kappa(\theta_i) = 0 \quad \text{for } i = 1, \ldots, m. \tag{6.36}$$

Let

$$\hat{C}' \coloneqq \{c - \kappa(c) \mid c \in \hat{C}\}.$$

This is a complement of $\ker \mathrm{d}$ in $\hat{V}$ perpendicular to $\hat{\mathcal{H}}$ because $\hat{C}$ is and $\operatorname{im} \kappa \subset \mathrm{d}\hat{V}$. Given homogenous $c_1, c_2 \in \hat{C}$ with $\deg c_1 + \deg c_2 = n$ and $\deg c_1 \leq \deg c_2$, write $c_1 = c_1^\perp + e_1$ and $c_2 = c_2^\perp + e_2$ for $c_1^\perp, c_2^\perp \in \hat{C}^\perp$ and $e_1, e_2 \in \hat{E}$, and compute

$$\begin{aligned} \langle c_1 - \kappa(c_1), c_2 - \kappa(c_2) \rangle &= \langle c_1, c_2 \rangle - \langle \kappa(c_1), c_2 \rangle - \langle c_1, \kappa(c_2) \rangle \\ &= \underbrace{\langle e_1, e_2 \rangle - \langle \kappa(e_1), e_2 \rangle - \langle e_1, \kappa(e_2) \rangle}_{=:(*)} \\ &\quad - \underbrace{\langle \kappa(e_1), c_2^\perp \rangle - \langle c_1^\perp, \kappa(e_2) \rangle}_{=:(**)}. \end{aligned}$$

Now, $(**) = 0$ because of (6.19). As for $(*)$, if $\deg c_1 < \deg c_2$, then

$$\begin{aligned} (**) &= \langle e_1, e_2 \rangle - \langle e_1, \kappa(e_2) \rangle \\ &= \langle e_1, e_2 \rangle - \langle e_1, \hat{\rho}(e_2) \rangle \end{aligned}$$

$$= 0$$

because of (6.18). If $\deg c_1 = \deg c_2 = k$ and k is even, we plug in the orthonormal basis and get using (6.35) that

$$\begin{aligned} e_1 = \eta_i,\ e_2 = \eta_j : \qquad (**) &= \langle \eta_i, \eta_j \rangle - \langle \kappa(\eta_i), \eta_j \rangle - \langle \eta_i, \kappa(\eta_j)\rangle \\ &= \langle \eta_i, \eta_j \rangle - \langle \eta_j, \kappa(\eta_i) \rangle - \langle \eta_i, \kappa(\eta_j)\rangle \\ &= \langle \eta_i, \eta_j \rangle - \frac{1}{2}\langle \eta_j, \hat{\rho}(\eta_i) \rangle - \frac{1}{2}\langle \eta_i, \hat{\rho}(\eta_j)\rangle \\ &= \langle \eta_i, \eta_j \rangle - \frac{1}{2}\langle \eta_j, \eta_i \rangle - \frac{1}{2}\langle \eta_i, \eta_j\rangle \\ &= 0. \end{aligned}$$

If k is odd, we plug in the symplectic basis and get using (6.36) that

$$\begin{aligned} e_1 = \eta_i,\ e_2 = \eta_j : \qquad (**) &= \langle \eta_j, \kappa(\eta_i) \rangle - \langle \eta_i, \kappa(\eta_j)\rangle \\ &= \langle \eta_j, \hat{\rho}(\eta_j) \rangle - \langle \eta_i, \hat{\rho}(\eta_j)\rangle \\ &= 0, \\ e_1 = \theta_i,\ e_2 = \eta_j : \qquad (**) &= \langle \theta_i, \eta_j \rangle - \langle \theta_i, \kappa(\eta_j)\rangle \\ &= \langle \theta_i, \eta_j \rangle - \langle \theta_i, \hat{\rho}(\eta_j)\rangle \\ &= 0, \\ e_1 = \theta_i,\ e_2 = \theta_j : \qquad (**) &= 0. \end{aligned}$$

This shows that $\hat{C} \perp \hat{C}$, and hence (6.34) is a Hodge decomposition.

Finally, because $\hat{V} = V \otimes \Lambda$ as a DGA, both the inclusion $\iota : V \to \hat{V}$ of V into $\hat{V}_0$ and the projection $\pi : \hat{V} \to V$ from $\hat{V}_0$ onto V are DGA morphisms. Because $\pi \circ \iota = \mathbb{1}$ and because ι_* is an orientation preserving isomorphism, π_* is an orientation preserving isomorphism as well. Therefore, V is a retract of $\mathrm{E}(V) := \hat{V}$ in the category of PDGA's. $\square$

The next example shows that the Sullivan minimal model is sometimes of Hodge type.

Example 6.2.7 (Sullivan minimal model of $\mathbb{S}^2$ is of Hodge type). The Sullivan minimal model of $\mathbb{S}^2$ is the free graded commutative algebra $\mathcal{M} := \Lambda(\eta_2, \eta_3)$ with $|\eta_2| = 2$, $|\eta_3| = 3$, $\mathrm{d}\eta_2 = 0$ and $\mathrm{d}\eta_3 = \eta_2 \wedge \eta_2$. From degree reasons, it holds $\mathcal{M} = \mathrm{span}\{\eta_2^k \eta_3^l \mid k \geq 0, l \in \{0,1\}\}$ as a graded vector space. We have a canonical decomposition $\mathcal{M} = \mathcal{H} \oplus \mathrm{d}\mathcal{M} \oplus C$, where $\mathcal{H} = \mathrm{span}\{\eta_2\}$, $\mathrm{d}\mathcal{M} = \mathrm{span}\{\eta_2^k \mid k \geq 2\}$ and $C = \mathrm{span}\{\eta_2^k \eta_3 \mid k \geq 0\}$. We define an orientation $\mathrm{o} : \mathcal{M} \to \mathbb{R}$ in degree 2 by $\mathrm{o}(\eta_2) := 1$ on $\mathcal{H}$ and by 0 on $\mathrm{d}\mathcal{M}$ and C.

It is easy to see that $C \perp \mathcal{H}$ and $C \perp C$ with respect to the induced cyclic structure $\langle\cdot,\cdot\rangle$. Consider the DGA-quasi-isomorphism $f : \mathcal{M} \to \Omega(\mathbb{S}^2)$ defined by $f(\eta_2) \coloneqq \mathrm{Vol}$ and $f(\eta_3) \coloneqq 0$. Clearly, it is orientation preserving. We have $\mathcal{M}/\mathcal{M}^\perp \simeq \Lambda(\eta_2)$ $\lhd$

The following proposition about the existence of a Poincaré dualiy model of a Poincaré DGA V with $\mathrm{H}^1(V) = 0$ was originally proven in [LS08, Theorem 1.1]. It was formulated in the category of DGA's, i.e., not checking whether the arrows are orientation preserving on homology. The idea was to construct an extension $\mathrm{E}(\Lambda U)$ of the Sullivan minimal model ΛU of V and an orientation on it such that the degenerate subspace is acyclic; the Poincaré duality model is then obtained by taking the quotient. The extension is constructed by adding elements which kill the so called orphans.

By Proposition 6.1.13, we know that $V^\perp$ is acyclic if and only if V is of Hodge type (V needs to be of finite type for the direct implication). Based on this, we give a new construction of $\mathrm{E}(\Lambda U)$ using Lemma 6.1.23, i.e., by adding exact partners to non-degenerates. Our construction works for $n \geq 5$, whereas the assumption of [LS08] is $n \geq 7$. We also do not need $\mathrm{d}(\Lambda U)^2 = 0$, although it follows from $\mathrm{H}^1(V) = 0$. It is also clear from our construction that the arrows preserve orientation on homology. However, this can be checked for the construction of [LS08] as well.

Proposition 6.2.8 (Existence of Poincaré duality model for $\mathrm{H}^1 = 0$)**.** *A Poincaré* DGA V *with* $\mathrm{H}^0(V) = \mathrm{span}\{1\}$ *and* $\mathrm{H}^1(V) = 0$ *admits a Poincaré duality model* $\mathcal{M}$*.*

Proof. If o^{H} comes from a pairing on V which is of Hodge type, then we can take $\mathcal{M} = \mathcal{Q}(\mathcal{S}(V))$. The weak homotopy equivalence of PDGA's looks like

$$\begin{tikzcd}
 & \mathcal{S}(V) \arrow[dl, two heads] \arrow[dr, hook] & \\
\mathcal{Q}(\mathcal{S}(V)) & & V.
\end{tikzcd} \tag{6.37}$$

If V is not of Hodge type, we proceed as follows. Let n be the degree of the orientation on $\mathrm{H}(V)$. If $n \leq 6$, then V is formal as a DGA by [Mil79], and we can take $\mathcal{M} = \mathrm{H}(V)$ by Proposition 6.2.5. The weak homotopy equivalence of PDGA's looks like

$$\begin{tikzcd}
 & \Lambda U \arrow[dl] \arrow[dr] & \\
\mathrm{H}(V) & & V,
\end{tikzcd} \tag{6.38}$$

where ΛU is the Sullivan minimal model of V. The Sullivan minimal model $W \coloneqq \Lambda U$ is of finite type and satisfies $W^0 = \mathbb{R}$, $W^1 = 0$ and $\mathrm{d}W^2 = 0$. Suppose that $n \geq 7$.

Let $\mathrm{E}(W)$ be the extension of W of Hodge type either from Proposition 6.2.6 or from [LS08, Section 4]. This extension is of finite type, the inclusion $W \hookrightarrow \mathrm{E}(W)$ is a quasi-isomorphism of DGA's and $\mathrm{E}(W)^{\perp}$ is acyclic. Moreover, $W \hookrightarrow \mathrm{E}(W)$ preserves orientation on homology. We take $\mathcal{M} = \mathcal{Q}(\mathrm{E}(\Lambda U))$ and obtain the following weak homotopy equivalence of PDGA's:

$$\begin{tikzcd}
 & \Lambda U \arrow[dl] \arrow[dr] & \\
\mathcal{Q}(\mathrm{E}(\Lambda V)) & & V.
\end{tikzcd} \tag{6.39}$$

This proves the proposition. ☐

The following is mostly [LS08, Theorem 7.1]. In addition, we check that the orientation on homology is preserved. Also, by using our extension of Hodge type, we can improve from $n \geq 7$ to $n \geq 5$.

Proposition 6.2.9 ("Weak uniqueness" of Poincaré duality model)**.** *Let V_1 and V_2 be differential Poincaré duality algebras of degree n which are weakly homotopy equivalent as* PDGA*'s. Suppose that $\mathrm{H}^0(V_1) = \mathrm{H}^0(V_2) = \mathrm{span}\{1\}$ and $\mathrm{H}^1(V_1) = \mathrm{H}^1(V_2) = 0$. In addition, suppose that $\mathrm{H}^2(V_1) = \mathrm{H}^2(V_2) = 0$, $V_1^1 = V_2^1 = 0$ and $n \geq 5$. Then there is a differential Poincaré duality algebra V_3 and* PDGA*-quasi-isomorphisms*[6]

$$\begin{tikzcd}
 & V_3 & \\
V_1 \arrow[ur] & & V_2. \arrow[ul]
\end{tikzcd} \tag{6.40}$$

Proof. By the assumption, there is $k \geq 1$ and a zig-zag of PDGA-quasi-isomorphisms

$$V_1 \longleftarrow Z_1 \longrightarrow Z_2 \longleftarrow Z_3 \longrightarrow Z_4 \longleftarrow \cdots \longleftarrow Z_k \longrightarrow V_2. \tag{6.41}$$

Consider the Sullivan minimal model $\Lambda U \to Z_2$ and use the Lifting Lemma [FOT08, Lemma 2.15] to construct DGA-quasi-isomorphisms $\Lambda U \to Z_1$ and $\Lambda U \to Z_3$ such that the diagram

$$\begin{tikzcd}
 & \Lambda U \arrow[dl] \arrow[d] \arrow[dr] & \\
Z_1 \arrow[r] & Z_2 & Z_3 \arrow[l]
\end{tikzcd}$$

---

[6] These are automatically injective and orientation preserving.

commutes up to homotopy of DGA's. It is easy to see that there is an orientation on $\mathrm{H}(\Lambda U)$ such that all morphisms preserve orientation on homology. Therefore, we can replace the segment $V_1 \longleftarrow Z_1 \longrightarrow Z_2 \longleftarrow Z_3 \longrightarrow Z_4$ in (6.41) by $V_1 \longleftarrow \Lambda U \longrightarrow Z_4$. Repeating this process, we can shorten (6.41) to

$$\begin{tikzcd} & \Lambda U \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ V_1 & & V_2, \end{tikzcd} \tag{6.42}$$

where f_1 and f_2 are PDGA-quasi-isomorphisms. In order to take $\mathcal{Q}(\mathrm{E}(\Lambda U))$ and obtain a zig-zag with three terms, we have to revert the arrows in (6.42). The trick from [LS08] is the following:

Consider the relative minimal model of the multiplication $\mu : \Lambda U \otimes \Lambda U \to \Lambda U$; from [FOT08, Example 2.48], it is given by

$$\begin{tikzcd} \Lambda U \otimes \Lambda U \arrow[rr, "\mu"] \arrow[drr, "i"] & & \Lambda U \\ & & M(\mu) \coloneqq \Lambda U \otimes \Lambda U \otimes \Lambda(U[1]), \arrow[u, "p"] \end{tikzcd} \tag{6.43}$$

where i is the inclusion into the first two factors, which is a cofibration, and p is a surjective quasi-isomorphism. Let $\iota_i : \Lambda U \to \Lambda U \otimes \Lambda U$ for $i = 1, 2$ be the inclusions to the first and the second factor, respectively. Because $\mu \circ \iota_i = \mathbb{1}$ and p is a quasi-isomorphism, the maps $i \circ \iota_i : \Lambda U \to M(\mu)$ for $i = 1, 2$ are quasi-isomorphisms. Moreover, it is easy to see that $\mathrm{H}(M(\mu))$ inherits an orientation such that p_* and $(i \circ \iota_i)_*$ are orientation preserving. To transfer this situation to V_i, we use the diagram

$$\begin{tikzcd} \Lambda U \arrow[r, "f_i"] \arrow[d, "\iota_i"] & V_1 \arrow[d, "\iota_i^V"] \\ \Lambda U \otimes \Lambda U \arrow[r, "f_1 \otimes f_2"] \arrow[d, "i"] & V_1 \otimes V_2 \arrow[d, "g_2"] \\ M(\mu) \arrow[r, "g_1"] & \tilde{V}_3 \coloneqq M(\mu) \otimes_{\Lambda U \otimes \Lambda U} (V_1 \otimes V_2), \end{tikzcd} \tag{6.44}$$

where $\iota_i^V : V_i \to V_1 \otimes V_2$ for $i = 1, 2$ are inclusions. The lower square, i.e., the maps g_1, g_2 and the DGA $\tilde{V}_3$, is a pushout diagram (see [Men15, Example 1.4]). According to [Law15], the model category of $\mathrm{CDGA}_1^{\geq 0}$ is proper, and hence pushouts along cofibrations preserve quasi-isomorphisms. Therefore, $f_1 \otimes f_2$ being a quasi-isomorphism implies that g_1 is a quasi-isomorphism. We push the orientation to $\mathrm{H}(\tilde{V}_3)$ via g_{1*}. Since $i \circ \iota_i$, g_1 and f_i are quasi-isomorphisms preserving orientation on homology, it follows that $h_i := g_2 \circ \iota_i^V : V_i \to V_3$ are quasi-isomorphisms preserving orientation of homology as well. It holds

$$\tilde{V}_3 \simeq \Lambda(U[1]) \otimes V_1 \otimes V_2.$$

Clearly, $\tilde{V}_3$ is of finite type. Using $\mathrm{H}^1(V_i) = \mathrm{H}^2(V_i) = 0$, we have $U^1 = U^2 = 0$, and hence $(\Lambda(U[1]))^1 = 0$. This together with $V_i^1 = 0$ implies that $\tilde{V}_3^1 = 0$. Therefore, all conditions for an application of the Hodge extension E from Proposition 6.2.6 are satisfied, and we can set

$$V_3 := \mathcal{Q}(\mathrm{E}(\tilde{V}_3)).$$

This finishes the proof. □

Conjecture 6.2.10. *The additional assumptions of Proposition 6.2.9 can be dropped.*

Remark 6.2.11 (Weak uniqueness in the case of $\mathrm{H}^1(V) = 0$ and $n \le 3$). For $n = 1$, there is no differential Poincaré duality algebra V with $\mathrm{H}^1(V) = 0$.

For $n = 2$, a general differential Poincaré duality algebra can be written in terms of its Hodge decomposition as

$$\begin{aligned} V^2 &= \mathrm{span}\{\mathrm{Vol}\} \oplus \mathrm{d}C^1 \\ V^1 &= \mathrm{d}C^0 \oplus C^1 \\ V^0 &= \mathrm{span}\{1\} \oplus C^0. \end{aligned}$$

Now, $\mathcal{H} = \mathrm{span}\{1\} \oplus \mathrm{span}\{\mathrm{Vol}\}$ is a dg-subalgebra which is itself a Poincaré duality algebra. Therefore, two differential Poincaré duality algebras with $\mathrm{H}(V_1) \simeq \mathrm{H}(V_2)$ and $\mathrm{H}^1(V_i) = 0$ are connected via the zig-zag

$$\begin{tikzcd} & \mathcal{H} \arrow[dl] \arrow[dr] & \\ V_1 & & V_2. \end{tikzcd}$$

For $n = 3$, we have

$$\begin{aligned} V^3 &= \mathrm{span}\{\mathrm{Vol}\} \oplus \mathrm{d}C^2 \\ V^2 &= \mathrm{d}C^1 \oplus C^2 \\ V^1 &= \mathrm{d}C^0 \oplus C^1 \end{aligned}$$

$$V^0 = \mathrm{span}\{1\} \oplus C^0,$$

and the same situation as for $n = 2$ occurs.

For $n = 4$ and $V^1 = 0$, we have $V \simeq \mathcal{H}$. $\triangleleft$

We would like to define a "minimal Poincaré duality model". We motivate this notion in the following remark.

Remark 6.2.12 (Model and minimal model). We shall understand models and minimal models in terms of model categories and their homotopy categories.

Let us illustrate this on Sullivan models. A Sullivan DGA is a free graded commutative algebra ΛU over a positively graded vector space U which admits a well-ordered homogenous basis (v_α) such that $\mathrm{d}v_\alpha \in \Lambda(v_\beta \mid \beta < \alpha)$ ($\coloneqq$ the subalgebra of ΛU generated by the v_β's) for all α. A Sullivan DGA is called minimal if $\mathrm{im}\,\mathrm{d} \subset \Lambda_{\geq 2} U$ ($\coloneqq$ the set of decomposable elements).

According to [BG76, Theorem 4.3], the category $\mathrm{CDGA}_1^{\geq 0}$ is a model category with weak equivalences being DGA-quasi-isomorphisms, fibrations being degree-wise surjective DGA-morphisms and cofibrations being retracts of relative Sullivan algebras (see [FOT08, Proposition 2.22 and Proposition 2.28]). Cofibrant objects are then precisely Sullivan algebras.

So, the homotopy extension property holds already for Sullivan DGA's. To see the role of minimality, we shall descent to the homotopy category. The homotopy category is constructed from a model category by localizing morphisms at weak equivalences. An isomorphism in the homotopy category, called weak homotopy equivalence, corresponds to a zig-zag of weak equivalences. If V_1 is weakly homotopy equivalent to V_2, we say that V_2 is a model of V_1. If there is a weak equivalence $V_2 \to V_1$, we say that V_2 is a resolution of V_1. We understand minimality as a condition which is in each weak homotopy equivalence class satisfied by at most one object up to isomorphism in the model category. If minimal models exist, they form a skeleton of the homotopy category. This is precisely the case of $\mathrm{CDGA}_1^{\geq 0}$ and minimal Sullivan algebras. Indeed, by [FOT08, Theorem 2.24], every connected $\mathrm{CDGA}_1^{\geq 0}$ is resolved by a minimal Sullivan algebra. Next, by [FOT08, Proposition 2.26], a DGA-morphism lifts to resolutions by Sullivan algebras, and by [FOT08, Corollary 2.13], quasi-isomorphic minimal Sullivan DGA's are isomorphic. Finally, this implies that weakly homotopy equivalent minimal Sullivan algebras are isomorphic, and hence are minimal in the sense above.

As another example, for an operad (or properad) $\mathcal{O}$, one wants to construct a dg-operad $\mathcal{O}_\infty$ which is a quasi-free resolution of $\mathcal{O}$ (see [Val12]). Quasi-free means that after forgetting the differential, the operad $\mathcal{O}_\infty$ is free over $\mathbb{S}$-bimodules. This is similar to Sullivan models which are free over vector spaces. For quadratic operads, $\mathcal{O}_\infty$ is often constructed as the cobar construction Ω of the Koszul dual cooperad $\mathcal{O}^{¡}$. This is the case of A_∞, L_∞ or of the properad IBL_∞. The differential on $\Omega\mathcal{O}^{¡}$ is the extension of the decomposition on $\mathcal{O}^{¡}$ to a derivative, and hence it has decomposable image (c.f., the explicit formula [Pek18, Formula (2)]). This is similar to minimal Sullivan models. It would be interesting to know whether $\mathcal{O}_\infty$ can be constructed using the same inductive method of "killing" and "adding" generators of homology as Sullivan minimal models.

Finally, let us note that in [CR19], they use the inductive method to construct (minimal) models of $\mathcal{O}$-algebras for a wide class of operads $\mathcal{O}$. Note that cyclic A_∞-algebras can be formulated in the language of cyclic operads. However, in the case of PDGA's, we have the non-degenerate pairing on homology and an operadic description is not clear. ◁

The following example shows that the zig-zag of a Poincaré duality model can not always be shortened to one arrow.

Example 6.2.13 (No Poincaré dualiy model with one arrow). (a) The closed genus 2 surface Σ_2 does not admit a Poincaré duality model A with just one arrow $A \to \Omega(\Sigma_2)$. Suppose the contrary. We consider the quasi-isomorphism $f : A \to \Omega(\Sigma_2)$ and compute for homogenous v_1, $v_2 \in A$ the following:

$$\begin{aligned}\langle f(v_1), f(v_2)\rangle &= \pm\langle f(v_1)\wedge f(v_2), 1\rangle \\ &= \pm\langle f(v_1\wedge v_2), 1\rangle \\ &= \pm\langle [f(v_1\wedge v_2)], [1]\rangle \qquad (*) \\ &= \pm\langle f_*[v_1\wedge v_2], f^*[1]\rangle \\ &= \pm\langle [v_1\wedge v_2], [1]\rangle \\ &= \pm\langle v_1\wedge v_2, 1\rangle \qquad (*) \\ &= \langle v_1, v_2\rangle.\end{aligned}$$

Stars hold because the only non-zero case is when $\deg(v_1)+\deg(v_2) = \deg(\langle\cdot,\cdot\rangle)$, and hence $\mathrm{d}(v_1\wedge v_2) = 0$ because non-degeneracy implies vanishing of higher degrees. We see that f preserves the pairing on the chain-level and is injective by non-degeneracy. We can thus assume that $A \subset \Omega(\Sigma_2)$ is a dg-subalgebra equipped with the restriction of the intersection pairing. Since $\dim(A) < \infty$, there is a Hodge decomposition

$$\begin{aligned}A^2 &= \mathcal{H}^2 \oplus \mathrm{d}C^1 \\ A^1 &= \mathcal{H}^1 \oplus \mathrm{d}C^0 \oplus C^1 \\ A^0 &= \mathcal{H}^0 \oplus C^0.\end{aligned}$$

If $1 \neq f \in C^\infty(\Sigma_2)$, then f^k, $k \in \mathbb{N}_0$ are linearly independent over $\mathbb{R}$. Therefore, it must hold $\mathcal{H}^0 = \operatorname{span}\{1\}$ and $C^0 = 0$. Duality implies $\mathrm{d}C^1 = 0$, and hence $\mathcal{H}^2$ is spanned by $\omega \in \Omega^2(\Sigma_2)$ with $\int_{\Sigma_2} \omega = 1$. It holds even $C^1 = 0$ as $\ker \mathrm{d} \cap C = 0$ in a Hodge decomposition. Since $A \simeq \mathrm{H}(\Sigma_2)$ as a DGA, there are closed α_1, β_1, α_2, $\beta_2 \in \Omega^1(\Sigma_2)$ such that $\mathcal{H}^1 = \operatorname{span}\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ and such that for all $x \in \Sigma_2$ the following holds:

$$\begin{aligned}\alpha_1(x) \wedge \alpha_2(x) &= \alpha_1(x) \wedge \beta_2(x) = 0\\ \alpha_1(x) \wedge \beta_1(x) &= \alpha_2(x) \wedge \beta_2(x) = \omega(x).\end{aligned}$$

Taking an $x \in \Sigma_2$ with $\omega(x) \neq 0$ gives a contradiction.

(b) The simply-connected 4-manifold $\mathbb{C}\mathrm{P}^{2\#7}$, where $\#$ denotes the connected sum, does not admit a Poincaré duality model A with just one arrow $A \to \Omega(M)$. Similarly as in the proof for Σ_2, we can restrict to the case $A \subset \Omega(\mathbb{C}\mathrm{P}^{2\#7})$. We obtain $A^4 = \mathcal{H}^4 = \langle \omega \rangle$ for a somewhere non-vanishing 4-form ω and $\mathcal{H}^2 = \langle K_0, \dots, K_6 \rangle$ such that for all $x \in \mathbb{C}\mathrm{P}^{2\#7}$ the following holds:

$$\begin{aligned}K_i(x) \wedge K_i(x) &= \pm\omega(x)\\ K_i(x) \wedge K_j(x) &= 0.\end{aligned}$$

We now view $K_i(x)$ as vectors in $\mathbb{R}^6$ so that $K_0(x) \wedge K_j(x) = 0$ corresponds to taking the scalar product. If $\sum_{i=0}^6 \lambda_i K_i(x) = 0$ for some $\lambda_i \in \mathbb{R}$ with $\lambda_{i_0} \neq 0$, then

$$0 = K_{i_0}(x) \wedge \Big(\sum_{i=0}^{6} \lambda_i K_i(x)\Big) = \pm\lambda_{i_0}\omega(x).$$

Therefore, $\omega(x) \neq 0$ implies that $K_i(x)$ are linearly independent. Hence, in this case, $K_0(x) \wedge K_i(x) = 0$ for all $i = 1, \dots, 6$ implies $K_0(x) = 0$, which is a contradiction with $K_0(x) \wedge K_0(x) = \omega(x)$. ◁

Questions 6.2.14. (i) Give an example of a Sullivan minimal model $f : \Lambda U \to V$ of a PDGA V which is not of Hodge type for any orientation such that $f_* : \mathrm{H}(\Lambda U) \to \mathrm{H}(V)$ is orientation preserving.

(ii) Can the additional assumptions in Lemma 6.1.23 and Proposition 6.2.6 be dropped?

(iii) Is smallness or minimal dimension the correct notion of “minimality” of a Poincaré duality model?

6.3. Consequences for IBL-infinity theory

Consider the construction $\mathrm{dIBL}^{\mathfrak{m}}(C(\cdot))$ on cyclic DGA's (= differential Poincaré duality algebras) from Section 3.4. We first need the following technical result.

Proposition 6.3.1 (Functoriality of $\mathrm{dIBL}^{\mathfrak{m}}$-construction up to homotopy)**.** *Suppose that V_1 and V_2 are differential Poincaré duality algebras, and let $f : V_1 \to V_2$ be a* PDGA*-quasi-isomorphism. Then there is an* IBL_∞*-quasi-isomorphism* $\mathfrak{f} : \mathrm{dIBL}(C(V_2)) \to \mathrm{dIBL}(C(V_1))$ *which is also an* IBL_∞*-quasi-isomorphism* $\mathfrak{f} : \mathrm{dIBL}^{\mathfrak{m}}(C(V_2)) \to \mathrm{dIBL}^{\mathfrak{m}}(C(V_1))$ *and satisfies* $\mathfrak{f}_{110} = f^*$*.*

Proof. Lemma 6.1.18 implies that f is injective, and hence we can consider the case when $V_1 \subset V_2$ is a dg-subalgebra and $f = \iota : V_1 \hookrightarrow V_2$ is the inclusion. In the proof of Lemma 6.1.20 it was argued that there are Hodge decompositions $V_1 = \mathcal{H} \oplus \mathrm{d}C_1 \oplus C_1$ and $V_1^\perp = \mathrm{d}C_2 \oplus C_2$ such that $V_2 = \mathcal{H} \oplus (\mathrm{d}C_1 \oplus \mathrm{d}C_2) \oplus (C_1 \oplus C_2)$ is a Hodge decomposition of V. Define $\mathrm{P} : V_2 \to V_2$ by

$$\mathrm{P}(\mathrm{d}c_2) \coloneqq -c_2 \quad \text{for all } c_2 \in C_2$$

and by 0 on other direct summands. Let $\pi : V_2 \to V_1$ be the canonical projection induced by $V_2 = V_1 \oplus V_1^\perp$. For v, $v' \in V_2$, write $v = h + \mathrm{d}c_{11} + c_{12} + \mathrm{d}c_{21} + c_{22}$, $v' = h' + \mathrm{d}c_{11}' + c_{12}' + \mathrm{d}c_{21}' + c_{22}'$ for h, $h' \in \mathcal{H}$, c_{11}, c_{11}', c_{12}, $c_{12}' \in C_1$, c_{21}, c_{21}', c_{22}, $c_{22}' \in C_2$ and compute

$$\begin{aligned}(\mathrm{P}\circ\mathrm{d} + \mathrm{d}\circ\mathrm{P})(v) &= \mathrm{P}(\mathrm{d}c_{21}) + \mathrm{d}\mathrm{P}(\mathrm{d}c_{22}) \\ &= -c_{21} - \mathrm{d}c_{22} \\ &= (\pi - \mathbb{1})(v)\end{aligned}$$

and

$$\begin{aligned}\langle \mathrm{P}v, v'\rangle &= \langle -c_{21}, \mathrm{d}c_{22}'\rangle \\ &= (-1)^{\deg c_{21}}\langle \mathrm{d}c_{21}, c_{22}'\rangle \\ &= (-1)^{\deg c_{21}+1}\langle \mathrm{d}c_{21}, \mathrm{P}v'\rangle \\ &= (-1)^{\deg v}\langle v, \mathrm{P}v'\rangle.\end{aligned}$$

These are conditions (11.2), resp. (11.3) from the beginning of [CFL15, Section 11]. We can now apply [CFL15, Theorem 11.3] to obtain an IBL_∞-quasi-isomorphism $\mathfrak{f} : \mathrm{dIBL}(C(V_1)) \to \mathrm{dIBL}(C(V_2))$ with $\mathfrak{f}_{110} = \iota^*$. Because ι is a morphism of algebras, ι^*

commutes also with the Hochschild differential, i.e., with $\mathfrak{q}^{\mathfrak{m}}_{110}$. The claim follows because $\mathrm{dIBL} = (\mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ and $\mathrm{dIBL}^{\mathfrak{m}} = (\mathfrak{q}^{\mathfrak{m}}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$. □

For any PDGA $(V, \mathrm{d}, \wedge, \mathrm{o}^{\mathrm{H}})$ with $\mathrm{H}^0(V) = \mathbb{R}$ and $\mathrm{H}^1(V) = 0$, we can pick its Poincaré duality model $\mathcal{M}(V)$ and associate to it the dIBL-algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}(V)))$. This extends the domain of $\mathrm{dIBL}^{\mathfrak{m}}(C(\cdot))$ to PDGA's. However, removing choices requires us to relax the codomain to IBL_∞-homotopy equivalence classes.

Proposition 6.3.2 (Uniqueness of $\mathrm{dIBL}^{\mathfrak{m}}$ for PDGA up to homotopy)**.** *Let V be a* PDGA *of degree n with* $\mathrm{H}^0(V) = \mathrm{span}\{1\}$ *and* $\mathrm{H}^1(V) = 0$. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Poincaré duality models of V. In addition, suppose that $\mathcal{M}_1^1 = \mathcal{M}_2^1 = 0$ if $n \geq 4$ and that* $\mathrm{H}^2(V) = 0$ *if $n \geq 5$. Then* $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_1))$ *and* $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_2))$ *are* IBL_∞*-quasi-isomorphic.*

Proof. By Proposition 6.2.9, there is a differential Poincaré duality algebra V_3 and PDGA-quasi-isomorphisms $f_1 : \mathcal{M}_1 \to V_3$ and $f_2 : \mathcal{M}_2 \to V_3$. Proposition 6.3.1 gives IBL_∞-quasi-isomorphisms $\mathfrak{f}_1 : \mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_1)) \to \mathrm{dIBL}^{\mathfrak{m}}(C(V_3))$ and $\mathfrak{f}_2 : \mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_2)) \to \mathrm{dIBL}^{\mathfrak{m}}(C(V_3))$ extending f_1^* and f_2^*, respectively. Application of [CFL15, Theorem 1.2] (existence of IBL_∞-homotopy inverses for IBL_∞-quasi-isomorphisms) finishes the proof. □

Conjecture 6.3.3. *The additional assumptions of Proposition 6.3.2 can be removed.*

If we had the notion of a minimal Poincaré duality model $\mathcal{M}_0(V)$ which is unique in every weak homotopy class of PDGA's (c.f., Questions 6.2.14), we could associate to a weak homotopy equivalence class of a PDGA V a canonical dIBL-algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}_0(V)))$ up to an IBL_∞-isomorphism.[7]

Definition 6.3.4 (IBL_∞-formality)**.** *We say that a differential Poincaré duality algebra V is* IBL_∞-formal *if* $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$ *and* $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}(V)))$ *are weakly* IBL_∞*-homotopy equivalent.*

Due to the existence of homotopy inverses of quasi-isomorphisms in the IBL_∞-category, being weakly IBL_∞-homotopy equivalent (the existence of zig-zag of quasi-isomorphisms) is equivalent to being IBL_∞-quasi-isomorphic (the existence of a direct quasi-isomorphism), which is equivalent to being IBL_∞-homotopy equivalent (the existence of a quasi-isomorphism with a homotopy inverse).

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[7] In order to think about functors, we would first have to find a canonical construction of an IBL_∞-morphism induced by a PDGA-morphism of cyclic DGA's. The construction of [CFL15, Section 11] applies only to quasi-isomorphisms, and even if we pick the standard Hodge propagator $\mathrm{P}_{\mathrm{std}}$, it still depends on the choice of the Hodge decomposition. Therefore, the best we can hope for is an assignment of PDGA-morphisms to IBL_∞-homotopy classes of IBL_∞-morphisms.

We formulate the following statement as a conjecture although a special case holds by Proposition 6.3.2. The proof might be easier than the proof of Conjecture 6.3.3.

Conjecture 6.3.5 (Algebraic formality conjecture)**.** *Let V be a* PDGA *with* $\mathrm{H}^0(V) = \operatorname{span}\{1\}$ *and* $\mathrm{H}^1(V) = 0$ *which is formal as a* DGA*. Then any its Poincaré duality model* $\mathcal{M}(V)$ *is* IBL_∞*-formal.*

Proof. By Proposition 6.2.5, DGA-formality is equivalent to PDGA-formality. Therefore, $\mathrm{H}(V)$ is a Poincaré duality model of V. The rest follows from Conjecture 6.3.3. ☐

Remark 6.3.6 (IBL_∞-formality of V and $C(V)$). Let $\mathcal{O}$ be a dg-operad. If V is an $\mathcal{O}$-algebra, then $\mathrm{H}(V)$ is an $\mathcal{O}$-algebra with zero differential. We say that V is $\mathcal{O}$-formal if V and $\mathrm{H}(V)$ are weakly homotopy equivalent as $\mathcal{O}$-algebras; i.e., if there is a zig-zag of $\mathcal{O}$-quasi-isomorphisms between V and $\mathrm{H}(V)$.

Consider DGA and A_∞. If a DGA V is formal, then it is also A_∞-formal because DGA is a subcategory of A_∞.[8] The converse is also true by the rectification procedure (see [Idr18]). Therefore, DGA-formality and A_∞-formality of a DGA V are equivalent. This should be true for any suitable (pro)perad $\mathcal{O}$ and its quasi-free resolution $\mathcal{O}_\infty$.

In the spirit above, we can speak about dIBL-, resp. IBL_∞-formality of the dIBL-algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$ on $C(V)$, which is about the existence of zig-zags of dIBL-, resp. IBL_∞-quasi-isomorphisms between $C(V)$ and $\mathrm{H}(C(V), \mathfrak{q}_{110}^{\mathfrak{m}})$. These notions are most likely equivalent as in the case of DGA- and A_∞-formality.

On the other hand, IBL_∞-formality of a differential Poincaré duality algebra V from Definition 6.3.4 is a different notion because V is not an IBL_∞-algebra.

We conclude that the following three types of "formalities" might be interesting:

1) DGA-formality of a PDGA V,

2) IBL_∞-formality of a PDGA V (resp. of its Poincaré duality model),

3) IBL_∞-formality of $C(V)$ as the dIBL-algebra $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$.

Except for the fact that (1) implies (2), we do not have any other guesses. Notions (1)–(3) should also be interpreted geometrically in the context of the String Topology Conjecture. ◁

Let M be a connected oriented closed n-manifold with $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$. We know that $\Omega(M)$ is of Hodge type and that a Riemannian metric on M induces a canonical Hodge

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[8] Note that A_∞, similarly as IBL_∞, admits homotopy inverses of quasi-isomorphisms, and hence A_∞-formality is equivalent to the existence of a direct A_∞-quasi-isomorphism $V \to \mathrm{H}(V)$.

decomposition $\Omega(M) = \mathcal{H} \oplus \operatorname{im} \mathrm{d} \oplus \operatorname{im} \mathrm{d}^*$, where $\mathcal{H}$ is the subspace of harmonic forms and d^* the codifferential. So, in this case, we have a canonical Poincaré duality model

$$\mathcal{M}(\Omega(M)) = \mathcal{Q}(\mathcal{S}(\Omega(M))).$$

Due to $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$, it is of finite type and hence finite-dimensional. Therefore, $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{M}(\Omega(M))))$ is well-defined. Recall that $(\Omega, \mathrm{d}, \wedge, \textstyle\int)$ is not of finite type and $\mathrm{dIBL}^{\mathfrak{m}}(C(\cdot))$ is not well-defined; that was the reason to define the Chern-Simons, aka formal pushforward, Maurer-Cartan element $\mathfrak{n}$ for $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ directly as a summation over Feynman integrals. The idea was that $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ is "homotopy equivalent" to "$\mathrm{dIBL}^{\mathfrak{m}}(C(\Omega(M)))$".

We conjecture the following:

Conjecture 6.3.7 (Equivalence of algebraic and geometric approach)**.** *Let M be a connected oriented closed Riemannian n-manifold with $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$. Then there is an admissible Hodge propagator* P *such that if we consider the Chern-Simons Maurer-Cartan element $\mathfrak{n}$, then the* IBL_∞*-algebras*

$$\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M))) \quad \textit{and} \quad \mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{Q}(\mathcal{S}(\Omega(M)))))$$

are IBL_∞*-homotopy equivalent.*

Comment on proof. By the vanishing result from Theorem B, we can find P such that the only non-zero component of $(\mathfrak{n}_{lg})$ is $\mathfrak{n}_{10}$. It satisfies $\mathfrak{n}_{10} = \sum \pm m_k^+$, where $m_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$ are the A_∞-operations obtained by homotopy transfer from Ω (we use the canonical Hodge isomorphism $\mathrm{H}_{\mathrm{dR}} \simeq \mathcal{H}$). By lifting the zig-zag of PDGA's to A_∞, we obtain an IBL_∞-quasi-isomorphism $\mathfrak{g} : (\mathcal{Q}(\mathcal{S}(\Omega)), \mathrm{d}, \wedge) \to (\mathrm{H}_{\mathrm{dR}}, (m_k))$. The idea is to show that $\mathfrak{g}_*\mathfrak{m}$ and $\mathfrak{n}$, where $\mathfrak{m}$ is the canonical Maurer-Cartan element, are gauge equivalent Maurer-Cartan elements (see [CFL15, Definition 9.7]). For this, it might be useful to describe a path object of the IBL-algebra $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}))$ (it is a dIBL-algebra), how it arises from the path object of a DGA, and how an overlying Maurer-Cartan element can be constructed from the A_∞-morphism. ☐

The following is a consequence of Conjecture 6.3.7. However, we formulate it separately as it might be easier to prove.

Conjecture 6.3.8 (Formality conjecture for geometric construction)**.** *In the situation of Conjecture 6.3.7, suppose that M is formal. Then* $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ *is* IBL_∞*-homotopy equivalent to* $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$.

Comment on proof. One has to prove the following. If there is an A_∞-isomorphism (f_i) of $(\mathrm{H}_{\mathrm{dR}}, \wedge)$ and $(\mathrm{H}_{\mathrm{dR}}, \wedge, m_3, \dots)$ with $f_1 = \mathbb{1}$, then $\mathfrak{m}_{10}$ and $\mathfrak{n}_{10} = \sum \pm \mathfrak{m}_k^+$ are gauge equivalent. □

Together with the String Topology Conjecture 4.5.1, this gives a canonical chain model for the equivariant Chas-Sullivan string topology for formal M with $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$.

7. Standard Hodge propagator

In this chapter, we tackle the question whether the standard Hodge propagator $\mathrm{P}_{\mathrm{std}}$ extends smoothly to the blow-up.

In Section 7.1, we recall the Schwartz kernel theorem (Proposition 7.1.1) and define the Schwartz form of an operator between sections of the exterior bundle of the cotangent bundle (Proposition 7.1.2 and Definition 7.1.3). We mention that the Schwartz form of a pseudo-differential operator is smooth outside of the diagonal (Proposition 7.1.4). On the example of the identity (Example 7.1.5) we illustrate that the smooth part does not always determine the pseudo-differential operator. We recall basic facts about regularity of the Laplace Green kernel and standard Hodge propagator (Proposition 7.1.6). We formulate the problem of smooth extension to the blow-up (Definition 7.1.7) and ask whether it is satisfied by G and $\mathrm{P}_{\mathrm{std}}$.

In Section 7.2, we study uniqueness of the Hodge homotopy (Proposition 7.2.1). We would like to characterize the standard Hodge propagator as a unique primitive to the harmonic kernel satisfying certain properties. We give some ideas how to do it. We also illustrate that one has to be careful with blow-ups (Proposition 7.2.2).

In Section 7.3, we consider the heat form approximation. We define the heat form and sum up its computational properties (Proposition 7.3.1). We write down formulas for the heat form approximations (Proposition 7.3). We postulate that the standard Hodge propagator is the codifferential of the Laplacian Green kernel with respect to one variable (Proposition 7.3.3).

In Section 7.4, we study the standard Hodge propagator on $\mathbb{R}^n$. We recall the well-known formulas for the heat form and the Green form on $\mathbb{R}^n$ (Proposition 7.4.1). We compute the standard Hodge propagator in two ways — as a coderivative of the Green form and as an integral using heat form approximation (Proposition 7.4.2 for $n \geq 2$ and Example 7.4.3 for $n = 1$).

In Section 7.5, we study the standard Hodge propagator on $\mathbb{S}^n$. We compute it explicitly for $\mathbb{S}^1$ (Example 7.5.1). Next, we study the structure of the space of Hodge propagators on $\mathbb{S}^n$ (Proposition 7.5.2). We show that the Hodge propagator for $\mathbb{S}^n$ constructed in Part I is coexact (Proposition 7.5.3). Finally, using the previous results, we prove that the standard Hodge propagator for $\mathbb{S}^2$ extends smoothly to the blow-up and give an explicit formula with an unknown constant (Proposition 7.5.4).

7.1. Schwartz form and smooth extension to blow-up

For a smooth vector bundle $\pi : E \to M$, let $\mathcal{D}(E)$ be the space of "test sections of E", and let $\mathcal{D}'(E^*)$ be the space of distributions. Let $\langle\cdot,\cdot\rangle : \mathcal{D}'(E^*) \otimes \mathcal{D}(E) \to \mathbb{R}$ be the natural pairing. Proper definitions can be formulated easily based on [Hör03, Section 6].

Proposition 7.1.1 (Schwartz kernel theorem, see [Hör03, Theorem 5.2.1] for the local version)**.** *Let E_1 and E_2 be smooth vector bundles over smooth manifolds M_1 and M_2, respectively. Then there is a one-to-one correspondence between continuous operators $T : \mathcal{D}(E_1) \to \mathcal{D}'(E_2)$ and elements $\mathrm{K}_T \in \mathcal{D}'(E_1 \boxtimes E_2^*)$ which is given by the equation*

$$\langle T s_1, \psi_2\rangle = \langle \mathrm{K}_T, s_1 \boxtimes \psi_2\rangle \quad \textit{for all } s_1 \in \mathcal{D}(E_1) \textit{ and } \psi_2 \in \mathcal{D}(E_2^*).$$

The distributional section K_T is called the Schwartz kernel *of T.*

An L^1_{loc}-integrable function $k : M \times M \to \mathrm{Hom}(E_1, E_2) = E_1^* \boxtimes E_2$ on an oriented Riemannian manifold M defines the distribution $\mathrm{K} \in \mathcal{D}'(E_1 \boxtimes E_2^*)$ by

$$\langle \mathrm{K}, s_1 \boxtimes \psi_2\rangle = \int_{x_1,x_2} \langle k(x_1,x_2)s_1(x_1), \psi_2(x_2)\rangle \mathrm{Vol}(x_1)\mathrm{Vol}(x_2)$$

In the case of exterior bundles, we introduce an equivalent notion of a Schwartz form. This name was proposed by Dr. A. Hermann in a discussion in Potsdam.

Proposition 7.1.2 (Schwartz kernel and Schwartz form)**.** *In the setting of Proposition 7.1.1, suppose that $M_1 = M_2 = M$ is a smooth oriented Riemannian manifold and $E_1 = E_2 = \Lambda T^*M$. We consider the isomorphism of vector bundles $\Psi : \Lambda T^*M \boxtimes (\Lambda T^*M)^* \to \Lambda T^*(M \times M)$ which is for every x_1, $x_2 \in M$ given by*

$$\begin{aligned} \Psi : \Lambda T^*_{x_1} M \otimes \left(\Lambda T^*_{x_2} M\right)^* &\longrightarrow \Lambda T^*_{(x_1,x_2)}(M \times M) \\ \omega_1 \otimes \xi_2 &\longmapsto \omega_1 \wedge \sharp\xi_2. \end{aligned}$$

Here, $\sharp : \Lambda T^*M \to \left(\Lambda T^*M\right)^*$ denotes the musical isomorphism with respect to the natural pointwise inner product on ΛT^*M. We obtain the isomorphism

$$\begin{aligned} \Psi_* : \Omega'(M \times M) &\longrightarrow \mathcal{D}'(E_1^* \boxtimes E_2) \\ \Omega_T &\longmapsto \mathrm{K}_T, \end{aligned}$$

where Ω' denotes the space of de Rham currents.

If T is homogenous of degree $|T|$, then the degree of Ω_T satisfies

$$\deg(\Omega_T) = \dim(M) + |T|.$$

If K_T is represented by an L^1_{loc}-integrable section of $\mathrm{Hom}(E_1, E_2) \simeq E_1^* \boxtimes E_2$, then Ω_T is represented by an L^1_{loc}-integrable form, and the other way round. In this case, for all $\omega_1 \in \Omega(M)$, $\omega_2 \in \Omega_c(M)$, we have

$$\begin{aligned}\int_{x_2} ((T\omega_1)(x_2))\,[\omega_2(x_2)] &= \int_{x_1,x_2} \mathrm{K}_T(x_1,x_2)[\omega_1(x_1)] \wedge \mathrm{Vol}(x_1) \wedge \omega_2(x_2) \\ &= \int_{x_1,x_2} \Omega_T(x_1,x_2) \wedge \omega_1(x_1) \wedge \omega_2(x_2).\end{aligned} \tag{7.1}$$

Proof. Straightforward computations similar to Example 7.1.5 below. □

Definition 7.1.3 (Schwartz form)**.** *The current $\Omega_T \in \Omega'(M \times M)$ from Proposition 7.1.2 is called the* Schwartz form *of T.*

We will consider pseudo-differential operators $T : \Omega(M) \to \Omega(M)$ on a Riemannian manifold M; this class of operators generalizes differential operators and contains generalized inverses of elliptic operators (see [Hör07] for thorough treatment).

Proposition 7.1.4 (Schwartz form of pseudo-differential operators)**.** *Let $T : \Omega(M) \to \Omega(M)$ be a pseudo-differential operator on a smooth oriented Riemannian manifold M. Then the Schwartz form Ω_T restricts to a smooth form on $M \times M \backslash \Delta$.*

Proof. Well known fact, proof based on [Hör07]. □

Example 7.1.5 (Schwartz form of $\mathbb{1}$)*.* The Schwartz form of the identity $\mathbb{1} : \Omega(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$ reads

$$\Omega_{\mathbb{1}}(x,y) = \delta(x-y)(\mathrm{dx}^1 - \mathrm{dy}^1) \cdots (\mathrm{dx}^n - \mathrm{dy}^n), \tag{7.2}$$

where δ denotes the Dirac delta function on $\mathbb{R}^n$ centered at 0. In order to prove this, we start by rewriting

$$\begin{aligned}(\mathrm{dx}^1 - \mathrm{dy}^1) \cdots (\mathrm{dx}^n - \mathrm{dy}^n) &= \sum_I (-1)^{|I|} \varepsilon(I^c I \mapsto [n])\, \mathrm{dx}^{I^c} \wedge \mathrm{dy}^I \\ &= \sum_I (-1)^{n|I|} (* \,\mathrm{dx}^I) \wedge \mathrm{dy}^I\,.\end{aligned}$$

Here, the sum is over all multiindices $I \subset \{1, \dots, n\}$, and we use that $*(\mathrm{dx}^I) = \varepsilon(I, I^c)\,\mathrm{dx}^{I^c}$, where $\varepsilon(I, I^c)$ denotes the sign to order II^c to $\{1, \dots, n\}$. Now, for any

$\omega \in \Omega_c(\mathbb{R}^n)$, which we write as $\omega(x) = \sum_K \omega_K(x)\,\mathrm{dx}^K$, we compute using $\mathrm{dx}^K \wedge *(\mathrm{dx}^I) = (\mathrm{dx}^K, \mathrm{dx}^I)\mathrm{Vol}(x) = \delta^{KI}\mathrm{Vol}(x)$ the following:

$$\begin{aligned}&\int_x \delta(x-y)(\mathrm{dx}^1 - \mathrm{dy}^1)\cdots(\mathrm{dx}^n - \mathrm{dy}^n)\omega(x)\\ &\quad= \sum_{I,K}\int_x \delta(x-y)(-1)^{n|I|}\omega_K(x)(*\,\mathrm{dx}^I)\,\mathrm{dy}^I\,\mathrm{dx}^K\\ &\quad= \sum_{I,K}\int_x \delta(x-y)(-1)^{n|I|+n|K|}\delta(x-y)\omega_K(x)\,\mathrm{dx}^K\,*(\mathrm{dx}^I)\,\mathrm{dy}^I\\ &\quad= \sum_{I}\int_x \delta(x-y)\omega_I(x)\mathrm{Vol}(x)\,\mathrm{dy}^I\\ &\quad= \omega(y).\end{aligned}$$

This shows (7.1), and (7.2) follows. Notice that

$$\Omega_{\mathbb{1}}\big|_{\mathbb{R}^n\times\mathbb{R}^n\backslash\Delta} = 0,$$

and thus the smooth part of $\Omega_{\mathbb{1}}$ does not recover the data of the operator $\mathbb{1}$. ◁

We consider the Green operator $\mathcal{G}$ for the Laplacian Δ (see [War83]) and the standard Hodge propagator $\mathcal{P}_{\mathrm{std}}$. They are both pseudo-differential operators, and it holds $\mathcal{P}_{\mathrm{std}} = -\mathrm{d}^*\,\mathcal{G}$. We will study their Schwartz forms G and $\mathrm{P}_{\mathrm{std}}$, which are called the *Green kernel* and the *standard Hodge propagator,* respectively.

Proposition 7.1.6 (Basic facts about G and $\mathrm{P}_{\mathrm{std}}$)**.** *The Green kernel G represents an L^2-integrable form. The standard Hodge propagator $\mathrm{P}_{\mathrm{std}}$ represent an L^1-integrable form.*

Proof. The fact that G is L^2 is an exercise in [War83]. The fact that $\mathrm{P}_{\mathrm{std}}$ defines an L^1-integrable form on $M \times M$ for a compact manifold M was proved in [Har04] using the heat kernel approximation (see the next section). □

A consequence of Proposition 7.1.6 is that the Schwartz forms G and $\mathrm{P}_{\mathrm{std}}$ are determined by their smooth restrictions to $M \times M\backslash\Delta$. This follows from (7.1) because the integral does not depend on sets of zero measure. Therefore, we will write G, $\mathrm{P}_{\mathrm{std}} \in \Omega^{n-1}(M \times M\backslash\Delta)$.

Definition 7.1.7 (Smooth extension to the blow-up)**.** *Let M be a smooth manifold, and let $\pi : \mathrm{Bl}_\Delta(M \times M) \to M \times M$ be the spherical blow-up of $M \times M$ at the diagonal Δ.*[1]

---

[1] Another name of this construction suggested to me by Dr. Oliver Lindblad Petersen after explaining him our setting should be "Melrose blow-up".

Consider the blow-up diagram

$$\begin{tikzcd}
 & \mathrm{Bl}_\Delta(M\times M) \arrow[d, two heads, "\pi"] \\
M\times M\backslash\Delta \arrow[ur, hook, "\tilde\iota"] \arrow[r, hook, "\iota"] & M\times M,
\end{tikzcd}$$

where ι is the inclusion and $\tilde\iota$ its unique smooth lift — the embedding of the interior. We say that a smooth form $\omega\in\Omega(M\times M\backslash\Delta)$ extends smoothly to the blow-up *if there is a smooth form $\tilde\omega\in\Omega(\mathrm{Bl}_\Delta(M\times M))$ such that*

$$\tilde\iota^*\tilde\omega=\omega.$$

Note that the extension, if it exists, is necessarily unique.

Question 7.1.8. Do G and $\mathrm{P}_{\mathrm{std}}$ extend smoothly to the blow-up? We expect that G does not and $\mathrm{P}_{\mathrm{std}}$ does.

7.2. Uniqueness of Hodge propagator

Proposition 7.2.1 (Uniqueness of Hodge homotopy)**.** *Let M be a closed oriented manifold, and let $\Omega(M)=\operatorname{im}\mathrm{d}\oplus\operatorname{im}\mathrm{d}^*\oplus\mathcal{H}$ be its Hodge decomposition. For any linear map $\mathcal{P}:\Omega(M)\to\Omega(M)$ satisfying*

$$\mathrm{d}\circ\mathcal{P}+\mathcal{P}\circ\mathrm{d}=\iota_{\mathcal{H}}\circ\pi_{\mathcal{H}}-\mathbb{1},\tag{7.3}$$

there exist linear maps $R_1:\operatorname{im}\mathrm{d}\to\operatorname{im}\mathrm{d}$, $R_2:\operatorname{im}\mathrm{d}^*\to\ker\mathrm{d}$ and $R_3:\mathcal{H}\to\ker\mathrm{d}$ such that

$$\mathcal{P}=\begin{cases}-(\mathrm{d}|_{\operatorname{im}\mathrm{d}^*})^{-1}+R_1 & \textit{on } \operatorname{im}\mathrm{d},\\ (\mathrm{d}|_{\operatorname{im}\mathrm{d}^*})^{-1}R_1\mathrm{d}+R_2 & \textit{on } \operatorname{im}\mathrm{d}^*,\\ R_3 & \textit{on } \mathcal{H}.\end{cases}\tag{7.4}$$

Note that $\mathrm{d}|_{\operatorname{im}\mathrm{d}^*}:\operatorname{im}\mathrm{d}^*\to\operatorname{im}\mathrm{d}$ is an isomorphism. Moreover, the following facts are equivalent for an operator $\mathcal{P}$ satisfying (7.3)*:*

(1) $\mathcal{P}=\mathcal{P}_{\mathrm{std}}$ *is the standard Hodge homotopy,*

(2) $R_1=R_2=R_3=0$*,*

(3) $\operatorname{im}\mathcal{P}\subset\operatorname{im}\mathrm{d}^*$*,*

Proof. Suppose first that $\omega \in \operatorname{im} \mathrm{d}$. Plugging ω in (7.3), we get that $\mathcal{P}\omega$ has to satisfy

$$\mathrm{d}\mathcal{P}\omega = \omega.$$

We see that

$$\mathcal{P}\omega \in \eta + \ker \mathrm{d},$$

where η is the unique coexact form with $\mathrm{d}\eta = \omega$. In other words, η is the preimage of ω under the isomorphism $\mathrm{d} : \operatorname{im} \mathrm{d}^* \to \operatorname{im} \mathrm{d}$, and we can write

$$\mathcal{P}\omega = (\mathrm{d}|_{\operatorname{im} \mathrm{d}^*})^{-1}\omega + R_1\omega, \tag{7.5}$$

where $R_1 : \operatorname{im} \mathrm{d} \to \ker \mathrm{d}$.

Suppose now that $\omega \in \operatorname{im} \mathrm{d}^*$. Using (7.3) and (7.5), we obtain

$$\mathrm{d}\mathcal{P}\omega = \omega - \mathcal{P}\mathrm{d}\omega = \underbrace{\omega - (\mathrm{d}|_{\operatorname{im} \mathrm{d}^*})^{-1}\mathrm{d}\omega}_{=0} - R_1\mathrm{d}\omega,$$

where the two terms cancel because ω is the unique coexact primitive to $\mathrm{d}\omega$. Notice that this equation restricts R_1 to $R_1 : \operatorname{im} \mathrm{d} \to \operatorname{im} \mathrm{d}$. Similarly as in the first case, we obtain

$$\mathcal{P}\omega = -(\mathrm{d}|_{\operatorname{im} \mathrm{d}^*})^{-1}R_1\mathrm{d}\omega + R_2\omega,$$

where $R_2 : \operatorname{im} \mathrm{d}^* \to \ker \mathrm{d}$.

If $\omega \in \mathcal{H}$, then (7.3) gives

$$\mathrm{d}\mathcal{P}\omega = 0,$$

and hence $\mathcal{P}\omega = R_3\omega$ for some $R_3 : \mathcal{H} \to \ker \mathrm{d}$.

Therefore, (7.3) implies (7.4). The other direction clearly holds as well.

As for the equivalent facts, because $\ker \mathrm{d} \perp \operatorname{im} \mathrm{d}^*$ (with respect to the L^2-inner product), it is clear from (7.4) that $\operatorname{im}(\mathcal{P}) \subset \operatorname{im}(\mathrm{d}^*)$ is equivalent to $R_1 = R_2 = R_3 = 0$. Therefore, if there is a $\mathcal{P}$ satisfying (7.3) and (3), then it is unique; it must be $\mathcal{P}_{\mathrm{std}} = -\mathrm{d}^* \mathcal{G}$. □

We see that the standard Hodge homotopy $\mathcal{P}_{\mathrm{std}}$ can be characterized as the unique Hodge homotopy with coexact image.

We would like to use the equation

$$\mathrm{dP}_{\mathrm{std}} = \pm \mathrm{H} \quad \text{on } (M \times M)\backslash\Delta \tag{7.6}$$

and characterize $\mathrm{P}_{\mathrm{std}}$ as its unique coexact solution. This is probably not enough and additional assumptions on the asymptotic behavior near Δ are needed.

Another idea is to lift the equation (7.6) to the blow-up and study primitives to $\pi^*\mathrm{H}$ on $\mathrm{Bl}_\Delta(M\times M)$. The advantage is that $\mathrm{Bl}_\Delta(M\times M)$ is a compact manifold with boundary, and hence for a given Riemannian metric, we have the Hodge decomposition with boundary conditions. However, the pull-back metric g along $\pi:\mathrm{Bl}_\Delta(M\times M)\to M\times M$ is singular at the boundary. We would have to approximate it with a family of metrics g_t in a correct way, use Hodge theory to find a d^*-primitive η_t to a unique coexact d-primitive P_t to $\pi^*\mathrm{H}$ with certain boundary conditions, and finally prove that P_t converges to a solution of (7.6) on the interior and η_t to its d^*-primitive.

Natural differential operators on $\Omega(M\times M)$ do not always pull-back to differential operators on $\Omega(\mathrm{Bl}_\Delta(M\times M))$ via the blow-down map. The total differential d does, but the operators $\mathbb{1}\otimes\mathrm{d}_y$ and $\mathrm{d}_x\otimes\mathbb{1}$ do not. We suppose that none of d^*, $\mathbb{1}\otimes\mathrm{d}_y^*$ and $\mathrm{d}_x^*\otimes\mathbb{1}$ does. We illustrate the consequences on the following seemingly pathological example.

Proposition 7.2.2 (Pathological example)**.** *For any $n\in\mathbb{N}$, there exists a smooth form $\eta\in\Omega(\mathrm{Bl}_\Delta(\mathbb{R}^n\times\mathbb{R}^n))$ with compact vertical support with respect to the fiber bundle $\tilde{\pi}_2=\mathrm{pr}_2\circ\pi:\mathrm{Bl}_\Delta(\mathbb{R}^n\times\mathbb{R}^n)\to\mathbb{R}^n$ such that $\mathrm{d}_y\eta=0$ on $(\mathbb{R}^n\times\mathbb{R}^n)\backslash\Delta$ but $\mathrm{d}\tilde{\pi}_{2*}\eta\neq 0$.*

If $\eta\in\Omega(\mathrm{Bl}_\Delta(\mathbb{R}\times\mathbb{R}))$ is as above and satisfies in addition $\tau^*\eta=\pm\eta$, then $\mathrm{d}_y\eta=0$ on $\mathbb{R}\times\mathbb{R}\backslash\Delta$ implies $\mathrm{d}\tilde{\pi}_{2*}\eta=0$.

Proof. Let us start with $n=1$. Let $f:\mathbb{R}\times\mathbb{R}\backslash\Delta\to\mathbb{R}$ be a function such that

1. $f:\mathbb{R}\times\mathbb{R}\backslash\Delta\to\mathbb{R}$ is smooth.
2. It holds $\frac{\partial f}{\partial y}(x,y)=0$ for all $(x,y)\in\mathbb{R}\times\mathbb{R}\backslash\Delta$.
3. For every $y\in\mathbb{R}$, the function $f(\cdot,y)$ defined on $\mathbb{R}\backslash\{y\}$ has compact support in $\mathbb{R}\backslash\{y\}$.

Clearly, (1) and (2) implies

$$f(x,y)=\begin{cases} f^-(x) & \text{for } x<y,\\ f^+(x) & \text{for } x>y,\end{cases}\quad\text{for all } (x,y)\in\mathbb{R}\times\mathbb{R}\backslash\Delta, \tag{7.7}$$

where f^+, $f^-:\mathbb{R}\to\mathbb{R}$ are smooth functions. It is easy to see that (3) implies the existence of x^+, $x^-\in\mathbb{R}$ such that

$$f^+(x)=0\quad\text{for all } x>x^+\quad\text{and}\quad f^-(x)=0\quad\text{for all } x<x^-.$$

We consider the form

$$\eta(x,y) = f(x,y)\,\mathrm{dx} \quad \text{on } \mathbb{R}\times\mathbb{R}\backslash\Delta. \tag{7.8}$$

In general, a form $\eta(x,y)$ on $\mathbb{R}^n\times\mathbb{R}^n\backslash\Delta$ is a restriction of a smooth form on $\mathrm{Bl}_\Delta(\mathbb{R}^n\times\mathbb{R}^n)$ if and only if the form $(\Phi^*\eta)(r,\omega,u)$ for the diffeomorphism $\Phi : (0,\infty)\times\mathbb{S}^{n-1}\times\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n\backslash\Delta$ given by $\Phi(r,\omega,u) = (u+r\omega,u)$ is a restriction of a smooth form on $[0,\infty)\times\mathbb{S}^{n-1}\times\mathbb{R}^n$. In our case, we have

$$(\Phi^*\eta)(r,u) = \begin{cases} f^+(u+r)\big(\mathrm{du}+\mathrm{dr}\big) & \text{on } D^+ := [0,\infty)\times\{1\}\times\mathbb{R}, \\ f^-(u-r)\big(\mathrm{du}-\mathrm{dr}\big) & \text{on } D^- := [0,\infty)\times\{-1\}\times\mathbb{R}, \end{cases}$$

where we used the fact that $\mathbb{S}^0 = \{\pm 1\}$ and splitted the domain in two connected components. We see that $\Phi^*\eta$ extends smoothly to $[0,\infty)\times\mathbb{S}^0\times\mathbb{R}$ in the obvious way. We denote the extension by $\widetilde{\Phi^*\eta}$ and the induced extension of η to $\mathrm{Bl}_\Delta(\mathbb{R}\times\mathbb{R})$ by $\tilde{\eta}$. It holds $\tilde{\Phi}^*\tilde{\eta} = \widetilde{\Phi^*\eta}$ under the extended diffeomorphism $\tilde{\Phi} : [0,\infty)\times\mathbb{S}^0\times\mathbb{R}\to\mathrm{Bl}_\Delta(\mathbb{R}\times\mathbb{R})$. The fiberwise integral along the smooth oriented fiber bundle $\tilde{\pi}_2 : \mathrm{Bl}_\Delta(\mathbb{R}\times\mathbb{R})\to\mathbb{R}$ transforms under $\tilde{\Phi}$ to the fiberwise integral along $\tilde{p}_3 : [0,\infty)\times\mathbb{S}^0\times\mathbb{R}\to\mathbb{R}$, and we get for all $y = u\in\mathbb{R}$ the following:

$$(\tilde{\pi}_{2*}\tilde{\eta})(y) = \tilde{p}_{3*}(\widetilde{\Phi^*\eta})(u) = \int_0^\infty \big(f^+(u+r) + f^-(u-r)\big)\,\mathrm{dr}\,.$$

The algebraic sign of the second term was canceled by the geometric sign coming from different orientations of $[0,\infty)$ in $[0,\infty)\times\{0\}$ and in $[0,\infty)\times\{1\}$. We think of $\tilde{\Phi}$ as of an isomorphism of fiber bundles $\tilde{p}_3$ and $\tilde{\pi}_2$ covering the identity, which explains the notation $y = u$. The fiberwise integration along $\tilde{p}_3 : [0,\infty)\times\mathbb{S}^0\times\mathbb{R}\to\mathbb{R}$ reduces to the Lebesgue integration of a smooth function g on $[0,\infty)\times\mathbb{S}^\omega\times\mathbb{R}$ with respect to (r,ω). Clearly, we can permute any differential operator acting on u with the integral. In our case, we use d and obtain

$$\begin{aligned}(\mathrm{d}\tilde{\pi}_{2*}\tilde{\eta})(y) &= (\mathrm{d}\tilde{p}_{3*}\widetilde{\Phi^*\eta})(u) \\ &= \frac{\partial}{\partial u}\Big(\int_0^\infty \big(f^+(u+r) + f^-(u-r)\big)\,\mathrm{dr}\Big)\,\mathrm{du} \\ &= \Big(\int_0^\infty (f^+)'(u+r) + (f^-)'(u-r)\,\mathrm{dr}\Big)\,\mathrm{du} \\ &= \Big(\int_u^\infty (f^+)'(z)\,\mathrm{dz} + \int_{-\infty}^u (f^-)'(z)\,\mathrm{dz}\Big)\,\mathrm{du} \\ &= \big(f^-(u) - f^+(u)\big)\,\mathrm{du}\end{aligned}$$

This is zero precisely when f extends continuously to $\mathbb{R} \times \mathbb{R}$.

A general form on $\mathbb{R} \times \mathbb{R} \backslash \Delta$ is

$$\eta(x,y) = f_0(x,y) + f_1(x,y)\,\mathrm{dx} + f_2(x,y)\,\mathrm{dy} + f_3(x,y)\,\mathrm{dx}\,\mathrm{dy}\,.$$

Imposing the condition $\mathrm{d}_y\eta = 0$, we get (7.7) for both f_0 and f_1. Imposing $\tau^*\eta = -\eta$, we get $f_3(x,y) = f_3(y,x)$, $f_2(x,y) = -f_1(y,x)$ and $f_0(x,y) = -f_0(y,x)$. Now, requiring that η has compact vertical support in x implies $f_0 = f_1 \equiv 0$ because of (7.7), and hence also $f_2 \equiv 0$. However, the form $\eta = f_3(x,y)\,\mathrm{dx}\,\mathrm{dy}$ satisfies $\mathrm{d}_y\eta = 0$ and $\mathrm{d}\tilde{\pi}_{2*}\eta = 0$ trivially.

As for the higher dimensions, let $n \geq 1$, and consider

$$\eta(x,y) := \frac{\lambda(x)}{|x-y|^n}\sum_{i=1}^{n}(-1)^{i+1}(x^i - y^i)\,\mathrm{dx}^1\cdots\mathrm{dx}^n\,\mathrm{dy}^1\cdots\widehat{\mathrm{dy}^i}\cdots\mathrm{dy}^n$$

for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \Delta$, where $\lambda : \mathbb{R}^n \to \mathbb{R}$ is a smooth bump function. Then η extends smoothly to $\mathrm{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$, it holds $\mathrm{d}_y\eta(x,y) = 0$, but

$$\mathrm{d}\int_x \eta(x,y) = \mathrm{Vol}(\mathbb{S}^n)\lambda(y)\,\mathrm{dy}^1\cdots\mathrm{dy}^n\,.$$

This finishes the proof. □

7.3. Approximation using heat form

Given an oriented Riemannian manifold (M,g), consider the *heat form*

$$\mathrm{K}_t(x,y) = \sum(-1)^{kn}e^{-\lambda_i t}(\star e_i)(x) \wedge e_i(y), \tag{7.9}$$

where (e_i) are eigenvectors of Δ and λ_i the corresponding eigenvalues. It is equivalently the Schwartz form of the operator $\exp(-\Delta t)$ (see [Har04, Chapter 3]) or a unique solution of the equation $\Delta\,\mathrm{K}_t(x,y) = -\frac{\partial}{\partial t}\mathrm{K}_t(x,y)$ with $\lim_{t\to 0}\mathrm{K}_t = \Omega_{\mathbb{1}}$, where $\Omega_{\mathbb{1}}$ is the Schwartz form of the identity (see [Hei06]).

Proposition 7.3.1 (Properties of the heat kernel). *Let M be an oriented Riemannian manifold. The heat form $\mathrm{K}_t(x,y)$ is smooth on $M \times M \times (0,\infty)$ and satisfies*

$$\mathrm{dK}_t = 0, \quad \tau^*\mathrm{K}_t = (-1)^n\mathrm{K}_t \quad \textit{and} \quad \frac{1}{2}\Delta\mathrm{K}_t = \Delta_x\mathrm{K}_t = -\frac{\partial}{\partial t}\mathrm{K}_t.$$

Proof. Straightforward computation and a nice combinatorial argument. □

Proposition 7.3.2 (Approximation using heat form)**.** *Let M be an oriented Riemannian manifold and $\mathrm{K}_t(x,y)$ the heat form. For all $(t,x,y)\in([0,\infty)\times M\times M)\backslash\{0\}\times\Delta =: D(\mathrm{K})$, define*

$$\begin{aligned} G_t(x,y) &\coloneqq \int_t^\infty \mathrm{K}_\tau(x,y)\,\mathrm{d}\tau \quad \textit{and} \\ \mathrm{P}_t(x,y) &\coloneqq (-1)^{n+1}\int_t^\infty (\mathbb{1}\mathbin{\hat{\otimes}}\mathrm{d}_y^*)\mathrm{K}_\tau(x,y)\,\mathrm{d}\tau\,. \end{aligned} \tag{7.10}$$

Then:

(a) The forms G_t and P_t are smooth on $D(\mathrm{K})$, the point-wise limits G' and P' as $t\to 0$ exist, and it holds $G_t \overset{t}{\rightrightarrows} G'$ and $\mathrm{P}_t \overset{t}{\rightrightarrows} \mathrm{P}'$ as $t\to 0$ (uniform convergence) in $C^\infty_{loc}(M\times M\backslash\Delta)$.

(b) On $D(\mathrm{K})$, the following relations hold:

$$\begin{aligned} \mathrm{d}G_t &= 0 & \mathrm{P}_t &= (-1)^{n+1}\frac{1}{2}\mathrm{d}^*G_t \\ \Delta\, G_t &= \mathrm{K}_t - \mathrm{H} & \mathrm{d}\mathrm{P}_t &= (-1)^n(\mathrm{H}-\mathrm{K}_t) \\ \tau^*G_t &= (-1)^nG_t & \tau^*\mathrm{P}_t &= (-1)^n\mathrm{P}_t. \end{aligned}$$

It follows that $G' = G$ is the (Laplace) Green form and $\mathrm{P}' = \mathrm{P}_{\mathrm{std}}$ the standard Hodge propagator.

Proof. The formal computation is clear. An honest proof uses the standard heat kernel estimates. ☐

Proposition 7.3.3 ($\mathrm{P}_{\mathrm{std}}$ is codifferential of G)**.** *Let M be an oriented Riemannian manifold, and let $G\in\Omega^n(M\times M\backslash\Delta)$ be the Green form. Then the standard Hodge propagator $\mathrm{P}_{\mathrm{std}}$ satisfies*

$$\mathrm{P}_{\mathrm{std}}(x,y) = (-1)^{n+1}(\mathbb{1}\otimes\mathrm{d}_y^*)G(x,y), \tag{7.11}$$

where $\mathbb{1}\otimes\mathrm{d}_y^*:\Omega^\bullet(M\times M)\to\Omega^{\bullet-1}(M\times M)$ is the differential operator defined in local coordinates by commuting d^* over the first factor with the Koszul sign and applying it to the second factor.

Proof. As for the signs, $(-1)^n$ comes from $T\,\mathcal{G}\,\omega(y) = (-1)^{nT}\int_x(\mathbb{1}\otimes T_y)G(x,y)\omega(x)$ with $T=\mathrm{d}^*$ and -1 from $\mathcal{P}_{\mathrm{std}} = -\mathrm{d}^*\,\mathcal{G}$. The rest can be proven using the heat kernel approximation and standard heat kernel estimates. There is an other method using the asymptotic expansion of G, which was shown to the author by Prof. Dr. Christian Bär. ☐

7.4. Standard Hodge propagator for Euclidean space

We will use Propositions 7.3.3 and 7.3.2 to prove in two ways that $\mathrm{P}_{\mathrm{std}}$ for $\mathbb{R}^n$ extends smoothly to the blow-up. We will start with the following well known formulas.

Proposition 7.4.1 (Green form and heat kernel for $\mathbb{R}^n$)**.** *The Green form for $\mathbb{R}^n$, which can be equivalently characterized as the unique solution $G \in \Omega^n(\mathbb{R}^n \times \mathbb{R}^n \backslash \Delta)$ of*

$$\Delta_y \, G(x,y) = \delta(x-y)(\mathrm{dx}^1 - \mathrm{dy}^1)\cdots(\mathrm{dx}^n - \mathrm{dy}^n), \tag{7.12}$$

where $\delta(x-y)$ is the Dirac delta function, satisfies

$$G(x,y) = \begin{cases} \dfrac{1}{(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})} \dfrac{1}{|x-y|^{n-2}}(\mathrm{dx}^1 - \mathrm{dy}^1)\cdots(\mathrm{dx}^n - \mathrm{dy}^n) & \text{for } n \geq 3, \\ -\dfrac{1}{2\pi} \ln|x-y|(\mathrm{dx}^1 - \mathrm{dy}^1)(\mathrm{dx}^2 - \mathrm{dy}^2) & \text{for } n = 2. \end{cases}$$

The heat kernel for $\mathbb{R}^n$, i.e., the solution of $\Delta_y \, Q_t(x,y) = -\frac{\partial}{\partial t} Q_t(x,y)$, is given by

$$Q_t(x,y) = (4\pi t)^{-\frac{n}{2}} \exp\Bigl(-\frac{|x-y|^2}{4t}\Bigr)(\mathrm{dx}^1 - \mathrm{dy}^1)\ldots(\mathrm{dx}^n - \mathrm{dy}^n). \tag{7.13}$$

Proof. See [BGV04]. □

Proposition 7.4.2 (Standard Hodge propagator for $\mathbb{R}^n$)**.** *The standard Hodge propagator for $\mathbb{R}^n$ with $n \geq 2$ satisfies*

$$\mathrm{P}_{\mathrm{std}}(x,y) = \frac{(-1)^{n+1}}{\mathrm{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^{n} (-1)^{i-1} \frac{x^i - y^i}{|x-y|^n} (\mathrm{dx}^1 - \mathrm{dy}^1)\cdots\widehat{(\mathrm{dx}^i - \mathrm{dy}^i)}\cdots(\mathrm{dx}^n - \mathrm{dy}^n)$$

and extends smoothly to $\mathrm{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof using the Green form. Consider the formulas from Proposition 7.4.1. Recall the formula for the codifferential

$$\mathrm{d}^* f(x)\,\mathrm{dx}^I = -\sum_{i \in I} \varepsilon(i,I) \frac{\partial f(x)}{\partial x^i}\,\mathrm{dx}^{I \backslash \{i\}}. \tag{7.14}$$

For $n \geq 3$, we compute

$$\begin{aligned} &(\mathbb{1} \otimes \mathrm{d}^*_y) G(x,y) \\ &\quad = \frac{1}{(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|} (\mathbb{1} \otimes \mathrm{d}^*_y)\Bigl(\frac{1}{|x-y|^{n-2}}(*\,\mathrm{dx}^I) \wedge \mathrm{dy}^I\Bigr) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|+|I|+n+1} \sum_{i\in I} \frac{x^i - y^i}{|x-y|^n} \varepsilon(i,I)(*\,\mathrm{dx}^I) \wedge \mathrm{dy}^{I\backslash\{i\}} \\
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|+|I|+n+1} \sum_{i\in I} \frac{x^i - y^i}{|x-y|^n} \varepsilon(i,I)\varepsilon(I,I^c)\,\mathrm{dx}^{I^c} \wedge \mathrm{dy}^{I\backslash\{i\}} \\
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n \frac{x^i - y^i}{|x-y|^n} \sum_{J\subset\{1,\dots,\hat{i},\dots,n\}} (-1)^{n|J|+|J|} \varepsilon(i,J)\varepsilon(J\cup\{i\},J^c\backslash\{i\}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{dx}^{J^c\backslash\{i\}} \wedge \mathrm{dy}^J \\
&= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n (-1)^{i-1} \frac{x^i - y^i}{|x-y|^n} (\mathrm{dx}^1 - \mathrm{dy}^1)\cdots\widehat{(\mathrm{dx}^i - \mathrm{dy}^i)}\cdots(\mathrm{dx}^n - \mathrm{dy}^n).
\end{aligned}
$$

In the last step, we used that

$$\varepsilon(i,J)\varepsilon(J\cup\{i\},J^c\backslash\{i\}) = (-1)^{i-1}\varepsilon(J,J^c\backslash\{i\})$$

and

$$
\begin{aligned}
&\mathrm{dx}^{J^c\backslash\{i\}} \wedge \mathrm{dy}^J \\
&= (-1)^{|J|}\varepsilon(J^c\backslash\{i\},J)[(\mathrm{dx}^1 - \mathrm{dy}^1)\cdots\widehat{(\mathrm{dx}^i - \mathrm{dy}^i)}\cdots(\mathrm{dx}^n - \mathrm{dy}^n)]_{J^c\backslash\{i\},J} \\
&= (-1)^{n|J|+|J|}[(\mathrm{dx}^1 - \mathrm{dy}^1)\cdots\widehat{(\mathrm{dx}^i - \mathrm{dy}^i)}\cdots(\mathrm{dx}^n - \mathrm{dy}^n)]_{J^c\backslash\{i\},J},
\end{aligned}
$$

where $[\cdot]_{I_1,I_2}$ denotes the part of the product which picks the first variable at positions I_1 and the second at positions I_2. The computation for $n = 2$ gives the same result, and the formula for $\mathrm{P}_{\mathrm{std}}$ is justified by Proposition 7.3.3.

We will now study whether $\mathrm{P}_{\mathrm{std}}$ extends smoothly to the blow-up. Consider the polar coordinates in one variable

$$
\begin{aligned}
\phi : [0,\infty) \times \mathbb{S}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\
(r,\omega,u) &\longmapsto (u + r\omega, u).
\end{aligned}
$$

There is a unique smooth map $\tilde{\phi}$ which fits in the blow-up diagram

$$\begin{tikzcd}
 & \mathrm{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n) \arrow[d, "\pi"] \\
{[0,\infty) \times \mathbb{S}^n \times \mathbb{R}^n} \arrow[ur, "\tilde{\phi}"] \arrow[r, "\phi"] & \mathbb{R}^n \times \mathbb{R}^n,
\end{tikzcd}$$

and it is a diffeomorphism of manifolds with boundary. We denote by

$$\phi_0 \coloneqq \phi\big|_{(0,\infty)\times\mathbb{S}^n\times\mathbb{R}^n} : (0,\infty)\times\mathbb{S}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}^n\times\mathbb{R}^n\backslash\Delta$$

the restriction of ϕ to the interior. The commutative diagram

$$\begin{tikzcd}
{[0,\infty)\times\mathbb{S}^n\times\mathbb{R}^n} \arrow[r, "\tilde{\phi}"] & \mathrm{Bl}_\Delta(\mathbb{R}^n\times\mathbb{R}^n) \\
(0,\infty)\times\mathbb{S}^n\times\mathbb{R}^n \arrow[u, hook] \arrow[r, "\phi_0"] & \mathbb{R}^n\times\mathbb{R}^n\backslash\Delta \arrow[u, hook, "\tilde{\imath}"]
\end{tikzcd}$$

shows that we can equivalently study whether the form

$$\mathrm{P}_{\mathrm{pol}} \coloneqq \phi_0^*\mathrm{P}_{\mathrm{std}}$$

admits a smooth extension $\widetilde{\mathrm{P}_{\mathrm{pol}}}$ to $[0,\infty)\times\mathbb{S}^{n-1}\times\mathbb{R}^n$. We have

$$\begin{aligned}
\mathrm{P}_{\mathrm{pol}}(r,\omega,u) &= \phi_0^*(\mathrm{d}_y^*G)(r,\omega,u) \\
&= \frac{(-1)^{n+1}}{r^{n-1}\mathrm{Vol}(\mathbb{S}^{n-1})}\sum_{i=1}^{n}(-1)^{i-1}\omega^i(r\,\mathrm{d}\omega^1+\omega^1\,\mathrm{dr})\cdots\widehat{(r\,\mathrm{d}\omega^i+\omega^i\,\mathrm{dr})}\cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r\,\mathrm{d}\omega^n+\omega^n\,\mathrm{dr}) \\
&= \frac{(-1)^{n+1}}{r^{n-1}\mathrm{Vol}(\mathbb{S}^{n-1})}\Big(r^{n-1}\sum_{1\le i\le n}(-1)^{i-1}\omega^i\,\mathrm{d}\omega^1\cdots\widehat{\mathrm{d}\omega^i}\cdots\mathrm{d}\omega^n \\
&\qquad -r^{n-2}\sum_{1\le i<j\le n}(-1)^{i+j}\omega^i\omega^j\,\mathrm{dr}\,\mathrm{d}\omega^1\cdots\widehat{\mathrm{d}\omega^i}\cdots\widehat{\mathrm{d}\omega^j}\cdots\mathrm{d}\omega^n \\
&\qquad +r^{n-2}\sum_{1\le j<i\le n}(-1)^{i+j}\omega^j\omega^i\,\mathrm{dr}\,\mathrm{d}\omega^1\cdots\widehat{\mathrm{d}\omega^j}\cdots\widehat{\mathrm{d}\omega^i}\cdots\mathrm{d}\omega^n\Big) \\
&= (-1)^{n+1}\frac{\mathrm{Vol}_{\mathbb{S}^{n-1}}(\omega)}{\mathrm{Vol}(\mathbb{S}^{n-1})},
\end{aligned}$$

where the last two sums canceled. The result extends smoothly beyond $r=0$ because it does not depend on r at all. $\square$

Proof using the heat kernel. Consider the polar coordinates with respect to the diagonal

$$\begin{aligned}
\varphi : [0,\infty)\times\mathbb{S}^{n-1}\times\mathbb{R}^n &\longrightarrow \mathbb{R}^n\times\mathbb{R}^n\backslash\Delta \\
(r,\omega,u) &\longmapsto (x,y) = (u+r\omega, u-r\omega).
\end{aligned}$$

There is a unique smooth map $\tilde{\varphi}$ which fits in the blow-up diagram

$$\begin{tikzcd}
 & \mathrm{Bl}_{\Delta}(\mathbb{R}^n \times \mathbb{R}^n) \arrow[d, "\pi"] \\
{[0,\infty) \times \mathbb{S}^n \times \mathbb{R}^n} \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"] & \mathbb{R}^n \times \mathbb{R}^n,
\end{tikzcd}$$

and it is a diffeomorphism of manifolds with boundary. The heat kernel (7.13) transforms under φ as

$$\begin{aligned} Q_t^{\mathrm{pol}}(r,\omega,u) &\coloneqq (\varphi^* Q_t)(r,\omega,u) \\ &= (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) 2^n r^{n-1} \,\mathrm{d}r \,\mathrm{Vol}(\omega) \\ &= (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) r^{n-1} \,\mathrm{d}r \,\mathrm{Vol}(\omega). \end{aligned}$$

It follows from Proposition 7.3.2 that for $(r,\omega,u) \in (0,\infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$ we have

$$\mathrm{P}_{\mathrm{pol}}(r,\omega,u) = (-1)^{n+1}\frac{1}{2}\lim_{t\to 0}\int_t^\infty \mathrm{d}^{*\mathrm{pol}} Q_\tau^{\mathrm{pol}}(r,\omega,u)\,\mathrm{d}\tau, \tag{7.15}$$

where $\mathrm{d}^{*\mathrm{pol}}$ is the codifferential on $\mathbb{R}^n \times \mathbb{R}^n$ computed with respect to the pullback of the product Riemannian metric. If $g : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ is an inner product, then the pullback Riemannian metric satisfies

$$g^{\mathrm{pol}} \coloneqq \varphi^*(g \oplus g)(r,\omega,u) = 2\begin{pmatrix} g(\omega,\omega) & 0 & 0 \\ 0 & r^2 g & 0 \\ 0 & 0 & g \end{pmatrix}; \tag{7.16}$$

it is degenerate at the boundary $r = 0$. Next, it is easy to check that the conformal transformation $g \mapsto \lambda g$ on vectors induces the conformal transformation $g \mapsto \frac{1}{\lambda^k} g$ on k-forms. Thus, if $\|\cdot\|^{\mathrm{pol}}$ denotes the point-wise norm with respect to g^{pol} for the standard Euclidean metric, then

$$\|\,\mathrm{d}r\,\|^{\mathrm{pol}} = 2^{-\frac{1}{2}}, \quad \|\mathrm{Vol}(u)\|^{\mathrm{pol}} = 2^{-\frac{n}{2}}, \quad \|\mathrm{Vol}(\omega)\|^{\mathrm{pol}} = 2^{-\frac{n-1}{2}} r^{-(n-1)},$$

and hence

$$\mathrm{Vol}^{\mathrm{pol}}(r,\omega,u) = 2^n r^{n-1} \,\mathrm{d}r \,\mathrm{Vol}(\omega)\mathrm{Vol}(u).$$

In order to get the Hodge star $*^{\mathrm{pol}}$, we compute

$$\begin{aligned}\mathrm{dr}\,\mathrm{Vol}(\omega)\wedge *^{\mathrm{pol}}(\mathrm{dr}\,\mathrm{Vol}(\omega)) &\overset{!}{=} \|\,\mathrm{dr}\,\mathrm{Vol}(\omega)\|^{\mathrm{pol}^2}\mathrm{Vol}^{\mathrm{pol}}(r,\omega,u)\\ &= r^{-(n-1)}\,\mathrm{dr}\,\mathrm{Vol}(\omega)\mathrm{Vol}(u),\\ \mathrm{dr}\,\mathrm{Vol}(u)\wedge *^{\mathrm{pol}}(\mathrm{dr}\,\mathrm{Vol}(u)) &\overset{!}{=} \|\,\mathrm{dr}\,\mathrm{Vol}(u)\|^{\mathrm{pol}^2}\mathrm{Vol}^{\mathrm{pol}}(r,\omega,u)\\ &= \frac{1}{2}r^{n-1}\,\mathrm{dr}\,\mathrm{Vol}(\omega)\mathrm{Vol}(u).\end{aligned}$$

Using the product structure (7.16), it follows that

$$\begin{aligned}*^{\mathrm{pol}}(\mathrm{dr}\,\mathrm{Vol}(\omega)) &= r^{-(n-1)}\mathrm{Vol}(u)\quad\text{and}\\ *^{\mathrm{pol}}(\mathrm{dr}\,\mathrm{Vol}(u)) &= \frac{1}{2}r^{n-1}\mathrm{Vol}(\omega).\end{aligned}$$

Recalling the definition of the codifferential $\mathrm{d}^*\alpha = (-1)^{n(k-1)+1} * \mathrm{d} * \alpha$, we compute

$$\begin{aligned}(*^{\mathrm{pol}}\mathrm{d}*^{\mathrm{pol}})\big(Q_t^{\mathrm{pol}}(r,\omega,u)\big) &= *^{\mathrm{pol}}\mathrm{d}\Big((\pi t)^{-\frac{n}{2}}\exp\Big(-\frac{r^2}{t}\Big)\mathrm{Vol}(u)\Big)\\ &= *^{\mathrm{pol}}\Big(\frac{-2r}{t}(\pi t)^{-\frac{n}{2}}\exp\Big(-\frac{r^2}{t}\Big)\,\mathrm{dr}\,\mathrm{Vol}(u)\Big)\\ &= -\frac{1}{t}(\pi t)^{-\frac{n}{2}}\exp\Big(-\frac{r^2}{t}\Big)r^n\mathrm{Vol}(\omega),\end{aligned}$$

and because the product dimension is even, we obtain

$$\mathrm{d}^{*\mathrm{pol}}Q_t^{\mathrm{pol}}(r,\omega,u) = \frac{1}{t}(\pi t)^{-\frac{n}{2}}\exp\Big(-\frac{r^2}{t}\Big)r^n\mathrm{Vol}(\omega).$$

In order to integrate this according to (7.15), we will make use of the Γ-function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,\mathrm{dx}$. Then, we have

$$\begin{aligned}(-1)^{n+1}\mathrm{P}_{\mathrm{pol}}(r,\omega,u) &= \frac{1}{2}\lim_{t\to 0}\int_t^\infty \mathrm{d}^{*\mathrm{pol}}Q_\tau^{\mathrm{pol}}(r,\omega,u)\,\mathrm{d}\tau\\ &= \frac{1}{2}\pi^{-\frac{n}{2}}r^n\Big(\lim_{t\to 0}\int_t^\infty \tau^{-\frac{n}{2}-1}\exp\Big(-\frac{r^2}{\tau}\Big)\,\mathrm{d}\tau\Big)\mathrm{Vol}(\omega)\\ &= \frac{1}{2}\pi^{-\frac{n}{2}}r^n\Big(\lim_{t\to 0}\int_{\frac{r^2}{t}}^0 \Big(\frac{r^2}{z}\Big)^{-\frac{n}{2}-1}\exp(-z)(-z^{-2}r^2\,\mathrm{dz})\Big)\mathrm{Vol}(\omega)\\ &= \frac{1}{2}\pi^{-\frac{n}{2}}\Big(\lim_{t\to 0}\int_0^{\frac{r^2}{t}} z^{\frac{n}{2}-1}\exp(-z)\,\mathrm{dz}\Big)\mathrm{Vol}(\omega)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}\mathrm{Vol}(\omega) \\
&= \frac{\mathrm{Vol}(\omega)}{\mathrm{Vol}(\mathbb{S}^{n-1})},
\end{aligned}
$$

which recovers the formula in the other proof of Proposition 7.4.2. □

Example 7.4.3 (The case $n = 1$). It is easy to check directly that

$$\mathrm{P}(x, y) \coloneqq \theta(x - y),$$

where θ is the Heavyside step function, is the Schwartz form of a Hodge homotopy $\mathcal{P}$. Indeed, for any smooth $f : \mathbb{R} \to \mathbb{R}$ with compact support, we have

$$\mathcal{P}(f\,\mathrm{dx})(y) = \int_x \mathrm{P}(x, y) f(x)\,\mathrm{dx} = \int_y^\infty f(x)\,\mathrm{dx},$$

and hence $\mathrm{d} \circ \mathcal{P} = -\mathbb{1}$. It is also easy to check that the following is the Laplace Green form:

$$G(x, y) = -\frac{1}{2}|x - y|(\mathrm{dx} - \mathrm{dy}).$$

Indeed, using $\mathrm{sgn}(x) = 2\theta(x) - 1$, we get

$$\frac{\partial^2}{\partial y^2}\frac{1}{2}|x - y| = \frac{\partial}{\partial y}\frac{1}{2}\,\mathrm{sgn}(y - x) = \delta(y - x) = \delta(x - y),$$

which shows (7.12). We carefully compute

$$
\begin{aligned}
(\mathbb{1} \otimes \mathrm{d}_y^*)G(x, y) &= \mathrm{d}_y^*\Big(\frac{1}{2}|x - y|\,\mathrm{dy}\Big) \\
&= -\frac{1}{2}\frac{\partial}{\partial y}|x - y| \\
&= -\frac{1}{2}(2\theta(y - x) - 1) \\
&= -\theta(y - x) + \frac{1}{2} \\
&= \theta(x - y) - \frac{1}{2}
\end{aligned}
$$

and apply Proposition 7.3.3 to get

$$\mathrm{P}_{\mathrm{std}}(x, y) = \theta(x - y) - \frac{1}{2}.$$

In particular, we see that the sign agrees with the direct computation of the Hodge propagator above. Smoothness on the blow-up is for $n = 1$ equivalent to smoothness on the closures of connected components of $\mathbb{R} \times \mathbb{R} \backslash \Delta$; this is readily satisfied. $\triangleleft$

Question 7.4.4. Is it possible to use the asymptotic heat kernel expansion and formula (4.16) or the asymptotic Green kernel expansion and formula (7.11) to infer from the case of $\mathbb{R}^n$ with the standard Euclidean metric g_0 that $\mathrm{P}_{\mathrm{std}}$ extends smoothly to the blow-up for flat manifolds, i.e., locally isometric to $(\mathbb{R}^n, g_0)$?

7.5. Standard Hodge propagator for 1- and 2-sphere

In this section, we denote the Hodge propagator for $\mathbb{S}^n$ constructed in Part I by $\mathrm{P}_{\mathrm{art}}$. We would like to use $\mathrm{P}_{\mathrm{art}}$ to study the standard Hodge propagator $\mathrm{P}_{\mathrm{std}}$ for $\mathbb{S}^n$.

For $\mathbb{S}^1$, we can compute K_t and hence $\mathrm{P}_{\mathrm{std}}$ explicitly.

Example 7.5.1 ($\mathrm{P}_{\mathrm{std}}$ for $\mathbb{S}^1$). We write $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and use the coordinate $x \in [0, 2\pi)$. Because $\mathbb{S}^1$ is flat, we have

$$\Delta = -\frac{\partial^2}{\partial x^2}.$$

Solving the eigenvalue problem $\Delta\,\omega = \lambda\omega$ for $\omega \in \Omega(\mathbb{S}^1)$ and $\lambda \in \mathbb{R}$, we get $\lambda \in \{0, n^2 \mid n \in \mathbb{N}\}$ and the corresponding eigenvectors

$$\Big\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\cos(nx)\,\mathrm{dx}, \frac{1}{\sqrt{\pi}}\sin(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\,\mathrm{dx} \mid n \in \mathbb{N}\Big\},$$

which we normalized in the L^2-norm. Plugging in (7.9), we get

$$\begin{aligned}\mathrm{K}_t(x_1, x_2) &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}(\cos(nx_1)\cos(nx_2) + \sin(nx_1)\sin(nx_2))(\mathrm{dx}_1 - \mathrm{dx}_2) \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos(nx_1 - nx_2)(\mathrm{dx}_1 - \mathrm{dx}_2).\end{aligned}$$

Applying the product codifferential, we get

$$\begin{aligned}\mathrm{d}^*\mathrm{K}_t(x_1, x_2) &= \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\Big[-\frac{\partial}{\partial x_1}\cos(nx_1 - nx_2) + \frac{\partial}{\partial x_2}\cos(nx_1 - nx_2)\Big] \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t} n \sin(nx_1 - nx_2).\end{aligned}$$

Finally, the integration gives

$$\mathrm{P}_{\mathrm{std}}(x_1, x_2) = \frac{1}{2}\int_0^{\infty} \mathrm{d}^*\mathrm{K}_t(x_1, x_2)\,\mathrm{dt}$$

$$\begin{aligned}
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\Big(\int_0^{\infty} e^{-n^2 t}\,\mathrm{d}t\Big) n\sin(nx_1 - nx_2)\\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\big(n(x_1-x_2)\big)}{n}\\
&= \frac{1}{2\pi}\begin{cases}\pi-(x_1-x_2) & x_1 > x_2,\\ -\pi-(x_1-x_2) & x_1 < x_2.\end{cases}\\
&= \frac{1}{2\pi}\big(\alpha(x_1,x_2)-\pi\big).
\end{aligned}$$

This is precisely $\mathrm{P}_{\mathrm{art}}$ from Part I. ◁

For $n \geq 2$, an explicit formula for any of $\mathrm{P}_{\mathrm{std}}$, G or K_t seems to be unknown. For $\mathbb{S}^2$, the formula for $\mathrm{P}_{\mathrm{std}}$ on functions was derived by Dr. A. Hermann.

Our idea to study $\mathrm{P}_{\mathrm{std}}$ via $\mathrm{P}_{\mathrm{art}}$ is to examine the uniqueness of Hodge propagators. Consider the Schwartz form $\mathrm{H}(x,y) = \frac{1}{V}(\mathrm{Vol}(x)+(-1)^n\mathrm{Vol}(y))$ of the harmonic projection for $\mathbb{S}^n$. Let $C_2(\mathbb{S}^n) \coloneqq \mathbb{S}^n \times \mathbb{S}^n\backslash\Delta$ denote the configuration space, and let

$$\mathcal{S}_n \coloneqq \{\mathrm{P} \in \Omega^{n-1}(C_2(\mathbb{S}^n)) \mid \mathrm{dP} = (-1)^n\mathrm{H}\} \tag{7.17}$$

be the space of primitives to $(-1)^n\mathrm{H}$. We know that $\mathrm{P}_{\mathrm{std}} \in \mathcal{S}_n$. The following holds.

Proposition 7.5.2 (The space of primitives to H for $\mathbb{S}^n$)**.** *Let*

$$V_n \coloneqq \begin{cases}\Omega^{n-2}(C_2(\mathbb{S}^n))/\mathrm{d}\Omega^{n-3}(C_2(\mathbb{S}^n)) & \text{for } n \geq 3,\\ \Omega^0(C_2(\mathbb{S}^n))/\mathbb{R} & \text{for } n = 2,\end{cases}$$

where $\mathbb{R} \subset \Omega^0(C_2(\mathbb{S}^n))$ *denotes the constants. The action* $\rho : V_n \times \mathcal{S}_n \to \mathcal{S}_n$, $(\lambda, \mathrm{P}) \mapsto \mathrm{P} + \mathrm{d}\lambda$ *of the additive group* V_n *on* $\mathcal{S}_n$ *defines the structure of an affine space on* $\mathcal{S}_n$ *for* $n \geq 2$*. If we require* $SO(n+1)$ *or* $(-1)^n\tau^*$*-invariance, then the same holds with* Ω *replaced by the correspondingly invariant forms.*

Proof. We have to check that the action ρ is free and transitive. For P_1, $\mathrm{P}_2 \in \mathcal{S}_n$, the difference $\eta \coloneqq \mathrm{P}_1 - \mathrm{P}_2$ is a closed $(n-1)$-form; it is exact because $C_2(\mathbb{S}^n)$ is homotopy equivalent to $\mathbb{S}^n$. A primitive λ_1 is an $n-2$ form. If λ_2 is another primitive, then $\lambda_1 - \lambda_2$ is closed, and hence it is a constant for $n = 2$ and an exact form for $n \geq 3$. Therefore, $\mathcal{S}_n$ is an affine space over V_n. As for the invariance, we can average a primitive of an invariant form over $SO(n+1)$ or take $\frac{1}{2}(\mathbb{1} + (-1)^n\tau^*)$. □

Note that $\mathcal{S}_1 \simeq \mathbb{R}$ by adding the constant and that all functions on $C_2(\mathbb{S}^1)$ are coexact.

Therefore, $\mathrm{P}_{\mathrm{std}}$ for $\mathbb{S}^1$ can not be characterized as a unique coexact solution of the differential equation for the Hodge propagator.

Proposition 7.5.3 (Coexactness of artificial Hodge propagator)**.** *The Hodge propagator* $\mathrm{P}_{\mathrm{art}} \in \Omega^{n-1}(\mathbb{S}^n \times \mathbb{S}^n \backslash \Delta)$ *constructed in Part I is coexact for every* $n \in \mathbb{N}$.

Proof. First of all, we rewrite

$$\begin{aligned}
\omega_k(x,y) &= \frac{1}{k!(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n+1}} x^{\sigma_1} y^{\sigma_1} \,\mathrm{d}x^{\sigma_3} \cdots \mathrm{d}x^{\sigma_{2+k}} \,\mathrm{d}y^{\sigma_{3+k}} \cdots \mathrm{d}y^{\sigma_{n+1}} \\
&= (-1)^k \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I| = k+1}} \iota_x(\mathrm{d}x^I) \wedge \underbrace{\iota_y *^{\mathbb{R}^{n+1}}}_{*^{\mathbb{S}^n}}(\mathrm{d}y^I).
\end{aligned}$$

Recall the formulas $\mathrm{d}^*\alpha = (-1)^{d(k-1)+1} * \mathrm{d} * \alpha$ and $* * \alpha = (-1)^{k(n-k)}\alpha$ for $\alpha \in \Omega^k(M)$, where $d = \dim(M)$. For all $(x,y) \in \mathbb{S}^n \times \mathbb{S}^n \backslash \Delta$, we compute

$$\begin{aligned}
\mathrm{d}_y^* \mathrm{P}_{\mathrm{art}}(x,y) &= \mathrm{d}_y^* \Big(\sum_{k=0}^{n-1} (-1)^k g_k(x \cdot y) \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I| = k+1}} (\iota_x \,\mathrm{d}x^I) \wedge *^{\mathbb{S}^n} (\mathrm{d}y^I) \Big) \\
&= \sum_{k=0}^{n-1} \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I| = k+1}} (\iota_x \,\mathrm{d}x^I) \wedge \mathrm{d}_y^* \big(g_k(x \cdot y) *^{\mathbb{S}^n} (\mathrm{d}y^I) \big) \\
&\underset{\substack{\uparrow \\ \mathrm{d}_y^* = (-1)^{n(n-k)+1} *^{\mathbb{S}^n} \mathrm{d} *^{\mathbb{S}^n} \\ *^{\mathbb{S}^n} *^{\mathbb{S}^n} = (-1)^{(k+1)(n-k-1)} \mathbb{1} \\ \text{tot. sign} = (-1)^k}}{=} \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I| = k+1}} (\iota_x \,\mathrm{d}x^I) \wedge *^{\mathbb{S}^n} \mathrm{d}_y \big(g_k(x \cdot y) \,\mathrm{d}y^I \big) \\
&= \sum_{k=0}^{n-1} (-1)^k g_k'(x \cdot y) \sum_{\substack{I \subset \{1,\dots,n+1\} \\ |I| = k+1}} \underbrace{\sum_{i \in I} \sum_{j \in \{1,\dots,n+1\} \backslash I} \varepsilon(i,I)\varepsilon(j,I) x^i x^j \,\mathrm{d}x^{I \backslash \{i\}} \wedge *^{\mathbb{S}^n} (\mathrm{d}y^{I \cup \{j\}})}_{=0} \\
&= 0.
\end{aligned}$$

The cancellation occurs because the summand (I, i, j) contains the same terms as the summand $(I' = I \backslash \{i\} \cup j, j, i)$, and the signs satisfy

$$\varepsilon(j, I')\varepsilon(i, I') = -\varepsilon(i, I)\varepsilon(j, I).$$

We have

$$\mathrm{H}_{n-1}(\Omega(\mathbb{S}^n \times \mathbb{S}^n \backslash \Delta), \mathrm{d}^*) \simeq \mathrm{H}^{n+1}_{\mathrm{dR}}(\mathbb{S}^n \times \mathbb{S}^n \backslash \Delta) = 0,$$

and hence any coclosed $(n-1)$-form is coexact. □

Proposition 7.5.4 (Smooth extension to the blow-up for $\mathbb{S}^2$)**.** *The standard Hodge propagator for $\mathbb{S}^2$ extends smoothly to the blow-up.*

Proof. Let P_1, $\mathrm{P}_2 \in \mathcal{S}_2$ be two SO(3)-symmetric solutions. Proposition 7.5.2 asserts that there is a smooth SO(3)-symmetric function $\lambda : C_2(\mathbb{S}^2) \to \mathbb{R}$ such that $\mathrm{P}_1 - \mathrm{P}_2 = \mathrm{d}\lambda$. Because SO(3) acts on $\mathbb{S}^2$ transitively, there is a smooth function $f : [-1, 1) \to \mathbb{R}$ such that

$$\lambda(x, y) = f(x \cdot y) \quad \text{for all } (x, y) \in C_2(\mathbb{S}^2).$$

Note that one can let f explode at 1 and obtain Hodge propagators which do not extend smoothly to the blow-up. Let us assume, in addition, that $\mathrm{P}_1 - \mathrm{P}_2$ is coexact. We obtain

$$0 = \mathrm{d}^*(\mathrm{P}_1 - \mathrm{P}_2) = \mathrm{d}^*\mathrm{d}\lambda = \Delta\,\lambda.$$

Therefore, λ is a harmonic function on $C_2(\mathbb{S}^2)$. Denoting

$$B(x, y) \coloneqq x \cdot y,$$

we can write $\lambda = f \circ B$, which implies

$$\Delta(f \circ B) = f'' \|\nabla B\|^2 + f'\,\Delta\,B.$$

The computation of $\|\nabla B\|$ and $\Delta\,B$ is straightforward and we will do it for any $n \in \mathbb{N}$. If $\tilde{f} : \mathbb{S}^n \to \mathbb{R}$ is a smooth function and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is defined by

$$f(x) \coloneqq \tilde{f}\Big(\frac{x}{|x|}\Big) \quad \text{for all } x \in \mathbb{R}^{n+1} \backslash \{0\},$$

then

$$\Delta^{\mathbb{S}^n}\,\tilde{f} = (\Delta^{\mathbb{R}^{n+1}}\,f)\Big|_{\mathbb{S}^n} \quad \text{and} \quad \nabla^{\mathbb{S}^n}\tilde{f} = (\nabla^{\mathbb{R}^{n+1}} f)\Big|_{\mathbb{S}^n}.$$

Here $\Delta^{\mathbb{S}^n}$, resp. $\nabla^{\mathbb{S}^n}$ are the Laplacian, resp. the gradient on $\mathbb{S}^n$ expressed in terms of the corresponding operators $\Delta^{\mathbb{R}^{n+1}}$ and $\nabla^{\mathbb{R}^{n+1}}$ on $\mathbb{R}^{n+1}$, where $\mathbb{S}^n$ is embedded into. We compute

$$\begin{aligned}
\Delta_x^{\mathbb{R}^{n+1}}\Big(\frac{x}{|x|}\cdot y\Big) &= \sum_{i=1}^{n+1} -\frac{\partial}{\partial x^i}\Big(\frac{y^i}{|x|} - \frac{x^i}{|x|^3}x\cdot y\Big) \\
&= \sum_{i=1}^{n+1} \frac{x^i y^i}{|x|^3} - 3\frac{x^i x^i}{|x|^5}x\cdot y + \frac{x\cdot y}{|x|^3} + \frac{x^i y^i}{|x|^3} \\
&= 0, \\
\nabla_x^{\mathbb{R}^{n+1}}\Big(\frac{x}{|x|}\cdot y\Big) &= \sum_{i=1}^{n+1}\Big(\frac{y^i}{|x|} - \frac{x^i}{|x|^3}x\cdot y\Big)\frac{\partial}{\partial x^i} \quad \text{and} \\
\|\nabla(x\cdot y)\|^2 &= \Big\|\sum_{i=1}^{n+1}(y^i - (x\cdot y)x^i)\frac{\partial}{\partial x^i} + \sum_{i=1}^{n+1}(x^i - (x\cdot y)y^i)\frac{\partial}{\partial y^i}\Big\|^2 \\
&= \sum_{i=1}^{n+1}(y^i - (x\cdot y)x^i)^2 + (x^i - (x\cdot y)y^i)^2 \\
&= 1 - 2(x\cdot y)^2 + (x\cdot y)^2 + 1 - 2(x\cdot y)^2 + (x\cdot y)^2 \\
&= 2(1 - (x\cdot y)^2).
\end{aligned}$$

Therefore, $\Delta B = 0$, $\|\nabla B(x_1, x_2)\|^2 = 2(1 - (x_1\cdot x_2)^2)$, and we arrive to the equation

$$2(1-u^2)f''(u) = 0 \quad \text{for all } u \in [-1,1)$$

and a smooth function $f : [-1,1) \to \mathbb{R}$. The only solution is a linear function, and it must hold

$$\lambda(x_1,x_2) = aB(x_1,x_2) + b \quad \text{for some } a,b\in\mathbb{R}.$$

We see that λ extends smoothly to the blow-up. □

If we determine the constant a in the proof of Proposition 7.5.4, then we get a formula relating $\mathcal{P}_{\mathrm{std}}$ to $\mathrm{P}_{\mathrm{art}}$ for $\mathbb{S}^2$.

8. BV-formalism for IBL-infinity theory on cyclic cochains

In this chapter, we use the BV-formalism to formulate the theory of dIBL-algebras on cyclic cochains of an odd symplectic vector space and its twisting with a Maurer-Cartan element. The “fields” are cyclic Hochschild chains and the BV-operator comes from the algebraic string bracket and cobracket. We hope that this point of view will help to understand relations between string topology, Chern-Simons theory, symplectic field theory and string field theory (let’s say “S(F)T”).

In Section 8.1, we summarize relevant IBL_∞-theory from [CFL15].

In Section 8.2, we define the canonical string BV-operator on the space of “observables” and the total action; it consists of the free part and of the interaction part which corresponds to the Maurer-Cartan element. We argue that the quantum master equation is satisfied for the canonical and the Chern-Simons Maurer-Cartan element and that the string BV-operator twisted by the action corresponds to the twisted dIBL-algebra (Proposition 8.2.1). In general, we show that the action satisfies the quantum master equation if and only if its interaction part encodes a Maurer-Cartan element and that the twistings in both formalisms agree (Proposition 8.2.2).

In Section 8.3, we sketch how to apply ideas from [DJP19] for quantum L_∞-algebras to IBL_∞-algebras. We ask whether their formulas for the effective action and the path integral which come from the Homological Perturbation Lemma (and Wick’s Theorem) agree with the formulas from [CFL15] with summations over ribbon graphs (Question 8.3.1).

8.1. Summary of IBL-infinity theory for cyclic DGA

Let (V, Π, m_1, m_2) be a cyclic DGA of degree n of finite type. We have:

- Cyclic cochains $C(V) = \hat{\mathrm{B}}^*_{\mathrm{cyc}} V[2-n]$,
- Canonical dIBL-algebra $\mathfrak{q}_{110}$, $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ on $C(V)$ denoted by $\mathrm{dIBL}(C(V))$,

- Canonical Maurer-Cartan element $\mathfrak{m} = (\mathfrak{m}_{10})$,
- Twisted dIBL-algebra $\mathfrak{q}^{\mathfrak{m}}_{110}$, $\mathfrak{q}_{210}$, $\mathfrak{q}_{120}$ on $C(V)$ denoted by $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$.

Recall from Appendix C that, up to signs and permutation of variables, $\mathfrak{q}_{210}$ is the cyclization $^+$ of the Gerstenhaber bracket $[\cdot,\cdot]$ and that it holds $\Delta_{\mathrm{cyc}} = \mu_{\mathrm{cyc}} \circ \mathfrak{q}_{120}$, where Δ_{cyc} is an extension of the Schwarz's BV-operator on polynomial functions on V to $C(V)$ and μ_{cyc} is the cyclic shuffle coproduct.

Let (V', Π', m_1', m_2') be another cyclic DGA of degree n of finite type, and let

$$\begin{tikzcd} (V, m_1) \arrow[loop left, "\mathcal{P}"'] \arrow[r, shift left, "\pi"] & (V', m_1') \arrow[l, shift left, "\iota"] \end{tikzcd} \tag{8.1}$$

be a deformation retraction. We have the following:

- IBL_∞-morphism $\mathfrak{f} : \mathrm{dIBL}(C(V)) \to \mathrm{dIBL}(C(V'))$ with components $\mathfrak{f}_{klg} : \hat{\mathrm{E}}_k C(V) \to \hat{\mathrm{E}}_l C(V')$ such that

  $$\mathfrak{f}_{110} = \iota^* : (C(V), \hat{\mathfrak{q}}_{110}) \longrightarrow (C(V'), \hat{\mathfrak{q}}'_{110})$$

  is a quasi-isomorphism. The number $\mathfrak{f}_{klg}(\mathrm{s}^k\psi_1 \otimes \cdots \otimes \psi_k)(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \omega_l)$ for ψ_1, …, $\psi_k \in \hat{\mathrm{B}}^*_{\mathrm{cyc}}V$ and $\omega_1 = \omega_{11} \cdots \omega_{1s_1}$, …, $\omega_l = \omega_{l1} \cdots \omega_{ls_l} \in \mathrm{B}^{\mathrm{cyc}}V'$, where s is the formal symbol of degree $n-3$, is computed via summation over ribbon graphs with interior vertices decorated with ψ_1, …, ψ_k, interior edges decorated with the propagator — the Schwartz kernel P of $\mathcal{P}$ — and exterior vertices decorated with ω_{ij} at the i-th boundary component and with j respecting the orientation. We will refer to such graphs shortly as *Feynman graphs.*

- Pushforward Maurer-Cartan element $\mathfrak{n} \coloneqq \mathfrak{f}_*\mathfrak{m}$ with components $\mathfrak{n}_{lg} \in \hat{\mathrm{E}}_l C(V)$. The number $\mathfrak{n}_{lg}(\mathrm{s}^l\omega_1 \otimes \ldots \otimes \omega_l)$ is computed via summation over Feynman graphs as above with trivalent internal vertices $\mathfrak{m}_{10}$.

- Twisted IBL_∞-morphism $\mathfrak{f}^{\mathfrak{m}} : \mathrm{dIBL}^{\mathfrak{m}}(C(V)) \to \mathrm{dIBL}^{\mathfrak{n}}(C(V'))$ with components $\mathfrak{f}^{\mathfrak{m}}_{klg} : \hat{\mathrm{E}}_k C(V) \to \hat{\mathrm{E}}_l C(V')$ such that

  $$\begin{aligned} \mathfrak{f}^{\mathfrak{m}}_{110} &: (C(V), \hat{\mathfrak{q}}^{\mathfrak{m}}_{110}) \longrightarrow (C(V'), \hat{\mathfrak{q}}'^{\mathfrak{n}}_{110}) \\ &= \iota^* + \mathfrak{f}_{210} \circ_1 \mathfrak{m}_{10} + \frac{1}{2!}\mathfrak{f}_{310} \circ_{1,1} (\mathfrak{m}_{10}, \mathfrak{m}_{10}) + \cdots \end{aligned}$$

  is a quasi-isomorphism. The number $\mathfrak{f}_{klg}(\mathrm{s}^k\psi_1 \otimes \cdots \otimes \psi_k)(\mathrm{s}^l\omega_1 \otimes \cdots \otimes \omega_l)$ is computed via summation over Feynmann graphs with interior vertices ψ_1, …, ψ_k and $\mathfrak{m}_{10}$.

Recall that an IBL_∞-quasi-isomorphism is automatically an IBL_∞-homotopy equivalence; this has been proven in [CFL15, Theorem 1.2] via obstruction theory. Obstruction theory also gives the existence of a minimal model of any IBL_∞-algebra, which is the content of [CFL15, Theorem 1.3]. What is not proven yet but what we suspect is true is the following:

(1) For a different choice of the deformation retraction (8.1), the corresponding IBL_∞-morphisms $\mathfrak{f}$ and $\mathfrak{f}'$ are IBL_∞-homotopic.

(2) If ι is in addition a morphism of Poincaré DGA's (perhaps simply-connected), then $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$ and $\mathrm{dIBL}^{\mathfrak{m}'}(C(V'))$ are IBL_∞-homotopy equivalent (perhaps even the Maurer-Cartan elements $\mathfrak{m}'$ and $\mathfrak{f}_*\mathfrak{m}$ are gauge equivalent).

8.2. BV-formulation for dIBL-algebra on cyclic cochains

In the setting of Section 8.1, we define

$$B(V) := \mathrm{B}^{\mathrm{cyc}}V[3-n] \quad \text{and} \quad \mathcal{F}(B(V)) := \hat{\mathrm{E}}C(V)((\hbar)) = \mathrm{E}C(V)\,\hat{\otimes}\,\mathbb{K}((\hbar)).$$

We call $\mathcal{F}(B(V))$ the *space of functions on* $B(V)$. This makes sense because $C(V) \subset (\mathrm{B}^{\mathrm{cyc}}V[2-n])^{*_{\mathrm{g}}}$, $\mathrm{E}C(V) = \mathrm{S}(C(V)[1])$ and the symmetric algebra of the dual can be viewed as polynomial functions. In contrast to [DJP19], we do not have any canonical odd symplecticform on $B(V)$, and hence there is no Schwarz's BV-operator on $\mathcal{F}(B(V))$. However, we have the following BV-operators from [CFL15]:

$$\begin{aligned}\Delta_0 &:= \hat{\mathfrak{q}}_{120} + \hbar\hat{\mathfrak{q}}_{210},\\ \Delta &:= \hat{\mathfrak{q}}_{110} + \Delta_0,\\ \Delta^{\mathfrak{m}} &:= \widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{10}} + \Delta\,.\end{aligned}$$

The first BV-operator is canonical for an odd symplectic vector space (V,Π), and we call it the *string* BV-*operator*. It is a BV-operator because $\hat{\mathfrak{q}}_{120}$ is a derivative of order 1, $\hat{\mathfrak{q}}_{210}$ a derivative of order 2, and it holds $\Delta_0 \circ \Delta_0 = 0$. The second BV-operator is canonical for a cyclic cochain complex (V,Π,m_1) and the third for a cyclic DGA (V,Π,m_1,m_2) (in both cases, (V,Π) is an odd symplectic vector space). Note that the perturbations $\hat{\mathfrak{q}}_{110}$, resp. $\widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{10}}$ are first order differential operators.

For Δ, we consider the associated Gerstenhaber bracket $\{\cdot,\cdot\} : \mathcal{F}(B(V))^{\otimes 2} \to \mathcal{F}(B(V))$, which is for all Ψ_1, $\Psi_2 \in \mathcal{F}(B(V))$ given by

$$\{\Psi_1, \Psi_2\} := (-1)^{|\Psi_1|}(\Delta(\Psi_1\Psi_2) - \Delta(\Psi_1)\Psi_2 - (-1)^{|\Psi_1|}\Psi_1\,\Delta(\Psi_2)).$$

It is easy to see that

$$\{\Psi_1, \Psi_2\} = (-1)^{1+(|\Psi_1|+1)(|\Psi_2|+1)}\{\Psi_2, \Psi_1\}.$$

If $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}^{\mathfrak{m}}$ are the Gerstenhaber brackets for Δ_0 and $\Delta^{\mathfrak{m}}$, respectively, then it holds

$$\{\cdot,\cdot\}_0 = \{\cdot,\cdot\}^{\mathfrak{m}} = \{\cdot,\cdot\}$$

because Δ, Δ_0 and $\Delta^{\mathfrak{m}}$ differ only by differential operators of order ≤ 1.

Consider the cyclizations m_1^+, $m_2^+ : \mathrm{B}^{\mathrm{cyc}}V \to \mathbb{R}$ defined for all v_1, v_2, $v_3 \in V[1]$ by

$$m_1^+(v_1, v_2) := \Pi(m_1(v_1), v_2) \quad \text{and} \quad m_2^+(v_1, v_2, v_3) := \Pi(m_2(v_1, v_2), v_3).$$

They have degree $3 - n$ as maps, thus degree $n - 3$ as elements of $\mathrm{B}^*_{\mathrm{cyc}}V$ with the cohomological grading, and thus $\mathrm{s}m_1^+$, $\mathrm{s}m_2^+ \in \mathcal{F}(B(V))$ have degree $2(n-3)$. They are "linear functions" in the sense that $\mathrm{s}m_1^+$, $\mathrm{s}m_2^+ \in \hat{\mathrm{E}}_1C(V)$. We define the *total action*

$$S := \underbrace{(-1)^{n-2}(\mathrm{s}m_1^+)\hbar^{-1}}_{=:\, S_{\mathrm{free}}} + \underbrace{(-1)^{n-2}(\mathrm{s}m_2^+)\hbar^{-1}}_{=:\, S_{\mathrm{int}}} \in \mathcal{F}(B(V)) \tag{8.2}$$

and call S_{free} the *free action* and S_{int} the *interaction.* The total action is linear and has degree 0. As a function, we can write

$$S(\mathrm{s}b) = (-1)^{n-2}m_1^+(b)\hbar^{-1} + (-1)^{n-2}m_2^+(b)\hbar^{-1} \quad \text{for } b \in \mathrm{B}^{\mathrm{cyc}}V.$$

Recall that $\mathfrak{m}_{10} = (-1)^{n-2}\mathrm{s}m_2^+$ is the canonical Maurer-Cartan element.

Proposition 8.2.1 (BV-action for canonical Maurer-Cartan element)**.** *Let* (V, Π, m_1, m_2) *be a cyclic* DGA *of degree* n *of finite type, and let* $\Delta_0 = \hat{\mathfrak{q}}_{120} + \hbar\hat{\mathfrak{q}}_{210}$ *be the string* BV-*operator on* $\mathcal{F}(B(V))$*. The total action* $S \in \mathcal{F}(B(V))$ *from* (8.2) *satisfies the* quantum master equation

$$\Delta_0\, S + \frac{1}{2}\{S, S\}_0 = 0, \tag{8.3}$$

and it holds

$$\Delta^{\mathfrak{m}}(f) = \Delta_0(f) + \{S, f\}_0 \quad \textit{for all } f \in \mathcal{F}(C(V)), \tag{8.4}$$

where $\Delta^{\mathfrak{m}} = \hat{\mathfrak{q}}_{110}^{\mathfrak{m}} + \hat{\mathfrak{q}}_{120} + \hbar\hat{\mathfrak{q}}_{210}$ *is the* BV-*operator associated to the twisted* dIBL-*algebra* $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$.

Proof. Let $f \in \mathcal{F}(B(V))$, and consider the left multiplication $L_f : \mathcal{F}(B(V)) \to \mathcal{F}(B(V))$ given by $g \mapsto fg$. If we view f as a linear map $\mathcal{F}(B(V)) \to \mathcal{F}(B(V))$ which maps the constant 1 to f and vanishes on the reduced part, then we can write $L_f = f \odot \mathbb{1}$, where $\odot$ is the convolution product on the symmetric algebra (see Appendix D). Using formulas from Proposition D.4.4, we compute

$$\begin{aligned}\hat{\mathfrak{q}}_{210} \circ L_f &= \hat{\mathfrak{q}}_{210}(f \odot \mathbb{1}) \\ &= \mathfrak{q}_{210} \circ_{0,2} (f, \mathbb{1}) + \mathfrak{q}_{210} \circ_{1,1} (f, \mathbb{1}) + \mathfrak{q}_{210} \circ_{2,0} (f, \mathbb{1}) \\ &= (-1)^{|f|} L_f \circ \hat{\mathfrak{q}}_{210} + \widehat{\mathfrak{q}_{210} \circ_1 f} + L_{\hat{\mathfrak{q}}_{210}(f)}.\end{aligned}$$

Therefore, it holds

$$\begin{aligned}\{f, \cdot\}_0 &= \hbar\big(\hat{\mathfrak{q}}_{210} \circ L_f - L_{\hat{\mathfrak{q}}_{210}(f)} - (-1)^{|f|} L_f \circ \hat{\mathfrak{q}}_{210}\big) \\ &= \hbar\widehat{\mathfrak{q}_{210} \circ_1 f}.\end{aligned}$$

We compute

$$\begin{aligned}&\big(\mathfrak{q}_{210} \circ_1 (\mathrm{s}m_1^+)\big)(\mathrm{s}\psi)(\omega) \\ &\quad = \mathfrak{q}_{210}(\mathrm{s}m_1^+, \mathrm{s}\psi)(\omega) \\ &\quad = (-1)^{|\mathrm{s}|}\mathfrak{q}_{210}(\mathrm{s}^2 m_1^+, \psi) \\ &\quad \underset{\substack{\uparrow \\ T^{ij}=(-1)^{|s|+|e_i||e_j|}T^{ji} \\ T^{ji}=(-1)^{|e_j|}\Pi(e^j,e^i) \\ \Pi(e^j,e^i)\neq 0 \implies |e_j|+|e_i|=|s|+1}}{=} (-1)^{|\mathrm{s}|} \sum \varepsilon(\omega \to \omega^1\omega^2)(-1)^{|e_j||\omega^1|} T^{ij} m_1^+(e_i\omega^1)\psi(e_j\omega^2) \\ &\quad \underset{\substack{\uparrow \\ \sum_i \Pi(v,e^i)e_i=v \\ m_1^+(e^j\omega^1)=(-1)^{|e^j||\omega^1|}m_1^+(\omega^1 e^j)}}{=} \sum \varepsilon(\omega \to \omega^1\omega^2)(-1)^{|e_j|(|\mathrm{s}|+1+|\omega^1|)+|e^j||\omega^1|}\Pi(m_1(\omega^1), e^j)\psi(e_j\omega^2) \\ &\quad \underset{\substack{\uparrow \\ \Pi(m_1(\omega^1),e^j)\neq 0 \implies 1+|\omega^1|+|e^j|=|\mathrm{s}|+1 \\ |e^j|+|e_j|=|\mathrm{s}|+1}}{=} (-1)^{|\mathrm{s}|+1} \sum \varepsilon(\omega^1\omega^2)\psi(m_1(\omega^1)\omega^2) \\ &\quad = (-1)^{|\mathrm{s}|+1} \sum_{i=1}^{k} (-1)^{|\omega_1|+\dots+|\omega_{i-1}|}\psi(\omega_1 \cdots m_1(\omega_i) \cdots \omega_k) \\ &\quad = (-1)^{|\mathrm{s}|+1}\mathfrak{q}_{110}(\mathrm{s}\psi)(\mathrm{s}\omega).\end{aligned} \tag{8.5}$$

Therefore, we have

$$\{S_{\mathrm{free}},\cdot\}_0=\hat{\mathfrak{q}}_{110}. \tag{8.6}$$

Since $m_1\circ m_1=0$ and since m_1^+ is cyclically symmetric, it holds

$$\{S_{\mathrm{free}},S_{\mathrm{free}}\}_0=(-1)^{n-2}\hbar^{-1}\mathfrak{q}_{110}(\mathrm{s}m_1^+)=0.$$

Because $m_1^+\in\mathrm{B}^*_{\mathrm{cyc}}V$ has weight 2, i.e., it vanishes on words of length 3 and more, $\mathfrak{q}_{120}$ decreases the weight by 2, and $\mathrm{B}^*_{\mathrm{cyc}}V$ is reduced, i.e., its weight-0 part is 0, we have

$$\Delta_0\, S_{\mathrm{free}}=(-1)^{n-2}\mathfrak{q}_{120}(\mathrm{s}m_1^+)=0.$$

We now compute

$$\begin{aligned}\{S,S\}_0&=\{S_{\mathrm{free}},S_{\mathrm{free}}\}_0+2\{S_{\mathrm{free}},S_{\mathrm{int}}\}_0+\{S_{\mathrm{int}},S_{\mathrm{int}}\}_0\\&=2\{S_{\mathrm{free}},S_{\mathrm{int}}\}_0+\{S_{\mathrm{int}},S_{\mathrm{int}}\}_0\\&=2\hbar^{-1}\Big(\mathfrak{q}_{110}(\mathfrak{m}_{10})+\frac{1}{2}\mathfrak{q}_{210}(\mathfrak{m}_{10},\mathfrak{m}_{10})\Big)\end{aligned} \tag{8.7}$$

and

$$\Delta_0\, S=\hbar^{-1}\mathfrak{q}_{120}(\mathfrak{m}_{10}). \tag{8.8}$$

The right-hand side of (8.7) vanishes due to [CFL15, Proposition 12.3], and the right-hand side of (8.8) vanishes by the discussion preceding [CFL15, Proposition 12.5] (for degree reasons). Therefore, S satisfies the quantum master equation (8.3). The twisting equation (8.4) is also clear. □

The inclusion of $\hbar^{-1}$ into S is the convention from [CFL15]. After the transformation $\Delta_0\to\Delta_0'=\Delta_0$, $S\to S'=\hbar S$, we get the well-known equations

$$\hbar\,\Delta_0'\,S'+\frac{1}{2}\{S',S'\}_0'=0\quad\text{and}\quad\hbar\Delta'^{\mathfrak{m}}=\hbar\,\Delta_0'+\{S',\cdot\}_0'.$$

Proposition 8.2.2 (BV-action for general Maurer-Cartan element)**.** *Let* (V,Π,m_1,m_2) *be a cyclic* DGA *of degree* n *of finite type, and let* $\Delta_0=\hat{\mathfrak{q}}_{120}+\hbar\hat{\mathfrak{q}}_{210}$ *be the string* BV*-operator on* $\mathcal{F}(B(V))$*. Let* $\mathfrak{m}_{lg}\in\hat{\mathrm{E}}_lC(V)$ *for all* l, $g\geq 0$ *satisfy the degree and filtration degree conditions of a Maurer-Cartan element for an* IBL_∞*-algebra of bidegree* $(n-3,2)$*.Then* $\mathfrak{m}=(\mathfrak{m}_{lg})$ *satisfies the Maurer-Cartan equation for* $\mathrm{dIBL}(C(V))$ *if and* *only if the total action*

$$S := S_{\text{free}} + \sum_{l,g\geq 0} \mathfrak{m}_{lg} \hbar^{g-1} \in \mathcal{F}(B(V)) \tag{8.9}$$

satisfies the quantum master equation (8.3)*. The* BV_∞*-operator* $\Delta^{\mathfrak{m}}$ *of the twisted* IBL_∞*-algebra* $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$ *is given by* (8.4)*. The elements* $\mathfrak{m}_{0g}$ *appear neither in* (8.3) *nor in* (8.4)*.*

Proof. Using $\Delta_0 S_{\text{free}} = \{S_{\text{free}}, S_{\text{free}}\}_0 = 0$, we compute

$$\begin{aligned}
\Delta_0 S + \frac{1}{2}\{S,S\}_0 &= \Delta_0 S_{\text{int}} + \{S_{\text{free}}, S_{\text{int}}\}_0 + \frac{1}{2}\{S_{\text{int}}, S_{\text{int}}\}_0 \\
&= \sum_{l,g\geq 0} \left(\hat{\mathfrak{q}}_{120}(\mathfrak{m}_{lg})\hbar^{g-1} + \hat{\mathfrak{q}}_{210}(\mathfrak{m}_{lg})\hbar^{g} + \hat{\mathfrak{q}}_{110}(\mathfrak{m}_{lg})\hbar^{g-1}\right) \\
&\qquad\qquad + \frac{1}{2} \sum_{l_1,l_2,g_1,g_2\geq 0} \widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{l_1g_1}}(\mathfrak{m}_{l_2g_2})\hbar^{g_1+g_2-1} \\
&= \sum_{l,g\geq 0} (\iota_l \pi_l)\Big(\mathfrak{q}_{120} \circ_1 \mathfrak{m}_{l-1,g} + \mathfrak{q}_{210} \circ_2 \mathfrak{m}_{l+1,g-1} + \mathfrak{q}_{110} \circ_1 \mathfrak{m}_{lg} \\
&\qquad\qquad\qquad + \frac{1}{2} \sum_{\substack{l_1,l_2,g_1,g_2\geq 0 \\ l_1+l_2-1=l \\ g_1+g_2=g}} \mathfrak{q}_{210} \circ_{1,1} (\mathfrak{m}_{l_1g_1}, \mathfrak{m}_{l_2g_2})\Big)\hbar^{g-1}.
\end{aligned}$$

Comparing to Proposition 3.2.10, we see that the (l,g)-components for $l \geq 1$, $g \geq 0$ give precisely the Maurer-Cartan equation in $\mathrm{dIBL}(C(V))$. We compute

$$\begin{aligned}
\Delta_0 + \{S,\cdot\}_0 &= \hat{\mathfrak{q}}_{120} + \hbar\hat{\mathfrak{q}}_{210} + \hat{\mathfrak{q}}_{110} + \sum_{l,g\geq 0} \widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{lg}}\hbar^{g} \\
&= (\hat{\mathfrak{q}}_{110} + \widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{10}}) + (\hat{\mathfrak{q}}_{120} + \widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{20}}) + (\widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{30}}) + \cdots \\
&\quad + \big[(\hat{\mathfrak{q}}_{210}) + (\widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{11}}) + (\widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{21}}) + \cdots\big]\hbar \\
&\quad + \sum_{l\geq 0, g\geq 2} (\widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{lg}})\hbar^{g} \\
&= (\hat{\mathfrak{q}}^{\mathfrak{m}}_{110}) + (\hat{\mathfrak{q}}^{\mathfrak{m}}_{120}) + (\hat{\mathfrak{q}}^{\mathfrak{m}}_{130}) + \cdots \\
&\quad + \big[(\hat{\mathfrak{q}}^{\mathfrak{m}}_{210}) + (\hat{\mathfrak{q}}^{\mathfrak{m}}_{111}) + (\hat{\mathfrak{q}}^{\mathfrak{m}}_{121}) + \cdots\big]\hbar \\
&\quad + \sum_{l\geq 0, g\geq 2} (\hat{\mathfrak{q}}^{\mathfrak{m}}_{1lg})\hbar^{g}
\end{aligned}$$

and see that this is indeed the BV_∞-operator $\Delta^{\mathfrak{m}}$ of $\mathrm{dIBL}^{\mathfrak{m}}(C(V))$. □

8.3. Homotopy transfer and effective action

In [DJP19], they consider multilinear operations $l_{lg} : V^{\otimes l} \to \mathbb{R}$ for $l \geq 1$, $g \geq 0$ on an odd symplectic vector space V and write down an action $S \in \mathcal{F}(V)$ in the form of (8.9) with $\mathfrak{m}_{lg}$ replaced by l^+_{lg}. We remark that their "vertices" are $l^+_{lg}(v, \dots, v)$ for a "field" $v \in V$, whereas ours are $\mathfrak{m}_{lg}(v_1, \dots, v_l)$ for a "string of fields" $v_1 \cdots v_l \in B(V)$. They consider the Schwarz's canonical BV-operator on $\mathcal{F}(V)$ from [Sch93] and show that S satisfies the quantum master equation if and only if (l_{lg}) satisfy the relations of a quantum L_∞-algebra.[1] On the other hand, we have the string BV-operator on $\mathcal{F}(B(V))$ and our action (8.9) satisfies the quantum master equation if and only if $\mathfrak{m} = (\mathfrak{m}_{lg})$ is a Maurer-Cartan element for $\mathrm{dIBL}(C(V))$. Next, they consider a deformation retract (=: DR) as in (8.1) and obtain explicit formulas for the homotopy transfered quantum L_∞-algebra on V' together with all maps and homotopies via the Homological Perturbation Lemma (=: HPL). In what follows, we will sketch how to apply their construction to IBL_∞-algebras. We stress that the details have NOT been done yet!

A DR (8.1) induces a DR

$$\begin{tikzcd}
(C(V), \mathfrak{q}_{110}) \arrow[loop left, "K_C"] \arrow[r, shift left, "P_C"] & (C(V'), \mathfrak{q}'_{110}), \arrow[l, shift left, "I_C"]
\end{tikzcd}$$

which further induces a DR

$$\begin{tikzcd}
(\mathcal{F}(B(V)), \hat{\mathfrak{q}}_{110}) \arrow[loop left, "K"] \arrow[r, shift left, "P"] & (\mathcal{F}(B(V')), \hat{\mathfrak{q}}'_{110}). \arrow[l, shift left, "I"]
\end{tikzcd} \tag{8.10}$$

In [DJP19, Remark 3], they write down a formula for K on $\mathcal{F}(V)$ given $\mathcal{P}$ using a "tensor trick" due to Eilenberg Mac-Lane; the same method may apply to get K_C from $\mathcal{P}$ on $C(V)$ and K on $\mathcal{F}(B(V))$ from K_C in our case. With such K's, one can take P_C and P, resp. I_C and I to be the natural extensions of ι^*, resp. π^*. Note that any surjective quasi-isomorphism over $\mathbb{R}$ is a deformation retraction, but their formula is explicit and preserves special DR's, i.e., DR's satisfying $\mathcal{P}^2 = \mathcal{P}\iota = \pi\mathcal{P} = 0$ (=: SDR). The crucial idea of [DJP19] translated to our situation is to view Δ and $\Delta^{\mathfrak{m}}$ as perturbations of $\{S_{\mathrm{free}}, \cdot\} = \hat{\mathfrak{q}}_{110}$, which we denote by $\delta^{(1)}$ and $\delta^{(2)}$, respectively, and apply the HPL from [Cra04]:

---

[1] This is equivalent to the notion of a loop homotopy algebra from [Mar01] and to string brackets in closed string field theory from [Zwi93].

For $i = 1, 2$, suppose that $\delta^{(i)}$ is "small", i.e., that $(\mathbb{1} - \delta^{(i)}K)$ is invertible, and consider the maps

$$\begin{aligned}
\Delta^{(i)} &:= \hat{\mathfrak{q}}'_{110} + P(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}I = \hat{\mathfrak{q}}'_{110} + P\delta^{(i)}I + P\delta^{(i)}K\delta^{(i)}I + \cdots\,,\\
I^{(i)} &:= I + K(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}I = I + K\delta^{(i)}I + K\delta^{(i)}K\delta^{(i)}I + \cdots\,,\\
P^{(i)} &:= P + P(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}K = P + P\delta^{(i)}K + P\delta^{(i)}K\delta^{(i)}K + \cdots\,,\\
K^{(i)} &:= K + K(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}K = K + K\delta^{(i)}K + K\delta^{(i)}K\delta^{(i)}K + \cdots\,.
\end{aligned}$$

The HPL asserts that if (8.10) is an SDR, then the following are SDR's as well:

$$\begin{tikzcd}
(\mathcal{F}(B(V)), \hat{\mathfrak{q}}_{110}) \arrow[loop left, "K"] \arrow[r, shift left, "P"] & (\mathcal{F}(B(V')), \hat{\mathfrak{q}}'_{110}) \arrow[l, shift left, "I"] \arrow[d, dashed, bend left=60, "\delta^{(1)} = \Delta_0"] \arrow[dd, dashed, bend left=30, "\delta^{(2)} = \Delta_0 + \{S_{\mathrm{int}}, \cdot\}"'] \\
(\mathcal{F}(B(V)), \Delta) \arrow[loop left, "K^{(1)}"] \arrow[r, shift left, "P^{(1)}"] & (\mathcal{F}(B(V')), \Delta^{(1)}) \arrow[l, shift left, "I^{(1)}"] \\
(\mathcal{F}(B(V)), \Delta^{\mathfrak{m}}) \arrow[loop left, "K^{(2)}"] \arrow[r, shift left, "P^{(2)}"] & (\mathcal{F}(B(V')), \Delta^{(2)}) \arrow[l, shift left, "I^{(2)}"]
\end{tikzcd}$$

One defines the *effective action*

$$W := \log(P^{(1)}(e^{S_{\mathrm{int}}})) \in \mathcal{F}(B(V'))$$

and the *path integral*

$$Z := L_{e^{-W}} \circ P^{(1)} \circ L_{e^{S_{\mathrm{int}}}} : \mathcal{F}(B(V)) \to \mathcal{F}(B(V')).$$

Under certain circumstances, the following formulas hold:

$$\Delta^{(1)} = P\,\Delta_0\,I, \quad \Delta^{(2)} = \Delta^{(1)} + \{W, \cdot\}^{(1)} \quad \text{and} \quad P^{(2)} = Z. \tag{8.11}$$

This is proven in [DJP19] on $\mathcal{F}(V)$ when $V' = \mathcal{H}$ is a "harmonic" subspace in a Hodge decomposition $V = \mathcal{H} \oplus C$ into odd symplectic subspaces, π and ι are the canonical projection and inclusion, respectively, the homotopy $\mathcal{P}$ is such that $(\pi, \iota, \mathcal{P})$ is an SDR, and the homotopy K was constructed from $\mathcal{P}$ via the tensor trick. Since we deal with the string BV-operator on $\mathcal{F}(B(V))$, we can not talk about this "symplectic compatibility" and the proof of (8.11) might be based on another arguments.

Question 8.3.1 (HPL and IBL_∞). If one picks an SDR of V onto a harmonic subspace $\mathcal{H}$ as above (basically equivalent to the setting of [CFL15, Section 11]) and constructs K_C and K in a particular way, can one achieve that (8.11) and the following identities hold?

$$\Delta^{(1)} = \Delta_0', \quad \Delta^{(2)} = \Delta^{\mathfrak{m}}, \quad W = S_{\mathfrak{f}_*\mathfrak{m}}, \quad P^{(1)} = e^{\mathfrak{f}}, \quad P^{(2)} = e^{\mathfrak{f}^{\mathfrak{m}}}.$$

Here, $S_{\mathfrak{f}_*\mathfrak{m}}$ denotes the action (8.9) for the Maurer-Cartan element $\mathfrak{f}_*\mathfrak{m}$.

Remark 8.3.2 (On BV-formalism for IBL_∞). (i) As summarized in [DJP19, Section 5], given a BV-action S, there are various approaches to obtain W and Z as summations over Feynman graphs (see [Mne17] for the stationary phase formula approach).

(ii) The appearance of Feynman graphs can be explained from the proof of [DJP19, Theorem 2], where they show that in the special setting above, it holds

$$P^{(1)} = Pe^{D_{\mathrm{P}}}$$

for an order ≤ 2 differential operator D_{P} which “connects” two legs with the propagator P (obtained by “rising one index” of $\mathcal{P}$ using the odd symplectic form). This is reminiscent of the Wick's Theorem for the (formal) perturbative expansion of the path integral (the classical approach to quantum field theories).

(iii) Having a BV-formulation of the IBL_∞-theory, it is intriguing to compare it to [MS13], where certain IBL_∞-structures are considered in the context of open-closed string field theory. $\triangleleft$

Part III.

Appendices

A. Evaluation of labeled ribbon graphs

In this appendix, we consider an algebraic propagator P and the graph pairing $\langle\cdot,\cdot\rangle_\Gamma^{\mathrm{P}}$ (Definition A.1.1), which encapsulates the contribution of a ribbon graph Γ to the map $f_{klg} : (\mathrm{B}^*_{\mathrm{cyc}}V)^{\otimes k} \to (\mathrm{B}^*_{\mathrm{cyc}}V)^{\otimes l}$ defined as a sum of contributions of ribbon graphs (Proposition A.1.2). Such maps were already defined in [CFL15, Section 11] using coordinates; here we use an invariant framework inspired by [Mne17]. As an example, we work out in details expressions for the canonical dIBL-operations $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ (Example A.1.5). We also explain the technicality of identifying symmetric maps with maps on symmetric powers (Remark A.1.3).

Next, we show that the matrix (T^{ij}) from Definition 3.4.1 corresponds to the algebraic Schwartz kernel (Definition A.1.4) of the identity $\mathbb{1}$ up to a sign. Assuming that the Hodge propagator P from Definition 4.2.4 is algebraic, we deduce natural candidates for the signs in Definition 4.3.6 based on the formula from [CFL15, Remark 12.10] for the genuine pushforward Maurer-Cartan element $\mathfrak{n}$ in the finite-dimensional case. Establishing the formal analogy between the de Rham case and the finite-dimensional case is our main application of the invariant framework. Finally, we sketch how to obtain signs for the Fréchet dIBL-structure on $\Omega(M)$ (Remark A.2.2).

Throughout this appendix, we will use Notation 3.3.3 without further remarks.

A.1. Finite dimensional case

Definition A.1.1 (Propagator & graph pairing). *Let V be a graded vector space. We will call a tensor $\mathrm{P} \in V[1]^{\otimes 2}$ an* algebraic propagator*. We call it* admissible *if the symmetry condition*

$$\tau(\mathrm{P}) = (-1)^{|\mathrm{P}|}\mathrm{P} \tag{A.1}$$

is satisfied, where $\tau : V[1]^{\otimes 2} \to V[1]^{\otimes 2}$ is the twist map defined by $\tau(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1$ for all v_1, $v_2 \in V[1]$.

For a ribbon graph $\Gamma \in \overline{\mathrm{RG}}_{klg}$ and its labeling L, consider the permutation σ_L from Definition 4.3.5. It acts on tensor powers according to Definition 3.1.6 and thus defines *the map*

$$\sigma_L : (V[1]^{\otimes 2})^{\otimes e} \otimes V[1]^{\otimes s_1} \otimes \cdots \otimes V[1]^{\otimes s_l} \longrightarrow V[1]^{\otimes d_1} \otimes \cdots \otimes V[1]^{\otimes d_k},$$

where d_i *and* s_i *are the valencies of internal vertices* $1, \ldots, k$ *and boundary components* $1, \ldots, l$*, respectively, and* e *is the number of internal edges. We extend* σ_L *by* 0 *to other combinations of tensor powers.*

The graph pairing

$$\langle \cdot, \cdot \rangle_\Gamma^{\mathrm{P}} \ : \ (\mathrm{B}^*_{\mathrm{cyc}} V)^{\otimes k} \otimes (\mathrm{B}^{\mathrm{cyc}} V)^{\otimes l} \longrightarrow \mathbb{R}$$

is defined for all $\psi_1, \ldots, \psi_k \in \mathrm{B}^*_{\mathrm{cyc}} V$ *and generating words* $w_i = v_{i1} \ldots v_{im_i}$ *with* $v_{ij} \in V[1]$ *for* $m_i \in \mathbb{N}$ *and* $i = 1, \ldots, l$ *by the following formula:*

$$\begin{aligned} &\langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_\Gamma^{\mathrm{P}} \\ &\quad \coloneqq \sum_{L_1, L_3^b} (\psi_1 \otimes \cdots \otimes \psi_k)\big(\sigma_L(\mathrm{P}^{\otimes e} \otimes (v_{11} \otimes \cdots \otimes v_{1m_1}) \otimes \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad \otimes (v_{l1} \otimes \cdots \otimes v_{lm_l})\big), \end{aligned}$$

where we use the pairing from Definition 3.3.4 and in every summand an L_2 *compatible with* L_1 *and an* L_3^v *are chosen arbitrarily to get a full labeling* L *of* Γ*. The graph pairing extends to* $\langle \cdot, \cdot \rangle_\Gamma^{\mathrm{P}} \ : \ \bar{\mathrm{T}}\mathrm{B}^*_{\mathrm{cyc}} V \otimes \bar{\mathrm{T}}\mathrm{B}^{\mathrm{cyc}} V \to \mathbb{R}$*, where* $\bar{\mathrm{T}} W = \bigoplus_{k=1}^\infty W^{\otimes k}$ *is the reduced tensor product.*

Proposition A.1.2. *In the setting of Definition A.1.1, we denote* $w = w_1 \otimes \cdots \otimes w_l$ *and* $\psi = \psi_1 \otimes \cdots \otimes \psi_k$*. If* P *is admissible, then the following holds:*

(a) The number $\psi(\sigma_L(\mathrm{P}^{\otimes e} \otimes w))$ *does not depend on the choice of* L_3^v *and an* L_2 *compatible with* L_1*. Moreover,* $\langle \cdot, \cdot \rangle_\Gamma^{\mathrm{P}}$ *does not depend on the representative of* $[\Gamma] \in \overline{\mathrm{RG}}_{klg}$*.*

(b) If V *is finite-dimensional, then for every* $k, l \geq 1$, $g \geq 0$ *there is a unique linear map*

$$f_{klg} : (\mathrm{B}^*_{\mathrm{cyc}} V)^{\otimes k} \longrightarrow (\mathrm{B}^*_{\mathrm{cyc}} V)^{\otimes l}$$

such that

$$\begin{aligned} &f_{klg}(\psi_1 \otimes \cdots \otimes \psi_k)(w_1 \otimes \cdots \otimes w_l) \\ &\qquad = \frac{1}{l!} \sum_{[\Gamma] \in \overline{\mathrm{RG}}_{klg}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_\Gamma^{\mathrm{P}}. \end{aligned}$$

(c) The following holds for the map f_{klg} from (b):

- *It is homogenous of degree*

$$|f_{klg}| = -|\mathrm{P}|(k+l-2+2g). \tag{A.2}$$

- *The filtration degree satisfies*

$$\|f_{klg}\| \geq -2(k+l-2+2g). \tag{A.3}$$

- *For all $\eta \in \mathbb{S}_l$ and $\mu \in \mathbb{S}_k$, we have*

$$\eta \circ f_{klg} \circ \mu = (-1)^{|\mathrm{P}|(\eta+\mu)} f_{klg}. \tag{A.4}$$

Proof. (a) Let us denote by $\bar{i}$ and ij the operations on L_2 given by $\mathrm{e}_i \mapsto -\mathrm{e}_i$ and $\mathrm{e}_i \leftrightarrow \mathrm{e}_j$, respectively. An even number of these operations does not change the orientation of the complex (4.18). Their effect in σ_L acting on $\mathrm{P}^{\otimes e} \otimes w$ is

$$\bar{i} : \mathrm{P}_i \mapsto \tau(\mathrm{P}_i) = (-1)^{|\mathrm{P}|}\mathrm{P}_i \quad \text{and} \quad ij : \mathrm{P}_i \dots \mathrm{P}_j \mapsto (-1)^{|\mathrm{P}|}\mathrm{P}_j \dots \mathrm{P}_i.$$

Therefore, an even number of them does not change $\sigma_L(\mathrm{P}^{\otimes e} \otimes w)$. This proves the independence of the choice of a compatible L_2. The independence of the choice of L_3^v is clear since ψ_i are cyclic symmetric.

An isomorphism of ribbon graphs $\eta : \Gamma \to \Gamma'$ induces the map of compatible labelings $L \mapsto L' = \eta_* L$ such that $\sigma_L = \sigma_{L'}$. The independence of the choice of a representative of $[\Gamma]$ follows.

(b) Suppose that $\psi = \psi_1 \otimes \dots \otimes \psi_k$ with $\psi_i \in (\mathrm{B}^*_{\mathrm{cyc}}V)^{c_i}_{r_i}$, where $r_i \in \mathbb{N}$ and $c_i \in \mathbb{Z}$ for $i = 1, \dots, k$. A general element of $(\mathrm{B}^*_{\mathrm{cyc}}V)^{\otimes k}$ is then a finite linear combination of such ψ's.

First of all, let us argue that the sum $\sum_{\overline{\mathrm{RG}}_{klg}}$ is finite. The number of internal edges e is fixed from the Euler formula (4.17). Therefore, the number of contributing graphs $(V_{\mathrm{int}}, E_{\mathrm{int}})$ is finite. In order to bound the number of external vertices, we notice that $d_1 = r_1, \dots, d_k = r_k$ must hold for $\psi(\sigma_L(\mathrm{P}^{\otimes e} \otimes w))$ to be non-zero. Therefore, the sum is finite.

We now have the linear functional

$$f_{klg}(\psi) \coloneqq \frac{1}{l!}\sum_{[\Gamma]\in\overline{\mathrm{RG}}_{klg}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\langle\psi\mid\cdot\rangle_\Gamma^{\mathrm{P}} : (\mathrm{B}^{\mathrm{cyc}}V)^{\otimes l} \longrightarrow \mathbb{R}$$

and need to show that $f_{klg}(\psi) \in (\mathrm{B}^*_{\mathrm{cyc}}V)^{\otimes l} \subset (\mathrm{B}^{\mathrm{cyc}}V)^{\otimes l *}$. Because V has finite dimension, the weight-filtration of $\mathrm{B}^{\mathrm{cyc}}V$ satisfies (WG1) & (WG2) (see (3.11) and Proposition 3.3.6), and hence we have

$$(\mathrm{B}^*_{\mathrm{cyc}}V)^{\otimes l} = (\mathrm{B}^{\mathrm{cyc}}V)^{*_{\mathrm{wg}}\otimes l} = \big((\mathrm{B}^{\mathrm{cyc}}V)^{\otimes l}\big)^{*_{\mathrm{wg}}}$$

for the weight-graded duals. Therefore, it suffices to show that $f_{klg}(\psi)$ vanishes on all but finitely many degrees and weights of $(\mathrm{B}^{\mathrm{cyc}}V)^{\otimes k}$. However, the relation $f_{klg}(\psi)(w) \neq 0$ for a generating word $w \in (\mathrm{B}^{\mathrm{cyc}}V)^{\otimes k}$ implies

$$\begin{aligned} |w| &= |\psi| - e|\mathrm{P}| \quad \text{and} \\ k(w) &= k(\psi) - 2e, \end{aligned} \tag{A.5}$$

where k denotes the weight, and hence $f_{klg}(\psi) \in (\mathrm{B}^*_{\mathrm{cyc}}V)^{\otimes l}$ indeed holds.

(c) The formulas (A.2) and (A.3) follow from (A.5) and from (4.17).

As for the symmetry (A.4), suppose that L and L' are compatible labelings of the same graph Γ such that L_1' differs from L_1 by a permutation $\mu \in \mathbb{S}_k$ of internal vertices and a permutation $\eta \in \mathbb{S}_l$ of boundary components. Viewing μ and η as block permutations in the vertex and edge order, respectively, we get

$$\sigma_{L'}(\mathrm{P}^{\otimes e} \otimes w) = (-1)^{|\mathrm{P}|(\eta+\mu)}\mu(\sigma_L(\mathrm{P}^{\otimes e}\otimes \eta(w))).$$

The sign comes from the difference of L_2 and L_2' which compensates the change of the orientation of (4.18) caused by μ and η. □

Given $\mu \in \mathbb{S}_k$ and $\psi = \psi_1 \otimes \cdots \otimes \psi_k \in (\mathrm{B}^*_{\mathrm{cyc}}V)^{\otimes k}$, it is easy to see that

$$\varepsilon(\mu,\Psi) = \varepsilon(\mu(\mathrm{s}),\mu(\psi))\varepsilon(\mu,\mathrm{s})\varepsilon(\mathrm{s},\psi)\varepsilon(\mu,\psi),$$

where $\Psi = (\mathrm{s}\psi_1)\otimes\cdots\otimes(\mathrm{s}\psi_k) \in (\mathrm{B}^*_{\mathrm{cyc}}V[A])^{\otimes k}$ and $\varepsilon(\mu,\mathrm{s}) = (-1)^{|s|\mu}$. If $A = -|\mathrm{P}|$, then we get from (A.4) that the degree shift $\mathfrak{f}_{klg} : (\mathrm{B}^*_{\mathrm{cyc}}V[A])^{\otimes k} \to (\mathrm{B}^*_{\mathrm{cyc}}V[A])^{\otimes l}$ has the following symmetries:

$$\forall \mu \in \mathbb{S}_k, \eta \in \mathbb{S}_l : \quad \eta \circ \mathfrak{f}_{klg} \circ \mu = \mathfrak{f}_{klg}. \tag{A.6}$$

Note that the degrees satisfy

$$|\mathfrak{f}_{klg}| = |f_{klg}| + (k-l)A. \tag{A.7}$$

Remark A.1.3 (Symmetric maps versus maps on symmetric powers)*.* In the situation above, we define $\tilde{\mathfrak{f}}_{klg}$ as the unique map such that the solid lines of the following diagram commute:

$$\begin{tikzcd}
(\mathrm{B}^*_{\mathrm{cyc}}V[A])^{\otimes k} \arrow[r, "\mathfrak{f}_{klg}"] \arrow[d, two heads, "\pi"] & (\mathrm{B}^*_{\mathrm{cyc}}V[A])^{\otimes l} \arrow[d, two heads, "\pi"'] \\
\mathrm{S}_k\mathrm{B}^*_{\mathrm{cyc}}V[A] \arrow[u, dotted, bend left, "\iota"] \arrow[r, "\tilde{\mathfrak{f}}_{klg}"] & \mathrm{S}_l\mathrm{B}^*_{\mathrm{cyc}}V[A]. \arrow[u, dotted, bend right, "\iota"']
\end{tikzcd}$$

The symmetry condition (A.6) provides the existence of $\tilde{\mathfrak{f}}_{klg}$ and implies commutativity of the dotted diagram as well. Moreover, for all $\psi_1, \ldots, \psi_k \in \mathrm{B}^*_{\mathrm{cyc}}V$ and $w_1, \ldots, w_l \in \mathrm{B}^{\mathrm{cyc}}V$, we have

$$\tilde{\mathfrak{f}}_{klg}(\mathrm{s}^k\psi_1\cdots\psi_k)(\mathrm{s}^l w_1\cdots w_l) = \mathfrak{f}_{klg}(\mathrm{s}^k\psi_1\otimes\cdots\otimes\psi_k)(\mathrm{s}^l w_1\otimes\cdots\otimes w_l),$$

where we use the pairing from Definition 3.3.4. We denote $\tilde{\mathfrak{f}}_{klg}$ again by $\mathfrak{f}_{klg}$. $\lhd$

Definition A.1.4 (Algebraic Schwartz kernel)**.** *Let V be a graded vector space and $\Pi : V\otimes V \to \mathbb{R}$ a non-degenerate pairing on V. We extend Π to a non-degenerate pairing $\Pi : V^{\otimes k}\otimes V^{\otimes k} \to \mathbb{R}$ for $k \geq 1$ by setting*

$$\Pi(v_{11}\otimes\cdots\otimes v_{1k}, v_{21}\otimes\cdots\otimes v_{2k}) \coloneqq \varepsilon(v_1,v_2)\Pi(v_{11},v_{21})\ldots\Pi(v_{1k},v_{2k})$$

for all $v_{11}, \ldots, v_{1k}, v_{21}, \ldots, v_{2k} \in V$, where ε is the Koszul sign (see Definition 3.1.2). For $k=0$, we let $\Pi : \mathbb{R}\otimes\mathbb{R}\to\mathbb{R}$ be the multiplication on $\mathbb{R}$.

For k, $l \geq 0$, we say that $\mathcal{K}_L \in V^{\otimes k+l}$ is the algebraic Schwartz kernel *of a linear operator $L : V^{\otimes k} \to V^{\otimes l}$ if the following is satisfied:*

$$\forall w_1 \in V^{\otimes k}, w_2 \in V^{\otimes l} : \quad \Pi(L(w_1), w_2) = \Pi(\mathcal{K}_L, w_1\otimes w_2). \tag{A.8}$$

We usually omit writing "algebraic" if it is clear from the context (i.e., if we do not consider any "extensions" of $V^{\otimes k}$).

In the situation of Definition A.1.4, let $(e_i) \subset V$ be a basis and (e^i) its dual basis such that $\Pi(e_i, e^j) = \delta_{ij}$. We define the coordinates $K_L^{ij} \in \mathbb{R}$ and $L^{ij} \in \mathbb{R}$ by

$$\mathcal{K}_L = \sum_{i,j} K_L^{ij} e_i \otimes e_j \quad \text{and} \quad L^{ij} \coloneqq \Pi(L(e^i), e^j).$$

From (A.8) we have

$$\mathcal{K}_L^{ij} = (-1)^{(|L|+1)(|\Pi|+|e_i|)} L^{ij} \quad \text{for all } i, j. \tag{A.9}$$

From now on, we will be in the situation of (A) and (B) in the Overview; in particular, we put $V[1]$ in place of V in Definition A.1.4. Let $\mathcal{K}_{\mathbb{1}} \in V[1]^{\otimes 2}$ be the Schwartz kernel of the identity $\mathbb{1} : V[1] \to V[1]$ and $\mathcal{K}_{\mathcal{P}} \in V[1]^{\otimes 2}$ the Schwartz kernel of the cochain homotopy $\mathcal{P} : V[1] \to V[1]$. From (A.9), we get

$$\mathcal{K}_{\mathcal{P}}^{ij} = \mathcal{P}^{ij} \quad \text{and} \quad {\mathcal{K}_{\mathbb{1}}}^{ij} = (-1)^{|e_i|+|\Pi|}\Pi(e^i, e^j) \quad \text{for all } i, j.$$

We see that the tensor $\mathrm{T} = \sum_{i,j} \mathrm{T}^{ij} e_i \otimes e_j$ from (3.34) can be expressed as

$$\mathrm{T} = (-1)^{n-2} \mathcal{K}_{\mathbb{1}}.$$

This is the invariant meaning of T. Note that the degrees satisfy

$$|\mathrm{T}| = n - 2 \qquad \text{and} \qquad |\mathcal{K}_{\mathcal{P}}| = n - 3.$$

The assumption (2.1) on $\mathcal{P}$ is equivalent to graded antisymmetry of the bilinear form $\mathcal{P}^+ \coloneqq \Pi \circ (\mathcal{P} \otimes \mathbb{1}) : V[1]^{\otimes 2} \to \mathbb{R}$. This is further equivalent to

$$\tau(\mathcal{K}_{\mathcal{P}}) = (-1)^{|\mathcal{K}_{\mathcal{P}}|} \mathcal{K}_{\mathcal{P}}.$$

Therefore, $\mathcal{K}_{\mathcal{P}}$ satisfies (A.1), and hence it defines an admissible propagator for the construction of f_{klg} for every k, $l \geq 1$, $g \geq 0$. We have from (A.4) that the degree shift $\mathfrak{f}_{klg} : (\mathrm{B}^*_{\mathrm{cyc}} V[3-n])^{\otimes k} \to (\mathrm{B}^*_{\mathrm{cyc}} V[3-n])^{\otimes l}$ is symmetric. Moreover, using (A.2), (A.3) and (A.7), we obtain

$$|\mathfrak{f}_{klg}| = -2d(k+g-1),$$
$$\|\mathfrak{f}_{klg}\| \geq \gamma(2-2g-k-l),$$

where $(d, \gamma) = (n-3, 2)$. These are the degree and filtration conditions on an IBL_∞-morphism from [CFL15, Definition 2.8 and (8.3)]. In fact, our $\mathfrak{f} = (\mathfrak{f}_{klg})_{k,l \geq 1, g \geq 0}$ is precisely the IBL_∞-homotopy from [CFL15, Theorem 11.3].

Graded antisymmetry of Π is equivalent to

$$\tau(\mathrm{T}) = (-1)^{|\mathrm{T}|+1}\mathrm{T}.$$

Visibly, T does not satisfy (A.1), and hence is not an admissible propagator. Nevertheless, we can still use it to define f_{210} and f_{120} since the corresponding graphs Γ (see Figure A.1) have only one internal edge e, and, for a given L_1, there is a unique compatible L_2 determined by the orientation of e (see Example A.1.5 for the compatibility condition). As for the symmetry of the resulting maps, a transposition of internal vertices or boundary components in (4.18) can be compensated only by $\mathrm{e} \mapsto -\mathrm{e}$, which produces $(-1)^{|\mathrm{T}|+1}$ (c.f. the proof of Proposition A.1.2 (a)). Therefore, if we shift the degrees by $A = -|\mathrm{T}| + 1 = n - 3$, we obtain symmetric maps $\mathfrak{q}_{210} : (\mathrm{B}^*_{\mathrm{cyc}}V[A])^{\otimes 2} \to \mathrm{B}^*_{\mathrm{cyc}}V[A]$ and $\mathfrak{q}_{120} : \mathrm{B}^*_{\mathrm{cyc}}V[A] \to (\mathrm{B}^*_{\mathrm{cyc}}V[A])^{\otimes 2}$. We show in Example A.1.5 below that these operations agree with those defined in Definition 3.4.1.

Example A.1.5 (The canonical dIBL-operations). We have

$$\begin{aligned} f_{210}(\psi_1 \otimes \psi_2)(w) &= \frac{1}{1!} \sum_{[\Gamma] \in \overline{\mathrm{RG}}_{210}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \langle \psi_1 \otimes \psi_2 \mid w \rangle_\Gamma^{\mathrm{P}} \quad \text{and} \\ f_{120}(\psi)(w_1 \otimes w_2) &= \frac{1}{2!} \sum_{[\Gamma] \in \overline{\mathrm{RG}}_{120}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \langle \psi \mid w_1 \otimes w_2 \rangle_\Gamma^{\mathrm{P}}. \end{aligned} \tag{A.10}$$

We parametrize RG_{210} by the ribbon graphs Γ_{k_1,k_2} with $1 \le k_1 \le k_2$ and RG_{120} by the ribbon graphs Γ^{s_1,s_2} with $0 \le s_1 \le s_2$; these graphs are depicted in Figure A.1. We have $\overline{\mathrm{RG}}_{210} = \mathrm{RG}_{210} \backslash \{[\Gamma_{1,1}]\}$ and $\overline{\mathrm{RG}}_{120} = \mathrm{RG}_{120} \backslash \{[\Gamma^{0,0}], [\Gamma^{0,1}]\}$. We also have

$$|\operatorname{Aut}(\Gamma_{k_1,k_2})| = \begin{cases} 1 & \text{if } k_1 \neq k_2, \\ 2 & \text{if } k_1 = k_2, \end{cases}$$

and likewise for Γ^{s_1,s_2}. We fix labelings L_3^v and parametrize L_3^b by $c = 1, \ldots, k_1 + k_2 - 2$ for Γ_{k_1,k_2} and by $c_1 = 1, \ldots, s_1$ and $c_2 = 1, \ldots, s_2$ for Γ^{s_1,s_2} as it is indicated in Figure A.1.

There are two possible labelings L_1^v for Γ_{k_1,k_2} and two possible labelings L_1^b for Γ^{s_1,s_2}; this is the only freedom in choosing a full labeling L because L_3 is fixed and L_2 is just the orientation of the single internal edge, which is uniquely determined by L_1. For both Γ_{k_1,k_2} and Γ^{s_1,s_2}, we will denote the two possible full labelings by L^1 and L^2. They can be depicted as follows:

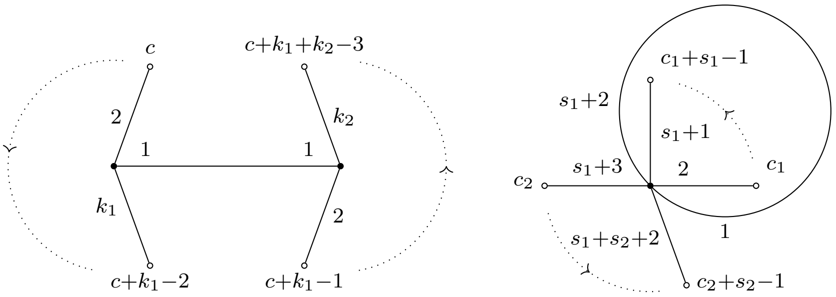
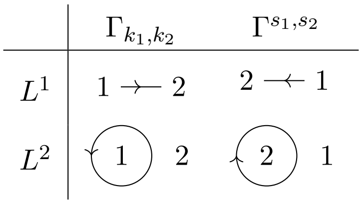


Figure A.1.: Graphs Γ_{k_1,k_2} and Γ^{s_1,s_2} with fixed labelings L_3.



(A.11)

Let us check that the indicated L_1 and L_2 are compatible. For the complexes $C_2 \to C_1 \to C_0$ from (4.18), we have the following:

$$\begin{aligned} \Gamma_{k_1,k_2}: \qquad & \langle \mathrm{b} \rangle \xrightarrow{\partial_2=0} \langle \mathrm{e} \rangle \xrightarrow{\partial_1} \langle \mathrm{v}_2 - \mathrm{v}_1 \rangle \oplus \langle \mathrm{v}_1 + \mathrm{v}_2 \rangle, \\ \Gamma^{s_1,s_2}: \qquad & \langle \mathrm{b}_1 - \mathrm{b}_2 \rangle \oplus \langle \mathrm{b}_1 + \mathrm{b}_2 \rangle \xrightarrow{\partial_2} \langle \mathrm{e} \rangle \xrightarrow{\partial_1=0} \langle \mathrm{v} \rangle. \end{aligned}$$

As for Γ_{k_1,k_2}, the basis $\mathrm{v}_2 - \mathrm{v}_1$, $\mathrm{v}_1 + \mathrm{v}_2$ of C_0 is positively oriented with respect to the basis v_2, v_1. Therefore, e has to be oriented such that $\partial_1 \mathrm{e} = \mathrm{v}_2 - \mathrm{v}_1$; i.e., it is a path from v_1 to v_2. As for Γ^{s_1,s_2}, the basis $\mathrm{b}_1 - \mathrm{b}_2$, $\mathrm{b}_1 + \mathrm{b}_2$ of C_2 is positively oriented with respect to b_1, b_2. Therefore, e has to be oriented such that $\mathrm{e} = \partial_2(\mathrm{b}_1 - \mathrm{b}_2)$. Recall that we orient the boundary of a 2-simplex by the "outer normal first" convention. We conclude that the labelings from (A.11) are indeed compatible.

As for f_{210}, the permutations $\sigma_1 \coloneqq \sigma_{L^1}$ and $\sigma_2 \coloneqq \sigma_{L^2}$ corresponding to the labelings L^1 and L^2, respectively, read

$$\sigma_1 = \left(\begin{array}{cc} 1 & 2 \\ 1 & k_1+1 \end{array} \middle| \underbrace{\begin{array}{ccc} \dots & c+2 & \dots \\ \dots & 2 & \dots \end{array}}_{k_1+k_1-2} \right) \quad \text{and}$$

$$\sigma_2 = \left(\begin{array}{cc} 1 & 2 \\ 1 & k_2+1 \end{array} \middle| \underbrace{\begin{array}{ccc} \dots & c+2 & \dots \\ \dots & k_2+2 & \dots \end{array}}_{k_1+k_2-2} \right).$$

The underbracket marks the block which represents a cyclic permutation of the remaining indices. We see that

$$\begin{aligned}\sigma_1 &: V^{\otimes 2} \otimes V^{\otimes s} \longrightarrow V^{\otimes k_1} \otimes V^{\otimes k_2}, \quad e_i e_j w \longmapsto e_i w^1 e_j w^2,\\ \sigma_2 &: V^{\otimes 2} \otimes V^{\otimes s} \longrightarrow V^{\otimes k_2} \otimes V^{\otimes k_1}, \quad e_i e_j w \longmapsto e_i w^2 e_j w^1,\end{aligned}$$

where $w^1 = w_c \dots w_{c+k_1-2}$, $w^2 = w_{c+k_1-1} \dots w_{c+k_1+k_2-3}$ and $s \coloneqq k_1 + k_2 - 2$. Defining $\tilde{w}^1 \coloneqq w^2$ and $\tilde{w}^2 \coloneqq w^1$, The Koszul sign of σ_2 can be written as

$$\varepsilon(w \mapsto w^1 w^2)(-1)^{|w^2||e_j|+|w^1||w^2|} = \varepsilon(w \mapsto \tilde{w}^1\tilde{w}^2)(-1)^{|\tilde{w}^1||e_j|}.$$

We use these facts to rewrite (A.10) as follows:

$$\begin{aligned} &f_{210}(\psi_1 \otimes \psi_2)(w)\\ &\quad= \sum_{1\le k_1<k_2} \sum_{i,j} \mathrm{T}^{ij}\Big(\sum_{k(w^1)=k_1-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1||e_j|}\psi_1(e_i w^1)\psi_2(e_j w^2)\\ &\qquad + \sum_{k(w^1)=k_2-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^2||e_j|+|w^1||w^2|}\psi_1(e_i w^2)\psi_2(e_j w^1)\Big)\\ &\qquad + \sum_{1<k_1=k_2} \frac{1}{2}\Big(\sum_{k(w^1)=k_1-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1||e_j|}\psi_1(e_i w^1)\psi_2(e_j w^2)\\ &\qquad + \sum_{k(w^1)=k_2-1} \varepsilon(w \mapsto w^1 w^2)(-1)^{|w^2||e_j|+|w^1||w^2|}\psi_1(e_i w^2)\psi_2(e_j w^1)\Big)\\ &\quad= \sum_{\substack{k_1,k_2\ge 1\\ k_1+k_2>2}} \sum_{\substack{k(w^1)=k_1-1\\ k(w^2)=k_2-1}} \mathrm{T}^{ij}\varepsilon(w \mapsto w^1 w^2)(-1)^{|w^1||e_j|}\psi_1(e_i w^1)\psi_2(e_j w^2). \end{aligned}$$

This coincides with the formula from Definition 3.4.1.

As for f_{120}, the permutations $\sigma_1 \coloneqq \sigma_{L^1}$ and $\sigma_2 \coloneqq \sigma_{L^2}$ corresponding to the labelings L^1 and L^2, respectively, read

$$\sigma_1 = \left(\begin{array}{cc|c|c} 1 & 2 & \dots\quad c_1+2\quad\dots & \dots\quad c_2+s_1+2\quad\dots\\ 1 & s_1+2 & \underbrace{\dots\qquad 2\qquad\dots}_{s_1} & \underbrace{\dots\qquad s_1+3\qquad\dots}_{s_2} \end{array}\right) \quad \text{and}$$

$$\sigma_2 = \left(\begin{array}{cc|c|c} 1 & 2 & \dots\quad c_2+2\quad\dots & \dots\quad c_1+s_2+2\quad\dots\\ s_1+2 & 1 & \underbrace{\dots\quad s_1+3\quad\dots}_{s_2} & \underbrace{\dots\qquad 2\qquad\dots}_{s_1} \end{array}\right),$$

where the underbracketed blocks denote cyclic permutations of consecutive indices on the corresponding boundary component. We see that

$$\begin{aligned}
\sigma_1 : V^{\otimes 2} \otimes V^{\otimes s_1} \otimes V^{\otimes s_2} &\longrightarrow V^{\otimes k}, \quad e_i e_j w_1 w_2 \longmapsto e_i w_1^1 e_j w_2^1, \\
\sigma_2 : V^{\otimes 2} \otimes V^{\otimes s_2} \otimes V^{\otimes s_1} &\longrightarrow V^{\otimes k}, \quad e_i e_j w_1 w_2 \longmapsto e_j w_2^1 e_i w_1^1,
\end{aligned}$$

where w_i^1 denotes a cyclic permutation and $k \coloneqq s_1 + s_2 + 2$. The Koszul sign of σ_2 can be written as

$$\begin{aligned}
&(-1)^{|e_i||e_j| + |w_1||w_2| + |e_i||w_2|} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \\
&\qquad = (-1)^{(|e_i| + |w_1|)(|e_j| + |w_2|) + |w_1||e_j|} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1).
\end{aligned}$$

We use this fact and the cyclic symmetry of ψ to rewrite (A.10) as follows:

$$\begin{aligned}
&f_{120}(\psi)(w_1 \otimes w_2) \\
&= \sum_{0 \le s_1 < s_2} \Big(\delta\tbinom{k(w_1)=s_1}{k(w_2)=s_2} \sum \mathrm{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|w_1||e_j|} \\
&\qquad \psi(e_i w_1^1 e_j w_2^1) + \delta\tbinom{k(w_1)=s_2}{k(w_2)=s_1} \sum \mathrm{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \\
&\qquad\qquad (-1)^{|e_i||e_j| + |w_2||w_1| + |e_i||w_2|} \psi(e_j w_2^1 e_i w_1^1) \Big) \\
&\quad + \sum_{0 < s_1 = s_2} \delta\tbinom{k(w_1)=k(w_1)=s_1}{k(w_2)=k(w_2)=s_2} \frac{1}{2} \Big(\sum \mathrm{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \\
&\qquad (-1)^{|w_1||e_j|} \psi(e_i w_1^1 e_j w_2^1) + \sum \mathrm{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \\
&\qquad\qquad (-1)^{|e_i||e_j| + |w_2||w_1| + |e_i||w_2|} \psi(e_j w_2^1 e_i w_1^1) \Big) \\
&= \sum_{\substack{s_1, s_2 \ge 0 \\ s_1 + s_2 > 0}} \delta\tbinom{k(w_1)=s_1}{k(w_2)=s_2} \sum \mathrm{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|w_1||e_j|} \\
&\qquad\qquad \psi(e_i w_1^1 e_j w_2^1).
\end{aligned}$$

This coincides with the formula from Definition 3.4.1. ◁

A.2. De Rham case

We will now establish a formal analogy between the finite-dimensional and the de Rham case, which will explain the signs in Definition 4.3.6.

The finite-dimensional case. Consider the situation of (A) – (D) in the Overview. To recall briefly, we have a finite-dimensional cyclic dga (V, Π, m_1, m_2) and a subcomplex $\mathcal{H} \subset V$ such that there is a projection $\pi : V[1] \to \mathcal{H}[1]$ chain homotopic to $\mathbb{1}$ via a chain homotopy $\mathcal{P} : V[1] \to V[1]$. Using m_2, one constructs the canonical Maurer-Cartan element $\mathfrak{m}$ for $\mathrm{dIBL}(C(V))$. The algebraic Schwartz kernel $\mathcal{K}_{\mathcal{P}}$ of $\mathcal{P}$ is an admissible propagator used to construct the IBL_∞-quasi-isomorphism $\mathfrak{f} = (\mathfrak{f}_{klg}) : \mathrm{dIBL}(C(V)) \to \mathrm{dIBL}(C(\mathcal{H}))$. The Maurer-Cartan element $\mathfrak{m}$ is then pushed forward along $\mathfrak{f}$ to obtain the Maurer-Cartan element $\mathfrak{n} \coloneqq \mathfrak{f}_*\mathfrak{m}$ for $\mathrm{dIBL}(C(\mathcal{H}))$ (see [CFL15, Lemma 9.5]). The formula for $\mathfrak{n}$ given in [CFL15, Remark 12.10] reads

$$\begin{aligned}&\mathfrak{n}_{lg}(\mathrm{s}^l w_1 \otimes \cdots \otimes w_l)\\ &\quad = \frac{1}{l!} \sum_{[\Gamma]\in\overline{\mathrm{RG}}^{(3)}_{klg}} \frac{1}{|\operatorname{Aut}(\Gamma)|}(-1)^{k(n-2)}\langle (m_2^+)^{\otimes k}, w_1 \otimes \cdots \otimes w_l\rangle_\Gamma^{\mathcal{K}_\mathcal{P}}.\end{aligned} \tag{A.12}$$

Here the artificial sign $(-1)^{k(n-2)}$ is added because our sign conventions for m_2^+ differ (see Remark 3.3.8).

The de Rham case. We are in the setting of Definition 4.3.6. To recall briefly, we have the cyclic dga $(\Omega(M), \Pi, m_1, m_2)$, the subspace of harmonic forms $\mathcal{H} \subset \Omega$, the harmonic projection $\pi_\mathcal{H} : \Omega \to \mathcal{H}$ and a Hodge propagator $\mathrm{P} \in \Omega(\mathrm{Bl}_\Delta(M \times M))$, which is the Schwartz kernel of a chain homotopy $\mathcal{P} : \Omega \to \Omega$ between $\pi_\mathcal{H}$ and $\mathbb{1}$. In analogy with the finite-dimensional case, the canonical Maurer-Cartan element (4.3) for $\mathrm{dIBL}(\mathcal{H})$ satisfies $\mathfrak{m}_{10} = (-1)^{n-2} m_2^+$ with $m_2^+ = \Pi(m_2 \otimes \mathbb{1})$. Because $\dim(\Omega)$ is not of finite type, Definition 3.4.1 does not give the canonical dIBL-structure on $C(\Omega)$, and hence we have neither $\mathfrak{f}$ nor $\mathfrak{n}$ in the standard sense.

In order to deduce the formal analogy, we embed $\Omega(M)^{\otimes 2}$ into $\Omega(\mathrm{Bl}_\Delta(M \times M))$ using the external wedge product $(\eta_1, \eta_2) \mapsto \tilde{\pi}_1^*\eta_1 \wedge \tilde{\pi}_2^*\eta_2$ and suppose that the Hodge propagator P satisfies $\mathrm{P} \in \Omega^{\otimes 2}$. This never happens, so what follows is just a formal computation whose purpose is to deduce candidates for signs.

Proposition A.2.1. *In the de Rham case, suppose that* $\mathrm{P} \in \Omega(M)^{\otimes 2}$. *Then* (A.12) *reduces to* (4.20).

Proof. Consider the intersection pairing $\tilde{\Pi}$ and its degree shift Π (see Proposition 4.1.2). According to Definition A.1.4, they extend to pairings on $\Omega(M)^{\otimes k}$ and $\Omega(M)[1]^{\otimes k}$ for all $k \geq 1$, respectively. For all η_{11}, η_{12}, η_{21}, $\eta_{22} \in \Omega(M)$, we have:

$$\begin{aligned}&\Pi(\theta^2\eta_{11} \otimes \eta_{12}, \theta^2\eta_{21} \otimes \eta_{22})\\ &\qquad = (-1)^{\eta_{11}+\eta_{21}}\Pi(\theta\eta_{11} \otimes \theta\eta_{12}, \theta\eta_{21} \otimes \theta\eta_{22})\\ &\qquad = (-1)^{\eta_{11}+\eta_{21}+(1+\eta_{12})(1+\eta_{21})}\Pi(\theta\eta_{11}, \theta\eta_{21})\Pi(\theta\eta_{12}, \theta\eta_{22})\\ &\qquad = (-1)^{1+\eta_{12}\eta_{21}}\tilde{\mathrm{P}}(\eta_{11}, \eta_{21})\tilde{\Pi}(\eta_{12}, \eta_{22})\\ &\qquad = -\tilde{\Pi}(\eta_{11} \otimes \eta_{12}, \eta_{21} \otimes \eta_{22}).\end{aligned} \tag{A.13}$$

One can also check that

$$\tilde{\Pi}(\eta_{11}\otimes\eta_{12},\eta_{21}\otimes\eta_{22})=\int_{x,y}\eta_{11}(x)\eta_{12}(y)\eta_{21}(x)\eta_{22}(y).$$

For the Hodge homotopy $\mathcal{P}:\Omega(M)\to\Omega(M)$ and its Hodge propagator $\mathrm{P}\in\Omega(M)^{\otimes 2}$, we have the following:

$$\forall\eta_1,\eta_2\in\Omega(M):\quad \tilde{\Pi}(\mathcal{P}(\eta_1),\eta_2)=\int_{x,y}\mathrm{P}(x,y)\eta_1(x)\eta_2(y)=\tilde{\Pi}(\mathrm{P},\eta_1\otimes\eta_2).$$

From this and (A.13), we obtain

$$\begin{aligned}\Pi(\mathcal{P}(\theta\eta_1),\eta_2)&=\Pi(\theta\mathcal{P}(\eta_1),\theta\eta_2)=(-1)^{1+\eta_1}\tilde{\Pi}(\mathcal{P}(\eta_1),\eta_2)\\&=(-1)^{1+\eta_1}\tilde{\Pi}(\mathrm{P},\eta_1\otimes\eta_2)=(-1)^{\eta_1}\Pi(\theta^2\mathrm{P},\theta^2\eta_1\otimes\eta_2)\\&=\Pi(\theta^2\mathrm{P},\theta\eta_1\otimes\mathrm{s}\eta_2).\end{aligned}$$

Therefore, the element $\theta^2\mathrm{P}\in V[1]^{\otimes 2}$ corresponds to the Schwartz kernel $\mathcal{K}_{\mathcal{P}}$ of $\mathcal{P}:V[1]\to V[1]$. We write this correspondence as

$$\mathcal{K}_{\mathcal{P}}\in V[1]^{\otimes 2}\ \sim\ \theta^2\mathrm{P}\in\mathrm{Bl}_{\Delta}(M\times M)[2].$$

Let us check that $\theta^2\mathrm{P}$ satisfies (A.1). First of all, if we embed $\Omega(M)^{\otimes k}$ into $\Omega(M^{\times k})$ using the external wedge product $\eta_1\otimes\dots\otimes\eta_k\mapsto\pi_1^*\eta_1\wedge\dots\wedge\pi_k^*\eta_k=:\eta_1(x_1)\wedge\dots\wedge\eta_k(x_k)$, then for all η_1, …, $\eta_k\in\Omega(M)$ we have

$$\sigma(\eta_1\otimes\dots\otimes\eta_k)(x_1,\dots,x_k)=\eta_1(x_{\sigma_1})\wedge\ldots\wedge\eta_k(x_{\sigma_k}),$$

where the action on the left-hand side is given by (3.8). Now, the symmetry property (4.7) implies

$$\tau(\theta^2\mathrm{P})=-\theta^2\tau^*(\mathrm{P})=(-1)^{n+1}\theta^2\mathrm{P}=(-1)^{|\theta^2\mathrm{P}|}\theta^2\mathrm{P}.$$

Therefore, the symmetry condition (A.1) is indeed satisfied.

Let $\Gamma\in\overline{\mathrm{RG}}^{(3)}_{klg}$, and let L be a labeling of Γ. We abbreviate $\sigma\coloneqq\sigma_L\in\mathbb{S}_{3k}$. Given $\eta_{ij}\in\Omega(M)$ for $j=1$, …, s_i and $i=1$, …, l, where s_i is the valency of the i-th boundary component, we set $\eta_i=\eta_{i1}\otimes\dots\otimes\eta_{is_i}$, $\eta=\eta_1\otimes\dots\otimes\eta_l$, $\alpha_{ij}=\theta\eta_{ij}$, $\omega_i=\alpha_{i1}\otimes\dots\otimes\alpha_{is_i}$ and $\omega=\omega_1\otimes\dots\otimes\omega_l$. We denote $s\coloneqq s_1+\dots+s_l$, so that $3k=2e+s$, where e is the number of internal edges. We have

$$\begin{aligned}(m_2^+)^{\otimes k}(\sigma((\theta^2\mathrm{P})^{\otimes e}\otimes\omega)) &= \varepsilon(\theta,\eta)(m_2^+)^{\otimes k}(\sigma((\theta^2\mathrm{P})^{\otimes e}\otimes\theta^s\eta))\\ &= (-1)^{se(n-1)}\varepsilon(\theta,\eta)(m_2^+)^{\otimes k}(\sigma(\theta^{2e+s}\mathrm{P}^{\otimes e}\otimes\eta))\\ &= \underbrace{(-1)^{\sigma+se(n-1)}\varepsilon(\theta,\eta)}_{=:\varepsilon_1}(m_2^+)^{\otimes k}(\theta^{2e+s}\sigma(\mathrm{P}^{\otimes e}\otimes\eta)),\end{aligned}$$

where $\varepsilon(\theta,\eta)$ is the Koszul sign to order $\theta^s\eta_{11}\dots\eta_{ls_l}\mapsto\theta\eta_{11}\dots\theta\eta_{ls_l}$ and the operation $m_2^+:\Omega(M)[1]^{\otimes 3}\to\mathbb{R}$ is given by $m_2^+=\Pi(m_2\otimes\mathbb{1})$. We denote $\kappa\coloneqq\mathrm{P}^{\otimes e}\otimes\eta=\kappa_1\otimes\dots\otimes\kappa_{3k}$, $\kappa_i\in\Omega(M)[1]$ and compute

$$\begin{aligned}&(m_2^+)^{\otimes k}(\theta^{3k}\sigma(\kappa))\\ &\quad\underset{\substack{\uparrow\\ |m_2^+|=3-n}}{=}\varepsilon(\sigma,\kappa)(m_2^+)^{\otimes k}(\theta^{3k}\kappa_{\sigma_1^{-1}}\otimes\dots\otimes\kappa_{\sigma_{3k}^{-1}})\\ &\quad= (-1)^{\frac{1}{2}k(k-1)n}\varepsilon(\sigma,\kappa)(m_2^+)^{\otimes k}(\theta^3(\kappa_{\sigma_1^{-1}}\otimes\kappa_{\sigma_2^{-1}}\otimes\kappa_{\sigma_3^{-1}})\otimes\cdots\\ &\qquad\qquad\otimes\theta^3(\kappa_{\sigma_{3k-2}^{-1}}\otimes\kappa_{\sigma_{3k-1}^{-1}}\otimes\kappa_{\sigma_{3k}^{-1}}))\\ &\quad= \overbrace{(-1)^{\frac{1}{2}k(k-1)n+\kappa_{\sigma_2^{-1}}+\dots+\kappa_{\sigma_{3k-1}^{-1}}}\varepsilon(\sigma,\kappa)}^{=:\varepsilon_2}(m_2^+)^{\otimes k}((\theta\kappa_{\sigma_1^{-1}}\otimes\theta\kappa_{\sigma_2^{-1}}\\ &\qquad\qquad\otimes\theta\kappa_{\sigma_3^{-1}})\otimes\dots\otimes(\theta\kappa_{\sigma_{3k-2}^{-1}}\otimes\theta\kappa_{\sigma_{3k-1}^{-1}}\otimes\theta\kappa_{\sigma_{3k}^{-1}})).\end{aligned}$$

Next, using the formula (2) for m_2^+, we get

$$\begin{aligned}&(m_2^+)^{\otimes k}((\theta\kappa_{\sigma_1^{-1}}\otimes\theta\kappa_{\sigma_2^{-1}}\otimes\theta\kappa_{\sigma_3^{-1}})\otimes\dots\otimes(\theta\kappa_{\sigma_{3k-2}^{-1}}\otimes\theta\kappa_{\sigma_{3k-1}^{-1}}\otimes\theta\kappa_{\sigma_{3k}^{-1}}))\\ &\quad= (-1)^{\kappa_{\sigma_2^{-1}}+\dots+\kappa_{\sigma_{3k-1}^{-1}}}\Big(\int_{x_1}\kappa_{\sigma_1^{-1}}(x_1)\kappa_{\sigma_2^{-1}}(x_1)\kappa_{\sigma_3^{-1}}(x_1)\Big)\cdots\\ &\qquad\qquad\Big(\int_{x_k}\kappa_{\sigma_{3k-2}^{-1}}(x_k)\kappa_{\sigma_{3k-1}^{-1}}(x_k)\kappa_{\sigma_{3k}^{-1}}(x_k)\Big)\\ &\quad= (-1)^{\kappa_{\sigma_2^{-1}}+\dots+\kappa_{\sigma_{3k-1}^{-1}}}\int_{x_1,\dots,x_k}\kappa_{\sigma_1^{-1}}(x_1)\kappa_{\sigma_2^{-1}}(x_1)\kappa_{\sigma_3^{-1}}(x_1)\cdots\\ &\qquad\qquad\kappa_{\sigma_{3k-2}^{-1}}(x_k)\kappa_{\sigma_{3k-1}^{-1}}(x_k)\kappa_{\sigma_{3k}^{-1}}(x_k)\\ &\quad= \overbrace{(-1)^{\kappa_{\sigma_2^{-1}}+\dots+\kappa_{\sigma_{3k-1}^{-1}}}\varepsilon(\sigma,\kappa)}^{=:\varepsilon_3}\int_{x_1,\dots,x_k}\kappa_1(x_{\xi(\sigma_1)})\kappa_2(x_{\xi(\sigma_2)})\kappa_3(x_{\xi(\sigma_3)})\cdots\\ &\qquad\qquad\kappa_{3k-2}(x_{\xi(\sigma_{3k-2})})\kappa_{3k-1}(x_{\xi(\sigma_{3k-1})})\kappa_{3k}(x_{\xi(\sigma_{3k})}),\end{aligned}$$

where $\xi(3j-2)=\xi(3j-1)=\xi(3j)=j$ for $j=1,\dots,k$ (see Definition 4.3.6). In total, we have

$$
\begin{aligned}
&(m_2^+)^{\otimes k}(\sigma((\theta^2\mathrm{P})^{\otimes e}\otimes\omega)) \\
&\quad = \varepsilon_1\varepsilon_2\varepsilon_3\int_{x_1,\dots,x_k}\mathrm{P}(x_{\xi(\sigma_1)},x_{\xi(\sigma_2)})\cdots\mathrm{P}(x_{\xi(\sigma_{2e-1})},x_{\xi(\sigma_{2e})}) \\
&\qquad\qquad\qquad\qquad\qquad\alpha_{11}(x_{\xi(\sigma_{2e+1})})\cdots\alpha_{ls_l}(x_{\xi(\sigma_{2e+s})}),
\end{aligned}
$$

where

$$\varepsilon_1\varepsilon_2\varepsilon_3=(-1)^{\sigma+se(n-1)+\frac{1}{2}k(k-1)n}\varepsilon(\theta,\eta).$$

Using (4.17) and (4.19) and $\varepsilon(\theta,\eta)=(-1)^{\mathrm{P}(\omega)}$, we get the total sign

$$(-1)^{k(n-2)}\varepsilon_1\varepsilon_2\varepsilon_3=(-1)^{s(k,l)+\sigma+\mathrm{P}(\omega)},$$

where $(-1)^{k(n-2)}$ is the artificial sign from (A.12). This proves the proposition. □

Remark A.2.2 (Signs for the Fréchet dIBL-structure on $\Omega(M)$). In [CFL15, Section 13], they consider the weight-graded nuclear Fréchet space $\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)_\infty\subset\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)$ generated by $\varphi\in\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)$ which have a smooth Schwartz kernel $k_\varphi\in\Omega(M^{\times k})$; they showed that there is a canonical Fréchet dIBL-structure on $\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)_\infty[2-n]$. In order to deduce the signs, we can consider the subspace $\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)_{\mathrm{alg}}\subset\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)_\infty$ generated by $\varphi\in\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)$ with an algebraic Schwartz kernel $\mathcal{K}_\varphi\in\Omega(M)[1]^{\otimes k}$, rewrite (A.10) in terms of $\mathcal{K}_\varphi$ and extend the obtained formulas to $\mathrm{B}^*_{\mathrm{cyc}}\Omega(M)_\infty$. ◁

B. Reduced cyclic homology of A-infinity-algebras

In this appendix, we prove Proposition 3.3.13. The idea from [Lod92] is to resolve cyclic (co)invariants degree-wise and obtain certain bicomplexes with better properties.

In Section B.1, given a strictly unital A_∞-algebra on a graded vector space V, we define the normalized and reduced Hochschild (co)chain complexes (Definition B.1.1) and prove the computational prerequisites CP1–CP4 (Lemmas B.1.4, B.1.5, B.1.6 and B.1.7); they are necessary for the development of the cyclic homology theory in the upcoming section. These prerequisites, like squaring to zero of the Hochschild differential, seem to be much harder computationally for A_∞-algebras than for DGA's. Proofs of some of these relations in different formalisms appeared already in [Mes16] and [Laz03].

In Section B.2, we define Loday's and Connes' cyclic half-plane bicomplexes for A_∞-algebras together with their normalized and reduced versions (Definition B.2.3). We then summarize some convergence results for spectral sequences associated to horizontal, vertical and diagonal filtrations (Proposition B.2.1). In a series of lemmas (Lemmas B.2.5, B.2.7, B.2.8 and B.2.9), we prove that some of the (co)homologies are isomorphic. These lemmas copy results for DGA's from [Lod92]; we just do them carefully for half-plane bicomplexes and more explicitly. There is a new phenomenon of long chains coming from completing the direct sum total complex; these long chains seem to disappear in homology if the degrees of V are bounded (Lemma B.2.6). Additionally, we point out some differences between first-quadrant and half-plane bicomplexes (Remark B.2.2) and mention the relation to mixed complexes (Remark B.2.4).

In Section B.3, we obtain short exact sequences for reduced Connes' bicomplexes (Lemma B.3.1), which replace, up to quasi-isomorphisms, the non-exact sequences for reduced cyclic Hochschild (co)chains. We summarize the isomorphisms of (co)homologies from Section B.2 (Figure B.3), finish the proof of Proposition 3.3.13 and formulate a few open question (Questions B.3.2).

B.1. Computational prerequisites

The heart of cyclic (co)homology theory, following [Lod92], are the following five *computational prerequisites* (CP):

CP0 (horizontal relations)**:** $\ker N = \operatorname{im}(\mathbb{1} - t)$, $\ker(\mathbb{1} - t) = \operatorname{im} N$,

CP1 (vertical relations)**:** $\mathrm{b} \circ \mathrm{b} = 0$, $\mathrm{b}' \circ \mathrm{b}' = 0$,

CP2 (vertical-horizontal relations)**:** $\mathrm{b}' \circ N = N \circ \mathrm{b}$, $(\mathbb{1} - t) \circ \mathrm{b}' = \mathrm{b} \circ (\mathbb{1} - t)$,

and in the strictly unital case

CP3 (null-homotopy of the bar resolution)**:** $\mathrm{b}' \circ \iota_1 + \iota_1 \circ \mathrm{b}' = \mathbb{1}$ and

CP4 (contraction onto normalized chains)**:** $\bar{p} : DV \to \bar{D}V$ is a quasi-isomorphisms.

The definitions of t (cyclic permutation), b (A_∞-Hochschild differential), b' (acyclic A_∞-Hochschild differential) and DV ((reduced) bar complex with reversed grading shifted by one) can be found in Section 3.3; in particular, consult Definition 3.3.9. The new players are the *counting operator*

$$N \coloneqq \sum_{k=1}^{\infty} \underbrace{\sum_{i=0}^{k-1} t_k^i}_{=: N_k} : DV \longrightarrow DV$$

and the projection $\bar{p} : DV \to \bar{D}V$ to normalized Hochschild chains — this we define below.

Definition B.1.1 (Normalized and reduced Hochschild complex)**.** *Let* $(V, (\mu_j), \imath)$ *be a strictly unital* A_∞*-algebra. Let* $\bar{V}[1] \coloneqq V[1]/\langle \imath \rangle$*. We define the* normalized Hochschild chain complex *by*

$$\bar{D}V \coloneqq \bigoplus_{l=0}^{\infty} V[1] \otimes \bar{V}[1]^{\otimes l}.$$

We consider the canonical projection $\bar{p} : V[1] \to \bar{V}[1]$ *and define* $\bar{p} : DV \to \bar{D}V$ *by*

$$\bar{p}\big|_{V[1]^{\otimes l}} \coloneqq \mathbb{1} \otimes \underbrace{\bar{p} \otimes \cdots \otimes \bar{p}}_{l-times}.$$

For every $l \geq 1$*, we define the operator* $\iota_l : DV \to DV$ *by inserting* $\imath$ *at the* l*-th position of a tensor product, where the position* $l = 1$ *is in front; i.e., we have*

$$\iota_1(v_1 \otimes \cdots \otimes v_i) = \imath \otimes v_1 \otimes \cdots \otimes v_i \quad \textit{for all } v_j \in V[1] \textit{ and } i \geq j \geq 1.$$

We define the normalized Hochschild cochain complex *by*

$$\bar{D}^*V \coloneqq \{\varphi \in D^*V \mid \varphi \circ \iota_l = 0 \text{ for all } l \geq 2\}.$$

If $u : D\mathbb{R} \to DV$ *is the unit map (in the strictly unital case,* $u|_{\mathbb{R}[1]^{\otimes k}} \coloneqq u^{\otimes k}$ *where* $u : \mathbb{R}[1] \to V[1]$ *is the canonical injection; see also Definition 3.3.10 and the discussion below) and* $\bar{\iota} : \bar{D}^*V \to D^*V$ *the inclusion, we define the* reduced Hochschild chain and cochain complexes $D^{\mathrm{red}}V$ *and* $D^*_{\mathrm{red}}V$ *by*

$$D^{\mathrm{red}}V \coloneqq \operatorname{coker}(\bar{p} \circ u) \quad \text{and} \quad D^*_{\mathrm{red}} \coloneqq \ker(u^* \circ \bar{\iota}), \quad \text{respectively.}$$

We denote by $p^{\mathrm{red}} : DV \to D^{\mathrm{red}}V$ *and* $\iota_{\mathrm{red}} : D^*_{\mathrm{red}}V \to D^*V$ *the canonical projection and inclusion, respectively.*

All chain complexes above are graded by degree and equipped with a boundary operator induced naturally from b *(see the remark below).*

Remark B.1.2 (Some details on normalized and reduced complexes)*.* Since $\mathbb{1}$ is a unit for μ_2, we have

$$\begin{aligned} &0 = (-1)^{|v_1|+\cdots+|v_{k-1}|}(v_1 \cdots \mu_2(v_k, \mathbb{1}) + (-1)^{|v_k|}\mu_2(\mathbb{1}, v_1) \cdots v_k) \quad \text{and} \\ &0 = (-1)^{|v|_1+\cdots+|v_{i-2}|}(v_1 \cdots \mu_2(v_{i-1}, \mathbb{1})v_i \cdots v_k + (-1)^{|v_{i-1}|}v_1 \cdots v_{i-1}\mu_2(\mathbb{1}, v_i) \cdots v_k) \end{aligned}$$

for all $i = 2, \ldots, k$. This fact and strict unitality implies

$$\mathrm{b}(\sum_{i\geq 2} \operatorname{im} \iota_i) \subset \sum_{i \geq 2} \operatorname{im} \iota_i = \ker \bar{p}.$$

Therefore, b induces a differential on $\bar{D}V$. Since $\bar{D}^*V = \{\varphi \in D^*V \mid \varphi(\sum_{i\geq 2} \operatorname{im} \iota_i) = 0\}$, the dual b^* restricts to $\bar{D}^*V$. Clearly, both $\bar{p}$ and $\bar{\iota}$ are chain maps, and they are compatible under the dualization from Definition 3.3.4; i.e., $\bar{\iota} \simeq \bar{p}^*$ under $\bar{D}^*V \simeq (\bar{D}V)^{*_{\mathrm{g}}}$ and $D^*V \simeq (DV)^{*_{\mathrm{g}}}$, where $^{*_{\mathrm{g}}}$ denotes the graded dual.

As for the reduced complexes, u is a chain map, and thus ker and coker are chain complexes. Again, it holds $D^*_{\mathrm{red}}V \simeq (D^{\mathrm{red}}V)^{*_{\mathrm{g}}}$ and $\iota_{\mathrm{red}} \simeq p^{\mathrm{red},*}$ under the dualization. $\lhd$

We will now prove CP1, CP2, CP3 and CP4 for strictly unital A_∞-algebras. We do not prove CP0 because it is a standard fact which does not depend on the algebra we work with (see [Lod92]). A proof of CP1 in a slightly different notation and in a more general setting (coefficients in a bimodule) can also be found in [Mes16]. The proofs of CP2 and CP3 work in the same way as the proofs for DGA's from [Lod92]. The computation is just a little longer. As for CP4, we can not use the proof for DGA's from [Lod92, Proposition 1.6.5] anymore because we do not have a simplicial module; instead of this, we consider an explicit homotopy inspired by [Laz03], where CP4 is also proven in a slightly different notation.

We first introduce some notation which simplifies computations:

Definition B.1.3 (Notation). *For the cyclic permutation* t_k^i *(*$:=$ *i-times* t_k*), we define* $c := k - i + 1$ *and write*

$$v_c \cdots v_{c-1} := t_k^i(v_1 \otimes \cdots \otimes v_k).$$

We compute indices modulo k *and often omit writing the tensor product.*

For every $i = 1, \ldots, k$ *and* $j = 1, \ldots, k - i + 1$, *we define the* closed bracket *by*

$$\begin{aligned} &v_1 \cdots \overbracket{v_i \cdots v_{i+j-1}} \cdots v_k \\ &\qquad := (-1)^{|v_1|+\cdots+|v_i-1|} v_1 \otimes \cdots \otimes \mu_j(v_i \otimes \cdots \otimes v_{i+j-1}) \otimes \cdots \otimes v_k. \end{aligned} \tag{B.1}$$

If we apply the closed bracket two-times, we write the first application as an underbracket *and the second as an* overbracket*; for instance, we have*

$$\begin{aligned} &v_1 \cdots \overbracket{v_{i_1} \cdots v_{i_2}} \cdots \underbracket{v_{i_3} \cdots v_{i_4}} \cdots v_k \\ &\quad = (-1)^{|v_{i_1}|+\cdots+|v_{i_3-1}|} v_1 \otimes \cdots \otimes \mu_{j_2}(v_{i_1} \otimes \cdots \otimes v_{i_2}) \otimes \cdots \\ &\qquad\qquad\qquad\qquad \otimes \mu_{j_1}(v_{i_3} \otimes \cdots \otimes v_{i_4}) \otimes \cdots \otimes v_k, \end{aligned}$$

where $j_1 = i_4 - i_3 + 1$, $j_2 = i_2 - i_1 + 1$. *Clearly, the difference is only in the sign. We denote*

$$v_1 \cdots v_{i-1} \overline{\mathbb{1}} v_i \cdots v_k := \iota_i(v_1 \cdots v_k) = (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \cdots v_{i-1} \mathbb{1} v_i \cdots v_k.$$

If ι_i *is composed with an other operation, we write* $\underline{\mathbb{1}}$ *if the corresponding* $\mathbb{1}$ *was inserted first and* $\overline{\mathbb{1}}$ *if it was inserted second. For example, we have*

$$v_1 \underline{\mathbb{1}} v_2 v_3 \overline{\mathbb{1}} v_4 = \iota_5(\iota_2(v_1 v_2 v_3 v_4)).$$

For $j \geq 1$ *and* $1 \leq i_1 \leq i_2 \leq k$ *with* $i_2 - i_1 \geq j$, *we define the* open bracket *as follows:*

$$v_1 \cdots \overset{j}{\overline{v_{i_1} \cdots v_{i_2}}} \cdots v_k := \sum_{\substack{i_1 \leq i_3 \leq i_4 \leq i_2 \\ i_4 - i_3 = j}} v_1 \cdots v_{i_1} \cdots \overbracket{v_{i_3} \cdots v_{i_4}} \cdots v_{i_2} \cdots v_k.$$

Using the notation above, it holds

$$\begin{aligned} b'(v_1 \cdots v_k) &= \sum_{1 \leq i_1 \leq i_2 \leq k} v_1 \cdots \overbracket{v_{i_1} \cdots v_{i_2}} \cdots v_k = \sum_{j=1}^{k} \overset{\mathrm{j}}{\overline{v_1 \cdots v_k}}, \\ R(v_1 \cdots v_k) &= \sum_{2 \leq c \leq k} \overbracket{v_c \cdots v_1} \cdots v_{c-1}, \end{aligned}$$

and the A_∞-relations simplify to

$$\sum_{1 \leq i_1 \leq i_2 \leq k} \overbracket{v_1 \cdots \underbracket{v_{i_1} \cdots v_{i_2}} \cdots v_k} = \sum_{j=1}^{k} \underset{\mathrm{j}}{\underline{\overbracket{v_1 \cdots v_k}}} = 0. \tag{B.2}$$

Because all signs are, in fact, Koszul signs for the symbols μ_{j_1}, μ_{j_2}, v_1, …, v_k, and because μ's have odd degree, we have for every $1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq k$ the following relation:

$$v_1 \cdots \underbracket{v_{i_1} \cdots v_{i_2}} \cdots \overbracket{v_{i_3} \cdots v_{i_4}} \cdots v_k + v_1 \cdots \overbracket{v_{i_1} \cdots v_{i_2}} \cdots \underbracket{v_{i_3} \cdots v_{i_4}} \cdots v_k = 0. \tag{B.3}$$

Lemma B.1.4 (CP1)**.** *For an* A_∞*-algebra* $(V, (\mu_j))$*, it holds*

$$\mathrm{b}' \circ \mathrm{b}' = 0 \quad \textit{and} \quad \mathrm{b} \circ \mathrm{b} = 0.$$

Proof. We write

$$\mathrm{b} \circ \mathrm{b} = (\mathrm{b}' + R) \circ (\mathrm{b}' + R) = \mathrm{b}' \circ \mathrm{b}' + \mathrm{b}' \circ R + R \circ \mathrm{b}' + R \circ R$$

and evaluate it on a tensor $v_1 \cdots v_k \in DV$. We claim that a summand of $\mathrm{b}(\mathrm{b}(v_1 \cdots v_k))$ coming from the subsequent application of the operations can be uniquely determined by the following data:

- the information whether it comes from $\mathrm{b}' \circ \mathrm{b}'$, $\mathrm{b}' \circ R$, $R \circ \mathrm{b}'$ or $R \circ R$;
- a cyclic permutation c of v_1, …, v_k;
- positions of the under- and upperbracket.

The reason for this is that both b' and R produce only Koszul signs, and hence the total sign of a summand in $\mathrm{b}(\mathrm{b}(v_1 \dots v_k))$ is the Koszul sign for the symbols μ_{j_1}, μ_{j_2}, v_1, …, v_k, which depends only on the start and final position of the symbols; this is precisely encoded in the data above.

For $c = 1$, only $\mathrm{b}' \circ \mathrm{b}'$ contributes; however, using (B.2) and (B.3), we get

$$\begin{aligned}(\mathrm{b}' \circ \mathrm{b}')(v_1 \cdots v_k) = &\sum_{1\le i_1\le i_2\le i_3\le i_4\le k} v_1 \cdots \overbracket{v_{i_1} \cdots \underbracket{v_{i_2} \cdots v_{i_3}} \cdots v_{i_4}} \cdots v_k \\ &+ \sum_{\substack{1\le i_1\le i_2\le k-1 \\ i_2+1\le i_3\le i_4\le k}} v_1 \cdots \underbracket{v_{i_1} \cdots v_{i_2}} \cdots \overbracket{v_{i_3} \cdots v_{i_4}} \cdots v_k \\ &+ \sum_{\substack{2\le i_3\le i_4\le k \\ 1\le i_1\le i_2\le i_3-1}} v_1 \cdots \overbracket{v_{i_1} \cdots v_{i_2}} \cdots \underbracket{v_{i_3} \cdots v_{i_4}} \cdots v_k \\ = 0.\end{aligned}$$

Let $c \geq 2$ and consider the summands from $\mathrm{b}' \circ R$, $R \circ \mathrm{b}'$ and $R \circ R$ based on $v_c \dots v_{c-1}$. The contribution of $R \circ \mathrm{b}'$ consists of the following three parts:

I. $$\sum_{\substack{c\le i_1\le i_2\le k \\ 1\le i_4\le c-1}} \overbracket{v_c \cdots \underbracket{v_{i_1} \cdots v_{i_2}} \cdots v_k v_1 \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1},$$

II. $$\sum_{1\le i_1\le i_2\le i_4\le c-1} \overbracket{v_c \cdots v_k v_1 \cdots \underbracket{v_{i_1} \cdots v_{i_2}} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1},$$

III. $$\sum_{1\le i_4\prec i_1\le i_2\le c-1} \overbracket{v_c \cdots v_k v_1 \cdots v_{i_4}} v_{i_4+1} \cdots \underbracket{v_{i_1} \cdots v_{i_2}} \cdots v_{c-1}.$$

The contribution of $\mathrm{b}' \circ R$ consists of the following two parts:

IV. $$\sum_{1\le i_2\le i_4\le c-1} \overbracket{\underbracket{v_c \cdots v_k v_1 \cdots v_{i_2}} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1},$$

V. $$\sum_{1\le i_2\prec i_3\le i_4\le c-1} \underbracket{v_c \cdots v_k v_1 \cdots v_{i_2}} v_{i_2+1} \cdots \overbracket{v_{i_3} \cdots v_{i_4}} \cdots v_{c-1}.$$

The contribution of $R \circ R$ is:

VI. $$\sum_{\substack{c\prec i_1\le k \\ 1\le i_2\le i_4\le c-1}} \overbracket{v_c \cdots \underbracket{v_{i_1} \cdots v_k v_1 \cdots v_{i_2}} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}.$$

Using (B.3), it is easy to see that III cancels with V. The sum of the other terms I, II, IV, VI vanishes for fixed $1 \leq i_4 \leq c-1$ due to (B.2). $\square$

Lemma B.1.5 (CP2)**.** *For an* A_∞*-algebra* $(V, (\mu_j))$*, the following relations hold:*

(a) $\mathrm{b}' \circ N = N \circ \mathrm{b}$,

(b) $(\mathbb{1} - t) \circ \mathrm{b}' = \mathrm{b} \circ (\mathbb{1} - t)$.

Proof. (a) We denote $z_j \coloneqq \mu_j \otimes \mathbb{1}^{k-j}$ and omit writing the composition $\circ$. We consider the components

$$\mathrm{b}'^{j}_{k} \coloneqq \sum_{i=0}^{k-j} t^{i}_{k-j+1} z_j t^{-i}_{k} \qquad \text{and} \qquad R^{j}_{k} \coloneqq \sum_{i=1}^{j-1} z_j t^{i}_{k}.$$

It holds

$$\begin{aligned}\mathrm{b}'^{j}_{k} N_k &= \sum_{l=0}^{k-1} \sum_{i=0}^{k-j} t^{i}_{k-j+1} z_j t^{-i+l}_{k} \\ &= \sum_{l=0}^{k-1} \sum_{i=0}^{k-j} t^{l}_{k-j+1} t^{i-l}_{k-j+1} z_j t^{-(i-l)}_{k} \\ &= \sum_{u=1-k}^{k-j} (\sum_{l \in L_u} t^{l}_{k-j+1}) t^{u}_{k-j+1} z_j t^{-u}_{k},\end{aligned}$$

where $u \coloneqq i - l$ and

$$L_u \coloneqq \{l \in \{0, \dots, k-1\} \mid \exists i \in \{0, \dots, k-j\} : u = i - l\}.$$

We distinguish the cases

$$L_u = \begin{cases} \{0, \dots, k-j-u\} & \text{for } 0 \leq u \leq k-j, \\ \{-u, \dots, k-j-u\} & \text{for } 1-j \leq u \leq -1 \quad \text{and} \\ \{-u, \dots, k-1\} & \text{for } 1-k \leq u \leq -j \end{cases}$$

and denote the corresponding sums by I, II and III, respectively. It holds

$$\mathrm{I} = \sum_{u=0}^{k-j} (\sum_{l=0}^{k-j-u} t^{l}_{k-j+1}) t^{u}_{k-j+1} z_j t^{-u}_{k}$$

and

$$\begin{aligned}\mathrm{III} &= \sum_{u=1-k}^{-j}\sum_{l=-u}^{k-1} t_{k-j+1}^{l} t_{k-j+1}^{u} z_j t_k^{-u} \\ &= \sum_{u=1-k}^{-j}\sum_{l=-u}^{k-1} t_{k-j+1}^{l} t_{k-j+1}^{u+k-j+1} z_j t_k^{-u-k} \\ &= \sum_{u=1}^{k-j}\sum_{l=k-u}^{k-1} t_{k-j+1}^{l-j+1} t_{k-j+1}^{u} z_j t_k^{-u} \\ &= \sum_{u=1}^{k-j}\Big(\sum_{l=k-j-u+1}^{k-j} t_{k-j+1}^{l}\Big) t_{k-j+1}^{u} z_j t_k^{-u}.\end{aligned}$$

Therefore, we have

$$\mathrm{I}+\mathrm{III} = \sum_{u=0}^{k-j}\Big(\sum_{l=0}^{k-j} t_{k-j+1}^{l}\Big) t_{k-j+1}^{u} z_j t_k^{-u} = N_{k-j+1}\mathrm{b}'^{j}_{k}$$

Next, we have

$$\mathrm{II} = \sum_{u=1-j}^{-1}\sum_{l=-u}^{k-j-u} t_{k-j+1}^{l} t_{k-j+1}^{u} z_j t_k^{-u} = \sum_{u=1}^{j-1}\Big(\sum_{l=0}^{k-j} t_{k-j+1}^{l}\Big) z_j t_k^{u} = N_{k-j+1} R_k^j.$$

We conclude that

$$\mathrm{b}'^{j}_{k} N_k = N_{k-j+1}\mathrm{b}'^{j}_{k} + N_{k-j+1} R_k^j = N_{k-j+1}\mathrm{b}_k.$$

This proves the claim.

(b) For every $k \geq 1$ and $1 \leq j \leq k$ we compute

$$\begin{aligned}
(\mathbb{1} - t)\mathrm{b}_k^j &= \mathrm{b}'^j_k - \sum_{i=0}^{k-j} t_{k-j+1}^{i+1} z_j t_k^{-(i+1)} t_k \\
&= \mathrm{b}'^j_k - \sum_{i=1}^{k-j+1} t_{k-j+1}^{i} z_j t_k^{-i} t_k \\
&= \mathrm{b}'^j_k - \sum_{i=0}^{k-j} t_{k-j+1}^{i} z_j t_k^{-i} t_k + t_{k-j+1}^{0} z_j t_k^{-0} t_k - t_{k-j+1}^{k-j+1} z_j t_k^{-k+j-1} t_k \\
&= \mathrm{b}'^j_k(\mathbb{1} - t_k) + z_j t_k - z_j t_k^j \\
&= \mathrm{b}'^j_k(\mathbb{1} - t_k) + \sum_{i=1}^{j-1} z_j t_k^i (\mathbb{1} - t_k) \\
&= (\mathrm{b}'^j_k + R_k^j)(\mathbb{1} - t_k).
\end{aligned}$$

This proves the claim. $\square$

Lemma B.1.6 (CP3)**.** *For a strictly unital* A_∞*-algebra* $(V, (\mu_k), 1)$*, it holds*

$$\mathrm{b}' \circ \iota_1 + \iota_1 \circ \mathrm{b}' = \mathbb{1}.$$

Proof. For any $k \geq 1$ and $v_1, \ldots, v_k \in V[1]$, we compute

$$\begin{aligned}
\mathrm{b}'\iota_1(v_1 \ldots v_k) + \iota_1 \mathrm{b}'(v_1 \ldots v_k) &= \overline{\underline{1} v_1} v_2 \ldots v_k + \underline{1}\,\overline{v_1 \ldots v_k} + \bar{1}\,\underline{v_1 \ldots v_k} \\
&= v_1 \ldots v_k.
\end{aligned}$$

This proves the claim. $\square$

Lemma B.1.7 (CP4)**.** *Let* $\mathcal{A} = (V, (\mu_k), 1)$ *be a strictly unital* A_∞*-algebra. For all* $k \geq 2$*, we define* $h_k : DV \to DV$ *by*

$$h_k \coloneqq \iota_k \circ \mathrm{b} + \mathrm{b} \circ \iota_k + \mathbb{1}.$$

Then the formulas

$$\begin{aligned}
s^* &\coloneqq s_2^* + s_3^* \circ h_2^* + s_4^* \circ h_3^* \circ h_2^* + \cdots \quad \text{and} \\
h^* &\coloneqq \cdots \circ h_k^* \circ \cdots \circ h_2^*
\end{aligned}$$

define homogenous linear maps s^* *and* $h^* : D^*V \to D^*V$ *of degrees* 1 *and* 0*, respectively.*

The map h^* is a projection onto $\bar{D}^*V$, and the following homotopy relation holds:

$$s^* \circ \mathrm{b}^* + \mathrm{b}^* \circ s^* = h^* - \mathbb{1}. \tag{B.4}$$

It implies that $\bar{\iota}$ and $\bar{\mathrm{p}}$ are quasi-isomorphisms.

Proof. We set $\bar{D}^*_{(1)}V := D^*V$, and for all $k \geq 2$, we define

$$\bar{D}^*_{(k)}V := \{\psi \in D^*V \mid \psi \circ \iota_i = 0 \text{ for all } i = 2, \dots, k\}.$$

We will show first that h^*_k restricts to a projection $\bar{D}^*_{(k-1)}V \to \bar{D}^*_{(k)}V$. Let $i \geq 1$ and $v_1, \dots, v_i \in V[1]$. We make the following computations:

(1) For $i < k-1$, we have
$$(\iota_i \mathrm{b} + \mathrm{b}\iota_i)(v_1 \dots v_i) = 0$$
by the definition of $\iota_i k$ and by the fact that b does not increase weights.

(2) For $i = k-1$, we have
$$\begin{aligned}
&(\iota_i \mathrm{b} + \mathrm{b}\iota_i)(v_1 \dots v_i) \\
&\quad = \underset{1}{\underline{v_1 \dots v_i}}\,\bar{1} + \overset{1}{\overline{v_1 \dots v_i}}\,\underline{1} + \sum_{j=2}^{i} \overset{j}{\overline{v_1 \dots v_i}}\,\underline{1} + v_1 \dots \overbracket{v_i \underline{1}} + \overbracket{\underline{1} v_1} \dots v_i \\
&\quad = \sum_{j=2}^{i} \overset{j}{\overline{v_1 \dots v_i}}\,\underline{1}.
\end{aligned}$$
Notice that 1 in the result is at positions $< k$.

(3) For $i > k-1$, we have

$$
\begin{aligned}
&(\iota_i \mathrm{b} + \mathrm{b}\iota_i)(v_1 \dots v_i) \\
&\quad = \sum_{j=1}^{i-k+2} \underset{j}{\underline{v_1 \dots v_{k+j-2}}}\bar{1} v_{k+j-1} \dots v_i + \sum_{j=1}^{i-k+1} v_1 \dots v_{k-1}\bar{1}\underset{j}{\underline{v_k \dots v_i}} \\
&\qquad + \sum_{m=1}^{i-k+1} \sum_{c=m+k-1}^{i} \underbracket{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2}\bar{1} v_{m+k-1} \dots v_{c-1} \\
&\qquad + \sum_{j=1}^{k-1} \overset{\mathrm{j}}{\overline{v_1 \dots v_{k-1}}}\underline{1} v_k \dots v_i + \sum_{j=1}^{i-k+1} v_1 \dots v_{k-1}\underline{1}\overset{\mathrm{j}}{\overline{v_k \dots v_i}} \\
&\qquad + v_1 \dots \overbracket{v_{k-1}\underline{1}} v_k \dots v_i + v_1 \dots v_{k-1}\overbracket{\underline{1} v_k} \dots v_i \\
&\qquad + \sum_{m=1}^{k-1} \sum_{c=k}^{i} \overbracket{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1}\underline{1} v_k \dots v_{c-1} \\
&\quad = \sum_{j=1}^{i-k+2} \underset{j}{\underline{v_1 \dots v_{k+j-2}}}\bar{1} v_{k+j-1} \dots v_i + \sum_{j=1}^{k-1} \overset{\mathrm{j}}{\overline{v_1 \dots v_{k-1}}}\underline{1} v_k \dots v_i \\
&\qquad + \sum_{m=1}^{i-k+1} \sum_{c=m+k-1}^{i} \underbracket{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2}\bar{1} v_{m+k-1} \dots v_{c-1} \\
&\qquad + \sum_{m=1}^{k-1} \sum_{c=k}^{i} \overbracket{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1}\underline{1} v_k \dots v_{c-1} \\
&\quad = \overbrace{\sum_{j=2}^{i-k+2} \underset{j}{\underline{v_1 \dots v_{k+j-2}}}\bar{1} v_{k+j-1} \dots v_i}^{=:\mathrm{I}} + \overbrace{\sum_{j=2}^{k-1} \overset{\mathrm{j}}{\overline{v_1 \dots v_{k-1}}}\underline{1} v_k \dots v_i}^{=:\mathrm{II}} \\
&\qquad + \overbrace{\sum_{m=2}^{i-k+1} \sum_{c=m+k-1}^{i} \underbracket{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2}\bar{1} v_{m+k-1} \dots v_{c-1}}^{=:\mathrm{III}} \\
&\qquad + \overbrace{\sum_{m=2}^{k-1} \sum_{c=k}^{i} \overbracket{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1}\underline{1} v_k \dots v_{c-1}}^{=:\mathrm{IV}}
\end{aligned}
$$

Notice that 1 is at the k-th position in I and III, whereas at positions $< k$ in II and IV.

Let $k \geq 2$, and let $\psi \in \bar{D}^*_{(k-1)}V$. In order to show that $\iota_j^* h_k^* \psi = 0$ for $2 \leq j \leq k$, let $i \geq j$, and let $v_1, \dots, v_i \in V[1]$ be such that $v_j = 1$. Clearly, $\psi(\mathrm{II}) = \psi(\mathrm{IV}) = 0$ for any v's. As for III, the vector $v_j = 1$ lies either inside μ_j with $j \geq 3$ or at a position $< k$. It follows that $\psi(\mathrm{III}) = 0$. As for I, we write

$$\mathrm{I} = \overbrace{\underset{2}{\underline{v_1 \cdots v_k}}\bar{\mathtt{1}}v_{k+1}\cdots v_i}^{\mathrm{Ia}} + \overbrace{\sum_{j\geq 3}^{i-k+2} \underset{\mathrm{j}}{\underline{v_1 \cdots v_{k+j-2}}}\bar{\mathtt{1}}v_{k+j-1}\cdots v_i}^{\mathrm{Ib}}\,.$$

It holds $\psi(\mathrm{Ib}) = 0$. For $2 \leq j < k$, it holds

$$\begin{aligned}\psi(\mathrm{Ia}) &= \psi(v_1 \ldots \underbracket{v_{j-1}\mathtt{1}}v_{j+1}\ldots v_k\bar{\mathtt{1}}v_{k+1}\ldots v_i) \\ &\qquad + \psi(v_1 \ldots v_{j-1}\underbracket{\mathtt{1}v_{j+1}}\ldots v_k\bar{\mathtt{1}}v_{k+1}\ldots v_i) \\ &= 0,\end{aligned}$$

whereas for $j = k$, we have

$$\psi(v_1 \cdots \underbracket{v_{k-1}\mathtt{1}}\bar{\mathtt{1}}v_{k+1}\cdots v_i) = -\psi(v_1 \ldots v_i).$$

It follows that

$$h_k^*\psi(v_1\cdots v_i) = \psi(v_1\cdots v_i) + \psi\big((\iota_i \mathrm{b} + \mathrm{b}\iota_i)(v_1\cdots v_i)\big) = 0. \tag{B.5}$$

Therefore, we have $h_k^*\psi \in \bar{D}^*_{(k)}V$. If $\psi \in \bar{D}^*_{(k)}V$, then clearly $h_k^*(\psi) = \psi$. Consequently, h_k^* is a projection $h_k^* : \bar{D}^*_{(k-1)}V \to \bar{D}^*_{(k)}V$.

For $k \geq 2$, we define

$$\begin{aligned}{}^k s^* &:= \iota_2^* + \iota_3^* \circ h_2^* + \cdots + \iota_k^* \circ h_{k-1}^* \circ \cdots \circ h_2^*, \\ {}^k h^* &:= h_k^* \circ \cdots \circ h_2^*.\end{aligned}$$

Let $\mathcal{F}^n_{\mathrm{w}}DV = \bigoplus_{k=1}^n D_kV$ be the filtration of DV by weights. For all $k \geq n+1$ it holds

$${}^k h^*\psi\Big|_{\mathcal{F}^n_{\mathrm{w}}DV} = {}^{n+1}h^*\psi \quad\text{and}\quad {}^k s^*\psi\Big|_{\mathcal{F}^n_{\mathrm{w}}DV} = {}^{n+1}s^*\psi.$$

It follows that h^*, $s^* : D^*V \to D^*V$ are well-defined and that h^* is a projection onto $\bar{D}^*V$. Also, it suffices to prove (B.4) with ${}^k s^*$ and ${}^k h^*$ instead of s^* and h^* for each $k \geq 2$.

For $k = 2$, it holds by definition. Suppose that (B.4) holds for some $k \geq 2$. Then

$$\begin{aligned}
&\mathrm{b}^* \circ ({}^{k+1}s^*) + ({}^{k+1}s^*) \circ \mathbb{1} \\
&\quad = \mathrm{b}^* \circ ({}^{k}s^*) + ({}^{k}s^*) \circ \mathrm{b}^* + \mathrm{b}^* \circ \iota_{k+1}^* \circ h_k^* \circ \cdots \circ h_2^* + \iota_{k+1}^* \circ h_k^* \circ \cdots \circ h_2^* \circ \mathrm{b}^* \\
&\quad = {}^{k}h^* - \mathbb{1} + (\mathrm{b}^* \circ \iota_{k+1}^* + \iota_{k+1}^* \circ \mathrm{b}^*) \circ h_k^* \circ \cdots \circ h_2^* \\
&\quad = {}^{k}h^* - \mathbb{1} + (h_{k+1}^* - \mathbb{1}) \circ h_k^* \circ \cdots \circ h_2^* \\
&\quad = {}^{k+1}h^* - \mathbb{1}.
\end{aligned}$$

The lemma is finally proven. □

B.2. Homological algebra of bicomplexes

We will consider homological and cohomological half-plane bicomplexes, which we depict as

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
 & B_{q,p+2} \arrow[d] & B_{q+1,p+2} \arrow[l] \arrow[d] & B_{q+2,p+2} \arrow[l] \arrow[d] & {} \arrow[l] \\
B : & B_{q,p+1} \arrow[d] & B_{q+1,p+1} \arrow[l] \arrow[d] & B_{q+2,p+1} \arrow[l] \arrow[d] & {} \arrow[l] \\
 & B_{q,p} \arrow[d] & B_{q+1,p} \arrow[l] \arrow[d] & B_{q+2,p} \arrow[l] \arrow[d] & {} \arrow[l] \\
 & {} & {} & {} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & {} & {} & {} & \\
 & B^{q,p+2} \arrow[u] \arrow[r] & B^{q+1,p+2} \arrow[u] \arrow[r] & B^{q+2,p+2} \arrow[u] \arrow[r] & {} \\
B^* : & B^{q,p+1} \arrow[u] \arrow[r] & B^{q+1,p+1} \arrow[u] \arrow[r] & B^{q+2,p+1} \arrow[u] \arrow[r] & {} \\
 & B^{q,p} \arrow[u] \arrow[r] & B^{q+1,p} \arrow[u] \arrow[r] & B^{q+2,p} \arrow[u] \arrow[r] & {} \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd},$$

respectively. The standard convention is that the squares anticommute (see [Lod92])!

We consider the *total complexes* $(\mathrm{Tot}_{\mathrm{I}}(B), \partial)$ and $(\mathrm{Tot}_{\mathrm{II}}(B), \partial)$, where for all $q \in \mathbb{Z}$, the chain groups are defined by

$$(\mathrm{Tot}_{\mathrm{I}}B)_q \coloneqq \bigoplus_{i+j=q} B_{i,j}, \quad \text{and} \quad (\mathrm{Tot}_{\mathrm{II}}B)_q \coloneqq \prod_{i+j=q} B_{i,j},$$

respectively, and where $\partial = \partial_v + \partial_h$ is the total boundary operator consisting of the vertical and horizontal boundary operators ∂_v and ∂_h, respectively. The homologies of $\mathrm{Tot}_{\mathrm{I}}$ and $\mathrm{Tot}_{\mathrm{II}}$ are denoted by $\mathrm{H}B$ and $\mathrm{H}\hat{B}$, respectively. We proceed similarly in cohomology.

For each B and B^*, we consider the vertical and horizontal filtrations which are defined in such a way that they are preserved by all the arrows and that the k-th group contains the k-th column and the k-th row, respectively. More precisely, the vertical filtration $\mathcal{F}^k_{\mathrm{v}}B$ of B consists of the columns $0, \dots, k$, whereas the vertical filtration $\mathcal{F}^k_{\mathrm{v}}B^*$ of B^* consists of the columns $k+1$, $k+2, \dots$; the horizontal filtration $\mathcal{F}^{\mathrm{h}}_k B$ of B consists of the rows k, $k-1, \dots$, whereas the horizontal filtration $\mathcal{F}^k_{\mathrm{h}}B^*$ of B^* consists of the rows k, $k+1, \dots$, and so on.

In order to check that morphisms of bicomplexes induce quasi-isomorphisms of total complexes, we will use the techniques of spectral sequences. Because we work with half-plane bicomplexes, our spectral sequences do not lie in the first quadrant, as in [Wei94], and the notion of conditional convergence from [Boa99] comes in handy. In the following, we recall some basic theory and formulate a proposition about the convergence of some unbounded spectral sequences.

A *cohomological spectral sequence* is a collection E_r of bigraded vector spaces and differentials $\mathrm{d}_r : \mathrm{E}_r^{\bullet\bullet} \to \mathrm{E}_r^{\bullet+r,\bullet-r+1}$ for $r \in \mathbb{N}$ such that $\mathrm{E}_{r+1} = \mathrm{H}(\mathrm{E}_r, \mathrm{d}_r)$. Let (C^*, d) be a cochain complex with a decreasing filtration $\mathcal{F}^s C^*$ (it has to be graded and preserved by d). For every $s \in \mathbb{Z}$, the short exact sequence

$$0 \longrightarrow \mathcal{F}^{s+1}C^* \xrightarrow{i} \mathcal{F}^s C^* \xrightarrow{j} \mathrm{gr}_s(C^*) \coloneqq \mathcal{F}^s C^* / \mathcal{F}^{s+1} C^* \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^\bullet(\mathcal{F}^{s+1}C^*) \xrightarrow{i} \mathrm{H}^\bullet(\mathcal{F}^s C^*) \xrightarrow{j} \mathrm{H}^\bullet(\mathrm{gr}_s C^*) \xrightarrow{\delta} \mathrm{H}^{\bullet+1}(\mathcal{F}^{s+1}C^*) \longrightarrow \cdots,$$

which wraps into the exact couple of bigraded vector spaces

$$\begin{tikzcd}
A_1 \coloneqq \bigoplus_{s\in\mathbb{Z}} \mathrm{H}(\mathcal{F}^s C^*)[s] \arrow[rr, "i"] & & A_1 \arrow[dl, "j"] \\
 & \mathrm{E}_1 \coloneqq \bigoplus_{s\in\mathbb{Z}} \mathrm{H}(\mathrm{gr}_s C^*)[s]. \arrow[ul, "\delta"] &
\end{tikzcd}$$

This is the so called *geometric grading* convention.[1] We also define $A^s_1 \coloneqq \mathrm{H}(\mathcal{F}^s C^*)$ and $\mathrm{E}^s_1 \coloneqq \mathrm{H}(\mathrm{gr}_s C^*)$. By deriving this triangle (see, e.g., [Cie13]), one obtains a spectral

---

[1] It is chosen such that $\mathrm{E}_1^{sd} = \mathrm{H}(B^{sd}, \mathrm{d}_{\mathrm{v}})$ for the vertical filtration.

sequence E_r associated to the filtration. One defines the E_∞ page (see [Boa99]) and studies the convergence of E_r to a filtered group G. In order to formulate this, one considers the limit $A^\infty := \lim_s A^s$, the colimit $A^{-\infty} := \operatorname{colim}_s A^s$ and the right derived module for the limit $RA^\infty := \lim^1_s A^s$. We will use the following notions of convergence:

1. *Strong convergence to a filtered group* G $:\Longleftrightarrow$ For each $s \in \mathbb{Z}$, we have $\mathrm{gr}_s G \simeq \mathrm{E}^s_\infty/\mathrm{E}^{s+1}_\infty$ and the filtration on G is exhaustive, Hausdorff and complete (i.e., $G^\infty = 0$, $G^{-\infty} = G$ and $RG^\infty = 0$ for $G^s := \mathcal{F}^s G$).
2. *Conditional convergence to the colimit* $G := A^{-\infty}$ $:\Longleftrightarrow$ It holds $A^\infty = 0$ and $RA^\infty = 0$.

Note that neither notion implies, in general, the other (see (b) of Remark B.2.2).

Proposition B.2.1 (Convergence of certain unbounded spectral sequences)**.** *The following statements about convergence of spectral sequences hold:*

(a) For any $\mathbb{Z}$*-graded cochain complex* (C^*, d) *with the canonical filtration* $\mathcal{F}^k_{\mathrm{can}} C^* := \bigoplus_{i\geq k} C^i$*, the associated spectral sequence converges strongly and conditionally to the colimit* $\mathrm{H}(C^*)$*.*

(b) The spectral sequence associated to the total complex $\mathrm{Tot}_{\mathrm{I}} B^*$ *of a cohomological half-plane bicomplex* B^* *with the filtration induced from the horizontal filtration* $\mathcal{F}^k_{\mathrm{h}} B^* = \bigoplus_{i\in\mathbb{Z}} \bigoplus_{j\geq k} B^{ij}$ *converges strongly to the colimit* $\mathrm{H}(\mathrm{Tot}_{\mathrm{I}} B^*)$*.*

(c) The spectral sequence associated to the total complex $\mathrm{Tot}_{\mathrm{I}} B^*$ *of a cohomological half-plane bicomplex* B^* *with the filtration induced from the diagonal filtration*

$$\mathcal{F}^k B^* = \bigoplus_{j-i>k} B^{ij} \oplus \bigoplus_{i\in\mathbb{N}_0} Z^{i,k+i},$$

where $Z^{ij} \subset B^{ij}$ *is such that* $\mathrm{d}_{\mathrm{h}} Z^{ij} = 0$ *and* $\mathrm{d}_{\mathrm{h}} B^{i-1j} \subset Z^{ij}$*, converges strongly to the colimit* $\mathrm{H}(\mathrm{Tot}_{\mathrm{I}} B^*)$*.*

(d) The spectral sequence associated the total complex $\mathrm{Tot}_{\mathrm{II}} B^*$ *of a cohomological half-plane bicomplex* B^* *with the filtration induced from the vertical filtration* $\mathcal{F}^k_{\mathrm{v}} B^* = \bigoplus_{i\geq k} \bigoplus_{j\in\mathbb{Z}} B^{ij}$ *converges conditionally to the colimit* $\mathrm{H}(\mathrm{Tot}_{\mathrm{II}} B^*)$*.*

The following statements about morphisms of spectral sequences hold:

(e) Let f *be a morphism of filtered complexes of types (a), (b) or (c). If it induces an isomorphism of* E_r *for some* r*, then it induces an isomorphism of the target groups.*

(f) *Let f be a morphism of filtered complexes of type (d). If it induces an isomorphism of* E_r *for some r, then it induces an isomorphism of the target groups.*

Proof. (a) Strong convergence follows from (b) by embedding C^* as the first column of B^*. Also, the computation there show that $A^{-\infty} = \mathrm{H}(C^*)$ and $A^{\infty} = 0$. The condition $RA^{\infty} = 0$ is equivalent to the degreewise completeness of the filtration, which is true as the filtration is degreewise trivial. This shows the conditional convergence.

(b) Let us compute the first page with the geometrical bigrading:

$$\begin{aligned}\mathrm{E}_1^{sd} &= \mathrm{H}(\mathrm{gr}_s\,\mathrm{Tot}_{\mathrm{I}}B^*, \mathrm{d}_{\mathrm{h}})[s]^d \\ &= \mathrm{H}^{d+s}(\mathrm{Tot}_{\mathrm{I}}(s\text{-th row of } B^*), \mathrm{d}_{\mathrm{h}}) \\ &= \mathrm{H}(B^{d,s}, \mathrm{d}_{\mathrm{h}}).\end{aligned}$$

We want to use the following theorem:

Theorem B.2.1 ([Boa99, Theorem 6.1]). Suppose that $\mathrm{E}^s = 0$ for all $s > 0$ and $A^{-\infty} = 0$. Then the spectral sequence converges strongly to the colimit A^{∞}.

The proof can be done degreewise (see [Rog19]), and it can be shown that Theorem B.2.1 generalizes appropriately under the weaker assumption of “exiting differentials”. This means that the pages occupy a half-plane and for any fixed (s, d), all but finitely many differentials $\mathrm{d}_r : \mathrm{E}_r^{sd} \to \mathrm{E}_r^{s+r,d-r+1}$ leave the half-plane (and thus vanish). In our case, E_r^{sd} occupy the half-plane $\{(s,d) \mid d \geq 0\}$, and because $d - r + 1 \to -\infty$ as $r \to \infty$, the condition of exiting differentials is satisfied.

We still have to check that $A^{\infty} = 0$ and compute $A^{-\infty}$. Because the colimit is an exact functor, it commutes with H, and we have

$$A^{-\infty} = \mathrm{colim}_s\,\mathrm{H}(\mathcal{F}^s\mathrm{Tot}_{\mathrm{I}}B^*) = \mathrm{H}(\mathrm{colim}_s\,\mathcal{F}^s\mathrm{Tot}_{\mathrm{I}}B^*) = \mathrm{H}\Big(\bigcup_s \mathcal{F}^s\mathrm{Tot}_{\mathrm{I}}B^*\Big) = \mathrm{H}(\mathrm{Tot}_{\mathrm{I}}B^*).$$

We used here that $\mathcal{F}$ is exhaustive. The limit A^{∞} can be represented as

$$A^{\infty} \simeq \Big\{([a_s]) \in \prod_{s\in\mathbb{Z}} A^s \mid [a_{s+1}] \mapsto [a_s]\Big\},$$

where $\mathrm{H}(\mathcal{F}^{s+1}\mathrm{Tot}_{\mathrm{I}}B^*) \to \mathrm{H}(\mathcal{F}^s\mathrm{Tot}_{\mathrm{I}}B)$ is induced by the inclusion $\mathcal{F}^{s+1}\mathrm{Tot}_{\mathrm{I}}B^* \hookrightarrow \mathcal{F}^s\mathrm{Tot}_{\mathrm{I}}B^*$. Pick $s_0 \in \mathbb{Z}$ and consider $[a_{s_0}] \in A^{s_0}$ with a fixed representative $a_{s_0} \in \mathcal{F}^{s_0}B^*$. Because the cohomological degrees of a_{s_0} are bounded, let's say that $d_0 \in \mathbb{Z}$ is an upper bound, and the filtration is degreewise bounded from below, there is an $s_1 \geq s_0$ such that

$\mathcal{F}^s\mathrm{Tot}_{\mathrm{I}}B^* \cap (\mathrm{Tot}_{\mathrm{I}}B^*)^d = 0$ for all $s \geq s_1$ and $d \leq d_0$. Now, we have $[a_{s_1}] \mapsto [a_{s_0}]$, and hence $[a_{s_0}] = 0$. Because s_0 was arbitrary, we get $([a_s]) = 0$. Therefore, it holds $A^\infty = 0$.

Alternatively, a direct proof of (b) can be found in [Cen98].

(c) The first page reads

$$\mathrm{E}_1^{sd} = \mathrm{H}^{s+d}(\mathrm{gr}_s\,\mathrm{Tot}_{\mathrm{I}}B^*) = \mathrm{H}(B^{\lfloor \frac{d}{2}\rfloor, s+\lceil \frac{d}{2}\rceil}, \mathrm{d}'),$$

where d' is the differential on $\mathrm{gr}\,\mathrm{Tot}_{\mathrm{I}}B^*$. We see that E_r^{sd} occupy the half-plane $\{(s,d) \mid d \geq 0\}$, and hence the condition of exiting differentials is satisfied. The groups A^∞ and $A^{-\infty}$ are computed as in (b). The strong convergence is again implied by a generalization of Theorem B.2.1.

Alternatively, one can modify the direct proof for the horizontal filtration from [Cen98].

(d) We want to use the following theorem:

Theorem B.2.2 ([Boa99, Theorem 9.2]). Let C^* be a cochain complex filtered by an exhaustive, Hausdorff and complete filtration. Then the spectral sequence converges conditionally to $\mathrm{H}(C^*)$.

We have

$$\mathcal{F}_{\mathrm{v}}^k(\mathrm{Tot}_{\mathrm{I}}B^*)^d = \bigoplus_{i\geq k} B^{i,d-i},$$

and hence $\mathrm{Tot}_{\mathrm{II}}B^*$ is the completion of $\mathrm{Tot}_{\mathrm{I}}B^*$ with respect to $\mathcal{F}_{\mathrm{v}}$. Hence, it is complete. Clearly, it is also Hausdorff and exhaustive, and we can apply Theorem B.2.2.

(e) This follows from (a), (b), (c) and the following theorem:

Theorem B.2.3 ([Boa99, Theorem 5.3]). Let $f : C^* \to \bar{C}^*$ be a morphism of filtered cochain complexes. Suppose that E_r converges strongly to a filtered group G and that $\bar{\mathrm{E}}_r$ converges (strongly) to a filtered group $\bar{G}$. If f induces an isomorphism $\mathrm{E}_r \simeq \bar{\mathrm{E}}_r$ for some r, then it induces an isomorphism $G \simeq \bar{G}$.

(f) We want to use the following theorem:

Theorem B.2.4 ([Boa99, Theorem 7.2]). Let $f : C^* \to \bar{C}^*$ be a morphism of filtered cochain complexes. Suppose that $\mathrm{E}^s = \bar{\mathrm{E}}^s = 0$ for all $s < 0$ and that the spectral sequences converge conditionally to the colimits $A^{-\infty}$ and $\bar{A}^{-\infty}$, respectively. If f induces isomorphisms $\mathrm{E}_\infty \simeq \bar{\mathrm{E}}_\infty$ and $R\mathrm{E}_\infty \simeq R\bar{\mathrm{E}}_\infty$, then it induces an isomorphism $A^\infty \simeq \bar{A}^\infty$.

The first page for the vertical filtration reads:

$$\begin{aligned}\mathrm{E}_1^{sd} &= \mathrm{H}(\mathrm{gr}_s \,\mathrm{Tot}_{\mathrm{II}} B^*, \mathrm{d}_{\mathrm{v}})[s]^d \\ &= \mathrm{H}^{d+s}(\mathrm{Tot}_{\mathrm{II}}(s\text{-th column of } B^*), \mathrm{d}_{\mathrm{v}}) \\ &= \mathrm{H}(B^{sd}, \mathrm{d}_{\mathrm{v}}).\end{aligned}$$

Therefore, the condition $\mathrm{E}^s = \bar{\mathrm{E}}^s = 0$ for all $s < 0$ is satisfied.[2] By (d), conditional convergence is guaranteed. Since both E_∞ and $R\mathrm{E}_\infty$ depend only on E_r for $r \geq r_0$ and any r_0 (see [Boa99, p. 16]), the rest of the assumptions of Theorem B.2.4 is fulfilled. □

Remark B.2.2 (Differences to first-quadrant bicomplexes). (i) Given a half-plane bicomplex B, we have

$$(\mathrm{Tot}_{\mathrm{I}} B)^{*_{\mathrm{g}}} \simeq \mathrm{Tot}_{\mathrm{II}} B^*,$$

where $B^* = \bigoplus_{i,j} B_{ij}^*$ is the “pointwise dual” to B. This is why we have to consider homology and cohomology separately and can not just dualize the results.

(ii) The vertical filtration of B^* might not converge strongly to $\mathrm{H}(\mathrm{Tot}_{\mathrm{I}} B^*)$. Indeed, let

$$B^* : \quad \begin{tikzcd} \mathbb{R} & 0 & 0 \\ \mathbb{R} \arrow[u, "\mathbb{1}"] \arrow[r, "\mathbb{1}"] & \mathbb{R} & 0 \\ 0 & \mathbb{R} \arrow[u, "\mathbb{1}"] \arrow[r, "\mathbb{1}"] & \cdots \end{tikzcd} .$$

Then $\mathrm{H}(\mathrm{Tot}_{\mathrm{I}} B^*) = \mathbb{R}$ (the $\mathbb{R}$ in the first column), but $\mathrm{E}_1 = 0$ because every column is exact. Notice that $\mathrm{H}(\mathrm{Tot}_{\mathrm{II}} B^*) = 0$. Taking 0’s instead of $\mathbb{1}$’s in the definition of B^*, we see that the horizontal filtration does not converge conditionally to $\mathrm{H}(\mathrm{Tot}_{\mathrm{I}} B^*)$ because its filtration by A^s is incomplete. ◁

We will work with the following bicomplexes.

Definition B.2.3 (Bicomplexes for cyclic (co)homology)**.** *Let* $\mathcal{A} = (V, (\mu_k))$ *be an* A_∞*-algebra.* Loday’s cyclic bicomplexes *are defined by*

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[2] It is again possible to relax this assumption and prove B.2.4 when the condition of “entering differentials” is satisfied. This means that the pages occupy a half-plane and for any fixed (s, d), all but finitely many differentials $\mathrm{d}_r : \mathrm{E}_r^{s-r, d+r-1} \to \mathrm{E}_r^{s,d}$ start outside of the half-plane (and thus vanish). See [Rog19].

$$CC(\mathcal{A}):\quad \begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
D_2V & D_2V \arrow[l, "\mathbb{1}-t"] & D_2V \arrow[l, "N"] & D_2V \arrow[l, "\mathbb{1}-t"] & {} \arrow[l, "N"] \\
D_1V \arrow[from=u, "\mathrm{b}"] & D_1V \arrow[from=u, "-\mathrm{b}'"] \arrow[l, "\mathbb{1}-t"] & D_1V \arrow[from=u, "\mathrm{b}"] \arrow[l, "N"] & D_1V \arrow[from=u, "-\mathrm{b}'"] \arrow[l, "\mathbb{1}-t"] & {} \arrow[l, "N"] \\
D_0V \arrow[from=u, "\mathrm{b}"] \arrow[d, "\mathrm{b}"] & D_0V \arrow[from=u, "-\mathrm{b}'"] \arrow[l, "\mathbb{1}-t"] \arrow[d, "-\mathrm{b}'"] & D_0V \arrow[from=u, "\mathrm{b}"] \arrow[l, "N"] \arrow[d, "\mathrm{b}"] & D_0V \arrow[from=u, "-\mathrm{b}'"] \arrow[l, "\mathbb{1}-t"] \arrow[d, "-\mathrm{b}'"] & {} \arrow[l, "N"] \\
{} & {} & {} & {} &
\end{tikzcd}$$

and

$$CC^*(\mathcal{A}):\quad \begin{tikzcd}
{} & {} & {} & {} & \\
D^2V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathbb{1}-t^*"] & D^2V \arrow[u, "-\mathrm{b}'^*"] \arrow[r, "N^*"] & D^2V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathbb{1}-t^*"] & D^2V \arrow[u, "-\mathrm{b}'^*"] \arrow[r, "N^*"] & {} \\
D^1V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathbb{1}-t^*"] & D^1V \arrow[u, "-\mathrm{b}'^*"] \arrow[r, "N^*"] & D^1V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathbb{1}-t^*"] & D^1V \arrow[u, "-\mathrm{b}'^*"] \arrow[r, "N^*"] & {} \\
D^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathbb{1}-t^*"] & D^0V \arrow[u, "-\mathrm{b}'^*"] \arrow[r, "N^*"] & D^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathbb{1}-t^*"] & D^0V \arrow[u, "-\mathrm{b}'^*"] \arrow[r, "N^*"] & {} \\
{} \arrow[u, "\mathrm{b}^*"] & {} \arrow[u, "-\mathrm{b}'^*"] & {} \arrow[u, "\mathrm{b}^*"] & {} \arrow[u, "-\mathrm{b}'^*"] &
\end{tikzcd} .$$

Clearly, CC^* is the "pointwise" graded dual to CC and analogously for other bicomplexes we are going to define.

Let $\mathbb{1}$ *be a strict unit for $\mathcal{A}$. We define the* Connes' operator $\mathrm{B} : DV \to DV$ *by*

$$\mathrm{B} := (\mathbb{1} - t) \circ \iota_1 \circ N. \tag{B.6}$$

Connes' cyclic bicomplexes *are defined by*

$$BC(\mathcal{A}):\quad \begin{tikzcd}
{} \arrow[d, "\mathrm{b}"] & {} \arrow[d, "\mathrm{b}"] & {} \arrow[d, "\mathrm{b}"] & \\
D_2V \arrow[d, "\mathrm{b}"] & D_1V \arrow[l, "\mathrm{B}"] \arrow[d, "\mathrm{b}"] & D_0V \arrow[l, "\mathrm{B}"] \arrow[d, "\mathrm{b}"] & {} \arrow[l, "\mathrm{B}"] \\
D_1V \arrow[d, "\mathrm{b}"] & D_0V \arrow[l, "\mathrm{B}"] \arrow[d, "\mathrm{b}"] & D_{-1}V \arrow[l, "\mathrm{B}"] \arrow[d, "\mathrm{b}"] & {} \arrow[l, "\mathrm{B}"] \\
D_0V \arrow[d, "\mathrm{b}"] & D_{-1}V \arrow[l, "\mathrm{B}"] \arrow[d, "\mathrm{b}"] & D_{-2}V \arrow[l, "\mathrm{B}"] \arrow[d, "\mathrm{b}"] & {} \arrow[l, "\mathrm{B}"] \\
{} & {} & {} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & {} & {} & {} & \\
BC^*(\mathcal{A}): & D^2V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & D^1V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & D^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & {} \\
 & D^1V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & D^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & D^{-1}V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & {} \\
 & D^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & D^{-1}V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & D^{-2}V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\mathrm{B}^*"] & {} \\
 & {} \arrow[u, "\mathrm{b}^*"] & {} \arrow[u, "\mathrm{b}^*"] & {} \arrow[u, "\mathrm{b}^*"] &
\end{tikzcd}$$

We define the normalized Connes' operator $\bar{\mathrm{B}} : \bar{D}V \to \bar{D}V$ *by*[3]

$$\bar{\mathrm{B}} \coloneqq \bar{p} \circ \mathrm{B} = \bar{p} \circ \iota_1 \circ N. \tag{B.7}$$

The normalized Connes' cyclic bicomplexes *are defined by*

$$\begin{tikzcd}
 & {} \arrow[d, "\mathrm{b}"] & {} \arrow[d, "\mathrm{b}"] & {} \arrow[d, "\mathrm{b}"] & \\
 & \bar{D}_2V \arrow[d, "\mathrm{b}"] & \bar{D}_1V \arrow[l, "\bar{\mathrm{B}}"] \arrow[d, "\mathrm{b}"] & \bar{D}_0V \arrow[l, "\bar{\mathrm{B}}"] \arrow[d, "\mathrm{b}"] & {} \arrow[l, "\bar{\mathrm{B}}"] \\
\overline{BC}(\mathcal{A}): & \bar{D}_1V \arrow[d, "\mathrm{b}"] & \bar{D}_0V \arrow[l, "\bar{\mathrm{B}}"] \arrow[d, "\mathrm{b}"] & \bar{D}_{-1}V \arrow[l, "\bar{\mathrm{B}}"] \arrow[d, "\mathrm{b}"] & {} \arrow[l, "\bar{\mathrm{B}}"] \\
 & \bar{D}_0V \arrow[d, "\mathrm{b}"] & \bar{D}_{-1}V \arrow[l, "\bar{\mathrm{B}}"] \arrow[d, "\mathrm{b}"] & \bar{D}_{-2}V \arrow[l, "\bar{\mathrm{B}}"] \arrow[d, "\mathrm{b}"] & {} \arrow[l, "\bar{\mathrm{B}}"] \\
 & {} & {} & {} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & {} & {} & {} & \\
 & \bar{D}^2V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & \bar{D}^1V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & \bar{D}^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & {} \\
\overline{BC}^*(\mathcal{A}): & \bar{D}^1V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & \bar{D}^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & \bar{D}^{-1}V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & {} \\
 & \bar{D}^0V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & \bar{D}^{-1}V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & \bar{D}^{-2}V \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & {} \\
 & {} \arrow[u, "\mathrm{b}^*"] & {} \arrow[u, "\mathrm{b}^*"] & {} \arrow[u, "\mathrm{b}^*"] &
\end{tikzcd}.$$

The reduced Connes' cyclic bicomplexes BC^{red} *and* BC^*_{red} *are defined by replacing* $\bar{D}V$ *and* $\bar{D}^*V$ *by* $D^{\mathrm{red}}V$ *and* $D^*_{\mathrm{red}}V$*, respectively.*

The coordinate $(0,0)$ *in the bicomplexes above always correspond to* D^0V *in the first column (bottom-left position in the figures).*

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[3] The definition does not depend on the chosen section of $\bar{p} : DV \to \bar{D}V$.

Note that another convention of drawing homological bicomplexes in the left half-plane and cohomological bicomplexes in the right half-plane might be more natural.

Remark B.2.4 (Mixed complexes). One can equivalently encode the data of a cohomological Connes bicomplex into that of a mixed complex $(D^*, \mathrm{b}^*, \mathrm{B}^*)$. In general, it is a graded vector space D^* with a differential b^*, $|\mathrm{b}^*| = 1$ and a boundary operator B^*, $|\mathrm{B}^*| = -1$ which anticommute. One introduces the formal symbol u of degree $|\mathrm{u}| = 2$ and considers the polynomial ring $D^*[\mathrm{u}]$ in u with values in D^* and the ring of power series $D^*[[\mathrm{u}]]$ with the differential $\mathrm{b}^* + \mathrm{B}^*\mathrm{u}$. Clearly, the former is quasi-isomorphic to $\mathrm{Tot}_{\mathrm{I}}(BC^*)$ and the latter to $\mathrm{Tot}_{\mathrm{II}}(BC^*)$ (columns of BC^* are indexed with non-negative powers of u). Altogether, there are seven versions $[\mathrm{u}]$, $[\mathrm{u}^{-1}]$, $[\mathrm{u}, \mathrm{u}^{-1}]$, $[[\mathrm{u}^{-1}, \mathrm{u}]$, $[\mathrm{u}^{-1}, \mathrm{u}]]$, $[[\mathrm{u}, \mathrm{u}^{-1}]]$ whose relation is studied in [CV20]. Some of these are related to periodic and negative versions of cyclic homology (see [Lod92]). ◁

In the following proofs, we might not need the full strength of Proposition B.2.1 since the spectral sequences for the bicomplexes from Definition B.2.3 mostly collapse already on the second page (see (iii) of Questions B.3.2).

Lemma B.2.5 (Loday's cyclic bicomplexes and cyclic homology). *Let $\mathcal{A} = (V, (\mu_k))$ be an A_∞-algebra. The projection $p^\lambda : CC \to D^\lambda$ to the first column modulo $\mathrm{im}(\mathbb{1} - t)$ is a chain map and induces an isomorphism $\mathrm{H}(\widehat{CC}) \simeq \mathrm{H}^\lambda(\mathcal{A})$. The inclusion $\iota_\lambda : D^*_\lambda \to CC^*$ into the first column is a chain map and induces an isomorphism $\mathrm{H}^*_\lambda(\mathcal{A}) \simeq \mathrm{H}(CC^*)$.*

Proof. The fact that p^λ and ι_λ are chain maps is obvious. We consider the horizontal filtration $\mathcal{F}^{\mathrm{h}}$ of CC^*. Because of CP1, the rows are acyclic, and we see that the only non-zero terms of the first page of the corresponding spectral sequence are

$$\mathrm{E}_1^{0d} = D^d V / \ker(\mathbb{1} - t^*).$$

The differential d_1 is easy to check to be b^*, and the inclusion ι_λ induces the isomorphism $D^d_\lambda \simeq D^d V / \ker(\mathbb{1} - t^*)$. Considering the canonical filtration on D^*_λ, claims (a), (b) and (e) of Proposition B.2.1 apply.

As for homology, we consider the degree reversed cochain complex $\mathrm{r}(\mathrm{Tot}_{\mathrm{II}} CC)$ and the reversed filtration $\mathrm{r}(\mathcal{F}^{\mathrm{h}})_s = \mathcal{F}^{\mathrm{h}}_{-s}$. For the corresponding cohomological spectral sequence $\tilde{\mathrm{E}}_r$, we have

$$\begin{aligned}\tilde{\mathrm{E}}_1^{sd} &= \mathrm{H}^{s+d}(\mathrm{r}(\mathcal{F})_s\, \mathrm{r}(\mathrm{Tot}_{\mathrm{II}} CC) / \mathrm{r}(\mathcal{F})_{s+1}\, \mathrm{r}(\mathrm{Tot}_{\mathrm{II}} CC)) \\ &= \mathrm{H}_{-s-d}(\mathrm{Tot}_{\mathrm{II}}(-s\text{-th row of } CC)) \\ &= \mathrm{H}(B_{-d,-s}, \partial_{\mathrm{h}}).\end{aligned}$$

Therefore, the only groups are $\tilde{\mathrm{E}}^{s0} = D_{-s}/\operatorname{im}(\mathbb{1}-t)$, and the spectral sequence converges conditionally to $\mathrm{r}(\mathrm{H}(\widehat{CC}))$. Clearly, ι_λ induces an isomorphism of the first pages, where on $\mathrm{r}(D^\lambda)$ we consider the canonical filtration. Proposition B.2.1 and its proof finishes the argument. □

Claim (a) of the following is similar to [CV20, Lemma 2.12].

Lemma B.2.6 (No long chains in homology for bounded degrees)**.** *Let* $\mathcal{A} = (V, (\mu_k), 1)$ *be a strictly unital* A_∞*-algebra. Suppose that* V *has bounded degrees. Then the canonical inclusion* $\mathrm{Tot}_{\mathrm{I}} \hookrightarrow \mathrm{Tot}_{\mathrm{II}}$ *induces the following isomorphism:*

(a) $\mathrm{H}(CC(\mathcal{A})) \simeq \mathrm{H}(\widehat{CC}(\mathcal{A}))$,

(b) $\mathrm{H}(\overline{BC}(\mathcal{A})) \simeq \mathrm{H}(\widehat{\overline{BC}}(\mathcal{A}))$.

Proof. (a) We will denote $CC(\mathcal{A})$, $\overline{BC}(\mathcal{A})$ and DV simply by CC, $\overline{BC}$ and D, respectively. Consider the (increasing) filtration $\mathcal{F}_{\mathrm{w}}$ of D by weights. We first prove the following subclaim:

Subclaim (Weight normalization). Let $c = (c_i)_{i=0}^\infty \in \mathrm{Tot}_{\mathrm{II}k}(CC)$ be a closed chain of degree k; i.e., for all $i \in \mathbb{N}_0$, we have $c_i \in D_{k-i}$, and the relations

$$\mathrm{b}c_{2i} + (\mathbb{1}-t)c_{2i+1} = 0 \quad \text{and} \quad -\mathrm{b}'c_{2i+1} + Nc_{2i+2} = 0$$

hold. Suppose that we are given $j \geq 1$ and $n_0 \in \mathbb{N}$ such that $c_{j-1} \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-j+1}$. Then we can construct $\tilde{c}_j \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-j}$, $\tilde{c}_{j+1} \in D_{k-j-1}$ and $\tilde{z}_{j+1} \in D_{k-j}$ such that if we define

$$\tilde{c}_i := \begin{cases} \tilde{c}_j & \text{for } i = j, \\ \tilde{c}_{j+1} & \text{for } i = j+1, \\ c_i & \text{otherwise} \end{cases} \quad \text{and} \quad z_i := \begin{cases} \tilde{z}_{j+1} & \text{for } u = j+1, \\ 0 & \text{otherwise}, \end{cases} \tag{B.8}$$

then $\tilde{c} := (\tilde{c}_i)_{i=0}^\infty$ is a closed chain and $z := (z_i)_{i=0}^\infty$ satisfies $\partial z = c - \tilde{c}$. By repeating this procedure inductively, we obtain chains $c' \in \mathrm{Tot}_{\mathrm{II}k}(CC)$ and $z' \in \mathrm{Tot}_{\mathrm{II}k+1}(CC)$ such that $c'_i \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-i}$ for all $i \geq j$ and $c - c' = \partial z'$.

Proof of the Subclaim. We will assume that j is odd; the proof is analogous for j even with the roles of $(\mathbb{1}-t)$ and N, resp. b and $-\mathrm{b}'$ switched. The situation is depicted in Figure B.1. Because $c_{j-1} \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-j+1}$ and b does not increase weights, we have $\mathrm{b}c_{j-1} \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-j}$. Since c is closed, we have $(\mathbb{1}-t)c_j = -\mathrm{b}c_{j-1} \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-j}$. Therefore, there is a $\tilde{c}_j \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-j}$ such that $(\mathbb{1}-t)\tilde{c}_j = (\mathbb{1}-t)c_j$. As $c_j - \tilde{c}_j \in \ker(\mathbb{1}-t) = \operatorname{im} N$,

$$\begin{tikzcd}
{} & c_j, \tilde{c}_j \in D_{k-j} \arrow[l, "\mathbb{1}-t"] \arrow[d, "-\mathrm{b}'"] & \tilde{z}_{j+1} \in D_{j-k} \arrow[l, "N"] \arrow[d, "\mathrm{b}"] \\
& {} & c_{j+1}, \tilde{c}_{j+1} \in D_{j-k-1} \arrow[l, "N"] \arrow[d, "\mathrm{b}"] \\
& & {}
\end{tikzcd}$$

Figure B.1.: Positions of element in CC for j odd.

we have $c_j - \tilde{c}_j = N\tilde{z}_{j+1}$ for some $\tilde{z}_{j+1} \in D_{k-j}$. We define $\tilde{c}_{j+1} \coloneqq c_{j+1} - \mathrm{b}\tilde{z}_{j+1}$. The following relations hold:

I. $(\mathbb{1} - t)\tilde{c}_j = (\mathbb{1} - t)c_j$,

II. $$\begin{aligned} -\mathrm{b}'\tilde{c}_j + N\tilde{c}_{j+1} &= -\mathrm{b}'\tilde{c}_j + Nc_{j+1} - N\mathrm{b}\tilde{z}_{j+1} \\ &= -\mathrm{b}'\tilde{c}_j + Nc_{j+1} - \mathrm{b}'N\tilde{z}_{j+1} \\ &= -\mathrm{b}'\tilde{c}_j + Nc_{j+1} - \mathrm{b}'(c_j - \tilde{c}_j) \\ &= -\mathrm{b}'c_j + Nc_{j+1} \\ &= 0, \end{aligned}$$

III. $\mathrm{b}\tilde{c}_{j+1} = \mathrm{b}c_{j+1}$,

IV. $N\tilde{z}_{j+1} = c_j - \tilde{c}_j$,

V. $\mathrm{b}\tilde{z}_{j+1} = c_{j+1} - \tilde{c}_{j+1}$.

The relations I–III show that $\tilde{c}$ is closed and the relations IV–V that $\partial z = c - \tilde{c}$.[4]

Starting with $c^1 \coloneqq \tilde{c}$ and $z^1 \coloneqq z$, we repeat the construction above to produce the telescopic sequence of homotopies

$$\begin{aligned} c - c^1 &= \partial z^1 \\ c^1 - c^2 &= \partial z^2 \\ \cdots &= \cdots \end{aligned}$$

such that $c_i^l \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-i}$ for all $j \le i \le j + l$. The limit chain $c' \coloneqq \sum_{k=0}^{\infty} c^k \in \mathrm{Tot}_{\mathrm{II}} CC$ has the property that $c_i' \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-i}$ for all $i \ge j$, and the limit homotopy $z' \coloneqq \sum_{k=1}^{\infty} z^k \in \mathrm{Tot}_{\mathrm{II}} CC$ converges and satisfies $\partial z' = c - c'$. *(Subclaim)* □

We will now prove surjectivity of the map on homology induced by the inclusion $\mathrm{Tot}_{\mathrm{I}} \hookrightarrow \mathrm{Tot}_{\mathrm{II}}$. Given $[c] \in \mathrm{H}_k(\widehat{CC})$, using the Subclaim, we can assume that there is an

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[4] In fact, $\partial\tilde{c} = 0$ follows from $\partial c = 0$ and $\partial z = c - \tilde{c}$.

$$\begin{tikzcd}
c_{j-1} \in \bar{D}_{k-2j+2} \arrow[d, "\mathrm{b}"] & z_j \in \bar{D}_{k-2j+1} \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d, "\mathrm{b}"] & \\
{} & c_j, \tilde{c}_j \in \bar{D}_{k-2j} \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d, "\mathrm{b}"] & z_{j+1} \in \bar{D}_{k-2j-1} \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d, "\mathrm{b}"] \\
 & {} & c_{j+1}, \tilde{c}_{j+1} \in \bar{D}_{k-2j-2} \arrow[l, "\bar{\mathrm{B}}"']
\end{tikzcd}$$

Figure B.2.: Positions of element in BC for j odd.

$n_0 \in \mathbb{N}$ such that $c_i \in \mathcal{F}_{\mathrm{w}}^{n_0} D_{k-i}$ for all $i \in \mathbb{N}_0$. However, we have $|c_i| = -k + i - 1$ for the degrees in $\mathrm{B}V$, and because the degrees of V are bounded, c_i eventually, as $i \to \infty$, reach degrees which can not be produced by n_0 vectors. Therefore, there is an $i_0 \in \mathbb{N}_0$ such that $c_i = 0$ for all $i \geq i_0$; this means that $c \in \mathrm{Tot}_{\mathrm{I}} CC$.

To show injectivity of the induced map on homology, suppose that $c \in \mathrm{Tot}_{\mathrm{I}} CC$ satisfies $c = \partial z$ for some $z \in \mathrm{Tot}_{\mathrm{II}} CC$. Let $i_0 \in N$ be such that $c_i = 0$ for all $i \geq i_0$. We use the Subclaim to alter z and obtain a chain $\tilde{z} \in \mathrm{Tot}_{\mathrm{I}} CC$ such that $\tilde{z}_i = z_i$ for $i \leq i_0$ and $\partial \tilde{z} = c$. This shows injectivity.

(b) We will prove an analogy of the Subclaim from (a):

Given a closed $c = (c_i)_{i=0}^{\infty} \in \mathrm{Tot}_{\mathrm{II}k} \overline{BC}$, every $c_i \in \bar{D}_{k-2i} V$ can be written as

$$c_i = \tilde{c}_i + 1 \hat{c}_i$$

for unique $\tilde{c}_i \in D_{q-2i} \bar{V}$ and $\hat{c}_i \in D_{q-2i-1} \bar{V}$. Using strict unitality, we have $\mathrm{b}(1 \hat{c}_i) = (\mathbb{1} - t) \hat{c}_i - 1 \mathrm{b}' \hat{c}_i$, and hence

$$\begin{aligned} \mathrm{b} c_i &= \mathrm{b} \bar{c}_i + (\mathbb{1} - t) \hat{c}_i - 1 \mathrm{b}' \hat{c}_i, \\ \bar{\mathrm{B}} c_i &= 1 N \bar{c}_i. \end{aligned}$$

For the second equality, recall the definition (B.7) and note that the $\bar{p}$ in front "kills" any input of $\bar{\mathrm{B}}$ containing at least one 1. We see that $\partial c = 0$ is equivalent to $\mathrm{b} c_i = -\bar{\mathrm{B}} c_{i+1}$ which is equivalent to

$$\begin{aligned} \mathrm{b} \bar{c}_i + (\mathbb{1} - t) \hat{c}_i &= 0 \quad \text{and} \\ \mathrm{b}' \hat{c}_i &= N \bar{c}_{i+1} \end{aligned}$$

for all $i \in \mathbb{N}_0$.

Suppose that $c_{j-1} \in \mathcal{F}_{\mathrm{w}}^{n_0} \bar{D}_{k-2j+2}$ for some $j \geq 1$ and $n_0 \in \mathbb{N}_0$. Then $\bar{c}_{j-1} \in$

$\mathcal{F}_{\mathrm{w}}^{n_0}\bar{D}_{k-2j+2}$ and $\hat{c}_{j-1} \in \mathcal{F}_{\mathrm{w}}^{n_0-1}\bar{D}_{k-2j+1}$. Because $\mathrm{b}'\hat{c}_{j-1} = N\bar{c}_j$, we can find $\bar{d}_j \in \mathcal{F}_{\mathrm{w}}^{n_0-1}D_{k-2j}\bar{V}$ such that $N\bar{d}_j = \mathrm{b}'\hat{c}_{j-1}$. Because $\bar{d}_j - \bar{c}_j \in \ker N = \mathrm{im}(\mathbb{1} - t)$, we can find $\hat{z}_j \in \bar{D}_{k-2j}V$ such that $(\mathbb{1} - t)\hat{z}_j = \bar{d}_j - \bar{c}_j$. We compute

$$\begin{aligned}
\mathrm{b}\bar{d}_j &= \mathrm{b}\big(\bar{c}_j + (\mathbb{1} - t)\hat{z}_j)\big) \\
&= -(\mathbb{1} - t)\hat{c}_j + \mathrm{b}(\mathbb{1} - t)\hat{z}_j \\
&= -(\mathbb{1} - t)\hat{c}_j + (\mathbb{1} - t)\mathrm{b}'\hat{z}_j \\
&= -(\mathbb{1} - t)\big(\hat{c}_j - \mathrm{b}'\hat{z}_j\big).
\end{aligned}$$

Because $\bar{d}_j \in \mathcal{F}_{\mathrm{w}}^{n_0-1}D_{k-2j}\bar{V}$ and b does not increase the filtration, we can find $\hat{d}_j \in \mathcal{F}_{\mathrm{w}}^{n_0-1}D_{k-2j-1}\bar{V}$ such that $(\mathbb{1} - t)\hat{d}_j = -\mathrm{b}\bar{d}_j$. Now, $\hat{d}_j - (\hat{c}_j - \mathrm{b}'\hat{z}_j) \in \ker(\mathbb{1} - t) = \mathrm{im}\, N$, and hence there is a $\bar{z}_{j+1} \in \bar{D}_{k-2j-1}$ such that $N\bar{z}_{j+1} = \hat{d}_j - (\hat{c}_j - \mathrm{b}'\hat{z}_j)$. We define the following elements:

$$\begin{aligned}
\tilde{c}_j &\coloneqq c_j + \bar{\mathrm{B}}\bar{z}_{j+1} + \mathrm{b}(\imath\hat{z}_j) \\
&= c_j + \bar{\mathrm{B}}\bar{z}_{j+1} + (\mathbb{1} - t)\hat{z}_j - \imath\mathrm{b}'\hat{z}_j \\
&= c_j + \imath N\bar{z}_{j+1} + (\mathbb{1} - t)\hat{z}_j - \imath\mathrm{b}'\hat{z}_j \\
&= c_j + \imath\hat{d}_j - \imath\hat{c}_j + \imath\mathrm{b}'\hat{z}_j + \bar{d}_j - \bar{c}_j - \imath\mathrm{b}'\hat{z}_j \\
&= \bar{d}_j + \imath\hat{d}_j, \\
\tilde{c}_{j+1} &\coloneqq c_{j+1} + \mathrm{b}\bar{z}_{j+1}, \\
\tilde{z}_j &\coloneqq \imath\hat{z}_j, \\
\tilde{z}_{j+1} &\coloneqq \bar{z}_{j+1}.
\end{aligned}$$

The following relations hold:

I. $\bar{\mathrm{B}}\tilde{c}_j = \bar{\mathrm{B}}c_j$,

II. $\mathrm{b}\tilde{c}_j = \mathrm{b}c_j + \mathrm{b}\bar{\mathrm{B}}\bar{z}_{j+1} = -\bar{\mathrm{B}}c_{j+1} - \bar{\mathrm{B}}\mathrm{b}\bar{z}_{j+1} = -\bar{\mathrm{B}}\tilde{c}_{j+1}$,

III. $\mathrm{b}\tilde{c}_{j+1} = \mathrm{b}c_{j+1}$,

IV. $\bar{\mathrm{B}}\tilde{z}_j = \bar{\mathrm{B}}\imath\hat{z}_j = 0$,

V. $\mathrm{b}\tilde{z}_j = \tilde{c}_j - c_j - \bar{\mathrm{B}}\tilde{z}_{j+1}$,

VI. $\mathrm{b}\tilde{z}_{j+1} = \tilde{c}_{j+1} - c_{j+1}$.

Relations I–III show that $\tilde{c}$ is closed, and relations IV–VI show that $\partial z = \tilde{c} - c$. Here $\tilde{c}$

is defined as in (B.8) and z has one more term:

$$z_i \coloneqq \begin{cases} \tilde{z}_j & \text{for } i = j, \\ \tilde{z}_{j+1} & \text{for } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{c}_j = \bar{d}_j + \mathbb{1}\hat{d}_j$, $\bar{d}_j \in \mathcal{F}_{\mathrm{w}}^{n_0-1} D_{k-2j}\bar{V}$ and $\hat{d}_j \in \mathcal{F}_{\mathrm{w}}^{n_0-1} D_{k-2j-1}\bar{V}$, we have $\tilde{c}_j \in \mathcal{F}_{\mathrm{w}}^{n_0} \bar{D}_{k-2j}$.

Having the recursive step, the rest can be done in the same way as in (a). ☐

Lemma B.2.7 (Loday's and Connes' bicomplexes are quasi-isomorphic)**.** *Let* $\mathcal{A} = (V, (\mu_k), \mathbb{1})$ *be a strictly unital* A_∞*-algebra. The map*

$$\begin{aligned} I : \mathrm{Tot}_{\Pi} BC &\longrightarrow \mathrm{Tot}_{\Pi} CC \\ (c_0, c_1, c_2, \dots) &\longmapsto (c_0, \iota_1 N c_1, c_1, \iota_1 N c_2, c_2, \dots) \end{aligned}$$

is a chain map inducing the isomorphisms

$$\mathrm{H}(\widehat{BC}) \simeq \mathrm{H}(\widehat{CC}) \quad \textit{and} \quad \mathrm{H}(BC) \simeq \mathrm{H}(CC).$$

Analogously, the map

$$\begin{aligned} P : \mathrm{Tot}_{\Pi} CC^* &\longrightarrow \mathrm{Tot}_{\Pi} BC^* \\ (\psi_0, \psi_1, \psi_2, \dots) &\longmapsto (\psi_0, \psi_2 + N^* \iota_1^* \psi_1, \psi_4 + N^* \iota_1^* \psi_3, \dots) \end{aligned}$$

is a chain map inducing the isomorphisms

$$\mathrm{H}(\widehat{BC}^*) \simeq \mathrm{H}(\widehat{CC}^*) \quad \textit{and} \quad \mathrm{H}(BC^*) \simeq \mathrm{H}(CC^*).$$

Proof. The following computation shows that ι is a chain map:

$$\begin{aligned} \partial_{CC} I(c_0, c_1, c_2 \dots) &= (\mathrm{b}c_0 + (\mathbb{1} - t)\iota_1 N c_1, -\mathrm{b}'\iota_1 N c_1 + N c_1, \mathrm{b}c_1 + (\mathbb{1} - t)\iota_1 N c_2, \dots) \\ &= (\mathrm{b}c_0 + \mathrm{B}c_1, -N c_1 + \iota_1 \mathrm{b}' N c_1 + N c_1, \mathrm{b}c_1 + \mathrm{B}c_2, \dots) \\ &= (\mathrm{b}c_0 + \mathrm{B}c_1, \iota_1 N \mathrm{b}c_1, \mathrm{b}c_1 + \mathrm{B}c_2, \dots) \\ &= I(\mathrm{b}c_0 + \mathrm{B}c_1, \mathrm{b}c_1 + \mathrm{B}c_2, \dots) \\ &= I\partial_{BC}(c_0, c_1, c_2, \dots). \end{aligned}$$

Clearly, I is injective, and hence it induces an isomorphisms of chain complexes

$$(\mathrm{Tot}_{\mathrm{II}}BC, \partial_{BC}) \simeq \big(\mathrm{im}(I), \partial_{CC}\big|_{\mathrm{im}(I)}\big) \subset (\mathrm{Tot}_{\mathrm{II}}CC, \partial_{CC}).$$

Consider the subcomplex $(\mathrm{Tot}_{\mathrm{II}}CC_{\mathrm{odd},\bullet}, -\mathrm{b}') \subset (\mathrm{Tot}_{\mathrm{II}}CC, \partial_{CC})$ which consists of odd columns of CC. It is a direct complement of $\mathrm{im}(I)$ in $\mathrm{Tot}_{\mathrm{II}}CC$. Indeed, $(0, c_1, 0, c_2, \dots) \in \mathrm{im}(I)$ implies $c_i = 0$ for all $i \in \mathbb{N}$, which gives $\mathrm{Tot}_{\mathrm{II}}(CC_{\mathrm{odd},\bullet}) \cap \mathrm{im}(I) = 0$; also, for any $(c_i) \in \mathrm{Tot}_{\mathrm{II}}CC$, we have

$$(c_0, c_1, c_2, c_3, c_4 \dots) = I\big((c_0, c_2, c_4, \dots)\big) - (0, \iota_1 N c_2 - c_1, 0, \iota_1 N c_4 - c_3, 0, \dots),$$

which gives $\mathrm{im}(I) + \mathrm{Tot}_{\mathrm{II}}(CC_{\mathrm{odd},\bullet}) = \mathrm{Tot}_{\mathrm{II}}CC$. Now, $\mathrm{Tot}_{\mathrm{II}}(CC_{\mathrm{odd},\bullet})$ is contractible by CP3, and hence $\mathrm{H}(\mathrm{Tot}_{\mathrm{II}}BC) \simeq \mathrm{H}(\mathrm{Tot}_{\mathrm{II}}CC)$ (using an argument with the long exact sequence in homology). Clearly, I restricts to short chains $\mathrm{Tot}_{\mathrm{I}}$, and thus $\mathrm{H}(\mathrm{Tot}_{\mathrm{I}}BC) \simeq \mathrm{H}(\mathrm{Tot}_{\mathrm{I}}CC)$ holds too.

A similar discussion applies in cohomology. The following computation shows that P is a chain map:

$$\begin{aligned}
&P\mathrm{d}_{CC}(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \dots) \\
&\quad = P(\mathrm{b}^*\psi_0, (\mathbb{1} - t^*)\psi_0 - \mathrm{b}'\psi_1, N^*\psi_1 + \mathrm{b}^*\psi_2, (\mathbb{1} - t^*)\psi_2 - \mathrm{b}'^*\psi_3, N^*\psi_4 + \mathrm{b}^*\psi_3, \dots) \\
&\quad = \big(\mathrm{b}^*\psi_0, N^*\psi_1 + \mathrm{b}^*\psi_2 + N^*\iota_1^*((\mathbb{1} - t^*)\psi_0 - \mathrm{b}'^*\psi_1), \\
&\qquad\qquad N^*\psi_4 + \mathrm{b}^*\psi_3 + N^*\iota_1^*((\mathbb{1} - t^*)\psi_2 - \mathrm{b}'^*\psi_3), \dots\big) \\
&\quad = \big(\mathrm{b}^*\psi_0, \mathrm{B}^*\psi_0 + \mathrm{b}^*\psi_2 + N^*(\psi_1 - \iota_1^*\mathrm{b}'^*\psi_1), \mathrm{B}^*\psi_2 + \mathrm{b}^*\psi_4 + N^*(\psi_3 - \iota_1^*\mathrm{b}'^*\psi_3), \dots\big) \\
&\quad = \big(\mathrm{b}^*\psi_0, \mathrm{B}^*\psi_0 + \mathrm{b}^*(\psi_2 + N^*\iota_1^*\psi_1), \mathrm{B}^*\psi_2 + \mathrm{b}^*(\psi_4 + N^*\iota_1^*\psi_3), \dots\big) \\
&\quad = \mathrm{d}_{BC}(\psi_0, \psi_2 + N^*\iota_1^*\psi_1, \psi_2, \psi_4 + N^*\iota_1^*\psi_3, \dots) \\
&\quad = \mathrm{d}_{BC}P(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \dots)
\end{aligned}$$

The fourth equality uses that $\iota_1\mathrm{b}' + \mathrm{b}'\iota_1 = \mathbb{1}$ and $\mathrm{b}'N = N\mathrm{b}$. Because $(\psi_0, \psi_1, \dots) = P(\psi_0, 0, \psi_1, 0, \dots)$, P is surjective, and hence it induces an isomorphism of cochain complexes $\mathrm{Tot}_{\mathrm{II}}CC^*/\ker(P) \simeq \mathrm{Tot}_{\mathrm{II}}BC^*$. It is easy to see that

$$\ker(P) = \big\{(0, \psi_1, -N^*\iota_1^*\psi_1, \psi_3, -N^*\iota_1^*\psi_3, \dots)\big\}$$

and that the map

$$\begin{aligned} Z : \mathrm{Tot}_{\mathrm{II}}(CC^{\mathrm{odd},\bullet}) &\longrightarrow \ker(p) \\ (\psi_1, \psi_3, \dots) &\longmapsto (0, \psi_1, -N^*\iota_1^*\psi_1, \psi_3, -N^*\iota_1^*\psi_3, \dots) \end{aligned}$$

is an isomorphism of the cochain complexes

$$(\mathrm{Tot}_{\mathrm{II}}(CC^{\mathrm{odd},\bullet}), -\mathrm{b}'^*) \simeq (\ker(P), \partial_{CC}|_{\ker(P)}) \subset (\mathrm{Tot}_{\mathrm{II}} CC, \partial_{CC}).$$

Indeed, we have

$$\begin{aligned} \partial_{CC} Z(\psi_1, \psi_3, \dots) &= (0, -\mathrm{b}'^*\psi_1, N^*\psi_1 - \mathrm{b}^* N^*\iota_1^*\psi_1, -(\mathbb{1} - t^*)N^*\iota_1^*\psi_1 - \mathrm{b}'^*\psi_3, \dots) \\ &= (0, -\mathrm{b}'^*\psi_1, -N^*\iota_1^*(-\mathrm{b}'^*\psi_1), -\mathrm{b}'^*\psi_3, \dots) \\ &= Z(-\mathrm{b}'^*)(\psi_1, \psi_3, \dots). \end{aligned}$$

Therefore, $\ker P$ is contractible, and the statement is implied by an argument with the long exact sequence in homology. □

Lemma B.2.8 (Connes' cyclic bicomplexes are quasi-iso to their normalized versions)**.** *Let $\mathcal{A} = (V, (\mu_k), 1)$ be a strictly unital A_∞-algebra. The projection $\bar{p}$ and the inclusion $\bar{\iota}$ (see Definition B.2.3) induce the isomorphisms $\mathrm{H}(BC) \simeq \mathrm{H}(\overline{BC})$ and $\mathrm{H}(\widehat{BC}^*) \simeq \mathrm{H}(\widehat{\overline{BC}}^*)$, respectively.*

Proof. It follows from CP4 using the spectral sequence associated to the vertical filtration. In cohomology, we use (d) and (f) of Proposition B.2.1.

In homology, we have $\tilde{\mathrm{E}}^{sd} = \mathrm{H}(BC_{-s,-d}, \partial_{\mathrm{v}})$ for the reversed spectral sequence (see the proof of Lemma B.2.5), and hence strong convergence is implied by Theorem B.2.1 from the proof of Proposition B.2.1. Claim (e) of Proposition B.2.1 finishes the proof. □

The following is based on [Lod92, Proposition 2.2.14] and its proof.

Lemma B.2.9 (Reduced Connes' cyclic bicomplexes and cyclic homology are quas-iso)**.** *Let $\mathcal{A} = (V, (\mu_k), 1)$ be a strictly unital A_∞-algebra. The projection $p^\lambda : BC^{\mathrm{red}} \to \bar{D}^\lambda$ to the first column modulo $\mathrm{im}(\mathbb{1} - t)$ is a chain map and induces an isomorphism $\mathrm{H}(\widehat{BC}^{\mathrm{red}}) \simeq \mathrm{H}(\bar{D}^\lambda)$ $(=: \mathrm{H}^{\lambda,\mathrm{red}}(\mathcal{A}))$. The inclusion $\iota_\lambda : \bar{D}^*_\lambda \to BC^*_{\mathrm{red}}$ into the first column of BC^*_{red} is a chain map and induces an isomorphism $\mathrm{H}(\bar{D}^*_\lambda) \simeq \mathrm{H}(BC^*_{\mathrm{red}})$.*

Proof. We start with the cohomology. It is easy to see that ι_1 induces $\iota_1^* : D^*_{\mathrm{red}} V \to D^*_{\mathrm{red}} V$

and that for this map, we have

$$Z \coloneqq \ker \iota_1^* = \operatorname{im} \iota_1^* \simeq D^* \bar{V}.$$

We consider the following diagonal filtration of BC^*_{red}:

$$\begin{tikzcd}
 & {} & {} & {} & \\
 & D^{s+2}_{\mathrm{red}} \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & D^{s+1}_{\mathrm{red}} \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & Z^s \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & {} \\
\mathcal{F}^s BC^*_{\mathrm{red}} : & D^{s+1}_{\mathrm{red}} \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & Z^s \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & 0 \arrow[u] \arrow[r] & {} \\
 & Z^s \arrow[u, "\mathrm{b}^*"] \arrow[r, "\bar{\mathrm{B}}^*"] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & {} \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

The first page of the corresponding spectral sequence consists of the columns

$$\mathrm{E}_1^s = \mathrm{H}(\mathrm{gr}_s \operatorname{Tot}_{\mathrm{I}}) = \mathrm{H}\big(Z^s \xrightarrow{\mathrm{b}^*} D^{s+1}_{\mathrm{red}}/Z^{s+1} \xrightarrow{\bar{\mathrm{B}}^*} Z^s \xrightarrow{\mathrm{b}^*} \cdots\big),$$

where the cochain complex starts in degree s. Because $\iota_1^* \mathrm{b}^* = -\mathrm{b}'^* \iota_1^* + (\mathbb{1} - t^*)$ and $\bar{\mathrm{B}}^* = N^* \iota_1^*$ on D_{red}, we have the commutative diagram

$$\begin{tikzcd}
Z^s \arrow[r, "\mathrm{b}^*"] \arrow[d, "\mathbb{1}"] & D^{s+1}_{\mathrm{red}}/Z^{s+1} \arrow[r, "\bar{\mathrm{B}}^*"] \arrow[d, "\iota_1^*"] & Z^s \arrow[r, "\mathrm{b}^*"] \arrow[d, "\mathbb{1}"] & \cdots \\
Z^s \arrow[r, "\mathbb{1}-t^*"] & Z^s \arrow[r, "N^*"] & Z^s \arrow[r, "\mathbb{1}-t^*"] & \cdots .
\end{tikzcd}$$

Therefore, the only non-zero terms of the first page are

$$\mathrm{E}_1^{s,0} = Z_s / \ker(\mathbb{1} - t^*) = \bar{D}^s_\lambda$$

with the differential $\mathrm{d}_1 = \mathrm{b}^*$. This is precisely the first page of the canonical filtration of $\bar{D}^*_\lambda$. The map induced by ι_λ on the first page is the identity, and Proposition B.2.1 implies the rest.

The situation in homology is analogous. We consider the restriction $\iota_1 : D^{\mathrm{red}} \to D^{\mathrm{red}}$ and the subspace

$$B \coloneqq \ker \iota_1 = \operatorname{im} \iota_1.$$

The diagonal filtration is now

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[l] \arrow[d] & B_s \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d, "\mathrm{b}"] & {} \arrow[l] \\
\mathcal{F}^s BC^{\mathrm{red}}: & 0 \arrow[d] & B_s \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d, "\mathrm{b}"] & D^{\mathrm{red}}_{s-1} \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d, "\mathrm{b}"] & {} \arrow[l] \\
 & B_s \arrow[d] & D^{\mathrm{red}}_{s-1} \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d] & D^{\mathrm{red}}_{s-2} \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d] & {} \arrow[l] \\
 & {} & {} & {} &
\end{tikzcd}$$

The reversed filtration $\mathrm{r}(\mathcal{F})_s = \mathcal{F}^{-s}$ of the degree reversed cochain comples $\mathrm{r}(\mathrm{Tot}_{\mathrm{I}})$ satisfies

$$\begin{aligned}\tilde{\mathrm{E}}_1^s &= \mathrm{H}(\mathrm{r}(\mathcal{F})_s \, \mathrm{r}(\mathrm{Tot}_{\mathrm{I}}) / \, \mathrm{r}(\mathcal{F})_{s+1} \, \mathrm{r}(\mathrm{Tot}_{\mathrm{I}})) \\ &= \mathrm{H}(D^{\mathrm{red}}_{-s-1}/B_{-s-1} \overset{\mathrm{b}}{\leftarrow} B_{-s} \overset{\bar{\mathrm{B}}}{\leftarrow} D^{\mathrm{red}}_{-s-1}/B_{-s-1} \overset{\mathrm{b}}{\leftarrow} \cdots)\end{aligned}$$

where the first group has degree s. We have the commutative diagram

$$\begin{tikzcd}
D^{\mathrm{red}}_{-s-1}/B_{-s-1} \arrow[d, "\iota_1"] & B_{-s} \arrow[l, "\mathrm{b}"'] \arrow[d, "\mathbb{1}"] & D^{\mathrm{red}}_{-s}/B_{-s} \arrow[l, "\bar{\mathrm{B}}"'] \arrow[d, "\iota_1"] & \cdots \arrow[l, "\mathrm{b}"'] \\
B^{-s} & B^{-s} \arrow[l, "\mathbb{1}-t"'] & B^{-s} \arrow[l, "N"'] & \cdots , \arrow[l, "\mathbb{1}-t"']
\end{tikzcd}$$

and thus $\tilde{\mathrm{E}}_1^{s,0} = B^{-s}/\,\mathrm{im}(\mathbb{1}-t) = D^{\lambda,\mathrm{red}}_{-s}$. The rest is as in the case of cohomology. □

B.3. Final argument and remarks

We are finally in position to prove Proposition 3.3.13. Precisely as in the proof sketch, we replace the unit-augmentation sequence (3.30), up to a quasi-isomorphism, by a short exact sequence of normalized Connes' cyclic bicomplexes.

Lemma B.3.1 (Short exact sequence of normalized Connes' cyclic bicomplexes)**.** *Let $\mathcal{A} = (V, (\mu_j), 1, \varepsilon)$ be a strictly augmented strictly unital A_∞-algebra. The short exact sequences of bicomplexes*

$$0 \longrightarrow \overline{BC}(\mathbb{R}) \xrightarrow{u} \overline{BC}(V) \xrightarrow{p^{\mathrm{red}}} BC^{\mathrm{red}} \longrightarrow 0$$

and

$$0 \longrightarrow BC^*_{\mathrm{red}} \xrightarrow{\iota_{\mathrm{red}}} \overline{BC}^*(V) \xrightarrow{u^*} \overline{BC}(\mathbb{R}) \longrightarrow 0$$

split. From this, we obtain the following isomorphisms:

$$\begin{aligned}\mathrm{H}(\overline{BC}) \simeq \mathrm{H}(BC^{\mathrm{red}}) \oplus \mathrm{H}^{\lambda}(\mathbb{R}), \quad & \mathrm{H}(\widehat{\overline{BC}}) \simeq \mathrm{H}(\widehat{BC}^{\mathrm{red}}) \oplus H^{\lambda}(\mathbb{R}),\\ \mathrm{H}(\overline{BC}^{*}) \simeq \mathrm{H}(BC^{*}_{\mathrm{red}}) \oplus H^{*}_{\lambda}(\mathbb{R}), \quad & \mathrm{H}(\widehat{\overline{BC}}^{*}) \simeq \mathrm{H}(\widehat{BC}^{*}_{\mathrm{red}}) \oplus H^{*}_{\lambda}(\mathbb{R}).\end{aligned} \tag{B.9}$$

Proof. Because we have a strict unit and a strict augmentation, the maps $u : D\mathbb{R} \to DV$ and $\varepsilon : DV \to D\mathbb{R}$ satisfy $\varepsilon \circ u = \mathbb{1}$. It follows that $DV = \ker \varepsilon \oplus \operatorname{im} u$. The same holds for the induced maps $\bar{u} : \bar{D}\mathbb{R} \to \bar{D}V$ and $\bar{\varepsilon} : \bar{D}V \to \bar{D}\mathbb{R}$. We can now define the splitting $r : \overline{BC}(V) \to \overline{BC}(\mathbb{R})$ of the homological short exact sequence by projecting to $\ker \bar{\varepsilon}$ along $\operatorname{im} \bar{u}$. It is easy to check that it is a morphism of bicomplexes. This dualizes “pointwisely” to cohomology.

Because

$$(\bar{D}\mathbb{R})_i = \begin{cases} 0 & \text{for } i \neq 0,\\ \mathbb{R} & \text{for } i = 0,\end{cases}$$

the bicomplex $\overline{BC}(\mathbb{R})$ is diagonal, and hence $\mathrm{Tot}_{\mathrm{I}}$ and $\mathrm{Tot}_{\mathrm{II}}$ are the same; both compute $\mathrm{H}^{\lambda}(\mathbb{R})$ (using results from the previous sections). The same is true in cohomology. This shows (B.9). ☐

We summarize our results in Figure B.3. Having $\mathrm{H}^{\lambda}(\mathcal{A}) \simeq \mathrm{H}^{\lambda,\mathrm{red}}(\mathcal{A}) \oplus \mathrm{H}^{\lambda}(\mathbb{R})$, Proposition 3.3.13 in Section 3.3 follows by dualization.

Questions B.3.2. (i) Suppose that V has bounded degrees and look at Figure B.3. Does $\mathrm{H}(\widehat{CC}^{*}) \simeq \mathrm{H}(CC^{*})$ hold? Does $\mathrm{H}(BC^{*}) \simeq \mathrm{H}(\overline{BC}^{*})$ hold?

(ii) Does Proposition 3.3.13 hold for homological unital and homological augmented A_∞-algebras? A strategy would be to construct a quasi-isomorphic strictly unital and strictly augmented A_∞-algebra. Does it exist?

(iii) How is it with the conditional and strong convergence of spectral sequences associated to different filtrations of bicomplexes from Definition B.2.3? Because of the simple internal data, lots of them collapse.

$$\begin{tikzcd}
\mathrm{H}^{\lambda}(\mathcal{A}) \arrow[r, dash, "Lem.\ B.2.5"] & \mathrm{H}(\widehat{CC}) \arrow[r, dash, dashed, "Lem.\ B.2.6"] \arrow[d, dash, "Lem.\ B.2.7"] & \mathrm{H}(CC) \arrow[d, dash, "Lem.\ B.2.7"] \\
 & \mathrm{H}(\widehat{BC}) & \mathrm{H}(BC) \arrow[d, dash, "Lem.\ B.2.8"] \\
 & \mathrm{H}(\widehat{\overline{BC}}) \arrow[r, dash, dashed, "Lem.\ B.2.6"] \arrow[d, dash, "Lem.\ B.3.1"] & \mathrm{H}(\overline{BC}) \arrow[d, dash, "Lem.\ B.3.1"] \\
 & \mathrm{H}(\widehat{BC}^{\mathrm{red}}) \oplus \mathrm{H}^{\lambda}(\mathbb{R}) \arrow[d, dash, "Lem.\ B.2.9"] & \mathrm{H}(BC^{\mathrm{red}}) \oplus \mathrm{H}^{\lambda}(\mathbb{R}) \\
 & \mathrm{H}^{\lambda,\mathrm{red}}(\mathcal{A}) \oplus \mathrm{H}^{\lambda}(\mathbb{R}) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{H}(\widehat{CC}^{*}) \arrow[d, dash, "Lem.\ B.2.7"] & \mathrm{H}(CC^{*}) \arrow[r, dash, "Lem.\ B.2.5"'] \arrow[d, dash, "Lem.\ B.2.7"] & \mathrm{H}^{*}_{\lambda} \arrow[ddddl, dash, dotted, bend left] \\
\mathrm{H}(\widehat{BC}^{*}) \arrow[d, dash, "Lem.\ B.2.8"] & \mathrm{H}(BC^{*}) & \\
\mathrm{H}(\widehat{\overline{BC}}^{*}) \arrow[d, dash, "Lem.\ B.3.1"] & \mathrm{H}(\overline{BC}^{*}) \arrow[d, dash, "Lem.\ B.3.1"] & \\
\mathrm{H}(\widehat{BC}^{*}_{\mathrm{red}}) \oplus \mathrm{H}^{*}_{\lambda}(\mathbb{R}) & \mathrm{H}(BC^{*}_{\mathrm{red}}) \oplus \mathrm{H}^{*}_{\lambda}(\mathbb{R}) \arrow[d, dash, "Lem.\ B.2.9"] & \\
 & \mathrm{H}^{*}_{\lambda,\mathrm{red}}(\mathcal{A}) \oplus \mathrm{H}^{*}_{\lambda}(\mathbb{R}) &
\end{tikzcd}$$

Figure B.3.: Isomorphisms of (co)homologies for a strictly unital strictly augmented A_∞-algebra $\mathcal{A}$ on a graded vector space V. A solid line denotes an isomorphism which is always valid and a dashed line an isomorphism which is valid provided that the degrees of V are bounded. The dotted line denotes the isomorphism obtained by dualizing the corresponding isomorphism in homology under the assumptions that the degrees of V are bounded.

C. Towards an invariant definition of canonical IBL-operations

The canonical IBL-operations $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ on cyclic cochains of an odd symplectic vector space V have been defined in coordinates in [CFL15]. In Part I, we used Definition 3.4.1; it takes and gives invariant objects but requires a choice of basis to describe the inner "trace" mechanism. Example A.1.5 shows that an invariant definition can be obtained from the invariant formalism for evaluation of ribbon graphs developed in Appendix A by plugging in the algebraic Schwarz kernel of the identity as a propagator. In this appendix, we ask the following question.

Question C.0.1. Do $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$ arise as a combination of some natural operations coming from the structure of Hochschild cochains on V or from odd symplectic geometry?

In Section C.1, we define a canonical Lie bracket $[\cdot,\cdot]$ on Hochschild cochains on V with values in V (Definition C.1.2 and Proposition C.1.3), which comes from a natural pre-Lie algebra structure which is visualized as grafting of trees (Equation (C.2), Figure C.1 and Lemma C.1.1). This is, in fact, the Gerstenhaber bracket from [Ger63]. Next, we use the symplectic form to define the operator $^+$ which "lowers indices" (Equation C.3) and show that it takes $[\cdot,\cdot]$ to a Lie bracket on cyclic cochains on V with values in $\mathbb{R}$ (Definition C.1.5 and Proposition C.1.6). By writing down everything in coordinates, we show that this Lie bracket is a degree shift of $\mathfrak{q}_{210}$ (Proposition C.1.8). This answers Question C.0.1 for $\mathfrak{q}_{210}$.

In Section C.2, we relate the cobracket $\mathfrak{q}_{120}$ to the canonical BV-operator Δ_{sym} on "functions" on an odd symplectic vector space (Proposition C.2.4). We do it by rewriting $\mathfrak{q}_{120}$ in terms of certain double derivative operators on the tensor algebra, which are associated to a given basis (Definition C.2.2). We prove that both definitions are equivalent (Proposition C.2.5). The BV-operator Δ_{sym} can be defined geometrically (see [DJP19]), and the cobracket $\mathfrak{q}_{120}$ is a factorization of an extension of Δ_{sym} to cyclic invariants with respect to the cyclic shuffle product. We do not know whether this characterization determines $\mathfrak{q}_{120}$ uniquely, and hence Question C.0.1 for $\mathfrak{q}_{120}$ remains open (see Questions C.2.6 for a list of related questions). This section was stimulated by a discussion with J. Pullman and L. Peksová at a winter school in Srní, 2019.

C.1. Bracket and grafting of trees

Let V be a $\mathbb{Z}$-graded vector space. We define the weight-graded vector spaces

$$\mathrm{L}V := \bigoplus_{k=0}^{\infty} \mathrm{Hom}(V^{\otimes k}, V) \quad \text{and} \quad \bar{\mathrm{L}}V := \bigoplus_{k=1}^{\infty} \mathrm{Hom}(V^{\otimes k}, V), \tag{C.1}$$

where Hom denotes the graded vector space generated by homogenous morphisms. The latter space is the weight-reduced version of the former (see Definition 3.1.1).

For homogenous $\psi_1 \in \mathrm{Hom}(V^{\otimes k_1}, V)$, $\psi_2 \in \mathrm{Hom}(V^{\otimes k_2}, V)$ with k_1, $k_2 \in \mathbb{N}$ and vectors $v_1, \ldots, v_{k_1+k_2-1} \in V$, we define[1]

$$\begin{aligned}(\psi_1 * \psi_2)&(v_1 \cdots v_{k_1+k_2-1}) \\ &:= \sum_{i=1}^{k_1} (-1)^{\psi_2(v_1+\cdots+v_{i-1})} \psi_1(v_1 \cdots v_{i-1} \psi_2(v_i \cdots v_{k_2+i-1}) v_{k_2+i} \cdots v_{k_1+k_2-1}).\end{aligned} \tag{C.2}$$

Here and almost everywhere in this section, we suppress writing the tensor product. We recall that we use the same symbol v to denote both a homogenous vector and its degree in the exponent. From (C.2), we get a bilinear operation

$$* : \mathrm{L}V \otimes \mathrm{L}V \longrightarrow \mathrm{L}V,$$

which preserves the degree and decreases the weight by 1. It can be visualized as grafting of trees (see Figure C.1).

Lemma C.1.1 (Pre-Lie algebra)**.** *Let V be a graded vector space. The pair $(\mathrm{L}V, *)$ is a graded pre-Lie algebra with respect to the degree, i.e., it holds*

$$(\psi_1 * \psi_2) * \psi_3 - \psi_1 * (\psi_2 * \psi_3) = (-1)^{\psi_2\psi_3} \left[(\psi_1 * \psi_3) * \psi_2 - \psi_1 * (\psi_3 * \psi_2)\right]$$

for all homogenous ψ_1, ψ_2, $\psi_3 \in \mathrm{L}V$.

Proof. In the following computations, U denotes a tensor product of homogenous vectors from the vector space V and U_i denotes a part of a decomposition of V, i.e., $U = U_1 \cdots U_k$.

---

[1] If $k_1 = 0$ or $k_2 = 0$, we define $*$ to be 0.

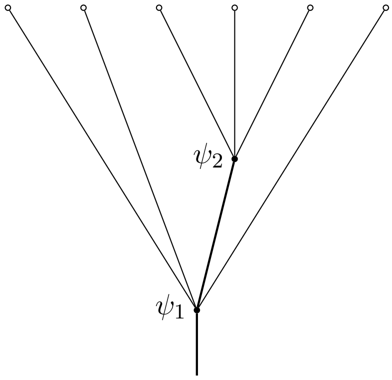

Figure C.1.: The Gerstenhaber bracket as grafting of trees.

We compute schematically

$$\begin{aligned}((\psi_1 * \psi_2) * \psi_3)(U) = &\sum(-1)^{\psi_3(U_1+U_2+U_3)+\psi_2 U_1}\psi_1(U_1\psi_2(U_2)U_3\psi_3(U_4)U_5)\\ &+\sum(-1)^{\psi_3 U_1+\psi_2(U_1+U_2+U_3+\psi_3)}\psi_1(U_1\psi_3(U_2)U_3\psi_2(U_4)U_5)\\ &+\sum(-1)^{\psi_3(U_1+U_2)+\psi_2 U_1}\psi_1(U_1\psi_2(U_2\psi_3(U_3)U_4)U_5)\end{aligned}$$

and

$$(\psi_1 * (\psi_2 * \psi_3))(U) = \sum(-1)^{(\psi_2-\psi_3)U_1+\psi_3 U_2}\psi_1(U_1\psi_2(U_2\psi_3(U_3)U_4)U_5).$$

Therefore, we have

$$\begin{aligned}&\big[((\psi_1 * \psi_2) * \psi_3) - (\psi_1 * (\psi_2 * \psi_3))\big](U)\\ &\qquad = \sum(-1)^{\psi_3(U_1+U_2+U_3)+\psi_2 U_1}\psi_1(U_1\psi_2(U_2)U_3\psi_3(U_4)U_5)\\ &\qquad\quad +\sum(-1)^{\psi_3 U_1+\psi_2(U_1+U_2+U_3+\psi_3)}\psi_1(U_1\psi_3(U_2)U_3\psi_2(U_4)U_5)\\ &\qquad = (-1)^{\psi_2\psi_3}\Big[\sum(-1)^{\psi_2(U_1+U_2+U_3)+\psi_3 U_1}\psi_1(U_1\psi_3(U_2)U_3\psi_2(U_4)U_5)\\ &\qquad\qquad +\sum(-1)^{\psi_2 U_1+\psi_3(U_1+U_2+U_3+\psi_2)}\psi_1(U_1\psi_2(U_2)U_3\psi_3(U_4)U_5)\Big]\\ &\qquad = (-1)^{\psi_2\psi_3}\big[((\psi_1 * \psi_3) * \psi_2) - (\psi_1 * (\psi_3 * \psi_2))\big](U).\end{aligned}$$

This finishes the proof. □

Definition C.1.2 (Canonical Lie algebra on Hochschild cochains)**.** *Let V be a graded* *vector space. For all ψ_1, $\psi_2 \in \mathrm{L}V$, we define the bracket by*

$$[\psi_1, \psi_2] \coloneqq \psi_1 * \psi_2 - (-1)^{\psi_1\psi_2}\psi_2 * \psi_1.$$

It is called the Gerstenhaber bracket*.*

Proposition C.1.3 (Canonical Lie algebra on Hochschild cochains)**.** *In the situation of Definition C.1.2, the pair* $(\mathrm{L}V, [\cdot,\cdot])$ *is a graded Lie algebra with bracket* $[\cdot,\cdot]$ *of degree* 0 *and weight* -1*.*

Proof. The proof is standard. ☐

Let $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{R}$ be a homogenous bilinear form of degree $-n$. This means that

$$\langle v_1, v_2\rangle \neq 0 \quad\Longrightarrow\quad v_1 + v_2 = n.$$

To every $\psi \in \mathrm{Hom}(V^{\otimes k}, V)$, we associate $\psi^+ \in \mathrm{Hom}(V^{\otimes k+1}, \mathbb{R})$ defined for all v_1, …, $v_{k+1} \in V$ by the formula

$$\psi^+(v_1 \cdots v_{k+1}) \coloneqq \langle \psi(v_1 \cdots v_k), v_{k+1}\rangle. \tag{C.3}$$

We define the weight-graded vector spaces

$$\mathrm{L}^+V \coloneqq \bigoplus_{k=0}^{\infty} \mathrm{Hom}(V^{\otimes k}, \mathbb{R}) \quad\text{and}\quad \bar{\mathrm{L}}^+V \coloneqq \bigoplus_{k=1}^{\infty} \mathrm{Hom}(V^{\otimes k}, \mathbb{R}). \tag{C.4}$$

We use here the cohomological grading convention; i.e., $\psi \in \mathrm{L}^+V$ has degree d and weight k if and only if for all homogenous v_1, …, $v_k \in V$, the following implication holds:

$$\psi(v_1 \cdots v_k) \neq 0 \quad\Longrightarrow\quad v_1 + \cdots + v_k = d.$$

We say that $\varphi \in \mathrm{L}V$ is cyclically symmetric if

$$\varphi(v_1 \cdots v_k) = \underbrace{(-1)^{v_k(v_1+\cdots+v_{k-1})}}_{=:\varepsilon(v_1\cdots v_k, v_k v_1 \cdots v_{k-1})} \varphi(v_k v_1 \cdots v_{k-1})$$

for all homogenous v_1, …, $v_k \in V$. We define the weight-graded vector spaces

$$\begin{aligned}&\mathrm{L}^+_{\mathrm{cyc}}V \coloneqq \{\varphi \in \mathrm{L}^+V \mid \varphi \text{ is cyclically symmetric}\} \quad\text{and}\\ &\mathrm{L}_{\mathrm{cyc}}V \coloneqq \{\psi \in \mathrm{L}V \mid \psi^+ \in \mathrm{L}^+_{\mathrm{cyc}}V\},\end{aligned}$$

and similarly for the reduced versions. Formula (C.3) defines the following linear maps of weight +1:

$$\begin{tikzcd} {}^+ : & \mathrm{L}V \arrow[r] & \bar{\mathrm{L}}^+V \\ {}^+ := {}^+|_{\mathrm{L}_{\mathrm{cyc}}V} : & \mathrm{L}_{\mathrm{cyc}}V \arrow[u, hook] \arrow[r] & \bar{\mathrm{L}}^+_{\mathrm{cyc}}V \arrow[u, hook] \end{tikzcd} \tag{C.5}$$

This might be related to Remark 3.3.2 about weight-reduced bar complexes.

Because L^+V has the cohomological grading and $\mathrm{L}V$ the standard grading for homogenous maps, $^+$ is not homogenous; instead, it satisfies

$$|\psi^+| = n - |\psi|.$$

It is also useful to note that if $\psi^+(v_1 \cdots v_{k+1}) \neq 0$, then

$$\varepsilon(v_1 \cdots v_{k+1}, v_{k+1}v_1 \cdots v_k) = (-1)^{v_{k+1}(\psi^+ - 1)} = (-1)^{v_{k+1}(\psi - n - 1)}.$$

Lemma C.1.4 (Restriction of Lie bracket to cyclic invariants)**.** *Let V be a graded vector space and $\langle \cdot, \cdot \rangle : V \otimes V \to \mathbb{R}$ a homogenous bilinear form of degree $-n$ which is graded antisymmetric. This means that for all homogenous v_1, $v_2 \in V$, we have*

$$\langle v_1, v_2 \rangle = (-1)^{1 + v_1 v_2} \langle v_2, v_1 \rangle.$$

Then the Lie bracket $[\cdot,\cdot]$ on $\mathrm{L}V$ satisfies $[\mathrm{L}_{\mathrm{cyc}}V, \mathrm{L}_{\mathrm{cyc}}V] \subset \mathrm{L}_{\mathrm{cyc}}V$, and thus it restricts to the Lie bracket (which we denote by the same symbol)

$$[\cdot,\cdot] : \mathrm{L}_{\mathrm{cyc}}V \otimes \mathrm{L}_{\mathrm{cyc}}V \longrightarrow \mathrm{L}_{\mathrm{cyc}}V.$$

Proof. For any homogenous $v_1, \ldots, v_{k_1+k_2} \in V$, we compute

$$\begin{aligned} &\langle [\psi_1, \psi_2](v_1, \ldots, v_{k_1+k_2-1}), v_{k_1+k_2} \rangle \\ &= \sum_{i=1}^{k_1} (-1)^{\psi_2(v_1 + \cdots + v_{i-1})} \langle \psi_1(v_1 \cdots v_{i-1} \psi_2(v_i \cdots v_{i+k_2-1}) v_{i+k_2} \cdots \\ &\quad v_{k_1+k_2-1}), v_{k_1+k_2} \rangle - (-1)^{\psi_1 \psi_2} \sum_{j=1}^{k_2} (-1)^{\psi_1(v_1 + \cdots + v_{j-1})} \langle \psi_2(v_1 \cdots \\ &\quad v_{j-1} \psi_1(v_j \cdots v_{j+k_1-1}) v_{j+k_1} \cdots v_{k_1+k_2-1})), v_{k_1+k_2} \rangle \end{aligned}$$

$$= (-1)^{v_{k_1+k_2}(v_1+\cdots v_{k_1+k_2-1})}$$

$$\Big[\sum_{i=1}^{k_1-1}(-1)^{\psi_2(v_{k_1+k_2}+v_1+\cdots+v_{i-1})}\langle\psi_1(v_{k_1+k_2}v_1\cdots\psi_2(v_i\cdots v_{i+k_2-1})$$

$$v_{i+k_2}\cdots v_{k_1+k_2-2}),v_{k_1+k_2-1}\rangle + (-1)^{\psi_2(v_1+\cdots+v_{k_1-1})+v_{k_1+k_2}\psi_2}$$

$$\langle\psi_1(v_{k_1+k_2}v_1\cdots v_{k_1-1}),\psi_2(v_{k_1}\cdots v_{k_1+k_2-1})\rangle$$

$$-(-1)^{\psi_1\psi_2}\Big(\sum_{j=1}^{k_2-1}(-1)^{\psi_1(v_{k_1+k_2}+v_1+\cdots+v_{j-1})}\langle\psi_2(v_1\cdots\psi_1(v_j\cdots$$

$$v_{j+k_1-1})v_{j+k_1}\cdots v_{k_1+k_2-1})),v_{k_1+k_2}\rangle + (-1)^{\psi_1(v_1+\cdots+v_{k_2-1})+v_{k_1+k_2}\psi_1}$$

$$\langle\psi_2(v_{k_1+k_2}v_1\cdots v_{k_2-1}),\psi_1(v_{k_2},\cdots,v_{k_1+k_2-1})\rangle\Big)\Big].$$

Using graded antisymmetry, we have for the last summand in square brackets

$$\begin{aligned}
&(-1)^{\psi_1(v_1+\cdots+v_{k_2-1}+v_{k_1+k_2})}\langle\psi_2(v_{k_1+k_2}v_1\cdots v_{k_2-1}),\psi_1(v_{k_2},\cdots,v_{k_1+k_2-1})\rangle\\
&= (-1)^{1+\psi_1\psi_2+(\psi_2+v_{k_1+k_2}+v_1+\cdots+v_{k_2-1})(v_{k_2}+\cdots+v_{k_1+k_2-1})}\\
&\qquad\qquad\langle\psi_1(v_{k_2}\cdots v_{k_1+k_2-1}),\psi_2(v_{k_1+k_2}v_1\cdots v_{k_2-1})\rangle\\
&= (-1)^{1+\psi_1\psi_2}\langle\psi_1(\psi_2(v_{k_1+k_2}v_1\cdots v_{k_2-1})v_{k_2}\cdots v_{k_1+k_2-2}),v_{k_1+k_2-1}\rangle
\end{aligned}$$

and similarly for the second summand

$$\begin{aligned}
&(-1)^{\psi_2(v_1+\cdots+v_{k_1-1}+v_{k_1+k_2})}\langle\psi_1(v_{k_1+k_2}v_1\cdots v_{k_1-1}),\psi_2(v_{k_1}\cdots v_{k_1+k_2-1})\rangle\\
&= (-1)^{1+\psi_1\psi_2}\langle\psi_2(\psi_1(v_{k_1+k_2}v_1\cdots v_{k_1-1})v_{k_1}\cdots v_{k_1+k_2-2}),v_{k_1+k_2-1}\rangle.
\end{aligned}$$

Therefore, the square bracket equals

$$\langle[\psi_1,\psi_2](v_{k_1+k_2}v_1\cdots v_{k_1+k_2-2}),v_{k_1+k_2-1}\rangle$$

and the lemma is proven. □

The horizontal arrows in (C.5) become isomorphisms provided that $\langle\cdot,\cdot\rangle$ is non-degenerate (injectivity) and V is of finite type (surjectivity). Therefore, the following definition is possible.

Definition C.1.5 (Candidate for IBL-product)**.** *Let V be a graded vector space of finite type, and let $\langle\cdot,\cdot\rangle : V\otimes V\to\mathbb{R}$ be a non-degenerate homogenous graded antisymmetric bilinear form of degree $-n$. Let ν be a formal symbol of degree n realizing the degree shift* *by* $-n$*. We define the operation*

$$\mathfrak{q}^*_{210} : (\bar{\mathrm{L}}^+_{\mathrm{cyc}}V)[-n] \otimes (\bar{\mathrm{L}}^+_{\mathrm{cyc}}V)[-n] \longrightarrow (\bar{\mathrm{L}}^+_{\mathrm{cyc}}V)[-n]$$

for all ψ_1*,* $\psi_2 \in \mathrm{L}_{\mathrm{cyc}}V$ *by*

$$\mathfrak{q}^*_{210}(\nu\psi_1^+ \otimes \nu\psi_2^+) \coloneqq \nu[\psi_2, \psi_1]^+.$$

Proposition C.1.6 (Candidate for IBL-product)**.** *In the situation of Definition C.1.5, the pair* $((\bar{\mathrm{L}}^+_{\mathrm{cyc}}V)[-n], \mathfrak{q}^*_{210})$ *is a graded Lie algebra with the bracket* $\mathfrak{q}^*_{210}$ *of degree* $-2n$ *and weight* -2*.*

Proof. As for the degree, we have

$$\begin{aligned}
|\nu[\psi_2, \psi_1]^+| &= n + |[\psi_2, \psi_1]^+| \\
&= n - |[\psi_2, \psi_1]| + n \\
&= n - |\psi_1| - |\psi_2| + n \\
&= n + |\psi_1^+| + |\psi_2^+| - n \\
&= -n + |\nu\psi_1^+| + |\nu\psi_2^+| - n \\
&= |\nu\psi_1^+| + |\nu\psi_2^+| - 2n.
\end{aligned}$$

The weights are clear. As for the graded anticommutativity, we have

$$\begin{aligned}
\mathfrak{q}^*_{210}(\nu\psi_2^+ \otimes \nu\psi_1^+) &= \nu[\psi_1, \psi_2]^+ \\
&= -(-1)^{\psi_1\psi_2}\nu[\psi_2, \psi_1]^+ \\
&= -(-1)^{|\nu\psi_1||\nu\psi_2|}\mathfrak{q}^*_{210}(\nu\psi_1^+ \otimes \nu\psi_2^+),
\end{aligned}$$

where we used that

$$|\nu\psi^+| = n + |\psi^+| = n + n - |\psi| = 2n - |\psi|.$$

From the same reason, the graded Jacobi identity for $\mathfrak{q}^*_{210}$ is implied by the graded Jacobi identity for $[\cdot, \cdot]$. ☐

In the situation of Definition C.1.5, let (e_i) be a basis of V, and let (e^i) be the dual basis of V such that $\langle e_i, e^j\rangle = \delta_i^j$. To an index i, we will assign the degree of e_i, so that, e.g., we can write $(-1)^i$ instead of $(-1)^{e_i}$. We will also use the Einstein summation convention.

We define the coordinates

$$\begin{aligned} g_{ij} &= \langle e_i, e_j \rangle, \\ g^{ij} &= \langle e^i, e^j \rangle, \\ \psi(e_{i_1} \cdots e_{i_{k_1}}) &= \psi^i_{i_1 \cdots i_{k_1}} e_i, \\ \psi^+(e_{i_1} \cdots e_{i_{k_1+1}}) &= \psi_{i_1 \cdots i_{k_1+1}}, \end{aligned}$$

where $\psi \in \mathrm{L}V$ and $\psi^+ \in \mathrm{L}^+V$. It holds

$$g_{ij} = (-1)^{ij+1} g_{ji}, \quad g^{ij} = (-1)^{ij+1} g^{ji} \quad \text{and} \quad g_{ik} g^{jk} = \delta^j_i.$$

Equation (C.3) is now equivalent to

$$\psi^i_{i_1 \cdots i_{k_1}} g_{i i_{k_1+1}} = \psi_{i_1 \cdots i_{k_1+1}}, \quad \text{or to} \quad \psi^i_{i_1 \cdots i_{k_1}} = \psi_{i_1 \cdots i_{k_1} k} g^{ik}. \tag{C.6}$$

We define the operation $\mu : (\bar{\mathrm{L}}^+_{\mathrm{cyc}} V)^{\otimes 2} \to \bar{\mathrm{L}}^+_{\mathrm{cyc}} V$ for $\varphi_1 \in \mathrm{Hom}_{\mathrm{cyc}}(V^{k_1+1}, \mathbb{R})$, $\varphi_2 \in \mathrm{Hom}_{\mathrm{cyc}}(V^{\otimes k_2+1}, \mathbb{R})$ with k_1, $k_2 \in \mathbb{N}_0$ such that $k_1 + k_2 \geq 1$ in coordinates by the formula

$$\begin{aligned} \mu(\varphi_1 \otimes \varphi_2)_{i_1 \cdots i_{k_1+k_2}} \coloneqq \sum_{i,j} \sum_{c=1}^{k_1+k_2} & \varepsilon(i_1 \cdots i_{k_1+k_2}, i_c \cdots i_{c-1}) (-1)^{(\varphi_1 - i) j} \\ & (-1)^i g^{ij} (\varphi_1)_{i i_c \cdots i_{c+k_1-1}} (\varphi_2)_{j i_{c+k_1} \cdots i_{c-1}}. \end{aligned} \tag{C.7}$$

Remark C.1.7 (Comparison to [CFL15]). In order to compare μ from (C.7) to μ from [CFL15, Section 10] (and to the definition of $\mathfrak{q}_{210}$ from Section 3.4), we must make the replacements

$$V \mapsto V[1], \quad n \mapsto n-2 \quad \text{and} \quad \bar{\mathrm{L}}^+_{\mathrm{cyc}} V \mapsto \mathrm{B}^*_{\mathrm{cyc}} V.$$

Then $\mathfrak{q}_{210}$ and $\mathfrak{q}^*_{210} : (\mathrm{B}^*_{\mathrm{cyc}} V)[2-n]^{\otimes 2} \to (\mathrm{B}^*_{\mathrm{cyc}} V)[2-n]$ clearly have the same degree $2(2-n)$.[2] Also, $\varphi_1 j + ij$ is precisely the sign needed to move j over $i_c \cdots i_{c+k_1-1}$. We also recall the identification of $\mathrm{B}^*_{\mathrm{cyc}} V$ with the weight-graded dual $(\mathrm{B}^{\mathrm{cyc}} V)^{*_{\mathrm{wg}}}$ from Remark 3.3.5 and remind that $\mathrm{B}^{\mathrm{cyc}} V$ and $\mathrm{B}^*_{\mathrm{cyc}} V$ were defined to be weight-reduced in order to ease the terminology and notation (see Definition 3.3.1). ◁

The equality $\mathfrak{q}^*_{210} = \mathfrak{q}_{210}$ is obtained from the next proposition (c.f., Remark C.1.7).

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[2] The degree of $\mathfrak{q}_{210}$ on tensor powers of $C = \mathrm{B}^*_{\mathrm{cyc}} V[2-n]$, is obtained from the degree on $\mathrm{E}C = \mathrm{S}(C[1])$ in Definition 3.4.1 by substracting 1 (and by adding 1 for the cobracket).

Proposition C.1.8 (Equivalence of definitions of IBL-product)**.** *Let V be a graded vector space of finite type, and let $\langle\cdot,\cdot\rangle : V \otimes V \to \mathbb{R}$ be a non-degenerate homogenous graded antisymmetric bilinear form of degree $-n$. Then the operation $\mathfrak{q}^*_{210} : (\mathrm{L}^+_{\mathrm{cyc}}V)[-n]^{\otimes 2} \to (\mathrm{L}^+_{\mathrm{cyc}}V)[-n]$ defined invariantly in Definition C.1.5 is a degree shift of the operation $\mu : (\bar{\mathrm{L}}^+_{\mathrm{cyc}}V)^{\otimes 2} \to \bar{\mathrm{L}}^+_{\mathrm{cyc}}V$ defined in coordinates above. More precisely, for all φ_1, $\varphi_2 \in \bar{\mathrm{L}}^+_{\mathrm{cyc}}V$, it holds*

$$\mathfrak{q}^*_{210}(\nu^2\varphi_1 \otimes \varphi_2) = \nu\mu(\varphi_1 \otimes \varphi_2) \qquad \textit{in } (\bar{\mathrm{L}}^+_{\mathrm{cyc}}V)[-n].$$

Proof. Consider ψ_1, $\psi_2 \in \mathrm{L}_{\mathrm{cyc}}V$ with k_1, k_2 inputs, respectively. Writing the definition of $[\cdot,\cdot]$ in coordinates, we get

$$\begin{aligned}
&[\psi_1,\psi_2]^i_{i_1\cdots i_{k_1+k_2-1}} \\
&\quad = \sum_{j=1}^{k_1}(-1)^{\psi_2(i_1+\cdots+i_{j-1})}(\psi_1)^i_{i_1\cdots i_{j-1}ki_{j+k_2}\cdots i_{k_1+k_2-1}}(\psi_2)^k_{i_j\cdots i_{j+k_2-1}} \\
&\qquad + \sum_{j=1}^{k_2}(-1)^{\psi_1(i_1+\cdots+i_{j-1}+\psi_2)+1}(\psi_2)^i_{i_1\cdots i_{j-1}ki_{j+k_1}\cdots i_{k_1+k_2-1}}(\psi_1)^k_{i_j\cdots i_{j+k_1-1}}.
\end{aligned}$$

Using (C.6) and the cyclic symmetry of ψ_1^+ and ψ_2^+, we compute

$$\begin{aligned}
&[\psi_1,\psi_2]_{i_1\cdots i_{k_1+k_2}} \\
&= [\psi_1,\psi_2]^i_{i_1\cdots i_{k_1+k_2-1}}g_{ii_{k_1+k_2}} \\
&= \sum_{j=1}^{k_1}(-1)^{\psi_2(i_1+\cdots+i_{j-1})}(\psi_1)_{i_1\ldots i_{j-1}ki_{j+k_2}\ldots i_{k_1+k_2}}(\psi_2)^k_{i_j\ldots i_{j+k_2-1}} \\
&\quad + \sum_{j=1}^{k_2}(-1)^{\psi_1(i_1+\cdots+i_{j-1}+\psi_2)+1}(\psi_2)_{i_1\ldots i_{j-1}ki_{j+k_1}\ldots i_{k_1+k_2}}(\psi_1)^k_{i_j\ldots i_{j+k_1-1}} \\
&= \sum_{j=1}^{k_1}(-1)^{\psi_2(i_1+\cdots+i_{j-1})}g^{ab}(\psi_1)_{i_1\ldots i_{j-1}ai_{j+k_2}\ldots i_{k_1+k_2}}(\psi_2)_{i_j\ldots i_{j+k_2-1}b} \\
&\quad + \sum_{j=1}^{k_2}(-1)^{\psi_1(i_1+\cdots+i_{j-1}+\psi_2)+1}g^{ab}(\psi_2)_{i_1\ldots i_{j-1}ai_{j+k_1}\ldots i_{k_1+k_2}}(\psi_1)_{i_j\ldots i_{j+k_1-1}b}
\end{aligned}$$

$$\begin{aligned} = \quad & \sum_{j=1}^{k_1} \underbrace{(-1)^{\psi_2(i_1+\dots+i_{j-1})+(i_1+\dots+i_{j-1})(\psi_1-n-1)+b(\psi_2-n-1)}}_{=:\varepsilon_1} \\ & \qquad g^{ab}(\psi_1)_{ai_{j+k_2}\cdots i_{k_1+k_2}i_1\cdots i_{j-1}}(\psi_2)_{bi_j\cdots i_{j+k_2-1}} \\ & + \sum_{j=1}^{k_2} \underbrace{(-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1+(i_1+\dots+i_{j-1})(\psi_2-n-1)+a(\psi_1-n-1)+ab+1}}_{=:\varepsilon_2} \\ & \qquad g^{ab}(\psi_1)_{ai_j\cdots i_{j+k_1-1}}(\psi_2)_{bi_{j+k_1}\cdots i_{k_1+k_2}i_1\cdots i_{j-1}}. \end{aligned}$$

Notice that the lower indices i_1, …, $i_{k_1+k_2}$ appear in cyclic permutations. The first sum consists of cyclic permutation starting with i_{k_2+1} and going up to $i_{k_1+k_2+1}$, and the second sum consists of cyclic permutations starting with i_1 and going up to i_{k_2}; hence, all cyclic permutations appear, and it just remains to check the signs. We see that

$$\begin{aligned} \varepsilon_1 &= (-1)^{\psi_2(i_1+\dots+i_{j-1})+(i_1+\dots+i_{j-1})(\psi_1-n-1)+b(\psi_2-n-1)} \\ &= \underbrace{(-1)^{(i_1+\dots+i_{j-1}+\psi_2-n-b)(\psi_1-n-a+\psi_2-n-b-1)}}_{=\varepsilon(i_1\cdots i_{k_1+k_2},i_{j+k_2}\cdots i_{k_1+k_2}i_1\cdots i_{j+k_2-1})}(-1)^{b(\psi_1-n-a)}(-1)^a \\ &\qquad\qquad (-1)^{n(\psi_1-n)+\psi_2\psi_1} \end{aligned}$$

and

$$\begin{aligned} \varepsilon_2 &= (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+(i_1+\dots+i_{j-1})(\psi_2-n-1)+a(\psi_1-n-1)+ab} \\ &= \underbrace{(-1)^{(i_1+\dots+i_{j-1})(\psi_1-n-a+\psi_2-n-b-1)}}_{=\varepsilon(i_1\cdots i_{k_1+k_2},i_j\cdots i_{k_1+k_2}i_1\cdots i_{j-1})}(-1)^{b(\psi_1-n-a)}(-1)^a \\ &\qquad\qquad (-1)^{n(\psi_1-n)+\psi_1\psi_2}. \end{aligned}$$

This finishes the proof. ☐

C.2. Cobracket and odd symplectic geometry

We start with a remark about the naive dualization of the bracket on the dual.

Remark C.2.1 (Naive dualization). If we ignore the fact that the dualization of spaces like $\mathrm{L}V$ is problematic and we might not get $(\mathrm{L}V^{*_\mathrm{g}})^{*_\mathrm{g}} \simeq \mathrm{L}V$,[3] a naive way to obtain a cobracket $\bar{\delta} : \mathrm{L}V \to \mathrm{L}V \otimes \mathrm{L}V$ would be to dualize the canonical bracket $[\cdot,\cdot]^* : \mathrm{L}V^{*_\mathrm{g}} \otimes$

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[3] It holds $V \simeq (V^{*_\mathrm{g}})^{*_\mathrm{g}}$ provided that V is of finite type. The tensor product of graded vector spaces of finite type is also of finite type provided that the grading is non-negative. Moreover, one has to require that $W^0 = W^1 = 0$ for W for which $V = W[1]$ or take suitable completions with respect to weights in order to deal with arbitrary long tensor products in the same degree in the dual of $\mathrm{L}V$.

$\mathrm{L}V^{*\mathrm{g}} \to \mathrm{L}V^{*\mathrm{g}}$ from the previous section (replacing V by $V^{*\mathrm{g}}$). Because $w([\cdot,\cdot]^*) = -1$, the dual would have $w(\bar{\delta}) = 1$, where w denotes the weights. However, the IBL-cobracket has $w(\mathfrak{q}_{120}) = -2$ (see Definition 3.4.1), and hence

$$w\big(\big(({}^{+})^{-1} \otimes ({}^{+})^{-1}\big) \circ \mathfrak{q}_{120} \circ {}^{+}\big) = w(\mathfrak{q}_{120}) - 1 = -3 \neq 1 = w(\bar{\delta}),$$

where the map ${}^{+} : \mathrm{L}V \to \mathrm{L}^{+}V$ of weight $+1$ was defined in (C.5). It might be interesting to write down the coordinate expression for such $\bar{\delta}$ which comes from $\mathfrak{q}_{120}$ and study its origin and properties on $\mathrm{L}V$. $\lhd$

In the following definition, we will use the terms “cobracket” and “BV-operator” to denote some operations without actually knowing that they satisfy the corresponding relations (see Questions C.2.6 at the end of this section).

Definition C.2.2 (Hochschild cobracket and BV-operator)**.** *Let V be a graded vector space, let $(e_i) \in V$ be its homogenous basis, and let $(\eta^i) \subset V^{*\mathrm{g}}$ be the dual basis, i.e., it holds $\eta^i(e_j) = \delta^i_j$. We denote by $\mathrm{T}(V^{*\mathrm{g}})$ the tensor algebra over $V^{*\mathrm{g}}$ and consider its basis $\eta^{\otimes I} := \eta^{i_1} \otimes \cdots \otimes \eta^{i_k}$ for all multiindices $I = (i_1, \ldots, i_k)$.*

For each i, j, we define the cyclic double derivative $\mathrm{Der}_{ij} : \mathrm{T}(V^{*\mathrm{g}}) \longrightarrow \mathrm{T}(V^{*\mathrm{g}}) \otimes \mathrm{T}(V^{*\mathrm{g}})$ *on the basis by the following formula, which we explain below:*

$$\mathrm{Der}_{ij}(\eta^{\otimes I}) := \sum_{(i,j)\in I} \sum_{\substack{c_1 \in \mathbb{S}^{\mathrm{cyc}}_{k_1} \\ c_2 \in \mathbb{S}^{\mathrm{cyc}}_{k_2}}} \frac{1}{k_1}\frac{1}{k_2} \varepsilon(I, ijI_1^{c_1} I_2^{c_2})(\eta^{\otimes I_1^{c_1}} \otimes \eta^{\otimes I_2^{c_2}}) \tag{C.8}$$

The first sum is over all pairs of indices (i,j) at two distinct positions in I. We consider the cyclic order on positions in I (of length k) starting from the left and define I_1 and I_2 as the substrings between i and j (of length k_1) and between j and i (of length k_2), respectively. We include neither i nor j in I_1 or I_2, i.e., $k = k_1 + k_2 + 2$. The second sum is an average over all cyclic permutations c_1 and c_2 of I_1 and I_2, respectively. As usual, ε stands for the Koszul sign and i has the degree of e_i. If k_1 or k_2 vanishes, we replace the average over c_1 or c_2, respectively, by 1 and set $\eta^{\emptyset} = 1$.

On $\mathrm{T}(V^{*\mathrm{g}})$, we consider the shuffle product $\mu^{\mathrm{sh}} : \mathrm{T}(V^{*\mathrm{g}}) \otimes \mathrm{T}(V^{*\mathrm{g}}) \to \mathrm{T}(V^{*\mathrm{g}})$ *which is defined by*

$$\mu^{\mathrm{sh}}(\eta^{\otimes I_1}, \eta^{\otimes I_2}) := \sum_{\mu \in \mathbb{S}_{k_1,k_2}} \frac{k_1! k_2!}{(k_1+k_2)!} \varepsilon(I_1 I_2, \mu(I_1 I_2)) \eta^{\otimes \mu(I_1 I_2)},$$

where k_1 and k_2 are the lengths of I_1 and I_2, respectively, and $\mathbb{S}_{k_1,k_2}$ denotes the shuffle permutations.

If V is of finite type and $\langle\cdot,\cdot\rangle$ is a non-degenerate homogenous bilinear form on V, we consider the coordinates $g^{ij} = \langle e^i, e^j\rangle$, where $(e^i) \subset V$ is the basis such that $\langle e^i, e_j\rangle = \delta^i_j$. We define the Hochschild cobracket $\zeta_{\mathrm{H}} : \mathrm{T}(V^{*\mathrm{g}}) \to \mathrm{T}(V^{*\mathrm{g}}) \otimes \mathrm{T}(V^{*\mathrm{g}})$ *by*

$$\zeta_{\mathrm{H}} = \frac{1}{2}\sum_{i,j}(-1)^{e_i} g^{ij}\mathrm{Der}_{ij} \tag{C.9}$$

and the Hochschild BV-operator $\Delta_{\mathrm{H}} : \mathrm{T}(V^{*\mathrm{g}}) \to \mathrm{T}(V^{*\mathrm{g}})$ *by*

$$\Delta_{\mathrm{H}} \coloneqq \mu^{\mathrm{sh}} \circ \zeta_{\mathrm{H}}. \tag{C.10}$$

It is easy to check that the definitions of μ^{sh}, ζ_{H} and Δ_{H} do not depend on the choice of the basis (e_i) of V.

We denote by $\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}$ the subspace of cyclic invariants of $\mathrm{T}(V^{*\mathrm{g}})$ and by $\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}$ the subspace of symmetric invariants. We consider the cyclization map $\pi_{\mathrm{cyc}} : \mathrm{T}(V^{*\mathrm{g}}) \to \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}$ and the symmetrization map $\pi_{\mathrm{sym}} : \mathrm{T}(V^{*\mathrm{g}}) \to \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}$; they are defined by the averages

$$\begin{aligned}\pi_{\mathrm{cyc}}(\eta^{\otimes I}) &= \sum_{c\in\mathbb{S}_k^{\mathrm{cyc}}} \frac{1}{k}\varepsilon(I, I^c)\eta^{\otimes I^c} \qquad \text{and}\\ \pi_{\mathrm{sym}}(\eta^{\otimes I}) &= \sum_{\sigma\in\mathbb{S}_k} \frac{1}{k!}\varepsilon(I, I^\sigma)\eta^{\otimes I^\sigma},\end{aligned}$$

respectively, where k is the length of I (if $k = 0$, we have $\eta^{\emptyset} = 1$ and $\pi_{\mathrm{cyc}}(1) \coloneqq 1$). We denote by $\mathrm{p}_{\mathrm{cyc}}$ the projection to cyclic coinvariants and by $\mathrm{p}_{\mathrm{sym}}$ the projection to symmetric coinvariants.

Definition C.2.3 (Cyclic and symmetric versions)**.** *In the situation of Definition C.2.2, we define the cyclic and symmetric versions of the operations* Der_{ij}*,* ζ_{H}*,* μ^{sh} *and* Δ_{H} *by precomposing with the inclusions* $\iota_{\mathrm{cyc}} : \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}} \to \mathrm{T}(V^{*\mathrm{g}})$ *and* $\iota_{\mathrm{sym}} : \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}} \to \mathrm{T}(V^{*\mathrm{g}})$ *and postcomposing with the projections* $\pi_{\mathrm{cyc}} : \mathrm{T}(V^{*\mathrm{g}}) \to \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}$ *and* $\pi_{\mathrm{sym}} : \mathrm{T}(V^{*\mathrm{g}}) \to \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}$*, respectively. In formulas, we have*

$$\begin{aligned}\mathrm{D}^{\mathrm{cyc}}_{ij} &\coloneqq (\pi_{\mathrm{cyc}}\otimes\pi_{\mathrm{cyc}})\circ\mathrm{Der}_{ij}\circ\iota_{\mathrm{cyc}}, & \mathrm{D}^{\mathrm{sym}}_{ij} &\coloneqq (\pi_{\mathrm{sym}}\otimes\pi_{\mathrm{sym}})\circ\mathrm{Der}_{ij}\circ\iota_{\mathrm{sym}},\\ \zeta_{\mathrm{cyc}} &\coloneqq (\pi_{\mathrm{cyc}}\otimes\pi_{\mathrm{cyc}})\circ\zeta_{\mathrm{H}}\circ\iota_{\mathrm{cyc}}, & \zeta_{\mathrm{sym}} &\coloneqq (\pi_{\mathrm{sym}}\otimes\pi_{\mathrm{sym}})\circ\zeta_{\mathrm{H}}\circ\iota_{\mathrm{sym}},\\ \mu_{\mathrm{cyc}} &\coloneqq \pi_{\mathrm{cyc}}\circ\mu^{\mathrm{sh}}\circ(\iota_{\mathrm{cyc}}\otimes\iota_{\mathrm{cyc}}), & \mu_{\mathrm{sym}} &\coloneqq \pi_{\mathrm{sym}}\circ\mu^{\mathrm{sh}}\circ(\iota_{\mathrm{sym}}\otimes\iota_{\mathrm{sym}}),\\ \Delta_{\mathrm{cyc}} &\coloneqq \pi_{\mathrm{cyc}}\circ\Delta_{\mathrm{H}}\circ\iota_{\mathrm{cyc}}, & \Delta_{\mathrm{sym}} &\coloneqq \pi_{\mathrm{sym}}\circ\Delta_{\mathrm{H}}\circ\iota_{\mathrm{sym}}.\end{aligned} \tag{C.11}$$

Proposition C.2.4 (Hochschild, cyclic and symmetric operations)**.** *In the situation of Definition C.2.3, we consider for every multiindex I the linear functional $\alpha^I : \mathrm{T}V \to \mathbb{R}$ given with respect to the basis $e_{\otimes J}$ by*

$$\alpha^I(e_{\otimes J}) \coloneqq \varepsilon(I, J).$$

We set

$$\alpha^I_{\mathrm{sym}} = \sum_{\sigma\in\mathbb{S}_k} \frac{1}{k!}\varepsilon(I, I^\sigma)\alpha^{I^\sigma} \quad \text{and} \quad \alpha^I_{\mathrm{cyc}} = \sum_{c\in\mathbb{S}_k^{\mathrm{cyc}}} \frac{1}{k}\varepsilon(I, I^c)\alpha^{I^c}.$$

We define the monomorphisms I, I_{cyc} and I_{sym} by

$$\begin{tikzcd}
TV^{*_{\mathrm{g}}} \arrow[r, hook, "I"] & \mathrm{L}^+V : & \eta^{\otimes I} \arrow[r, mapsto] & \alpha^I, \\
(TV^{*_{\mathrm{g}}})_{\mathrm{cyc}} \arrow[r, hook, "I_{\mathrm{cyc}}"] \arrow[u, hook] & \mathrm{L}^+_{\mathrm{cyc}}V : \arrow[u, hook] & \pi_{\mathrm{cyc}}(\eta^{\otimes I}) \arrow[r, mapsto] & \alpha^I_{\mathrm{cyc}}, \\
(TV^{*_{\mathrm{g}}})_{\mathrm{sym}} \arrow[r, hook, "I_{\mathrm{sym}}"] \arrow[u, hook] & \mathrm{L}^+_{\mathrm{sym}}V : \arrow[u, hook] & \pi_{\mathrm{sym}}(\eta^{\otimes I}) \arrow[r, mapsto] & \alpha^I_{\mathrm{sym}}.
\end{tikzcd} \tag{C.12}$$

The vertical arrows are inclusions and the diagram commutes. Moreover, I, I_{cyc} and I_{sym} become isomorphisms provided that V is of finite type with non-negative degrees. In this case, we transfer the operations and obtain the following commutative diagram:

$$\begin{tikzcd}
\Delta_{\mathrm{H}} : & \mathrm{L}^+V \arrow[r, "\zeta_{\mathrm{H}}"] & \mathrm{L}^+V \otimes \mathrm{L}^+V \arrow[r, "\mu^{\mathrm{sh}}"] & \mathrm{L}^+V \\
\Delta_{\mathrm{cyc}} : & \mathrm{L}^+_{\mathrm{cyc}}V \arrow[r, "\zeta_{\mathrm{cyc}}"] \arrow[u, hook] & \mathrm{L}^+_{\mathrm{cyc}}V \otimes \mathrm{L}^+_{\mathrm{cyc}}V \arrow[r, "\mu_{\mathrm{cyc}}"] \arrow[u, hook] & \mathrm{L}^+_{\mathrm{cyc}}V \arrow[u, hook] \\
\Delta_{\mathrm{sym}} : & \mathrm{L}^+_{\mathrm{sym}}V \arrow[r, "\zeta_{\mathrm{sym}}"] \arrow[u, hook] & \mathrm{L}^+_{\mathrm{sym}}V \otimes \mathrm{L}^+_{\mathrm{sym}}V \arrow[r, "\mu_{\mathrm{sym}}"] \arrow[u, hook] & \mathrm{L}^+_{\mathrm{sym}}V. \arrow[u, hook]
\end{tikzcd} \tag{C.13}$$

The following formulas hold:

$$\begin{aligned}
\zeta_{\mathrm{cyc}}(\alpha^I_{\mathrm{cyc}}) &= \frac{1}{2}\sum_{i,j}(-1)^{e_i}g^{ij}\sum_{(i,j)\in I}\varepsilon(I, ijI_1I_2)(\alpha^{I_1}_{\mathrm{cyc}} \otimes \alpha^{I_2}_{\mathrm{cyc}}), \\
\mu_{\mathrm{sym}}(\alpha^{I_1}_{\mathrm{sym}}, \alpha^{I_2}_{\mathrm{sym}}) &= \alpha^{I_1I_2}_{\mathrm{sym}}, \\
\Delta_{\mathrm{sym}}(\alpha^I_{\mathrm{sym}}) &= \frac{1}{2}\sum_{i,j}(-1)^{e_i}g^{ij}\sum_{(i,j)\in I}\varepsilon(I, ijI_1I_2)\alpha^{I_1I_2}_{\mathrm{sym}}.
\end{aligned} \tag{C.14}$$

Proof. The commutativity of (C.12) is clear from the definitions. The commutativity of (C.13) is equivalent to the commutativity of the same diagram with L^+V, $\mathrm{L}^+_{\mathrm{cyc}}V$, $\mathrm{L}^+_{\mathrm{sym}}V$ replaced by $\mathrm{T}(V^{*\mathrm{g}})$, $\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}$, $\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}$, respectively. This commutativity then follows from the defining relations (C.11) if we show the following inclusions:

1. $\mathrm{Der}_{ij}(\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}) \subset \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}} \otimes \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}$,
2. $\mathrm{Der}_{ij}(\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}) \subset \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}} \otimes \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}$,
3. $\mu^{\mathrm{sh}}(\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}} \otimes \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}) \subset \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}$,
4. $\mu^{\mathrm{sh}}(\mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}} \otimes \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}) \subset \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{sym}}$.

Inclusion (1) is clear from (C.8). In fact, it holds even $\mathrm{Der}_{ij}(\mathrm{T}(V^{*\mathrm{g}})) \subset \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}} \otimes \mathrm{T}(V^{*\mathrm{g}})_{\mathrm{cyc}}$. As for inclusion (2), we have

$$\mathrm{Der}_{ij}(\pi_{\mathrm{sym}}(\eta^{\otimes I})) = \sum_{\sigma \in \mathbb{S}_k} \frac{1}{k!} \sum_{(i,j)\in I^\sigma} \sum_{\substack{c_1 \in \mathbb{S}^{\mathrm{cyc}}_{k_1} \\ c_2 \in \mathbb{S}^{\mathrm{cyc}}_{k_2}}} \frac{1}{k_1} \frac{1}{k_2} \varepsilon(I, ijI_1^{\sigma c_1} I_2^{\sigma c_2}) \eta^{\otimes I_1^{\sigma c_1}} \otimes \eta^{\otimes I_2^{\sigma c_2}}.$$

A pair $(\sigma_1, \sigma_2) \in \mathbb{S}_{k_1} \times \mathbb{S}_{k_2}$ induces a bijection of the domain of summation, which consists of the elements σ, (i,j), c_1, c_2, by keeping (i,j), c_1, c_2 fixed and defining a new $\tilde{\sigma}$ by altering σ to compensate for the effect of (σ_1, σ_2). The signs fit since we only consider the Koszul sign. Therefore, we have

$$(\sigma_1 \otimes \sigma_2)\mathrm{Der}_{ij}(\pi_{\mathrm{sym}}(\eta^{\otimes I})) = \mathrm{Der}_{ij}(\pi_{\mathrm{sym}}(\eta^{\otimes I})),$$

and (2) follows. As for (3), we have

$$\begin{aligned} &\mu^{\mathrm{sh}}(\pi_{\mathrm{cyc}}(\eta^{\otimes I_1}), \pi_{\mathrm{cyc}}(\eta^{\otimes I_2})) \\ &\quad = \sum_{\mu \in \mathbb{S}_{k_1,k_2}} \frac{k_1! k_2!}{(k_1+k_2)!} \sum_{\substack{c_1 \in \mathbb{S}^{\mathrm{cyc}}_{k_1} \\ c_2 \in \mathbb{S}^{\mathrm{cyc}}_{k_2}}} \frac{1}{k_1} \frac{1}{k_2} \varepsilon(I_1 I_2, \mu(I_1^{c_1}, I_2^{c_2})) \eta^{\otimes \mu(I_1^{c_1}, I_2^{c_2})}. \end{aligned}$$

A permutation $c \in \mathbb{S}_k$ again induces a bijection of the domain of summation, which consists of the elements μ, c_1, c_2. A similar argument holds for (4).

We now derive (C.14). Because

$$\mathrm{Der}_{ij}(\alpha^I_{\mathrm{cyc}}) = \sum_{(i,j)\in I} \varepsilon(I, ijI_1 I_2) \alpha^{I_1}_{\mathrm{cyc}} \otimes \alpha^{I_2}_{\mathrm{cyc}},$$

the formula for ζ_{cyc} is clear. As for Δ_{sym}, because $\mu_{\mathrm{sym}} = \pi_{\mathrm{sym}} \circ \mu_{\mathrm{con}}$ for the concatenation product $\mu_{\mathrm{con}}(\eta^{\otimes I_1}, \eta^{\otimes I_2}) = \eta^{\otimes I_1 I_2}$, we have

$$\begin{aligned}(\mu_{\mathrm{sym}} \circ \mathrm{Der}_{ij})(\pi_{\mathrm{sym}}(\eta^{\otimes I})) &= \sum_{\sigma \in \mathbb{S}_k} \frac{1}{k!} \sum_{(i,j) \in I^\sigma} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\mathrm{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\mathrm{cyc}}}} \frac{1}{k_1} \frac{1}{k_2} \pi_{\mathrm{sym}}\big(\varepsilon(I, ij I_1^{\sigma c_1} I_2^{\sigma c_2}) \eta^{\otimes I_1^{\sigma c_1} I_2^{\sigma c_2}}\big) \\ &= \sum_{(i,j) \in I} \varepsilon(I, ij I_1 I_2) \pi_{\mathrm{sym}}(\eta^{\otimes I_1 I_2}),\end{aligned}$$

where we used that $\pi_{\mathrm{sym}} \circ \sigma = \pi_{\mathrm{sym}}$ for every permutation σ. The formula for Δ_{sym} follows. The formula for μ_{sym} is also clear. □

In the formula (C.14) for Δ_{sym}, we recognize the canonical BV-operator on $\mathrm{S}V^{*\mathrm{g}}$ from [DJP19, Definition 4].[4]

We will now relate ζ_{cyc} to $\mathfrak{q}_{120}$. The steps are similar to Section C.1. We define the coproduct $\mathfrak{q}_{120}^* : (\mathrm{L}_{\mathrm{cyc}}^+ V)[-n] \to (\mathrm{L}_{\mathrm{cyc}}^+ V)[-n]^{\otimes 2}$, where $-n$ is the degree of $\langle \cdot, \cdot \rangle$, by

$$\mathfrak{q}_{120}^*(\nu \varphi) = \nu^2 \zeta_{\mathrm{cyc}}(\varphi) \qquad \text{for all } \varphi \in \mathrm{L}_{\mathrm{cyc}}^+ V, \tag{C.15}$$

where ν is a formal symbol of degree n.

For $\varphi \in \mathrm{Hom}_{\mathrm{cyc}}(V^{\otimes k}, \mathbb{R})$ with $k \geq 4$, we define $(\delta\varphi) \in (\bar{\mathrm{L}}_{\mathrm{cyc}}^+ V)^{\otimes 2}$ using the coordinates

$$\varphi_I \coloneqq \varphi(e_{\otimes I}) \quad \text{and} \quad (\delta\varphi)_{I_1;I_2} = (\delta\varphi)(e_{\otimes I_1} \otimes e_{\otimes I_2}) \tag{C.16}$$

by

$$(\delta\varphi)_{I_1;I_2} \coloneqq \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\mathrm{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\mathrm{cyc}}}} \sum_{i,j} (-1)^{e_i} g^{ij} \varepsilon(I_1, I_1^c) \varepsilon(I_2, I_2^c) (-1)^{e_j I_1} \varphi_{i I_1^{c_1} j I_2^{c_2}}. \tag{C.17}$$

We set $\delta\varphi = 0$ for $k \leq 3$. In order to obtain $(\delta\varphi) \in (\bar{\mathrm{L}}_{\mathrm{cyc}}^+ V)^{\otimes 2}$, we use the embedding $(\mathrm{L}_{\mathrm{cyc}}^+ V)^{\otimes k} \subset (\mathrm{T}V)^{\otimes k *}$ from Remark 3.3.5. Note that we can also use the canonical identification $(\mathrm{T}V/\mathrm{cyc})^{*\mathrm{wg}} \simeq \mathrm{L}_{\mathrm{cyc}}^+ V$ and work with the cyclic words $e_I \coloneqq \mathrm{p}_{\mathrm{cyc}}(e_{\otimes I})$ instead. We see that (C.17) is precisely the defining equation [CFL15, Equation (10.6)] for the IBL-cobracket $\delta : \bar{\mathrm{L}}_{\mathrm{cyc}}^+ V \to (\bar{\mathrm{L}}_{\mathrm{cyc}}^+ V)^{\otimes 2}$, and $\mathfrak{q}_{120} : (\bar{\mathrm{L}}_{\mathrm{cyc}}^+ V)[-n] \to (\bar{\mathrm{L}}_{\mathrm{cyc}}^+ V)[-n]^{\otimes 2}$ is the degree shift of δ.

The equality $\mathfrak{q}_{120}^* = \mathfrak{q}_{120}$ on the reduced part is obtained from the next proposition (c.f., Remark C.1.7 for the translation from the $\bar{\mathrm{L}}_{\mathrm{cyc}}^+$- to the $\mathrm{B}_{\mathrm{cyc}}^*$-notation).

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[4] In fact, they work on a bigger "algebra of functions" on V, which is a completion of $\mathrm{S}V^{*\mathrm{g}}$ with respect to a suitable filtration. Their statement is that the algebra of functions on an odd symplectic vector space carries a canonical BV-operator. It is given by the divergence of the Hamiltonian vector field.

Proposition C.2.5 (Equivalence of definitions of IBL-cobracket)**.** *Let V be a graded vector space of finite type, and let $\langle\cdot,\cdot\rangle$ be a non-degenerate homogenous bilinear form of degree $-n$. For every $\varphi \in \bar{\mathrm{L}}^+_{\mathrm{cyc}}V$, it holds*

$$\mathfrak{q}^*_{120}(\nu\varphi) = \nu^2\delta\varphi \qquad \textit{in} \quad (\bar{\mathrm{L}}^+_{\mathrm{cyc}}V)[-n]^{\otimes 2}.$$

Proof. We consider the basis $(\alpha^I_{\mathrm{cyc}})$ of $\mathrm{L}^+_{\mathrm{cyc}}V$ from Proposition C.2.4. Given $\varphi \in \mathrm{L}^+_{\mathrm{cyc}}V$, we can write $\varphi = \sum_I \frac{1}{|I|}\varphi_I\alpha^I_{\mathrm{cyc}}$, where we sum over all multiindices I and $\varphi_I \in \mathbb{R}$ are coefficients cyclically symmetric in I. For every I of length k, we have

$$\varphi(e_{\otimes I}) = \sum_{c\in\mathbb{S}^{\mathrm{cyc}}_k} \frac{1}{k}\varphi_{I^c}\alpha^{I^c}_{\mathrm{cyc}}(e_{\otimes I}) = \sum_{c\in\mathbb{S}^{\mathrm{cyc}}_k} \frac{1}{k}\varphi_{I^c}\varepsilon(I^c, I) = \varphi_I,$$

and hence φ_I correspond to the coefficients (C.16). Using (C.14), we compute

$$\begin{aligned}
\zeta_{\mathrm{cyc}}(\varphi) &= \frac{1}{2}\sum_I \sum_{(i,j)\in I} \frac{1}{|I|}\varphi_I(-1)^{e_i}g^{ij}\varepsilon(I, ijI_1I_2)\alpha^{I_1}_{\mathrm{cyc}} \otimes \alpha^{I_2}_{\mathrm{cyc}} \\
&= \frac{1}{2}\sum_J \sum_{c\in\mathbb{S}^{\mathrm{cyc}}_{|J|}} \sum_{j\in J\setminus\{j_1\}} \frac{1}{|J|}\varphi_{J^c}(-1)^{e_{j_1}}g^{j_1j}(-1)^{jJ_1}\varepsilon(J, J^c)\alpha^{J_1}_{\mathrm{cyc}} \otimes \alpha^{J_2}_{\mathrm{cyc}} \\
&= \frac{1}{2}\sum_J \sum_{j\in J\setminus\{j_1\}} \varphi_J(-1)^{e_{j_1}}g^{j_1j}(-1)^{jJ_1}\alpha^{J_1}_{\mathrm{cyc}} \otimes \alpha^{J_2}_{\mathrm{cyc}} \\
&= \sum_{J_1,J_2} \underbrace{\Big(\frac{1}{2}\sum_{i,j}(-1)^{e_i}g^{ij}(-1)^{jJ_1}\varphi_{iJ_1jJ_2}\Big)}_{=:(*)_{J_1;J_2}} \alpha^{J_1}_{\mathrm{cyc}} \otimes \alpha^{J_2}_{\mathrm{cyc}}.
\end{aligned}$$

On the second line, we used the bijection of the summation domains consisting of I, (i,j), resp. J, c, b, which assigns to I its cyclic permutation J such that the position of i corresponds to the first position of J, i.e., it holds $j_1 = i$. On the third line, we used that φ_I are cyclic symmetric, and on the fourth line, we just rewrote the summation using $J = iJ_1jJ_2$. We now relate $(*)_{J_1;J_2}$ to $(\delta\varphi)_{J_1;J_2}$. We compute (notice that there is no Koszul sign from the application of the tensor product, c.f., Remark 3.3.5)

$$\begin{aligned}
\zeta_{\mathrm{cyc}}(\varphi)(e_{I_1} \otimes e_{I_2}) &= \Big(\sum_{\substack{c_1\in\mathbb{S}^{\mathrm{cyc}}_{k_1} \\ c_2\in\mathbb{S}^{\mathrm{cyc}}_{k_2}}} (*)_{I_1^{c_1};I_2^{c_2}}\alpha^{I_1^{c_1}}_{\mathrm{cyc}} \otimes \alpha^{I_2^{c_2}}_{\mathrm{cyc}}\Big)(e_{I_1} \otimes e_{I_2}) \\
&= \sum_{\substack{c_1\in\mathbb{S}^{\mathrm{cyc}}_{k_1} \\ c_2\in\mathbb{S}^{\mathrm{cyc}}_{k_2}}} \varepsilon(I_1^{c_1}, I_1)\varepsilon(I_2^{c_2}, I_2)(*)_{I_1^{c_1};I_2^{c_2}}
\end{aligned}$$

$$= (\delta\varphi)_{J_1;J_2}.$$

This shows the proposition. □

Questions C.2.6. (i) Are $(\mathrm{L}^+_{\mathrm{cyc}}V, \Delta_{\mathrm{cyc}}, \mu_{\mathrm{cyc}})$ and $(\mathrm{L}^+V, \Delta_{\mathrm{H}}, \mu^{\mathrm{sh}})$ also BV-algebras like $(\mathrm{L}^+_{\mathrm{sym}}V, \Delta_{\mathrm{sym}}, \mu_{\mathrm{sym}})$? Recall the 2nd-derivative identity (or the "seven term identity") which is used to define a BV-algebra:

$$\begin{aligned}\Delta(abc) = \Delta(ab)c + (-1)^{a(b+c)}\,\Delta(bc)a + (-1)^{bc}\,\Delta(ac)b \\ - \Delta(a)bc - (-1)^{ab}\,\Delta(b)ac - (-1)^{c(a+b)}\,\Delta(c)ab.\end{aligned} \tag{C.18}$$

It clearly uses associativity (and in this form also commutativity). It is a known fact that μ^{sh} is both associative and commutative (see [Lod92]). Therefore, the question for Δ_{H} makes sense. However, μ_{cyc} might not be associative (is it?), and hence Δ_{cyc} might not be a BV-operator in the classical sense. Is it possible to define a second order derivative, and hence the notion of a BV-operator, on a non-associative algebra?

(ii) Do the factorizations $\Delta_{\mathrm{cyc}} = \mu_{\mathrm{cyc}} \circ \zeta_{\mathrm{cyc}}$, resp. $\Delta_{\mathrm{sym}} = \mu_{\mathrm{sym}} \circ \zeta_{\mathrm{sym}}$ define ζ_{cyc}, resp. ζ_{sym} uniquely? Do other BV-operators factorize like this? Is ζ_{sym} a cobracket and is there a bracket on "symmetric Hochschild cochains" inducing a bialgebra structure on $\mathrm{L}^+_{\mathrm{sym}}V$?

(iii) Are the new operations chain maps with respect to b and $\mathrm{b} + \mathrm{d}$?

D. Filtered IBL-infinity-algebras as filtered MV-algebras

In this appendix, we want to understand the BV-formalism for IBL_∞-algebras and the equivalence

$$\mathrm{IBL}_\infty : \quad \begin{array}{c} \underline{\text{Surface calculus}} \\ (\mathfrak{q}_{klg}), (f_{klg}), \text{gluing relations} \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \underline{\text{BV-formalismus}} \\ \Delta, \mathfrak{f}, \text{ relations } \Delta^2 = 0, e^{\mathfrak{f}}\,\Delta^+ = \Delta^-\,e^{\mathfrak{f}}. \end{array}$$

This was originally done in [CFL15] using filtrations and formal power series in $\hbar$ as a “bookkeeping”. Nevertheless, the author thinks that some details were not fully addressed and was also curious about extending the MV-formalism from [MV17] to the filtered setting.

As a matter of fact, the main motivation for better understanding of these details was a brief discussion with Kyler Siegel at a workshop about BV-quantization in Stony Brook, 2019. It was pointed out that a twisted ∞-algebra is, in some sense, “isomorphic” to the untwisted one because the twisting is just conjugation with the exponential of the Maurer-Cartan element. The case of A_∞-algebras from [Fuk+09] was mentioned. The author of this text was confused because, by Part I, the twisting with the Chern-Simons Maurer-Cartan element “adds” the information about string topology to the canonical structure and one expects to obtain a non-isomorphic structure. The precise notions of isomorphisms needed to be clarified.

In Section D.1, we start by dealing with filtrations and completions in terms of series in more details. We formulate the Resummation Lemma (Lemma D.1.1), define (complete) filtered algebras (Definition D.1.2) and comment on units and augmentations (Remark D.1.4). We then consider combinations of two filtrations (Equation (D.1)) and show that the symmetric bialgebra with the filtration which is the union of the induced filtration and the filtration by weights might not be a filtered bialgebra (Example D.1.5); it is, however, under a boundedness assumption on the filtration (Lemma D.1.6). We then consider the exponential and the logarithm on a complete filtered algebra (Lemma D.1.7) and generalize it for the convolution product of morphisms with codomain a complete filtered algebra and domain a complete filtered coalgebra satisfying the limit conilpotency property (Proposition D.1.8). This is the case of completions of conilpotent coalgebras, e.g., the symmetric bialgebra (Lemma D.1.9). We remark that the exponential of a morphism might not be invertible as a map (Remark D.1.10).

In Section D.2, we recall MV-algebras from [MV17] and introduce their complete filtered version (Definition D.2.1). Based on this, in analogy to [MV17], we define the notion of a complete filtered IBL_∞-algebra in MV-formalism and its morphisms (Definition D.2.3). We study the equivalent formulation in components via the surface calculus (Proposition D.2.6). We show that the notion of a filtered IBL_∞-algebra in MV-formalism is equivalent to the notion of a filtered IBL_∞-algebra from [CFL15] in some cases, e.g., for the dual cyclic bar complex from Part I (Proposition D.2.7). The formalism of [CFL15] is symmetric on exchanging inputs and outputs whereas the MV-formalism is not; we illustrate how bubblings at inputs and outputs are handled differently (Example D.2.9). Finally, we consider twisting with a Maurer-Cartan element (Proposition D.2.10).

In Section D.3, we define the BV-chain complexes associated to a complete filtered IBL_∞-algebra in MV-formalism (Definition D.3.1) and observe that morphisms and the twisting with the Maurer-Cartan element induce chain maps of these chain complexes (Proposition D.3.3). We formulate some open questions (Questions D.3.4).

In Section D.4, we study the composition of maps which are convolutions of other maps (Lemma D.4.1). This leads to the definition of compositions controlling the number of "veins" between the individual maps (Definition D.4.2). This can be used to formulate the surface calculus algebraically (Proposition D.4.3). We give proofs of some formulas for partial compositions from Section 3.2 (Proposition D.4.4).

D.1. Filtered bialgebras and exponential in convolution product

We use the definitions of a filtration, completion and the induced filtration on the completion from Section 3.1. We formulate the statements in the category of vector spaces, but similar statements hold in the category of graded vector spaces (writing "homogenous" and "graded" everywhere). All filtrations are decreasing, i.e., $\mathcal{F}^{\lambda_1} \supseteq \mathcal{F}^{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$. We work over a field $\mathbb{K}$ of characteristic zero.

We start with the following technical lemma, which shows that convergence in completion is very much like absolute convergence in analysis.[1]

---

[1] In fact, for complex numbers $v_k \in \mathbb{C}$ for $k \in \mathbb{N}_0$, property (b) of Lemma D.1.1 is equivalent to the absolute convergence of $\sum_{k=0}^{\infty} v_k$ in $\mathbb{C}$. Thanks to Jiří Zeman for noting that.

Lemma D.1.1 (Resummation Lemma)**.** *Let V be a vector space filtered by a decreasing filtration $\mathcal{F}$, and let $\hat{V}$ be its completion.*

(a) Let $v_{ij} \in V$ for all i, $j \in \mathbb{N}_0$ be vectors such that $\sum_{j=0}^{\infty} v_{ij}$ converges for every $i \geq 0$ (i.e., $\|v_{ij}\| \to \infty$ as $j \to \infty$, so that $\sum_{j=0}^{\infty} v_{ij} \in \hat{V}$) and $\|\sum_{j=0}^{\infty} v_{ij}\| \to \infty$ as $i \to \infty$. Then for any bijection $r : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$, the sum $\sum_{i=0}^{\infty} v_{r(i)}$ converges and the limit does not depend on r; we denote it by $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{ij} \in \hat{V}$.[2]

(b) Suppose that $\sum_{k=0}^{\infty} v_k$ for $v_k \in V$ converges, and let $s : Z \subset \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be a bijection. If we define $w_{ij} := v_{s(i,j)}$ for $(i,j) \in Z$ and $w_{ij} := 0$ otherwise, then $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_{ij}$ converges (in the sense of (a)) and equals $\sum_{k=0}^{\infty} v_k$.

(c) Let V and V' be vector spaces filtered by exhaustive filtrations, and let $\sum_{i=0}^{\infty} v_i$ for $v_i \in V$ and $\sum_{i=0}^{\infty} v_i'$ for $v_i' \in V'$ converge. Then for any bijection $r : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$, $i \to (r_1(i), r_2(i))$, the sum $\sum_{i=0}^{\infty} v_{r_1(i)} \otimes v'_{r_2(i)}$ converges and the limit does not depend on r; we denote it by $\sum_{i=0}^{\infty} v_i \otimes \sum_{j=0}^{\infty} v_j' \in V \hat{\otimes} V'$.

Moreover, we have the following statements about maps of filtered vector spaces:

(d) If $f : \hat{V} \to \hat{V}$ is a linear map of finite filtration degree, i.e., $\|f\| > -\infty$, and $\sum_{i=0}^{\infty} v_i$ with $v_i \in V$ a convergent series, then $\sum_{i=0}^{\infty} f(v_i)$ converges to $f(\sum_{i=0}^{\infty} v_i)$.

(e) Given linear maps $f_i : \hat{V} \to \hat{V}$ for $i \in \mathbb{N}_0$, we say that the sum $\sum_{i=0}^{\infty} f_i$ converges if $\sum_{i=0}^{\infty} f_i(v)$ converges for all $v \in \hat{V}$. In this case, $f(v) := \sum_{i=0}^{\infty} f_i(v)$ defines a linear map $\hat{V} \to \hat{V}$ which satisfies $\|f\| \geq \inf_{i \in \mathbb{N}_0} \|f_i\|$.

Proof. (a) Let $r : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ be a bijection. We first prove the convergence of $\sum_{i=0}^{\infty} v_{r(i)}$. Given $K > 0$, let $i_0 \in \mathbb{N}_0$ be such that $\|v_{ij}\| \geq K$ for $(i,j) \in \{i \geq i_0, j \geq 0\}$. Such i_0 exists by the assumption that $\|\sum_{j=0}^{\infty} v_{ij}\| \to \infty$ as $i \to \infty$. Let $j_0 \in \mathbb{N}_0$ be such that $\|v_{ij}\| \geq K$ for $(i,j) \in \{i \leq i_0, j \geq j_0\}$. Such j_0 exists from the convergence of $\sum_{j=0}^{\infty} v_{0j}$, $\dots$, $\sum_{j=0}^{\infty} v_{i_0 j}$. We can now pick $k_0 \in \mathbb{N}$ such that $r(\{k < k_0\}) \supset \{i \leq i_0, j \leq j_0\}$, and hence $\|v_{r(k)}\| \geq K$ for all $k \geq k_0$. This implies the convergence of $\sum_{i=0}^{\infty} v_{r(i)}$.

Let now $r' : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ be another bijection. Given $K > 0$, we find i_0, $j_0 \in \mathbb{N}_0$ so that $\|v_{ij}\| \geq K$ for all $(i,j) \not\in \{i \leq i_0, j \leq j_0\}$ (like in the previous paragraph). There exits $k_0 \in \mathbb{N}$ such that $r(\{k < k_0\}), r'(\{k \leq k_0\}) \supset \{i \leq i_0, j \leq j_0\}$. Hence, for any $k \geq k_0$, the contribution of v_{ij} with $(i,j) \in \{i \leq i_0, j \leq j_0\}$ cancels in $\Delta_k := \sum_{i=0}^{k} v_{r(i)} - \sum_{i=0}^{k} v_{r'(i)}$, and we get $\|\Delta_k\| \geq K$. It follows that the sums are equal in the limit.

---

[2] It is an exercise to prove a similar statement for nested infinite sums $\sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} v_{i_1,\dots,i_n}$.

(b) It is easy to see that w_{ij} satisfy the assumptions of (a). The claim follows from (a) by defining $r \coloneqq s^{-1} \sqcup \mathbb{1} : \mathbb{N}_0 = \mathbb{N}_0 \sqcup (\mathbb{N}_0 \times \mathbb{N}_0 \backslash Z) \to \mathbb{N}_0 \times \mathbb{N}_0$, where the notation $\mathbb{N}_0 = \mathbb{N}_0 \sqcup (\mathbb{N}_0 \times \mathbb{N}_0 \backslash Z)$ means an infinite shuffle permutation. Clearly, $\sum_{i=0}^\infty w_{r(i)}$ is precisely $\sum_{i=0}^\infty v_i$ after crossing out possibly infinitely many zeros.

(c) We start with convergence. Let $K > 0$. Pick indices $i_0 \in \mathbb{N}_0$ and $i_0' \in \mathbb{N}_0$ such that $\|v_i\| \geq K - \inf_j \|v_j'\|$ and $\|v_i'\| \geq K - \inf_j \|v_j\|$ for all $i \geq i_0$ and $i \geq i_0'$, respectively. Note that the right-hand sides of the inequalities are finite because the filtrations are exhaustive. Pick $k_0 \in \mathbb{N}$ such that $r(\{k < k_0\}) \supset \{i \leq i_0, j \leq j_0\}$. Then, for $k \geq k_0$, we have

$$\begin{aligned}\|v_{r_1(k)} \otimes v'_{r_2(k)}\| &\geq \|v_{r_1(k)}\| + \|v'_{r_1(k)}\| \\ &\geq \begin{cases} (K - \inf_j \|v_j'\|) + \inf_j \|v_j'\| = K & \text{if } r_1(k) \geq i_0, \\ \inf_j \|v_j\| + (K - \inf_j \|v_j\|) = K & \text{if } r_2(k) \geq i_0'. \end{cases}\end{aligned}$$

It follows that $\sum_{i=0}^\infty v_{r_1(i)} \otimes v'_{r_2(i)}$ converges.

If $r' : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ is another bijection, we write $r' = r \circ s$ for a bijection $s : \mathbb{N}_0 \to \mathbb{N}_0$ and apply (b) to $V \otimes V'$ with the bijection $s : Z \coloneqq \mathbb{N}_0 \times \{0\} \simeq \mathbb{N}_0 \to \mathbb{N}_0$. It follows that $\sum_{i=0}^\infty v_{r(i)} = \sum_{i=0}^\infty v_{r'(i)}$.

(d) This is clear because $f(\sum_{i=0}^\infty v_i) = \sum_{i=0}^n f(v_i) + f(\sum_{i=n+1}^\infty v_i)$ for any $n \in \mathbb{N}_0$ and $\|f(\sum_{i=n+1}^\infty v_i)\| \geq \|f\| + \inf_{i>n} \|v_i\| \to \infty$ as $n \to \infty$.

(e) Linearity is clear. The inequality $\|f\| \geq \inf_i \|f_i\|$ follows immediately from $\|f(v)\| \geq \inf_i \|f_i(v)\| \geq \inf_i \|f_i\| + \|v\|$ for all $v \in \hat{V}$. ☐

Note that (a) and (c) of the Resummation Lemma are, in fact, reformulations of the facts that the canonical inclusions induce the isomorphisms $\hat{V} \simeq \hat{\hat{V}}$ and $V \hat{\otimes} V \simeq \hat{V} \hat{\otimes} \hat{V}$, respectively.

We will work with (complete) filtered algebras, coalgebras and bialgebras, which we now define schematically. They are basically algebras over the corresponding (pr)operad in the category of (complete) filtered vector spaces.

Definition D.1.2 ((Complete) filtered algebras, coalgebras, bialgebras)**.** *A* filtered algebra, coalgebra or bialgebra *is a filtered vector space V with linear operations $\mu_{klg} : V^{\otimes k} \to V^{\otimes l}$ of non-negative filtration degree for k, l, $g \in \mathbb{N}_0$ such that $(V, (\mu_{klg}))$ is an algebra, coalgebra or bialgebra, respectively.*

A complete filtered algebra, coalgebra or bialgebra *is a complete filtered vector space V with linear operations $\mu_{klg} : V^{\hat{\otimes} k} \to V^{\hat{\otimes} l}$ of non-negative filtration degree for k, l, $g \in \mathbb{N}_0$ such that μ_{klg} satisfy the relations of an algebra, coalgebra or bialgebra with $\otimes$ replaced by $\hat{\otimes}$, respectively.*

A morphism of (complete) filtered algebras, coalgebras or bialgebras is a linear map $f : V_1 \to V_2$ with non-negative filtration degree intertwining μ_{klg}.

Remark D.1.3 (On (complete) filtered algebras). (i) Clearly, completion is a functor from filtered algebras to complete filtered algebras. In deformation theory, we may deal with complete filtered algebras which are not in the image of this functor.

(ii) The IBL_∞-algebra on C from Definition 3.2.4 for $\gamma = 0$ is, in fact, a "complete filtered IBL_∞-algebra on $\hat{C}$" in the sense of Definition D.1.2 plus the data of the filtered vector space C. It might not be the completion of a "filtered IBL_∞-algebra on C"; in fact, we called an IBL_∞-algebra "completion-free" if the operations on $\hat{C}$ arose as continuous extensions of operations on C. Definition 3.2.4 seemed natural when we were trying to compute examples of the twisted structure, hoping that we do not encounter infinite sums. After dealing with completions, however, Definition D.1.2 is more logical as an abstract definition.

Notice that for $\gamma > 0$, the norm $\|\mathfrak{q}_{klg}\| \geq \gamma(2 - 2g - k - l)$ is allowed to be negative; in particular, $\|\mathfrak{q}_{210}\| \geq -\gamma$ and $\|\mathfrak{q}_{120}\| \geq -\gamma$. Therefore, one has to allow finite filtration degree in Definition D.1.2 in order to accommodate it to IBL_∞-algebras with $\gamma \geq 0$.

$\lhd$

Remark D.1.4 (Units and augmentations). The ground field $\mathbb{K}$ is filtered by the trivial complete filtration $\mathcal{F}^{\lambda \leq 0}\mathbb{K} = \mathbb{K}$ and $\mathcal{F}^{\lambda > 0}\mathbb{K} = 0$ (see also (3.9)). Suppose that $\eta : \mathbb{K} \to V$ is a unit and $\varepsilon : V \to \mathbb{K}$ an augmentation of a complete filtered algebra V; in particular, we have $\varepsilon \circ \eta = \mathbb{1}$ and $\|\eta\|$, $\|\varepsilon\| \geq 0$. From this, we deduce the following implications for $v \in V$:

$$\begin{aligned} \|v\| > 0 \quad &\Longrightarrow \quad v \in \ker \varepsilon =: \bar{V}, \\ v \in \operatorname{im} \eta \quad &\Longrightarrow \quad \|v\| = 0. \end{aligned}$$

Recall that $V = \operatorname{im} \eta \oplus \bar{V}$. $\lhd$

Given a vector space U over $\mathbb{K}$, we will work with the *symmetric bialgebra* $\mathrm{S}U = \bigoplus_{k=0}^{\infty} \mathrm{S}_k U$ from Definition 3.1.7. It has the concatenation product $\mu : \mathrm{S}U \otimes \mathrm{S}U \to \mathrm{S}U$ and the shuffle coproduct $\delta : \mathrm{S}U \to \mathrm{S}U \otimes \mathrm{S}U$ which are given for all u_{ij}, $u_i \in U$ and k,

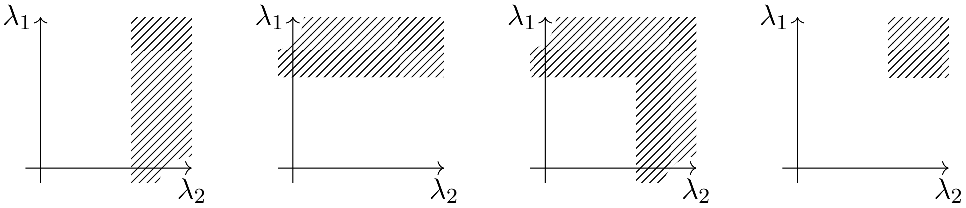

Figure D.1.: For a bigraded vector space $V = \bigoplus_i \bigoplus_j V_{ij}$, let $\mathcal{F}_{\mathrm{v}}^{\lambda_1} := \bigoplus_{i\ge\lambda_1}\bigoplus_j V_{ij}$ be the vertical and $\mathcal{F}_{\mathrm{h}}^{\lambda_2} := \bigoplus_i \bigoplus_{j\ge\lambda_2} V_{ij}$ the horizontal filtration. The illustrations above depict $\mathcal{F}_{\mathrm{v}}$, $\mathcal{F}_{\mathrm{h}}$, $\mathcal{F}_{\mathrm{v}}\cup\mathcal{F}_{\mathrm{h}}$ and $\mathcal{F}_{\mathrm{v}}\cap\mathcal{F}_{\mathrm{h}}$, respectively. We imagine the component V_{ij} sitting at the position (i,j) in the plane. Given $v_{ij}\in V_{ij}$, the sum $\sum_{i,j} v_{ij}$ converges with respect to the given filtration if and only if for every λ, only finitely many v_{ij}'s in the white region are non-zero.

$k'\in\mathbb{N}_0$ by

$$\mu(u_{11}\cdots u_{1k}\otimes u_{21}\cdots u_{2k'}) = u_{11}\cdots u_{1k}u_{21}\cdots u_{2k'} \quad \text{and}$$
$$\delta(u_1\cdots u_k) = \sum_{\substack{k_1,k_2\ge 0\\ k_1+k_2=k}} \sum_{\sigma\in\mathbb{S}_{k_1,k_2}} \varepsilon(\sigma,u) u_{\sigma_1^{-1}}\cdots u_{\sigma_{k_1}^{-1}}\otimes u_{\sigma_{k_1+1}^{-1}}\cdots u_{\sigma_{k_1+k_2}^{-1}},$$

respectively. Here $\mathbb{S}_{k_1,k_2}$ denotes the shuffle permutations and $\varepsilon(\sigma,u)$ the Koszul sign. The unit $\eta:\mathbb{K}\to \mathrm{S}U$ is given by $\eta(1)=1\in\mathrm{S}_0U=\mathbb{K}$, and the augmentation $\varepsilon:\mathrm{S}U\to\mathbb{K}$ is determined by its reduced algebra $\bar{\mathrm{S}}U=\bigoplus_{k=1}^\infty \mathrm{S}_kU$. If U is filtered, then μ, δ, η and ε preserve the induced filtration on $\mathrm{S}U$ (see Definition 3.1.8), and hence $\mathrm{S}U$ is a filtered bialgebra. We will also use the canonical projection $\pi_k:\mathrm{S}U\to\mathrm{S}_kU$ and the canonical inclusion $\iota_l:\mathrm{S}_lU\to\mathrm{S}U$ for $k\in\mathbb{N}_0$ and $l\in\mathbb{N}_0$. They too preserve the filtration.

The symmetric bialgebra of a filtered vector space has naturally two filtrations — the induced filtration and the filtration by weights $\mathcal{F}_{\mathrm{w}}^\lambda\mathrm{S}U := \bigoplus_{k\ge\lambda}\mathrm{S}_kU$.[3] In general, having two filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$ on a vector space V, we define the filtrations

$$\mathcal{F}_\cup^\lambda := \mathcal{F}_1^\lambda + \mathcal{F}_2^\lambda \quad \text{and} \quad \mathcal{F}_\cap^\lambda := \mathcal{F}_1^\lambda\cap\mathcal{F}_2^\lambda \tag{D.1}$$

and call them the *combined filtrations* — the union and the intersection. It is easy to see that

$$\|\cdot\|_\cup \ge \max(\|\cdot\|_1,\|\cdot\|_2) \quad \text{and} \quad \|\cdot\|_\cap\le\min(\|\cdot\|_1,\|\cdot\|_2)$$

hold for the corresponding filtration degrees.

---

[3] Note that the filtration by weights can be also viewed as the induced filtration from the filtration on U defined by $\mathcal{F}^{\le 1}U := U$, $\mathcal{F}^{>1}U := 0$.

The following example shows that $\mathrm{S}U$ might not be a filtered bialgebra with respect to combined filtrations.

Example D.1.5 (Combined filtrations of symmetric bialgebra $\mathrm{S}U$)*.* Let U be a vector space filtered by a filtration $\mathcal{F}$. Consider the symmetric bialgebra $\mathrm{S}U$ with the induced filtration $\mathcal{F}$ and the filtration by weights $\mathcal{F}_{\mathrm{w}}^{\lambda}$. It is easy to see that the operations μ, δ, η and ε preserve both $\mathcal{F}$ and $\mathcal{F}_{\mathrm{w}}$. Consider the combined filtrations $\mathcal{F}_{\cup}$ and $\mathcal{F}_{\cap}$. Because

$$\begin{aligned} \min(\lambda_1+\lambda_2, k_1+k_2) &\geq \min(\lambda_1, k_1) + \min(\lambda_2, k_2) \quad \text{and} \\ \max(\lambda_1+\lambda_2, k_1+k_2) &\leq \max(\lambda_1, k_1) + \max(\lambda_2, k_2), \end{aligned}$$

it holds $\|\mu\|_{\cap} \geq 0$ and $\|\delta\|_{\cup} \geq 0$, respectively.

We now demonstrate that it might happen that $\|\mu\|_{\cup} = -\infty$ and $\|\delta\|_{\cap} = -\infty$. Let $U = \mathrm{T}W$ be the tensor algebra over a non-zero vector space W. On U, consider the filtration by weights $\mathcal{F}^{\lambda}U = \bigoplus_{k\geq\lambda} W^{\otimes k}$. Let $0 \neq w \in W$, and define

$$v_k \coloneqq \underbrace{w \otimes \cdots \otimes w}_{k\text{-times}} \in \mathcal{F}^k \mathrm{S}_1 U \quad \text{for all } k \in \mathbb{N},$$

where $\otimes$ denotes the tensor product on $\mathrm{T}W$. For $0 \neq u \in \mathrm{T}_0 W = \mathbb{K}$, define

$$v_k' \coloneqq \underbrace{u \cdots u}_{k\text{-times}} \in \mathcal{F}^0 \mathrm{S}_k U \quad \text{for all } k \in \mathbb{N},$$

where $\cdot$ denotes the concatenation product on $\mathrm{S}U$. It follows that

$$\|v_k\|_{\cup} = k, \quad \|v_k'\|_{\cup} = k \quad \text{and} \quad \|v_k v_k'\|_{\cup} = k+1.$$

Therefore, for all $k \in \mathbb{N}$, it holds

$$\begin{aligned} \|\mu\|_{\cup} &\leq \|v_k v_k'\|_{\cup} - \|v_k \otimes v_k'\|_{\cup} \\ &\leq k+1 - \|v_k\|_{\cup} - \|v_k'\|_{\cup} \\ &= 1-k. \end{aligned}$$

Consequently $\|\mu\|_{\cup} = -\infty$. Next, it holds

$$\|v_k v_k'\|_{\cap} = k \quad \text{and} \quad \|v_k \otimes v_k'\|_{\cap} = 1,$$

and we compute

$$\begin{aligned} \|\delta\|_\cap &\leq \|\delta(v_k v_k')\|_\cap - \|v_k v_k'\|_\cap \\ &= 1 - k. \end{aligned}$$

We used here that the summands of $\delta(v_k v_k')$ are tensor products of $v_1'' = u^i v_k \in \mathcal{F}^1\mathrm{S}_{i+1}U$ and $v_2'' = u^{k-i} \in \mathcal{F}^0\mathrm{S}_{k-i}U$ for $i = 0, \ldots, k$, so that $\|v_1'' \otimes v_2''\|_\cap = 1$. It follows that $\|\delta\|_\cap = -\infty$. A heuristic explanation of these phenomenons is that the tensor product of combined filtrations is not the combined filtration of the tensor product.

In the next paragraph, we will argue that it might still be possible to extend μ to $\mathrm{S}U \mathbin{\hat{\otimes}_\cup} \mathrm{S}U$ — the completion of $\mathrm{S}U \otimes \mathrm{S}U$ with respect to the tensor product of the union filtrations — even though $\|\mu\| = -\infty$. A similar discussion applies for δ.

Suppose that the filtration on U is bounded from above. Let $v_i \in \mathcal{F}^{\lambda_i}\mathrm{S}_{k_i}U$ and $v_i' \in \mathcal{F}^{\lambda_i'}\mathrm{S}_{k_i'}U$ for all $i \in \mathbb{N}_0$ be such that $\max(\lambda_i, k_i) + \max(\lambda_i', k_i') \to \infty$ as $i \to \infty$, so that $\sum_{i=0}^\infty v_i \otimes v_i'$ converges in $\mathrm{S}U \mathbin{\hat{\otimes}_\cup} \mathrm{S}U$ (in fact, any element of $\mathrm{S}U \mathbin{\hat{\otimes}_\cup} \mathrm{S}U$ can be written in this way). We have $\mu(v_i, v_i') \in \mathcal{F}^{\lambda_i + \lambda_i'}\mathrm{S}_{k_i + k_i'}U$ for all $i \in \mathbb{N}_0$. Suppose that $\sum_{i=0}^\infty v_i v_i'$ does not converge with respect to $\mathcal{F}_\cup$. Thus, $\max(\lambda_i + \lambda_i', k_i + k_i')$ is bounded, hence $\lambda_i + \lambda_i'$ and $k_i + k_i'$ are bounded, and since k_i, $k_i' \geq 0$, also k_i and k_i' are bounded. Nevertheless, in order to comply with the assumption on the convergence of $\sum_{i=0}^\infty v_i \otimes v_i'$, we need one of λ_i or λ_i', let's say λ_i, to diverge to ∞. Because $\lambda_i + \lambda_i'$ is bounded, λ_i' has to diverge to $-\infty$. But this is not allowed by the boundedness assumption. ◁

It turns out that the union filtration is often identical with the induced filtration.

Lemma D.1.6 (Combined filtration and boundedness condition)**.** *Let U be a vector space filtered by a decreasing filtration $\mathcal{F}$. Suppose that there is $\gamma > 0$ such that*

$$\mathcal{F}^\gamma U = U. \tag{D.2}$$

Then the following holds:

$$\begin{aligned} \gamma \geq 1 \quad &\Longrightarrow \quad \forall \lambda \in \mathbb{R}: \ \mathcal{F}_\cup^\lambda \mathrm{S}U = \mathcal{F}^\lambda \mathrm{S}U, \\ 0 < \gamma < 1 \quad &\Longrightarrow \quad \forall v \in \mathrm{S}U: \ \|v\|_\cup \geq \|v\| \geq \gamma \|v\|_\cup. \end{aligned}$$

Proof. Under the assumption (D.2), we have for any $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ the following:

$$\lambda \leq k\gamma \quad \Longrightarrow \quad \mathcal{F}^\lambda \mathrm{S}_k U = \sum_{\lambda_1 + \cdots + \lambda_k = \lambda} \mathcal{F}^{\lambda_1} U \otimes \cdots \otimes \mathcal{F}^{\lambda_k} U \Big/ \mathbb{S}_k = \mathrm{S}_k U. \tag{D.3}$$

We compute

$$\begin{aligned}\mathcal{F}^{\lambda}_{\cup}\mathrm{S}U &= \bigoplus_{k<\lambda}\mathcal{F}^{\lambda}\mathrm{S}_kU \oplus \bigoplus_{k\geq\lambda}\mathrm{S}_kU\\ &= \begin{cases}\displaystyle\bigoplus_{k=0}^{\infty}\mathcal{F}^{\lambda}\mathrm{S}_kU = \mathcal{F}^{\lambda}\mathrm{S}U & \text{for } \gamma\geq 1,\\ \displaystyle\bigoplus_{k<\lambda \text{ or } k\geq\frac{\lambda}{\gamma}}\mathcal{F}^{\lambda}\mathrm{S}_kU \oplus \bigoplus_{\lambda\leq k<\frac{\lambda}{\gamma}}\mathrm{S}_kU \subset \mathcal{F}^{\lambda\gamma}\mathrm{S}U & \text{for } 0<\gamma<1.\end{cases}\end{aligned}$$

The first case holds because if $\gamma \geq 1$, then $k \geq \lambda$ implies $\gamma k \geq \lambda$, and thus $\mathrm{S}_kU = \mathcal{F}^{\lambda}\mathrm{S}_kU$ by (D.3). The second case holds from the following reasons. If $0 < \gamma < 1$ and $k \geq \frac{\lambda}{\gamma}$, so that $\mathrm{S}_kU = \mathcal{F}^{\lambda}\mathrm{S}_kU$ by (D.3), then $k > \lambda$. Now, $\mathcal{F}^{\lambda}\mathrm{S}_kU \subset \mathcal{F}^{\lambda\gamma}\mathrm{S}_kU$ because the filtration is decreasing and because $\lambda \geq \lambda\gamma$, and if $v \in \mathrm{S}_kU$ with $\lambda \leq k < \frac{\lambda}{\gamma}$, then $k\gamma \geq \lambda\gamma$, and hence $\mathrm{S}_kU = \mathcal{F}^{\lambda\gamma}\mathrm{S}_kU$ by (D.3). The claim follows. □

We consider the exponential and the logarithm on filtered algebras.

Lemma D.1.7 (Exponential and logarithm on filtered algebras)**.** *Let V be a complete filtered associative algebra with unit* 1*. Given $v \in V$ with $\|v\| > 0$, we define the exponential*

$$e^v := 1 + v + \frac{1}{2!}v^2 + \frac{1}{3!}v^3 + \cdots \in V.$$

It holds $e^v e^{-v} = 1$. Given $v \in V$ with $\|v - 1\| > 0$, we define the logarithm

$$\log(v) = \sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r}(v-1)^r \in V.$$

It holds $\log(e^v) = v$ for $v \in V$ with $\|v\| > 0$ and $e^{\log v} = v$ for $v \in V$ with $\|v - 1\| > 0$.

Proof. For any $v \in V$, we have

$$\begin{aligned}e^{-v}e^{v} &= e^{-v}\Big(\sum_{i=0}^{\infty}\frac{1}{i!}v^i\Big)\\ &= \sum_{i=0}^{\infty}\frac{1}{i!}e^{-v}v^i\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}(-1)^j v^{i+j}\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\sum_{k=0}^{n}(-1)^k\binom{n}{k}}_{=0 \text{ for } n>0}v^n.\end{aligned}$$

On the second line, we used that the multiplication with e^{-v} from the left has finite filtration degree, and hence it commutes with infinite sums; on the third line, we used that the multiplication with v^i from the right has finite filtration degree; on the fourth line, we used the Resummation Lemma (Lemma D.1.1). The facts about log are proven similarly, the equations reducing to known combinatorial identities. $\square$

Let (V, δ) be a coalgebra and (V', μ') an algebra. The *convolution product* $*$ on $\mathbb{K}$-linear maps $f_1,\ f_2 : V \to V'$ is defined by

$$f_1 * f_2 := \mu' \circ (f_1 \otimes f_2) \circ \delta.$$

See also [LV12, Section 1.6]. If $\varepsilon : V \to \mathbb{K}$ is a counit for V and $\eta' : \mathbb{K} \to V'$ a unit for V', then

$$\mathrm{e} := \eta' \circ \varepsilon : V \to V' \tag{D.4}$$

is the unit for $*$. The convolution product is associative and commutative provided that the algebra and coalgebra are.

Let us recall the conilpotency property of a coaugmented counital coassociative coalgebra $(V, \delta, \varepsilon, \eta)$, where $\varepsilon : V \to \mathbb{K}$ is the counit and $\eta : \mathbb{K} \to V$ the coaugmentation. We write

$$\delta = \bar{\delta} + \delta_0,$$

where $\delta_0 : V \to V \otimes V$ is defined by

$$\delta_0(v) := \begin{cases} 1 \otimes 1 & \text{if } v = 1, \\ 1 \otimes v + v \otimes 1 & \text{if } v \in \bar{V}, \end{cases} \tag{D.5}$$

with respect to the decomposition $V = \langle 1 \rangle \oplus \bar{V}$. The other map is then defined simply as the difference

$$\bar{\delta} := \delta - \delta_0 : V \longrightarrow V \otimes V.$$

We call δ_0 the *trivial coproduct* and $\bar{\delta}$ the *reduced diagonal.* They are both coassociative, and δ_0 is even cocommutative. The *conilpotency property* reads: for every $v \in V$, there exists an $n \in \mathbb{N}$ such that $\bar{\delta}^{(n)}(v) = 0$, where $\bar{\delta}^{(n)} := (\bar{\delta} \otimes \mathbb{1}^{\otimes n-2}) \circ \cdots \circ \bar{\delta}$ is the iterated reduced diagonal with n outputs.

Using the conilpotency property, it is possible to weaken the condition $\|f\| > 0$ on the convergence of a power series in $f \in \mathrm{Hom}(V, V')$ in the convolution algebra $(\mathrm{Hom}(V, V'), *, \mathrm{e})$ to $\|f\| \geq 0$ and $\|f(1)\| > 0$. The latter condition can not be weakened because $e^f(1) = e^{f(1)}$ is the exponential in (V', μ', η').

Proposition D.1.8 (Power series in $*$ with coefficients in $\mathbb{K}$)**.** *Let $(V, \delta, \varepsilon, \eta)$ be a complete filtered coaugmented counital cocommutative coassociative coalgebra such that the following* limit conilpotency property *holds:*

$$\bar{\delta}^{(n)}(v) \to 0 \quad \textit{as} \quad n \to \infty \quad \textit{for all } v \in V. \tag{D.6}$$

Let $(V', \mu', \eta', \varepsilon')$ be a complete filtered augmented unital commutative associative algebra. Let R be a complete filtered $\mathbb{K}$-algebra. Then the following holds:

(a) The convolution product $*$ extends naturally to R-linear maps $\mathfrak{f}_i : V \hat{\otimes} R \to V' \hat{\otimes} R$ ($\otimes = \otimes_{\mathbb{K}}$) of finite filtration degrees, and for its iterations, we have

$$\mathfrak{f}_1 * \cdots * \mathfrak{f}_n = {\mu'}^{(n)} \circ (\mathfrak{f}_1 \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n) \circ \delta^{(n)}, \tag{D.7}$$

where ${\mu'}^{(n)}$ and $\delta^{(n)}$ are the continuous R-linear extensions of the iterated product and coproduct, respectively. (Recall that the continuous extension of a linear map f of finite filtration degree on V to the completion $\hat{V}$ is defined by $f(\sum_{i=0}^{\infty} v_i) := \sum_{i=0}^{\infty} f(v_i)$ for all $v_i \in V$ with $\|v_i\| \to \infty$.)

(b) Let $\mathfrak{f} : V \hat{\otimes} R \to V' \hat{\otimes} R$ be an R-linear map of finite filtration degree such that

$$\|\mathfrak{f}\| \geq 0 \quad \textit{and} \quad \|\mathfrak{f}(1)\| > 0. \tag{D.8}$$

Then any power series $\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k}$ with coefficients $\alpha_k \in \mathbb{K}$, where $\mathfrak{f}^{*0} := \mathrm{e}$ is the unit of the convolution product, converges to an R-linear map $V \hat{\otimes} R \to V' \hat{\otimes} R$ of non-negative filtration degree, and it holds

$$(\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k})(1) = \sum_{k=0}^{\infty} \alpha_k \mathfrak{f}(1)^k.$$

Proof. (a) Given a $\mathbb{K}$-linear map $u : V^{\hat{\otimes} k} \to \hat{V}^{\otimes l}$ of finite filtration degree for some k, $l \geq 0$, we consider its R-linear extension $u \otimes \mathbb{1} : V^{\hat{\otimes} k} \otimes R \to V^{\hat{\otimes} l} \otimes R$ and extend it continuously to $V^{\hat{\otimes} k} \hat{\otimes} R \to V^{\hat{\otimes} l} \hat{\otimes} R$. Because $V^{\hat{\otimes} n} \hat{\otimes} R \simeq (V \hat{\otimes} R)^{\hat{\otimes}_R n}$ via canonical maps (this can be proven explicitly using Proposition 3.1.11), we get the desired continuous R-linear extension $(V \hat{\otimes} R)^{\hat{\otimes}_R k} \to (V \hat{\otimes} R)^{\hat{\otimes}_R l}$. We apply this construction to $u = {\mu'}^{(n)}$, $\delta^{(n)}$ and define $\mathfrak{f}_1 * \cdots * \mathfrak{f}_n$ using (D.7) for all $n \geq 2$. It is then easy to see that

$$\begin{aligned}
&(\cdots((\mathfrak{f}_1 * \mathfrak{f}_2) * \mathfrak{f}_3) * \cdots) * \mathfrak{f}_n \\
&\quad = \mu'(\mu'(\cdots \mu'(\mathfrak{f}_1 \hat{\otimes}_R \mathfrak{f}_2)\delta \hat{\otimes}_R \mathfrak{f}_3)\delta \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n)\delta
\end{aligned}$$

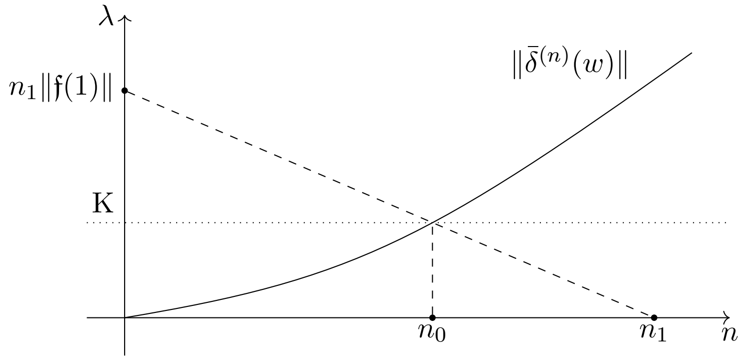

Figure D.2.: Given $K > 0$, pick $n_0 \in \mathbb{N}$ such that $\|\bar{\delta}^{(n)}(w)\| \geq K$ for $n \geq n_0$. Find $n_1 \geq n_0$ such that $n_1\|\mathfrak{f}(1)\| \geq K + n_0\|\mathfrak{f}(1)\|$. Then $(n-i)\|\mathfrak{f}(1)\| + \|\bar{\delta}^{(i)}(w)\| \geq K$ for $i = 0, \ldots, n$ for any $n \geq n_1$.

$$= \underbrace{\mu'(\mu' \mathbin{\hat{\otimes}_R} \mathbb{1}) \cdots (\mu' \mathbin{\hat{\otimes}_R} \mathbb{1}^{n-2})}_{=\mu'^{(n)}}(\mathfrak{f}_1 \mathbin{\hat{\otimes}_R} \cdots \mathbin{\hat{\otimes}_R} \mathfrak{f}_n) \underbrace{(\delta \mathbin{\hat{\otimes}_R} \mathbb{1}^{n-2}) \cdots (\delta \mathbin{\hat{\otimes}_R} \mathbb{1})\delta}_{\delta^{(n)}}$$

$$= \mathfrak{f}_1 * \cdots * \mathfrak{f}_n.$$

(b) Let $w \in V \hat{\otimes} R$ and $n \geq 2$. For the map $\delta^{(n)} : V \hat{\otimes} R \to (V \hat{\otimes} R)^{\hat{\otimes}_R n}$, we write

$$\delta^{(n)}(w) = \sum_{i=0}^{n} \sum_{(n-i)\times 1} \sum_{\bar{\delta}^{(i)}} w_{(1)} \mathbin{\hat{\otimes}_R} 1 \mathbin{\hat{\otimes}_R} 1 \mathbin{\hat{\otimes}_R} w_{(2)} \mathbin{\hat{\otimes}_R} 1 \mathbin{\hat{\otimes}_R} \cdots \mathbin{\hat{\otimes}_R} w_{(i)} \mathbin{\hat{\otimes}_R} 1, \tag{D.9}$$

where the second sum is over all insertions of $n-i$ units 1 into the tensor product and the third sum denotes the sum from the Sweedler's notation

$$\bar{\delta}^{(i)}(w) = \sum w_{(1)} \mathbin{\hat{\otimes}_R} \cdots \mathbin{\hat{\otimes}_R} w_{(i)}.$$

Formula (D.9) is easy to see from $\delta = \delta_0 + \bar{\delta}$.

Before we continue, let us check that $\bar{\delta} : V \hat{\otimes} R \to (V \hat{\otimes} R)^{\hat{\otimes}_R 2}$ also satisfies the limit conilpotency property (the proof is similar to the proof of Lemma D.1.9 below). Every element of $V \hat{\otimes} R$ can be written as $\sum_{i=0}^{\infty} r_i v_i$ for $r_i \in R$ and $v_i \in V$ such that $\|r_i v_i\| \to \infty$ as $i \to \infty$; this follows from the Resummation Lemma (Lemma D.1.1). Given $K > 0$, we pick $i_0 \in \mathbb{N}$ such that $\|r_i v_i\| \geq K$ for all $i \geq i_0$. We pick $n_0 \in \mathbb{N}$ such that $\|\bar{\delta}^{(n)}(v_0)\| + \|r_0\|, \ldots, \|\bar{\delta}^{(n)}(v_{i_0-1})\| + \|r_{i_0-1}\| \geq K$ for all $n \geq n_0$. According to the construction in (a), we have

$$\bar{\delta}^{(n)}(\sum_{i=0}^{\infty} r_i v_i) = \sum_{i=0}^{\infty} r_i \bar{\delta}^{(n)}(v_i).$$

For $n \geq n_0$, it holds $\|r_i\bar{\delta}^{(n)}(v_i)\| \geq \|r_i\| + \|\bar{\delta}^{(n)}(v_i)\| \geq K$ for $i < i_0$ and $\|r_i\bar{\delta}^{(n)}(v_i)\| \geq \|r_i v_i\| \geq K$ for $i \geq i_0$. It follows that $\bar{\delta}^{(n)}(\sum_{i=0}^{\infty} r_i v_i) \to 0$ as $n \to \infty$.

Going on with the main proof, using (D.7) and (D.9), we have

$$\begin{aligned}\|\mathfrak{f}^{*n}(w)\| &= \|\mu^{(n)}(\mathfrak{f} \mathbin{\hat{\otimes}_R} \cdots \mathbin{\hat{\otimes}_R} \mathfrak{f})\delta^{(n)}(w)\| \\ &= \|\sum_{i=0}^{n} \sum_{(n-i)\times 1} \sum_{\bar{\delta}^{(i)}} \mathfrak{f}(w_{n,(1)}) \mathbin{\hat{\otimes}_R} \mathfrak{f}(1) \mathbin{\hat{\otimes}_R} \mathfrak{f}(1) \mathbin{\hat{\otimes}_R} \mathfrak{f}(w_{n,(2)}) \mathbin{\hat{\otimes}_R} \mathfrak{f}(1) \mathbin{\hat{\otimes}_R} \cdots \\ &\qquad\qquad \hat{\otimes}_R\, \mathfrak{f}(w_{n,(n-i)}) \mathbin{\hat{\otimes}_R} \mathfrak{f}(1)\| \\ &\geq \min_{i=0,\dots,n} \left((n-i)\|\mathfrak{f}(1)\| + i\|\mathfrak{f}\| + \|\bar{\delta}^{(i)}(w)\|\right) \\ &\geq \underbrace{\min_{i=0,\dots,n} \left((n-i)\|\mathfrak{f}(1)\| + \|\bar{\delta}^{(i)}(w)\|\right)}_{=:(*)_n}.\end{aligned}$$

Let $K > 0$ be arbitrary. Because $\|\mathfrak{f}\| \geq 0$, $\|\mathfrak{f}(1)\| > 0$ and $\bar{\delta}^i(w) \to 0$ as $i \to \infty$ by the assumptions, we obtain an $n_1 \in \mathbb{N}$ such that $(*)_n \geq K$ for all $n \geq n_1$ (see Figure D.2). This implies the convergence of $\sum_{n=0}^{\infty} \alpha_n \mathfrak{f}^{*n}(w)$ in $V \mathbin{\hat{\otimes}} R$. Using (e) of the Resummation Lemma, we get a well-defined $\mathbb{K}$-linear map $\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k} : V \mathbin{\hat{\otimes}} R \to V \mathbin{\hat{\otimes}} R$ with $\| \sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k}\| \geq \inf_{k=0,\dots,\infty} \|\mathfrak{f}^{*k}\| \geq 0$. It is easy to see that it is R-linear as well. □

Lemma D.1.9 (Completion of conilpotent coalgebra satisfies limit conilpotency)**.** *Let $(V, \delta, \varepsilon, \eta)$ be a filtered coaugmented counital coassociative coalgebra which is conilpotent. Then the completion $(\hat{V}, \delta, \varepsilon, \eta)$ is a complete filtered coaugmented counital coassociative coalgebra which satisfies the limit conilpotency property* (D.6)*.*

Proof. Let $v \in \hat{V}$, and write $v = \sum_{i=0}^{\infty} v_i$ for $v_i \in V$ with $\|v_i\| \to \infty$. Given $K > 0$, find $i_0 \in \mathbb{N}$ such that $\|v_i\| \geq K$ for all $i \geq i_0$. From the conilpotency, we can find $n_0 \in \mathbb{N}$ such that $\bar{\delta}^{(n)}(v_0) = \ldots = \|\bar{\delta}^{(n)}(v_{i_0-1})\| = 0$ for all $n \geq n_0$. Because $\bar{\delta}$ has finite filtration degree, $\bar{\delta}^{(n)}$ has finite filtration degree too, and so we can permute it with the infinite sum and write

$$\bar{\delta}^{(n)}(v) = \sum_{i=0}^{\infty} \bar{\delta}^{(n)}(v_i) = \sum_{i=i_0}^{\infty} \bar{\delta}^{(n)}(v_i).$$

Because $\|\bar{\delta}\| \geq 0$, we have $\|\bar{\delta}^{(n)}\| \geq 0$, and hence

$$\|\bar{\delta}^{(n)}(v)\| \geq \inf_{i=i_0,\ldots,\infty} \|\bar{\delta}^{(n)}(v_i)\| \geq \inf_{i=i_0,\ldots,\infty} \|v_i\| \geq K$$

for all $n \geq n_0$. This shows the limit conilpotency property. $\square$

Remark D.1.10 (Exponential and logarithm in $*$). (i) It is obvious that Lemma D.1.7 holds for $*$ with the weakened conditions (D.8). To sum up, the conditions for the existence of $\exp(\mathfrak{f})$ are $\|\mathfrak{f}\| \geq 0$ and $\|\mathfrak{f}(1)\| > 0$, and the conditions for the existence of $\log(\mathfrak{f})$ are $\|\mathfrak{f}\| \geq 0$ and $\|\mathfrak{f}(1) - 1\| > 0$.

As for $\log(\mathfrak{f})$, we will explain how $\|\mathfrak{f}\| \geq 0$ and $\|\mathfrak{f}(1)\| > 0$ together imply $\|\mathfrak{f} - \mathrm{e}\| \geq 0$. Let $\lambda < 0$, $\mu \in \mathbb{R}$ and $v \in \mathcal{F}_\mu(V \hat{\otimes} R)$ be arbitrary. If v is not a multiple of 1, then $(\mathfrak{f} - \mathrm{e})(v) = \mathfrak{f}(v) \in \mathcal{F}_{\mu+\lambda}$ as $\lambda < \|\mathfrak{f}\|$. If $v = \tau$ for $0 \neq \tau \in \mathbb{K}$, then it must hold $\mu \leq 0$, and further because of $\|\mathfrak{f}(1) - 1\| > 0$, there is an $\varepsilon > 0$ such that $(\mathfrak{f} - \mathrm{e})(\tau) \in \mathcal{F}_\varepsilon \subset \mathcal{F}_{\mu+\lambda}$. It follows that $\|\mathfrak{f} - \mathrm{e}\| \geq 0$.

(ii) Given $\mathfrak{f} : V \hat{\otimes} R \to V \hat{\otimes} R$ satisfying (D.8), it holds $e^{-\mathfrak{f}} * e^{\mathfrak{f}} = \mathrm{e}$ by Lemma D.1.7 because e is the unit for the convolution product. However, there might not be any $\mathfrak{g} : V \hat{\otimes} R \to V \hat{\otimes} R$ such that $\mathfrak{g} \circ e^{\mathfrak{f}} = \mathbb{1}$ (or $e^{\mathfrak{f}} \circ \mathfrak{g} = \mathbb{1}$). To see this, let $V = \hat{\mathrm{S}}U$, $R = \mathbb{K}$, and let $\mathfrak{f}_{110} : U \to U$ be a linear map with $\|\mathfrak{f}_{110}\| \geq 0$ such that there is $0 \neq v \in U$ with $\mathfrak{f}_{110}(v) = 0$. Consider the trivial extension $\mathfrak{f} \coloneqq \mathfrak{f}_{110} : \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ (it equals $\mathfrak{f}_{110}$ on $\hat{\mathrm{S}}_1U \to \hat{\mathrm{S}}_1U$ and 0 otherwise). Because

$$\mathfrak{f}^{*n}(\underbrace{v \cdots v}_{k\text{-times}}) = \begin{cases} n!\mathfrak{f}_{110}(v)^n & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

for all k, $n \in \mathbb{N}_0$, it holds $e^{\mathfrak{f}}(v) = \mathfrak{f}_{110}(v) = 0$, and so $e^{\mathfrak{f}} : \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ is not invertible.

(iii) In the case of $V = \hat{\mathrm{S}}U$, one can check that $\log(\mathbb{1})$ is the trivial extension of $\mathbb{1}_{110} : \hat{U} \to \hat{U}$ (see also [MV17, Example 21] in the non-filtered setting). $\triangleleft$

D.2. Filtered IBL-infinity-algebras in filtered MV-formalism

In [MV17], MV-algebras were introduced; they are precisely the algebras governed by the BV-relations $\Delta^2 = 0$ and $e^{\mathfrak{f}}\,\Delta^+ = \Delta^-\,e^{\mathfrak{f}}$ (following [CFL15], we will denote by $^+$ the source and by $^-$ the target). Schematically, there is the following inclusion of categories (explanations will be given below):

$$\begin{array}{llllll} & \underline{\text{MV-algebras}} & \supset & \underline{\text{BV}_\infty\text{-algebras}^*} & \supset & \underline{\text{IBL}_\infty\text{-algebras}^{**}} \\ \text{Obj.:} & (V,\mu,\delta,\eta,\varepsilon), R, & & R=\mathbb{K}[[\hbar]], & & V = \mathrm{S}U,\\ & \Delta: V\hat\otimes R \to V\hat\otimes R, & & \Delta = \Delta_1 + \hbar\,\Delta_2+\hbar^2\,\Delta_3+\cdots, & & \mathrm{S}_{k>i}(U)\subset \ker \mathfrak{f}_i.\\ & \Delta^2=0, |\,\Delta\,|=1, & & \Delta_i : V\to V \text{ differential} & & \\ & \Delta(1)=0. & & \quad\text{operator of order} \le i, & & \\ \text{Mor.:} & \mathfrak{f}: V^+\hat\otimes R\to V^-\hat\otimes R, & & \mathfrak{f}=\mathfrak{f}_1+\hbar\mathfrak{f}_2+\hbar^2\mathfrak{f}_3+\cdots . & & \\ & e^{\mathfrak{f}}\circ\Delta^+ = \Delta^-\circ e^{\mathfrak{f}}, & & & & \\ & \mathfrak{f}(1)\subset V^-\hat\otimes \mathfrak{m}^-. & & & & \end{array} \tag{D.10}$$

Here, R is a complete local Noetherian ring with residue field $\mathbb{K}$ of characteristic 0 and maximal ideal $\mathfrak{m}$, (V,μ,η,ε) is a graded commutative associative algebra over $\mathbb{K}$ with unit $\eta : \mathbb{K}\to V$ and augmentation $\varepsilon : V\to\mathbb{K}$, $(V,\delta,\varepsilon,\eta)$ is a *conilpotent* graded cocommutative coassociative coalgebra with counit ε and coaugmentation η (the same maps as ε and η for (V,μ)), $V\hat\otimes R \coloneqq \varprojlim_n (V\otimes R/V\otimes\mathfrak{m}^n)$, where $\mathfrak{m}^n = \mathfrak{m}\cdots\mathfrak{m}$, is the completed tensor product, the operators Δ and $\mathfrak{f}$ are R-linear and have non-negative filtration degrees with respect to the filtration by $V\hat\otimes\mathfrak{m}^n$.

If R is a $\mathbb{K}$-algebra,[4] then this is precisely the same setting as that of Proposition D.1.8 for the filtrations $\mathcal{F}^{\le 0}V = V$, $\mathcal{F}^{>0}V = 0$ and $\mathcal{F}^{\lambda\le 0}R = R$, $\mathcal{F}^{\lambda>0}R = \mathfrak{m}^{\lceil\lambda\rceil}$. Clearly, $\mathfrak{f}(1)\subset V^-\hat\otimes\mathfrak{m}^-$ is equivalent to $\|\mathfrak{f}(1)\| > 0$. Therefore, $e^{\mathfrak{f}}$ exists by Proposition D.1.8.

Recall from [MV17, p. 5] that a linear operator $D : V\to V$ on a commutative associative algebra V with unit 1 is called a *differential operator* of order $\le k$ for $k\in\mathbb{N}_0$ if it holds

$$\psi^D_{k+1}(v_1,\dots,v_{k+1}) \coloneqq [[\cdots[[D,L_{v_1}],L_{v_2}],\dots],L_{v_{k+1}}] = 0 \quad\text{for all } v_1,\dots,v_{k+1}\in V,$$

where

$$L_v(w)\coloneqq vw \quad\text{for all } w\in V$$

is the left-multiplication with $v\in V$ and $[\cdot,\cdot]$ is the graded commutator.

The categories of BV_∞- and IBL_∞-algebras contained in the MV-formalism are not the most general ones; this is what $*$ and $**$ indicate. Before we explain this, let us agree on calling $(\mathfrak{q}_{klg})$ a *strict* IBL_∞*-algebra* if $\mathfrak{q}_{0lg} = \mathfrak{q}_{k0g} = 0$ for all k, l, $g\in\mathbb{N}_0$, an

---

[4] The ring $\mathbb{Z}_4$ is a local ring which is not a $\mathbb{K}$-algebra over its residue field $\mathbb{K}=\mathbb{Z}_2$. The ring of polynomials $\mathbb{R}[x]$ in a single variable x is a $\mathbb{K}$-algebra over its residue field $\mathbb{K}=\mathbb{R}$, but it is not a local ring because both (x) and $(x+1)$ are maximal ideals. In contrast to this, the ring of power series $\mathbb{K}((\hbar))$ is both a local ring and a $\mathbb{K}$-algebra. Thanks to Thorsten Hertl for pointing this out.

input-strict IBL_∞*-algebra* if $\mathfrak{q}_{0lg} = 0$ for all l, $g \in \mathbb{N}_0$, and a *weak* IBL_∞*-algebra* if we want to emphasize that operations with no input and output are allowed. The same terminology will be used for morphisms.

* It is the category of *augmented strictly commutative* BV_∞*-algebras* (see [CL09, Section 5] for the definition of a commutative BV_∞-algebra). The augmentation ε is an additional data to get an MV-algebra. The coproduct is defined by $\delta := \delta_0$ (see (D.5)).

** It is the category of *(non-filtered) input-strict* IBL_∞*-algebras* $(\mathfrak{q}_{klg})_{k\geq 1, l,g\geq 0}$ and input-strict morphisms $(\mathfrak{f}_{klg})_{k\geq 1,l,g\geq 0}$ which satisfy the following *finiteness condition* for all $k \geq 1$ and $g \geq 0$: for any $v \in \mathrm{S}U$, we have

$$\mathfrak{q}_{klg}(v) = 0 \quad \text{and} \quad \mathfrak{f}_{klg}(v) = 0 \quad \text{for all but finitely many } l \in \mathbb{N}_0. \tag{D.11}$$

See [CFL15, Remark (6), p. 14] for the same finiteness condition. This is precisely what symplectic field theory for exact cobordisms gives.

(The transformation formulas between $\mathfrak{q}_{klg}$ and Δ_i and $\mathfrak{f}_{klg}$ and $\mathfrak{f}_i$ were written down in Remark 3.2.9. We will repeat them in Proposition D.2.6 below.)

The following filtered version of MV-algebras will allow us to describe more general weak IBL_∞-algebras. In particular, we will be able to remove the restriction (D.11).

Definition D.2.1 (Complete filtered MV-algebra)**.** *A* complete filtered MV-algebra *over a complete filtered graded algebra R over a field $\mathbb{K}$ is a complete filtered graded vector space V over $\mathbb{K}$, a homogenous R-linear map $\Delta : V \mathbin{\hat{\otimes}} R \to V \mathbin{\hat{\otimes}} R$ of finite filtration degree satisfying*

$$|\,\Delta\,| = -1, \quad \|\,\Delta\,\| \geq 0, \quad \|\,\Delta(1)\| > 0 \quad \textit{and} \quad \Delta \circ \Delta = 0,$$

and operations $\mu : V \mathbin{\hat{\otimes}} V \to V$, $\delta : V \to V \mathbin{\hat{\otimes}} V$, $\eta : \mathbb{K} \to V$ and $\varepsilon : V \to \mathbb{K}$ such that $(V, \mu, \eta, \varepsilon)$ is a complete filtered augmented unital commutative associative algebra and $(V, \delta, \varepsilon, \eta)$ is a complete filtered coaugmented counital cocommutative coassociative coalgebra satisfying the limit conilpotency property (D.6)*. We often denote a complete filtered* MV*-algebra simply by (V, Δ).*

A morphism of complete filtered MV-algebras (V^+, Δ^+) *and* (V^-, Δ^-) *over R is a homogenous R-linear map $\mathfrak{f} : V^+ \mathbin{\hat{\otimes}} R \to V^- \mathbin{\hat{\otimes}} R$ of finite filtration degree such that*

$$|\mathfrak{f}| = 0, \quad \|\mathfrak{f}\| \geq 0, \quad \|\mathfrak{f}(1)\| > 0 \quad \textit{and} \quad e^{\mathfrak{f}} \circ \Delta^+ = \Delta^- \circ e^{\mathfrak{f}}.$$

The composition $\diamond$ *of two composable morphisms* $\mathfrak{f}_1$ *and* $\mathfrak{f}_2$ *(i.e., the target of* $\mathfrak{f}_2$ *is the source of* $\mathfrak{f}_1$*) is defined by*

$$\mathfrak{f}_1 \diamond \mathfrak{f}_2 \coloneqq \log\Big(e^{\mathfrak{f}_1} \circ e^{\mathfrak{f}_2}\Big). \tag{D.12}$$

(The exponential and logarithm are well-defined by Proposition D.1.8.)

Remark D.2.2 (On complete filtered MV-algebras). (i) The condition $\|\mathfrak{f}(1)\| > 0$ is required for the exponential to converge; i.e., it might be seen as a technical condition. On the other hand, the condition $\|\,\Delta(1)\| > 0$ is optional, and we impose it in the manner of [MV17] and [CFL15].

(ii) We do not define "non-complete filtered MV-algebras" because it is not clear whether $\mathfrak{f}_1 \diamond \mathfrak{f}_2 : V^+ \hat{\otimes} R \to V^- \hat{\otimes} R$ for $V^+ \otimes R \xrightarrow{\mathfrak{f}_2} V^{+-} \otimes R \xrightarrow{\mathfrak{f}_1} V^- \otimes R$ restricts to a map $V^+ \otimes R \to V^- \otimes R$. This forces us to define morphisms as maps of completions $V^+ \hat{\otimes} R \to V^- \hat{\otimes} R$. In such category, two filtered MV-algebras would be isomorphic if and only if their completions were. This is not desired, and thus we define only "complete filtered MV-algebras". $\lhd$

Inspired by (D.10), we define the following version of weak IBL_∞-algebras.

Definition D.2.3 (Complete filtered IBL_∞-algebra in MV-formalism). *Let $d \in \mathbb{Z}$ and $\gamma > 0$. A* complete filtered IBL_∞-algebra of bidegree (d, γ) in MV-formalism *is a complete filtered* MV-*algebra* $(V, \Delta, R, \mu, \delta, \varepsilon, \eta)$ *satisfying the following conditions:*

(1) $R = \mathbb{K}((\hbar))$ *is the ring of Laurent series*[5] *in a formal variable $\hbar$ of degree $|\hbar| = 2d$ equipped with the complete, Hausdorff and exhaustive filtration*

$$\mathcal{F}^\lambda \mathbb{K}((\hbar)) \coloneqq \Big\{ \sum_{i=-\infty}^{\infty} a_i \hbar^i \in \mathbb{K}((\hbar)) \mid a_i = 0 \textit{ for } i < \frac{\lambda}{2\gamma} \Big\} \quad \textit{for } \lambda \in \mathbb{R}. \tag{D.13}$$

(2) There is a complete filtered graded vector space W such that if we define

$$U \coloneqq W[1] \quad \textit{and} \quad \mathcal{F}^\lambda U \coloneqq (\mathcal{F}^{\lambda-\gamma} W)[1] \quad \textit{for all } \lambda \in \mathbb{R}, \tag{D.14}$$

then $(V = \hat{\mathrm{S}}U, \mu, \delta, \eta, \varepsilon)$ *is the completion of the symmetric bialgebra on U filtered by the induced filtration.*

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[5] Another notation for $\mathbb{K}((\hbar))$ is $\mathbb{K}[[\hbar]][\hbar^{-1}]$ to emphasize that it is the localization of the ring of power series $\mathbb{K}[[\hbar]]$ at the powers of $\hbar$. An element of $\mathbb{K}((\hbar))$ is a formal power series $\sum_{i=-\infty}^{\infty} a_i \hbar^i$ with only finitely many non-zero $a_i \in \mathbb{K}$ for $i \leq 0$.

(3) The BV*-operator* $\Delta : \hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar))$*, where* $\hat{\mathrm{S}}U((\hbar)) \coloneqq \mathrm{S}U \mathbin{\hat{\otimes}} \mathbb{K}((\hbar))$*,*[6] *decomposes as*

$$\Delta = \hbar^{-1}\,\Delta_0 + \Delta_1 + \hbar\,\Delta_2 + \hbar^2\,\Delta_3 + \cdots , \tag{D.15}$$

where for all $i \geq 0$ *the operator* $\Delta_i : \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ *is a linear differential operator of order* $\leq i$*.*[7]

We often denote the data of a complete filtered IBL_∞*-algebra in* MV*-formalism simply by* (W, Δ)*.*

A morphism of complete filtered IBL_∞-algebras (W^+, Δ^+) and (W^-, Δ^-) of bidegree (d, γ) in MV-formalism *is a morphism of complete filtered* MV*-algebras* $\mathfrak{f} : \hat{\mathrm{S}}U^+((\hbar)) \to \hat{\mathrm{S}}U^-((\hbar))$ *which decomposes as*

$$\mathfrak{f} = \hbar^{-1}\mathfrak{f}_0 + \mathfrak{f}_1 + \hbar\mathfrak{f}_2 + \hbar^2\mathfrak{f}_3 + \cdots , \tag{D.16}$$

where for all $i \geq 0$ *the map* $\mathfrak{f}_i : \hat{\mathrm{S}}U^+ \to \hat{\mathrm{S}}U^-$ *is* $\mathbb{K}$*-linear and satisfies*

$$\hat{\mathrm{S}}_j U^+ \subset \ker \mathfrak{f}_i \quad \textit{for all } j > i. \tag{D.17}$$

Notation D.2.4 (Replacing $*$ with $\odot$ for $V = \mathrm{S}U$)**.** *From now on, we will denote the convolution product* $*$ *on morphisms of* $\mathrm{S}U$ *by* $\odot$*. It is namely the same operation on morphisms which is denoted by* $\odot$ *in Section 3.2 and in [CFL15].*

For a map $\mathfrak{f} = \sum_{i=-\infty}^{\infty} \mathfrak{f}_{i+1}\hbar^i : \hat{\mathrm{S}}U^+((\hbar)) \to \hat{\mathrm{S}}U^-((\hbar))$, where $\mathfrak{f}_i : \hat{\mathrm{S}}U^+ \to \hat{\mathrm{S}}U^-$ is $\mathbb{K}$-linear, we denote

$$\langle \mathfrak{f} \rangle_{klg} \coloneqq \pi_l \circ \mathfrak{f}_{k+g} \circ \iota_k : \hat{\mathrm{S}}_k U^+ \to \hat{\mathrm{S}}_l U^- \quad \text{for all } k, l \geq 0, g \in \mathbb{Z}.$$

This is the notation introduced in [CFL15, Equation (2.14)].

Based on [CFL15], we will now give an equivalent characterization of complete filtered IBL_∞-algebras in MV-formalism and their morphisms in terms of their components $(\mathfrak{q}_{klg})$ and $(\mathfrak{f}_{klg})$ within the surface calculus. We will not repeat the interpretation of the algebraic relations in terms of gluing of surface; for this, see [CFL15]. We need the following lemma.

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[6] In contrast to $\mathbb{K}((\hbar))$, an element of $\hat{\mathrm{S}}U((\hbar))$, when seen as a power series, can have a non-zero coefficient at every power $\hbar^i$.

[7] The definition of a differential operator on p. 300 generalizes in a straightforward way to complete filtered algebras.

Lemma D.2.5 (Differential operators on filtered symmetric bialgebras)**.** *Let U be a complete filtered graded vector space. Then the following holds for the complete filtered symmetric bialgebra $\hat{\mathrm{S}}U$ and all $k \in \mathbb{N}_0$:*

(a) A linear homogenous map $D : \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ of finite filtration degree is a differential operator of order $\leq k$ if and only if it can be written as

$$D = \sum_{i=0}^{k} D_i \odot \mathbb{1} \tag{D.18}$$

for linear homogenous maps $D_i : \hat{\mathrm{S}}_i U \subset \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ of finite filtration degrees (here $D_i = 0$ on $\hat{\mathrm{S}}_j U$ for $j \neq i$ is the trivial extension). The maps D_i are uniquely determined by D.

(b) A linear homogenous map $\mathfrak{f} : \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ of finite filtration degree satisfies $\hat{\mathrm{S}}_{i>k}U \subset \ker \mathfrak{f}$ if and only if it can be written as $\mathfrak{f} = \mathfrak{f}_0 + \cdots + \mathfrak{f}_k$ for linear homogenous maps $\mathfrak{f}_i : \hat{\mathrm{S}}_i U \to \hat{\mathrm{S}}U$ of finite filtration degrees. The maps $\mathfrak{f}_i$ are uniquely determined by $\mathfrak{f}$.

Proof. (a) The claim is implied by the following subclaim and proposition, which generalize in a straightforward way to the filtered case. Recall that an operator $D : \mathrm{S}U \to \mathrm{S}U$ is called a *derivative* of order $\leq k$ if

$$D(1) = 0 \quad \text{and} \quad \psi_i^D(v_1, \ldots, v_i)(1) = 0 \quad \text{for all } i \geq k+1 \text{ and } v_1, \ldots, v_i \in V,$$

where ψ_i^D were defined on page 300.

Subclaim (Derivatives and differential operators)*.* A homogenous linear operator $D : \mathrm{S}U \to \mathrm{S}U$ is a differential operator of order $\leq k$ if and only if the linear operator $D' \coloneqq D - D_0 \odot \mathbb{1}$, where $D_0 = D \circ \iota_0$, is a derivative of order $\leq k$ (recall that $\iota_k : \mathrm{S}_k U \to \mathrm{S}U$ is the inclusion).

Proof. Given a derivative D' of order $\leq k$ and a linear map $D_0 : \mathrm{S}_0 U \subset \mathrm{S}U \to \mathrm{S}U$, we will check that $D \coloneqq D' + D_0 \odot \mathbb{1}$ is a differential operator of order $\leq k$. We have

$$\begin{aligned} \underbrace{\psi_{k+2}^{D'}(v_1, \ldots, v_{k+2})(1)}_{=0} &= [\psi_{k+1}^{D'}(v_1, \ldots, v_{k+1}), L_{v_{k+2}}](1) \\ &= \psi_{k+1}^{D'}(v_1, \ldots, v_{k+1})(v_{k+2}) - \underbrace{\psi_{k+1}^{D'}(v_1, \ldots, v_{k+1})(1)}_{=0}\, v_{k+2} \end{aligned}$$

for all v_1, …, $v_{k+2} \in V$, and hence D' is also a differential operator of order $\leq k$. For all $v \in \mathrm{S}U$, it holds

$$(D_0 \odot \mathbb{1})(v) = D_0(1)v,$$

and hence for all v, $v_1 \in \mathrm{S}U$, we have

$$\begin{aligned}\psi_1^{D_0\odot\mathbb{1}}(v_1)(v) &= (D_0 \odot \mathbb{1})(v_1 v) - (-1)^{D_0 v_1} v_1 (D_0 \odot \mathbb{1})(v)\\ &= D_0(1) v_1 v - (-1)^{D_0 v_1} v_1 D_0(1) v\\ &= 0.\end{aligned}$$

It follows that $D_0 \odot \mathbb{1}$ is a differential operator of order 0. Clearly, a sum of differential operators of orders $\le k_1$ and $\le k_2$ is a differential operator of order $\le \max(k_1, k_2)$. Therefore, $D = D' + D_0 \odot \mathbb{1}$ is a differential operator of order $\le k$. Moreover, since $D'(1) = 0$ by definition, we have $D(1) = D_0(1)$, i.e., $D_0 = D \circ \iota_0$.

Given a differential operator D of order $\le k$, we define $D_0 \coloneqq D \circ \iota_0$ and $D' \coloneqq D - D_0 \odot \mathbb{1}$. Firstly, it holds

$$D'(1) = D(1) - (D_0 \odot \mathbb{1})(1) = 0.$$

Secondly, we have

$$\psi_i^D(v_1, \ldots, v_i) = \psi^{D'}(v_1, \ldots, v_i) = 0,$$

and hence $\psi_i^D(v_1, \ldots, v_i)(1) = 0$ for all $i \ge k+1$ and $v_1, \ldots, v_i \in V$. Therefore, D' is a derivative of order $\le k$. *(Subclaim)* □

Proposition D.2.1 ([Mar01, Proposition 3.2]). A linear operator $D : \mathrm{S}U \to \mathrm{S}U$ is a derivative of order $\le k$ if and only if it can be written as $D = \sum_{i=1}^{k} D_i \odot \mathbb{1}$ for unique $D_i : \mathrm{S}_i U \to \mathrm{S}U$.

(b) This is clear. We have $\mathfrak{f}_i = \mathfrak{f} \circ \iota_i$ for all $i = 0, \ldots, k$. □

Proposition D.2.6 (Filtered IBL_∞-algebras in MV-formalism in components)**.** *We have the following equivalences of structures:*

(a) The data of a complete filtered IBL_∞*-algebra of bidegree* (d, γ) *in* MV*-formalism* (W, Δ) *is equivalent to the data of a complete filtered graded vector space* W *and a collection of* $\mathbb{K}$*-linear maps* $\mathfrak{q}_{klg} : \hat{\mathrm{S}}_k U \to \hat{\mathrm{S}}_l U$ *for all* k, l, $g \in \mathbb{N}_0$ *which are homogenous, have finite filtration degrees and satisfy the following conditions for all* k, l, $g \ge 0$*:*

- *(1)* $|\mathfrak{q}_{klg}| = -2d(k+g-1) - 1$.
- *(2)* $\|\mathfrak{q}_{klg}\| \ge -2\gamma(k+g-1)$.
- *(3) The inequality in (2) is strict whenever* $k = 0$.
- *(4) The sum of maps* $Q_{kg} \coloneqq \sum_{l=0}^{\infty} \mathfrak{q}_{klg} : \hat{\mathrm{S}}_k U \to \hat{\mathrm{S}} U$ *converges (in the sense of Lemma D.1.1).*

(5) The following equation holds:[8]

$$\sum_{\substack{h\geq 1\\ k_1,k_2,l_1,l_2,g_1,g_2\geq 0\\ k_1+k_2-h=k\\ l_1+l_2-h=l\\ g_1+g_2+h-1=g}} \mathfrak{q}_{k_1l_1g_1}\circ_h \mathfrak{q}_{k_2l_2g_2}=0. \tag{D.19}$$

(b) The data of a strict morphism $\mathfrak{f}:(W^+,\Delta^+)\to(W^-,\Delta^-)$ *of complete filtered* IBL_∞*-algebras of bidegree* (d,γ) *in* MV*-formalism, strict meaning that* $\varepsilon^-\circ\mathfrak{f}=\mathfrak{f}\circ\eta^+=0$*, is equivalent to a collection of* $\mathbb{K}$*-linear maps* $\mathfrak{f}_{klg}:\hat{\mathrm{S}}_kU^+\to\hat{\mathrm{S}}_lU^-$ *for all* k*,* l*,* $g\in\mathbb{N}_0$ *which are homogenous, have finite filtration degrees and satisfy the following conditions for all* k*,* l*,* $g\geq 0$*:*

(1) $|\mathfrak{f}_{klg}|=-2d(k+g-1)$*.*

(2) $\|\mathfrak{f}_{klg}\|\geq -2\gamma(k+g-1)$*.*

(3) The inequality in (2) is strict whenever $k=0$*.*

(4) The sum $F_{kg}:=\sum_{l=0}^\infty \mathfrak{f}_{klg}:\hat{\mathrm{S}}_kU^+\to\hat{\mathrm{S}}U^-$ *converges.*

(5) It holds $\mathfrak{f}_{k0g}=\mathfrak{f}_{0lg}=0$*.*

(6) The following equation holds:

$$\begin{aligned}&\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{\substack{h_1,\dots,h_r\geq 1\\ k_1,\dots,k_r,k^-,l_1,\dots,l_r,l^-,g_1,\dots,g_r,g^-\geq 0\\ k_1+\dots+k_r=k\\ l_1+\dots+l_r-k^-+l^-=l\\ g_1+\dots+g_r+g^-+k^--r=g}} \mathfrak{q}^-_{k^-l^-g^-}\circ_{h_1,\dots,h_r}(\mathfrak{f}_{k_1l_1g_1},\dots,\mathfrak{f}_{k_rl_rg_r})\\ &=\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{\substack{h_1,\dots,h_r\geq 1\\ k_1,\dots,k_r,k^+,l_1,\dots,l_r,l^+,g_1,\dots,g_r,g^+\geq 0\\ k_1+\dots+k_r+k^+-l^+=k\\ l_1+\dots+l_r=l\\ g_1+\dots+g_r+g^++l^+-r=g}} (\mathfrak{f}_{k_1l_1g_1},\dots,\mathfrak{f}_{k_rl_rg_r})\circ_{h_1,\dots,h_r}\mathfrak{q}^+_{k^+l^+g^+}.\end{aligned} \tag{D.20}$$

The term $r=0$ *on the left- or right-hand side is possible only for* $k=0$ *or* $l=0$ *and equals* $\mathfrak{q}^-_{0lg}\circ\mathrm{e}:\hat{\mathrm{S}}_0U^+\to\hat{\mathrm{S}}_lU^-$ *or* $\mathrm{e}\circ\mathfrak{q}^+_{k0g}:\hat{\mathrm{S}}_kU^+\to\hat{\mathrm{S}}_0U^-$*, respectively, where* e *is the* $\hat{\mathrm{S}}_0U^+\to\hat{\mathrm{S}}_0U^-$ *component of the unit for the*

---

[8] Notice that (D.19) does not contain $\mathfrak{q}_{00g}$ with $g\geq 0$ for any k, l, $g\geq 0$ (otherwise all $\mathfrak{q}_{klg}$ appear). In fact, it will be clear from the proof that there is no condition on $\mathfrak{q}_{00g}$ coming from $\Delta^2=0$; if we work over a general ring, then $\mathfrak{q}_{00g}(1)$ for $g\geq 0$ can be arbitrary odd elements in it. Note that this contradicts [CFL15, p. 47, bottom-most paragraph].

convolution product (D.4) *(the identity under* $\hat{\mathrm{S}}_0 U^{\pm} \simeq \mathbb{K}$ *via* $\eta^{\pm}$*).* [9]

The equivalences (a) and (b) are given by the formulas

$$\Delta_i = \sum_{\substack{k,g\geq 0\\ k+g=i}} Q_{kg} \odot \mathbb{1} \quad \text{and} \quad \mathfrak{f}_i = \sum_{\substack{k,g\geq 0\\ k+g=i}} F_{kg} \quad \text{for all } i \geq 0, \ \text{respectively.} \tag{D.21}$$

We remark that the operations $\circ_{h_1,\ldots,h_r}$ *are defined in Definition D.4.2 in the next section or in Definition 3.2.2.*

Suppose that $\mathfrak{f}^+$ *and* $\mathfrak{f}^-$ *are strict composable morphisms of complete filtered* IBL_∞*-algebras in* MV*-formalism. Then we have the following:*

(c) The composition $\mathfrak{f} \coloneqq \mathfrak{f}^- \diamond \mathfrak{f}^+$ *is a strict morphism of complete filtered* IBL_∞*-algebras in* MV*-formalism and its components* $\mathfrak{f}_{klg}$ *for* k, $l \geq 1$, $g \geq 0$ *are given by*

$$\mathfrak{f}_{klg} = \sum_{\substack{r^-,r^+\geq 0\\ k_1^-,l_1^-,\ldots,k_{r^-}^-,l_{r^-}^-\geq 1\\ k_1^+,l_1^+,\ldots,k_{r^+}^+,l_{r^+}^+\geq 1\\ g_1^+,\ldots,g_{r^+}^+,g_1^-,\ldots,g_{r^-}^-\geq 0\\ k_1^-+\cdots+k_{r^-}^-=l_1^++\cdots+l_{r^+}^+\\ k_1^++\cdots+k_{r^+}^+=k\\ l_1^-+\cdots+l_{r^-}^-=l\\ g_1^++\cdots+g_{r^+}^++g_1^-+\cdots+g_r^--r^+-r^-\\ +k_1^-+\cdots+k_{r^-}^-+1=g}} \frac{1}{r^+!r^-!}(\mathfrak{f}^-_{k_1^- l_1^- g_1^-},\ldots,\mathfrak{f}^-_{k_{r^-}^- l_{r^-}^- g_{r^-}^-}) \circ_{\mathrm{con}} (\mathfrak{f}^+_{k_1^+ l_1^+ g_1^+},\ldots,\mathfrak{f}^+_{k_{r^+}^+ l_{r^+}^+ g_{r^+}^+}), \tag{D.22}$$

where $\circ_{\mathrm{con}}$ *denotes the connected composition (see Definition D.4.2 in the next section).*

Proof. (a) Suppose first that (W, Δ) is a complete filtered IBL_∞-algebra of bidegree (d, γ) in MV-formalism; i.e., we have $\Delta = \Delta_0\,\hbar^{-1} + \Delta_1 + \Delta_2\,\hbar + \cdots$ for $\Delta_i : \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ differential operators of order $\leq i$, $|\,\Delta\,| = -1$, $\|\,\Delta\,\| \geq 0$ and $\|\,\Delta(1)\| > 0$. Using Lemma D.2.5, we write $\Delta_i = \sum_{j=0}^{i} D_{ij} \odot \mathbb{1}$ for $\mathbb{K}$-linear maps $D_{ij} : \hat{\mathrm{S}}_j U \subset \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ of finite filtration degrees which are uniquely determined by Δ_i, and we define $Q_{kg} \coloneqq D_{k+g,k} : \hat{\mathrm{S}}_k U \to \hat{\mathrm{S}}U$ for all k, $g \in \mathbb{N}_0$. Next, we define $\mathfrak{q}_{klg} \coloneqq \pi_l \circ Q_{kg} : \hat{\mathrm{S}}_k U \to \hat{\mathrm{S}}_l U$ for all k, l, $g \in \mathbb{N}_0$.

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[9] If $\mathfrak{f}$ is not strict, it would be interesting to know whether $e^{\mathfrak{f}}\,\Delta^+ = \Delta^-\,e^{\mathfrak{f}}$ is still equivalent to the connected calculus (D.20) or whether there are some other equations possibly involving disconnected gluing. Note that trying to incorporate $\mathfrak{f}_{0lg}$ to the right-hand side and $\mathfrak{f}_{k0g}$ to the left-hand side of (D.20) always leads to the disconnected calculus. Also note that $\mathfrak{f}_{00g}$ for $g \geq 0$ do not appear in (D.20) for any k, l, $g \geq 0$; otherwise all $\mathfrak{f}_{klg}$ appear. Finally, note that $\mathfrak{q}^+_{klg}$ and $\mathfrak{q}^-_{klg}$ appear for all k, l, $g \geq 0$ and that it follows from (D.20) that $\mathrm{e} \circ \mathfrak{q}^+_{00g}$ and $\mathfrak{q}^-_{00g} \circ \mathrm{e}$ have to be equal for all $g \geq 0$.

In the following computations, we will keep in mind that $|\hbar| = 2d$ and $\|\hbar\| = 2\gamma$. Using the formula $|f \circ g| = |f| + |g|$, which is valid for homogenous linear maps f and g, we have

$$\begin{aligned} |\,\Delta\,| = -1 \quad &\iff \quad \forall i \geq -1 : |\,\Delta_{i+1}\,| = -1 - di \\ &\iff \quad \forall k, g \geq 0, k+g = i+1 : |Q_{kg}| = -1 - di \\ &\iff \quad \forall k, l, g \geq 0, k+g = i+1 : |\mathfrak{q}_{klg}| = -1 - d(k+g-1). \end{aligned}$$

Thus (1) follows. Using that $\|f \circ g\| \geq \|f\| + \|g\|$ and $\|\sum_i v_i\| \geq \inf_i \|v_i\|$, we have

$$\begin{aligned} \|\,\Delta\,\| \geq 0 \quad &\iff \quad \forall i \geq -1 : \|\,\Delta_{i+1}\,\| \geq -2\gamma i \\ &\iff \quad \forall k, g \geq 0, k+g = i+1 : \|Q_{kg}\| \geq -2\gamma i \\ &\iff \quad \forall k, l, g \geq 0, k+g = i+1 : \|\mathfrak{q}_{klg}\| \geq -2\gamma(k+g-1). \end{aligned}$$

Thus (2) follows. The argument for "$\Longrightarrow$" on the second line is inductive using that

$$Q_{kg} = \Big(\Delta_{k+g} - \sum_{j=0}^{k-1} Q_{j,k+g-j} \odot \mathbb{1}\Big) \circ \iota_k$$

for all k, $g \geq 0$. Evaluation of $\Delta(1)$ gives

$$\begin{aligned} \Delta(1) &= \Delta_0(1)\hbar^{-1} + \Delta_1(1) + \Delta_2(1)\hbar + \cdots \\ &= Q_{00}(1)\hbar^{-1} + Q_{01}(1) + Q_{02}(1)\hbar + \cdots \\ &= \Big(\sum_{l=0}^{\infty} \mathfrak{q}_{0l0}(1)\Big)\hbar^{-1} + \Big(\sum_{l=0}^{\infty} \mathfrak{q}_{0l1}(1)\Big) + \hbar\Big(\sum_{l=0}^{\infty} \mathfrak{q}_{0l2}(1)\Big) + \cdots, \end{aligned}$$

and we see that $\|\,\Delta(1)\| > 0$ is equivalent to (3). As for (4), it is implied by the Resummation Lemma (Lemma D.1.1). Namely, we have

$$\sum_{l=0}^{\infty} \mathfrak{q}_{klg}(v) = \sum_{l=0}^{\infty} \pi_l(Q_{kg}(v)) \overset{D.1.1}{=} Q_{kg}(v) \quad \text{for all } v \in \hat{\mathrm{S}}_k U.$$

Thus $\sum_{l=0}^{\infty} \mathfrak{q}_{klg}$ converges to Q_{kg}.

On the other hand, given $\mathfrak{q}_{klg}$ satisfying (1), (2), (3) and (4), we clearly get an operator $\Delta : \hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar))$ which has the decomposition (D.15) and which satisfies $|\,\Delta\,| = 0$, $\|\,\Delta\,\| \geq 0$ and $\|\,\Delta(1)\| > 0$.

Assuming (1), (2), (3) and (4), it remains to check that $\Delta^2 = 0$ is equivalent to (5).

This is the same computation as in [CFL15, Section 2], just in the filtered setting and allowing $k = 0$ or $l = 0$ (it will turn out that it does not change anything). Recall the notation from Section 3.2 that $\hat{D} \coloneqq D \odot \mathbb{1} = \mu(D \mathbin{\hat{\otimes}} \mathbb{1})\delta$. We will be using formulas from Remark 3.2.3 in that section, which are proven in Proposition D.4.4 in the next section. Note that by the Resummation Lemma, elements of the completion can be summed up in any order, even as nested infinite sums, provided one (and hence all) of these sums converges. Therefore, we could, in principle, write just one sum $\sum$ and think of Δ as of the sum $\sum \hat{\mathfrak{q}}_{klg}\hbar^{k+g-1}$ over all k, l, $g \geq 0$.

We have

$$\begin{aligned}\Delta \circ \Delta &= \sum_{i=-2}^{\infty}\Big(\sum_{\substack{k_1,l_1,g_1,k_2,l_2,g_2\geq 0\\ k_1+k_2+g_1+g_2=i+2}} \hat{\mathfrak{q}}_{k_1l_1g_1} \circ \hat{\mathfrak{q}}_{k_2l_2g_2}\Big)\hbar^i \\ &= \sum_{i=-2}^{\infty}\Big(\sum_{\substack{k_1,l_1,g_1,k_2,l_2,g_2\geq 0\\ k_1+k_2+g_1+g_2=i+2}} \sum_{h=0}^{\min(k_1,l_2)} \widehat{\mathfrak{q}_{k_1l_1g_1} \circ_h \mathfrak{q}_{k_2l_2g_2}}\Big)\hbar^i,\end{aligned}$$

and we see that

$$\langle \Delta^2 \rangle_{klg} = \sum_{\substack{k_1,l_1,g_1,k_2,l_2,g_2\geq 0\\ k_1+k_2+g_1+g_2=k+g+1}} \sum_{h=0}^{\min(k_1,l_2)} \pi_l \circ \widehat{\mathfrak{q}_{k_1l_1g_1} \circ_h \mathfrak{q}_{k_2l_2g_2}} \circ \iota_k \quad \text{for all } k, l \geq 0, g \geq -1.$$

Because $\mathfrak{q}_{k_1l_1g_1} \circ_h \mathfrak{q}_{k_2l_2g_2} : \hat{\mathrm{S}}_{k_1+k_2-h}U \to \hat{\mathrm{S}}_{l_1+l_2-h}U$, it follows from the definition $\hat{f} \coloneqq \mu(f \otimes \mathbb{1})\delta$ that $\pi_l \circ \widehat{\mathfrak{q}_{k_1l_1g_1} \circ_h \mathfrak{q}_{k_2l_2g_2}} \circ \iota_k = 0$ unless $0 \leq k_1 + k_2 - h \leq k$, $0 \leq l_1 + l_2 - h \leq l$ and $l_1 + l_2 + h - (k_1 + k_2 + h) = l - k$. For fixed k, l, $h \geq 0$, $g \geq -1$, it holds

$$\left\{\begin{aligned}&k_1, l_1, g_1, k_2, l_2, g_2 \geq 0\\ &0 \leq k_1 + k_2 - h \leq k\\ &0 \leq l_1 + l_2 - h \leq l\\ &l_1 + l_2 - k_1 - k_2 = l - k\\ &k_1 + k_2 + g_1 + g_2 = k + g + 1\end{aligned}\right\} = \bigsqcup_{s=0,\dots,k} \left\{\begin{aligned}&k_1, k_2, l_1, l_2, g_1, g_2 \geq 0\\ &k_1 + k_2 - h = k - s\\ &l_1 + l_2 - h = l - s\\ &g_1 + g_2 + h - 1 = g + s\end{aligned}\right\},$$

and hence

$$\begin{aligned}\langle\Delta^2\rangle_{klg} &= \sum_{s=0}^{k}\sum_{h=0}^{g+s+1}\sum_{\substack{k_1,k_2,l_1,l_2,g_1,g_2\geq 0\\ k_1+k_2-h=k-s\\ l_1+l_2-h=l-s\\ g_1+g_2+h-1=g+s}} \pi_l\circ\widehat{\mathfrak{q}_{k_1l_1g_1}\circ_h\mathfrak{q}_{k_2l_2g_2}}\circ\iota_k\\ &= \underbrace{\sum_{h=0}^{g+1}\sum_{\substack{k_1,k_2,l_1,l_2,g_1,g_2\geq 0\\ k_1+k_2-h=k\\ l_1+l_2-h=l\\ g_1+g_2+h-1=g}} \mathfrak{q}_{k_1l_1g_1}\circ_h\mathfrak{q}_{k_2l_2g_2}}_{=:\,(\square)_{klg}} + \sum_{s=1}^{k}\pi_l\circ\widehat{(\square)_{k-s,l-s,g+s}}\circ\iota_k.\end{aligned} \tag{D.23}$$

We see immediately that

$$\langle\Delta^2\rangle_{klg} = 0 \quad \text{for all } k,l\geq 0, g\geq -1, \tag{D.24}$$

which is equivalent to $\Delta^2 = 0$, follows from $(\square)_{klg} = 0$ for all k, $l\geq 0$, $g\geq -1$. The reverse implication is proven by induction on the linear order $\prec$ on signatures (k,l,g) defined in [CFL15, Definition 2.4]. It holds $(k',l',g')\prec(k,l,g)$, by definition, if one of the following conditions is satisfied:

(i) $k'+l'+2g' < k+l+2g$,

(ii) $k'+l'+2g' = k+l+2g$ and $g' > g$, or

(iii) $k'+l'+2g' = k+l+2g$ and $g' = g$ and $k' < k$.

For the last sum in (D.23), we denote $k_s \coloneqq k-s$, $l_s\coloneqq l-s$ and $g_s\coloneqq g+s$ and compute

$$(k-k_s)+(l-l_s)+2(g-g_s) = s+s+2(-s) = 0.$$

Therefore, case (ii) applies, and so $(k_s,l_s,g_s)\prec(k,l,g)$ for all $s=1,\ldots,k$. This shows the equivalence of $\Delta^2=0$ and $(\square)_{klg}=0$ for all k, $l\geq 0$, $g\geq -1$. Finally, $(\square)_{k,l,-1}=0$ holds automatically because $g_1+g_2+h=0$ implies $h=0$, and all terms $\mathfrak{q}_{k_1l_1g_1}\circ_h\mathfrak{q}_{k_2l_2g_2}$ with $h=0$ vanish. This is because $\mathfrak{q}_{klg}$ are odd and $f_1\circ_0 f_2 = (-1)^{|f_1||f_2|}f_2\circ_0 f_1$.

(b) Exactly as in (a), we first prove the equivalence of $|\mathfrak{f}|=0$, $\|\mathfrak{f}\|\geq 0$, $\|\mathfrak{f}(1)\|>0$ and the conditions (D.16) and (D.17) to (1), (2), (3) and (4) under the correspondence (D.21). Clearly, $\varepsilon^-\circ\mathfrak{f} = \mathfrak{f}\circ\eta^+ = 0$ is equivalent to (5).

Assuming (1), (2), (3) and (4), it remains to check that the equation $e^{\mathfrak{f}}\,\Delta^+ = \Delta^-\,e^{\mathfrak{f}}$ is equivalent to (6). This is again the same computation as in [CFL15, Section 2]. We will do it in the weak case and then restrict to the strict case for the induction. We have

$$
\begin{aligned}
&e^{\mathfrak{f}} \circ \Delta^+ \\
&\quad = \Big(\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i_1,\dots,i_r \geq -1} \sum_{\substack{k_1,l_1,g_1,\dots,k_r,l_r,g_r \geq 0 \\ k_1+g_1=i_1+1,\dots,k_r+g_r=i_r+1}} \mathfrak{f}_{k_1 l_1 g_1} \odot \cdots \odot \mathfrak{f}_{k_r l_r g_r} \hbar^{i_1+\cdots+i_r} \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \circ \Big(\sum_{i^+=-1}^{\infty} \sum_{\substack{k^+,l^+,g^+ \geq 0 \\ k^++g^+=i^++1}} \hat{\mathfrak{q}}^+_{k^+ l^+ g^+} \hbar^{i^+} \Big) \\
&\quad = \sum_{i=-\infty}^{\infty} \Big(\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{k^+,l^+,g^+,k_1,l_1,g_1,\dots,k_r,l_r,g_r \geq 0 \\ k^++k_1+\cdots+k_r+g^++g_1+\cdots+g_r=i+r+1}} (\mathfrak{f}_{k_1 l_1 g_1} \odot \cdots \odot \mathfrak{f}_{k_r l_r g_r}) \circ \hat{\mathfrak{q}}^+_{k^+ l^+ g^+} \Big) \hbar^i \\
&\quad = \sum_{i=-\infty}^{\infty} \Big(\sum_{r=0}^{\infty} \frac{1}{r!} \underbrace{\sum_{\substack{k^+,l^+,g^+ \geq 0 \\ k_1,l_1,g_1,\dots,k_r,l_r,g_r \geq 0 \\ k^++k_1+\cdots+k_r+g^+ \\ +g_1+\cdots+g_r=i+r+1}} \sum_{\substack{h_1,\dots,h_r \geq 0 \\ h_1+\cdots+h_r=l^+}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1,\dots,h_r} \mathfrak{q}^+_{k^+ l^+ g^+} \Big) \hbar^i}_{=: (*)^+_{i,r} \,:\, \hat{\mathrm{S}}U^+ \to \hat{\mathrm{S}}U^-}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\Delta^- \circ e^{\mathfrak{f}} \\
&\quad = \sum_{i=-\infty}^{\infty} \Big(\sum_{r=0}^{\infty} \frac{1}{r!} \underbrace{\sum_{\substack{k^-,l^-,g^- \geq 0 \\ k_1,l_1,g_1,\dots,k_r,l_r,g_r \geq 0 \\ k^-+k_1+\cdots+k_r+g^- \\ +g_1+\cdots+g_r=i+r+1}} \sum_{\substack{h_1,\dots,h_r \geq 0 \\ h_1+\cdots+h_r=k^-}} \mathfrak{q}^-_{k^- l^- g^-} \circ_{h_1,\dots,h_r} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \Big) \hbar^i}_{=: (*)^-_{i,r} \,:\, \hat{\mathrm{S}}U^+ \to \hat{\mathrm{S}}U^-}.
\end{aligned}
$$

We will consider $\pi_l \circ (*)^+_{i,r} \circ \iota_k$ for fixed $i \in \mathbb{Z}$ and k, $l \geq 0$. Because of the definition of $\circ_{h_1,\dots,h_r}$, only the terms with $l_1 + \cdots + l_r = l$ and $k^+ + k_1 + \cdots + k_r - h_1 - \cdots - h_r = k^+ + k_1 + \cdots + k_r - l^+ = k$ survive in $\pi_l \circ (*)^+_{i,r} \circ \iota_k$. Denoting $g := i - k + 1$, we have

$$
\begin{aligned}
k^+ + k_1 + \cdots + k_r + g^+ + g_1 + \cdots + g_r = i + r + 1 &\qquad & g_1 + \cdots + g_r + g^+ + l^+ - r = g \\
k^+ + k_1 + \cdots + k_r - l^+ = k &\quad \Longleftrightarrow \quad & k^+ + k_1 + \cdots + k_r - l^+ = k \\
l_1 + \cdots + l_r = l &\qquad & l_1 + \cdots + l_r = l.
\end{aligned}
$$

Therefore, it holds

$$\pi_l \circ (*)^+_{i,r} \circ \iota_k = \sum_{\substack{k^+,l^+,g^+\geq 0\\ k_1,l_1,g_1,\dots,k_r,l_r,g_r\geq 0\\ k^++k_1+\dots+k_r-l^+=k\\ l_1+\dots+l_r=l\\ g_1+\dots+g_r+g^++l^+-r=g}} \sum_{\substack{h_1,\dots,h_r\geq 0\\ h_1+\dots+h_r=l^+}} (\mathfrak{f}_{k_1l_1g_1},\dots,\mathfrak{f}_{k_rl_rg_r}) \circ_{h_1,\dots,h_r} \mathfrak{q}^+_{k^+l^+g^+}.$$

Collecting the terms with $h_i=0$ and using the graded commutativity of $\odot$, we obtain

$$\begin{aligned}\pi_l \circ (*)^+_{i,r} \circ \iota_k = {} & \underbrace{\sum_{\substack{h_1,\dots,h_r\geq 1\\ k^+,l^+,g^+,k_1,l_1,g_1,\dots,k_r,l_r,g_r\geq 0\\ h_1+\dots+h_r=l^+\\ k_1+\dots+k_r+k^+-l^+=k\\ l_1+\dots+l_r=l\\ g_1+\dots+g_r+g^++l^+-r=g}} (\mathfrak{f}_{k_1l_1g_1},\dots,\mathfrak{f}_{k_rl_rg_r}) \circ_{h_1,\dots,h_r} \mathfrak{q}^+_{k^+l^+g^+}}_{=:\,(\Box^+)^r_{klg}} \\ & + \sum_{r'=0}^{r-1} \binom{r}{r'} \sum_{\substack{h_1,\dots,h_{r'}\geq 1\\ k^+,l^+,g^+\geq 0\\ k_1,l_1,g_1,\dots,k_r,l_r,g_r\geq 0\\ h_1+\dots+h_{r'}=l^+\\ k_1+\dots+k_r+k^+-l^+=k\\ l_1+\dots+l_r=l\\ g_1+\dots+g_r+g^++l^+-r=g}} (\mathfrak{f}_{k_1l_1g_1},\dots,\mathfrak{f}_{k_{r'}l_{r'}g_{r'}}) \circ_{h_1,\dots,h_{r'}} \mathfrak{q}^+_{k^+l^+g^+} \\ & \qquad \odot \mathfrak{f}_{k_{r'+1}l_{r'+1}g_{r'+1}} \odot \dots \odot \mathfrak{f}_{k_rl_rg_r} \\ = {} & (\Box^+)^r_{klg} + \sum_{r'=0}^{r-1} \binom{r}{r'} \sum_{\substack{0\leq k'\leq k\\ 0\leq l'\leq l\\ 0\leq g'\leq g+r-r'}} \Bigg((\Box^+)^{r'}_{k'l'g'} \\ & \qquad \odot \underbrace{\sum_{\substack{k_{r'+1},l_{r'+1},g_{r'+1},\dots,k_r,l_r,g_r\geq 0\\ k_{r'+1}+\dots+k_r=k-k'\\ l_{r'+1}+\dots+l_r=l-l'\\ g_{r'+1}+\dots+g_r-(r-r')=g-g'}} \mathfrak{f}_{k_{r'+1}l_{r'+1}g_{r'+1}} \odot \dots \odot \mathfrak{f}_{k_rl_rg_r}}_{=:\,(\triangle)^{r-r'}_{k-k',l-l',g-g'}} \Bigg).\end{aligned}$$

We used here that for fixed $r \geq 0$, $0 \leq r' \leq r-1$, k, $l \geq 0$ and $g \in \mathbb{Z}$, we have

$$\left\{\begin{array}{l} h_1, \dots, h_{r'} \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ \hline h_1 + \dots + h_{r'} = l^+ \\ k_1 + \dots + k_r + k^+ - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g \end{array}\right\} = \bigsqcup_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ -r' \leq g' \leq g + r - r'}} \left\{\begin{array}{l} h_1, \dots, h'_r \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_{r'}, l_{r'}, g_{r'} \geq 0 \\ \hline h_1 + \dots + h_{r'} = l^+ \\ k_1 + \dots + k_{r'} + k^+ - l^+ = k' \\ l_1 + \dots + l_{r'} = l' \\ g_1 + \dots + g_{r'} + g^+ + l^+ - r' = g' \end{array}\right\}$$

$$\times \left\{\begin{array}{l} k_{r'+1}, l_{r'+1}, g_{r'+1}, \dots, k_r, l_r, g_r \geq 0 \\ \hline k_{r'+1} + \dots + k_r = k - k' \\ l_{r'+1} + \dots + l_r = l - l' \\ g_{r'+1} + \dots + g_r - (r - r') = g - g' \end{array}\right\},$$

where the notation is the vertical version of $\{\cdot \mid \cdot\}$. In fact, the summation starts from $g' = 0$ because if $g' < 0$, then $(\Box^+)^{r'}_{k'l'g'} = 0$. Summing over $r \in \mathbb{N}_0$, we get

$$\begin{aligned} &\sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)^+_{i,r} \circ \iota_k \\ &\quad = \sum_{r=0}^{\infty} \frac{1}{r!} (\Box^+)^r_{klg} + \sum_{r=0}^{\infty} \sum_{r'=0}^{r-1} \frac{1}{r'!} \frac{1}{(r-r')!} \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ 0 \leq g' \leq g + r - r'}} (\Box^+)^{r'}_{k'l'g'} \odot (\triangle)^{r-r'}_{k-k', l-l', g-g'} \\ &\quad = \sum_{r=0}^{\infty} \frac{1}{r!} (\Box^+)^r_{klg} + \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ g' \geq 0}} \Big(\sum_{r'=0}^{\infty} \frac{1}{r'!} (\Box^+)^{r'}_{k'l'g'} \Big) \odot \Big(\sum_{t=1}^{\infty} \frac{1}{t!} (\triangle)^t_{k-k', l-l', g-g'} \Big), \end{aligned} \tag{D.25}$$

where we used the substitution $t = r - r'$ and the fact that if $g - g' < -t$, then $(\triangle)^t_{k-k', l-l', g-g'} = 0$. Similarly, we obtain

$$\begin{aligned} &\sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)^-_{i,r} \circ \iota_k \\ &\quad = \sum_{r=0}^{\infty} \frac{1}{r!} (\Box^-)^r_{klg} + \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ g' \geq 0}} \Big(\sum_{t=1}^{\infty} \frac{1}{t!} (\triangle)^t_{k-k', l-l', g-g'} \Big) \odot \Big(\sum_{r'=0}^{\infty} \frac{1}{r'!} (\Box^-)^{r'}_{k'l'g'} \Big). \end{aligned} \tag{D.26}$$

Now, $e^{\mathfrak{f}}\,\Delta^+ = \Delta^-\,e^{\mathfrak{f}}$ is equivalent to

$$\sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)^+_{i,r} \circ \iota_k = \sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (*)^-_{i,r} \circ \iota_k \quad \text{for all } k, l \geq 0 \text{ and } i \in \mathbb{Z}. \tag{D.27}$$

By our previous computations, this follows from

$$\sum_{r=0}^{\infty} \frac{1}{r!} (\Box^+)^r_{klg} = \sum_{r=0}^{\infty} \frac{1}{r!} (\Box^-)^r_{klg} \quad \text{for all } k, l, g \geq 0,$$

which are precisely equations (D.20). In order to prove the reverse implication, one needs to do induction on signatures (k,l,g) as in (a). It is possible in the strict case by the following lemma.

Subclaim (Induction in strict case I). Suppose that $\mathfrak{f}_{0lg} = \mathfrak{f}_{k0g} = 0$ for all k, l, $g \geq 0$. Then for any k, $l \geq 0$ and $i \in \mathbb{Z}$, $g = i - k + 1$, only the terms with $(k',l',g') \prec (k,l,g)$ contribute to the sums on the right-hand side of (D.25) and (D.26). These sums vanish if $(k,l,g) = (0,0,0)$.

Proof. Recall the definition of $\prec$ on p. 310. First of all, $(\triangle)^t_{k-k',l-l',g-g'} \neq 0$ implies

$$D \coloneqq (k+l+2g) - (k'+l'+2g') = (k-k') + (l-l') + 2(g-g') \geq t + t + 2(-t) = 0,$$

where $k - k'$, $l - l' \geq t$ holds due to strictness. Suppose that $D = 0$. Then $g' \geq g$ must hold because $k - k' \geq 0$ and $l - l' \geq 0$. If $g' = g$, then $l = l'$ and $k = k'$ must hold; this is a contradiction with $k - k' \geq t \geq 1$. Therefore, it holds $g' > g$, and case (ii) of the definition of $\prec$ applies.

If $(k,l,g) = (0,0,0)$, then $(\triangle)^t_{0,0,-g'} = 0$ because $t \geq 1$ and $k - k' \geq t$. *(Subclaim)* □

(c) We have

$$e^{\mathfrak{f}} = \sum_{k,l,g \geq 0} \langle e^{\mathfrak{f}} \rangle_{klg} \hbar^{k+g-1},$$

where

$$\langle e^{\mathfrak{f}} \rangle_{klg} = \sum_{\substack{r \geq 0 \\ k_1,l_1,g_1,\ldots,k_r,l_r,g_r \geq 0 \\ k_1+\cdots+k_r=k \\ l_1+\cdots+l_r=l \\ g_1+\cdots+g_r-r+1=g}} \frac{1}{r!} \mathfrak{f}_{k_1 l_1 g_1} \odot \cdots \odot \mathfrak{f}_{k_r l_r g_r}$$

$$= \mathfrak{f}_{klg} + \sum_{\substack{r\geq 2\\ k_1,l_1,g_1,\dots,k_r,l_r,g_r\geq 0\\ k_1+\dots+k_r=k\\ l_1+\dots+l_r=l\\ g_1+\dots+g_r-r+1=g}} \frac{1}{r!}\mathfrak{f}_{k_1 l_1 g_1}\odot\dots\odot\mathfrak{f}_{k_r l_r g_r} \tag{D.28}$$

for all $(k,l,g)\neq(0,0,1)$ with $k,\, l\geq 0,\, g\in\mathbb{Z}$. For $(k,l,g)=(0,0,1)$, one has to add the unit e coming from the summand with $r=0$. We find that

$$\langle e^{\mathfrak{f}^-}e^{\mathfrak{f}^+}\rangle_{klg} = \sum_{\substack{r^-,r^+\geq 0\\ k_1^-,l_1^-,g_1^-,\dots,k_{r^-}^-,l_{r^-}^-,g_{r^-}^-\geq 0\\ k_1^+,l_1^+,g_1^+,\dots,k_{r^+}^+,l_{r^+}^+,g_{r^+}^+\geq 0\\ k_1^-+\dots+k_{r^-}^-=l_1^++\dots+l_{r^+}^+\\ k_1^++\dots+k_{r^+}^+=k\\ l_1^-+\dots+l_{r^-}^-=l\\ g_1^++\dots+g_{r^+}^++g_1^-+\dots+g_r^--r^+-r^-+k_1^-+\dots+k_{r^-}^-+1=g}} \frac{1}{r^+!r^-!}(\mathfrak{f}^-_{k_1^-l_1^-g_1^-}\odot\dots\odot\mathfrak{f}^-_{k_{r^-}^-l_{r^-}^-g_{r^-}^-})\circ(\mathfrak{f}^+_{k_1^+l_1^+g_1^+}\odot\dots\odot\mathfrak{f}^+_{k_{r^+}^+l_{r^+}^+g_{r^+}^+})$$

for all $k,\, l\geq 0,\, g\in\mathbb{Z}$. The composition $\mathfrak{f}^-\diamond\mathfrak{f}^+=\log\Big(e^{\mathfrak{f}^-}\circ e^{\mathfrak{f}^+}\Big)$ is the unique $\mathfrak{f}$ such that $e^{\mathfrak{f}}=e^{\mathfrak{f}^-}\circ e^{\mathfrak{f}^+}$; this is equivalent to

$$\langle e^{\mathfrak{f}}\rangle_{klg}=\langle e^{\mathfrak{f}^-}e^{\mathfrak{f}^+}\rangle_{klg}\quad\text{for all } k,l\geq 0, g\in\mathbb{Z}. \tag{D.29}$$

In the strict case, this can be solved for $(\mathfrak{f}_{klg})$ by induction on signatures.

Subclaim (Induction in strict case II)*.* Suppose that $\mathfrak{f}^+$ and $\mathfrak{f}^-$ are strict. Then it holds $(k_i,l_i,g_i)\prec(k,l,g)$ for all $i=1,\,\dots,\,r$ and $r\geq 2$ in (D.28) for all $k,\, l\geq 0,\, g\in\mathbb{Z}$.

Proof. We compute

$$k+l+2g-2=(k_1+l_1+2g_1-2)+\dots+(k_r+l_r+2g_r-2),$$

where $k_i+l_i+2g_i-2\geq 0$ by strictness. It follows that $k+l+2g\geq k_i+l_i+2g_i$ holds for all $i=1,\,\dots,\,r$. If, e.g., $k+l+2g=k_1+l_1+2g_1$, then $k_2=l_2=\dots=k_r=l_r=1$, $g_2=\dots=g_r=0$, and hence $g=1-r$. However, this can not happen as $r\geq 2$ and $g\geq 0$. Therefore, case (i) of $\prec$ always occurs. *(Subclaim)* □

Using this, we can set $\mathfrak{f}_{110}=\mathfrak{f}^-_{110}\circ\mathfrak{f}^+_{110}$ and $\mathfrak{f}_{0lg}=\mathfrak{f}_{k0g}\coloneqq 0$ for all $k,\, l,\, g\geq 0$ and solve (D.29) for $(\mathfrak{f}_{klg})$ for all $k,\, l\geq 1,\, g\geq 0$ by induction over (k,l,g). The solution (D.22) will solve (D.29) for all $k,\, l\geq 0,\, g\in\mathbb{Z}$ (the equations for $g<0$ consist of disconnected gluings and can be checked by splitting into connected components).Conditions on the filtration degree are easy to check as in [CFL15, Lemma 8.5]. □

We expect that weak complete filtered IBL_∞-algebras in MV-formalism with weak morphisms form a subcategory of the category of complete filtered MV-algebras (to see this, it remains to prove a version of (c) of Proposition D.2.6 for weak morphisms). The identity morphism in this category is the continuous $\mathbb{K}((\hbar))$-linear extension of the trivial extension of the identity $\mathbb{1} : \hat{U} \to \hat{U}$ to $\hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar))$ (see (iii) of Remark D.1.10).

In the following, we will compare Definition D.2.3 to the definition of filtered IBL_∞-algebras from [CFL15, Section 8].

A *filtered* IBL_∞*-algebra of bidegree* (d, γ) *over* $\mathbb{K}$ *on a complete filtered graded vector space* W *according to [CFL15]* is a collection of homogenous $\mathbb{K}$-linear maps $\mathfrak{q}_{klg} : \hat{\mathrm{S}}_k U \to \hat{\mathrm{S}}_l U$ of finite filtration degrees for all k, l, $g \in \mathbb{N}_0$, where U was defined in (D.14), which satisfy, firstly, (1) and (2) of (a) of Proposition D.2.6 with the strict inequality in (2) for all (k, l, g) from the set

$$\{(0,0,0),(1,0,0),(0,1,0),(2,0,0),(0,2,0),(0,0,1)\} \tag{D.30}$$

and, secondly, the equations (D.24). A *morphism of filtered* IBL_∞*-algebras of bidegree* (d, γ) *over* $\mathbb{K}$ *on complete filtered graded vector spaces* W^+ *and* W^- *according to [CFL15]* is a collection of homogenous $\mathbb{K}$-linear maps $\mathfrak{f}_{klg} : \hat{\mathrm{S}}_k U^+ \to \hat{\mathrm{S}}_l U^-$ of finite filtration degrees for all k, l, $g \in \mathbb{N}_0$ which satisfy, firstly, (1) and (2) of (b) of Proposition D.2.6 with the strict inequality in (2) for $(k, l, g) \in$ (D.30) and, secondly, the equations (D.27).

Recall (D.14) and notice that if we define the filtration $\mathcal{F}^\lambda_W U := (\mathcal{F}^\lambda W)[1]$ for all $\lambda \in \mathbb{R}$ and denote the corresponding filtration degree by $\|\cdot\|_W$, then it holds $\|\cdot\| = \|\cdot\|_W + \gamma$ on U, and for a map $\mathfrak{q}_{klg} : \hat{\mathrm{S}}_k U \to \hat{\mathrm{S}}_l U$, we have

$$\|\mathfrak{q}_{klg}\| \geq -2\gamma(k + g - 1) \quad \Longleftrightarrow \quad \|\mathfrak{q}_{klg}\|_W \geq \gamma(2 - 2g - k - l).$$

Signatures (D.30) together with $(1, 1, 0)$ correspond to unstable surfaces, i.e., those (k, l, g) for which $\chi_{klg} = 2 - 2g - k - l \geq 0$. Allowing morphisms $\mathfrak{f}$ which have non-zero components of these signatures leads to the appearance of an infinite number of summands in $\langle \Delta^- e^{\mathfrak{f}} \rangle_{klg}$, $\langle e^{\mathfrak{f}} \Delta^+ \rangle_{klg}$ and $\langle e^{\mathfrak{f}^-} e^{\mathfrak{f}^+} \rangle_{klg}$ (and in the Maurer-Cartan equation (D.38) later); the strict inequalities for filtration degrees of these components seem to be the minimal condition to algebraically handle this situation. Again, the author does not see any technical reason for imposing strict filtration degree conditions for $\mathfrak{q}_{klg}$. See Figure D.3 for the graphical explanation of the *bubbling*.

We observe the following differences between our approach to weak IBL_∞-algebras and the original approach from [CFL15]:

(i) *Bubbling.* The definition of [CFL15] is symmetric in inputs and outputs, whereas our definition is not (compare (D.30) and (3) of Proposition D.2.6). We will illustrate this in Example D.2.9 below.

(ii) *General ring.* The theory of [CFL15] is formulated over a filtered graded commutative ring R, i.e., W is a filtered R-module and the maps are R-linear. We can tweak our formalism to handle this situation if R is a $\mathbb{K}$-algebra (e.g., the Novikov ring) by replacing $\mathbb{K}((\hbar))$ with $R((\hbar))$ in Definition D.2.3.

(iii) *Completions.* The BV-formalism of [CFL15] uses the completion $\hat{\mathrm{S}}U$ with respect to the union filtration $\mathcal{F}_\cup$ of the induced filtration and the filtration by weights (see Section D.1); however, this is not necessarily a filtered bialgebra (see Example D.1.5), Proposition D.1.8 might not apply, and it is not clear whether $e^{\mathfrak{f}}$ (or the multiplication with the exponential of the Maurer-Cartan element later) are well-defined operators on $\hat{\mathrm{S}}U((\hbar))$.

In the following case, our theory and the theory of [CFL15] agree.

Proposition D.2.7 (Equivalence of definitions in bounded case over $\mathbb{K}$)**.** *Let $d \in \mathbb{Z}$ and $\gamma > 0$. For $\zeta \in \mathbb{R}$, we denote by $\mathcal{W}_\zeta$ the class of complete filtered graded vector spaces W for which there is an $\alpha > \zeta$ such that*

$$\mathcal{F}^\alpha W = W. \tag{D.31}$$

For such W, we will consider weak IBL_∞-algebras of bidegree (d,γ) over $\mathbb{K}$ and their weak IBL_∞-morphisms. We have the following:

(a) Filtered IBL_∞-algebras and morphisms from [CFL15] over $\mathcal{W}_{-\gamma}$ are also complete filtered MV-algebras and morphisms from Definition D.2.3, respectively.

(b) The two definitions agree over $\mathcal{W}_0$. If in addition $\gamma \geq 1$, then also the BV-formalisms are identical.

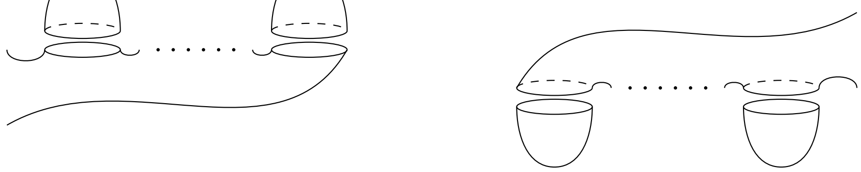
(c) The canonical dIBL-structure on the (reduced) dual cyclic bar complex

$$W = \hat{\mathrm{B}}^*_{\mathrm{cyc}} V[2-n]$$

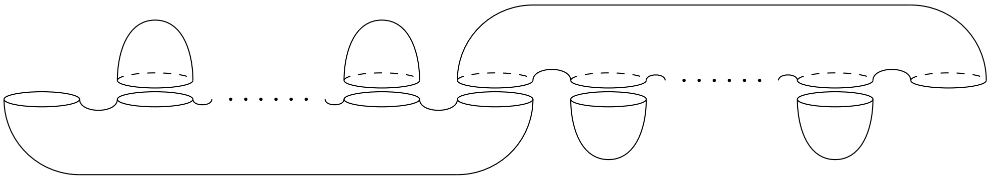
for a Poincaré duality algebra V of degree n from Section 3.4 satisfies (b).

Proof. For $W \in \mathcal{W}_\zeta$, there is an $\alpha > \zeta$ such that

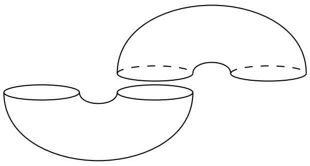
$$\|\cdot\| = \|\cdot\|_W + \gamma \geq \alpha + \gamma \quad \text{on } U,$$



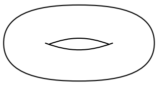
(a) Connected bubbling of $\mathfrak{f}_{010}$ in $\Delta^- e^{\mathfrak{f}}$. (b) Connected bubbling of $\mathfrak{f}_{100}$ in $e^{\mathfrak{f}}\,\Delta^+$.



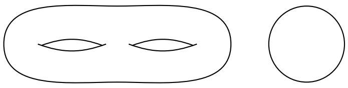
(c) Connected bubbling involving $\mathfrak{f}_{010}$ or $\mathfrak{f}_{100}$ in the composition $e^{\mathfrak{f}^- \diamond \mathfrak{f}^+} = e^{\mathfrak{f}^-} e^{\mathfrak{f}^+}$.



(d) Connected bubbling in the composition with no $\mathfrak{f}_{010}$ and $\mathfrak{f}_{100}$.



(e) Disconnected bubbling of $\mathfrak{f}_{001}$ in the relation $\Delta^- e^{\mathfrak{f}}$, $e^{\mathfrak{f}}\,\Delta^+$ or in the composition.



(f) A possible disconnected bubbling involving $\mathfrak{f}_{000}$ in $\Delta^- e^{\mathfrak{f}}$, $e^{\mathfrak{f}}\,\Delta^+$ or in the composition.

Figure D.3.: Bubbling in IBL_∞-relations. If we glue any of the components above to a surface of signature (k, l, g), the signature remains the same. Note that since g is defined via the Euler characteristic, adding a disconnected component without inputs and outputs decreases g by one.

and hence

$$\|\mathfrak{q}_{klg}(v)\|_U \geq l(\alpha+\gamma) \quad \text{for all } v \in \hat{\mathrm{S}}_k U \tag{D.32}$$

and any $k \geq 0$, $l \geq 1$, $g \geq 0$. It holds also for $l=0$ because the filtration on $\hat{S}_0 U \simeq \mathbb{K}$ is non-negative. We see that if $\zeta \geq -\gamma$, which is the case of both (a) and (b), then the sums $\sum_{l=0}^{\infty} \mathfrak{q}_{klg}$ converge automatically; i.e., (4) of (a) of Proposition D.2.6 holds. It remains to study the relation of the strict filtration degree conditions

$$\forall l, g \in \mathbb{N}_0 : \quad \|\mathfrak{q}_{0lg}\| = \|\mathfrak{q}_{0lg}(1)\| > -2\gamma(g-1) \tag{D.33}$$

and

$$\forall (k,l,g) \in \text{(D.30)} : \quad \|\mathfrak{q}_{klg}\| > -2\gamma(k+g-1). \tag{D.34}$$

Here, $\|\mathfrak{q}_{0lg}\| = \|\mathfrak{q}_{0lg}(1)\|$ holds because the trivial filtration on $\mathbb{K}$ is non-negative and it holds $\|1\| = 0$. For morphisms, the story is the same, and the rest is implied by Proposition D.2.6.

(a) Assuming (D.34), we have to prove (D.33). It is easy to see that if $\zeta \geq -\gamma$, then (D.33) follows from (D.32) except for $(k,l,g) = (0,0,0)$, $(0,1,0)$, $(0,2,0)$ and $(0,0,1)$. These are precisely the signatures from (D.30) with no input; in particular, they are implied by (D.34).

(b) Assuming (D.33), we have to prove (D.34). Clearly, (D.33) implies the strict inequality for all $(k,l,g) \in$(D.30) with no input. It remains to check it for $(1,0,0)$ and $(2,0,0)$. Using $\zeta \geq -\gamma$, we obtain

$$\|\mathfrak{f}_{100}(v)\| \geq \|\mathfrak{f}_{100}\| + \|v\| \geq 0 + (\gamma+\alpha) > 0 \quad \text{for all } v \in \hat{\mathrm{S}}_1 V,$$

and using $\zeta \geq 0$, we obtain

$$\|\mathfrak{f}_{200}(v)\| \geq \|\mathfrak{f}_{200}\| + \|v\| \geq -2\gamma + 2(\gamma+\alpha) > 0 \quad \text{for all } v \in \hat{\mathrm{S}}_2 V.$$

Because $\operatorname{im}\mathfrak{f}_{100}$, $\operatorname{im}\mathfrak{f}_{200} \subset \mathbb{K}$, it must hold $\mathfrak{f}_{100} = \mathfrak{f}_{200} = 0$, and hence the strict filtration degree condition is automatically satisfied.

The equivalence of BV-formalisms for $\gamma \geq 1$ follows from Lemma D.1.6 because $\mathcal{F}^1 U = (\mathcal{F}^{1-\gamma} W)[1] = W[1] = U$, where $\alpha > 0 \geq 1 - \gamma$.

(c) For any $k \in \mathbb{N}_0$ and $k \leq \lambda < k+1$, we have

$$\mathcal{F}^{\lambda}_{\mathrm{w}} \mathrm{B}^{\mathrm{cyc}} V = \mathrm{B}^{\mathrm{cyc}}_1 V \oplus \cdots \oplus \mathrm{B}^{\mathrm{cyc}}_k V,$$

$$\begin{aligned}\mathcal{F}^\lambda \mathrm{B}^*_{\mathrm{cyc}} V &= \{\psi \in \mathrm{B}^*_{\mathrm{cyc}} V \mid \psi|_{\mathcal{F}^\lambda \mathrm{B}^{\mathrm{cyc}} V} = 0\} \\ &\simeq (\mathrm{B}^*_{\mathrm{cyc}} V)_{k+1} \oplus (\mathrm{B}^*_{\mathrm{cyc}} V)_{k+2} \oplus \cdots ,\end{aligned}$$

where $\mathcal{F}_{\mathrm{w}}$ is the increasing filtration by weights, $\mathcal{F}$ the dual filtration and $(\mathrm{B}^*_{\mathrm{cyc}} V)_i$ the graded dual to $\mathrm{B}^{\mathrm{cyc}}_i V$. We also see that $\mathcal{F}^\lambda \mathrm{B}^*_{\mathrm{cyc}} V = \mathrm{B}^*_{\mathrm{cyc}} V$ for all $\lambda < 1$. Since $\hat{\mathrm{B}}^*_{\mathrm{cyc}} V$ is the completion of $\mathrm{B}^*_{\mathrm{cyc}} V$ in the graded category, we have $\hat{\mathrm{B}}^*_{\mathrm{cyc}} V \in W_0$. Finally, we know that $\gamma = 2$ for the canonical dIBL-algebra. □

Remark D.2.8 (Generalization over algebra). It is easy to see that (a) of Proposition D.2.7 also holds when we work over a non-negatively filtered augmented unital graded $\mathbb{K}$-algebra R. However, (b) should not generalize as there are no conditions in the filtered MV-formalism implying the strict filtration degree condition for $(1,0,0)$ and $(2,0,0)$. ◁

Example D.2.9 (Asymmetry of MV-formalism). How is it possible that there are no strict filtration degree conditions on $(1,0,0)$ and $(2,0,0)$ in the filtered MV-formalism even though these unstable surfaces can obviously bubble off (see Figure D.3)?[10]

Heuristically, increasing the number of $(1,0,0)$'s glued to the outputs increases the number of times $\bar{\delta}$ has to be applied to the input to split it and feed it into the new $(1,0,0)$'s. The limit conilpotency property (D.6) steps in, increases the filtration degree and provides convergence of the infinite sum. As for the bubbling of $(2,0,0)$, we look at (c) and (d) of Figure D.3 and see that there is always an increasing number of $(0,2,0)$'s or $(1,0,0)$'s in the adjacent components. Now, $(0,2,0)$ increases the filtration degree due to the strict filtration degree condition, and $(1,0,0)$ increases the filtration degree due to the limit conilpotency property as explained above. The convergence is again established.

We will now illustrate the bubbling of $\mathfrak{f}_{100}$ and $\mathfrak{f}_{010}$ in the relations for morphisms with a concrete computation. Let $\mathfrak{f}_{100} : \hat{\mathrm{S}}_1 U \to \hat{\mathrm{S}}_0 U$, $\mathfrak{f}_{010} : \hat{\mathrm{S}}_0 U \to \hat{\mathrm{S}}_1 U$, $\mathfrak{q}_{1l0} : \hat{\mathrm{S}}_1 U \to \hat{\mathrm{S}}_l U$ and $\mathfrak{q}_{k10} : \hat{\mathrm{S}}_k U \to \hat{\mathrm{S}}_1 U$ for k, $l \geq 0$ be $\mathbb{K}$-linear maps such that the following sums converge to $\mathbb{K}((\hbar))$-linear operators $\hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar))$:

$$\Delta \coloneqq \sum_{l=1}^{\infty} \hat{\mathfrak{q}}_{1l0}, \qquad \mathfrak{f} \coloneqq \mathfrak{f}_{100},$$

$$\Delta' \coloneqq \sum_{k=1}^{\infty} \hat{\mathfrak{q}}_{k10} \hbar^{k-1}, \qquad \mathfrak{f}' \coloneqq \mathfrak{f}_{010} \hbar^{-1}.$$

---

[10] Note that this question is relevant only when we work over a general $\mathbb{K}$-algebra R, so that $\mathfrak{f}_{100}$ and $\mathfrak{f}_{200}$ do not necessarily vanish.

Assume that $\|\mathfrak{f}\| = \|\mathfrak{f}_{100}\| \geq 0$. Because $\mathfrak{f}_{100}(1) = 0$, convergence of the exponential

$$e^{\mathfrak{f}} = \sum_{r=0}^{\infty} \frac{1}{r!} \mathfrak{f}_{100}^{\odot r} : \hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar))$$

relies purely on the limit conilpotency property (see the proof of Proposition D.1.8). Assume that $\|\mathfrak{f}_{010}\| = \|\mathfrak{f}_{010}(1)\| > 2\gamma$, i.e., $\|\mathfrak{f}'\| = \|\mathfrak{f}'(1)\| = \|\mathfrak{f}_{010}(1)\| - 2\gamma > 0$. Clearly, convergence of the exponential

$$e^{\mathfrak{f}'} = \sum_{r=0}^{\infty} \frac{1}{r!} \mathfrak{f}_{010}^{\odot r} \hbar^{-r} : \hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar)) \tag{D.35}$$

relies purely on the strict filtration degree condition.

For all $v \in \mathrm{S}_1 U$, we have

$$\begin{aligned} (e^{\mathfrak{f}}\, \Delta)(v) &= \sum_{l=1}^{\infty} e^{\mathfrak{f}} \hat{\mathfrak{q}}_{1l0}(v) \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \mu^{(l)} \mathfrak{f}_{100}^{\otimes l} \bar{\delta}^{(l)} \mathfrak{q}_{1l0}(v) \\ &= \sum_{l=1}^{\infty} \mathfrak{f}_{100}^{\otimes l} \big(\mathfrak{q}_{1l0}(v)\big). \end{aligned} \tag{D.36}$$

In this simplest case, the limit conilpotency property takes the form $\bar{\delta}^{(k)}(\mathrm{S}_l U) = 0$ for $k > l$, which bounds the number of $\mathfrak{f}_{100}$'s applied to individual summands. Next, we have

$$\begin{aligned} (\Delta'\, e^{\mathfrak{f}'})(1) &= \sum_{r=0}^{\infty} \frac{1}{r!}\, \Delta'\, \mathfrak{f}_{010}(1)^r \hbar^{-r} \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i=1}^{\infty} \hat{\mathfrak{q}}_{i10} \mathfrak{f}_{010}(1)^r \\ &= \sum_{r=1}^{\infty} \sum_{i=1}^{r} \frac{1}{i!(r-i)!} \mathfrak{q}_{i10}\big(\mathfrak{f}_{010}(1)^i\big) \mathfrak{f}_{010}(1)^{r-i} \hbar^{-r+i-1} \\ &= \Big(\sum_{i=1}^{\infty} \frac{1}{i!} \mathfrak{q}_{i10}\big(\mathfrak{f}_{010}(1)^i\big)\Big) \underbrace{\Big(\sum_{t=0}^{\infty} \frac{1}{t!} \mathfrak{f}_{010}(1)^t \hbar^{-t-1}\Big)}_{= e^{\mathfrak{f}'(1)} \hbar^{-1}}. \end{aligned} \tag{D.37}$$

Clearly, the condition $\|\mathfrak{f}_{010}(1)\| > 0$ is required for the convergence of the sum in the first bracket. ◁

Let us now consider the twisting of a complete filtered IBL$_\infty$-algebra in MV-formalism $(\mathfrak{q}_{klg})$ on W with a Maurer-Cartan element $\mathfrak{n}$. The *Maurer-Cartan element* is, by definition, a morphism from the trivial IBL_∞-algebra 0 to W. Because $\hat{\mathrm{S}}0 = \hat{\mathrm{S}}_0 0 = \mathbb{K}$, only the $(0, l, g)$-components denote by $\mathfrak{n}_{lg} : \mathbb{K} \to \hat{\mathrm{S}}_l U$ for l, $g \geq 0$ might be non-zero, and thus

$$\mathfrak{n} = \sum_{l,g \geq 0} \mathfrak{n}_{lg} \hbar^{g-1} : \ \hat{\mathrm{S}}0((\hbar)) \longrightarrow \hat{\mathrm{S}}U((\hbar)).$$

Clearly, $e^{\mathfrak{n}}(1) = e^{\mathfrak{n}(1)} \in \hat{\mathrm{S}}U((\hbar))$. Consider the left multiplication $L_{e^{\mathfrak{n}(1)}} : \hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar))$, and let Δ be the BV-operator for $(\mathfrak{q}_{klg})$. The BV-operator for the twisted IBL_∞-algebra $(\mathfrak{q}^{\mathfrak{n}}_{klg})$ is defined by

$$\Delta^{\mathfrak{n}} := L_{e^{-\mathfrak{n}(1)}} \circ \Delta \circ L_{e^{\mathfrak{n}(1)}} : \hat{\mathrm{S}}U((\hbar)) \longrightarrow \hat{\mathrm{S}}U((\hbar)). \tag{D.38}$$

The twisting always produces an input-strict IBL_∞-algebra. Indeed, we have

$$\Delta \circ e^{\mathfrak{n}} = 0 \quad \iff \quad \Delta(e^{\mathfrak{n}(1)}) = 0 \quad \iff \quad \Delta^{\mathfrak{n}}(1) = 0.$$

This is called the *Maurer-Cartan equation.*

Proposition D.2.10 (Twisted IBL_∞-algebra)**.** *Let (W, Δ) be a complete filtered IBL_∞-algebra in* MV*-formalism, and let $\mathfrak{n}$ be a Maurer-Cartan element. Then $\Delta^{\mathfrak{n}}$ defined by* (D.38) *is an input-strict complete filtered IBL_∞-algebra in* MV*-formalism of the same bidegree as (W, Δ), and for its components $(\mathfrak{q}^{\mathfrak{n}}_{klg})$, it holds*

$$\mathfrak{q}^{\mathfrak{n}}_{klg} = \sum_{\substack{r \geq 0 \\ k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ k' - h_1 - \dots - h_r = k \\ l' + l_1 + \dots + l_r - h_1 - \dots - h_r = l \\ g' + h_1 + \dots + h_r - r + g_1 + \dots + g_r = g}} \mathfrak{q}_{k'l'g'} \circ_{h_1, \dots, h_r} (\mathfrak{n}_{l_1 g_1}, \dots, \mathfrak{n}_{l_r, g_r}) \quad \textit{for all } k \geq 1, l, g \geq 0,$$

where $(\mathfrak{q}_{klg})$ and $(\mathfrak{n}_{lg})$ are components of Δ and $\mathfrak{n}$, respectively.

Proof. Using $L_{e^{\mathfrak{n}(1)}} = e^{\mathfrak{n}} \odot \mathbb{1}$, we compute

$$\begin{aligned} &\Delta \circ L_{e^{\mathfrak{n}(1)}} \\ &\quad = \sum_{\substack{r \geq 0 \\ k', l', g', l_1, g_1, \dots, l_{r'}, g_{r'} \geq 0}} \frac{1}{r!} \hat{\mathfrak{q}}_{k'l'g'} \circ (\mathfrak{n}_{l_1 g_1} \odot \cdots \odot \mathfrak{n}_{l_r g_r} \odot \mathbb{1}) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \\ &\quad = \sum_{\substack{r \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_{r'}, g_{r'} \geq 0 \\ h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r + k = k'}} \frac{1}{r!} \mathfrak{q}_{k'l'g'} \circ_{h_1, \dots, h_r, k} (\mathfrak{n}_{l_1 g_1}, \cdots, \mathfrak{n}_{l_r g_r}, \mathbb{1}) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{r\geq 0\\ 0\leq r'\leq r\\ k,k',l',g',l_1,g_1,\dots,l_{r'},g_{r'}\geq 0\\ h_1,\dots,h_{r'}\geq 1\\ h_1+\dots+h_{r'}+k=k'}} \frac{1}{(r-r')!}\frac{1}{r'!}\mathfrak{n}_{l_{r'+1}g_{r'+1}}\odot\dots\odot\mathfrak{n}_{l_rg_r}\\
&\qquad\qquad \odot\big(\mathfrak{q}_{k'l'g'}\circ_{h_1,\dots,h_{r'},k}(\mathfrak{n}_{l_1g_1},\dots,\mathfrak{n}_{l_{r'}g_{r'}},\mathbb{1})\big)\hbar^{g_1+\dots+g_r-r+k'+g'-1}\\
&= \Big(\sum_{t\geq 0}\frac{1}{t!}\mathfrak{n}_{l'_1g'_1}\odot\dots\odot\mathfrak{n}_{l'_tg'_t}\hbar^{g'_1+\dots+g'_t-t}\Big)\\
&\quad \odot\Big(\sum_{k,l,g\geq 0}\sum_{\substack{r'\geq 0\\ k,k',l',g',l_1,g_1,\dots,l_r,g_r\geq 0\\ h_1,\dots,h_{r'}\geq 1\\ k'-h_1-\dots-h_{r'}=k\\ l'+l_1+\dots+l_{r'}-h_1-\dots-h_{r'}=l\\ g'+h_1+\dots+h_{r'}-r'+g_1+\dots+g_{r'}=g}}\frac{1}{r'!}\mathfrak{q}_{k'l'g'}\circ_{h_1,\dots,h_{r'},k}(\mathfrak{n}_{l_1g_1},\dots,\mathfrak{n}_{l_{r'}g_{r'}},\mathbb{1})\hbar^{k+g-1}\Big)\\
&= \Big(\sum_{t\geq 0}\frac{1}{t!}(\mathfrak{n}_{l'_1g'_1}\odot\dots\odot\mathfrak{n}_{l'_tg'_t}\odot\mathbb{1})\hbar^{g'_1+\dots+g'_t-t}\Big)\\
&\quad \circ\Big(\sum_{k,l,g\geq 0}\sum_{\substack{r'\geq 0\\ k,k',l',g',l_1,g_1,\dots,l_r,g_r\geq 0\\ h_1,\dots,h_{r'}\geq 1\\ k'-h_1-\dots-h_{r'}=k\\ l'+l_1+\dots+l_{r'}-h_1-\dots-h_{r'}=l\\ g'+h_1+\dots+h_{r'}-r'+g_1+\dots+g_{r'}=g}}\frac{1}{r'!}\mathfrak{q}_{k'l'g'}\circ_{h_1,\dots,h_{r'}}(\mathfrak{n}_{l_1g_1},\dots,\mathfrak{n}_{l_{r'}g_{r'}})\odot\mathbb{1}\hbar^{k+g-1}\Big)\\
&= L_{e^{\mathfrak{n}(1)}}\circ\Big(\sum_{k,l,g\geq 0}\sum_{\substack{r'\geq 0\\ k,k',l',g',l_1,g_1,\dots,l_r,g_r\geq 0\\ h_1,\dots,h_{r'}\geq 1\\ k'-h_1-\dots-h_{r'}=k\\ l'+l_1+\dots+l_{r'}-h_1-\dots-h_{r'}=l\\ g'+h_1+\dots+h_{r'}-r'+g_1+\dots+g_{r'}=g}}\frac{1}{r'!}\widehat{\mathfrak{q}_{k'l'g'}\circ_{h_1,\dots,h_{r'}}(\mathfrak{n}_{l_1g_1},\dots,\mathfrak{n}_{l_{r'}g_{r'}})}\hbar^{k+g-1}\Big).
\end{aligned}
$$

The claim follows. □

D.3. BV-complexes for IBL-infinity-algebras

We plan to use the filtered MV-formalism from the previous section to study chain-maps between the following chain complexes.

Definition D.3.1 (Chain complexes for IBL$_\infty$-algebras)**.** *Given a complete filtered* IBL$_\infty$*-algebra in* MV*-formalism* (U,Δ)*, we call the chain complex* $(\hat{\mathrm{S}}U((\hbar)),\Delta)$ *the* BV-complex*. If* (U,Δ) *is input-strict, i.e., if* $\Delta(1)=0$*, then* $(\hat{\mathrm{S}}U[[\hbar]],\Delta)$ *is a chain complex as well, and we call it the* strict BV-complex*. If* (U,Δ) *is strict, we also have the chain complex* $(\hat{\mathrm{S}}_1U,\mathfrak{q}_{110})$*, and we call it the* IBL$_\infty$-chain complex *(c.f., Definition 3.2.8).*

Clearly, the definition works also for filtered IBL$_\infty$-algebras from [CFL15] (we are just not sure about Proposition D.3.3 below, and thus we rather stick to our formalism).

Remark D.3.2 (BV-bicomplex for surfaces). Writing $\Delta = \Delta_1 + \Delta_2\,\hbar + \Delta_3\,\hbar^2 + \cdots$ in the input-strict case, we have

$$0 \overset{!}{=} \Delta^2 = \Delta_1^2 + \hbar(\Delta_1\,\Delta_2 + \Delta_2\,\Delta_1) + \hbar^2(\Delta_2^2 + \Delta_1\,\Delta_3 + \Delta_3\,\Delta_1) + \hbar^3(\cdots) + \cdots .$$

We see that $\Delta_1^2 = 0$ and $[\Delta_1, \Delta_2] = 0$, where $[\cdot,\cdot]$ is the graded commutator, always hold. Moreover, if $[\Delta_1, \Delta_3] = 0$, then also $\Delta_2^2 = 0$. Now, $|\,\Delta_1\,| = -1$ always holds, and if $d = -1$, then $|\hbar| = -2$ and $|\,\Delta_2\,| = -1 - 2d = 1$. We see that if $[\Delta_1, \Delta_3] = 0$ and $d = -1$, then $(\hat{\mathrm{S}}U, \Delta_1, \Delta_2)$ is a (homological) mixed complex. If $\Delta_k = 0$ for $k \geq 3$, then the $[\hbar^{-1}, \hbar]]$- and $[[\hbar]]$-versions of the total complex (see Remark B.2.4) compute the BV- and strict BV-homology, respectively. Note that by [CV20, Proposition 2.2], these versions are quasi-isomorphism invariants of mixed complexes.[11]

In the case of the canonical dIBL-algebra on cyclic cochains of a Poincaré duality algebra of degree n, we have $d = n - 3$, and so $d = -1$ corresponds to $n = 2$. $\lhd$

Proposition D.3.3 (Morphisms of BV-complexes)**.** *In the category of complete filtered* IBL_∞*-algebras in* MV*-formalism, the following holds:*

(a) A weak morphism of weak IBL_∞*-algebras induces a chain map* $e^{\mathfrak{f}}$ *of* BV*-complexes.*

(b) An input-strict morphism of input-strict IBL_∞*-algebras induces a chain map of both the* BV*-complex and its strict version.*

(c) The $(1,1,0)$*-part of a strict* IBL_∞*-morphism of strict* IBL_∞*-algebras induces a chain map of* IBL_∞*-chain complexes.*

(d) For a Maurer-Cartan element $\mathfrak{n}$*, the left multiplication* $L_{e^{\mathfrak{n}(1)}} : \hat{\mathrm{S}}U((\hbar)) \to \hat{\mathrm{S}}U((\hbar))$ *is an isomorphism of the* BV*-complexes* $(\hat{\mathrm{S}}U((\hbar)), \Delta) \simeq (\hat{\mathrm{S}}U((\hbar)), \Delta^{\mathfrak{n}})$*.*

Proof. Clear. $\square$

Notice that whereas $L_{e^{\mathfrak{n}(1)}} : \hat{\mathrm{S}}U \to \hat{\mathrm{S}}U$ is an isomorphism of chain complexes and it has a well-defined logarithm (i.e., it holds $\|L_{e^{\mathfrak{n}(1)}}\| \geq 0$ and $\|L_{e^{\mathfrak{n}(1)}}(1) - 1\| = \|e^{\mathfrak{n}(1)} - 1\| > 0$), it is typically not equal to $e^{\mathfrak{f}}$ for any (weak) IBL_∞-morphism $\mathfrak{f}$.

As an example, suppose that $\mathfrak{n}(1) = \hbar^{-1}\mathfrak{n}_{10}(1) \in \hat{\mathrm{S}}U$ (recall that we interpret $\mathfrak{n}_{10}$ as a map $\mathbb{K} \to \hat{\mathrm{S}}U$) and assume that $L_{e^{\mathfrak{n}(1)}} = e^{\mathfrak{f}}$ for $\mathfrak{f} = \mathfrak{f}_0\hbar^{-1} + \mathfrak{f}_1 + \mathfrak{f}_2\hbar + \cdots$ with $\hat{\mathrm{S}}_i U \subset \ker(\mathfrak{f}_k)$

---

[11] A morphism of mixed complexes commutes with both the boundary operator and the differential. A quasi-isomorphism of homological mixed complexes is a morphism of mixed complexes which is a quasi-isomorphism with respect to the boundary operator.

for all $i > k$. In particular, for $u \in U$, there is no negative power of $\hbar$ in $\mathfrak{f}(u)$. On the other hand, there is precisely one negative power of $\hbar$ in $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}(L_{e^{\mathfrak{n}(1)}} - \mathrm{e})^{\odot k} u$, namely $\mathfrak{n}_{10}(1)\hbar^{-1}$ for $k = 1$. Therefore, $L_{e^{\mathfrak{n}(1)}}$ does not come from a weak IBL_∞-morphism as long as $\mathfrak{n}_{10}(1) \neq 0$. This sheds light on the matter of the author's confusion from the preamble about twisting with a Maurer-Cartan element.

Questions D.3.4. (i) What is the geometric meaning of

$$\mathrm{H}(\hat{\mathrm{S}}U((\hbar)), \Delta) \simeq \mathrm{H}(\hat{\mathrm{S}}U((\hbar)), \Delta^{\mathfrak{n}}), \quad \mathrm{H}(\hat{\mathrm{S}}U[[\hbar]], \Delta) \quad \text{and} \quad \mathrm{H}(\hat{\mathrm{S}}U[[\hbar]], \Delta^{\mathfrak{n}}), \tag{D.39}$$

where $U = \mathrm{B}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}(M)[3-n]$ is the (reduced) dual cyclic bar complex of the de Rham cohomology of a closed oriented n-manifold M with the filtration dual to the (homological, i.e., increasing) filtration by weights, Δ represents the canonical dIBL-structure for the intersection pairing on $\mathrm{H}_{\mathrm{dR}}(M)$ and $\mathfrak{n}$ is the Chern-Simons Maurer-Cartan element? Are some of the homologies (D.39) isomorphic? How about the bicomplex from Remark D.3.2 for surfaces?

(ii) Are there any implications between the following statements (a)–(c)? A strict IBL_∞-morphism of strict IBL_∞-algebras induces

(a) a quasi-isomorphism of IBL_∞-complexes,

(b) a quasi-isomorphism of strict BV-complexes and

(c) a quasi-isomorphism of BV-complexes?

(iii) For a Maurer-Cartan element $\mathfrak{n}$, is there a formula for

$$\log L_{e^{\mathfrak{n}(1)}} : \ \hat{\mathrm{S}}U((\hbar)) \longrightarrow \hat{\mathrm{S}}U((\hbar))?$$

D.4. Composition of polynomials in convolution product

We will work on a (non-filtered) $\mathbb{N}_0$-graded bialgebra V over $\mathbb{K}$ and consider $\mathbb{K}$-linear maps $V \to V$. Nevertheless, the results extend in a straightforward way to weight-graded complete filtered bialgebras $\hat{V}$ and $\mathbb{K}$-linear maps of finite filtration degrees, and ultimately to the formal series $\hat{V}((\hbar))$ and $\mathbb{K}((\hbar))$-linear maps of finite filtration degrees.

The theory is based on the following observation.

Lemma D.4.1 (Iterated compatibility condition). *Let $(V, \mu, \delta, \eta, \varepsilon)$ be an $\mathbb{N}_0$-graded bialgebra. Write $V = \bigoplus_{i=0}^{\infty} V_i$,*[12] *and for every $i \in \mathbb{N}_0$, let $\iota_i : V_i \to V$ and $\pi_i : V \to V_i$ be*

---

[12] We use the lower index because it should correspond to the weight in the weight-graded setting of $V = \mathrm{S}U$.

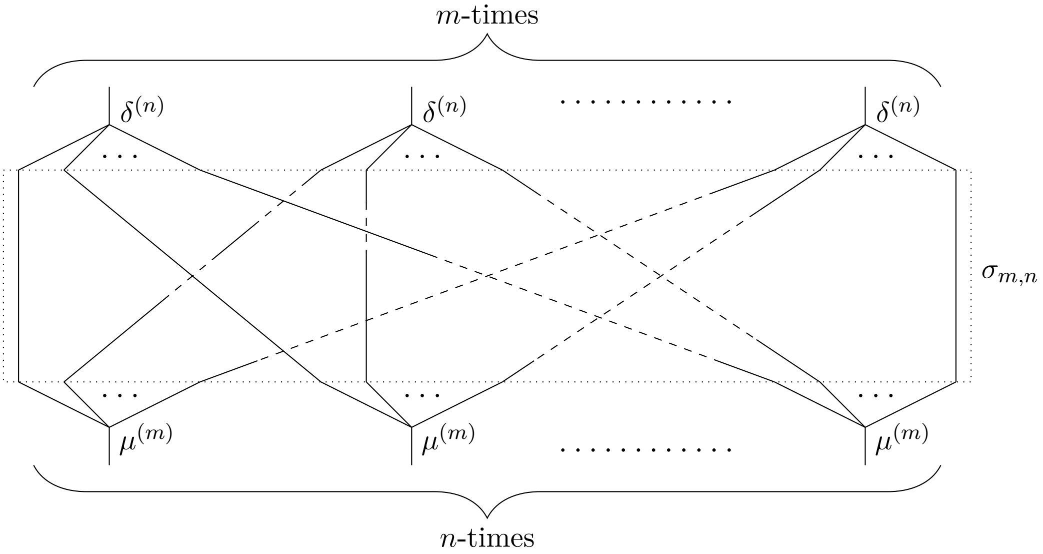


Figure D.4.: The "heart and veins" of the iterated bialgebra compatibility condition $\delta^{(n)}\mu^{(m)}$, and the definition of $\sigma_{m,n}$.

the canonical inclusion and projection, respectively. Let m, $n \in \mathbb{N}$, *and let* $\sigma_{m,n} \in \mathbb{S}_{mn}$ *be the permutation of* mn *elements depicted in Figure D.4. For* $A \in \mathbb{N}_0^{m\times n}$, *let*

$$F_A := \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{n} \pi_{A_{ij}} \circ \iota_{A_{ij}} : V^{\otimes mn} \longrightarrow V^{\otimes mn}.$$

Then for the iterated product $\mu^{(m)} : V^{\otimes m} \to V$ *and the iterated coproduct* $\delta^{(n)} : V \to V^{\otimes n}$, *the following holds:*

$$\delta^{(n)}\mu^{(m)} = \mu^{(m)\otimes n}\sigma_{m,n}\delta^{(n)\otimes m} = \sum_{A\in\mathbb{N}_0^{m\times n}} \mu^{(m)\otimes n}\sigma_{m,n}F_A\delta^{(n)\otimes m}. \tag{D.40}$$

For $n = m = 2$, *this reduces to the well-known compatibility relation*

$$\delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{1} \otimes \tau \otimes \mathbb{1}) \circ (\delta \otimes \delta). \tag{D.41}$$

Proof. Clearly, the second equality of (D.40) holds because $\sum_{A\in\mathbb{N}_0^{m\times n}} F_A = \mathbb{1}^{mn}$. As for the first equality, it holds for $m = n = 2$ from the definition of a bialgebra; the rest is obtained by induction on m, n, distinguishing the two cases $\delta^{(n)}\mu^{(m)} = (\delta \otimes \mathbb{1}^{\otimes n-2})\delta^{(n-1)}\mu^{(m)}$ and $\delta^{(n)}\mu^{(m)} = \delta^{(n)}\mu^{(m-1)}(\mu \otimes \mathbb{1}^{\otimes n-2})$, respectively. In the first case, the induction step reads

$$\begin{aligned}\delta^{(n)}\mu^{(m)} &= (\delta\otimes\mathbb{1}^{\otimes n-2})\delta^{(n-1)}\mu^{(m)}\\ &= (\delta\otimes\mathbb{1}^{\otimes n-2})\mu^{(m)\otimes n-1}\sigma_{m,n-1}\delta^{(n-1)\otimes m}\\ &= \mu^{(m)\otimes n}(\sigma_{m,2}\delta^{\otimes m}\otimes\mathbb{1}^{\otimes m(n-2)})\sigma_{m,n-1}\delta^{(n-1)\otimes m}\\ &= \mu^{(m)\otimes n}\sigma_{m,n}\delta^{(n)\otimes m}.\end{aligned}$$

On the third line, we used that $\delta\mu^{(m)} = (\mu^{(m)}\otimes\mu^{(m)})\sigma_{m,2}\delta^{\otimes m}$, which equals (D.41) for $m = 2$, and for $m \geq 3$, it is proven by induction with the induction step

$$\begin{aligned}\delta\mu^{(m)} &= \delta\mu^{(m-1)}(\mu\otimes\mathbb{1}^{\otimes m-2})\\ &= (\mu^{(m-1)}\otimes\mu^{(m-1)})\sigma_{m-1,2}\delta^{\otimes m-1}(\mu\otimes\mathbb{1}^{\otimes m-2})\\ &= (\mu^{(m-1)}\otimes\mu^{(m-1)})\sigma_{m-1,2}(\mu\otimes\mu\otimes\mathbb{1}^{m-2})(\mathbb{1}\otimes\tau\otimes\mathbb{1}\otimes\mathbb{1}^{m-2})\delta^{\otimes m}\\ &= (\mu^{(m)}\otimes\mu^{(m)})\sigma_{m,2}\delta^{\otimes m}.\end{aligned}$$

The second case is analogous. □

Definition D.4.2 (Compositions of monomials in convolution product)**.** *Consider an $\mathbb{N}_0$-graded bialgebra $(V,\mu,\delta,\eta,\varepsilon)$. Given linear maps $\mathfrak{f}_1$, …, $\mathfrak{f}_r$, $\mathfrak{f}'_1,\dots,\mathfrak{f}'_{r'} : V\to V$ and a matrix $A\in\mathbb{N}_0^{r'\times r}$ for r, $r'\in\mathbb{N}$, we define the A-*composition *$(\mathfrak{f}_1,\dots,\mathfrak{f}_r)\,\square_A\,(\mathfrak{f}_1,\dots,\mathfrak{f}_{r'}) : V\to V$ by the formula*

$$\begin{aligned}&(\mathfrak{f}_1,\dots,\mathfrak{f}_r)\,\square_A\,(\mathfrak{f}_1,\dots,\mathfrak{f}_{r'})\\ &\qquad\coloneqq \mu^{(r)}\circ(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r)\circ\mu^{(r')\otimes r}\circ\sigma_{r',r}\circ F_A\circ\delta^{(r)\otimes r'}\circ(\mathfrak{f}'_1\otimes\cdots\otimes\mathfrak{f}'_{r'})\circ\delta^{(r')},\end{aligned}$$

where $\sigma_{r',r}$ and F_A were defined in Lemma D.4.1.

We call a matrix $A\in\mathbb{N}_0^{r'\times r}$ connected *if for the matrix*

$$B \coloneqq \begin{pmatrix} 0 & A\\ A^T & 0\end{pmatrix},$$

the matrix product $B^{r+r'}$ has at least one row with all entries non-zero.[13] In this case, we say that the A-composition $(\mathfrak{f}_1,\dots,\mathfrak{f}_r)\,\square_A\,(\mathfrak{f}_1,\dots,\mathfrak{f}_{r'})$ is connected.

Suppose that V is connected, i.e., $V_0 = \langle 1\rangle$. Given linear maps $\mathfrak{f}_1$, …, $\mathfrak{f}_r$, $\mathfrak{f}'_1$, …, $\mathfrak{f}'_{r'} : V\to V$ for r, $r'\in\mathbb{N}$, we define the connected composition $(\mathfrak{f}_1,\dots,\mathfrak{f}_r)\circ_{\mathrm{con}}(\mathfrak{f}'_1,\dots,\mathfrak{f}'_{r'}) :$

---

[13] Such A represents a connected weighted bipartite graph.

$V \to V$ *by*

$$(\mathfrak{f}_1, \ldots, \mathfrak{f}_r) \circ_{\mathrm{con}} (\mathfrak{f}_1, \ldots, \mathfrak{f}_{r'}) = \sum_{\substack{A \in \mathbb{N}_0^{(r'+1)\times(r+1)} \\ A \text{ connected} \\ A_{r'+1,r+1}=0}} (\mathfrak{f}_1, \ldots, \mathfrak{f}_r, \mathbb{1}) \,\Box_A\, (\mathfrak{f}'_1, \ldots, \mathfrak{f}'_{r'}, \mathbb{1}).$$

Given linear maps $\mathfrak{f}$, $\mathfrak{f}_1$, …, $\mathfrak{f}_r : V \to V$ *and parameters* h_1, …, $h_r \in \mathbb{N}_0$ *for* $r \in \mathbb{N}$, *we define the* partial compositions $\mathfrak{f} \circ_{h_1,\ldots,h_r} (\mathfrak{f}_1, \ldots, \mathfrak{f}_r)$, $(\mathfrak{f}_1, \ldots, \mathfrak{f}_r) \circ_{h_1,\ldots,h_r} \mathfrak{f} : V \to V$ *by*

$$\begin{aligned}
&\mathfrak{f} \circ_{h_1,\ldots,h_r} (\mathfrak{f}_1, \ldots, \mathfrak{f}_r) \\
&\qquad := \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)\big([(\mu^{(r)} \otimes \mu^{(r)})(F_{h_1,\ldots,h_r} \otimes \mathbb{1}^{\otimes r})\sigma_r \delta^{\otimes r}] \otimes \mathbb{1}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1})\delta^{(r+1)}, \\
&(\mathfrak{f}_1, \ldots, \mathfrak{f}_r) \circ_{h_1,\ldots,h_r} \mathfrak{f} \\
&\qquad := \mu^{(r+1)}(\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1})\big([\mu^{\otimes r}\sigma_r^{-1}(F_{h_1,\ldots,h_r} \otimes \mathbb{1}^{\otimes r})(\delta^{(r)} \otimes \delta^{(r)})] \otimes \mathbb{1}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathbb{1} \otimes \tau)(\delta \otimes \mathbb{1})(\mathfrak{f} \otimes \mathbb{1})\delta,
\end{aligned}$$

where we set $F_{h_1,\ldots,h_r} := F_{(h_1,\ldots,h_r)} = F_{(h_1,\ldots,h_r)^T} = \iota_{h_1}\pi_{h_1} \otimes \cdots \otimes \iota_{h_r}\pi_{h_r}$ *and* $\sigma_r := \sigma_{r,2}$, *and we omit writing the composition* $\circ$. *For* $r = 1$, *we have* $\mu^{(1)} = \delta^{(1)} := \mathbb{1}$ *by definition, and both equations above reduce to*

$$\begin{aligned}
\mathfrak{f} \circ_h \mathfrak{f}_1 &= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)(F_h \otimes \mathbb{1}^{\otimes 2})(\delta \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \mathbb{1})\delta \\
&\Big(= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(F_h \otimes \mathbb{1}^{\otimes 2})(\mathbb{1} \otimes \tau)(\delta \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \mathbb{1})\delta\Big).
\end{aligned}$$

(c.f., Definition 3.2.2 in Section 3.2 for the case of $V = \mathrm{S}(C[1])$*)*

Proposition D.4.3 (Partial compositions)**.** *Let* $\mathfrak{f}$, $\mathfrak{f}_1$, …, $\mathfrak{f}_r : V \to V$ *be linear maps on a connected* $\mathbb{N}_0$*-graded bialgebra* V, *and let* h_1, …, $h_r \in \mathbb{N}_0$ *for* $r \in \mathbb{N}$. *Then the following formulas hold:*

$$\begin{aligned}
\mathfrak{f} \circ_{h_1,\ldots,h_r} (\mathfrak{f}_1, \ldots, \mathfrak{f}_r) &= \sum_{\substack{A \in \mathbb{N}_0^{(r+1)\times 2} \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} (\mathfrak{f}, \mathbb{1}) \,\Box_A\, (\mathfrak{f}_1, \ldots, \mathfrak{f}_r, \mathbb{1}), \\
(\mathfrak{f}_1, \ldots, \mathfrak{f}_r) \circ_{h_1,\ldots,h_r} \mathfrak{f} &= \sum_{\substack{A \in \mathbb{N}_0^{2\times(r+1)} \\ A = \begin{pmatrix} h_1 & \cdots & h_r & \bullet \\ \bullet & \cdots & \bullet & 0 \end{pmatrix}}} (\mathfrak{f}_1, \ldots, \mathfrak{f}_r, \mathbb{1}) \,\Box_A\, (\mathfrak{f}, \mathbb{1}).
\end{aligned}$$

Proof. We compute

$$\begin{aligned}
&\sum_{\substack{A\in\mathbb{N}_0^{(r+1)\times 2}\\ A=\begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0\end{pmatrix}}} (\mathfrak{f} * \mathbb{1}) \,\Box_A\, (\mathfrak{f}_1 * \cdots * \mathfrak{f}_r * \mathbb{1}) \\
&= \sum_{\substack{A\in\mathbb{N}_0^{(r+1)\times 2}\\ A=\begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0\end{pmatrix}}} \mu(\mathfrak{f}\otimes\mathbb{1})\,\mu^{(r+1)\otimes 2}\,\sigma_{r+1,2}F_A\delta^{\otimes r+1}(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r\otimes\mathbb{1})\delta^{(r+1)} \\
&= \mu(\mathfrak{f}\otimes\mathbb{1})\,\mu^{(r+1)\otimes 2}\,\sigma_{r+1}(\iota_{h_1}\pi_{h_1}\otimes\mathbb{1}\otimes\cdots\otimes\iota_{h_r}\pi_{h_r}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\iota_0\pi_0)\delta^{\otimes r+1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r\otimes\mathbb{1})\delta^{(r+1)} \\
&= \mu(\mathfrak{f}\otimes\mathbb{1})\,\mu^{(r+1)\otimes 2}(F_{h_1,\dots,h_r}\otimes\mathbb{1}\otimes\mathbb{1}^{\otimes r}\otimes\iota_0\pi_0)\sigma_{r+1}\delta^{\otimes r+1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r\otimes\mathbb{1})\delta^{(r+1)} \\
&= \mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mathbb{1})(\mathbb{1}\otimes\tau)\big(\big[\mu^{(r)\otimes 2}(F_{h_1,\dots,h_r}\otimes\mathbb{1}^{\otimes r})\sigma_r\delta^{\otimes r}\big]\otimes\mathbb{1}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r\otimes\mathbb{1})\delta^{(r+1)} \\
&= \mathfrak{f}\circ_{h_1,\dots,h_r}(\mathfrak{f}_1,\dots,\mathfrak{f}_r).
\end{aligned}$$

On the line before the last line we used that for a connected bialgebra V, it holds

$$(\mathbb{1}\otimes\iota_0\pi_0)\delta(v) = v\otimes 1 \quad\text{and}\quad (\iota_0\pi_0\otimes\mathbb{1})\delta(v) = 1\otimes v \quad\text{for all } v\in V.$$

The case of $(\mathfrak{f}_1,\dots,\mathfrak{f}_r)\circ_{h_1,\dots,h_r}\mathfrak{f}$ is treated analogously, using that $\sigma_{m,n}=\sigma_{n,m}^{-1}$ (this is visible by looking at the highlighted square in Figure D.4, which is symmetric under the rotation by 180°). ☐

We see that the operations $\circ_{h_1,\dots,h_r}$ and $\circ_{\mathrm{con}}$, which are the cornerstone of the surface calculus for SU in Section D.2, originate naturally from $\Box_A$. Clearly, we can now replace SU by any connected weight-graded bialgebra V and develop a surface calculus for BV$_\infty$-algebras over V.

The following formulas were stated in Remark 3.2.3 in Section 3.2 and come from [CFL15], where they were used in a slightly different form (and without a proof).

Proposition D.4.4 (Formulas involving partial compositions)**.** *For $r\in\mathbb{N}$, let $\mathfrak{f}$, $\mathfrak{f}_1$, …, $\mathfrak{f}_r : V\to V$ be linear maps on a connected $\mathbb{N}_0$-graded bialgebra V. Then the following* *formulas hold:*

$$\hat{\mathfrak{f}} \circ \hat{\mathfrak{f}}_1 = \sum_{h\geq 0} \widehat{\mathfrak{f} \circ_h \mathfrak{f}_1}, \tag{D.42}$$

$$\hat{\mathfrak{f}} \circ (\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) = \sum_{k\geq 0} \sum_{\substack{h_1,\ldots,h_r\geq 0\\ h_1+\cdots+h_r=k}} (\mathfrak{f}\iota_k\pi_k) \circ_{h_1,\ldots,h_r} (\mathfrak{f}_1,\ldots,\mathfrak{f}_r), \tag{D.43}$$

$$(\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) \circ \hat{\mathfrak{f}} = \sum_{l\geq 0} \sum_{\substack{h_1,\ldots,h_r\geq 0\\ h_1+\cdots+h_r=l}} (\mathfrak{f}_1,\ldots,\mathfrak{f}_r) \circ_{h_1,\ldots,h_r} (\iota_l\pi_l\mathfrak{f}), \tag{D.44}$$

$$\mathfrak{f} \circ_{h_1,\ldots,h_{r-1},0} (\mathfrak{f}_1,\ldots,\mathfrak{f}_r) = \mathfrak{f} \circ_{h_1,\ldots,h_{r-1}} (\mathfrak{f}_1,\ldots,\mathfrak{f}_{r-1}) * \mathfrak{f}_r, \tag{D.45}$$

$$(\mathfrak{f}_1,\ldots,\mathfrak{f}_r) \circ_{0,h_2,\ldots,h_r} \mathfrak{f} = \mathfrak{f}_1 * (\mathfrak{f}_2,\ldots,\mathfrak{f}_r) \circ_{h_2,\ldots,h_r} \mathfrak{f}. \tag{D.46}$$

(Recall that $\hat{\mathfrak{f}} \coloneqq \mathfrak{f} * \mathbb{1} : V \to V$*.)*

Proof. As for (D.42), we compute

$$\begin{aligned}
\hat{\mathfrak{f}} \circ \hat{\mathfrak{f}}_1 &= \mu(\mathfrak{f}\otimes\mathbb{1})\delta\mu(\mathfrak{f}_1\otimes\mathbb{1})\delta\\
&= \mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mu)(\mathbb{1}\otimes\tau\otimes\mathbb{1})(\delta\otimes\delta)(\mathfrak{f}_1\otimes\mathbb{1})\delta\\
&= \mu(\mu\otimes\mathbb{1})(\mathfrak{f}\otimes\mathbb{1}^{\otimes 2})(\mu\otimes\mathbb{1}^{\otimes 2})(\mathbb{1}\otimes\tau\otimes\mathbb{1})(\delta\otimes\mathbb{1}^{\otimes 2})(\mathfrak{f}_1\otimes\mathbb{1}^{\otimes 2})(\delta\otimes\mathbb{1})\delta\\
&= \mu(\underbrace{[\mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mathbb{1})(\mathbb{1}\otimes\tau)(\delta\otimes\mathbb{1})(\mathfrak{f}_1\otimes\mathbb{1})\delta]}_{=\sum_{h\geq 0}\mathfrak{f}\circ_h\mathfrak{f}_1}\otimes\mathbb{1})\delta\\
&= \sum_{h\geq 0}\widehat{\mathfrak{f}\circ_h\mathfrak{f}_1}.
\end{aligned}$$

On the second line, we used the compatibility condition; on the third line, we used associativity and coassociativity.

In order to see (D.43), the easiest is to use Proposition D.4.3:

$$\begin{aligned}
\hat{\mathfrak{f}}(\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) &= \sum_{k\geq 0} \sum_{\substack{h_1,\ldots,h_r\geq 0\\ h_1+\cdots+h_r=k}} \sum_{\substack{A\in\mathbb{N}_0^{r\times 2}\\ A=\begin{pmatrix} h_1 & \bullet\\ \vdots & \vdots\\ h_r & \bullet\end{pmatrix}}} (\mathfrak{f}\iota_k\pi_k,\mathbb{1}) \,\square_A\, (\mathfrak{f}_1,\ldots,\mathfrak{f}_r)\\
&= \sum_{k\geq 0} \sum_{\substack{h_1,\ldots,h_r\geq 0\\ h_1+\cdots+h_r=k}} \sum_{\substack{A'\in\mathbb{N}_0^{(r+1)\times 2}\\ A'=\begin{pmatrix} h_1 & \bullet\\ \vdots & \vdots\\ h_r & \bullet\\ \bullet & 0\end{pmatrix}}} (\mathfrak{f}\iota_k\pi_k,\mathbb{1}) \,\square_{A'}\, (\mathfrak{f}_1,\ldots,\mathfrak{f}_r,\mathbb{1})
\end{aligned}$$

$$= \sum_{k\geq 0} \sum_{\substack{h_1,\ldots,h_r\geq 0\\ h_1+\cdots+h_r=k}} (\mathfrak{f}\iota_k\pi_k) \circ_{h_1,\ldots,h_r} (\mathfrak{f}_1,\ldots,\mathfrak{f}_r).$$

On the second line, the bottom-left $\bullet$, i.e., how many "veins" of $\mathbb{1}$ go into $\mathfrak{f}\iota_k\pi_k$, is forced to be 0 because $\mathfrak{f}\iota_k\pi_k$ has k inputs and $h_1+\cdots+h_r=k$. Equation (D.44) is proven analogously.

As for (D.45), we prefer to manipulate the expression with bialgebra operations:

$$\begin{aligned}
&\mathfrak{f} \circ_{h_1,\ldots,h_{r-1},0} (\mathfrak{f}_1,\ldots,\mathfrak{f}_r)\\
&\quad = \mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mathbb{1})(\mathbb{1}\otimes\tau)\big(\big[\mu^{(r)\otimes 2}(F_{h_1,\ldots,h_{r-1},0}\otimes\mathbb{1}^{\otimes r})\sigma_r\delta^{\otimes r}\big]\otimes\mathbb{1}\big)\\
&\qquad\qquad (\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r\otimes\mathbb{1})\delta^{(r+1)}\\
&\quad = \mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mathbb{1})(\mathbb{1}\otimes\tau)\big(\big[(\mu^{(r-1)}\otimes\mu^{(r)})(F_{h_1,\ldots,h_{r-1}}\otimes\mathbb{1}^{\otimes r})(\sigma_{r-1}\otimes\mathbb{1})\\
&\qquad\qquad (\delta^{\otimes r-1}\otimes\mathbb{1})\big]\otimes\mathbb{1}\big)(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r\otimes\mathbb{1})\delta^{(r+1)}\\
&\quad = \mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mathbb{1})(\mathbb{1}\otimes\tau)(\mathbb{1}\otimes\mu\otimes\mathbb{1})\big(\big[\mu^{(r-1)\otimes 2}(F_{h_1,\ldots,h_{r-1}}\otimes\mathbb{1}^{\otimes r-1})\sigma_{r-1}\\
&\qquad\qquad \delta^{\otimes r-1}\big]\otimes\mathbb{1}^{\otimes 2}\big)(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_r\otimes\mathbb{1})\delta^{(r+1)}\\
&\quad = \mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mathbb{1})(\mathbb{1}\otimes\mu)(\mathbb{1}\otimes\tau\otimes\mathbb{1})\big(\big[\mu^{(r-1)\otimes 2}(F_{h_1,\ldots,h_{r-1}}\otimes\mathbb{1}^{\otimes r-1})\sigma_{r-1}\\
&\qquad\qquad \delta^{\otimes r-1}\big]\otimes\mathbb{1}^{\otimes 2}\big)(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_{r-1}\otimes\mathbb{1}\otimes\mathfrak{f}_r)\delta^{(r+1)}\\
&\quad = \mu\big(\big[\mu(\mathfrak{f}\otimes\mathbb{1})(\mu\otimes\mathbb{1})(\mathbb{1}\otimes\tau)\big(\big[\mu^{(r-1)\otimes 2}(F_{h_1,\ldots,h_{r-1}}\otimes\mathbb{1}^{\otimes r-1})\sigma_{r-1}\\
&\qquad\qquad \delta^{\otimes r-1}\big]\otimes\mathbb{1}\big)(\mathfrak{f}_1\otimes\cdots\otimes\mathfrak{f}_{r-1}\otimes\mathbb{1})\delta^{(r)}\big]\otimes\mathfrak{f}_r\big)\delta\\
&\quad = \mu\big(\mathfrak{f}\circ_{h_1,\ldots,h_{r-1}}(\mathfrak{f}_1,\ldots,\mathfrak{f}_{r-1})\otimes\mathfrak{f}_r\big)\delta\\
&\quad = \mathfrak{f}\circ_{h_1,\ldots,h_{r-1}}(\mathfrak{f}_1,\ldots,\mathfrak{f}_{r-1}) * \mathfrak{f}_r
\end{aligned}$$

Equation (D.46) is proven analogously. □

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