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1.)git: Do not forget to regularly commit to git!

2.)todonotes: You can use **Correct**, **Modify**, **Remove**, **Add**, **Reminder**, **ToDo**.

# IBL-Infinity Model of String Topology from Perturbative Chern-Simons Theory

**Dissertation**

zur Erlangung des akademischen Grades

Dr. rer. nat.

eingereicht an der

Mathematisch-Naturwissenschaftlich-Technischen Fakultät

der Universität Augsburg

von

**Pavel Hájek**

Augsburg, Oktober 2019

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1. Gutachter: Prof. Dr. Kai Cieliebak, Universität Augsburg
2. Gutachter: Prof. Dr. Janko Latschev, Universität Hamburg

Tag der mündlichen Prüfung: Freitag, 20. Dezember, 2019

# Abstract

Pavel Hájek: *IBL-Infinity Model of String Topology from Perturbative Chern-Simons Theory*

Following Cieliebak, Fukaya, Latschev and Volkov, we construct an IBL-infinity chain model for equivariant string topology on cyclic Hochschild cochains of de Rham cohomology. We study its properties and perform explicit computations.

*Keywords:* chain model for equivariant string topology, IBL-infinity algebra, Maurer-Cartan element, Hodge propagator, perturbative Chern-Simons theory, Poincaré duality models, BV-formalism, cyclic homology

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DONE Check who has Dr. in name!

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DONE Add cross: DO NOT Add cross to Anna Uhlíkova

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DONE Add more names: Add Ushi for giving me cookies and cakes. Tennis partners: Thomas, Lukas, Tobi

DONE Add picture: Add the picture of Mikesch with snails and Augsburg

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# Structure of the thesis

The thesis consists of the *Introduction* and the following three parts:

- *Part I—IBL-infinity chain model*: the goal is to compute the twisted  $IBL_\infty$ -algebra on de Rham cohomology of  $\mathbb{S}^n$  explicitly and compare it to string topology (the original task). A version of Part I was previously made available online and can be found under the following reference:

P. Hájek. *Twisted IBL-infinity-algebra and string topology: First look and examples*. Nov. 2018. arXiv: 1811.05281 [math-ph]

- *Part II—Follow-up topics*: additional topics discovered by the author while working on Part I; they may lead to further research.
- *Part III—Appendices*: algebraic details of Part I and related questions.

The three parts were written chronologically and reflect how the author’s view of the theory developed. The mathematical notation is consistent but the terminology and understanding might vary slightly from part to part. All in all, Part I is more about “getting things done” while Parts II and III are more about “optimizing”.

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Completed tensor product: Add a remark that Koszul convention does not matter for $\odot$ because tensor products of more morphisms appear only in morphisms which have even degree. Polemize once more that this pairing is important in order to make Chapter 10 work, ie, the cyclic bar complex. One might change the definitions of canonical operations and the evaluation of ribbon graphs and redo do the proofs in those section. However, the author failed. But maybe just because of his newbie. Now it seems more plausible.
DONE Replace overline with widebar:
DONE The correct definition equality $:=$ : Replace $:=$ with $\coloneqq$
DONE Chern-Simons Maurer-Cartan element: Change formal pushforward Maurer-Cartan element to Chern-Simons Maurer-Cartan element
DONE Filtration by weights: The filtration on the dual cyclic bar complex is the dual to the filtration by weights on the bar complex.
DONE $\circ_{h_1, \dots, h_r}$ not defined for bialgebra: In fact, $q_{klg}$ and the operations $\circ_{h_1, \dots, h_r}$ can not be defined on an arbitrary bialgebra. We still need the filtration by weights. It might be possible to extend this to weight-graded bialgebras (operations are continuous with respect to the grading and weights). Also resolve the notion of "continuous extension" and what is, in fact, continuous. Filtration preserving versus non-negative filtration degree. Maybe call all guys with finite-filtration degree continuous.
DONE Hodge htpy: Change Green operator to Hodge homotopy. Also GOp to HOp or something like that. Add admissible to symmetric Hodge propagator which extends smoothly to the blow-up. Add special is it satisfies the additional properties.
Melrose Blowup: I was told by Oliver that we do Melrose blowups.
Name of MaurerCarta: We should call $\int$ Chern-Simons Maurer-Cartan element and $n$ effective Chern-Simons element. We previously used also formal pushforward Maurer-Cartan element. However, in fact, $\int$ is not a Maurer-Cartan element of the Fréchet algebra.
Questions: Make questions with ending QED Sign

# 1. Introduction

## 1.1. String topology and Chen's iterated integrals

String topology of a manifold  $M$  is the study of the *free loop space*

$$LM = \{\gamma : \mathbb{S}^1 \rightarrow M \text{ continuous}\}$$

with the compact-open topology and natural structures on it. Each loop  $\gamma$  is parametrized, with base-point 1, and there is a natural  $\mathbb{S}^1$ -action changing the base-point. Therefore, we distinguish two homology theories:

$$\begin{aligned} H(LM) &\dots \text{ the } \textit{singular homology} \text{ and} \\ H^{\mathbb{S}^1}(LM) &\dots \text{ the } \textit{equivariant homology} \text{ — “the singular homology of the} \\ &\quad \text{space of parametrized loops with the base-point forgotten.”} \end{aligned}$$

In this thesis, we consider  $H^{\mathbb{S}^1}(LM)$  with coefficients in  $\mathbb{R}$  only.

If  $M = \Sigma$  is an oriented surface, we consider immersed loops with transverse double points and define a bracket and cobracket by Figure 1.1. In words:

$\mathfrak{m}_2$ : Imagine putting one base-point  $b_1$  on the first loop  $\gamma_1$  and another base-point  $b_2$  on the second loop  $\gamma_2$  in all possible positions. Whenever  $\gamma_1(b_1) = \gamma_2(b_2) = p$ , construct a new loop  $\gamma = \gamma_1 *_p \gamma_2$  by running first along  $\gamma_1$  with double speed starting and ending at  $b_1$  and continuing along  $\gamma_2$  starting and ending at  $b_2$ . Forget the base-points and multiply  $\gamma$  with the sign of the intersection  $\varepsilon(p; \gamma_1, \gamma_2)$ . Should more intersections occur, take the sum.

$\mathfrak{c}_2$ : Imagine putting the basepoints  $b_1$  and  $b_2$  on  $\gamma$  in all possible positions such that  $b_1 \neq b_2$ . Whenever  $\gamma(b_1) = \gamma(b_2) = p$ , split  $\gamma$  into  $\gamma_1$  and  $\gamma_2$  as follows. The first loop  $\gamma_1$  is the portion of  $\gamma$  from  $b_1$  to  $b_2$  and the second loop  $\gamma_2$  is the portion from  $b_2$  to  $b_1$  ran along with the correspondingly scaled speed. Forget the base-points, form the tensor product  $\gamma_1 \otimes \gamma_2$  and multiply it with  $\varepsilon(p; \gamma)$ . Should more self-intersections occur, take the sum of the tensors.

How is it with loop parametrization: How is it precisely with the speeds?

## 1. Introduction

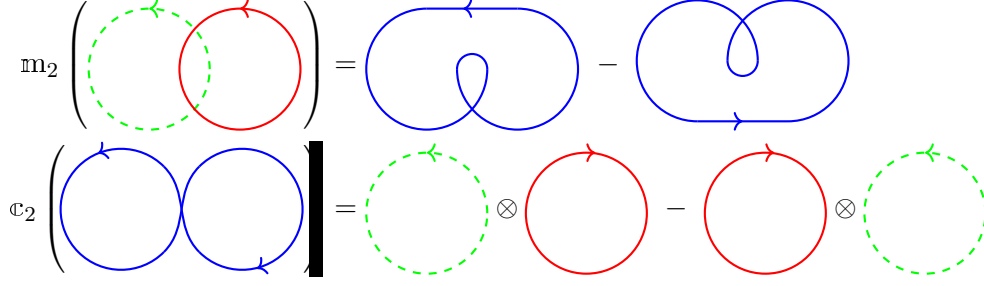


Figure 1.1.: Bracket and cobracket in equivariant string topology. Imagine some genus so that the operations are homotopically non-trivial.

The operations  $m_2$  and  $c_2$  are known as the *Goldman bracket* and the *Turaev cobracket* and were defined and studied in [Gol86] and [Tur91], respectively.

In [CS99], it was demonstrated that the construction of  $m_2$  extends to families of loops and to an arbitrary dimension  $n$  of an oriented manifold  $M$ ; it produces a Lie bracket on the equivariant homology. The construction of  $c_2$  generalizes too, which is explained for instance in [CL09], and gives a Lie cobracket. The picture is always 1.1, just the intersection points come from transverse intersections of smooth parameter spaces of points on loops in  $M$ ; thus, both  $m_2$  and  $c_2$  have degrees  $2 - n$ . Another definition of the coproduct is in [Bas11], where loops in  $M$  are viewed as open strings in  $M \times M$  with endpoints at the diagonal. In order to make these geometric constructions rigorous, the most straightforward way is to use a *geometric homology theory* of  $M$  based on smooth chains, so that the transverse intersection of two smooth chains is again a smooth chain. Such theory for smooth manifolds was constructed in [Lip14]. A version for general topological spaces is proposed in [Cie13], and some details regarding triangulations are addressed in [Háj14] (see also the discussion at [Par14]). Note that  $m_2$  and  $c_2$  are only “transversally defined” on the chain level and for the definition on homology it is important that we can homotop to a generic situation within the homology class.

The main theorem of [CS04] asserts that  $m_2$  and  $c_2$  induce the structure of an *involutive bi-Lie algebra* (abbreviated IBL) of degree  $2 - n$  on the equivariant homology  $H^{S^1}(LM, M)$  relative to constant loops  $M \hookrightarrow LM$ . Modding out constant loops is necessary for  $c_2$  to be well defined because of the phenomenon of “*vanishing of small loops*” illustrated in [CL09]: Let  $\sigma \in C_1(LM)$  be a 1-chain supported at  $[0, 1]$  which for  $t = 0$  agrees with the loop in the argument of  $c_2$  in 1.1, for  $t \in (0, 1)$  the left knot  $L$  contracts to the mid-point  $p$  (thus the name) and for  $t = 1$  only the right knot  $R$  remains. It is

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then easy to see that

$$0 \neq \partial \mathfrak{c}_2(\sigma) - \mathfrak{c}_2(\partial \sigma) = p \otimes R - L \otimes p \in C(M) \otimes C(LM) + C(LM) \otimes C(M).$$

The string bracket  $\mathfrak{m}_2$  restricts to a Lie bracket on the relative homology because applied to a chain of constant loops it gives a degenerate chain.<sup>1</sup> In the work in progress [CHOon], Poincaré duality on the Rabinowitz-Floer homology of the unit cotangent bundle of  $M$  is introduced and related to the (non-equivariant) string topology via a long exact sequence. This shall give a canonical extension of  $\mathfrak{c}_2$  to  $H^{\mathbb{S}^1}(LM, \text{pt})$ , the relative homology modulo one point, which together with  $\mathfrak{m}_2$  makes  $H^{\mathbb{S}^1}(LM, \text{pt})$  into a bi-Lie algebra (in fact, this is what our chain model computes). We will use  $\overline{H}^{\mathbb{S}^1}(LM)$  as an avatar for either  $H^{\mathbb{S}^1}(LM, M)$  or  $H^{\mathbb{S}^1}(LM, \text{pt})$ .

It is expected that the IBL-structure on  $\overline{H}^{\mathbb{S}^1}(LM)$  is induced from a much richer (and in some context natural) algebraic structure on the chain level whose homotopy type is an invariant of  $M$ . In fact, there is the notion of a *strong homotopy involutive bi-Lie algebra* (abbreviated  $\text{IBL}_\infty$ ) which was developed in [CFL15]. It consists of operations  $(\mathfrak{q}_{klg})$  for  $k, l \geq 1$  and  $g \geq 0$ , where  $\mathfrak{q}_{110}$  is a boundary operator,  $\mathfrak{q}_{210}$  a bracket and  $\mathfrak{q}_{120}$  a cobracket, which satisfy the IBL-relations up to a coherent system of higher homotopies  $(\mathfrak{q}_{klg})$ . Consider the string space

$$L_{\mathbb{S}^1} M := (E\mathbb{S}^1 \times LM)/\mathbb{S}^1$$

(the homotopically correct version of the quotient  $LM/\mathbb{S}^1$ ), and let  $(C(L_{\mathbb{S}^1} M), \partial)$  be the singular chain complex of  $L_{\mathbb{S}^1} M$  where  $\mathfrak{m}_2$  and  $\mathfrak{c}_2$  are partially defined on transverse smooth chains. An  $\text{IBL}_\infty$ -chain model for equivariant string topology is an  $\text{IBL}_\infty$ -algebra  $(\mathcal{M}(L_{\mathbb{S}^1} M), (\mathfrak{q}_{klg}))$  ( $\mathcal{M}$  stands for “model”) together with a weak homotopy equivalence ( $:=$  zig-zag of quasi-isomorphisms) of  $(C(L_{\mathbb{S}^1} M), \partial)$  and  $(\mathcal{M}(L_{\mathbb{S}^1} M), \mathfrak{q}_{110})$  which induces an isomorphism of IBL-algebras

$$(\overline{H}^{\mathbb{S}^1}(LM), \mathfrak{m}_2, \mathfrak{c}_2) \simeq (H(\mathcal{M}(L_{\mathbb{S}^1} M), \mathfrak{q}_{110}), \mathfrak{q}_{210}, \mathfrak{q}_{120}).$$

Note that there can be various non-homotopically equivalent  $\text{IBL}_\infty$ -chain models. On the other hand, the properad  $\text{IBL}_\infty$  is a quasi-free resolution of the properad IBL and as

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<sup>1</sup>This is in contrast to the situation on the non-equivariant homology  $H(LM)$ , where constant loops do not always form an ideal for the associative loop product; this is easy to see in the case of the torus  $\mathbb{T}^2$ . They do, however, provided that the Euler characteristics  $\chi(M)$  is non-zero (see [Tam10]). On the other hand, if  $\chi(M) = 0$ , then the loop coproduct admits an extension to  $H(LM)$ , which is possibly dependent on the choice of a non-vanishing vector field (see [Bas11]).

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such has convenient homotopy theoretical properties, e.g., homotopy inverses of quasi-isomorphisms exist. These properties imply that any two weakly homotopy equivalent  $\text{IBL}_\infty$ -algebras are (strongly) homotopy equivalent.

In this thesis, we use a version of *perturbative Chern-Simons theory* to construct an  $\text{IBL}_\infty$ -chain model for the equivariant string topology starting with an oriented compact Riemannian manifold  $M$ . Chains are cyclic Hochschild cochains of the de Rham cohomology  $H_{\text{dR}}$  of  $M$ , and the homotopy type of the model is supposed to be an invariant of  $M$  (perhaps topological). As the construction involves a version of Feynman integrals, the proof of well-definedness relies on Stokes' theorem on compactifications of configuration spaces, which is currently being developped in [CVona] and [CVonb]. The concrete form of this chain model was sketched in [CFL15]. It is expected that the evaluation at boundaries of pseudo-holomorphic curves in the compactification induces an  $\text{IBL}_\infty$ -quasi-isomorphism of the SFT of the unit cotangent bundle of  $M$  and the  $\text{IBL}_\infty$ -chain model of string topology, see [CL09].

We now describe the underlying chain complex and the quasi-isomorphism to string topology in more details. Let  $B^{\text{cyc}}\Omega$ , where  $\Omega$  is the space of de Rham forms on  $M$ , be the graded vector space generated by cyclic words

$$\omega_1 \cdots \omega_k = (-1)^{|\omega_k|(|\omega_1| + \cdots + |\omega_{k-1}|)} \omega_k \omega_1 \cdots \omega_{k-1}$$

with homogenous  $\omega_1, \dots, \omega_k \in \Omega$  for  $k \geq 1$ . The grading satisfies  $|\omega_i| = \deg(\omega_i) - 1$ , where  $\deg(\omega_i)$  denotes the form-degree of  $\omega_i$ . This  $B^{\text{cyc}}\Omega$  is called the *(reduced) cyclic bar complex of  $\Omega$*  (reduced because we omit  $k = 0$ ). On  $B^{\text{cyc}}\Omega$ , we consider the *cyclic Hochschild differential*

$$\begin{aligned} b(\omega_1 \cdots \omega_k) = & \sum_{i=1}^k (-1)^{|\omega_1| + \cdots + |\omega_{i-1}|} \omega_1 \cdots d\omega_i \cdots \omega_k \\ & + \sum_{i=1}^{k-1} (-1)^{|\omega_1| + \cdots + |\omega_{i-1}| + |\omega_i| + 1} \omega_1 \cdots \omega_i \wedge \omega_{i+1} \cdots \omega_k \\ & + (-1)^{|\omega_k|(|\omega_1| + \cdots + |\omega_{k-1}|) + |\omega_k| + 1} \omega_k \wedge \omega_1 \cdots \omega_{k-1}. \end{aligned}$$

It descends from the Hochschild differential on the bar construction  $B\Omega$ , which is standardly defined as the sum of the unique extensions of degree shifts of  $d$  and  $\wedge$  to coderiva-

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tives of  $B\Omega$ . The *Chen's iterated integrals map* is the map

$$I : B^{\text{cyc}}\Omega \longrightarrow C^*(L_{\mathbb{S}^1}M)$$

$$\omega_1 \cdots \omega_k \longmapsto \left( \sigma \mapsto \varepsilon(\omega) \int_{K_\sigma \times \Delta^k} \omega_1(\tilde{\sigma}(x, t_1)) \cdots \omega_k(\tilde{\sigma}(x, t_k)) \right),$$

where  $\varepsilon(\omega)$  is the sign

$$\varepsilon(\omega) = (-1)^{(k-1)(|\omega_1|+1)+(k-2)(|\omega_2|+1)+\cdots+|\omega_{k-1}|+1},$$

the map  $\tilde{\sigma} : K_\sigma \times \mathbb{S}^1 \rightarrow M$ , where  $K_\sigma$  is a smooth chain in  $M$  (a manifold with corners), is the projection of a lift  $K_\sigma \xrightarrow{\sigma} L_{\mathbb{S}^1}M \dashrightarrow E\mathbb{S}^1 \times LM$  to the second factor, and we identify  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ . The map  $I$  is a chain map (of degree 0 with respect to the cohomological grading of  $B^{\text{cyc}}\Omega$  by  $|\omega_1 \cdots \omega_k| = |\omega_1| + \cdots + |\omega_k|$ ), and if  $\pi_1(M) = \{1\}$ , it induces an isomorphism

$$H(B^{\text{cyc}}\Omega, b) / \text{span}\{[1^{2k-1}] \mid k \in \mathbb{N}\} \simeq H^*(L_{\mathbb{S}^1}M) / \mathbb{R}[[u]], \quad (1.1)$$

where  $\text{span}\{[1^{2k-1}] \mid k \in \mathbb{N}\} = H(B^{\text{cyc}}\mathbb{R}, b)$  (we prove later that this agrees with the homology of  $\text{coker}(B^{\text{cyc}}\mathbb{R} \hookrightarrow B^{\text{cyc}}\Omega)$ ) and the polynomial ring  $\mathbb{R}[u]$  with  $|u| = 2$  comes from the module structure on  $H^*(L_{\mathbb{S}^1}M)$  induced from  $L_{\mathbb{S}^1}M = (E\mathbb{S}^1 \times LM)/\mathbb{S}^1 \rightarrow E\mathbb{S}^1/\mathbb{S}^1 = \mathbb{CP}^\infty$ . The proof of this isomorphism is the goal of [CVonb], which is again under development; it is based on [GJP91] after identifying  $(B^{\text{cyc}}\Omega, b)$  with the correct totalization of the Connes' cyclic bicomplex of  $\Omega$  up to weak equivalence.

## 1.2. IBL-infinity chain model and Chern-Simons theory

We consider the *dual cyclic bar complex*

$$\hat{B}_{\text{cyc}}^*\Omega = \bigoplus_{d \in \mathbb{Z}} \prod_{k=1}^{\infty} (B_k^{\text{cyc}}\Omega)^{d*},$$

where  $(B_k^{\text{cyc}}\Omega)^{d*}$  denotes the linear dual to the degree  $d$  component of the weight  $k$  component  $B_k^{\text{cyc}}\Omega$  of  $B^{\text{cyc}}\Omega$ , equipped with the dual Hochschild differential  $b^*$ . By taking  $\prod$ , we allow bubbling of degree 1 forms and constants 1; namely, if  $\omega \in \Omega^1$ , then  $|\omega| = 0$ , and  $|1| = -1$ . Notice that if we take  $H_{\text{dR}}$  instead of  $\Omega$ , then the completion is relevant

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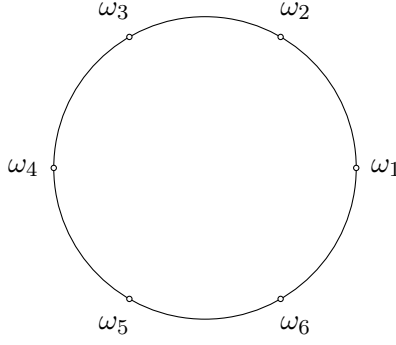


Figure 1.2.: Inserting fields on strings.

only if  $H_{\text{dR}}^1 \neq 0$ . The dual of the Chen's iterated integral map provides a chain map

$$I^* : (C(L_{\mathbb{S}^1} M), \partial) \rightarrow (\hat{B}_{\text{cyc}}^* \Omega, b^*)$$

(in fact,  $\hat{B}_{\text{cyc}}^* \Omega$  is the graded dual of  $B^{\text{cyc}} \Omega$ ).

We introduce the following *physical interpretation*. Beware, the discussion is imprecise! We think of

- elements of  $\Omega$  as *fields*,
- elements of  $B^{\text{cyc}} \Omega$  as *field strings* (not to confuse with string-fields) and
- elements of the space

$$\mathcal{F}(B^{\text{cyc}} \Omega[1]) := \hat{S}(B_{\text{cyc}}^* \Omega[1]) \hat{\otimes} \mathbb{R}((\hbar)),$$

where  $\mathbb{R}((\hbar))$  is the ring of Laurent series in Planck's constant  $\hbar$ ,  $\hat{S}$  denotes the completed symmetric algebra and  $\hat{\otimes}$  the completed tensor product, as *observables on field strings*.

If  $\sigma : \mathbb{S}^1 \rightarrow M$  is a string, the observable  $I^*(\sigma)$  “localizes” on field strings which approximate  $\sigma$ ; for instance, it holds  $I^*(\omega) = \int_{\sigma} \omega$ , and so  $I^*(\omega)$  “localizes” at fundamental forms of  $\sigma(\mathbb{S}^1)$ . We imagine that we decorate  $\sigma$  with a field string  $\omega_1 \dots \omega_k$  as in Figure 1.2 and get a number  $I(\sigma)(\omega_1 \dots \omega_k)$ . The isomorphism (1.1) guarantees that the observable  $I(\sigma)$  determines  $\sigma$  up to a boundary. We will be dealing with *two “dynamical” theories*: one is the theory of fields  $\omega \in \Omega$  and one of field strings  $\omega_1 \dots \omega_k \in B^{\text{cyc}} \Omega$ .

The field theory is at hand. We know that  $(\Omega, d, \wedge, \langle \cdot, \cdot \rangle)$ , where  $\langle \omega_1, \omega_2 \rangle = \int_M \omega_1 \wedge \omega_2$  for  $\omega_1, \omega_2 \in \Omega$ , is a *symmetric dg-Frobenius algebra*. It is well-known that finite-dimensional Frobenius algebras  $V$  are equivalent to 2-dimensional topological quantum



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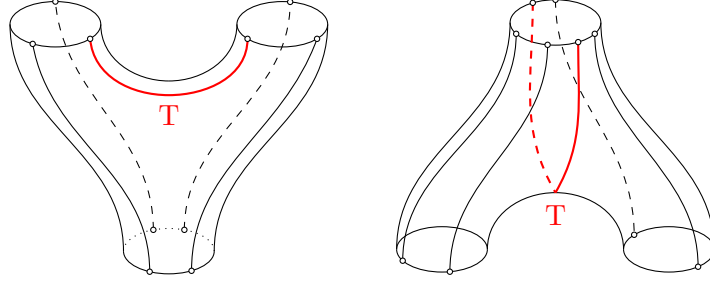


Figure 1.3.: Operations  $q_{210}$  and  $q_{120}$ . Fields propagate from the bottom to the top and an additional dualization, equivalently contraction with the identity propagator, is required when two outputs are connected; hence the red color.

field theories (TQFT). Finite-dimensionality is a sufficient (and necessary) condition so that the trivial cylinder corresponding to the identity satisfies  $\mathbb{1} = \sum \langle \cdot, e^i \rangle e_i$ . For now, we will ignore this issue and imagine  $V = \Omega$ , although we will soon transfer to  $H_{dR}$ , where everything works just fine. We will represent interactions of fields via Feynman graphs drawn on surfaces — the trivial cylinder for fields, i.e., the free propagation, will be a line and the pair of pants, i.e., the interaction via the intersection  $\wedge$ , will be a point with 3 segments emanating from it (we do not distinguish input and outputs by cyclic symmetry).

Let us now consider field strings. Figure 1.3 defines the operations

$$q_{210} : \hat{E}_2 B_{cyc}^* V \rightarrow \hat{E}_1 B_{cyc}^* V \quad \text{and} \quad q_{120} : \hat{E}_1 B_{cyc}^* V \rightarrow \hat{E}_2 B_{cyc}^* V,$$

where  $E(B_{cyc}^* V)$  is the exterior algebra canonically isomorphic to  $S(B_{cyc}^* V[1])$ . We read the diagram from the top to the bottom but imagine fields  $\omega$  being fed into  $\psi$ 's from the bottom to the top. We think of these digrams as of *string interaction diagrams* for strings freely moving in a topological space  $M$ , connecting and disconnecting. This suggests that  $q_{210}$  and  $q_{120}$  are related to  $m_2$  and  $c_2$ , respectively. Formulas for  $q_{210}$  and  $q_{120}$  for a finite dimensional symmetric dg-Frobenius algebra  $V$  were written down in [CFL15] and it was proven that they indeed constitute an IBL-algebra on  $B_{cyc}^* V$  (note that this is not a TQFT!). As for the formulas, one simply decorates the world-lines of fields with the identity propagator  $T = \sum_i e^i \otimes e_i$  and evaluates. As a mathematical remark, we will show that  $q_{210}$  is obtained from the Gerstenhaber bracket on Hochschild cochains via cyclization by  $\langle \cdot, \cdot \rangle$  and that  $q_{120}$  is a factorization of an extension of the canonical Schwarz's BV-operator on  $\mathcal{F}(V[1])$  (odd degree shift of a finite-dimensional symmetric dg-Frobenius algebra is an odd symplectic vector space) to cyclic invariants with respect to the cyclic shuffle product.

Which degree shift: Need to sort out which degree shift for bislgebra one needs! How is it with  $2 - n$ .

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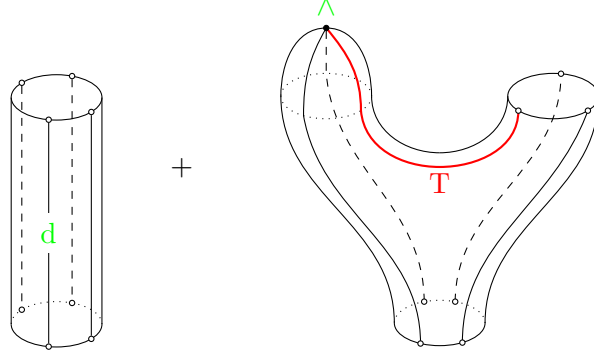


Figure 1.4.: Adding  $d$  and  $\wedge$  via  $S$ .

Since  $B_{\text{cyc}}^* V[1]$  is not naturally an odd symplectic vector space, there is no Schwarz's BV-operator on  $\mathcal{F}(B_{\text{cyc}}^* V[1])$ . However, the following canonical operator

$$\Delta_s := \hat{q}_{120} + \hbar \hat{q}_{210} : \mathcal{F}(B_{\text{cyc}}^* V[1]) \rightarrow \mathcal{F}(B_{\text{cyc}}^* V[1]),$$

where  $\hat{\cdot}$  denote canonical extension to (co)derivatives, is a BV-operator with respect to the function multiplication; we call it the *string BV-operator*. Recall that in physics, a BV-operator  $\Delta$  on  $\mathcal{F}(U)$ , where  $U$  is the space of fields, is related to the measure in the path integral  $\int_U \mu$ . An action  $S \in \mathcal{F}(U)$  satisfying the *quantum master equation* (QME)

$$\Delta S + \frac{1}{2} \{S, S\} = 0$$

defines a new measure  $e^{-S} \mu$ , and the corresponding twisted BV-operator (or rather  $BV_\infty$ -operator) satisfies  $\Delta^S = e^{-S} \Delta e^S$ . For field strings, we define the following *actions*  $S_{\text{free}}, S_{\text{int}} \in \mathcal{F}(B^{\text{cyc}} V[1])$ , which remind us of the *Chern-Simons functional*:

$$S_{\text{free}}(\omega_1 \omega_2) := \pm \hbar^{-1} \int_M \omega_1 \wedge d\omega_2 \quad \text{and} \quad S_{\text{int}}(\omega_1 \omega_2 \omega_3) := \pm \hbar^{-1} \int_M \omega_1 \wedge \omega_2 \wedge \omega_3.$$

Note that  $S_{\text{free}}$  and  $S_{\text{int}}$  are linear functions on  $B^{\text{cyc}} V[1]$  which vanish everywhere but field strings of lengths 2 and 3, respectively. It turns out that  $S_{\text{free}}$  and  $S := S_{\text{free}} + S_{\text{int}}$  satisfy the QME for  $\Delta_s$ . The twisted BV-operators look like

$$\Delta_s^{S_{\text{free}}} = \hat{q}_{110} + \Delta_s \quad \text{and} \quad \Delta_s^S = \hat{q}_{110} + \widehat{q_{210}(S_{\text{int}}, \cdot)} + \Delta_s.$$

Figure 1.4 depicts the new terms  $q_{110}$  and  $q_{210}(S_{\text{int}}, \cdot)$  in  $\Delta_s^S$ . The corresponding dIBL-

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algebra reads

$$\left( B_{\text{cyc}}^* V, \mathfrak{q}_{110}^{\mathfrak{m}} := \mathfrak{q}_{110} + \mathfrak{q}_{210}(S_{\text{int}}, \cdot), \mathfrak{q}_{210}, \mathfrak{q}_{120} \right),$$

where we recognize the Hochschild differential  $\mathfrak{q}_{110}^{\mathfrak{m}}$ . If this was well-defined for  $V = \Omega$ , then it would give the  $\text{IBL}_{\infty}$ -chain model of string topology.

As in quantum field theories, we are going to “formally” *integrate out redundant degrees of freedom in the path integral* of our ill-defined theory and obtain a well-defined theory on  $B_{\text{cyc}}^* H_{\text{dR}}$ , which is “formally” homotopy equivalent to the original one. We pick a Riemannian metric on  $M$  and consider the Hodge decomposition  $\Omega = \mathcal{H} \oplus \text{im } d \oplus \text{im } d^*$ , where  $\mathcal{H} \simeq H_{\text{dR}}$  are the harmonic forms  $d\omega = d^*\omega = 0$ . For Chern-Simons fields  $\omega \in \Omega$ , one thinks of  $d\omega = 0$  as of the Euler-Lagrange equation and of  $d^*\omega = 0$  as of the Lorentz gauge. The “inverse” of  $d$  as a map  $\text{im } d \rightarrow \text{im } d^*$  is the standard Hodge homotopy  $\mathcal{P}_{\text{std}}$ , i.e., the unique coexact solution of  $d\mathcal{P} + \mathcal{P}d = \mathbb{1} - \pi_{\mathcal{H}}$ , where  $\pi_{\mathcal{H}} : \Omega \rightarrow \mathcal{H}$  is the orthogonal projection. The Schwartz kernel of  $\mathcal{P}_{\text{std}}$  is the *standard Hodge propagator*  $\mathcal{P}_{\text{std}}$ . A formula for the *effective action*  $W \in \mathcal{F}(B^{\text{cyc}} H_{\text{dR}}[1])$  was given in [CFL15]; in their terminology, it would be called the *(formal) pushforward Maurer-Cartan element*. We have

$$W = \hbar^{-1} \sum_{l \geq 1, g \geq 0} \mathfrak{n}_{lg} \hbar^g,$$

where  $\mathfrak{n}_{lg} \in \hat{E}_l B_{\text{cyc}}^* H_{\text{dR}}$  is computed from  $(l + g - 1)$ -loop Feynman diagrams with interaction vertices  $\wedge$  and propagator  $\mathcal{P}_{\text{std}}$ . One can think of trying to push the field strings to their representatives consisting of harmonic forms. The twisted string BV-operator on  $\mathcal{F}(B^{\text{cyc}} H_{\text{dR}}[1])$  reads

$$\Delta_s^W = \hat{\mathfrak{q}}_{110}^{\mathfrak{n}} + \hbar \mathfrak{q}_{210} + \sum_{l \geq 2, g \geq 0} \hat{\mathfrak{q}}_{1lg}^{\mathfrak{n}} \hbar^g,$$

where  $\mathfrak{q}_{110}^{\mathfrak{n}} = \mathfrak{q}_{210}(\mathfrak{n}_{10}, \cdot) = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$  and  $\mathfrak{q}_{1lg}^{\mathfrak{n}} = \mathfrak{q}_{210} \circ \mathfrak{n}_{lg}$ . The resulting  $\text{IBL}_{\infty}$ -structure on  $\hat{B}_{\text{cyc}}^* H_{\text{dR}}$ , i.e., our chain model of equivariant string topology, is, in fact, a *quantum  $\text{coL}_{\infty}$ -algebra*  $(\mathfrak{q}_{1lg}^{\mathfrak{n}})$  with *Drinfeld-compatible Lie bracket*  $\mathfrak{q}_{210}$ . The weak homotopy equivalence of  $(\hat{B}_{\text{cyc}}^* H_{\text{dR}}, \mathfrak{q}_{110}^{\mathfrak{n}})$  and  $(C(L_{\mathbb{S}^1} M), \partial)$  inducing an isomorphism of the IBL-structure on homology is given by the composition  $F \circ I^* : C(L_{\mathbb{S}^1} M) \rightarrow \hat{B}_{\text{cyc}}^* H_{\text{dR}}$ , where

$$F = \mathfrak{f}_{110} + \mathfrak{f}_{210} \circ_1 \mathfrak{m}_{10} + \frac{1}{2!} \mathfrak{f}_{310} \circ_{1,1} (\mathfrak{m}_{10}, \mathfrak{m}_{10}) + \cdots$$

with  $\mathfrak{f}_{110} = \iota^*$  for  $\iota : H_{\text{dR}} \simeq \mathcal{H} \hookrightarrow \Omega$ , and where  $\mathfrak{f}_{k10} \circ_{1, \dots, 1} (\mathfrak{m}_{10}, \dots, \mathfrak{m}_{10})$  contributes by trivalent trees as in Figure 1.5. Note that in order to evaluate trees, we do not need the Schwartz kernel  $\mathcal{P}$  and hence any pairing. The homotopy  $\mathcal{P}$  is enough because the graph

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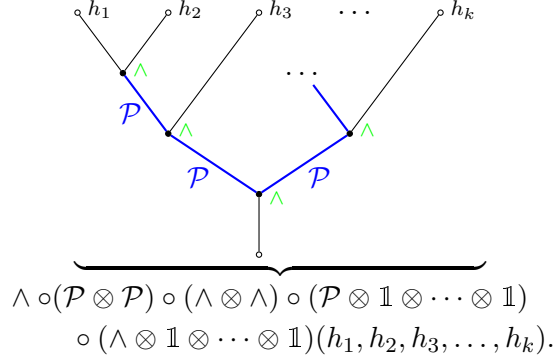


Figure 1.5.: Kontsevich-Soibelman evaluation of a decorated tree.

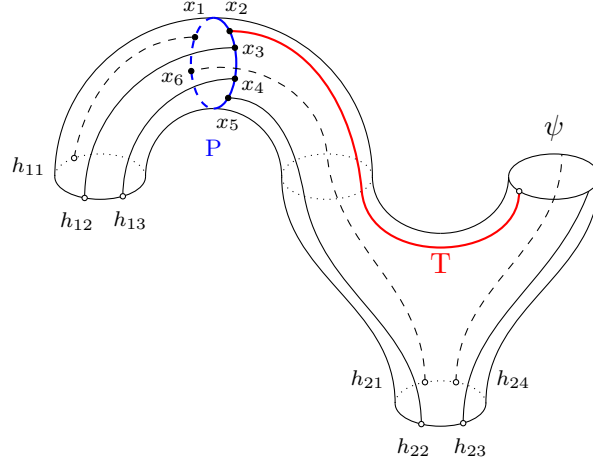
is directed and we can distinguish inputs and outputs. On the other hand, evaluation of the 1-loop Feynman graph in Figure 1.6 contributing to  $\mathbf{q}_{120}^n$ , for instance, requires  $\mathcal{P}$ , and hence  $\langle \cdot, \cdot \rangle$ . From Sullivan's minimal model theory it is well-known that the homotopy type of the homotopy transferred  $A_\infty$ -structure  $(H_{dR}, (m_k))$  is a topological invariant which encodes the rational homotopy theory of a simply-connected manifold  $M$ . Our construction can be seen as being associated to the Poincaré DGA  $(\Omega, d, \wedge, \langle \cdot, \cdot \rangle)$ , i.e, a DGA whose homology is a Poincaré duality algebra. It is not clear to the author what kind of invariant is the weak homotopy type of the Poincaré DGA and what kind of invariant is the homotopy type of the twisted  $IBL_\infty$ -algebra on  $\hat{B}_{cyc}^* H_{dR}$ . However, if  $M$  is formal in the sense of DGA's, then  $M$  is formal in the sense of Poincaré DGA's, and we conjecture that it is formal also in the sense of  $IBL_\infty$ -algebras; i.e., the twisted  $IBL_\infty$ -algebra on  $\hat{B}_{cyc}^* H_{dR}$  is homotopy equivalent to the canonical  $IBL$ -algebra.

Finally, note that from the theory of Koszul (pr)operads it is well-known that  $IBL$  is Koszul dual to  $\text{Frob}$ , i.e.,  $IBL^\perp = \text{Frob}$  and  $\text{Frob}^\perp = IBL$ , where  $\text{Frob}$  is the properad of Frobenius bialgebras. Therefore, it holds  $IBL_\infty = \Omega(\text{Frob}^*)$ , where  $\Omega$  denotes the cobar construction and  $*$  the linear dual coproperad; this precisely reflects our situation of having a Frobenius structure on  $H_{dR}$  (the coproduct is obtained from the product via dualization) and  $IBL_\infty$ -structure on  $\hat{B}_{cyc}^* H_{dR} = \widehat{B^{cyc} H_{dR}^*}$ , where  $\hat{\cdot}$  denotes a suitable completion, since  $B^{cyc}$  is a cyclic version of the cobar construction.

### 1.3. Relevant work of others

In [CJ02], there is a homotopy theoretical realization of string topology operations  $\mathfrak{m}_2$  and  $\mathfrak{c}_2$ . This is another approach to their rigorous definition.

## 1. Introduction



$$= \sum_{a,b} \sum_{c=1}^4 \pm T^{ab} \psi(e_a h_{2,c+2} h_{2,c+3}) \left( \int_{x_1 x_2 x_3 x_4 x_5 x_6} P(x_1, x_2) P(x_2, x_3) P(x_3, x_4) \right. \\ \left. P(x_4, x_5) P(x_5, x_6) P(x_6, x_1) (h_{11}(x_1) h_{12}(x_3) h_{13}(x_4)) (e_b(x_2) h_{2,c}(x_6) h_{2,c+1}(x_5)) \right)$$

Figure 1.6.: A 1-loop diagram and its contribution to the twisted cobracket.

In [CL09], they sketch a proof that symplectic field theory of the unit cotangent bundle of  $M$  and equivariant string topology of  $M$  are  $\text{IBL}_\infty$ -quasi-isomorphic via a map induced from evaluations at boundaries of holomorphic curves. String topology operations then arise naturally from the structure of codimension 1 boundary stratas of the moduli space of holomorphic curves with certain boundary conditions. In fact, a precise formulation of this correspondence was perhaps the main reason for Cieliebak & Latschev to think about the  $\text{IBL}_\infty$ -chain model of string topology.

An  $\text{IBL}_\infty$ -structure on the singular equivariant chain complex was proposed by [Sul07]. It is constructed using “diffusion intersection product” which realizes the intersection on the chain level. In fact, there is much more string operations coming from stacks of Sullivan diagrams than just the bracket and cobracket.

There is a recent construction of chain level non-equivariant string topology on the singular chain complex of  $LM$  using fat graphs [DPR15] and the diffusion intersection. It was shown in [CHV06] that there is a TQFT of string operations on the non-equivariant homology. The aforementioned reference constructs a chain model for this entire gadget.

There is an algebraic chain model for non-equivariant string topology in [Che12]. An algebraic chain model is also constructed in [Iri18].

Chain level string topology is required, e.g., in Fukaya’s work on Lagrangian subman-

Precise formulation of SFT: How is it precisely, what kind of boundary conditions and punctures?

ifolds.

What paper?:  
In what paper  
is this?

## 1.4. Summary of my results

- 1) The biggest part of the work was about setting up a formalism and deducing signs for a definition of the formal-pushforward Maurer-Cartan element, aka Chern-Simons Maurer-Cartan element, and the  $\text{IBL}_\infty$ -chain model in de Rham setting. Another big part was about understanding what is happening and discovering and formulating the structure and possible claims.

- 2) We compute the  $\text{IBL}_\infty$ -chain model for  $\mathbb{S}^n$  with  $n \neq 2$  by finding an explicit Hodge propagator and computing Feynman integrals. In fact, for  $n \geq 3$ , all integrals which are relevant for the  $\text{IBL}_\infty$ -theory vanish.

A trick from [CM10] is based on modifying an abstract Hodge propagator to obtain special properties implying vanishing of the integrals. The author of this thesis was not aware of this trick and tried to compute integrals with an explicit propagator in spherical coordinates for around 3 years until he rediscovered a part of this trick himself. The interesting thing is that the discovery was made via explicit computations, and it was a coincidence that the constructed Hodge propagator satisfied the special properties.

- 3) Using the trick from [CM10], we generalize the previous computation to geometrically formal manifolds and show that the Feynman integrals vanish provided that  $H_{\text{dR}}^1(M) = 0$ . For a general manifold, all higher coproducts vanish unless  $M$  is a surface or a 3-manifold with  $H_{\text{dR}}^1(M) \neq 0$ . In fact, the homotopy type of the  $\text{IBL}_\infty$ -chain model for a manifold with  $H_{\text{dR}}^1(M) = 0$  is determined by the tree-level perturbative Chern-Simons theory for a special Hodge propagator.
- 4) We formulate a conjecture that the  $\text{IBL}_\infty$ -chain model for formal manifolds with  $H_{\text{dR}}^1(M) = 0$  is homotopy equivalent to the canonical dIBL-structure. We propose a strategy of proof based on Poincaré duality models.
- 5) We study DGA's of Hodge type and give an alternative proof of the existence of a Poincaré duality model in the category of PDGA's; in the DGA category, this is originally due to Lambrechts & Stanley. The new method is based on adding exact partners to non-degenerates rather than adding killers of orphans and works for  $n \geq 5$  rather than  $n \geq 7$ .

## 1. Introduction

- 6) We prove a proposition that the cyclic homology of a strictly unital  $A_\infty$ -algebra can be computed from its reduced version. We do it by rewriting the Loday's cyclic homology theory for DGA's.
- 7) We relate  $\mathfrak{q}_{210}$  to Gerstenhaber bracket and cyclization and  $\mathfrak{q}_{120}$  to the Schwarz's BV-operator and cyclic shuffle product.
- 8) We extend the MV-formalism to filtered MV-formalism and use it to construct a BV-formulation of the weak  $IBL_\infty$ -theory. This has the advantage that the exponentials are honest exponentials and honest maps. This will be useful for studying BV-chain complexes.
- 9) We formulate the composition at  $k$ -common channels  $\circ_k$  using "heart with veins" which appears in the iterated bialgebra compatibility condition.
- 10) We understood the BV-formulation of  $IBL_\infty$ -theory with actions, effective action and quantum master equation.
- 11) We find the standard Hodge propagator for  $\mathbb{S}^2$  up to constants and prove that it smoothly extends to blow-up.

**Part I.**

## **IBL-infinity chain model**



## 2. Overview

An  $\text{IBL}_\infty$ -algebra is essentially a collection of multilinear operations  $\mathfrak{q}_{klg}$  with  $k$  inputs,  $l$  outputs and “genus”  $g$  satisfying certain relations; in particular,  $\mathfrak{q}_{110}$  is a boundary operator, and the pair  $\mathfrak{q}_{210}, \mathfrak{q}_{120}$  induces the structure of an involutive Lie bialgebra on the homology of  $\mathfrak{q}_{110}$ . It was introduced in [CFL15] and applications to string topology, symplectic field theory and higher genus Lagrangian Floer theory were proposed.

This part of the thesis is an attempt to understand the application to *string topology*. The idea was to carry out some explicit computations according to the plan sketched in [CFL15, Section 13] and test the string topology conjecture (see below).

The following results from [CFL15, Corollary 11.9] are our starting point (precise definitions of all the notions will be given in the next chapters;  $\text{IBL}_\infty$ -algebras in Part I will be strict and filtered in the terminology of [CFL15]):

- (A) For a finite-dimensional cyclic cochain complex  $(V, \Pi, m_1)$  of degree  $2 - n$ , there is a canonical dIBL-structure  $\mathfrak{p}_{110}, \mathfrak{p}_{210}, \mathfrak{p}_{120}$  of bidegree  $(n - 3, 2)$  on the degree shifted dual cyclic bar complex

$$C(V) := B_{\text{cyc}}^* V[2 - n] \simeq \left( \bigoplus_{k \geq 1} (V[1]^{\otimes k} / \text{cyc})' \right) [2 - n],$$

where  $\text{cyc}$  stands for cyclic permutations with the Koszul sign,  $'$  denotes the graded dual and  $[\cdot]$  the degree shift. This structure is denoted by  $\text{dIBL}(C(V))$ .

- (B) Let  $(\mathcal{H}, \Pi, m_1) \subset (V, \Pi, m_1)$  be a subcomplex such that the restriction of  $\Pi$  to  $\mathcal{H}[1]$  is non-degenerate. We apply (A) to  $(\mathcal{H}, \Pi, m_1)$  to get the canonical dIBL-algebra  $\text{dIBL}(C(\mathcal{H})) = (C(\mathcal{H}), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$ . Suppose that  $\pi : V[1] \rightarrow V[1]$  is a projection to  $\mathcal{H}[1]$  which satisfies

$$\begin{aligned} \pi \circ m_1 &= m_1 \circ \pi \quad \text{and} \\ \Pi(\pi(v_1), v_2) &= \Pi(v_1, \pi(v_2)) \end{aligned}$$

for all  $v_1, v_2 \in V[1]$ , and let  $\iota : \mathcal{H}[1] \rightarrow V[1]$  be the inclusion. A linear map

## 2. Overview

$\mathcal{P} : V[1] \rightarrow V[1]$  of degree  $-1$  such that

$$\begin{aligned} m_1 \circ \mathcal{P} + \mathcal{P} \circ m_1 &= \iota \circ \pi - \mathbb{1}_{V[1]} \quad \text{and} \\ \Pi(\mathcal{P}(v_1), v_2) &= (-1)^{|v_1|} \Pi(v_1, \mathcal{P}(v_2)) \end{aligned} \tag{2.1}$$

for all  $v_1, v_2 \in V[1]$  induces the  $\text{IBL}_\infty$ -homotopy equivalence

$$\mathfrak{f} = (\mathfrak{f}_{klg}) : \text{dIBL}(C(V)) \longrightarrow \text{dIBL}(C(\mathcal{H}))$$

such that  $\mathfrak{f}_{110} : C(V)[1] \rightarrow C(\mathcal{H})[1]$  is the map given by the precomposition with  $\iota$  in every component. We recall from [CFL15] that  $\mathfrak{f}_{klg} : E_k C(V) \rightarrow E_l C(\mathcal{H})$  is a linear map between exterior powers.

The map  $\mathfrak{f}_{klg}$  is constructed as a sum of contributions coming from isomorphism classes of *ribbon graphs* (=: multigraphs with a cyclic ordering of half-edges at every internal vertex) with  $k$  internal vertices,  $l$  boundary components and genus  $g$ . To compute the contribution of a labeled ribbon graph  $\Gamma$  to the value

$$\mathfrak{f}_{klg}(\Psi_1 \otimes \cdots \otimes \Psi_k)(W_1 \otimes \cdots \otimes W_l)$$

for  $\Psi_1, \dots, \Psi_k \in B_{\text{cyc}}^* V[3-n]$  and  $W_1, \dots, W_l \in B^{\text{cyc}} \mathcal{H}[3-n]$ , we decorate the  $i$ -th internal vertex of  $\Gamma$  with  $\Psi_i$ , external vertices lying on the  $i$ -th boundary component with components  $v_{i1}, \dots, v_{is_i} \in V[1]$  of  $W_i = s(v_{i1} \otimes \cdots \otimes v_{is_i} / \text{cyc})$ , where  $s$  is a formal symbol of degree  $n-3$ , and internal edges with the Schwartz kernel  $\mathcal{P}$  of  $\mathcal{P}$  with respect to  $\Pi$ . Decorated ribbon graphs are then evaluated in a consistent way to obtain real numbers (see Appendix A for an invariant formalism or [CFL15, Section 10] for a coordinate version of this construction).

We will also use the following results from [CFL15, Proposition 12.5 and Theorem 12.9] about deformations of  $\text{IBL}_\infty$ -algebras:

- (C) If in addition to (A) there is the product  $m_2 : V[1] \otimes V[1] \rightarrow V[1]$  making  $(V, m_1, m_2)$  into a cyclic dga, then  $(-1)^{n-2} m_2^+$  defines a canonical Maurer-Cartan element  $\mathfrak{m} := (\mathfrak{m}_{10})$  for  $\text{dIBL}(C(V))$ . The twisted  $\text{IBL}_\infty$ -algebra is again a  $\text{dIBL}$ -algebra of bidegree  $(n-3, 2)$ ; it is denoted by  $\text{dIBL}^{\mathfrak{m}}(C(V))$  and satisfies

$$\begin{aligned} \text{dIBL}^{\mathfrak{m}}(C(V)) \\ = (C(V), \mathfrak{p}_{110}^{\mathfrak{m}} = \mathfrak{p}_{110} + \mathfrak{p}_{210} \circ_1 \mathfrak{m}_{10}, \mathfrak{p}_{210}^{\mathfrak{m}} = \mathfrak{p}_{210}, \mathfrak{p}_{120}^{\mathfrak{m}} = \mathfrak{p}_{120}). \end{aligned} \tag{2.2}$$

- (D) The  $\text{IBL}_\infty$ -morphism  $\mathfrak{f}$  from (B) can be used to pushforward  $\mathfrak{m}$  and obtain the

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Maurer-Cartan element  $\mathbf{n} = (\mathbf{n}_{lg})$  for  $\mathrm{dIBL}(C(\mathcal{H}))$ . The twist by  $\mathbf{n}$  is an  $\mathrm{IBL}_\infty$ -algebra of bidegree  $(n-3, 2)$ ; it is denoted by  $\mathrm{dIBL}^{\mathbf{n}}(C(\mathcal{H}))$  and satisfies

$$\begin{aligned} \mathrm{dIBL}^{\mathbf{n}}(C(\mathcal{H})) \\ = (C(\mathcal{H}), \mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110} + \mathbf{q}_{210} \circ_1 \mathbf{n}_{10}, \mathbf{q}_{210}^{\mathbf{n}} = \mathbf{q}_{210}, \mathbf{q}_{120}^{\mathbf{n}} = \mathbf{q}_{120} \\ + \mathbf{q}_{210} \circ_1 \mathbf{n}_{20}, \text{ plus the higher operations } \mathbf{q}_{1lg}^{\mathbf{n}} = \mathbf{q}_{210} \circ_1 \mathbf{n}_{lg}). \end{aligned}$$

This  $\mathrm{IBL}_\infty$ -algebra is  $\mathrm{IBL}_\infty$ -homotopy equivalent to  $\mathrm{dIBL}^{\mathbf{m}}(C(V))$  via the twisted  $\mathrm{IBL}_\infty$ -morphism

$$\mathbf{f}^{\mathbf{m}} = (\mathbf{f}_{klg}^{\mathbf{m}}) : \mathrm{dIBL}^{\mathbf{m}}(C(V)) \longrightarrow \mathrm{dIBL}^{\mathbf{n}}(C(\mathcal{H})).$$

The pushforward Maurer-Cartan element  $\mathbf{n} = \mathbf{f}_* \mathbf{m}$  can be expressed as a sum of contributions of isomorphism classes of *trivalent ribbon graphs* ( $m_2^+$  has namely three inputs), where a labeled ribbon graph  $\Gamma$  is decorated with  $m_2^+$  at internal vertices, with the components of the  $i$ -th argument of  $\mathbf{n}_{lg}$ , i.e., elements of  $\mathcal{H}(V)[1]$ , at the  $i$ -th boundary component and with  $P$  at internal edges. Note that whereas (A) – (C) can be formulated without completions, infinite sums appear in  $\mathbf{n}_{lg}$ , and hence filtration and completions necessarily come into play.

The application to string topology of an oriented closed manifold  $M$  of dimension  $n$  comes from studying generalizations of (A) – (D) to the infinite-dimensional cyclic dga  $(\Omega^*(M), \Pi, m_1, m_2)$ . Here  $\Omega^*(M)$  is the de Rham complex of  $M$  and the maps  $\Pi : \Omega(M)[1]^{\otimes 2} \rightarrow \mathbb{R}$ ,  $m_1 : \Omega(M)[1] \rightarrow \Omega(M)[1]$  and  $m_2 : \Omega(M)[1]^{\otimes 2} \rightarrow \Omega(M)[1]$  are defined for all  $\eta, \eta_1, \eta_2 \in \Omega(M)$  as follows:

$$\text{de Rham cyc. dga} \left\{ \begin{array}{l} \Pi(\theta\eta_1, \theta\eta_2) := (-1)^{\eta_1} \int_M \eta_1 \wedge \eta_2, \\ m_1(\theta\eta) := \theta d\eta, \\ m_2(\theta\eta_1, \theta\eta_2) := (-1)^{\eta_1} \theta(\eta_1 \wedge \eta_2), \end{array} \right. \quad (2.3)$$

where  $d$  is the de Rham differential,  $\wedge$  the wedge product,  $\theta$  a formal symbol of degree  $-1$  and  $\eta_1$  in the exponent denotes the form-degree of  $\eta_1$ . By picking a Riemannian metric on  $M$ , we obtain the subcomplex of harmonic forms

$$(\mathcal{H}^*(M), \Pi, m_1 \equiv 0)$$

with the projection  $\pi_{\mathcal{H}} : \Omega(M) \rightarrow \mathcal{H}(M)$  coming from the Hodge decomposition. This

## 2. Overview

cyclic cochain complex shall be taken as the subcomplex in (B).

From technical reasons stemming from the fact that the non-degenerate pairing  $\Pi$  on  $\Omega(M)[1]$  is not perfect, one has to restrict the construction in (A) to the subspace  $B_{\text{cyc}}^*\Omega(M)_\infty$  of elements with a smooth Schwartz kernel. Then (A) and (B) work in the setting of the so called Fréchet IBL $_\infty$ -algebras introduced in [CFL15, Section 13]. However, the element  $\mathfrak{m}_{10} \in B_{\text{cyc}}^*\Omega(M)[3-n]$ , which translates into the *Chern-Simons term*

$$m_2^+(\theta\eta_1, \theta\eta_2, \theta\eta_3) := (-1)^{\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \eta_1, \eta_2, \eta_3 \in \Omega(M),$$

does not define the canonical Maurer-Cartan element  $\mathfrak{m}$  in (C) directly because  $m_2^+ \notin \hat{B}_{\text{cyc}}^*\Omega(M)_\infty$ . This also means that one cannot use (D) to conclude the existence of the pushforward Maurer-Cartan element  $\mathfrak{n}$ .

Nevertheless, it was proposed to define  $\mathfrak{n}$  formally using the summation over trivalent ribbon graphs as in the finite-dimensional case. We call such  $\mathfrak{n}$  a *Chern-Simons Maurer-Cartan element*. In order to compute the contribution of a labeled trivalent ribbon graph  $\Gamma$  with  $k$  internal vertices,  $l$  boundary components and genus  $g$  to the value

$$\mathfrak{n}_{lg}(\Omega_1 \otimes \cdots \otimes \Omega_l),$$

where  $\Omega_i = \text{sw}_i$  for  $\omega_1, \dots, \omega_l \in B^{\text{cyc}}\mathcal{H}(M)$ , one starts by decorating internal vertices with integration variables  $x_1, \dots, x_k$  on the  $k$ -fold product  $M \times \cdots \times M$ , external vertices on the  $i$ -th boundary component with the components  $\alpha_{i1}, \dots, \alpha_{is_i} \in \mathcal{H}(M)[1]$  of  $\omega_i$  and internal edges with the Hodge propagator  $P$ . In this setting,  $P$  becomes the Schwartz kernel of  $\mathcal{P}$  in the sense of pseudo-differential operators; this  $P$  is necessarily singular at the diagonal  $\Delta$ , so that we have only  $P \in \Omega^{n-1}(M \times M \setminus \Delta)$ . One then takes the wedge product of all forms in the decorated graph in the order and with the sign deduced from the labeling of  $\Gamma$  and computes the integral over  $x_1, \dots, x_k$ . Similar integrals appear in *perturbative Chern-Simons quantum field theory*.

Because of the singularity of  $P$  at  $\Delta$ , the integrand described above is smooth only on the  $k$ -th configuration space of  $M$ . It is not clear that all the integrals converge and that the resulting  $\mathfrak{n}_{lg}$  are well-defined and satisfy the Maurer-Cartan equation. The idea of work in progress [CVona] of K. Cieliebak and E. Volkov is to use iterated spherical blow-ups of the diagonals to resolve the singularities and obtain integrals of smooth forms on compact manifolds with corners; this guarantees integrability. The Maurer-Cartan equation for  $\mathfrak{n} = (\mathfrak{n}_{lg})$  is then proven with the help of Stokes' formula and by showing that the contributions of hidden codimension-1 faces cancel. This method is

## 2. Overview

similar to the method from [AS92] and [AS94], where Feynman integrals of perturbative Chern-Simons theory were considered.

Having  $\mathfrak{n}$ , the twisted  $\mathrm{IBL}_\infty$ -algebra  $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(M)))$ , which can be equivalently written as  $\mathrm{dIBL}^{\mathfrak{n}}(C(H_{\mathrm{dR}}(M)))$  using the Hodge isomorphism  $\mathcal{H}(M) \simeq H_{\mathrm{dR}}(M)$ , should satisfy the following conjecture:

**String topology conjecture** (Conjecture 1.12 in [CFL15]). *Let  $M$  be a closed oriented manifold of dimension  $n$  and  $H_{\mathrm{dR}}(M)$  its de Rham cohomology. Then there exists an  $\mathrm{IBL}_\infty$ -structure on (a suitable version of)  $B_{\mathrm{cyc}}^* H_{\mathrm{dR}}[2-n]$  whose homology equals the cyclic cohomology of the de Rham complex of  $M$ .*

The idea is that the  $\mathbb{S}^1$ -equivariant homology of the free loop space  $H_*^{\mathbb{S}^1}(LM)$  is isomorphic to a version of Connes' cyclic cohomology of the de Rham algebra  $H_\lambda^*(\Omega^*(M))$ , at least for simply-connected  $M$ . The precise relation will be established in yet another work in progress [CVona] of K. Cieliebak and E. Volkov using a chain-map coming from a cyclic version of Chen's iterated integrals. Now, a suitable degree shift of  $H_\lambda^*(\Omega^*(M))$  is isomorphic to the homology of the boundary operator  $\mathfrak{q}_{110}^{\mathfrak{m}}$  of the only formally defined  $\mathrm{dIBL}$ -algebra  $\mathrm{dIBL}^{\mathfrak{m}}(C(\Omega(M)))$ , which is according to (D) (formally) quasi-isomorphic to  $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(M)))$  via the twisted  $\mathrm{IBL}_\infty$ -morphism  $\mathfrak{f}^{\mathfrak{m}}$ .

The space  $H_*^{\mathbb{S}^1}(LM)$  is equipped with an  $\mathrm{IBL}$ -structure coming from the Chas-Sullivan string bracket  $\mathfrak{m}_2$  and string cobracket  $\mathfrak{c}_2$ ; these operations were defined geometrically on suitably transverse smooth chains in [CS99] and [CS04], respectively.<sup>1</sup> The natural question is: How is the  $\mathrm{IBL}$ -structure  $\mathfrak{m}_2, \mathfrak{c}_2$  related to the  $\mathrm{IBL}$ -structure  $\mathfrak{q}_{210}^{\mathfrak{n}}, \mathfrak{q}_{120}^{\mathfrak{n}}$  induced on  $H_*^{\mathbb{S}^1}(LM)$  via the isomorphism from the string topology conjecture? The *extended string topology conjecture* asserts that these structures agree, and hence the operations  $\mathfrak{q}_{210}^{\mathfrak{n}}, \mathfrak{q}_{120}^{\mathfrak{n}}$  defined on cyclic cochains provide a *chain model* for  $\mathfrak{m}_2, \mathfrak{c}_2$ . Based on our observations and explicit computations, we formulate an up-to-date version of the string topology conjecture for simply-connected manifolds (see Conjecture 4.5.1).

DONE Reduced loop space: Write that they are actually defined on the homology relative to constant loops. Actually, write here modulo problems with modding out constant loops or one constant loop.

A large part of this text consists of setting up the algebraic base for the work with  $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathcal{H}(M)))$ . In addition to repeating the theory from [CFL15] in a slightly different formalism, we also include the following topics:

- A formula for the partial composition  $\circ_s$  in terms of operations of the canonical associative bialgebra on the symmetric algebra (Definition 3.2.2); formulas for  $\mathfrak{q}_{klg}^{\mathfrak{n}}$  (Proposition 3.4.4).

<sup>1</sup>In fact,  $\mathfrak{c}_2$  is geometrically defined only on the homology relative to constant loops and  $\mathfrak{m}_2$  does not always restrict to it.

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- Definition of the cyclic cohomology of  $A_\infty$ -algebras (Definition 3.3.9) and its relation to the homology of  $\mathfrak{q}_{110}^n$  (Proposition 3.4.6); definitions of the reduced versions (Definitions 3.3.12, 3.4.7 and 3.4.8) and their relation to the unreduced versions (Propositions 3.4.10 and 3.3.13).
- A coordinate-free framework for the evaluation of labeled ribbon graphs (Definition A.1.1 and Proposition A.1.2); formal analogy of the finite-dimensional and the de Rham case which we use to obtain signs for the definition of  $\mathfrak{n}$  (Proposition A.2.1).
- Definition of the Hodge propagator (Definition 4.2.4) and of the Chern-Simons Maurer-Cartan element  $\mathfrak{n}$  (Definition 4.3.6).

Our first result is an explicit computation of  $\mathrm{dBL}^n(C(H_{\mathrm{dR}}(\mathbb{S}^n)))$  by finding a particular Hodge propagator and showing that all integrals which contribute to  $\mathfrak{n}$  vanish for  $n \geq 3$ ; for  $n = 1$ , there is a non-vanishing integral whose value we compute (see Section 5.1); for  $n = 2$ , the existence of a non-vanishing integral remains open.

**Theorem A** (Explicit computation for  $\mathbb{S}^n$ ). *Consider the round sphere  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ . Define  $\mathbf{1} := \theta \mathbf{1}$ ,  $\mathbf{v} := \theta \mathrm{Vol} \in H_{\mathrm{dR}}(\mathbb{S}^n)[1]$ , where  $\mathrm{Vol}$  is the volume form,  $\mathbf{1}$  the constant one and  $\theta$  a formal symbol of degree  $-1$ . There exists a Hodge propagator  $\mathbf{P}$  such that the following holds. For the homology of the twisted boundary operator  $\mathfrak{q}_{110}^n$ , we have:*

$$\begin{aligned} \mathbb{H}^n(C(H_{\mathrm{dR}}(\mathbb{S}^n)))[1] &:= H(\hat{\mathbf{B}}_{\mathrm{cyc}}^* H_{\mathrm{dR}}(\mathbb{S}^n)[3-n], \mathfrak{q}_{110}^n) \\ &= \begin{cases} \langle \mathbf{sv}^{i*}, \mathbf{s1}^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \geq 3 \text{ odd,} \\ \langle \mathbf{sv}^{2i-1*}, \mathbf{s1}^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \text{ even,} \\ \langle \mathbf{s} \sum_{k=1}^{\infty} c_k \mathbf{v}^{k*}, \mathbf{s1}^{2j-1*} \mid c_k \in \mathbb{R}, j \geq 1 \rangle & \text{for } n = 1. \end{cases} \end{aligned}$$

Here  $\langle \cdot \rangle$  denotes the linear span over  $\mathbb{R}$ ,  $*$  the dual and  $\mathbf{s}$  is a formal symbol of degree  $n-3$ . The product  $\mathfrak{q}_{210}^n$  vanishes on  $\mathbb{H}^n$  except for the following relations for  $n \geq 3$  odd

$$\mathfrak{q}_{210}^n(\mathbf{s1}^* \otimes \mathbf{sv}^{k*}) = \mathfrak{q}_{210}^n(\mathbf{sv}^{k*} \otimes \mathbf{s1}^*) = -(k-1)\mathbf{v}^{k-1*}$$

and the following relations for  $n = 1$ :

$$\mathfrak{q}_{210}^n\left(\mathbf{s1}^* \otimes \mathbf{s} \sum_{k=1}^{\infty} c_k \mathbf{v}^{k*}\right) = -\mathbf{s} \sum_{k=1}^{\infty} k c_{k+1} \mathbf{v}^{k*}.$$

The coproduct  $\mathfrak{q}_{120}^n$  as well as all higher operations  $\mathfrak{q}_{1lg}^n$  vanish on  $\mathbb{H}^n$  in every dimension

## 2. Overview

$n$ . For  $\mathbb{S}^1$ , we have  $\mathfrak{q}_{120}^n \neq \mathfrak{q}_{120}$  on the chain level; i.e., the twisting is non-trivial. For  $n \neq 2$ , all higher operations vanish on the chain level.

If we mod out  $s_1^{2j-1*}$ , i.e., if we consider the point-reduced version, then, after dropping  $s$ , the results agree with the string topology of  $M$  relative to one constant loop and with Chas-Sullivan operations. The only exception is  $M = \mathbb{S}^1$ . This supports the string topology conjecture for simply-connected manifolds and provides a counterexample for non-simply connected manifolds.

DONE Point reduced: Write Point reduced version.

Our second result generalizes the previous explicit computation and shows that in many cases, the twists with  $\mathfrak{n}$  and  $\mathfrak{m}$  coincide. Its proof is a combination of facts from Section 4.4.

DONE Degree shift: Write here that we have to drop  $s$ !

**Theorem B** (Triviality of the twist with  $\mathfrak{n}$  on the chain level). *Let  $M$  be a closed oriented  $n$ -manifold. There exists a Hodge propagator  $P$ , the so called special Hodge propagator, such that the following holds for the twisted  $\text{IBL}_\infty$ -structure  $\text{dIBL}^n(C(H_{\text{dR}}(M)))$ :*

- (1) For the basic operations  $\mathfrak{q}_{110}^n = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$ ,  $\mathfrak{q}_{210}^n = \mathfrak{q}_{210}$ ,  $\mathfrak{q}_{120}^n = \mathfrak{q}_{120} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$ , we have:
  - a) If  $H_{\text{dR}}^1(M) = 0$ , then  $\mathfrak{n}_{20} = 0$ , and hence  $\mathfrak{q}_{120}^n = \mathfrak{q}_{120}$ .
  - b) If  $M$  is geometrically formal, then  $\mathfrak{n}_{10} = \mathfrak{m}_{10}$ , and hence  $\mathfrak{q}_{110}^n = \mathfrak{q}_{110}^{\mathfrak{m}}$ . (In fact, if in addition  $H_{\text{dR}}^1(M) = 0$ , then  $\mathfrak{n} = \mathfrak{m}$ , at least for  $n \neq 2$ .)
- (2) For the higher operations  $\mathfrak{q}_{1lg}^n = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$  with  $(l, g) \neq (1, 0), (2, 0)$ , we have  $\mathfrak{n}_{lg} = 0$ , and hence  $\mathfrak{q}_{1lg}^n = 0$  with the possible exception of surfaces and 3-manifolds with  $H_{\text{dR}}^1(M) \neq 0$ .

The following are some topics for future work:

- (1) We would like to find out which field theory has  $P$  as a propagator and the trivalent ribbon graphs with harmonic forms as Feynman graphs. Note that due to the polarization  $P = \sum_{k=0}^{n-1} P^{(k)}(x, y)$ , the theory has a field in every degree. This is in contrast to the Chern-Simons theory for  $n = 3$ , whose field is a 1-form. Some actions were suggested to the author by Dr. Puzzio and Prof. A. Cattaneo.
- (2) We would like to improve Theorem B by showing that the higher operations for  $\mathbb{S}^2$  vanish. If this is the case, then the statement that all higher operations vanish for every manifold  $M$  with  $H_{\text{dR}}^1(M) = 0$  is true.

## 2. Overview

- (3) For a formal simply-connected manifold  $M$ , we would like to investigate whether  $\mathrm{dIBL}^n(C(H_{\mathrm{dR}}(M)))$  and  $\mathrm{dIBL}^m(C(H_{\mathrm{dR}}(M)))$  are homotopy equivalent as  $\mathrm{IBL}_\infty$ -algebras. If not, we would like to understand the obstruction.
- (4) We would like to compute  $\mathrm{dIBL}^n(C(H_{\mathrm{dR}}(M)))$  for surfaces  $\Sigma_g$  with  $g \geq 1$  and formulate a string topology conjecture for non-simply connected manifolds.
- (5) We would like to know whether the Schwartz kernel  $P_{\mathrm{std}}$  of  $\mathcal{P}_{\mathrm{std}} = -d^* \Delta^{-1}$  (the so called standard Hodge propagator), where  $d^*$  is the codifferential and  $\Delta$  the Hodge-de Rham Laplacian, extends smoothly to a blow-up. If yes, then it is a canonical special Hodge propagator, and hence the statement of Theorem B holds.
- (6) We would like to define a generalization of an  $\mathrm{IBL}_\infty$ -algebra — a weak, non-reduced  $\mathrm{IBL}_\infty$ -algebra with a gauge group — and understand its precise relation to perturbative Chern-Simons theory within the BV-formalism.
- (7) Identify the theory from which the integrals come from, i.e., write down an action.

In the end, let us summarize some existing work on  $\mathrm{IBL}_\infty$ -algebras which helped us to understand  $\mathrm{IBL}_\infty$ -algebras in broader context: In [MS13], they find an  $\mathrm{IBL}_\infty$ -structure in open-closed string field theory. In [DJP17], they view  $\mathrm{IBL}_\infty$ -algebras as algebras over the cobar construction of the closed Frobenius properad. In [MV17], they consider  $\mathrm{IBL}_\infty$ -algebras as a particular case of  $\mathrm{BV}_\infty$ - or, more generally,  $\mathrm{MV}$ -algebras.

DONE cobar complex of properad: Algebra over the cobar complex of the closed Frobenius properad.



### 3. Algebraic structures

In Section 3.1, we recall weight-grading (Definition 3.1.1), Koszul sign (Definition 3.1.2), degree shift (Definition 3.1.3), filtrations (Definition 3.1.8) and completions (Definition 3.1.9). We prove the Künneth formula for completed symmetric cohomology (Proposition 3.1.13).

In Section 3.2, we review basics of  $\text{IBL}_\infty$ -algebras from [CFL15]. We define the exterior algebra  $EC$  over a graded vector space  $C$  as the symmetric algebra  $S$  over  $C[1]$  (Definition 3.2.1) and use the operations  $\mu$  and  $\Delta$  of the structure of an associative bialgebra on  $S(C[1])$  to give explicit formulas for the partial compositions  $\circ_{h_1, \dots, h_k}$  (Definition 3.2.2). We use the compositions to define the notion of an  $\text{IBL}_\infty$ -algebra  $(\mathfrak{q}_{klg})$  on  $C$  (Definition 3.2.4), a Maurer-Cartan element  $(\mathfrak{n}_{lg})$  (Definition 3.2.6) and twisted operations  $(\mathfrak{q}_{klg}^n)$  (Definition 3.2.7). We mention that an  $\text{IBL}$ -algebra according to our definition is an odd degree shift of a classical  $\text{IBL}$ -algebra (Proposition 3.2.5). We define the induced  $\text{IBL}$ -structure on homology (Definition 3.2.8), briefly discuss the BV-formalism (Remark 3.2.9) and mention weak  $\text{IBL}_\infty$ -algebras (Remark 3.2.10). Finally, we summarize the situation for twisted  $\text{dIBL}$ -algebras (Proposition 3.2.11) and briefly discuss higher operations (Remark 3.2.12).

In Section 3.3, we define the (weight-reduced) dual cyclic bar-complex  $B_{\text{cyc}}^*V$  of a graded vector space  $V$  (Definition 3.3.1) and introduce some notation (Notation 3.3.3). We then summarize some facts about the completions  $\hat{B}_{\text{cyc}}^*V$  and  $\hat{E}_k B_{\text{cyc}}^*V$  (Proposition 3.3.6). We define the notion of a cyclic  $A_\infty$ -structure on  $V$  (Definition 3.3.7) and its Hochschild and cyclic cohomology (Definition 3.3.9). We recall strict units and strict augmentations (Definition 3.3.10), define the reduced dual cyclic bar complex  $B_{\text{cyc}, \text{red}}^*V$  (Definition 3.3.12) and sketch a proof of the fact that the cyclic cohomology is a direct sum of the reduced cyclic cohomology and the cyclic cohomology of the ground field (Proposition 3.3.13). We relate our version of the cyclic cohomology for dga's to the classical version from [Lod92] (Proposition 3.3.14). We also show that the reduced spaces for a simply connected and connected  $V$  are complete (Proposition 3.3.15).

In Section 3.4, we review the construction of the canonical  $\text{dIBL}$ -structure  $\text{dIBL}(C(V))$  (Definition 3.4.1) and the canonical Maurer-Cartan element  $\mathfrak{m}$  (Definition 3.4.2) starting

### 3. Algebraic structures

from a cyclic dga  $(V, \Pi, m_1, m_2)$ . We give formulas for the operations  $(q_{1lg}^n)$  of the  $\text{IBL}_\infty$ -algebra  $\text{dIBL}^n(C(V))$  twisted by a Maurer-Cartan element  $\mathbf{n}$  (Proposition 3.4.4). We consider the  $A_\infty$ -structure induced on  $V$  by  $\mathbf{n}_{10}$  (Definition 3.4.5) and relate its cyclic cohomology to the homology of  $q_{110}^n$  (Proposition 3.4.6). We define the reduced canonical  $\text{dIBL}$ -algebra  $\text{dIBL}(C_{\text{red}}(V))$  (Definition 3.4.7) and the notion of a strictly reduced Maurer-Cartan element (Definition 3.4.8). The twisted  $\text{IBL}_\infty$ -structure then splits into the reduced part and the part generated by  $1^{i*}$ , which we can explicitly compute (Proposition 3.4.10).

#### 3.1. Gradings, degree shifts and completions

We will work with vector spaces over  $\mathbb{R}$ , possibly infinite-dimensional, graded by the *degree*  $d \in \mathbb{Z}$  and the *weight*  $k \in \mathbb{N}_0$ .

**Definition 3.1.1** (Weight-graded vector spaces). *A graded vector space is a vector space  $W$  together with a collection of subspaces  $W^d \subset W$  for all  $d \in \mathbb{Z}$  such that*

$$W = \bigoplus_{d \in \mathbb{Z}} W^d.$$

*Elements of  $W^d$  are called homogenous of degree  $d$ ; given  $w \in W^d$ , we denote the degree of  $w$  by  $|w| := d$ .*

*A linear map of graded vector spaces  $f : W_1 \rightarrow W_2$  is called homogenous of degree  $|f| \in \mathbb{Z}$  if it holds*

$$f(W_1^d) \subset W_2^{d+|f|} \quad \text{for all } d \in \mathbb{Z}. \quad (3.1)$$

*A weight-graded vector space is a graded vector space  $W$  together with a collection of subspaces  $W_k^d \subset W^d$  for all  $k \in \mathbb{N}_0$  and  $d \in \mathbb{Z}$  such that*

$$W^d = \bigoplus_{k \in \mathbb{N}_0} W_k^d \quad \text{for all } d \in \mathbb{Z}.$$

*We define the weight- $k$  component of  $W$  by*

$$W_k := \bigoplus_{d \in \mathbb{Z}} W_k^d \quad \text{for all } k \in \mathbb{N}_0.$$

*If  $W_0^d = 0$  for all  $d \in \mathbb{Z}$ , we say that  $W$  is weight-reduced. We define the weight-reduced*

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subspace  $\overline{W}$  of a weight-graded vector space  $W$  by

$$\overline{W} := \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}} W_k^d.$$

The superscript  $*$  usually denotes the dual in the category we work in, e.g., chain complexes. For a (weight-)graded vector space  $W$ , we also introduce the following notation to avoid confusion:

$$\begin{aligned} W^{*\text{lin}} &:= \{\psi : W \rightarrow \mathbb{R} \text{ linear}\} \quad \dots \quad \text{linear dual,} \\ W^{*\text{g}} &:= \bigoplus_{d \in \mathbb{Z}} \prod_{k \in \mathbb{N}_0} W_k^{d*} \quad \dots \quad \text{graded dual,} \\ W^{*\text{wg}} &:= \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}_0} W_k^{d*} \quad \dots \quad \text{weight-graded dual.} \end{aligned} \tag{3.2}$$

Here,  $W^{*\text{lin}}$  is a vector space,  $W^{*\text{g}}$  a graded vector space and  $W^{*\text{wg}}$  a weight-graded vector space. The grading convention from (3.2) is the cohomological grading convention, which differs from the convention (3.1) for maps  $f : W \rightarrow \mathbb{R}$  by degree reversal (for this, see Definition 3.1.3).

We identify  $W^{*\text{g}}$  with the subspace of  $W^{*\text{lin}}$  generated by homogenous maps and  $W^{*\text{wg}}$  with the subspace of  $W^{*\text{lin}}$  generated by maps which are non-zero only on finitely many  $W_k^d$ ; hence, under this identification, we have the tower of inclusions

$$W^{*\text{wg}} \subset W^{*\text{g}} \subset W^{*\text{lin}}.$$

**Definition 3.1.2** (Koszul sign). Let  $k \geq 1$ , and let  $\sigma \in \mathbb{S}_k$  be a permutation on  $k$  elements, i.e., a bijection of the set  $\{1, \dots, k\}$ . For  $i = 1, \dots, k$ , let  $a_i$  and  $b_i$  be graded symbols of degrees  $|a_i|$  and  $|b_i|$ , respectively. We denote by

$$\varepsilon(\sigma, a) \quad \text{and} \quad \varepsilon(a, b)$$

the Koszul signs of the transformations

$$a_1 \dots a_k \mapsto a_{\sigma_1^{-1}} \dots a_{\sigma_k^{-1}} \quad \text{and} \quad a_1 \dots a_k b_1 \dots b_k \mapsto a_1 b_1 \dots a_k b_k,$$

respectively. Here  $\sigma_i^{-1} := \sigma^{-1}(i)$ . The Koszul sign is computed by permuting the left-hand side to the right-hand side via transpositions of two adjacent elements such that whenever we transpose two graded symbols, e.g.,  $a_i \longleftrightarrow a_j$ , we multiply with  $(-1)^{|a_i||a_j|}$ .

We emphasize that the Koszul sign depends only on the initial and the final order of

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the graded symbols and not on the sequence of transpositions; this is easy to show.

**Definition 3.1.3** (Degree shift and grading reversal). *Let  $A \in \mathbb{Z}$ . The degree shift by  $A$  is a functor which associates to a graded vector space  $W$  the graded vector space  $W[A]$  with degree- $d$  components*

$$W[A]^d := W^{d+A} \quad \text{for all } d \in \mathbb{Z}.$$

*There is the canonical degree shift morphism*

$$W \longrightarrow W[A] \tag{3.3}$$

*of degree  $-A$  mapping  $W^d$  identically to  $W[A]^{d-A}$ . We view this morphism as multiplication from the left with a formal symbol  $s_A$  of degree  $|s_A| = -A$ , so that (3.3) can be written as  $w \in W \mapsto s_A w \in W[A]$ .*

*Given graded vector spaces  $W_1, W_2$  and constants  $A_1, A_2 \in \mathbb{Z}$ , we associate to a morphism  $f : W_1 \rightarrow W_2$  its degree shift  $f : W_1[A_1] \rightarrow W_2[A_2]$  by defining<sup>1</sup>*

$$f(s_{A_1} w) := s_{A_2} f(w) \quad \text{for all } w \in W_1. \tag{3.4}$$

*Notice that if  $f : W_1 \rightarrow W_2$  has degree  $|f|$ , then  $f : W_1[A_1] \rightarrow W_2[A_2]$  has degree  $|f| + A_1 - A_2$ .*

*The grading reversal  $r$  is a functor which associates to a graded vector space  $W$  the graded vector space  $r(W)$  with*

$$r(W)^d := W^{-d} \quad \text{for all } d \in \mathbb{Z}.$$

*There is the canonical morphism  $W \rightarrow r(W)$  mapping  $W^d$  identically to  $W^{-d}$  for every  $d \in \mathbb{Z}$ . The degree reversal of a morphism  $f : W_1 \rightarrow W_2$  is the morphism  $f : r(W_1) \rightarrow r(W_2)$  defined by conjugating  $f$  with the canonical morphism. If  $|f|$  is the degree of  $f : W_1 \rightarrow W_2$ , then  $-|f|$  is the degree of  $f : r(W_1) \rightarrow r(W_2)$ .*

In our main reference [CFL15], they view  $W$  and  $W[A]$  as one vector space with two different gradings  $\deg(\cdot)$  and  $|\cdot|$ , respectively; these are related by

$$|w| = \deg(w) - A \quad \text{for all homogenous } w \in W.$$

On the other hand, we think of  $W$  and  $W[A]$  as of two different graded vector spaces

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<sup>1</sup>This convention is not optimal; see Remark 3.1.4.

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and never use the same symbol for an element  $w \in W$  and its degree shift  $s_A w \in W[A]$ . It allows us to use just one notation  $|\cdot|$  for the gradings on both  $W$  and  $W[A]$ . However, in order to preserve compatibility with [CFL15], we will also sometimes use the notation  $\deg(w)$  (in the exponent just  $(-1)^w$ ) for the degrees on  $W$ .

For graded vector spaces  $W_1, \dots, W_k$  and constants  $A_1, \dots, A_k \in \mathbb{Z}$ , we identify

$$W_1[A_1] \otimes \dots \otimes W_k[A_k] \simeq (W_1 \otimes \dots \otimes W_k)[A_1 + \dots + A_k]$$

using the *Koszul convention for the tensor product*; for homogenous elements  $w_1 \in W_1, \dots, w_k \in W_k$ , it reads

$$\begin{aligned} s_{A_1} w_1 \otimes \dots \otimes s_{A_k} w_k &= \varepsilon(s_A, w) \underbrace{s_{A_1} \dots s_{A_k}}_{=: s_{A_1 + \dots + A_k}} w_1 \otimes \dots \otimes w_k. \end{aligned} \quad (3.5)$$

If  $A_1 = \dots = A_k =: A$  is fixed in the context, which is our usual case, we omit the subscript  $A$  and write just  $s$ .

In the case of the multilinear map  $f : W_1 \otimes \dots \otimes W_k \rightarrow V_1 \otimes \dots \otimes V_l$ , the combination of (3.4) and (3.5) gives for  $f : W_1[A_1] \otimes \dots \otimes W_k[A_k] \rightarrow V_1[B_1] \otimes \dots \otimes V_l[B_l]$  the following:

$$f(s_{A_1 + \dots + A_k} w_1 \otimes \dots \otimes w_k) = s_{B_1 + \dots + B_l} f(w_1 \otimes \dots \otimes w_k). \quad (3.6)$$

*Remark 3.1.4* (Why is this sign convention bad?). Let us illustrate that (3.6) is not compatible with the following standard Koszul rule:

$$(K) : \quad (f_1 \otimes f_2)(w_1 \otimes w_2) = (-1)^{|f_2||w_1|} f_1(w_1) \otimes f_2(w_2).$$

On one hand, we get

$$\begin{aligned} (f_1 \otimes f_2)(s^2 w_1 \otimes w_2) &\stackrel{(3.6)}{=} s^2 (f_1 \otimes f_2)(w_1 \otimes w_2) \\ &\stackrel{(K)}{=} (-1)^{|f_2||w_1|} s^2 f_1(w_1) \otimes f_2(w_2) \\ &\stackrel{(3.5)}{=} (-1)^{|f_2||w_1| + A(|f_1| + |w_1|)} s f_1(w_1) \otimes s f_2(w_2). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} (f_1 \otimes f_2)(s^2 w_1 \otimes w_2) &\stackrel{(3.5)}{=} (-1)^{A|w_1|} (f_1 \otimes f_2)(s w_1 \otimes s w_2) \\ &\stackrel{(K)}{=} (-1)^{A|w_1| + |f_2|(A + |w_1|)} f_1(s w_1) \otimes f_2(s w_2) \\ &\stackrel{(3.6)}{=} (-1)^{A|w_1| + |f_2|(A + |w_1|)} s f_1(w_1) \otimes s f_2(w_2). \end{aligned}$$

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The results differ by  $(-1)^{A(|f_1|+|f_2|)}$ . Therefore, we can not use (K) to identify the tensor product  $\text{Hom}(W_1, V_1) \otimes \text{Hom}(W_2, V_2)$  with a subspace of  $\text{Hom}(W_1 \otimes W_2, V_1 \otimes V_2)$  in general. We will rather define an ad-hoc pairing in the case where we need it (see Definition 3.3.4).

Another caveat is that in the case of the tensor product, the degree shift by  $A_1$  followed by the degree shift by  $A_2$  is not the same as the degree shift by  $A_1 + A_2$ . Indeed, we compute

$$\begin{aligned} (s_{A_1+A_2} w_1) \otimes (s_{A_1+A_2} w_2) &= (s_{A_2} s_{A_1} w_1) \otimes (s_{A_2} s_{A_1} w_2) \\ &= (-1)^{A_2(A_1+|w_1|)} s_{A_2}^2 (s_{A_1} w_1) \otimes (s_{A_1} w_2) \\ &= (-1)^{A_2 A_1 + (A_1+A_2)|w_1|} s_{A_2}^2 s_{A_1}^2 (w_1 \otimes w_2) \\ &= (-1)^{A_2 A_1 + (A_1+A_2)|w_1|} s_{2(A_1+A_2)} (w_1 \otimes w_2), \end{aligned}$$

which differs by  $(-1)^{A_1 A_2}$  from the direct degree shift by  $A_1 + A_2$ . Therefore, we have to always remember the vector spaces which we started with and the sequence of degree shifts.

Note that we also have the unnatural “ $s_{A_1} s_{A_2} = s_{A_1+A_2} = s_{A_2} s_{A_1}$ ” due to (3.5).  $\triangleleft$

*Remark 3.1.5* (Is there a better sign convention?). The author originally respected the Koszul rule for the algebra with formal symbols and considered the following map  $s_*^l \bar{s}^{k*} f : W[A]^{\otimes k} \rightarrow V[A]^{\otimes l}$  as the degree shift of  $f : W^{\otimes k} \rightarrow V^{\otimes l}$ :

$$(s_*^l \bar{s}^{k*} f)(s^k w_1 \otimes \cdots \otimes w_k) = (-1)^{k|f|A + \frac{1}{2}k(k-1)A} s^l f(w_1 \otimes \cdots \otimes w_k). \quad (3.7)$$

Here  $\bar{s}$  denotes the “inverse” of  $s$  with  $|\bar{s}| = -|s|$ ,  $s_*^l f = s^l \circ f$  is the post-composition,  $\bar{s}^{k*} f = (-1)^{kA|f|} f \circ s^k$  the pre-composition, and the sign  $\varepsilon(s, \bar{s}) = (-1)^{\frac{1}{2}k(k-1)A}$  comes from the “collision”  $\bar{s}_1 \dots \bar{s}_k s_1 \dots s_k \mapsto \bar{s}_1 s_1 \dots \bar{s}_k s_k$ .

However, the author did not manage to reprove the theory in [CFL15] using (3.7) (because of too many “external” signs appearing and a problem with disconnected graphs). A motivation to try a different sign convention was to explain some artificial signs in [CFL15] and formulate their coordinate constructions invariantly in order to generalize them to the “continuous” de Rham case.

It might be possible to deduce a “universal” sign convention “respecting” the Koszul rules by considering the category of chain complexes and graded morphisms  $\mathcal{C}$  as the category enriched in the closed monoidal category of chain complexes and chain maps of degree 0. One can then define the enriched degree shift functor  $s_A : \mathcal{C} \rightarrow \mathcal{C}$ , embed  $\mathcal{C}^{\otimes k} \subset \mathcal{C}$  using (K) and study enriched natural transformations in the algebra of

### 3. Algebraic structures

functors consisting of tensor products and compositions of  $s_A$ ,  $\text{Hom}(\cdot, \cdot)$  and the dual  $*$ . The question is whether the sign rules for degree shifts, in particular (3.7), are uniquely determined by the requirements on enriched functors and enriched natural transformations.

Another idea for how to specify the degree shift convention (3.7) is the observation that maps  $f : W^{\otimes k} \rightarrow W^{\otimes l}$  under consideration usually form an algebra  $\mathcal{P} \rightarrow \text{End}_W$  of a certain PROP  $\mathcal{P}$  over  $W$ ; here  $\text{End}_W$  denotes the endomorphism PROP. Suppose that there is a natural notion of the degree shift  $\mathcal{P}[A]$  of a PROP  $\mathcal{P}$  and a canonical degree shift morphism  $\mathcal{P} \rightarrow \mathcal{P}[A]$ . The PROPs  $(\text{End}_W)[A]$  and  $\text{End}_{W[A]}$  are better to be isomorphic, and any intelligent degree shift convention  $\psi_{k,l} : \text{Hom}(W^{\otimes k}, W^{\otimes l})[A(k-l)] \rightarrow \text{Hom}(W[A]^{\otimes k}, W[A]^{\otimes l})$  should induce an isomorphism of the PROPs  $(\text{End}_W)[A]$  and  $\text{End}_{W[A]}$ .  $\triangleleft$

**Definition 3.1.6** (Standard action of permutations). *For  $k \geq 1$  and  $\sigma \in \mathbb{S}_k$  ( $:=$  the group of permutations on  $k$  elements), we define the standard action of  $\sigma$  on  $W^{\otimes k}$  by*

$$\sigma(w_1 \otimes \cdots \otimes w_k) := \varepsilon(\sigma, w) w_{\sigma_1^{-1}} \otimes \cdots \otimes w_{\sigma_k^{-1}} \quad (3.8)$$

for all homogenous  $w_1, \dots, w_k \in W$ .

Notice that the  $i$ -th vector is permuted to the  $\sigma_i$ -th place — this is the “active” convention for permutations.

**Definition 3.1.7** (Symmetric algebra). *Let  $T(V) := \bigoplus_{k \geq 0} V^{\otimes k}$  be the tensor algebra over a graded vector space  $V$ . The symmetric algebra over  $V$  is defined by  $S(V) := \bigoplus_{k \geq 0} S_k(V)$ , where*

$$S_k(V) := V^{\otimes k} / \sum_{\sigma \in \mathbb{S}_k} \text{im}(\mathbb{1} - \sigma) \quad (=:\mathbb{S}_k\text{-coinvariants}).$$

*It is a weight-graded vector space with components denoted by  $(S_k V)^d$  for all  $d \in \mathbb{Z}$  and  $k \in \mathbb{N}_0$ . Note that  $S_0 V = \mathbb{R}$  has degree 0 by definition. Consider the canonical projection*

$$\begin{aligned} \pi : T(V) &\longrightarrow S(V) \\ v_1 \otimes \cdots \otimes v_k &\longmapsto v_1 \cdots v_k. \end{aligned}$$

*If  $v_i \in V$  are homogenous, we call  $v_1 \cdots v_k$  a generating word; we have*

$$v_1 \cdots v_k = \varepsilon(\sigma, v) v_{\sigma_1^{-1}} \cdots v_{\sigma_k^{-1}} \quad \text{for every } \sigma \in \mathbb{S}_k.$$

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Let  $\iota : S(V) \rightarrow T(V)$  be the section of  $\pi$  defined by

$$\iota(v_1 \cdots v_k) := \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \varepsilon(\sigma, v) v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_k^{-1}}.$$

We use it to identify  $S(V)$  with the subspace of symmetric tensors

$$\iota(S_k(V)) = \bigcap_{\sigma \in \mathbb{S}_k} \ker(\mathbb{1} - \sigma) \subset T_k(V) \quad (=:\mathbb{S}_k\text{-invariants}).$$

**Definition 3.1.8** (Filtrations). *Let  $W$  be a graded vector space. A filtration of  $W$  is a collection of linear subspaces  $\mathcal{F}^\lambda W \subset W$  for  $\lambda \in \mathbb{R}$  such that we have either*

- $\mathcal{F}^{\lambda_1} W \subset \mathcal{F}^{\lambda_2} W$  for all  $\lambda_1 \leq \lambda_2 \iff$  increasing filtration, or
- $\mathcal{F}^{\lambda_1} W \supset \mathcal{F}^{\lambda_2} W$  for all  $\lambda_1 \leq \lambda_2 \iff$  decreasing filtration.

We will assume that our filtrations are graded in the following sense:

$$\forall \lambda \in \mathbb{R} : \quad \mathcal{F}^\lambda W = \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^\lambda W^d, \quad \text{where} \quad \mathcal{F}^\lambda W^d := \mathcal{F}^\lambda W \cap W^d.$$

A filtration  $\mathcal{F}^\lambda W$  is called:

$$\begin{aligned} \text{exhaustive} & \iff \bigcup_{\lambda \in \mathbb{R}} \mathcal{F}^\lambda W = W; \\ \text{Hausdorff} & \iff \bigcap_{\lambda \in \mathbb{R}} \mathcal{F}^\lambda W = 0; \\ \mathbb{Z}\text{-gapped} & \iff \mathcal{F}^\lambda W = \mathcal{F}^{\lfloor \lambda \rfloor} W \text{ for all } \lambda \in \mathbb{R}; \\ \text{bounded from below} & \iff \exists \lambda \in \mathbb{R} : \mathcal{F}^\lambda W = 0; \\ \text{bounded from above} & \iff \exists \lambda \in \mathbb{R} : \mathcal{F}^\lambda W = W. \end{aligned}$$

Given a graded vector space  $W$  filtered by a  $\mathbb{Z}$ -gapped filtration  $\mathcal{F}^\lambda W$ , we associate to it the bi-graded vector space

$$\text{gr}(W) = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{\lambda \in \mathbb{Z}} \text{gr}(W)_\lambda^d$$

called the graded module whose components are given as follows:<sup>2</sup>

<sup>2</sup>The definition is made in such a way that if  $r$  is the functor which reverses  $\lambda$ , i.e.,  $r(\mathcal{F})^\lambda = \mathcal{F}^{-\lambda}$ , then it holds  $r(\text{gr}(\mathcal{F})) = \text{gr}(r(\mathcal{F}))$ .

DONE Graded module: Should not the grading of gr be shifted? I propose  $gr_k = F_k / F_{k+1}$  because  $r(gr(\tilde{F})) = gr(F)$  for the degree reversed filtration.



### 3. Algebraic structures

$$\forall d, \lambda \in \mathbb{Z} : \quad \text{gr}(W)_\lambda^d := \begin{cases} \mathcal{F}^\lambda W^d / \mathcal{F}^{\lambda-1} W^d & \text{for increasing } \mathcal{F}^\lambda W, \\ \mathcal{F}^\lambda W^d / \mathcal{F}^{\lambda+1} W^d & \text{for decreasing } \mathcal{F}^\lambda W. \end{cases}$$

We naturally extend a filtration over degree shifts, graded duals, direct sums, tensor products and symmetric products as follows:

$$\mathcal{F}^\lambda W[A]^d := \mathcal{F}^\lambda W^{d+A},$$

$$\mathcal{F}^\lambda (W^{*g})^d := \{\psi \in W^{d*} \mid \psi|_{\mathcal{F}^\lambda W} = 0\},$$

$$\mathcal{F}^\lambda \left( \bigoplus_{i \in I} W_i \right)^d := \bigoplus_{i \in I} \mathcal{F}^\lambda W_i^d,$$

$$\mathcal{F}^\lambda (W_1 \otimes \cdots \otimes W_k)^d := \bigoplus_{\substack{d_1, \dots, d_k \in \mathbb{Z} \\ d_1 + \dots + d_k = d}} \sum_{\substack{\lambda_1, \dots, \lambda_k \in \mathbb{R} \\ \lambda_1 + \dots + \lambda_k = \lambda}} \mathcal{F}^{\lambda_1} W_1^{d_1} \otimes \cdots \otimes \mathcal{F}^{\lambda_k} W_k^{d_k},$$

$$\mathcal{F}^\lambda (S_k V)^d := \pi(\mathcal{F}^\lambda (V^{\otimes k})^d),$$

where  $\pi : T(V) \rightarrow S(V)$  is the canonical projection. The ground field  $\mathbb{R}$  is filtered by the trivial filtration:

$$\mathcal{F}^\lambda \mathbb{R} := \begin{cases} \mathbb{R} & \lambda \leq 0, \\ 0 & \lambda > 0. \end{cases} \quad (3.9)$$

If  $(W, \partial)$  is a filtered chain complex, we filter the homology as follows:

$$\forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda H_d(W, \partial) := \{\alpha \in H_d(C, \partial) \mid \exists w \in \alpha : w \in \mathcal{F}^\lambda W^d\}.$$

**Definition 3.1.9** (Completions). *Let  $W$  be a graded vector space filtered by a decreasing filtration  $\mathcal{F}^\lambda W$ . The filtration degree of  $w \in W$  is defined by*

$$\|w\| := \sup\{\lambda \in \mathbb{R} \mid w \in \mathcal{F}^\lambda W\}.$$

The filtration degree of a linear map  $f : W_1 \rightarrow W_2$  is defined by

$$\|f\| := \sup\{\lambda \in \mathbb{R} \mid \|f(w)\| \geq \|w\| + \lambda \ \forall w \in W_1\}.$$

We say that the filtration degree is finite if  $\|f\| > -\infty$ . Note that  $\|0\| = \infty$ .

### 3. Algebraic structures

The completion of  $W$  is the graded vector space

$$\hat{W} := \bigoplus_{d \in \mathbb{Z}} \hat{W}^d,$$

where for all  $d \in \mathbb{Z}$  we define

$$\hat{W}^d := \left\{ \sum_{i=0}^{\infty} w_i \mid \forall i \in \mathbb{N}_0 : w_i \in W^d, \|w_i\| \rightarrow \infty \text{ as } i \rightarrow \infty \right\} / \sim.$$

Here  $\sum_{i=0}^{\infty} w_i \sim \sum_{i=0}^{\infty} w'_i$  if and only if  $\|\sum_{i=0}^n (w_i - w'_i)\| \rightarrow \infty$  as  $n \rightarrow \infty$ .<sup>3</sup> The completion  $\hat{W}$  is canonically filtered by the filtration  $\mathcal{F}^\lambda \hat{W}$  defined as follows:

$$\forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda \hat{W}^d := \left\{ \sum_{i=0}^{\infty} w_i \in \hat{W}^d \mid \forall i \in \mathbb{N}_0 : w_i \in \mathcal{F}^\lambda W^d \right\}.$$

We denote the completion of  $W_1 \otimes \cdots \otimes W_k$  by  $W_1 \hat{\otimes} \cdots \hat{\otimes} W_k$  and the completion of  $S_k V$  by  $\hat{S}_k V$ .

A map  $f : W_1 \rightarrow W_2$  of finite filtration degree extends continuously to a linear map  $f : \hat{W}_1 \rightarrow \hat{W}_2$ ; this continuous extension is defined by

$$f\left(\sum_{i=0}^{\infty} w_i\right) := \sum_{i=0}^{\infty} f(w_i) \quad \text{for all } \sum_{i=0}^{\infty} w_i \in \hat{W}.$$

*Remark 3.1.10* (Completed tensor product). Using Proposition 3.1.11 below, one can show that the *completed tensor product*  $\hat{\otimes}$  is associative and that  $W_1 \hat{\otimes} W_2 \simeq \hat{W}_1 \hat{\otimes} \hat{W}_2$ . By refining this argument, one can show that  $\hat{S}_k V \simeq \hat{S}_k \hat{V}$  for any  $k \in \mathbb{N}$ .  $\triangleleft$

A weight-graded vector space  $W$  is canonically *filtered by weights*:

$$\forall \lambda \in \mathbb{R}, d \in \mathbb{Z} : \quad \mathcal{F}^\lambda W^d := \bigoplus_{k \leq \lambda} W_k^d. \quad (3.10)$$

This filtration is  $\mathbb{Z}$ -gapped, exhaustive, Hausdorff, increasing and bounded from below. The induced filtration on the graded dual  $W^{*\mathfrak{g}}$  is  $\mathbb{Z}$ -gapped, Hausdorff, decreasing and bounded from above (and thus automatically exhaustive). It holds  $\text{gr}(W) \simeq W$ , and it

---

<sup>3</sup>In fact,  $\hat{W}$  is the inverse limit  $\varprojlim_{\lambda}^{\text{gr}} (W / \mathcal{F}^\lambda W)$  in the category of graded vector spaces and  $\hat{W}^d$  the inverse limit  $\varprojlim_{\lambda} (W^d / \mathcal{F}^\lambda W^d)$  in the category of vector spaces. As a side-remark, if we forget the grading on  $W$ , we might also consider  $\varprojlim_{\lambda} (W / \mathcal{F}^\lambda W)$ , which would be a vector space containing  $\hat{W}$  as a subspace.

### 3. Algebraic structures

is easy to see from (3.2) that the canonical map  $W^{*\text{wg}} \rightarrow W^{*\text{g}}$  induces the isomorphism

$$\widehat{W^{*\text{wg}}} \simeq W^{*\text{g}}.$$

We also see that the condition

$$(WG0) : \quad \forall d \in \mathbb{Z} \exists J \subset \mathbb{N}_0, |J| < \infty \forall k \in \mathbb{N}_0 \setminus J : \quad W_k^d = 0$$

is equivalent to  $W^{*\text{wg}} = W^{*\text{g}}$ .

A useful tool to compare completions is the following proposition:

**Proposition 3.1.11** ([Fre17, Proposition 7.3.7], Isomorphism criterion). *Let  $W_1$  and  $W_2$  be graded vector spaces filtered by  $\mathbb{Z}$ -gapped filtrations which are decreasing and bounded from above. Suppose that  $f : W_1 \rightarrow W_2$  is a filtration preserving homogenous linear map. Then the continuous extension  $f : \hat{W}_1 \rightarrow \hat{W}_2$  is an isomorphism if and only if the induced map  $f : \text{gr}(W_1) \rightarrow \text{gr}(W_2)$  is an isomorphism.*

*Proof.* The implication from the right to the left is obtained from the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{gr}(W_1)_\lambda & \hookrightarrow & W_1/\mathcal{F}^{\lambda+1}W_1 & \twoheadrightarrow & W_1/\mathcal{F}^\lambda W_1 \longrightarrow 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ 0 & \longrightarrow & \text{gr}(W_2)_\lambda & \hookrightarrow & W_2/\mathcal{F}^{\lambda+1}W_2 & \twoheadrightarrow & W_2/\mathcal{F}^\lambda W_2 \longrightarrow 0 \end{array}$$

by induction using the definition of  $\hat{W}$  as the inverse limit of  $W/\mathcal{F}^\lambda W$  (see Footnote 3 on page 33). The other implication follows from  $\mathcal{F}^\lambda \hat{W}/\mathcal{F}^{\lambda-1} \hat{W} \simeq \mathcal{F}^\lambda W/\mathcal{F}^{\lambda-1} W$ .  $\square$

For a graded vector space  $W$  filtered by a  $\mathbb{Z}$ -gapped filtration, consider the following conditions:

$$\begin{aligned} (WG1) : \quad & \forall \lambda \in \mathbb{Z} \exists I \subset \mathbb{Z}, |I| < \infty \forall d \in \mathbb{Z} \setminus I : \quad \text{gr}(W)_\lambda^d = 0, \\ (WG2) : \quad & \forall d, \lambda \in \mathbb{Z} : \quad \dim(\text{gr}(W)_\lambda^d) < \infty. \end{aligned} \tag{3.11}$$

**Lemma 3.1.12** (Completion of symmetric powers of the graded dual). *Let  $W$  be a graded vector space filtered by an exhaustive  $\mathbb{Z}$ -gapped filtration  $\mathcal{F}^\lambda W$  which is increasing and bounded from below. If (WG1) & (WG2) are satisfied, then the natural map  $S_k(W^{*\text{g}}) \rightarrow (S_k W)^{*\text{g}}$  induces the isomorphism*

$$\hat{S}_k(W^{*\text{g}}) \simeq (S_k W)^{*\text{g}} \quad \text{for every } k \in \mathbb{N}.$$

*Note that we filter graded duals by the induced filtration from Definition 3.1.8.*

### 3. Algebraic structures

*Proof.* The natural map  $S_k(W^{*\mathfrak{g}}) \rightarrow (S_k W)^{*\mathfrak{g}}$  is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space  $(S_k W)^{*\mathfrak{g}}$  is already complete (the dual space  $W^{*\mathfrak{g}}$  is complete, provided that the filtration of  $W$  is exhaustive), and thus we obtain the map  $\hat{S}_k(W^{*\mathfrak{g}}) \rightarrow (S_k W)^{*\mathfrak{g}}$ . According to Proposition 3.1.11, this map is an isomorphism if and only if the induced map  $\text{gr}(S_k(W^{*\mathfrak{g}})) \rightarrow \text{gr}((S_k W)^{*\mathfrak{g}})$  is. This is shown by the following computation (the maps involved are natural in at least one direction):

$$\begin{aligned}
\frac{\mathcal{F}^\lambda(W^{\otimes k * \mathfrak{g}})^d}{\mathcal{F}^{\lambda+1}(W^{\otimes k * \mathfrak{g}})^d} &\simeq \frac{\mathcal{F}^\lambda(W^{\otimes k})^{d*}}{\mathcal{F}^{\lambda+1}(W^{\otimes k})^{d*}} \simeq \left( \frac{\mathcal{F}^{\lambda+1}(W^{\otimes k})^d}{\mathcal{F}^\lambda(W^{\otimes k})^d} \right)^* \\
&\simeq \left( \frac{\bigoplus_{|\vec{d}|=d} \sum_{|\vec{\lambda}|=\lambda+1} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}}{\bigoplus_{|\vec{d}|=d} \sum_{|\vec{\lambda}|=\lambda} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \left( \bigoplus_{|\vec{d}|=d} \frac{\sum_{|\vec{\lambda}|=\lambda+1} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}}{\sum_{|\vec{\lambda}|=\lambda} \mathcal{F}^{\lambda_1} W^{d_1} \otimes \dots \otimes \mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \left( \bigoplus_{|\vec{d}|=d} \bigoplus_{|\vec{\lambda}|=\lambda} \frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \left( \bigoplus_{|\vec{\lambda}|=\lambda} \bigoplus_{|\vec{d}|=d} \frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\stackrel{\mathbb{Z}\text{-gapped}}{\& \text{ bounded below}} \rightarrow \simeq \bigoplus_{|\vec{\lambda}|=\lambda} \bigoplus_{|\vec{d}|=d} \left( \frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\stackrel{(WG1)}{\& (WG1)} \rightarrow \simeq \bigoplus_{|\vec{\lambda}|=\lambda} \bigoplus_{|\vec{d}|=d} \left( \frac{\mathcal{F}^{\lambda_1+1} W^{d_1}}{\mathcal{F}^{\lambda_1} W^{d_1}} \right)^* \otimes \dots \otimes \left( \frac{\mathcal{F}^{\lambda_k+1} W^{d_k}}{\mathcal{F}^{\lambda_k} W^{d_k}} \right)^* \\
&\simeq \bigoplus_{|\vec{d}|=d} \bigoplus_{|\vec{\lambda}|=\lambda} \frac{\mathcal{F}^{\lambda_1}(W^{*\mathfrak{g}})^{d_1}}{\mathcal{F}^{\lambda_1+1}(W^{*\mathfrak{g}})^{d_1}} \otimes \dots \otimes \frac{\mathcal{F}^{\lambda_k}(W^{*\mathfrak{g}})^{d_k}}{\mathcal{F}^{\lambda_k+1}(W^{*\mathfrak{g}})^{d_k}} \\
&\simeq \frac{\mathcal{F}^\lambda(W^{*\mathfrak{g} \otimes k})^d}{\mathcal{F}^{\lambda+1}(W^{*\mathfrak{g} \otimes k})^d}.
\end{aligned}$$

In fact, this computation shows that  $\hat{T}_k(W^{*\mathfrak{g}}) \simeq (T_k W)^{*\mathfrak{g}}$ . The conclusion for  $S_k$  follows by checking that the maps above are  $S_k$ -equivariant.  $\square$

Given a chain complex  $(W, \partial)$ , the boundary operator  $\partial$  induces the boundary operator  $\partial_k : W^{\otimes k} \rightarrow W^{\otimes k}$  for all  $k \in \mathbb{N}$ ; for all  $w_1, \dots, w_k \in W$ , it is defined by

$$\partial_k(w_1 \otimes \dots \otimes w_k) := \sum_{i=1}^k (-1)^{|w_1| + \dots + |w_{i-1}|} w_1 \otimes \dots \otimes \partial w_i \otimes \dots \otimes w_k. \quad (3.12)$$

### 3. Algebraic structures

The map  $\partial_k$  is clearly  $\mathbb{S}_k$ -equivariant, and thus induces the boundary operator  $\partial_k : S_k W \rightarrow S_k W$ .

**Proposition 3.1.13** (Künneth formula for completed symmetric cohomology). *Let  $(W, \partial)$  be a  $\mathbb{Z}$ -graded chain complex over  $\mathbb{R}$  filtered by an exhaustive  $\mathbb{Z}$ -gapped filtration  $\mathcal{F}^\lambda W$  which is increasing and bounded from below. Consider the dual cochain complex  $(W^{*\mathbb{g}}, d := \partial^*)$ . Suppose that  $d$  has finite filtration degree, so that  $d_k : S_k(W^{*\mathbb{g}}) \rightarrow S_k(W^{*\mathbb{g}})$  extends continuously to  $d_k : \hat{S}_k(W^{*\mathbb{g}}) \rightarrow \hat{S}_k(W^{*\mathbb{g}})$  for every  $k \in \mathbb{N}$ . If (WG1) & (WG2) are satisfied, then the natural map  $S_k H(W^{*\mathbb{g}}, d) \rightarrow H(\hat{S}_k(W^{*\mathbb{g}}), d_k)$  induces the isomorphism*

$$\hat{S}_k H(W^{*\mathbb{g}}, d) \simeq H(\hat{S}_k(W^{*\mathbb{g}}), d_k) \quad \text{for all } k \in \mathbb{N}.$$

*Proof.* The natural map  $S_k H(W^{*\mathbb{g}}, d) \rightarrow H(\hat{S}_k W^{*\mathbb{g}}, d_k)$  is clearly filtration preserving, and hence it extends continuously to a map of completions. The target space  $H(\hat{S}_k W^{*\mathbb{g}}, d_k)$  is already complete (the homology of a complete space is complete), and hence we obtain the map  $\hat{S}_k H(W^{*\mathbb{g}}, d) \rightarrow H(\hat{S}_k W^{*\mathbb{g}}, d_k)$ . The following facts are easy to verify:

- (1) The isomorphism from Lemma 3.1.12 is an isomorphism of cochain complexes

$$(\hat{S}_k W^{*\mathbb{g}}, d_k) \simeq ((S_k W)^{*\mathbb{g}}, \partial_k^*).$$

- (2) If the filtration on  $W$  satisfies (WG1) and (WG2), then the filtration on  $H(W)$  also satisfies (WG1) and (WG2), respectively. Consequently, Lemma 3.1.12 holds for symmetric powers of  $H(W, \partial)^{*\mathbb{g}}$  as well.
- (3) The Künneth formula  $H(W^{\otimes k}) \simeq H(W)^{\otimes k}$  implies  $H(S_k W) \simeq S_k H(W)$  for any  $\mathbb{Z}$ -graded chain complex  $W$  over  $\mathbb{R}$ .
- (4) We have  $(H(W))^{*\mathbb{g}} \simeq H(W^{*\mathbb{g}})$  over  $\mathbb{R}$  by the universal coefficient theorem.

Now, we compute

$$\begin{aligned} H(\hat{S}_k W^{*\mathbb{g}}, d_k) &\simeq H((S_k W)^{*\mathbb{g}}, \partial_k^*) \simeq H(S_k W, \partial_k)^{*\mathbb{g}} \simeq (S_k H(W, \partial))^{*\mathbb{g}} \\ &\quad \begin{array}{c} \uparrow \\ (1) \end{array} \quad \begin{array}{c} \uparrow \\ (4) \end{array} \quad \begin{array}{c} \uparrow \\ (3) \end{array} \\ &\simeq \hat{S}_k (H(W, \partial)^{*\mathbb{g}}) \simeq \hat{S}_k H(W^{*\mathbb{g}}, d). \\ &\quad \begin{array}{c} \uparrow \\ (2) \end{array} \quad \begin{array}{c} \uparrow \\ (4) \end{array} \end{aligned}$$

This proves the proposition. □

### 3.2. Basics of IBL-infinity-algebras

**Definition 3.2.1** (Exterior algebra). *Given a graded vector space  $C$  over  $\mathbb{R}$ , we define the exterior algebra over  $C$  by*

$$EC := S(C[1]).$$

*The weight- $k$  component is denoted by  $E_k C$  and the weight-reduced part by  $\bar{E}C$ . If  $C$  is in addition filtered, then  $E_k C$  is filtered by the induced filtration and its completion is denoted by  $\hat{E}_k C$ .*

We have the concatenation product  $\mu : EC \otimes EC \rightarrow EC$  and the shuffle coproduct  $\delta : EC \rightarrow EC \otimes EC$  defined by

$$\begin{aligned} \mu(c_{11} \dots c_{1k} \otimes c_{21} \dots c_{2k'}) &:= c_{11} \dots c_{1k} c_{21} \dots c_{2k'} \quad \text{and} \\ \delta(c_1 \dots c_k) &:= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = k}} \sum_{\sigma \in \mathbb{S}_{k_1, k_2}} \varepsilon(\sigma, c) c_{\sigma_1^{-1}}^{-1} \dots c_{\sigma_{k_1}^{-1}}^{-1} \otimes c_{\sigma_{k_1+1}^{-1}}^{-1} \dots c_{\sigma_{k_1+k_2}^{-1}}^{-1} \end{aligned}$$

for all homogenous  $c_{ij}$ ,  $c_i \in C[1]$  and  $k, k' \geq 0$ , respectively, where  $\mathbb{S}_{k_1, k_2} \subset \mathbb{S}_{k_1+k_2}$  denotes the set of shuffle permutations with blocks of lengths  $k_1$  and  $k_2$ . These operations satisfy the relations of an *associative bialgebra* (see [LV12]):

$$\text{Assoc. bialg.} \quad \begin{cases} \mu \circ (\mathbb{1} \otimes \mu) = \mu \circ (\mu \otimes \mathbb{1}), \\ (\mathbb{1} \otimes \delta) \circ \delta = (\delta \otimes \mathbb{1}) \circ \delta, \\ \delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{1} \otimes \tau \otimes \mathbb{1}) \circ (\delta \otimes \delta). \end{cases} \quad (3.13)$$

Here  $\tau : C_1 \otimes C_2 \rightarrow C_2 \otimes C_1$ ,  $c_1 \otimes c_2 \mapsto (-1)^{|c_1||c_2|} c_2 \otimes c_1$  denotes the *twist map*.

We will use the bialgebra calculus ( $\text{:=}$  relations (3.13)) to write down explicit formulas for the operations  $\circ_{h_1, \dots, h_r}$  which were briefly introduced in [CFL15]; these operations take symmetric maps  $f_1, \dots, f_r$  and connect  $h_1, \dots, h_r$  of their outputs to the inputs of a symmetric map  $f$  in all possible ways, so that the result, which we denote by  $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r)$ , becomes a symmetric map again.

**Definition 3.2.2** (Partial compositions). *Let  $C$  be a graded vector space. For  $i, j \geq 0$ , we denote by*

$$\begin{aligned} \pi_i : EC &\longrightarrow E_i C, \quad \iota_i : E_i C \longrightarrow EC, \\ \mathbb{1}_i : E_i C &\longrightarrow E_i C, \quad \tau_{i,j} : E_i C \otimes E_j C \longrightarrow E_j C \otimes E_i C \end{aligned}$$

DONE Bad picture: There might be wrong dimensions on one side of the maurer cartan element. Compare to the section about filtrations. REMARK THAT THE DEFINITION NEEDS THE WEIGHT STRUCTURE OF THE BIALGEBRA. No Problem.

DONE New algebraic setting in Appendix: There is a new algebraic description of the iterated compatibility condition in the appendix.

### 3. Algebraic structures

the components of the canonical projection  $\pi$ , the canonical inclusion  $\iota$ , the identity  $\mathbb{1}$  and the twist map  $\tau$ , respectively. We also set

$$\delta_{i,j} := (\pi_i \otimes \pi_j) \circ \delta \circ \iota_{i+j} \quad \text{and} \quad \mu_{i,j} := \pi_{i+j} \circ \mu \circ (\iota_i \otimes \iota_j).$$

For  $k', k_1, l', l_1 \geq 0$ , let  $f : E_{k'}C \rightarrow E_{l'}C$  and  $f_1 : E_{k_1}C \rightarrow E_{l_1}C$  be linear maps, and let  $0 \leq h \leq \min(k', l_1)$ . We set

$$k := k' + k_1 - h \quad \text{and} \quad l := l' + l_1 - h$$

and define the composition of  $f$  and  $f_1$  at  $h$  common outputs to be the linear map  $f \circ_h f_1 : E_kC \rightarrow E_lC$  given by

$$\begin{aligned} f \circ_h f_1 &:= \mu_{l', l_1 - h} \circ (f \otimes \mathbb{1}_{l_1 - h}) \circ (\mu_{h, k' - h} \otimes \mathbb{1}_{l_1 - h}) \circ (\mathbb{1}_h \otimes \tau_{l_1 - h, k' - h}) \\ &\quad \circ (\delta_{h, l_1 - h} \otimes \mathbb{1}_{k' - h}) \circ (f_1 \otimes \mathbb{1}_{k' - h}) \circ \delta_{k_1, k' - h}. \end{aligned} \quad (3.14)$$

More generally, we define the composition of  $f : E_{k'} \rightarrow E_{l'}$  with  $r \geq 1$  linear maps  $f_i : E_{k_i} \rightarrow E_{l_i}$  with  $k_i, l_i \geq 0$  for  $i = 1, \dots, r$  at  $0 \leq h_i \leq l_i$  common outputs such that  $h := h_1 + \dots + h_r \leq k'$  as follows. We set

$$k := k' + k_1 + \dots + k_r - h \quad \text{and} \quad l := l' + l_1 + \dots + l_r - h$$

and define  $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) : E_kC \rightarrow E_lC$  by

$$\begin{aligned} &f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) \\ &:= \mu \circ (f \otimes \mathbb{1}) \circ (\mu \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \tau) \\ &\quad \circ ([(\mu^{(r)} \otimes \mu^{(r)}) \circ (F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r}) \circ \sigma_r \circ \delta^{\otimes r}] \otimes \mathbb{1}) \\ &\quad \circ (f_1 \otimes \dots \otimes f_r \otimes \mathbb{1}) \circ \delta^{(r+1)}, \end{aligned} \quad (3.15)$$

where we have:

- The operation  $\mu^{(r)}$  is the “product with  $r$  inputs” and the operation  $\delta^{(r)}$  the “co-product with  $r$  outputs”; they are defined by

$$\begin{aligned} \mu^{(r)} &:= \mu(\mathbb{1} \otimes \mu) \dots (\mathbb{1}^{\otimes r-2} \otimes \mu), \quad \mu^{(1)} := \mathbb{1}, \\ \delta^{(r)} &:= (\mathbb{1}^{\otimes r-2} \otimes \delta) \dots (\mathbb{1} \otimes \delta) \delta, \quad \delta^{(1)} := \mathbb{1}. \end{aligned}$$

- $F_{h_1, \dots, h_r} := (\iota_{h_1} \pi_{h_1}) \otimes \dots \otimes (\iota_{h_r} \pi_{h_r})$ .

DONE Add the other composition: Add the definition of  $(f_1, \dots, f_r) \circ_{h_1, \dots, h_r} f$ . IT IS DONE IN THE APPENDIX.

### 3. Algebraic structures

- The permutation  $\sigma_r \in \mathbb{S}_{2r}$  is given by

$$\sigma_r : (1, 2, 3, 4, \dots, 2r-1, 2r) \mapsto (1, r+1, 2, r+2, \dots, r, 2r).$$

DONE Better definition!!!:  
This needs a better definition of  $\sigma_r$ !!!

- The symbols  $f$  and  $f_i$  inside the formula denote the trivial extensions of  $f$  and  $f_i$ , respectively; we extend a linear map  $f : E_{k'}C \rightarrow E_{l'}C$  trivially to  $f : EC \rightarrow EC$  by defining  $f(E_iC) = 0$  for  $i \neq k'$ .

*Remark 3.2.3* (On partial compositions). (i) Defining  $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) : E_kC \rightarrow E_lC$  using (3.15) makes sense because the right hand side is a trivial extension of its component  $E_kC \rightarrow E_lC$ . In fact, all  $\mu, \delta, \pi, \iota$  in (3.15) can be replaced with  $\mu_{i,j}, \delta_{i,j}, \pi_i, \iota_i$  for unique  $i, j$ , so that trivial extensions do not have to be used at all. In this way, it can be seen that (3.14) is indeed a special case of (3.15).

(ii) If  $h = k' = l_1$ , then  $f \circ_h f_1 = f \circ f_1$ .

(iii) It holds  $f \circ_0 f_1 = (-1)^{|f||f_1|} f_1 \circ_0 f$  and

$$f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) = \varepsilon(\sigma, f) f \circ_{h_{\sigma_1^{-1}}, \dots, h_{\sigma_r^{-1}}} (f_{\sigma_1^{-1}}, \dots, f_{\sigma_r^{-1}}).$$

(iv) Consider the (“non-trivial”) extension  $\hat{f} := \mu(f \otimes \mathbb{1}) \delta : EC \rightarrow EC$  and the symmetric product  $f_1 \odot \dots \odot f_r := \mu^{(r)}(f_1 \otimes \dots \otimes f_r) \delta^{(r)} : EC \rightarrow EC$ . The following formulas from [CFL15] hold:

$$\begin{aligned} f \circ_{h_1, \dots, h_{r-1}, 0} (f_1, \dots, f_r) &= f \circ_{h_1, \dots, h_{r-1}} (f_1, \dots, f_{r-1}) \odot f_r, \\ \hat{f} \circ \hat{f}_1 &= \sum_{h=0}^{\min(k', l_1)} \widehat{f \circ_h f_1}, \\ \hat{f} \circ (f_1 \odot \dots \odot f_r) &= \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k'}} f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r). \end{aligned} \tag{3.16}$$

We also have the “weak associativity”

$$\sum_{\substack{0 \leq h_2 \leq \min(f_3^-, f_2^+) \\ 0 \leq h_1 \leq \min(f_1^+, f_2^- + f_3^- - h_2) \\ h_1 + h_2 = h}} f_1 \circ_{h_1} (f_2 \circ_{h_2} f_3) = \sum_{\substack{0 \leq h_1 \leq \min(f_1^+, f_2^-) \\ 0 \leq h_2 \leq \min(f_1^+ + f_2^+ - h_1, f_3^-) \\ h_1 + h_2 = h}} (f_1 \circ_{h_1} f_2) \circ_{h_2} f_3 \tag{3.17}$$

for every  $0 \leq h \leq \min(k_1 + k_2 + k_3, l_1 + l_2 + l_3)$ , where  $f^+$  denotes the number of inputs and  $f^-$  the number of outputs of  $f$ . The weak associativity of  $\circ_h$  can be proven using



### 3. Algebraic structures

the associativity of  $\hat{\cdot}$  and the second relation of (3.16).

(v) We refer to Section D.4 of Appendix D for a thorough treatment of partial compositions. We show there that  $\circ_{h_1, \dots, h_r}$  can be defined on maps on any connected weight-graded bialgebra using natural bilinear operations  $\square_A$  on polynomials in the convolution product. In Proposition D.4.4 in the appendix, we prove the relations above.  $\triangleleft$

If  $C$  is filtered by a decreasing filtration, then the bialgebra operations extend continuously to

$$\begin{aligned} \mu : \hat{E}_{k_1} C \hat{\otimes} \hat{E}_{k_2} C &\longrightarrow \hat{E}_{k_1+k_2} C \quad \text{and} \\ \delta : \hat{E}_k C &\longrightarrow \bigoplus_{\substack{l_1, l_2 \geq 0 \\ l_1 + l_2 = k}} \hat{E}_{l_1} C \hat{\otimes} \hat{E}_{l_2} C \end{aligned}$$

for all  $k_1, k_2, k \in \mathbb{N}_0$  because they preserve the filtration degree (see [Fre17] for a similar construction). Next, if  $f_1 : \hat{E}_{k_1} C \rightarrow \hat{E}_{l_1} C$  and  $f_2 : \hat{E}_{k_2} C \rightarrow \hat{E}_{l_2} C$  have finite filtration degrees, then  $f_1 \otimes f_2 : \hat{E}_{k_1} C \otimes \hat{E}_{k_2} C \rightarrow \hat{E}_{l_1} C \otimes \hat{E}_{l_2} C$  has finite filtration degree too, and hence it extends continuously to  $f_1 \otimes f_2 : \hat{E}_{k_1} C \hat{\otimes} \hat{E}_{k_2} C \rightarrow \hat{E}_{l_1} C \hat{\otimes} \hat{E}_{l_2} C$ . Using these facts, we can canonically extend Definition 3.2.2 to maps  $f : \hat{E}_{k'} C \rightarrow \hat{E}_{l'} C$  and  $f_i : \hat{E}_{k_i} C \rightarrow \hat{E}_{l_i} C$  of finite filtration degrees. The resulting map  $f \circ_{h_1, \dots, h_r} (f_1, \dots, f_r) : \hat{E}_k C \rightarrow \hat{E}_l C$  will have finite filtration degree too. Moreover, the formulas in Remark 3.2.3 will still hold.

We will now rephrase the definitions of an  $\text{IBL}_\infty$ -algebra, a Maurer-Cartan element and twisted operations from [CFL15] in terms of  $\circ_{h_1, \dots, h_r}$ .

**Definition 3.2.4** ( $\text{IBL}_\infty$ -algebra). *Let  $C$  be a graded vector space equipped with a decreasing filtration, and let  $d \in \mathbb{Z}$  and  $\gamma \geq 0$  be fixed constants. An  $\text{IBL}_\infty$ -algebra of bidegree  $(d, \gamma)$  on  $C$  is a collection of linear maps  $\mathfrak{q}_{klg} : \hat{E}_k C \rightarrow \hat{E}_l C$  for all  $k, l \geq 1$ ,  $g \geq 0$  which are homogenous, of finite filtration degree and satisfy the following conditions:*

- 1)  $|\mathfrak{q}_{klg}| = -2d(k + g - 1) - 1$ .
- 2)  $\|\mathfrak{q}_{klg}\| \geq \gamma \chi_{klg}$ , where  $\chi_{klg} := 2 - 2g - k - l$ .
- 3) The  $\text{IBL}_\infty$ -relations hold: for all  $k, l \geq 1$ ,  $g \geq 0$ , we have

$$\sum_{h=1}^{g+1} \sum_{\substack{k_1, k_2, l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ k_1 + k_2 = k + h \\ l_1 + l_2 = l + h \\ g_1 + g_2 = g + 1 - h}} \mathfrak{q}_{k_2 l_2 g_2} \circ_h \mathfrak{q}_{k_1 l_1 g_1} = 0. \quad (3.18)$$

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We denote a given  $\text{IBL}_\infty$ -algebra structure on  $C$  by  $\text{IBL}_\infty(C)$ ; i.e., we write  $\text{IBL}_\infty(C) = (C, (\mathbf{q}_{klg}))$ .

If  $\mathbf{q}_{klg} \equiv 0$  for all  $(k, l, g) \neq (1, 1, 0), (2, 1, 0), (1, 2, 0)$ , then we call  $\text{IBL}_\infty(C)$  a dIBL-algebra and denote it by  $\text{dIBL}(C)$ . If in addition  $\mathbf{q}_{110} \equiv 0$ , then we have an IBL-algebra  $\text{IBL}(C)$ .

If the operations on the completed exterior powers  $\hat{E}_k C$  arise as continuous extensions of operations  $\mathbf{q}_{klg} : E_k C \rightarrow E_l C$ , then we call the  $\text{IBL}_\infty$ -algebra completion-free and denote  $C$  together with the operations  $\mathbf{q}_{klg} : E_k C \rightarrow E_l C$  by  $\text{IBL}_\infty^0(C)$ .

The acronym IBL stands for an *involutive Lie bialgebra*. It follows namely from the  $\text{IBL}_\infty$ -relations (3.18) that for  $\text{IBL}(C) = (C, \mathbf{q}_{210}, \mathbf{q}_{120})$  the following holds:

$$\text{Lie bialg.} \quad \begin{cases} 0 = \mathbf{q}_{210} \circ_1 \mathbf{q}_{210} & \leftarrow \text{Jacobi id.} \\ 0 = \mathbf{q}_{120} \circ_1 \mathbf{q}_{120} & \leftarrow \text{co-Jacobi id.} \\ 0 = \mathbf{q}_{120} \circ_1 \mathbf{q}_{210} + \mathbf{q}_{210} \circ_1 \mathbf{q}_{120} & \leftarrow \text{Drinfeld id.} \\ 0 = \mathbf{q}_{210} \circ_2 \mathbf{q}_{120} & \leftarrow \text{Involutivity} \end{cases}$$

The acronym dIBL stands for a *differential involutive Lie bialgebra* — an involutive Lie bialgebra together with a differential (a boundary operator in our case) such that the bracket and cobracket are chain maps.

**Proposition 3.2.5** (Odd degree shift of an IBL-algebra). *Let  $(C, \mathbf{q}_{210}, \mathbf{q}_{120})$  be an IBL-algebra of degree  $d$  from Definition 3.2.4, and let  $\tilde{\mathbf{q}}_{210} : C^{\otimes 2} \rightarrow C$  and  $\tilde{\mathbf{q}}_{120} : C \rightarrow C^{\otimes 2}$  be the linear maps defined by*

$$\begin{aligned} \theta \tilde{\mathbf{q}}_{210}(x_1 \otimes x_2) &:= \mathbf{q}_{210}(\pi(\theta^2 x_1 \otimes x_2)) \quad \text{and} \\ \theta^2 \tilde{\mathbf{q}}_{120}(x) &:= \iota(\mathbf{q}_{120}(\theta x)) \end{aligned} \tag{3.19}$$

for all  $x_1, x_2, x \in C$ , where  $\iota : S_2(C[1]) \rightarrow C[1]^{\otimes 2}$  is the section of  $\pi : C[1]^{\otimes 2} \rightarrow S_2(C[1])$  from Definition 3.1.7 and  $\theta$  is a formal symbol of degree  $|\theta| = -1$ . Then the degrees satisfy

$$|\tilde{\mathbf{q}}_{210}| = |\mathbf{q}_{210}| - 1 = -2d - 2 \quad \text{and} \quad |\tilde{\mathbf{q}}_{120}| = |\mathbf{q}_{120}| + 1 = 0,$$

the operations  $\tilde{\mathbf{q}}_{210}$  and  $\tilde{\mathbf{q}}_{120}$  are graded antisymmetric, i.e., we have

$$\tilde{\mathbf{q}}_{210} \circ \tau = -\tilde{\mathbf{q}}_{210} \quad \text{and} \quad \tau \circ \tilde{\mathbf{q}}_{120} = -\tilde{\mathbf{q}}_{120}$$

### 3. Algebraic structures

for the twist map  $\tau$ , and the relations

$$\begin{aligned} 0 &= \tilde{\mathbf{q}}_{210} \circ (\tilde{\mathbf{q}}_{210} \otimes \mathbb{1}) \circ (\mathbb{1}^{\otimes 3} + t_3 + t_3^2), \\ 0 &= (\mathbb{1}^{\otimes 3} + t_3 + t_3^2) \circ (\tilde{\mathbf{q}}_{120} \otimes \mathbb{1}) \circ \tilde{\mathbf{q}}_{120}, \\ 0 &= x_1 \cdot \tilde{\mathbf{q}}_{120}(x_2) - (-1)^{x_1 x_2} x_2 \cdot \tilde{\mathbf{q}}_{120}(x_1) - \tilde{\mathbf{q}}_{120}(\tilde{\mathbf{q}}_{210}(x_1, x_2)), \\ 0 &= \tilde{\mathbf{q}}_{210} \circ \tilde{\mathbf{q}}_{120}, \end{aligned}$$

hold for all  $x_1, x_2 \in C$ . Here,  $t_3 \in \mathbb{S}_3$  denotes the cyclic permutation with  $t_3(1) = 2$  acting on  $C^{\otimes 3}$ , and we define

$$x \cdot (y_1 \otimes y_2) := \tilde{\mathbf{q}}_{210}(x, y_1) \otimes y_2 + (-1)^{xy_1} y_1 \otimes \tilde{\mathbf{q}}_{210}(x, y_2)$$

for all  $x, y_1, y_2 \in C$ .

*Proof.* The proof is a lengthy but straightforward computation.  $\square$

Consider the *sign-action* of  $\mathbb{S}_k$  on  $C^{\otimes k}$  given by  $\sigma \mapsto \bar{\sigma}$ , where

$$\bar{\sigma}(c_1 \otimes \cdots \otimes c_k) := (-1)^\sigma \varepsilon(\sigma, c) c_{\sigma_1^{-1}} \otimes \cdots \otimes c_{\sigma_k^{-1}}$$

for all  $c_1, \dots, c_k \in C$  and  $\sigma \in \mathbb{S}_k$ . We define

$$\Lambda C := \bigoplus_{k=0}^{\infty} \Lambda_k C \quad \text{with} \quad \Lambda_k C := C^{\otimes k} / \sim,$$

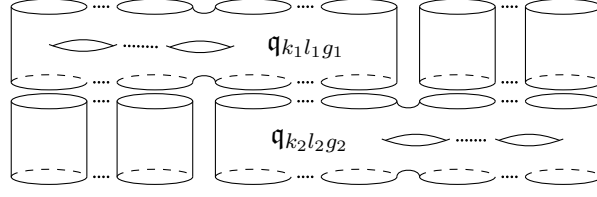
where  $c \sim \bar{\sigma}(c)$  for all  $c \in C^{\otimes k}$  and  $\sigma \in \mathbb{S}_k$ . It is easy to see that the degree-shift map  $\theta^{\otimes k} : c_1 \otimes \cdots \otimes c_k \in C^{\otimes k} \mapsto \varepsilon(c, \theta)(\theta c_1) \otimes \cdots \otimes (\theta c_k) \in C[1]^{\otimes k}$  is equivariant with respect to the sign-action of  $\mathbb{S}_k$  on  $C^{\otimes k}$  and the standard action of  $\mathbb{S}_k$  on  $C[1]^{\otimes k}$  for all  $k$ , and thus it induces an isomorphism of vector spaces  $\Lambda C$  and  $EC$ . We use the following notation:

$$\begin{array}{ccc} \mathbf{q}_{klg} : \hat{\mathbf{E}}_k C & \longrightarrow & \hat{\mathbf{E}}_l C \\ \downarrow \theta^{\otimes k} & & \downarrow \theta^{\otimes l} \\ \tilde{\mathbf{q}}_{klg} : \hat{\Lambda}_k C & \longrightarrow & \hat{\Lambda}_l C. \end{array}$$

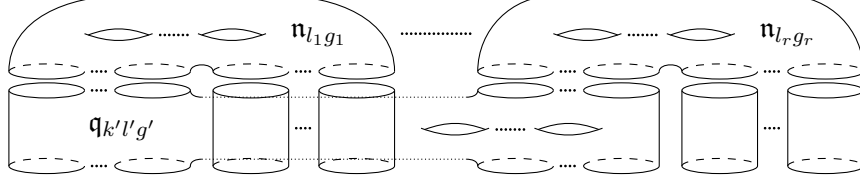
In fact,  $\mathbf{q}_{klg}$  and  $\tilde{\mathbf{q}}_{klg}$  are related precisely by the degree shift (3.6).

**Definition 3.2.6** (Maurer-Cartan element). *A Maurer-Cartan element for an  $\text{IBL}_\infty$ -algebra  $\text{IBL}_\infty(C)$  from Definition 3.2.4 is a collection  $\mathbf{n} := (\mathbf{n}_{lg})_{l \geq 1, g \geq 0}$  of elements  $\mathbf{n}_{lg} \in \hat{\mathbf{E}}_l C$  which are homogenous, of finite filtration degree and satisfy the following conditions:*

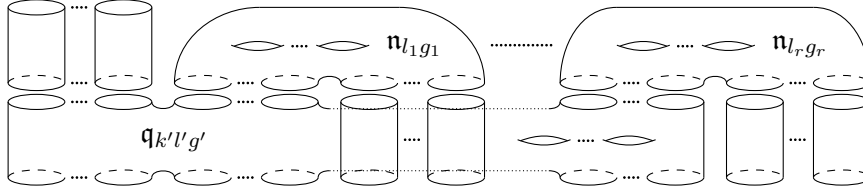
### 3. Algebraic structures



(a) The term  $q_{k_2 l_2 g_2} \circ_h q_{k_1 l_1 g_1}$  in the  $IBL_\infty$ -equation (3.18).



(b) The term  $q_{k' l' g'} \circ_{h_1, \dots, h_r} (n_{l_1 g_1}, \dots, n_{l_r g_r})$  in the Maurer-Cartan equation (3.20). We remark that the contour of the surface corresponding to  $q_{k' l' g'}$  starts on the left and continues to the right along the dotted line behind the two trivial cylinders.



(c) The term  $q_{k' l' g'} \circ_{l_1, \dots, l_r} (n_{l_1 g_1}, \dots, n_{l_r g_r})$  in the twisted operation (3.21). The remark to Figure (b) applies too.

Figure 3.1.: Graphical representation of compositions appearing in Definitions 3.2.4, 3.2.6 and 3.2.7 as gluing of connected Riemannian surfaces. The figure is to be read from the top to the bottom, the empty cylinder represents the identity, and the resulting surface must be connected. We emphasize that the gluing is not associative (c.f., weak associativity (3.17)).

$$1) \quad |n_{lg}| = -2d(g-1).$$

$$2) \quad \|n_{lg}\| \geq \gamma \chi_{0lg} \text{ with } > \text{ for } (l, g) = (1, 0), (2, 0) \text{ (see Definition 3.2.4 for } \chi_{klg}).$$

3) The Maurer-Cartan equation holds: for all  $l \geq 1, g \geq 0$ , we have

$$\sum_{r \geq 1} \frac{1}{r!} \sum_{\substack{l', k', l_1, \dots, l_r \geq 1 \\ g', g_1, \dots, g_r \geq 0 \\ h_1, \dots, h_r \geq 1 \\ l_1 + \dots + l_r + l' - k' = l \\ g_1 + \dots + g_r + g' + k' = g + r \\ h_1 + \dots + h_r - k' = 0}} q_{k' l' g'} \circ_{h_1, \dots, h_r} (n_{l_1 g_1}, \dots, n_{l_r g_r}) = 0, \quad (3.20)$$

where we view  $n_{lg}$  as a linear map  $n_{lg} : \hat{E}_0 C = \mathbb{R} \rightarrow \hat{E}_l C$  with  $n_{lg}(1) = n_{lg}$ .

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**Definition 3.2.7** (Twisted operations). *In the setting of Definition 3.2.6, the twisted operations  $\mathfrak{q}_{klg}^{\mathfrak{n}} : \hat{E}_k C \rightarrow \hat{E}_l C$  for  $k, l \geq 1, g \geq 0$  are defined by*

$$\mathfrak{q}_{klg}^{\mathfrak{n}} = \sum_{r \geq 0} \frac{1}{r!} \sum_{\substack{k', l', l_1, \dots, l_r \geq 1 \\ g', g_1, \dots, g_r \geq 0 \\ h_1, \dots, h_r \geq 1 \\ l_1 + \dots + l_r + l' - k' = l - k \\ g_1 + \dots + g_r + g' + k' = g + r + k \\ h_1 + \dots + h_r - k' = -k}} \mathfrak{q}_{k'l'g'} \circ_{h_1, \dots, h_r} (\mathfrak{n}_{l_1 g_1}, \dots, \mathfrak{n}_{l_r g_r}). \quad (3.21)$$

In [CFL15, Proposition 9.3], they prove that  $(\mathfrak{q}_{klg}^{\mathfrak{n}})_{k, l \geq 1, g \geq 0}$  is again an  $\text{IBL}_{\infty}$ -algebra of bidegree  $(d, \gamma)$  on  $C$  — the twisted  $\text{IBL}_{\infty}$ -algebra. We denote it by  $\text{IBL}_{\infty}^{\mathfrak{n}}(C)$ .

Let  $(\mathfrak{q}_{klg})$  be an  $\text{IBL}_{\infty}$ -algebra on  $C$ . The boundary operator  $\mathfrak{q}_{110} : C[1] \rightarrow C[1]$  induces the boundary operator  $\partial_k : E_k C \rightarrow E_k C$  for every  $k \in \mathbb{N}$  (see (3.12)). Because of the finite filtration degree,  $\partial_k$  continuously extends to  $\partial_k : \hat{E}_k C \rightarrow \hat{E}_k C$ . The following is easy to see using (3.14):

$$\begin{aligned} \mathfrak{q}_{klg} \circ_1 \mathfrak{q}_{110} &= \mathfrak{q}_{klg} \circ \partial_k, \\ \mathfrak{q}_{110} \circ_1 \mathfrak{q}_{klg} &= \partial_l \circ \mathfrak{q}_{klg}. \end{aligned}$$

Because  $\mathfrak{q}_{klg}$  are odd ( $:=$  have odd degree), we have

$$\begin{aligned} [\partial, \mathfrak{q}_{klg}] &:= \partial_l \circ \mathfrak{q}_{klg} - (-1)^{|\partial||\mathfrak{q}_{klg}|} \mathfrak{q}_{klg} \circ \partial_k \\ &= \partial_l \circ \mathfrak{q}_{klg} + \mathfrak{q}_{klg} \circ \partial_k \\ &= \mathfrak{q}_{110} \circ_1 \mathfrak{q}_{klg} + \mathfrak{q}_{klg} \circ_1 \mathfrak{q}_{110}. \end{aligned}$$

With this notation, the  $\text{IBL}_{\infty}$ -relations (3.18) for  $\mathfrak{q}_{210} : \hat{E}_2 C \rightarrow \hat{E}_1 C$  and  $\mathfrak{q}_{120} : \hat{E}_1 C \rightarrow \hat{E}_2 C$  become  $[\partial, \mathfrak{q}_{210}] = 0$  and  $[\partial, \mathfrak{q}_{120}] = 0$ , respectively. If moreover the canonical maps  $E_k H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \rightarrow H(\hat{E}_k C, \partial_k)$  induce the isomorphisms  $\hat{E}_k H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \simeq H(\hat{E}_k C, \partial_k)$  for  $k = 1, 2$ , e.g., when Proposition 3.1.13 holds, then we obtain the maps

$$\mathfrak{q}_{210} : \hat{E}_2 H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \rightarrow \hat{E}_1 H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \quad \text{and} \quad \mathfrak{q}_{120} : \hat{E}_1 H(\hat{C}, \tilde{\mathfrak{q}}_{110}) \rightarrow \hat{E}_2 H(\hat{C}, \tilde{\mathfrak{q}}_{110}),$$

and  $(H(\hat{C}, \tilde{\mathfrak{q}}_{110}), \mathfrak{q}_{210}, \mathfrak{q}_{120})$  becomes an  $\text{IBL}$ -algebra according to Definition 3.2.4 — the induced  $\text{IBL}$ -algebra on homology.

**Definition 3.2.8** (Homology). *We define the homology of an  $\text{IBL}_{\infty}$ -algebra  $\text{IBL}_{\infty}(C)$  by*

$$\mathbb{H}(C)[1] := H(\hat{C}[1], \mathfrak{q}_{110}).$$

*It is a graded vector space with the induced filtration. If  $\mathfrak{n}$  is a Maurer-Cartan element*

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for  $\mathrm{IBL}_\infty(C)$ , we denote by  $\mathbb{H}^n(C)$  the homology of  $\mathrm{IBL}_\infty^n(C)$ .

*Remark 3.2.9* (BV-formalism). Consider the weight-reduced exterior algebra  $\bar{\mathrm{EC}}$ . Let  $\hat{\mathrm{EC}}[[\hbar]]$ , resp.  $\hat{\mathrm{EC}}((\hbar))$  be the spaces of formal power, resp. Laurent series in the variable  $\hbar$  of degree  $|\hbar| = 2d$  with coefficients in  $\bar{\mathrm{EC}}$  completed with respect to a suitable completion. Operations of an  $\mathrm{IBL}_\infty$ -algebra on  $C$  can be encoded in a degree  $-1$  operator  $\Delta : \hat{\mathrm{EC}}[[\hbar]] \rightarrow \hat{\mathrm{EC}}[[\hbar]]$  called the  $\mathrm{BV}_\infty$ -operator, while the data of a Maurer-Cartan element  $(\mathbf{n}_l)_g$  give rise to an operator  $e^\mathbf{n} : \hat{\mathrm{EC}}[[\hbar]] \rightarrow \hat{\mathrm{EC}}((\hbar))$  — the multiplication with the exponential of  $\mathbf{n}$  with respect to  $\odot$ . These operators are given by

$$\Delta := \sum_{i \geq 0} \Delta_{i+1} \hbar^i \quad \text{and} \quad e^\mathbf{n} := \sum_{j \in \mathbb{Z}} (e^\mathbf{n})_j \hbar^j,$$

where the maps  $\Delta_i, (e^\mathbf{n})_j : \hat{\mathrm{EC}} \rightarrow \hat{\mathrm{EC}}$  for  $i \geq 1, j \in \mathbb{Z}$  are defined by

$$\begin{aligned} \Delta_i &:= \sum_{\substack{k \geq 1, g \geq 0 \\ k+g=i}} \sum_{l \geq 1} \hat{\mathbf{q}}_{klg} \quad \text{and} \\ (e^\mathbf{n})_j &:= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{g_1, \dots, g_r \geq 0 \\ g_1 + \dots + g_r = j}} \sum_{\substack{l_1, \dots, l_r \geq 1}} \mathbf{n}_{l_1 g_1} \odot \dots \odot \mathbf{n}_{l_r g_r}. \end{aligned}$$

It can be shown that the  $\mathrm{IBL}_\infty$ -relations (3.18) and the Maurer-Cartan equation (3.20) are equivalent to

$$\Delta \circ \Delta = 0 \quad \text{and} \quad \Delta \circ e^\mathbf{n} = 0, \tag{3.22}$$

respectively, and that the  $\mathrm{BV}_\infty$ -operator  $\Delta^\mathbf{n}$  for the twisted  $\mathrm{IBL}_\infty$ -structure  $(\mathbf{q}_{klg}^\mathbf{n})$  satisfies

$$\Delta^\mathbf{n} = e^{-\mathbf{n}} \Delta(e^\mathbf{n} \cdot) \tag{3.23}$$

These facts were shown in [CFL15] using (3.16).<sup>4</sup>

There are some things which are unclear in the above. We do it properly in the Appendix about filtrations.  $\triangleleft$

*Remark 3.2.10* (Weak  $\mathrm{IBL}_\infty$ -algebras). A possible generalization of the  $\mathrm{IBL}_\infty$ -theory above is to allow  $k = 0$  and  $l = 0$ , so that  $\mathrm{EC}$  must be used instead of  $\bar{\mathrm{EC}}$  in Remark 3.2.9. Such structures would be called *weak  $\mathrm{IBL}_\infty$ -algebras*.  $\triangleleft$

In our application in string topology, a canonical  $\mathrm{dIBL}$ -algebra  $\mathrm{dIBL}(C)$  with a natural Maurer-Cartan element  $\mathbf{n}$  are given, and we want to study  $\mathrm{dIBL}^\mathbf{n}(C)$ ; in particular, we

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<sup>4</sup>One has to check that the compositions (3.22) and (3.23) are well-defined and pick a suitable completion  $\hat{\mathrm{EC}}$  so that all the constructions work.

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are interested in  $\mathbb{H}^n(C)$ ,  $\text{IBL}(\mathbb{H}^n(C))$  and possible higher operations on  $\mathbb{H}^n(C)$  induced by  $\mathfrak{q}_{klg}^n$  (these are not chain maps in general). The following proposition summarizes some observations in this situation:

**Proposition 3.2.11** (Twist of a dIBL-algebra). *Let  $\text{dIBL}(C) = (C, \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120})$  be a dIBL-algebra, and let  $\mathfrak{n} = (\mathfrak{n}_l)$  be a Maurer-Cartan element. The Maurer-Cartan equation (3.20) reduces to the following:*

$$0 = \mathbf{q}_{110} \circ_1 \mathbf{n}_{lg} + \mathbf{q}_{120} \circ_1 \mathbf{n}_{l-1,g} + \mathbf{q}_{210} \circ_2 \mathbf{n}_{l+1,g-1} \\ + \frac{1}{2} \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l+1 \\ g_1 + g_2 = g}} \mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{l_1 g_1}, \mathbf{n}_{l_2 g_2}) \quad \forall l \geq 1, g \geq 0.$$

In particular, the “lowest” equation is given by<sup>5</sup>

$$(l, g) = (1, 0) : \quad \mathbf{q}_{110}(\mathbf{n}_{10}) + \frac{1}{2}\mathbf{q}_{210}(\mathbf{n}_{10}, \mathbf{n}_{10}) = 0. \quad (3.24)$$

*This can be visualized as*

$$0 = \begin{array}{c} \text{n}_{10} \\ \text{q}_{110} \end{array} + \frac{1}{2} \begin{array}{cc} \text{n}_{10} & \text{n}_{10} \\ & \text{q}_{210} \end{array}.$$

The twisted  $\text{IBL}_\infty$ -algebra  $\text{dIBL}^n(C)$  consists of the operations  $\mathfrak{q}_{110}^n, \mathfrak{q}_{210}^n$  and  $\mathfrak{q}_{120}^n$ , which we call the basic operations, and of the operations  $\mathfrak{q}_{1lg}^n$  for the pairs  $(l, g) \in \mathbb{N} \times \mathbb{N}_0 \setminus \{(1, 0), (2, 0)\}$ , which we call the higher operations. These operations are given by

$$\begin{aligned}\mathbf{q}_{110}^{\mathbf{n}} &= \mathbf{q}_{110} + \mathbf{q}_{210} \circ_1 \mathbf{n}_{10}, \\ \mathbf{q}_{210}^{\mathbf{n}} &= \mathbf{q}_{210}, \\ \mathbf{q}_{120}^{\mathbf{n}} &= \mathbf{q}_{120} + \mathbf{q}_{210} \circ_1 \mathbf{n}_{20}, \\ \mathbf{q}_{1lg}^{\mathbf{n}} &= \mathbf{q}_{210} \circ_1 \mathbf{n}_{lg}.\end{aligned}$$

<sup>5</sup>In [CFL15, Definition 2.4.], they define a partial ordering on the signatures  $(k, l, g)$ .

### 3. Algebraic structures

This can be visualized as

$$\begin{aligned}
 \mathfrak{q}_{110}^{\mathfrak{n}} &= \text{diagram of two cylinders stacked vertically, labeled } \mathfrak{q}_{110} \text{ on the bottom one} + \text{diagram of two cylinders with a cap labeled } \mathfrak{n}_{10} \text{ on top and a base labeled } \mathfrak{q}_{210} \text{ on the bottom}, \\
 \mathfrak{q}_{210}^{\mathfrak{n}} &= \text{diagram of two cylinders with a cap labeled } \mathfrak{n}_{20} \text{ on top and a base labeled } \mathfrak{q}_{210} \text{ on the bottom}, \\
 \mathfrak{q}_{120}^{\mathfrak{n}} &= \text{diagram of two cylinders with a cap labeled } \mathfrak{n}_{120} \text{ on top and a base labeled } \mathfrak{q}_{120} \text{ on the bottom} + \text{diagram of two cylinders with a cap labeled } \mathfrak{n}_{20} \text{ on top and a base labeled } \mathfrak{q}_{210} \text{ on the bottom}, \\
 \mathfrak{q}_{1lg}^{\mathfrak{n}} &= \text{diagram of a long horizontal cylinder with multiple caps labeled } \mathfrak{n}_{lg} \text{ and a base labeled } \mathfrak{q}_{210} \text{ on the right side}.
 \end{aligned}$$

The  $\text{IBL}_{\infty}$ -relations satisfied by  $(\mathfrak{q}_{klg}^{\mathfrak{n}})$  read for all  $l \geq 1, g \geq 0$  as follows:

$$\begin{aligned}
 (3, 1, 0) : \quad 0 &= \mathfrak{q}_{210}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{210}^{\mathfrak{n}}, \\
 (2, l, g) : \quad 0 &= \mathfrak{q}_{1lg}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{210}^{\mathfrak{n}} + \mathfrak{q}_{210}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{1lg}^{\mathfrak{n}}, \\
 (1, l, g) : \quad 0 &= \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l + 1 \\ g_1 + g_2 = g}} \mathfrak{q}_{1l_1g_1}^{\mathfrak{n}} \circ_1 \mathfrak{q}_{1l_2g_2}^{\mathfrak{n}} + \mathfrak{q}_{210}^{\mathfrak{n}} \circ_2 \mathfrak{q}_{1, l+1, g-1}^{\mathfrak{n}}.
 \end{aligned} \tag{3.25}$$

We call the relations for  $(k, l, g) = (1, 1, 0), (2, 1, 0), (1, 2, 0), (3, 1, 0), (1, 3, 0), (2, 2, 0), (1, 1, 1)$  basic relations because they contain all compositions of basic operations. In the



### 3. Algebraic structures

order above, they read:

$$\begin{aligned}
0 &= \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{110}^n, \\
0 &= \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{210}^n + \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{110}^n, \\
0 &= \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{120}^n + \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{110}^n, \\
0 &= \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{210}^n, & \leftarrow \text{Jacobi identity} \\
0 &= \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{120}^n + \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{130}^n + \mathfrak{q}_{130}^n \circ_1 \mathfrak{q}_{110}^n, & \leftarrow \text{co-Jacobi id. up to htpy.} \\
0 &= \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{210}^n + \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{120}^n, & \leftarrow \text{Drinfeld identity} \\
0 &= \mathfrak{q}_{210}^n \circ_2 \mathfrak{q}_{120}^n + \mathfrak{q}_{111}^n \circ_1 \mathfrak{q}_{110}^n + \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{111}^n. & \leftarrow \text{Involutivity up to htpy.}
\end{aligned}$$

The last four equations can be visualized as

The first equation shows a diagram with two cylinders, one labeled  $\mathfrak{q}_{210}^n$  and one labeled  $\mathfrak{q}_{210}^n$ , connected by a horizontal line, representing the identity  $0 = \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{210}^n$ .

The second equation shows three diagrams: the first has two cylinders labeled  $\mathfrak{q}_{120}^n$  and  $\mathfrak{q}_{120}^n$ ; the second has two cylinders labeled  $\mathfrak{q}_{130}^n$  and  $\mathfrak{q}_{130}^n$ ; the third has three cylinders labeled  $\mathfrak{q}_{110}^n$ ,  $\mathfrak{q}_{110}^n$ , and  $\mathfrak{q}_{110}^n$ . These are summed to represent the identity  $0 = \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{120}^n + \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{130}^n + \mathfrak{q}_{130}^n \circ_1 \mathfrak{q}_{110}^n$ .

The third equation shows two diagrams: the first has two cylinders labeled  $\mathfrak{q}_{120}^n$  and  $\mathfrak{q}_{120}^n$ ; the second has two cylinders labeled  $\mathfrak{q}_{120}^n$  and  $\mathfrak{q}_{210}^n$ . These are summed to represent the identity  $0 = \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{120}^n + \mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{210}^n + \mathfrak{q}_{210}^n \circ_1 \mathfrak{q}_{120}^n$ .

The fourth equation shows three diagrams: the first has two cylinders labeled  $\mathfrak{q}_{120}^n$  and  $\mathfrak{q}_{210}^n$ ; the second has two cylinders labeled  $\mathfrak{q}_{110}^n$  and  $\mathfrak{q}_{111}^n$ ; the third has two cylinders labeled  $\mathfrak{q}_{111}^n$  and  $\mathfrak{q}_{110}^n$ . These are summed to represent the identity  $0 = \mathfrak{q}_{210}^n \circ_2 \mathfrak{q}_{120}^n + \mathfrak{q}_{111}^n \circ_1 \mathfrak{q}_{110}^n + \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{111}^n$ .

*Proof.* The proof is clear by specializing (3.18), (3.20) and (3.21).  $\square$

*Remark 3.2.12* (Higher operations). (i) We see from Proposition 3.2.11 that if  $\mathfrak{q}_{120}^n \circ_1 \mathfrak{q}_{120}^n = 0$  and  $\mathfrak{q}_{210}^n \circ_2 \mathfrak{q}_{120}^n = 0$ , then  $[\partial^n, \mathfrak{q}_{130}^n] = 0$  and  $[\partial^n, \mathfrak{q}_{111}^n] = 0$ , respectively, and hence the operations  $\mathfrak{q}_{130}^n : \hat{E}_1 \mathbb{H}^n \rightarrow \hat{E}_3 \mathbb{H}^n$  and  $\mathfrak{q}_{111}^n : \hat{E}_1 \mathbb{H}^n \rightarrow \hat{E}_1 \mathbb{H}^n$  are well-defined (provided

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that the assumption of Definition 3.2.8 holds). Likewise, the higher operation  $\mathfrak{q}_{1lg}^n$  defines a map  $\hat{E}_1\mathbb{H}^n \rightarrow \hat{E}_l\mathbb{H}^n$ , provided that the following equation holds:

$$\mathfrak{q}_{210}^n \circ_2 \mathfrak{q}_{1,l+1,g-1}^n + \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l+1 \\ g_1 + g_2 = g \\ (l_i, g_i) \neq (1, 0)}} \mathfrak{q}_{1l_1g_1}^n \circ_1 \mathfrak{q}_{1l_2g_2}^n = 0.$$

This expression is just the left-over after subtracting the commutator  $[\mathfrak{q}_{1lg}^n, \mathfrak{q}_{110}^n] = \mathfrak{q}_{110}^n \circ_1 \mathfrak{q}_{1lg}^n + \mathfrak{q}_{1lg}^n \circ_1 \mathfrak{q}_{110}^n$  from (3.25).

(ii) In the genus-0 case, i.e.,  $\mathfrak{q}_{1lg}^n = 0$  whenever  $g \geq 1$ , relations (3.25) reduce to

$$\begin{aligned} 0 &= \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{210}, \\ 0 &= \mathfrak{q}_{1lg}^n \circ_1 \mathfrak{q}_{210} + \mathfrak{q}_{210} \circ_1 \mathfrak{q}_{1lg}^n, \\ 0 &= \sum_{\substack{l_1, l_2 \geq 1 \\ g_1, g_2 \geq 0 \\ l_1 + l_2 = l+1 \\ g_1 + g_2 = g}} \mathfrak{q}_{1l_1g_1}^n \circ_1 \mathfrak{q}_{1l_2g_2}^n. \end{aligned}$$

The first relation is the Jacobi identity for  $\mathfrak{q}_{210}$ , the second relation is a generalization of the Drinfeld identity to higher coproducts  $\mathfrak{q}_{1l0}^n$ , and the third relations are  $\text{coL}_\infty$ -relations for  $\mathfrak{q}_{1lg}^n$ . Therefore, allowing  $g \geq 0$ , we see that the twisted dIBL-algebra  $\mathfrak{q}_{110}^n, \mathfrak{q}_{210}, (\mathfrak{q}_{1lg}^n)$  is, in fact, a *quantum coL<sub>∞</sub>-algebra*.

◁

### 3.3. Dual cyclic bar complex and cyclic cohomology

DONE Reduced is defined wrt. normalized!: Reduced is defined as coker or ker in normalized and not in the full complex! It is OK because we have the cyclic symmetry!

DONE Convention for duals: The convention for cochain complexes will be as follows. Chain complex will be denoted by  $C$  and cochain complex by  $C^*$ . The components will be denoted by  $C_q$  and  $C^q$ .

DONE Notation for cyclic Hochschild: Notation for cyclic Hochschild

**Definition 3.3.1** (Bar complexes). *Let  $V$  be a graded vector space. The bar- and dual bar-complex of  $V$  are the weight-graded vector spaces defined by*

$$BV := \bar{T}(V[1]) \quad \text{and} \quad B^*V := (BV)^{*_{\text{wg}}},$$

*respectively, where  $\bar{T}V := \bigoplus_{k=1}^{\infty} V^{\otimes k}$  is the weight-reduced tensor algebra. For every  $k \in \mathbb{N}$ , let  $t_k \in \mathbb{S}_k$  be the cyclic permutation  $t_k : (1, \dots, k) \mapsto (2, \dots, k, 1)$ , so that for all*

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$v_1, \dots, v_k \in V[1]$  we have

$$t_k(v_1 \otimes \dots \otimes v_k) = (-1)^{|v_k|(|v_1|+\dots+|v_{k-1}|)} v_k \otimes v_1 \otimes \dots \otimes v_{k-1}.$$

We set

$$t := \sum_{k=1}^{\infty} t_k : BV \longrightarrow BV.$$

The cyclic bar-complex is defined by

$$B^{\text{cyc}}V := BV / \text{im}(1 - t).$$

We denote the image of  $v_1 \otimes \dots \otimes v_k \in BV$  under the canonical projection  $\pi : BV \rightarrow B^{\text{cyc}}V$  by  $v_1 \dots v_k$ . If  $v_i \in V[1]$  are homogenous, then  $v_1 \dots v_k$  is called a generating word; we have

$$v_1 \dots v_k = (-1)^{|v_k|(|v_1|+\dots+|v_{k-1}|)} v_k v_1 \dots v_{k-1}.$$

We define the section  $\iota : B^{\text{cyc}}V \rightarrow BV$  of  $\pi$  by

$$\iota(v_1 \dots v_k) := \frac{1}{k} \sum_{i=0}^{k-1} \underbrace{t_k^i}_{=: t_k \circ \dots \circ t_k \text{ } i\text{-times}}(v_1 \otimes \dots \otimes v_k)$$

and use it to identify  $B^{\text{cyc}}V$  with the subspace  $\text{im } \iota = \ker(1 - t) \subset BV$  consisting of cyclic symmetric tensors.

We define the dual cyclic bar-complex by

$$B_{\text{cyc}}^*V := \{\psi \in B^*V \mid \psi \circ t = \psi\}.$$

*Remark 3.3.2* (Non-weight-reduced bar complex). In fact, our  $B_{\text{cyc}}^*V$  is weight-reduced. The non-weight-reduced version would be  $B_{\text{cyc}}^*V \oplus \mathbb{R}$  with  $\mathbb{R}$  of degree 0. This might play a role in the theory of weak  $A_\infty$ -algebras ( $\mu_0$  added; c.f., Definition 3.3.7), and it might also be possible to consider  $\text{IBL}_\infty$ -algebras on non-weight-reduced cyclic cochains (c.f., Section 3.4).  $\triangleleft$

Notice that  $\psi \in B^*V$  is homogenous of degree  $|\psi| \in \mathbb{Z}$  if and only if for all homogenous  $v_1, \dots, v_k \in V[1]$  the following implication holds:

$$|v_1| + \dots + |v_k| \neq |\psi| \implies \psi(v_1 \otimes \dots \otimes v_k) = 0.$$

This is the cohomological grading convention.

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**Notation 3.3.3** (Degree shifts of bar complexes). *Let  $A \in \mathbb{Z}$ . In the following, we write  $B_{\text{cyc}}^*V$ , but the convention applies to all complexes from Definition 3.3.1. We denote by  $s_A$  and  $\theta$  the formal symbols of degrees*

$$|s_A| = -A \quad \text{and} \quad |\theta| = -1,$$

*respectively. The degree shift  $V \mapsto V[1]$  will be realized as multiplication with  $\theta$  and the degree shift  $B_{\text{cyc}}^*V \mapsto B_{\text{cyc}}^*V[A]$  as multiplication with  $s_A$ . In addition, the following notation will be used consistently:*

- $\tilde{v} \in V \longleftrightarrow v = \theta\tilde{v} \in V[1]$   
*To clarify this, given  $\tilde{v} \in V$ , then  $v$  automatically means  $v = \theta\tilde{v} \in V[1]$ , and the other way round. Recall that the degree of  $\tilde{v} \in V$  is denoted by  $\deg(\tilde{v})$  or simply by  $\tilde{v}$  in the exponent, e.g.,  $(-1)^{\tilde{v}}$ .*
- $\psi \in B_{\text{cyc}}^*V \longleftrightarrow \Psi = s_A\psi \in B_{\text{cyc}}^*V[A]$ .
- *A generating word of  $B^{\text{cyc}}V$  of weight  $k$  will be denoted by the symbol  $w$  and written as  $w = v_1 \dots v_k$ , where  $v_i = \theta\tilde{v}_i \in V[1]$ . A generating word of  $E_k B^{\text{cyc}}V$  is an element  $w_1 \dots w_k \in E_k B^{\text{cyc}}V$  such that each  $w_i$  is a generating word of  $B^{\text{cyc}}V$ .*
- $w \in B^{\text{cyc}}V \longleftrightarrow W = s_A w \in B^{\text{cyc}}V[A]$ .

We abbreviate

$$B_{\text{cyc}}^*V[A] := (B_{\text{cyc}}^*V)[A].$$

*In contrast to this, we would write  $B_{\text{cyc}}^*(V[A])$  for the dual cyclic bar-complex of  $V[A]$ . We also identify  $(B_{\text{cyc}}^*V[A])[1] = B_{\text{cyc}}^*V[A+1]$  in  $EB_{\text{cyc}}^*V[A]$ .*

**Definition 3.3.4** (Pairing of tensor powers of bar complexes). *For every  $A \in \mathbb{Z}$  and  $k \in \mathbb{N}$ , we define the pairing as follows:*

$$\begin{aligned} (B^*V[A])^{\otimes k} \otimes (BV[A])^{\otimes k} &\longrightarrow \mathbb{R} \\ (\Psi_1 \otimes \dots \otimes \Psi_k, W_1 \otimes \dots \otimes W_k) &\longmapsto \underbrace{\psi_1(w_1) \dots \psi_k(w_k)}_{(\Psi_1 \otimes \dots \otimes \Psi_k)(W_1 \otimes \dots \otimes W_k)} := \end{aligned} \quad (3.26)$$

*This means that we evaluate elements from the left-hand side on the elements from the right-hand side in this way without any signs (see the discussion in Remark 3.1.4). We extend the pairing by 0 if the number of  $\Psi_i$ 's and the number of  $W_i$ 's differ.*

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*Remark 3.3.5* (Dual bar complex and dual of the bar complex). Because the pairing (3.26) is non-degenerate, we can embed the space on the left into the linear dual of the space on the right. From Definition 3.3.1 we have  $B_{\text{cyc}}^*V \subset B^*V$ , and  $B^{\text{cyc}}V$  is identified with  $\text{im } \iota \subset BV$ . Therefore, we can restrict (3.26) to obtain the pairing of  $B_{\text{cyc}}^*V$  and  $B^{\text{cyc}}V$ . It is easy to see that for any  $\psi \in B_{\text{cyc}}^*V$  and any generating word  $v_1 \dots v_k \in B^{\text{cyc}}V$ , we have

$$\psi(v_1 \dots v_k) = \psi(v_1 \otimes \dots \otimes v_k).$$

The subspace of  $(B^{\text{cyc}}V)^{*_{\text{lin}}}$  corresponding to  $B_{\text{cyc}}^*V$  is then precisely  $(B^{\text{cyc}}V)^{*_{\text{wg}}}$ .

More generally, for every  $k \in \mathbb{N}$ , the spaces  $E_k B_{\text{cyc}}^*V$  and  $E_k B^{\text{cyc}}V$  are embedded into  $(B_{\text{cyc}}^*V[1])^{\otimes k}$  and  $(B^{\text{cyc}}V[1])^{\otimes k}$ , respectively, using  $\iota$  and  $\pi$  from Definition 3.1.7. Therefore, the restriction of (3.26) gives the pairing of  $E_k B_{\text{cyc}}^*V$  and  $E_k B^{\text{cyc}}V$ . It is easy to see that for any generating word  $w_1 \dots w_k \in E_k B^{\text{cyc}}V$  and any  $\psi_1 \dots \psi_k \in E_k B_{\text{cyc}}^*V$ , we have

$$(\psi_1 \dots \psi_k)(w_1 \dots w_k) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} \varepsilon(\sigma, w) \psi_1(w_{\sigma_1^{-1}}) \dots \psi_k(w_{\sigma_k^{-1}}).$$

The subspace of  $(E_k B^{\text{cyc}}V)^{*_{\text{lin}}}$  corresponding to  $E_k B_{\text{cyc}}^*V$  lies in  $(E_k B^{\text{cyc}}V)^{*_{\text{wg}}}$ ; it is equal to  $(E_k B^{\text{cyc}}V)^{*_{\text{wg}}}$ , provided that  $V$  is finite-dimensional.<sup>6</sup>  $\triangleleft$

The weight-graded vector spaces  $BV$  and  $B^{\text{cyc}}V$  are canonically filtered by the filtration by weights (3.10). Their weight-graded duals  $B^*V$  and  $B_{\text{cyc}}^*V$  are filtered by the dual filtrations and the exterior powers  $E_k B^*V$  and  $E_k B_{\text{cyc}}^*V$  by the induced filtration from Definition 3.1.8.

**Proposition 3.3.6** (Completed dual cyclic bar complex). *Let  $V$  be a graded vector space and  $A \in \mathbb{Z}$ . The filtration of  $B_{\text{cyc}}^*V$  dual to the weight-filtration of  $B^{\text{cyc}}V$  is  $\mathbb{Z}$ -gapped, Hausdorff, decreasing and bounded from above. Moreover, the following holds:*

$$\dim(V) < \infty \implies (WG1) \text{ \& } (WG2) \text{ are satisfied.}$$

*The same holds for the induced filtration of  $E_k B_{\text{cyc}}^*V[A]$ .*

*In the sense of Remark 3.3.5, we have*

$$\hat{B}_{\text{cyc}}^*V \simeq (B^{\text{cyc}}V)^{*_{\text{g}}} \quad \text{and} \quad \hat{E}_k B_{\text{cyc}}^*V[A] \subset (E_k B^{\text{cyc}}V[A+1])^{*_{\text{g}}},$$

*where “=” holds if  $V$  is finite-dimensional.*

E: Here should be  $A$  instead of  $A+1$

<sup>6</sup>The problem is that if  $\dim(V) = \infty$ , then  $(V \otimes V)^* \neq V^* \otimes V^*$ .

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The filtration degree of  $\Psi \in \hat{E}_m B_{\text{cyc}}^* V[A]$  satisfies

$$\|\Psi\| = \min\{k \in \mathbb{N}_0 \mid \exists W \in (E_m B^{\text{cyc}} V[A])_k : \Psi(W) \neq 0\}.$$

*Proof.* The proof is clear. □

**Definition 3.3.7** (Cyclic  $A_\infty$ -algebra). A graded vector space  $V$  together with a pairing

$$\Pi : V[1] \otimes V[1] \rightarrow \mathbb{R}$$

of degree  $d \in \mathbb{Z}$  and a collection of homogenous linear maps

$$\mu_k : V[1]^{\otimes k} \rightarrow V[1] \quad \text{for } k \geq 1$$

is called a cyclic  $A_\infty$ -algebra of degree  $d$  if the following conditions are satisfied:

(1) The pairing  $\Pi$  is non-degenerate and graded antisymmetric; i.e., we have

$$\Pi(v_1, v_2) = (-1)^{1+|v_1||v_2|} \Pi(v_2, v_1) \quad \text{for all } v_1, v_2 \in V[1].$$

(2) The degrees satisfy  $|\mu_k| = 1$  for all  $k \geq 1$ .

(3) The  $A_\infty$ -relations are satisfied: for all  $k \geq 1$ , we have

$$\sum_{\substack{k_1, k_2 \geq 1 \\ k_1 + k_2 = k+1}} \sum_{p=1}^{k_1} \mu_{k_1} \circ_1^p \mu_{k_2} = 0, \quad (3.27)$$

where for all  $p = 1, \dots, k$  and  $v_1, \dots, v_k \in V[1]$  we define

$$\begin{aligned} (\mu_{k_1} \circ_1^p \mu_{k_2})(v_1, \dots, v_k) &:= (-1)^{|v_1| + \dots + |v_{p-1}|} \mu_{k_1}(v_1, \dots, v_{p-1}, \\ &\quad \mu_{k_2}(v_p, \dots, v_{p+k_2-1}), v_{p+k_2}, \dots, v_k). \end{aligned}$$

(4) The operations  $\mu_k^+ : V[1]^{\otimes k+1} \rightarrow \mathbb{R}$  defined by

$$\mu_k^+ := \Pi \circ (\mu_k \otimes \mathbb{1})$$

for all  $k \geq 1$  are cyclic symmetric; i.e., we have

$$\mu_k^+ \circ t_{k+1} = \mu_k^+.$$

Degree of pairing: This should be standardized with the degree of Poincare duality algebra, canonical dIBL algebra, .... The degree should be probably minus the degree of the pairing on  $V$  (not  $V[1]$ )

### 3. Algebraic structures

We denote by  $\tilde{\Pi} : V \otimes V \rightarrow \mathbb{R}$  and  $\tilde{\mu}_k : V^{\otimes k} \rightarrow \mathbb{R}$  the operations before the degree shift; i.e., for all  $k \geq 1$  and  $\tilde{v}_1, \dots, \tilde{v}_k \in V$  with  $v_i = \theta \tilde{v}_i$ , we have

$$\begin{aligned}\tilde{\Pi}(\tilde{v}_1, \tilde{v}_2) &:= (-1)^{\tilde{v}_1} \Pi(v_1, v_2) \quad \text{and} \\ \tilde{\mu}_k(\tilde{v}_1, \dots, \tilde{v}_k) &:= \varepsilon(\theta, \tilde{v}) \mu_k(v_1, \dots, v_k).\end{aligned}$$

We define  $\tilde{\mu}_k^+ : V^{\otimes k+1} \rightarrow \mathbb{R}$  similarly.

If  $\mu_k \equiv 0$  for all  $k \geq 2$ , then  $(V, \Pi, \mu_1)$  is called a cyclic cochain complex. If  $\mu_k \equiv 0$  for all  $k \geq 3$ , then  $(V, \Pi, \mu_1, \mu_2)$  is called a cyclic dga. We use the same terminology but omit “cyclic” if there is no pairing  $\Pi$  and 1) and 4) are thus irrelevant.

*Remark 3.3.8* (A difference in sign conventions). Our definition of  $\mu_k^+$  differs from the definition of  $m_k^+$  in [CFL15, Definition 12.1] by a sign. To compensate this, we have to add this artificial sign in the definitions of Maurer-Cartan elements later; e.g., in Definition 3.4.2 or in the formula (A.12).  $\triangleleft$

**Definition 3.3.9** ((cyclic) (co)homology of an  $A_\infty$ -algebra). Let  $\mathcal{A} = (V, (\mu_k))$  be an  $A_\infty$ -algebra. For every  $k \geq 1$ , we consider the maps  $b'_k, R_k : V[1]^{\otimes k} \rightarrow BV$  given by

$$\begin{aligned}b'_k &:= \sum_{j=1}^k \sum_{i=0}^{k-j} t_{k-j+1}^i \circ (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^{-i} \quad \text{and} \\ R_k &:= \sum_{j=2}^k \sum_{i=1}^{j-1} (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^i,\end{aligned}\tag{3.28}$$

respectively, and define the following maps  $BV \rightarrow BV$ :

$$b' := \sum_{k=1}^{\infty} b'_k, \quad R := \sum_{k=2}^{\infty} R_k \quad \text{and} \quad b := b' + R.$$

We denote by  $b^* : \hat{B}^*V = (BV)^{*g} \rightarrow \hat{B}^*V$  the dual map to  $b : BV \rightarrow BV$ . The following holds:<sup>7</sup>

$$|b| = 1 \quad (|b^*| = -1), \quad b \circ b = 0 \quad \text{and} \quad b(1-t) = (1-t)b'.\tag{3.29}$$

From the last equation we see that  $b$  restricts to  $B^{\text{cyc}}V = BV/\text{im}(1-t)$ . We define the

<sup>7</sup>The facts (3.29) are generally known in some form (see [Mes16] or [Laz03]).

DONE Change  
sub to sup:  
Change  $b^k$  to  
 $b_k$  and so on.  
Why are the in-  
dices upstairs?

### 3. Algebraic structures

following graded vector spaces:

$$\begin{aligned} D(V) &:= r(\mathbf{B}V)[1], & D^*(V) &:= r(\hat{\mathbf{B}}^*V)[1], \\ D^\lambda(V) &:= r(\mathbf{B}^{\text{cyc}}V)[1], & D_\lambda^*(V) &:= r(\hat{\mathbf{B}}_{\text{cyc}}^*V)[1]. \end{aligned}$$

For instance, we have

$$D_\lambda^q(V) = r(\hat{\mathbf{B}}_{\text{cyc}}^*V)^{q+1} = (\hat{\mathbf{B}}_{\text{cyc}}^*V)^{-q-1} \quad \text{for all } q \in \mathbb{Z}.$$

Then  $(D(V), \mathbf{b})$  and  $(D^\lambda(V), \mathbf{b})$  are chain complexes and  $(D^*(V), \mathbf{b}^*)$  and  $(D_\lambda^*(V), \mathbf{b}^*)$  the dual cochain complexes, respectively. We define the following (co)homologies:

$$\begin{aligned} \text{HH}(\mathcal{A}; \mathbb{R}) &:= \text{H}(D(V), \mathbf{b}), & \text{HH}^*(\mathcal{A}; \mathbb{R}) &:= \text{H}(D^*(V), \mathbf{b}^*), \\ \text{H}^\lambda(\mathcal{A}; \mathbb{R}) &:= \text{H}(D^\lambda(V), \mathbf{b}), & \text{H}_\lambda^*(\mathcal{A}; \mathbb{R}) &:= \text{H}(D_\lambda^*(V), \mathbf{b}^*). \end{aligned}$$

We call  $\text{HH}$  the Hochschild homology and  $\text{H}^\lambda$  the cyclic homology of the  $\mathbf{A}_\infty$ -algebra  $\mathcal{A}$ . We call  $\text{HH}^*$  the Hochschild cohomology and  $\text{H}_\lambda^*$  the cyclic cohomology of  $\mathcal{A}$ .

For a dga  $\mathcal{A} = (V, \mu_1, \mu_2)$ , we have for all  $v_1, \dots, v_k \in V[1]$  the formula

$$\begin{aligned} \mathbf{b}(v_1 \dots v_k) &= \sum_{i=1}^k (-1)^{|v_1| + \dots + |v_{i-1}|} v_1 \dots \mu_1(v_i) \dots v_k \\ &\quad + \sum_{i=1}^{k-1} (-1)^{|v_1| + \dots + |v_{i-1}|} v_1 \dots \mu_2(v_i, v_{i+1}) \dots v_k \\ &\quad + (-1)^{|v_k|(|v_1| + \dots + |v_{k-1}|)} \mu_2(v_k, v_1) v_2 \dots v_{k-1}. \end{aligned}$$

**Definition 3.3.10** (Strict units and strict augmentations). *Let  $\mathcal{A} = (V, (\mu_k))$  be an  $\mathbf{A}_\infty$ -algebra. A non-zero homogenous element  $\mathbf{1} \in V[1]$  with  $|\mathbf{1}| = -1$  is called a strict unit for  $\mathcal{A}$  if the following holds:*

$$\begin{aligned} \mu_2(\mathbf{1}, v) &= (-1)^{|v|+1} \mu_2(v, \mathbf{1}) = v & \forall v \in V[1], \\ \mu_k(v_1, \dots, v_{i-1}, \mathbf{1}, v_{i+1}, \dots, v_k) &= 0 & \forall k \neq 2, 1 \leq i \leq k, v_j \in V[1]. \end{aligned}$$

The pair  $(\mathcal{A}, \mathbf{1})$  is called a strictly unital  $\mathbf{A}_\infty$ -algebra.

A strictly unital  $\mathbf{A}_\infty$ -algebra  $(\mathcal{A}, \mathbf{1})$  is called strictly augmented if it is equipped with a linear map  $\varepsilon : V[1] \rightarrow \mathbb{R}[1]$  which satisfies

$$\varepsilon(\mathbf{1}_V) = \mathbf{1}_{\mathbb{R}}, \quad \varepsilon \circ \mu_1 = 0 \quad \text{and} \quad \varepsilon \circ \mu_2 = \mu_2 \circ (\varepsilon \otimes \varepsilon),$$



### 3. Algebraic structures

where  $1_{\mathbb{R}}$  is the strict unit for  $\mathbb{R}$  endowed with the standard multiplication. The map  $\varepsilon$  is called a strict augmentation..

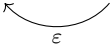
If the homological dga  $H(\mathcal{A}) := (H(V, \tilde{\mu}_1), \mu_1 \equiv 0, \mu_2)$  of  $\mathcal{A}$  is strictly unital and strictly augmented, then  $\mathcal{A}$  is called homologically unital and homologically augmented, respectively. A strictly unital and strictly augmented cochain complex  $(V, \mu_1, 1, \varepsilon)$  is called just augmented.

We denote by  $u : \mathbb{R}[1] \rightarrow V[1]$  the injective linear map defined by  $u(1_{\mathbb{R}}) := 1_V$ , and by  $u^* : B_{\text{cyc}}^* V \rightarrow B_{\text{cyc}}^* \mathbb{R}$  and  $\varepsilon^* : B_{\text{cyc}}^* \mathbb{R} \rightarrow B_{\text{cyc}}^* V$  the precompositions with  $u^{\otimes k}$  and  $\varepsilon^{\otimes k}$  in every weight- $k$  component, respectively.

*Remark 3.3.11* (On units and augmentations). (i) A strict unit  $1_V$  for  $\mathcal{A}$  induces an  $A_{\infty}$ -morphism  $(u_k) : \mathbb{R} \rightarrow V$  given by  $u_1(1_{\mathbb{R}}) := 1_V$  and  $u_k \equiv 0$  for all  $k \geq 2$ . A (general) augmentation of  $(\mathcal{A}, 1_V)$  is by definition any  $A_{\infty}$ -morphism  $(\varepsilon_k) : V \rightarrow \mathbb{R}$  such that  $(\varepsilon_k) \circ (u_k) = \mathbb{1}$  as  $A_{\infty}$ -morphisms (see [Kel01]). Strict augmentations are precisely the maps  $\varepsilon_1$  coming from augmentations  $(\varepsilon_k)$  with  $\varepsilon_k \equiv 0$  for all  $k \geq 2$ .

(ii) As for  $(V, \mu_1, 1, \varepsilon)$ , we need the chain map  $\varepsilon$  to provide the splitting of the short exact sequence of chain complexes

$$0 \longrightarrow \mathbb{R}[1] \xrightarrow{u} V[1] \twoheadrightarrow \text{coker}(u) \longrightarrow 0,$$



so that we get  $H(V) \simeq H_{\text{red}}(V) \oplus \mathbb{R}$ , where  $H_{\text{red}}(V) := H(\text{coker}(u))$ . If  $(V, \mu_1)$  is non-negatively graded and we are given an injective chain map  $u : \mathbb{R}[1] \rightarrow V[1]$  (=the classical augmentation), then one can show that such  $\varepsilon$  always exists.  $\triangleleft$

**Definition 3.3.12** (Reduced dual cyclic bar complex). *Let  $(\mathcal{A}, 1)$  be a strictly unital  $A_{\infty}$ -algebra. Consider the injection  $\iota_1 : BV \rightarrow BV$ ,  $v_1 \otimes \cdots \otimes v_k \mapsto 1 \otimes v_1 \otimes \cdots \otimes v_k$ . We define the reduced dual cyclic bar-complex by*

$$B_{\text{cyc}, \text{red}}^* V := \{\psi \in B_{\text{cyc}}^* V \mid \psi \circ \iota_1 = 0\}.$$

*Under the assumption of strict unitality,  $b^*$  preserves  $B_{\text{cyc}, \text{red}}^* V$ , and hence we can consider the reduced cyclic cochain complex*

$$D_{\lambda, \text{red}}^*(V) := r(\hat{B}_{\text{cyc}, \text{red}}^* V)[1]$$

DONE Missing  
red: Correct  
missing red in  
the definition  
of  $D_r$ .

### 3. Algebraic structures

and define the reduced cyclic cohomology of  $\mathcal{A}$  by

$$H_{\lambda, \text{red}}^*(\mathcal{A}; \mathbb{R}) := H(D_{\lambda, \text{red}}^*(V), b^*).$$

**Proposition 3.3.13** (Reduction to the reduced cyclic cohomology). *Let  $\mathcal{A} = (V, (\mu_k))$  be an  $A_\infty$ -algebra with a strict unit  $1$  and a strict augmentation  $\varepsilon$ . Then the inclusions  $B_{\text{cyc}, \text{red}}^* V, \varepsilon^*(B_{\text{cyc}}^* \mathbb{R}) \subset B_{\text{cyc}}^* V$  induce the decomposition*

$$H_\lambda^*(\mathcal{A}; \mathbb{R}) \simeq H_{\lambda, \text{red}}^*(\mathcal{A}; \mathbb{R}) \oplus H_\lambda^*(\mathbb{R}; \mathbb{R}).$$

Here we have

$$H_\lambda^q(\mathbb{R}; \mathbb{R}) = \begin{cases} \langle 1^{q+1*} \rangle & \text{for } q \geq 0 \text{ even,} \\ 0 & \text{for } q > 0 \text{ odd and } q < 0, \end{cases}$$

where  $1^{i*} : \mathbb{R}[1]^{\otimes i} \rightarrow \mathbb{R}$  is defined by  $1^{i*}(1^i) := 1$ .

*Sketch of the proof.* The maps  $\varepsilon^* : D_\lambda(\mathbb{R}) \rightarrow D_\lambda(V)$  and  $u^* : D_\lambda(V) \rightarrow D_\lambda(\mathbb{R})$  are chain maps with  $u^* \circ \varepsilon^* = \mathbb{1}$ . Therefore, we have the sequence of cochain complexes

$$0 \longrightarrow D_{\lambda, \text{red}}(V) \hookrightarrow D_\lambda(V) \xrightarrow{u^*} D_\lambda(\mathbb{R}) \longrightarrow 0, \quad (3.30)$$

$\xleftarrow{\varepsilon^*}$

which is exact everywhere except for the middle, and where  $\varepsilon^*$  is a splitting map. The idea of [Lod92] is to replace these cochain complexes with quasi-isomorphic bicomplexes consisting of normalized Hochschild cochains  $\bar{D}(V)$  such that the sequence becomes exact. The work then reduces to proving that  $\bar{D}(V)$  computes  $\text{HH}(\mathcal{A}; \mathbb{R})$ ; a variant of this result for  $A_\infty$ -algebras was proven in [Laz03].  $\square$

We will now compare our version of the cyclic cohomology of a dga  $(V, \mu_1, \mu_2)$  to a version based on [Lod92, Section 5.3.2]. Let  $\tilde{b}, \tilde{\delta} : \bar{T}V \rightarrow \bar{T}V$  be the linear maps defined for all  $\tilde{v}_1, \dots, \tilde{v}_k \in V$  by

$$\begin{aligned} \tilde{b}(\tilde{v}_1 \otimes \dots \otimes \tilde{v}_k) &:= \sum_{i=1}^{k-1} (-1)^{i-1} \tilde{v}_1 \otimes \dots \otimes \tilde{\mu}_2(\tilde{v}_i, \tilde{v}_{i+1}) \otimes \dots \otimes \tilde{v}_k \\ &\quad + (-1)^{k-1+\tilde{v}_k(\tilde{v}_1+\dots+\tilde{v}_{k-1})} \tilde{\mu}_2(\tilde{v}_k, \tilde{v}_1) \otimes \tilde{v}_2 \otimes \dots \otimes \tilde{v}_{k-1}, \\ \tilde{\delta}(\tilde{v}_1 \otimes \dots \otimes \tilde{v}_k) &:= \sum_{i=1}^k (-1)^{\tilde{v}_1+\dots+\tilde{v}_{i-1}} \tilde{v}_1 \otimes \dots \otimes \tilde{\mu}_1(\tilde{v}_i) \otimes \dots \otimes \tilde{v}_k. \end{aligned}$$

### 3. Algebraic structures

For all  $q \geq 0$ , we define

$$\tilde{D}_q(V) := \bigoplus_{\substack{k \geq 1 \\ d \in \mathbb{Z} \\ k-d=q+1}} (V^{\otimes k})^d$$

and  $\tilde{\partial} : \tilde{D}_{q+1}(V) \rightarrow \tilde{D}_q(V)$  by

$$\tilde{\partial}(\tilde{v}_1 \cdots \tilde{v}_k) = \tilde{b}(\tilde{v}_1 \cdots \tilde{v}_k) + (-1)^{k+1} \tilde{\delta}(\tilde{v}_1 \cdots \tilde{v}_k).$$

It can be checked that  $\tilde{\partial} \circ \tilde{\partial} = 0$  and  $\tilde{\partial}(\text{im}(1 - \tilde{t})) \subset \text{im}(1 - \tilde{t})$ , so that  $\tilde{\partial}$  induces a boundary operator on the chain complexes

$$\tilde{D}(V) := \bigoplus_{q \in \mathbb{Z}} \tilde{D}_q(V) \quad \text{and} \quad \tilde{D}^\lambda(V) := \tilde{D}(V) / \text{im}(1 - \tilde{t}).$$

DONE  
Wrong cyclic  
permutation:  
Here the  $t$  is  
modified i.e.  
 $\tilde{t}(v_1 \dots v_k) =$   
 $(-1)^{k-1} t(v_1 \dots v_k)$  ■

Here, we have  $\tilde{t}(\tilde{v}_1 \cdots \tilde{v}_k) := (-1)^{k+|\tilde{v}_k|(|\tilde{v}_1|+\dots+|\tilde{v}_{k-1}|)} \tilde{v}_k \tilde{v}_1 \cdots \tilde{v}_{k-1}$ . We call  $(\tilde{D}(V), \tilde{\partial})$  the *classical Hochschild complex* and  $(\tilde{D}^\lambda(V), \tilde{\partial})$  the *classical cyclic complex* of the dga  $(V, \mu_1, \mu_2)$ . The chain complex  $(\tilde{D}(V), \tilde{\partial})$  is the total complex of the bicomplex

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow & \\ \leftarrow & (V^{\otimes 3})^1 & \xleftarrow{\tilde{\delta}} & (V^{\otimes 3})^0 & \xleftarrow{\tilde{\delta}} & (V^{\otimes 3})^{-1} & \leftarrow \\ & \downarrow \tilde{b} & & \downarrow \tilde{b} & & \downarrow \tilde{b} & \\ \leftarrow & (V^{\otimes 2})^1 & \xleftarrow{-\tilde{\delta}} & (V^{\otimes 2})^0 & \xleftarrow{-\tilde{\delta}} & (V^{\otimes 2})^{-1} & \leftarrow \\ & \downarrow \tilde{b} & & \downarrow \tilde{b} & & \downarrow \tilde{b} & \\ \leftarrow & V^1 & \xleftarrow{\tilde{\delta}} & V^0 & \xleftarrow{\tilde{\delta}} & V^{-1} & \leftarrow, \end{array}$$

which differs from the bicomplex [Lod92, Equation (5.3.2.1)] by the reversed grading and by the fact that it lies in the whole upper half-plane and not just in the first quadrant. Their convention for a dga is namely  $|\tilde{\mu}_1| = -1$ , whereas ours is  $|\tilde{\mu}_1| = 1$ , and they consider  $\mathbb{N}_0$ -grading, whereas we have  $\mathbb{Z}$ -grading.

**Proposition 3.3.14** (The classical case). *Let  $\mathcal{A} = (V, \mu_1, \mu_2)$  be a dga. Then the degree shift map*

$$U : \tilde{D}_q(V) \longrightarrow D_q(V),$$

$$\tilde{v}_1 \otimes \cdots \otimes \tilde{v}_k \longmapsto \varepsilon(\theta, \tilde{v}) v_1 \otimes \cdots \otimes v_k,$$

where we denote  $v_i = \theta \tilde{v}_i$ , is an isomorphism of the chain complexes  $(\tilde{D}(V), \tilde{\partial}) \simeq (D(V), b)$  and  $(\tilde{D}^\lambda(V), \tilde{\partial}) \simeq (D^\lambda(V), b)$ , respectively.

### 3. Algebraic structures

*Proof.* First of all, for the degrees holds  $|\tilde{\mu}_j| = 2 - j$  for every  $j \geq 1$ . For every  $j, k, l \geq 1$  such that  $j + l \leq k + 1$  and for every  $\tilde{v}_1, \dots, \tilde{v}_k \in V$ , we compute

$$\begin{aligned} & [U^{-1}(\mathbb{1}^{l-1} \otimes \mu_j \otimes \mathbb{1}^{k-j-l+1})U](\tilde{v}_1 \cdots \tilde{v}_k) \\ &= (-1)^{l-1+(j-2)(\tilde{v}_1+\dots+\tilde{v}_{l-1}+k-l-j+1)} \tilde{v}_1 \cdots \tilde{v}_{l-1} \tilde{\mu}_j(\tilde{v}_l \cdots \tilde{v}_{l+j-1}) \tilde{v}_{l+j} \cdots \tilde{v}_k, \\ & [U^{-1}t_k U](\tilde{v}_1 \cdots \tilde{v}_k) = (-1)^{k-1} \tilde{v}_1 \cdots \tilde{v}_k, \end{aligned}$$

where we use the Koszul convention  $(f_1 \otimes f_2)(v_1 \otimes v_2) = (-1)^{|f_2||v_1|} f_1(v_1) \otimes f_2(v_2)$ . Using this, we obtain

$$\begin{aligned} U^{-1}b'_k U &= \sum_{j=1}^k \sum_{i=0}^{k-1} (-1)^{i+j(i+k+1)} t_{k-j+1}^i (\tilde{\mu}_j \otimes \mathbb{1}^{k-j}) t_k^{-i} \quad \text{and} \\ U^{-1}R_k U &= \sum_{j=1}^k \sum_{i=1}^{j-1} (-1)^{(i+j)(k+1)} (\tilde{\mu}_j \otimes \mathbb{1}^{k-j}) t_k^i. \end{aligned}$$

It is now easy to check that  $U^{-1} \circ b \circ U = \tilde{\partial}$ .

If  $k \in \mathbb{N}$  is a weight and  $d \in \mathbb{Z}$  a degree such that  $k - d - 1 = q$  for some  $q \in \mathbb{Z}$ , we have schematically  $U : (k, d) \mapsto (k, d - k) = (k, -q - 1)$ . Therefore,  $U$  preserves the grading of chain complexes. This finishes the proof.  $\square$

**Proposition 3.3.15** (Reduced cochains are complete in 0, 1-connected case). *Suppose that  $V = \bigoplus_{d \geq 0} V^d$  is a non-negatively graded vector space with  $V^0 = \langle 1 \rangle$  for some  $1 \in V$  ( $=: V$  is connected) and  $V^1 = 0$  ( $=: V$  is simply-connected). Then for all  $m \geq 1$ , we have*

$$\hat{E}_m B_{\text{cyc,red}}^* V = E_m B_{\text{cyc,red}}^* V.$$

*Proof.* Let  $\bar{V} := \bigoplus_{d \geq 2} V^d$ . We clearly have  $B_{\text{cyc,red}}^* V \simeq B_{\text{cyc}}^* \bar{V}$ . Since  $\bar{V}[1]$  is positively graded, we have  $(B\bar{V})_k^d = 0$  whenever  $k > d$ . Therefore, a map  $\Psi \in \hat{E}_m \bar{V}$ , which is non-zero only on finitely many homogenous components of  $B^{\text{cyc}} V[1]^{\otimes m}$ , will be non-zero only on finitely many weights. This implies that  $\Psi \in E_m \bar{V}$ .  $\square$

*Remark 3.3.16* (Universal coefficient theorem). We have

$$BV = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k=1}^{\infty} (V[1]^{\otimes k})^d \quad \text{and} \quad B^* V = \bigoplus_{d \in \mathbb{Z}} \bigoplus_{k=1}^{\infty} (V[1]^{\otimes k})^{d*},$$

and hence

$$(BV)^{*_{\text{g}}} = \bigoplus_{d \in \mathbb{Z}} \prod_{k=1}^{\infty} (V[1]^{\otimes k})^{d*} = \bigoplus_{d \in \mathbb{Z}} \widehat{(B^* V)^d} = \hat{B}^* V.$$

### 3. Algebraic structures

Therefore,  $(D_\lambda^*(V), b^*)$  is dual to  $(D^\lambda(V), b)$  as a chain complex. Now, because we work over  $\mathbb{R}$ , the universal coefficient theorem gives

$$H_\lambda^q(\mathcal{A}, b^*) \simeq [H_q^\lambda(\mathcal{A}, b)]^* \quad \text{for all } q \in \mathbb{Z}.$$

Suppose that we have found closed homogenous elements  $(w_i)_{i \in I} \subset D^\lambda(V)$  for some index set  $I$  which induce a basis of  $H^\lambda(\mathcal{A}; \mathbb{R})$ . For every  $i \in I$ , we define the linear map  $w_i^* : D^\lambda(V) \rightarrow \mathbb{R}$  by prescribing

$$w_i^*(w_j) = \delta_{ij} \quad \text{for all } j \in I$$

and  $w_i^* \equiv 0$  on  $\text{im } b$  and on a complement of  $\ker(b)$  in  $D^\lambda(V)$ . Then  $(w_i^*)_{i \in I} \subset D_\lambda^*(V)$  are closed homogenous elements which generate linearly independent cohomology classes in  $H_\lambda^*(\mathcal{A}; \mathbb{R})$ ; if we denote  $I_q := \{i \in I \mid w_i \in C_q^\lambda(V)\}$ , then we can write

$$H_\lambda^q(\mathcal{A}; \mathbb{R}) = \left\{ \sum_{i \in I_q} \alpha_i w_i^* \mid \alpha_i \in \mathbb{R} \right\} \quad \text{for all } q \in \mathbb{Z}.$$

DONE Universal coefficient theorem: Here is enough an arbitrary complement of  $\ker(b)$ . That means that for every  $i$ , we can have a different complement  $Z_i$

### 3.4. Canonical dIBL-structure on cyclic cochains

In this section, we will consider a finite-dimensional cyclic dga  $(V, \Pi, m_1, m_2)$  of degree  $2 - n$  for some  $n \in \mathbb{N}$ . This means that for all  $v_1, v_2, v_3 \in V[1]$ , the following relations holds:

$$\text{cyc. dga} \left\{ \begin{array}{l} \Pi(v_1, v_2) = (-1)^{1+|v_1||v_2|} \Pi(v_2, v_1), \\ m_1(m_1(v_1)) = 0, \\ m_1^+(v_1, v_2) = (-1)^{|v_1||v_2|} m_1^+(v_2, v_1), \\ m_1(m_2(v_1, v_2)) = -m_2(m_1(v_1), v_2) \\ \quad - (-1)^{|v_1|} m_2(v_1, m_1(v_2)), \\ m_2(m_2(v_1, v_2), v_3) = (-1)^{|v_1|+1} m_2(v_1, m_2(v_2, v_3)), \\ m_2^+(v_1, v_2, v_3) = (-1)^{|v_3|(|v_1|+|v_2|)} m_2^+(v_3, v_1, v_2). \end{array} \right. \quad \begin{array}{l} \text{cyc.} \\ \text{cochain} \\ \text{complex} \end{array} \quad (3.31)$$

The facts (A) and (C) from the Overview apply, and we get the canonical dIBL-algebra  $\text{dIBL}(B_{\text{cyc}}^* V[2 - n])$  of bidegree  $(n - 3, 2)$  and the canonical Maurer-Cartan element  $\mathbf{m} = (\mathbf{m}_{10})$ . We will denote

$$C(V) := B_{\text{cyc}}^* V[2 - n]$$

and call it the space of *cyclic cochains on  $V$* . If  $V$  is fixed, we will write just  $C$ .

Definition of cyclic cochains: Define cyclic cochains as completion, and then distinguish long and short cyclic cochains.

### 3. Algebraic structures

**Definition 3.4.1** (The canonical dIBL-algebra). *Let  $(V, \Pi, m_1)$  be a cyclic cochain complex of degree  $2 - n$  which is finite-dimensional. Let  $(e_0, \dots, e_m) \subset V[1]$  be a basis of  $V[1]$ , and let  $(e^0, \dots, e^m)$  be the dual basis with respect to  $\Pi$ ; this means that*

$$\Pi(e_i, e^j) = \delta_{ij} \quad \text{for all } i, j = 0, \dots, m.$$

We define the tensor  $T = \sum_{i,j=0}^m T^{ij} e_i \otimes e_j \in V[1]^{\otimes 2}$  by<sup>8</sup>

$$T^{ij} = (-1)^{|e_i|} \Pi(e^i, e^j) \quad \text{for all } i, j = 0, \dots, m. \quad (3.32)$$

The canonical dIBL-algebra on  $C(V)$  is the quadruple

$$\text{dIBL}(C(V)) := (C(V), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}),$$

where the operations  $\mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}$  are defined for all  $\psi, \psi_1, \psi_2 \in \hat{B}_{\text{cyc}}^* V$  and generating words  $w = v_1 \dots v_k, w_1 = v_{11} \dots v_{1k_1}, w_2 = v_{21} \dots v_{2k_2} \in B^{\text{cyc}} V$  with  $k, k_1, k_2 \geq 1$  as follows:

- The dIBL-boundary operator  $\mathfrak{q}_{110} : \hat{E}_1 C \rightarrow \hat{E}_1 C$  of degree  $|\mathfrak{q}_{110}| = -1$  is defined by

$$\mathfrak{q}_{110}(s\psi)(sw) := s \sum_{i=1}^k (-1)^{|v_1| + \dots + |v_{i-1}|} \psi(v_1 \dots v_{i-1} m_1(v_i) v_{i+1} \dots v_k). \quad (3.33)$$

Should here be s?: There should not be s, it is a number.

- The product  $\mathfrak{q}_{210} : \hat{E}_2 C \rightarrow \hat{E}_1 C$  of degree  $|\mathfrak{q}_{210}| = -2(n - 3) - 1$  is written schematically as

$$\mathfrak{q}_{210}(s^2 \psi_1 \otimes \psi_2)(sw) := \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{|e_j| |w^1|} T^{ij} \psi_1(e_i w^1) \psi_2(e_j w^2)$$

and defined “algorithmically” as follows:

For every cyclic permutation  $\sigma \in \mathbb{S}_k$ , consider the tensor

$$\sigma(w) := \varepsilon(\sigma, w) v_{\sigma_1^{-1}} \otimes \dots \otimes v_{\sigma_k^{-1}},$$

and split it into two parts  $w^1$  and  $w^2$  of possibly zero length such that  $v_{\sigma_1^{-1}} \otimes \dots \otimes v_{\sigma_k^{-1}} = w^1 \otimes w^2$ . Feed  $w^1$  and  $w^2$  into  $\psi_1$  and  $\psi_2$  preceded by  $e_i$  and  $e_j$ , respectively,

---

<sup>8</sup>See Appendix A for the invariant meaning of  $T$  as the Schwartz kernel of  $\pm \mathbb{1}$ .

### 3. Algebraic structures

and multiply the result with the sign  $(-1)^{|e_j||w^1|}$ , which is the Koszul sign to order

$$e_i e_j w^1 w^2 \mapsto e_i w^1 e_j w^2.$$

Finally, sum over all  $\sigma \in \mathbb{S}_k$ , all splittings of  $\sigma(w)$  and all indices  $i, j = 0, \dots, m$ . The sign  $\varepsilon(\sigma, w)$  is denoted by  $\varepsilon(w \mapsto w^1 w^2)$  to indicate the splitting.

- The coproduct  $\mathfrak{q}_{120} : \hat{\mathbb{E}}_1 C \longrightarrow \hat{\mathbb{E}}_2 C$  of degree  $|\mathfrak{q}_{120}| = -1$  is written schematically as

$$\begin{aligned} & \mathfrak{q}_{120}(\mathfrak{s}\psi)(\mathfrak{s}^2 w_1 \otimes w_2) \\ &= \frac{1}{2} \sum \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|e_j||w_1^1|} T^{ij} \psi(e_i w_1^1 e_j w_2^1) \end{aligned}$$

and defined “algorithmically” as follows:

For all cyclic permutations  $\sigma \in \mathbb{S}_{k_1}$  and  $\mu \in \mathbb{S}_{k_2}$ , denote  $w_1^1 := \sigma(w_1)$  and  $w_2^1 := \mu(w_2)$  and let  $\varepsilon(w_1 \mapsto w_1^1)$  and  $\varepsilon(w_2 \mapsto w_2^1)$  be the corresponding Koszul signs, respectively. Feed  $w_1^1$  and  $w_2^1$  into  $\psi$  in the indicated order interleaved by  $e_i$  and  $e_j$  and multiply the result with the sign  $(-1)^{|e_j||w_1^1|}$ , which is the Koszul sign to order

$$e_i e_j w_1^1 w_2^1 \mapsto e_i w_1^1 e_j w_2^1.$$

Finally, sum over all  $\sigma \in \mathbb{S}_{k_1}$ ,  $\mu \in \mathbb{S}_{k_2}$  and all indices  $i, j = 0, \dots, m$ .

The operations are extended continuously to the completion.

**Definition 3.4.2** (Canonical Maurer-Cartan element). *Let  $(V, \Pi, m_1, m_2)$  be a finite-dimensional cyclic dga of degree  $2 - n$ . The canonical Maurer-Cartan element  $\mathfrak{m}$  for  $\text{dIBL}(C(V))$  consists of only one element  $\mathfrak{m}_{10} \in \hat{\mathbb{E}}_1 C$  of degree  $|\mathfrak{m}_{10}| = 2(n - 3)$  which is defined by*

$$\mathfrak{m}_{10}(\mathfrak{s}v_1 v_2 v_3) := (-1)^{n-2} \mu_2^+(v_1, v_2, v_3) \quad \text{for all } v_1, v_2, v_3 \in V[1]$$

on the weight-three component of  $\text{B}^{\text{cyc}} V[3 - n]$  and extended by 0 to other weight- $k$  components.

**Remark 3.4.3** (On canonical dIBL-structure). (i) Elements of the completion  $\hat{C}(V)$  which are not in  $C(V)$  will be called *long cyclic cochains*. Because there are no infinite sums in Definition 3.4.1,  $\text{dIBL}(C)$  is completion-free. Clearly, the twist  $\text{dIBL}^n(C)$  remains completion-free as long as  $\mathfrak{n}_{lg} \in \mathbb{E}_l C$  for all  $l, g$ .

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(ii) The constructions of  $\mathbf{q}_{210}$  and  $\mathbf{q}_{120}$  do not depend on the choice of a basis and can be rephrased in terms of summation over ribbon graphs (see Example A.1.5).

(iii) According to Proposition 3.3.6, the filtration on  $C(V)$  satisfies (WG1) & (WG2), and hence the IBL-structures  $\text{IBL}(\mathbb{H}(C))$  and  $\text{IBL}(\mathbb{H}^m(C))$  are well-defined (see Definition 3.2.8).  $\triangleleft$

**Proposition 3.4.4** (Formulas for twisted operations). *Let  $\text{dIBL}(C(V))$  be the canonical dIBL-algebra for a finite-dimensional cyclic cochain complex  $(V, \Pi, m_1)$  of degree  $2 - n$ , and let  $\mathbf{n} = (\mathbf{n}_{lg})$  be a Maurer-Cartan element. Then for all  $l \geq 1, g \geq 0, \Psi \in \hat{\mathbf{B}}_{\text{cyc}}^* V[3 - n]$  and generating words  $W_1, \dots, W_l \in \mathbf{B}^{\text{cyc}} V[3 - n]$ , we have*

$$\begin{aligned} & [(\mathbf{q}_{210} \circ_1 \mathbf{n}_{lg})(\Psi)](W_1 \otimes \dots \otimes W_l) \\ &= \sum_{j=1}^l \sum \varepsilon' \varepsilon(w_j \mapsto w_j^1 w_j^2) T^{ab} \Psi(se_a w_j^1) \mathbf{n}_{lg}(W_1 \otimes \dots \otimes W_{j-1} \otimes (se_b w_j^2) \\ & \quad \otimes W_{j+1} \otimes \dots \otimes W_l), \end{aligned} \quad (3.34)$$

where the sum without limits is the sum in Definition 3.4.1 for  $\mathbf{q}_{210}$  and  $\varepsilon'$  is the Koszul sign of the following operation:

$$\begin{aligned} & (se_a e_b) W_1 \dots W_{j-1} (sw_j^1 w_j^2) W_{j+1} \dots W_l \\ & \mapsto (se_a w_j^1) W_1 \dots W_{j-1} (se_b w_j^2) W_{j+1} \dots W_l. \end{aligned}$$

In particular, for  $l = 1, g \geq 0$  and  $W \in \mathbf{B}^{\text{cyc}} V[3 - n]$ , we have

$$(\mathbf{q}_{210} \circ_1 \mathbf{n}_{1g})(W) = (-1)^{n-3} \sum T^{ab} \varepsilon(w \mapsto w^1 w^2) \mathbf{n}_{1g}(se_a w^1) \psi(e_b w^2), \quad (3.35)$$

and for  $l = 2, g \geq 0$  and  $W_1, W_2 \in \mathbf{B}^{\text{cyc}} V[3 - n]$ , we have

$$\begin{aligned} & [(\mathbf{q}_{210} \circ_1 \mathbf{n}_{2g})(\Psi)](W_1 \otimes W_2) \\ &= (-1)^{(n-3)(|\Psi|+1)} \left[ \sum T^{ab} \varepsilon(w_1 \mapsto w_1^1 w_1^2) (-1)^{|e_b||w_1^1|} \Psi(se_a w_1^1) \right. \\ & \quad \mathbf{n}_{20}(se_b w_1^2 \otimes W_2) + (-1)^{|W_1||W_2|} \sum T^{ab} \varepsilon(w_2 \mapsto w_2^1 w_2^2) \\ & \quad \left. (-1)^{|e_b||w_2^1|} \Psi(se_a w_2^1) \mathbf{n}_{20}(se_b w_2^2 \otimes W_1) \right]. \end{aligned} \quad (3.36)$$

*Proof.* Let us first discuss the completions. Given  $\mathbf{n}_{lg} \in \hat{\mathbf{E}}_l C$ , we can write it as  $\mathbf{n}_{lg} = \sum_{i=1}^{\infty} \Phi_1^i \dots \Phi_l^i$  with generating words  $\Phi_1^i \dots \Phi_l^i \in \mathbf{E}_l C$  of weights approaching  $\infty$ . The canonical extension of  $\circ_h$  to maps with finite filtration degree commutes with convergent infinite sums, and hence we have  $\mathbf{q}_{klg} \circ_h \mathbf{n}_{lg} = \sum_{i=1}^{\infty} \mathbf{q}_{klg} \circ_h (\Phi_1^i \dots \Phi_l^i)$ . Therefore, it suffices



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to prove the formulas for generating words  $\Phi_1^i \cdots \Phi_l^i \in E_l C$ .

From (3.14), we get for every  $\Psi, \Phi_1, \dots, \Phi_l \in C$  the equation

$$[\mathfrak{q}_{210} \circ_1 (\Phi_1 \cdots \Phi_l)](\Psi) = \sum_{i=1}^l (-1)^{|\Phi_i|(|\Phi_1|+\dots+|\Phi_{i-1}|)} \mathfrak{q}_{210}(\Psi, \Phi_i) \Phi_1 \cdots \hat{\Phi}_i \cdots \Phi_l,$$

where  $\Phi_1 \cdots \Phi_l$  on the left-hand-side is considered as a map  $E_0 C = \mathbb{R} \rightarrow E_l C$  mapping 1 to  $\Phi_1 \cdots \Phi_l$ . For  $W_1, \dots, W_l \in B^{\text{cyc}} V[3-n]$  and  $\sigma \in \mathbb{S}_l$ , we use

$$[\sigma(\Phi_1 \otimes \cdots \otimes \Phi_l)](W_1 \otimes \cdots \otimes W_l) = (\Phi_1 \otimes \cdots \otimes \Phi_l)[\sigma^{-1}(W_1 \otimes \cdots \otimes W_l)]$$

and Definition 3.4.1 to get

$$\begin{aligned} & ([\mathfrak{q}_{210} \circ_1 (\Phi_1 \cdots \Phi_l)](\Psi))(W_1 \otimes \cdots \otimes W_l) = \\ &= \sum_{i=1}^l (-1)^{|\Phi_i|(|\Phi_1|+\dots+|\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in \mathbb{S}_l} \varepsilon(\sigma^{-1}, W) [\mathfrak{q}_{210}(\Psi, \Phi_i)](W_{\sigma_1}) \\ & \quad \Phi_1(W_{\sigma_2}) \cdots \hat{\Phi}_i(\emptyset) \cdots \Phi_l(W_{\sigma_l}) \\ &= \sum_{i=1}^l (-1)^{|\Phi_i|(|\Phi_1|+\dots+|\Phi_{i-1}|)} \frac{1}{l!} \sum_{\sigma \in \mathbb{S}_l} \varepsilon(\sigma^{-1}, W) (-1)^{|s|\Psi} \\ & \quad \sum \varepsilon(w_{\sigma_1} \mapsto w_{\sigma_1}^1 w_{\sigma_1}^2) (-1)^{|e_b||w_{\sigma_1}^1|} T^{ab} \Psi(e_a w_{\sigma_1}^1) \Phi_i(e_b w_{\sigma_1}^2) \\ & \quad \Phi_1(W_{\sigma_2}) \cdots \hat{\Phi}_i(\emptyset) \cdots \Phi_l(W_{\sigma_l}) \\ &=: (*), \end{aligned}$$

where  $\hat{\Phi}_i(\emptyset)$  means omission of the corresponding term. Consider the bijection

$$I : \{1, \dots, l\} \times \mathbb{S}_l \longrightarrow \{1, \dots, l\} \times \mathbb{S}_l$$

$$(i, \sigma) \longmapsto \left( j := \sigma_1, \mu := \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & l \\ \sigma_2 & \dots & \sigma_i & \sigma_1 & \sigma_{i+1} & \dots & \sigma_l \end{pmatrix} \right).$$

Given  $(i, \sigma) \in \{1, \dots, l\} \times \mathbb{S}_l$  and  $b \in \{1, \dots, m\}$ , let  $(j, \mu) := I(i, \sigma)$  and

$$W' := W_1 \otimes \cdots \otimes W_{j-1} \otimes (se_b w_j^2) \otimes W_{j+1} \otimes \cdots \otimes W_l.$$

Suppose that  $(\Phi_1 \otimes \cdots \otimes \Phi_l)(W') \neq 0$ . We compute the Koszul sign  $\varepsilon(\mu^{-1}, W')$  in the

This can't be correct!: There must be a sign problem, set  $l = 1!$

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following way:

$$\begin{aligned}
W' &\mapsto (-1)^{(|w_j^1|+|e_b|+|W_j|)(|W_1|+\dots+|W_{j-1}|)} (se_b w_j^2) W_1 \dots \hat{W}_j \dots W_l \\
&\mapsto \underbrace{(-1)^{(|w_j^1|+|e_b|)(|W_1|+\dots+|W_{j-1}|)}}_{=:\varepsilon_1} \varepsilon(\sigma^{-1}, W) (se_b w_j^2) W_{\sigma_2} \dots W_{\sigma_l} \\
&\mapsto \underbrace{\varepsilon_1 \varepsilon(\sigma^{-1}, W) (-1)^{|\Phi_i|(|\Phi_1|+\dots+|\Phi_{i-1}|)}}_{=\varepsilon(\mu^{-1}, W')} \underbrace{W_{\sigma_2} \dots W_{\sigma_i} (se_b w_j^2) W_{\sigma_{i+1}} \dots W_{\sigma_l}}_{=W'_{\mu_1} \dots W'_{\mu_l}}.
\end{aligned}$$

Using this, we can rewrite (\*) as

$$\begin{aligned}
(*) &= (-1)^{|s||\Psi|} \sum_{j=1}^l \sum \varepsilon(w_j \mapsto w_j^1 w_j^2) (-1)^{|e_b||w_j^1|} T^{ab} \Psi(e_a w_j^1) \\
&\quad \varepsilon_1 \frac{1}{l!} \sum_{\mu \in \mathbb{S}_l} \varepsilon(\mu^{-1}, W') \Phi_1(W'_{\mu_1}) \dots \Phi_l(W'_{\mu_l}) \\
&= \sum_{j=1}^l \sum \varepsilon(w_j \mapsto w_j^1 w_j^2) (-1)^{|s||\Psi|+|e_b||w_j^1|+(|w_j^1|+|e_b|)(|W_1|+\dots+|W_{j-1}|)} T^{ab} \\
&\quad \Psi(se_a w_j^1) (\Phi_1 \dots \Phi_l) (W_1 \otimes \dots \otimes W_{j-1} \otimes (se_b w_j^2) \otimes W_{j+1} \otimes \dots \otimes W_l).
\end{aligned}$$

Finally, we use

$$T^{ab} \neq 0 \implies |e_a| + |e_b| = n - 2$$

to write

$$\begin{aligned}
|s||\Psi| &= |s|(|w_j^1| + |e_a|) = (n - 3)(|w_j^1| + n - 2 - |e_b|) \\
&= |s|(|w_j^1| + |e_b|) \pmod{2},
\end{aligned}$$

and the formula (3.34) follows.

As for (3.35), we first compute  $\varepsilon'$  for  $l = 1$  as follows:

$$\begin{aligned}
\ln_{-1}(\varepsilon') &= |w_1||e_b| + (|e_b| + |w_1|)|s| = |w^1||w^2| + |s||e_b| \\
&\quad \uparrow \\
&\quad 2(n-3)=|n_{10}|=|s|+|e_b|+|w^2| \\
&= |w^1||w^2| + |e_a||e_b| \pmod{2} \\
&\quad \uparrow \\
&\quad |e_a|+|e_b|=|s|+1
\end{aligned}$$

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Using this, we obtain

$$\begin{aligned}
[(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{1g})(\Psi)](W) &= \sum \varepsilon' \varepsilon(w \mapsto w^1 w^2) T^{ab} \Psi(se_a w^1) \mathfrak{n}_{1g}(se_b w^2) \\
&= (-1)^{|s|} \sum \varepsilon(w \mapsto w^2 w^1) T^{ba} \mathfrak{n}_{1g}(se_b w^2) \Psi(se_a w^1), \\
&\quad \uparrow \\
&\quad T^{ab} = (-1)^{|s|+|e_a||e_b|} T^{ba} \\
\varepsilon(w \mapsto w^1 w^2) &= (-1)^{|w^1||w^2|} \varepsilon(w \mapsto w^2 w^1)
\end{aligned}$$

which implies (3.35).

The proof of (3.36) is a combination of the same arguments.  $\square$

We will now relate homology of the twisted boundary operator  $\mathfrak{q}_{110}^{\mathfrak{n}}$  to cohomology of an  $A_\infty$ -algebra on  $V$  induced by  $\mathfrak{n}_{10}$ .

**Definition 3.4.5** ( $A_\infty$ -operations and compatible Maurer-Cartan element). *Suppose that  $(V, \Pi, m_1)$  is a finite-dimensional cyclic cochain complex of degree  $2 - n$ , and let  $\mathfrak{n} = (\mathfrak{n}_{lg})$  be a Maurer-Cartan element for  $\text{dIBL}(C(V))$ . We define the operations  $\mu_k : V[1]^{\otimes k} \rightarrow V[1]$  for all  $k \geq 1$  by*

$$\mu_k(v_1, \dots, v_k) := (-1)^{n-3} \sum_{i,j} T^{ij} \mathfrak{n}_{10}(se_i v_1 \dots v_k) e_j$$

for all  $v_1, \dots, v_k \in V[1]$ , where  $T^{ij}$  is the matrix from Definition 3.4.1.

If  $(V, \Pi, m_1, m_2)$  is in addition a cyclic dga and  $\mathfrak{m}$  the canonical Maurer-Cartan element for  $\text{dIBL}(C(V))$ , then we say that  $\mathfrak{n}$  is compatible with  $\mathfrak{m}$  if

$$\mathfrak{n}_{10}(sv_1 v_2 v_3) = \mathfrak{m}_{10}(sv_1 v_2 v_3) \quad \text{for all } v_1, v_2, v_3 \in V[1].$$

**Proposition 3.4.6** (Twisted boundary operator  $\mathfrak{q}_{110}^{\mathfrak{n}}$  and  $A_\infty$ -cyclic cohomology). *In the setting of Definition 3.4.5, the triple  $\mathcal{A}_{\mathfrak{n}}(V) := (V, \Pi, (\mu_k))$  is a cyclic  $A_\infty$ -algebra. We always have  $\mu_1 = m_1$ , and if  $\mathfrak{n}$  is compatible with  $\mathfrak{m}$  for a cyclic dga  $(V, \Pi, m_1, m_2)$ , then also  $\mu_2 = m_2$ .*

The following holds for the homologies:

$$\mathbb{H}^n(C(V)) = r(H_\lambda^*(\mathcal{A}_{\mathfrak{n}}(V); \mathbb{R}))[3 - n].$$

*Proof.* First of all, according to Definition 3.2.6 we must have  $\|\mathfrak{n}_{10}\| > 2$ , and hence

$$\mathfrak{n}_{10}(sv_1 v_2) = \mathfrak{n}_{10}(sv_1) = 0 \quad \text{for all } v_1, v_2 \in V[1].$$

This implies  $\mu_1 = m_1$ .

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Now, let  $e_0, \dots, e_m$  be a basis of  $V[1]$  and let  $e^0, \dots, e^m$  be the dual basis with respect to  $\Pi$ . For all  $k \geq 2$  and  $v_1, \dots, v_k \in V[1]$ , we compute the following:

$$\begin{aligned}
& \Pi(\mu_k(v_1, \dots, v_k), v_{k+1}) \\
&= (-1)^{n-3} \sum_{i,j} (-1)^{|e_i|} \Pi(e^i, e^j) \mathbf{n}_{10}(se_i v_1 \dots v_k) \Pi(e_j, v_{k+1}) \\
&= (-1)^{n-3} \sum_i (-1)^{|e_i|} \mathbf{n}_{10}(se_i v_1 \dots v_k) \Pi(e^i, v_{k+1}) \\
&\quad \uparrow \\
&\forall v \in V[1]: \sum_j \Pi(v, e^j) e_j = v \\
&= (-1)^{n-3} \sum_i (-1)^{|e_i|(n-3)} \mathbf{n}_{10}(sv_1 \dots v_k e_i) \Pi(e^i, v_{k+1}) \\
&\quad \uparrow \\
&(|v|_1 + \dots + |v_k|)|e_i| = \\
&(|\mathbf{n}_{10}| + |s| + |e_i|)|e_i| = (|s| + 1)|e_i| \\
&= (-1)^{n-2} \sum_i \mathbf{n}_{10}(sv_1 \dots v_k e_i) \Pi(v_{k+1}, e^i) \\
&\quad \uparrow \\
&1 + |v_{k+1}| |e^i| = 1 + (|e^i| + 2 - n) |e^i| \\
&= 1 + (3 - n) |e^i| = 1 + (3 - n) |e_i| \\
&= (-1)^{n-2} \mathbf{n}_{10}(sv_1 \dots v_{k+1}).
\end{aligned}$$

Therefore, we have

$$\mathbf{n}_{10} = (-1)^{n-2} \sum_{k \geq 2} \mu_k^+.$$

In this case, [CFL15, Proposition 12.3] asserts that the  $A_\infty$ -relations (3.27) for  $(\mu_k)_{k \geq 1}$  are equivalent to the “lowest” Maurer-Cartan equation (3.24) for  $\mathbf{n}_{10}$ . The degree condition  $|\mu_k| = 1$  and the cyclic symmetry of  $\mu_k^+$  are easy to check. Therefore,  $\mathcal{A}_n(V)$  is a cyclic  $A_\infty$ -algebra.

As for the compatibility with  $\mathbf{m}$ , we have for all  $v_1, v_2 \in V[1]$  the following:

$$\begin{aligned}
m_2(v_1, v_2) &= \sum_i \Pi(e_i, m_2(v_1, v_2)) e^i \\
&= \sum_{i,j} (-1)^{|e_i|} T^{ij} \Pi(e_i, m_2(v_1, v_2)) e_j \\
&\quad \uparrow \\
&T^{ij} = (-1)^{|e_i|} \Pi(e^i, e^j) \\
&= \sum_{i,j} (-1)^{1+(n-2)|e_i|} T^{ij} \underbrace{\Pi(m_2(v_1, v_2), e_i)}_{= (-1)^{n-2} \mathbf{m}_{10}(sv_1 v_2 e_i)} e_j \\
&\quad \uparrow \\
\Pi(v_1, v_2) &= (-1)^{1+(n-3)|v_1|} \Pi(v_2, v_1) \\
&= (-1)^{n-3} \sum_{i,j} T^{ij} \mathbf{m}_{10}(se_i v_1 v_2) e_j \\
&\quad \uparrow \\
(|v_1| + |v_2|)|e_i| &= (|\mathbf{m}_{10}| - |s| - |e_i|)|e_i| \\
&= (n-2)|e_i| \\
&= \mu_2(v_1, v_2).
\end{aligned}$$

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We will now clarify the relation to the cyclic cohomology of  $\mathcal{A}_n(V)$ . Recall from Proposition 3.2.11 that  $\mathfrak{q}_{110}^n(\Psi) = \mathfrak{q}_{110}(\Psi) + \mathfrak{q}_{210}(\mathfrak{n}_{10}, \Psi)$  for  $\Psi \in \hat{B}_{\text{cyc}}^* V[3-n]$ , where the first term is given by (3.33) and the second by (3.35). Consider now  $b'^k$  and  $R^k$  from (3.28), whose sum gives the Hochschild boundary operator  $b$ . Using the cyclic symmetry, we can rewrite a summand of  $b'^k$  for  $j = 1, \dots, k$  and  $i = 0, \dots, k-j$  applied to a generating word  $v_1 \dots v_k \in B^{\text{cyc}} V$  as follows:

$$\begin{aligned} [t_{k-j+1}^i \circ (\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^{-i}](\underbrace{v_1 \dots v_k}_{=:w}) &= \\ &= (-1)^{|v_1|+\dots+|v_i|} v_1 \dots v_i \mu_j(v_{i+1} \dots v_{i+j}) v_{i+j+1} \dots v_k \\ &= \varepsilon(w \mapsto w^1 w^2) \underbrace{\mu_j(v_{i+1} \dots v_{i+j})}_{=:w^1} \underbrace{v_{i+j+1} \dots v_k v_1 \dots v_i}_{=:w^2} \end{aligned} \quad (3.37)$$

Clearly, summing (3.37) over  $j = 1$  and  $i = 0, \dots, k-1$  gives the dual to  $\mathfrak{q}_{110}$ . For  $j = 2, \dots, k$ , we can write (3.37) as

$$(-1)^{n-3} \sum_{i,j} \varepsilon(w \mapsto w^1 w^2) T^{ij} \mathfrak{n}_{10}(se_i w^1) e_j w^2.$$

Therefore, the sum over  $j = 2, \dots, k$  and  $i = 0, \dots, k-j$  gives the part of the dual to  $\mathfrak{q}_{210}(\mathfrak{n}_{10}, \Psi)$  corresponding to the cyclic permutations  $\sigma \in \mathbb{S}_k$  with  $\sigma_1 = 1, j+1, \dots, k$ . The rest, i.e., the cyclic permutations with  $\sigma_1 = 2, \dots, j$ , is obtained analogously from the summands  $(\mu_j \otimes \mathbb{1}^{k-j}) \circ t_k^i$  of  $R^k$  for  $j = 2, \dots, k$  and  $i = 1, \dots, j-1$ . We conclude that  $\mathfrak{q}_{110}^n : \hat{B}_{\text{cyc}}^* V[3-n] \rightarrow \hat{B}_{\text{cyc}}^* V[3-n]$  is a degree shift of  $b^* : \hat{B}_{\text{cyc}}^* V \rightarrow \hat{B}_{\text{cyc}}^* V$ . As for the gradings, we have:

$$\begin{aligned} r(D_\lambda(V))[3-n]^i &= r(D_\lambda(V))^{i+3-n} = (D_\lambda(V))^{-i-3+n} = (\hat{B}_{\text{cyc}}^* V)^{i+3-n-1} \\ &= \hat{B}_{\text{cyc}}^* V[2-n]^i. \end{aligned}$$

This finishes the proof. □

We will now turn to units and augmentations.

**Definition 3.4.7** (Reduced canonical dIBL-algebra). *Let  $(V, \Pi, m_1, \mathbf{1}, \varepsilon)$  be an augmented cyclic cochain complex of degree  $2-n$  from Definition 3.3.10. We define the space of reduced cyclic cochains on  $V$  by*

$$C_{\text{red}}(V) := B_{\text{cyc,red}}^* V[2-n].$$

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We define the reduced canonical dIBL-algebra by

$$\text{dIBL}(C_{\text{red}}(V)) := (C_{\text{red}}(V), \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}),$$

where  $\mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}$  are restrictions of the operations of  $\text{dIBL}(C(V))$ .

**Definition 3.4.8** (Strictly reduced Maurer-Cartan element). *In the setting of Definition 3.4.7, we call a Maurer-Cartan element  $\mathfrak{n} = (\mathfrak{n}_{lg})$  for  $\text{dIBL}(C(V))$  strictly reduced if  $\mathfrak{n}_{lg} \in \hat{\mathbb{E}}_l C_{\text{red}}(V)$  for all  $(l, g) \neq (1, 0)$  and if the  $A_\infty$ -algebra  $(\mathcal{A}_{\mathfrak{n}}(V), \mathfrak{1}, \varepsilon)$  induced by  $\mathfrak{n}_{10}$  is strictly unital and strictly augmented. Given a strictly reduced Maurer-Cartan element  $\mathfrak{n}$ , we can define the twisted  $\text{IBL}_\infty$ -algebra*

$$\text{dIBL}^{\mathfrak{n}}(C_{\text{red}}(V)) := (C_{\text{red}}(V), (\mathfrak{q}_{klg}^{\mathfrak{n}})),$$

where  $\mathfrak{q}_{klg}^{\mathfrak{n}}$  are the restrictions of the operations of  $\text{dIBL}^{\mathfrak{n}}(C(V))$ . We denote the homology of  $\text{dIBL}^{\mathfrak{n}}(C_{\text{red}})$  by  $\mathbb{H}^{\mathfrak{n}}(C_{\text{red}})$  or  $\mathbb{H}^{\mathfrak{n}, \text{red}}(C)$ .<sup>9</sup>

*Remark 3.4.9* (On strictly reduced Maurer-Cartan element). (i) We can say that the  $\text{IBL}_\infty$ -algebra  $\text{dIBL}^{\mathfrak{n}}(C_{\text{red}})$  is a *subalgebra* of  $\text{dIBL}^{\mathfrak{n}}(C)$ , which means that the inclusion  $C_{\text{red}} \hookrightarrow C$  induces the following commutative diagram for all  $k, l \geq 1, g \geq 0$ :

$$\begin{array}{ccc} \hat{\mathbb{E}}_k C & \xrightarrow{\mathfrak{q}_{klg}^{\mathfrak{n}}} & \hat{\mathbb{E}}_l C \\ \uparrow & & \uparrow \\ \hat{\mathbb{E}}_k C_{\text{red}} & \xrightarrow{\mathfrak{q}_{klg}^{\mathfrak{n}}} & \hat{\mathbb{E}}_l C_{\text{red}}. \end{array}$$

We denote this fact by  $\text{dIBL}^{\mathfrak{n}}(C_{\text{red}}) \subset \text{dIBL}^{\mathfrak{n}}(C)$ .

(ii) The canonical Maurer-Cartan element  $\mathfrak{m}$  of a strictly augmented strictly unital dga  $(V, m_1, m_2, \mathfrak{1}, \varepsilon)$  is strictly reduced (this follows from Proposition 3.4.6).

(iii) In the situation of Definition 3.4.8, we denote

$$\bar{V}[1] := \ker(\varepsilon),$$

so that  $V = \bar{V} \oplus \langle 1 \rangle$ . We use the canonical projection  $\pi : V \rightarrow \bar{V}$  to identify  $\hat{B}_{\text{cyc}}^* \bar{V} \xrightarrow{\sim} \hat{B}_{\text{cyc}, \text{red}}^* V$  via the componentwise pullback  $\pi^*$ . In this way, we obtain the

<sup>9</sup>The latter option suggests that it might be possible to define the reduced homology with the induced  $\text{IBL}$ -algebra even if  $\mathfrak{n}$  is not strictly reducible, e.g., if  $(\mathcal{A}_{\mathfrak{n}}(V), \mathfrak{1}, \varepsilon)$  is only homologically unital and augmented.

DONE Put remark to text: Include this remark to text because it is in fact a definition.

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$\text{IBL}_\infty$ -algebras  $\text{dIBL}(C(\bar{V}))$  and  $\text{dIBL}^n(C(\bar{V}))$ , which are isomorphic to  $\text{dIBL}(C_{\text{red}}(V))$  and  $\text{dIBL}^n(C_{\text{red}}(V))$ , respectively.  $\triangleleft$

In the following list, we sum up our main reasons for considering units, augmentations and reduced Maurer-Cartan elements. Suppose that we are in the situation of Definition 3.4.8, then:

- Proposition 3.3.13 implies the splitting

$$\mathbb{H}^n(C)[1] = \mathbb{H}^n(C_{\text{red}})[1] \oplus \langle s1^{2q-1*} \mid q \in \mathbb{N} \rangle. \quad (3.38)$$

Here  $1^{i*} \in B_{\text{cyc}}^* V$  is the componentwise pullback  $\varepsilon^*$  of  $1^{i*} \in B_{\text{cyc}}^*(\mathbb{R})$ . To get this, we used

$$\mathbb{H}^n(C_{\text{red}}) = r(H_{\lambda, \text{red}}^*(\mathcal{A}_n))[3 - n],$$

which can be seen by redoing the proof of Proposition 3.4.6 with reduced cochains.

- The subalgebra  $\text{dIBL}^n(C_{\text{red}}) \subset \text{dIBL}^n(C)$  induces the subalgebra

$$\text{IBL}(\mathbb{H}^n(C_{\text{red}})) \subset \text{IBL}(\mathbb{H}^n(C)),$$

and any higher operation  $\mathfrak{q}_{1lg}^n$  which induces a map  $\hat{E}_1\mathbb{H}(C) \rightarrow \hat{E}_l\mathbb{H}(C)$  induces a map  $\hat{E}_1\mathbb{H}(C_{\text{red}}) \rightarrow \hat{E}_l\mathbb{H}(C_{\text{red}})$  as well.

- If  $V$  is non-negatively graded, connected and simply-connected, then we have  $\hat{E}_k C_{\text{red}} \simeq E_k C_{\text{red}}$  for all  $k \in \mathbb{N}_0$  by Proposition 3.3.15, and hence  $\text{dIBL}^n(C_{\text{red}})$  is completion-free.

**Proposition 3.4.10** (Operations on units). *Suppose that  $(V, \Pi, m_1, 1, \varepsilon)$  is a finite-dimensional augmented cyclic cochain complex of degree  $2 - n$  such that  $n \geq 1$ , and let  $\mathfrak{n}$  be a strictly reduced Maurer-Cartan element for  $\text{dIBL}(C(V))$ . The following relations are the only relations containing  $1^{i*}$  which may be non-zero on the homology  $\mathbb{H}^n(C)$ :*

*For all  $\Psi \in C_{\text{red}}(V)$  and  $l \geq 1, g \geq 0$ , we have*

$$\mathfrak{q}_{210}(s1^* \otimes \Psi) = (-1)^{(n-2)|\Psi|} \mathfrak{q}_{210}(\Psi \otimes s1^*) = (-1)^{n-2} \Psi \circ \iota_v \quad \text{and}$$

$$\mathfrak{q}_{1lg}^n(s1^*) = -\mathfrak{n}_{lg} \circ \iota_v,$$

where  $\iota_v$  is defined as follows:

- The element  $v \in V[1]$  is the unique vector such that  $\Pi(1, v) = 1$  and  $v \perp \bar{V}[1]$  with respect to  $\Pi$ . Note that  $|v| = n - 1$  and that such  $v$  always exists due to non-degeneracy.

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- We start by defining  $\iota_v : B^{\text{cyc}}V \rightarrow B^{\text{cyc}}V$  by

$$\iota_v(v_1 \dots v_k) := \sum_{i=1}^k (-1)^{|\mathbf{v}|(|v_1| + \dots + |v_{i-1}|)} v_1 \dots v_{i-1} v v_i \dots v_k$$

for all generating words  $v_1 \dots v_k \in B^{\text{cyc}}V$ . Next, for all  $k \geq 1$ , we define  $\iota_v : (B^{\text{cyc}}V)^{\otimes k} \rightarrow (B^{\text{cyc}}V)^{\otimes k}$  by

$$\begin{aligned} \iota_v(w_1 \otimes \dots \otimes w_k) &:= (-1)^{|\mathbf{v}|k} \sum_{j=1}^k (-1)^{|\mathbf{v}|(|w_1| + \dots + |w_{j-1}|)} w_1 \otimes \dots \otimes w_{j-1} \\ &\quad \otimes \iota_v(w_j) \otimes w_{j+1} \otimes \dots \otimes w_k \end{aligned}$$

for all generating words  $w_1, \dots, w_k \in B^{\text{cyc}}V$ . Finally, we take the degree shift  $\iota_v : (B^{\text{cyc}}V[3-n])^{\otimes k} \rightarrow (B^{\text{cyc}}V[3-n])^{\otimes k}$  according to the degree shift convention (3.4).

*Proof.* Pick a basis  $(e_0, \dots, e_m)$  of  $V[1]$  such that  $e_0 = 1$  and  $\bar{V}[1] = \langle e_1, \dots, e_m \rangle$ . If  $(e^0, \dots, e^m)$  is the dual basis, then we have  $\mathbf{v} = e^0$ . We will often use the following relation:

$$\sum_{j=0}^m T^{1j} e_j = \sum_{j=0}^m (-1)^{|\mathbf{1}|} \Pi(\mathbf{v}, e^j) e_j = -\mathbf{v}. \quad (3.39)$$

We consider only those generating words  $w = v_1 \dots v_k$  of  $B^{\text{cyc}}V$  with either  $v_i \in \bar{V}$  for each  $i$  (shortly  $w \in B^{\text{cyc}}\bar{V}$ ) or  $v_i = 1$  for each  $i$  with  $k$  odd (i.e.,  $w = 1^{2j-1}$  for some  $j$ ). Let  $w_1, \dots, w_l$  with  $w_j = v_{j1} \dots v_{jk_j}$  denote such generating words. Clearly, if  $\Phi \in \hat{E}_l C(V)$  is a  $\mathfrak{q}_{110}^n$ -closed element which vanishes on all  $w_1 \otimes \dots \otimes w_l$ , then (3.38) implies that  $[\Phi] = 0$  in  $\hat{E}_l \mathbb{H}(C)$ .

For  $\Psi \in C_{\text{red}}(V)$  and  $q \geq 1$  odd, we compute using (3.39) the following:

$$\begin{aligned} \mathfrak{q}_{210}(s^2 1^{q*} \otimes \psi)(sw) &= \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{(n-1)|w^1|} T^{1j} 1^{q*} (1 w^1) \psi(e_j w^2) \\ &= - \sum \varepsilon(w \mapsto w^1 w^2) (-1)^{(n-1)|w^1|} 1^{q*} (1 w^1) \psi(\mathbf{v} w^2) \\ &=: (*). \end{aligned}$$

Now, in order to get  $(*) \neq 0$ , we need either  $q = 1$  and  $w \in B^{\text{cyc}}\bar{V}$ , in which case

$$(*) = - \sum \varepsilon(w \mapsto \underbrace{w^1}_{=\emptyset} w^2) \psi(\mathbf{v} w^2)$$



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$$\begin{aligned}
&= - \sum_{j=1}^k (-1)^{|v|(|v_1|+\dots+|v_{j-1}|)} \psi(v_1 \dots v_{j-1} v v_{j+1} \dots v_k) \\
&= -(\psi \circ \iota_v)(w) = (-1)^{n-2} (\Psi \circ \iota_v)(W),
\end{aligned}$$

or  $q > 1$  odd and  $w = 1^{q-1}$ , in which case

$$\begin{aligned}
(*) &= \sum \varepsilon(w \mapsto w^1 \underbrace{w^2}_{=\emptyset}) 1^{q*} (1^q) \psi(v) \\
&= \psi(v) \sum_{j=1}^{q-1} (-1)^j \\
&= 0.
\end{aligned}$$

Next, because  $n \geq 1$ , we get  $T^{11} = 0$ , and hence

$$\mathfrak{q}_{120}(1^{q*}) = 0 \quad \text{for all } q \in \mathbb{N}$$

on the chain level. Therefore, we have  $\mathfrak{q}_{1lg}^n = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg}$  for all  $l \geq 1$ ,  $g \geq 0$ , and using Proposition 3.4.4 and (3.39), we obtain

$$\begin{aligned}
&[(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{lg})(1^{q*})](W_1 \otimes \dots \otimes W_l) \\
&= - \sum_{j=1}^l \sum \varepsilon' \varepsilon(w_j \mapsto w_j^1 w_j^2) 1^{q*} (1 w_j^1) \mathfrak{n}_{lg}(W_1 \otimes \dots \otimes W_{j-1} \otimes (sv w_j^2) \\
&\quad \otimes W_{j+1} \otimes \dots \otimes W_l) \\
&=: (**).
\end{aligned}$$

In order to get  $(**) \neq 0$ , we need either  $q = 1$  and  $w_j \in B^{\text{cyc}} \bar{V}$  for all  $j$ , in which case

$$\begin{aligned}
(**) &= - \sum_{j=1}^l \sum_{i=1}^{k_j} (-1)^{|v|(|W_1|+\dots+|W_{j-1}|+|s|)} (-1)^{|v|(|v_1|+\dots+|v_{j-1}|)} \mathfrak{n}_{lg}(W_1 \otimes \dots \\
&\quad \otimes W_{j-1} \otimes (sv_1 \dots v_{i-1} v v_i \dots v_{k_j}) \otimes W_{j+1} \otimes \dots \otimes W_l) \\
&= -(\mathfrak{n}_{lg} \circ \iota_v)(W_1 \otimes \dots \otimes W_l),
\end{aligned}$$

or  $q > 1$  odd and  $w_j = 1^{q-1}$  for some  $j$ , in which case

$$\begin{aligned}
(**) &= - \sum_{\substack{1 \leq j \leq l \\ w_j = 1^{q-1}}} \varepsilon' \left( \sum \varepsilon(w_j \mapsto w_j^1 \underbrace{w_j^2}_{=\emptyset}) 1^{q*} (1 w_j^1) \right) \mathfrak{n}_{lg}(W_1 \otimes \dots \\
&\quad \otimes W_{j-1} \otimes (sv) \otimes W_{j+1} \otimes \dots \otimes W_l)
\end{aligned}$$

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$$\begin{aligned}
&= - \sum_{\substack{1 \leq j \leq l \\ w_j = 1^{q-1}}} \varepsilon' \left( \underbrace{\sum_{i=1}^{q-1} (-1)^i}_{=0} \right) \mathfrak{n}_{lg}(W_1 \otimes \cdots \otimes W_{j-1} \otimes (sv) \otimes W_{j+1} \otimes \cdots \otimes W_l) \\
&= 0.
\end{aligned}$$

The only relation left to check is

$$\mathfrak{q}_{210}(s_1^{q_1*}, s_1^{q_1*}) = 0 \quad \text{for all } q_1, q_2 \in \mathbb{N}.$$

However, this is easy to see, and the proof is done. □

## 4. Chern-Simons Maurer-Cartan element and string topology

In Section 4.1, we consider the cyclic dga's  $\Omega(M)$ ,  $H_{\text{dR}}(M)$  and  $\mathcal{H}(M)$  for a closed oriented  $n$ -manifold  $M$  (Proposition 4.1.2) and apply the theory from Section 3.4 to the last two, which are finite-dimensional.

In Section 4.2, we define the admissible Hodge propagator  $P$  (Definition 4.2.4). It is a primitive to the Schwartz kernel  $H$  of the harmonic projection  $\pi_{\mathcal{H}}$  (see Proposition 4.2.7) outside the diagonal and extends smoothly to the spherical blow-up of the diagonal. These ideas come from an early version of [CVona]. We consider conditions (P1)–(P5) on a linear operator  $\mathcal{P}$  and its Schwartz kernel  $P$  (see p. 84) and show that  $P$  satisfying all these conditions always exists (Proposition 4.2.10). We also mention the standard Hodge propagator  $P_{\text{std}}$  (see (4.15)), which might be a canonical Hodge propagator satisfying (P1)–(P5).

In Section 4.3, we review ribbon graphs, labelings, compatibility of the order and orientation of internal edges, and the edge and vertex order from [CFL15] (Definitions 4.3.1, 4.3.3, 4.3.4 and 4.3.5). We then define  $\mathbf{n}$  as a signed sum of integrals of products of Hodge propagators and harmonic forms which are associated to labeled trivalent ribbon graphs (Definition 4.3.6). We do not show that these integrals converge and that  $\mathbf{n}$  satisfies the Maurer-Cartan equation, but we do show all other properties of a Maurer-Cartan element (Lemma 4.3.7 and Proposition 4.3.10). We define the  $Y$ -graph, trees, circular graphs, vertices of types  $A$ ,  $B$ ,  $C$  and their contributions  $A_{\alpha_1, \alpha_2}$ ,  $B_{\alpha}$ ,  $C$ , respectively (Definitions 4.3.8 and 4.3.11).

In Section 4.4, we observe that vanishing of some special vertices in the graphs implies  $\mathbf{n}_{lg} = \mathbf{m}_{lg}$ . For example, if all graphs, except for the  $Y$ -graph, with  $1$  at an external vertex vanish, which holds if  $P$  satisfies (P4) and (P5) (Proposition 4.4.2), then all higher operations  $\mathbf{q}_{lg}^{\mathbf{n}}$  vanish on the chain level in dimensions  $n > 3$  (Proposition 4.4.1). Next, if all graphs with an  $A$ -vertex vanish, then  $\mathbf{n}_{10} = \mathbf{m}_{10}$ , and hence  $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110}^{\mathbf{m}}$  (Proposition 4.4.3). We show that  $\mathbf{n} = \mathbf{m}$  for simply-connected geometrically formal manifolds with  $n \neq 2$  (Proposition 4.4.5). Using the results of [CVona], we argue that the

chain complexes of  $\mathfrak{q}_{110}^m$  and  $\mathfrak{q}_{110}^n$  are quasi-isomorphic provided  $M$  is simply-connected and formal (Proposition 4.4.7).

In Section 4.5, we recall basic facts about the Chas-Sullivan operations  $m_2$  and  $c_2$  on the  $\mathbb{S}^1$ -equivariant homology of the free loop space and formulate a version of the string topology conjecture for simply-connected manifolds (Conjecture 4.5.1).

## 4.1. Canonical dIBL-structures on cyclic cochains of de Rham cohomology

Let  $M$  be an oriented closed Riemannian manifold of dimension  $n$ . We consider the following graded vector spaces:

$$\begin{aligned}\Omega^*(M) & \dots \text{ smooth de Rham forms,} \\ \mathcal{H}^*(M) & \dots \text{ harmonic forms,} \\ H_{\text{dR}}^*(M) & \dots \text{ de Rham cohomology.}\end{aligned}$$

Since  $M$  is fixed, we often write just  $\Omega$ ,  $\mathcal{H}$  and  $H_{\text{dR}}$ . We consider the Hodge decomposition  $\Omega = \mathcal{H} \oplus \text{im } d \oplus \text{im } d^*$ , where  $d$  is the de Rham differential and  $d^*$  the codifferential. We call the corresponding projection

$$\pi_{\mathcal{H}} : \Omega^*(M) \longrightarrow \mathcal{H}^*(M) \tag{4.1}$$

the *harmonic projection* and the induced isomorphism  $\pi_{\mathcal{H}} : H_{\text{dR}} \rightarrow \mathcal{H}$  mapping a cohomology class into its unique harmonic representative the *Hodge isomorphism*.

**Notation 4.1.1** (Updated notation for bar complexes). *We use Notation 3.3.3 for  $V = \Omega, \mathcal{H}, H_{\text{dR}}$  and  $A = n - 3$  with the following changes:*

$$\tilde{v} \sim \eta \in V, \quad v \sim \alpha \in V[1], \quad w \sim \omega \in B^{\text{cyc}}V, \quad W \sim \Omega \in B^{\text{cyc}}V[n - 3].$$

*We use the formal symbols  $s$  and  $\theta$  with  $|s| = n - 3$  and  $|\theta| = -1$ , so that  $\alpha = \theta\eta$  and  $\Omega = s\omega$ .*

**Proposition 4.1.2** (De Rham cyclic dga's). *Let  $M$  be an oriented closed Riemannian manifold of dimension  $n$ . The quadruple  $(\Omega(M), \Pi, m_1, m_2)$  with the operations from (2.3) is a cyclic dga of degree  $2 - n$ . For the operations before the degree shift, we*

DONE Mistake in reduced notation: Correct the notation here, if it is not already corrected. Anyway, define something like Red-NTwistHIBL and replace the construction.

#### 4. Chern-Simons Maurer-Cartan element and string topology

have

$$\begin{aligned}\tilde{m}_1(\eta_1) &= d\eta_1, \\ \tilde{m}_2(\eta_1, \eta_2) &= \eta_1 \wedge \eta_2, \\ \tilde{\Pi}(\eta_1, \eta_2) &= \int_M \eta_1 \wedge \eta_2 =: (\eta_1, \eta_2),\end{aligned}$$

where  $d$  is the de Rham differential,  $\wedge$  the wedge product and  $\tilde{\Pi}$  the intersection pairing. The operations restrict to  $H_{dR}(M)$  and make  $(H_{dR}(M), \Pi, m_1 \equiv 0, m_2)$  into a cyclic dga. If we define  $\mu_1 \equiv 0$  and

$$\mu_2(\alpha_1, \alpha_2) := \pi_{\mathcal{H}}(m_2(\alpha_1, \alpha_2)) \quad \text{for all } \alpha_1, \alpha_2 \in \mathcal{H}(M)[1], \quad (4.2)$$

then  $(\mathcal{H}(M), \Pi, \mu_1, \mu_2)$  is a cyclic dga as well, and  $\pi_{\mathcal{H}} : H_{dR} \rightarrow \mathcal{H}$  is an isomorphism of cyclic dga's. All three dga's  $\Omega$ ,  $H_{dR}$  and  $\mathcal{H}$  are strictly unital and strictly augmented with the unit  $1 := \theta 1 \in \Omega[1]$ , where  $1$  is the constant one.

*Proof.* The relations (3.31) follow from the classical properties of  $d$  and  $\wedge$  and from the Stokes' theorem for oriented closed manifolds. The Poincaré duality asserts that  $(\cdot, \cdot)$  is non-degenerate on  $H_{dR}$  and  $\mathcal{H}$ , and thus they are cyclic dga's as well. The fact that  $\pi_{\mathcal{H}} : H_{dR} \rightarrow \mathcal{H}$  is an isomorphism of vector spaces follows from the Hodge theory. As for compatibility with the product, given  $\eta_1, \eta_2 \in \mathcal{H}$ , then  $\eta_1 \wedge \eta_2$  is closed, and since  $\ker d = \mathcal{H} \oplus \text{im } d$ , we see that  $\pi_{\mathcal{H}}(\eta_1 \wedge \eta_2) = \eta_1 \wedge \eta_2 + d\eta$  for some  $\eta \in \Omega$  is a harmonic representative of the cohomology class  $[\eta_1 \wedge \eta_2] = [\eta_1] \wedge [\eta_2]$ . Unitality is obvious, and the construction of an augmentation map clear. Note that a strict augmentation for  $\Omega(M)$  is the evaluation at a point, for instance.  $\square$

The facts (A) and (C) from the Overview apply to the cyclic dga's  $\mathcal{H}$  and  $H_{dR}$  (not to  $\Omega$  because it is infinite-dimensional!), and we get the canonical dIBL-algebras  $\text{dIBL}(C(\mathcal{H}))$  and  $\text{dIBL}(C(H_{dR}))$  of bidegrees  $(n-3, 2)$  with the canonical Maurer-Cartan element  $\mathbf{m} = (\mathbf{m}_{10})$ . The Hodge isomorphism induces an isomorphism of these dIBL-algebras, and hence we can use  $\mathcal{H}$  and  $H_{dR}$  interchangeably. We have  $\mathfrak{q}_{110} \equiv 0$ , and hence  $\text{dIBL}(C(\mathcal{H}))$  is, in fact, an IBL-algebra. However, we will denote it by dIBL and call it a dIBL algebra as a reminder of the canonical dIBL-structure. The canonical Maurer-Cartan element  $\mathbf{m}$  satisfies

$$\mathbf{m}_{10}(s\alpha_1\alpha_2\alpha_3) = (-1)^{n-2+\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{for all } \alpha_1, \alpha_2, \alpha_3 \in \mathcal{H}[1]. \quad (4.3)$$

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We get the canonical twisted dIBL-algebra  $\text{dIBL}^m(C(\mathcal{H}))$  from (2.2) with, in general, non-trivial boundary operator  $\mathfrak{q}_{110}^m$  whose homology is the cyclic homology of  $H_{\text{dR}}$  up to degree shifts.

### 4.2. Hodge propagator

DONE Notation for piH: We should define the special deformation retract by requiring only  $\mathcal{P}\iota_{\mathcal{H}} = 0!!$  The notation  $\mathcal{P}\pi_{\mathcal{H}} = 0$  does not make much sense! It makes.

We will use fiberwise integration and spherical blow-ups, which we now recall.

**Definition 4.2.1** (Fiberwise integration). *Let  $\text{pr} : E \rightarrow B$  be a smooth oriented fiber bundle with an oriented fiber  $F$  over an oriented manifold  $B$  with  $\partial B = \emptyset$ . We orient  $E$  as  $F \times B$ . Let  $\Omega_c(B)$  denote the space of forms with compact support and  $\Omega_{\text{cv}}(E)$  the space of forms with compact vertical support. For any  $\kappa \in \Omega_{\text{cv}}(E)$ , let  $\int^F \kappa \in \Omega(B)$  be the unique smooth form such that*

$$\int_E \kappa \wedge \text{pr}^* \eta = \int_B \left( \int^F \kappa \right) \wedge \eta \quad \text{for all } \eta \in \Omega_c(B).$$

**Definition 4.2.2** (Spherical blow-up). *Let  $X$  be a smooth  $n$ -dimensional manifold and  $Y \subset X$  a smooth  $k$ -dimensional submanifold. The blow-up of  $X$  at  $Y$  is as a set defined by*

$$\text{Bl}_Y X := X \setminus Y \sqcup P^+ NY,$$

where  $P^+ NY$  is the real oriented projectivization of the normal bundle  $NY$  of  $Y$  in  $X$ . This means that  $P^+ NY$  is the quotient of  $\{v \in NY \mid v \neq 0\}$  by the relation  $v \sim av$  for all  $a \in (0, \infty)$ . The blow-down map is defined by

$$\begin{aligned} \pi : \text{Bl}_Y X &\longrightarrow X \\ p \in X \setminus Y &\longmapsto p, \\ [v]_p \in P^+ NY &\longmapsto p. \end{aligned}$$

In the following, we will equip the blow-up with the structure of a smooth manifold with boundary such that its interior becomes diffeomorphic to  $X \setminus Y$  via the blow-down map and the boundary becomes  $P^+ NY$ . Consider an adapted chart  $(U, \psi)$  for  $Y$  in  $X$  with  $\psi(U) = \mathbb{R}^n$  and  $\psi(U \cap Y) = \{(0, y) \mid y \in \mathbb{R}^k\}$ . It induces the bijection

DONE Blow up charts: It is not a bijection if  $\psi(U \cap Y) \neq \emptyset$ ! We need it centered at  $(0, 0)$ ! No we need actually only the intersection with  $Y$ !

#### 4. Chern-Simons Maurer-Cartan element and string topology

$$\begin{aligned}\tilde{\psi} : \text{Bl}_{U \cap Y} U &\longrightarrow [0, \infty) \times \mathbb{S}^{n-k-1} \times \mathbb{R}^k \\ p \in U \setminus Y &\longmapsto \left( |\pi_1 \psi(p)|, \frac{\pi_1 \psi(p)}{|\pi_1 \psi(p)|}, \pi_2 \psi(p) \right), \\ [v] \in P^+ N_p Y &\longmapsto \left( 0, \frac{\pi_1 d\psi(v)}{|\pi_1 d\psi(v)|}, \pi_2 \psi(p) \right),\end{aligned}$$

where  $\pi_1$  and  $\pi_2$  are the canonical projections to the factors of  $\mathbb{R}^{n-k} \times \mathbb{R}^k$ . Notice that we have the canonical inclusion  $\text{Bl}_{U \cap Y} U \subset \text{Bl}_Y X$ . It can be checked that for any two overlapping adapted charts  $(U_1, \psi_1)$  and  $(U_2, \psi_2)$ , the transition function  $\tilde{\psi}_1 \circ \tilde{\psi}_2^{-1}$  is a diffeomorphism of manifolds with boundary. Therefore, we can use the charts  $(\text{Bl}_{U \cap Y} U, \tilde{\psi})$  to define a smooth atlas on  $\text{Bl}_Y X$ . If  $X$  is oriented, we orient  $\text{Bl}_Y X$  so that  $\pi$  restricts to an orientation preserving diffeomorphism of the interior.

An important fact is that if  $X$  is compact, then  $\text{Bl}_Y X$  is compact.

We are interested in the case when  $X = M \times M$  for an oriented closed manifold  $M$  and  $Y = \Delta := \{(m, m) \mid m \in M\}$  is the *diagonal*. Given a chart  $\varphi : U \rightarrow \mathbb{R}^n$  on  $M$ , the following is a smooth chart on  $\text{Bl}_\Delta(M \times M)$ :

$$\begin{aligned}\tilde{\varphi} : \text{Bl}_\Delta(U \times U) &\longrightarrow [0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n \\ (x, y) \in (U \times U) \setminus \Delta &\longmapsto (r, w, u) := \left( \frac{1}{2} |\varphi(x) - \varphi(y)|, \frac{\varphi(x) - \varphi(y)}{|\varphi(x) - \varphi(y)|}, \right. \\ &\quad \left. \frac{1}{2} (\varphi(x) + \varphi(y)) \right), \\ [(v, -v)]_{(x, x)} &\longmapsto \left( 0, \frac{d\varphi_x(v)}{|d\varphi_x(v)|}, \varphi(x) \right).\end{aligned}\tag{4.4}$$

The inverse relations for  $r > 0$  read

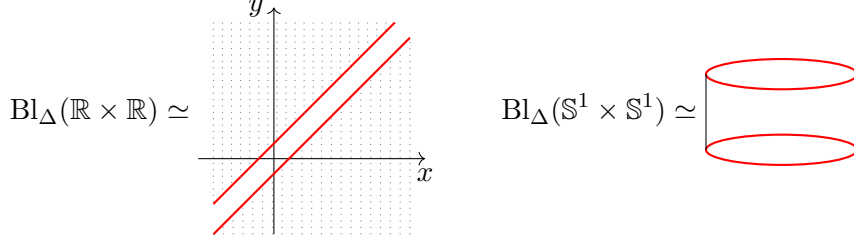
$$\varphi(x) = u + wr \quad \text{and} \quad \varphi(y) = u - wr.$$

We will denote by  $M_i$  the  $i$ -th factor of  $M \times M$ ; i.e., we will write  $M \times M = M_1 \times M_2$ . We denote the corresponding projection by  $\text{pr}_i$ . We define  $\widetilde{\text{pr}}_i := \text{pr}_i \circ \pi$ , where  $\pi : \text{Bl}_\Delta(M \times M) \rightarrow M \times M$  is the blow-down map. We also identify  $(M \times M) \setminus \Delta$  with the interior of  $\text{Bl}_\Delta(M \times M)$  via  $\pi$ .

*Example 4.2.3* (Product minus open thickening of diagonal). We have the following

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examples:



In fact, the manifold  $\text{Bl}_\Delta(M \times M)$  is always diffeomorphic to  $M \times M$  minus an open thickening of the diagonal. The blow-down map is therefore an essential part of the blow-up construction.  $\triangleleft$

The map  $\widetilde{\text{pr}}_2 : \text{Bl}_\Delta(M \times M) \rightarrow M_2$  is an oriented fiber bundle with fiber  $\text{Bl}_*(M_1)$ , which is the blow-up of  $M_1$  at a point (we shall assume that  $M$  is connected). The fiberwise integration along  $\widetilde{\text{pr}}_2$  will be denoted by  $\int^{\text{Bl}_*M_1}$ .

**Definition 4.2.4** (Hodge propagator). *Let  $M$  be an oriented closed  $n$ -dimensional Riemannian manifold. Consider the harmonic projection  $\pi_{\mathcal{H}}$  from (4.1), and let  $\iota_{\mathcal{H}} : \mathcal{H}(M) \hookrightarrow \Omega(M)$  be the inclusion. A smooth  $(n-1)$ -form  $P$  on  $(M \times M) \setminus \Delta$  is called an admissible Hodge propagator if the following conditions are satisfied:*

- (1) *The form  $P$  admits a smooth extension to  $\text{Bl}_\Delta(M \times M)$ . More precisely, the pullback  $(\pi|_{\text{int}})^*P$  along the blow-down map restricted to the interior is a restriction of a smooth form on  $\text{Bl}_\Delta(M \times M)$ . We denote this form by  $P$  again by uniqueness.*
- (2) *The operator  $\mathcal{P} : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$  defined by*

$$\mathcal{P}(\eta) := \int^{\text{Bl}_*M_1} P \wedge \widetilde{\text{pr}}_1^* \eta \quad \text{for all } \eta \in \Omega(M) \quad (4.5)$$

*satisfies*

$$\text{d} \circ \mathcal{P} + \mathcal{P} \circ \text{d} = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1}. \quad (4.6)$$

*Any homogenous linear operator  $\mathcal{P} : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$  satisfying (4.6) will be called a Hodge homotopy.*

- (3) *For the twist map  $\tau : M \times M \rightarrow M \times M$  defined by  $(x, y) \mapsto (y, x)$ , the following symmetry property holds:*

$$\tau^*P = (-1)^n P. \quad (4.7)$$



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*Remark 4.2.5* (On Hodge propagator). (i) Given a homogenous linear operator  $\mathcal{P} : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$ , if there is a  $P \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M))$  such that (4.5) holds, then it is unique.

(ii) Because  $\tau : M \times M \rightarrow M \times M$  preserves  $\Delta$ , it extends to a diffeomorphism  $\tilde{\tau}$  of  $\text{Bl}_\Delta(M \times M)$ . The condition (4.7) is then equivalent to  $\tilde{\tau}^* \tilde{P} = (-1)^n \tilde{P}$  for the extension  $\tilde{P}$  of  $P$  to  $\text{Bl}_\Delta(M \times M)$ . We denote both extensions by  $\tau$  and  $P$ , respectively.

(iii) Using the intersection pairing  $(\cdot, \cdot)$ , we have

$$\begin{aligned} (\mathcal{P}(\eta_1), \eta_2) &= \int_{\text{Bl}_\Delta(M \times M)} P \wedge \widetilde{\text{pr}}_1^* \eta_1 \wedge \widetilde{\text{pr}}_2^* \eta_2 \\ &= (-1)^n \int_{\text{Bl}_\Delta(M \times M)} \tau^* P \wedge \widetilde{\text{pr}}_2^* \eta_1 \wedge \widetilde{\text{pr}}_1^* \eta_2 \end{aligned}$$

and

$$\begin{aligned} (\eta_1, \mathcal{P}(\eta_2)) &= (-1)^{\eta_1(\eta_2-1)} (\mathcal{P}(\eta_2), \eta_1) \\ &= (-1)^{\eta_1} \int_{\text{Bl}_\Delta(M \times M)} P \wedge \widetilde{\text{pr}}_2^* \eta_1 \wedge \widetilde{\text{pr}}_1^* \eta_2 \end{aligned}$$

for all  $\eta_1, \eta_2 \in \Omega(M)$ . This implies the following:

$$\begin{aligned} \tau^* P = (-1)^n P \quad \Longleftrightarrow \quad (\mathcal{P}(\eta_1), \eta_2) &= (-1)^{\eta_1} (\eta_1, \mathcal{P}(\eta_2)) \\ &\quad \forall \eta_1, \eta_2 \in \Omega(M). \end{aligned} \tag{4.8}$$

(iv) Because  $\text{Bl}_\Delta(M \times M)$  is compact,  $P \in \Omega(\text{Bl}_\Delta(M \times M))$  induces an  $L^1$ -integrable form on  $M \times M$ .

(v) In the original article [CFL15], Hodge propagator is called “Green kernel”. However, this term is standardly used to denote the Schwartz kernel of the generalized inverse of an elliptic pseudo-differential operator, e.g., of the Laplacian  $\Delta$ . Because this caused confusion in discussions, we decided to change it to Hodge propagator. This turned out to appear already in [CMR18]. This is how we discovered [CM10], where this problematic was also treated, and where we took the trick in the proof of Proposition 4.2.10 from.  $\triangleleft$

We will now prove three propositions which will allow us to rewrite (4.6) equivalently as a differential equation for  $P$  on  $M \times M \setminus \Delta$ .

**Proposition 4.2.6** (Identities for fiberwise integration). *In the situation of Definition 4.2.1, assume that  $F$  has a boundary  $\partial F$ . We orient  $\partial F$  using  $T_p F = N(p) \oplus T_p \partial F$*

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for  $p \in \partial F$ , where  $N$  is an outward pointing normal vector field. The following formulas hold for all  $\kappa \in \Omega_{\text{cv}}(E)$  and  $\eta \in \Omega_{\text{c}}(B)$ :

- The projection formula

$$\int^F (\kappa \wedge \pi^* \eta) = \left( \int^F \kappa \right) \wedge \eta,$$

- Stokes' formula

$$(-1)^F d \int^F \kappa = \int^F d\kappa - \int^{\partial F} \kappa,$$

where  $F$  in the exponent denotes the dimension of  $F$ .

*Proof.* The projection formula is proven by a straightforward calculation from the definition.

As for Stokes' formula, we get the oriented fiber bundle  $\partial E \rightarrow B$  with fiber  $\partial F$  by restricting an oriented trivialization of  $E$ . There are two orientations on  $\partial E$  — as the total space of  $\partial E \rightarrow B$  and as the boundary of  $E$ . They agree due to our orientation convention. Using standard Stokes' theorem, we get

$$\begin{aligned} (-1)^F \int_B d \left( \int^F \kappa \right) \wedge \eta &= (-1)^{\kappa+1} \int_B \left( \int^F \kappa \right) \wedge d\eta \\ &= (-1)^{\kappa+1} \int_E \kappa \wedge d\pi^* \eta \\ &= \int_E (d\kappa \wedge \pi^* \eta - d(\kappa \wedge \pi^* \eta)) \\ &= \int_B \left( \int^F d\kappa \right) \wedge \eta - \int_{\partial E} \kappa \wedge \pi^* \eta \\ &= \int_B \left( \int^F d\kappa - \int^{\partial F} \kappa \right) \wedge \eta. \end{aligned}$$

This proves the proposition.  $\square$

In what comes next, we will view the canonical projection  $\text{pr}_2 : M_1 \times M_2 \rightarrow M_2$  as an oriented fiber bundle such that the orientation of the total space agrees with the product orientation. The fiberwise integration for this bundle will be denoted by  $\int^{M_1}$ .

**Proposition 4.2.7** (Schwartz kernel of the harmonic projection). *Let  $M$  be an oriented closed  $n$ -dimensional Riemannian manifold. Let  $\nu_1, \dots, \nu_m$  be a homogenous basis of  $\mathcal{H}(M)$  which is orthonormal with respect to the  $L^2$ -inner product*

$$(\eta_1, \eta_2)_{L^2} := \int_M \eta_1 \wedge * \eta_2 \quad \text{for } \eta_1, \eta_2 \in \Omega(M),$$

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where  $*$  denotes the Hodge star. The smooth form  $H \in \Omega^n(M \times M)$  defined by

$$H := \sum_{i=1}^m (-1)^{n\nu_i} \text{pr}_1^*(\nu_i) \wedge \text{pr}_2^*(\nu_i) \quad (4.9)$$

satisfies the following properties:

(a) For all  $\eta \in \Omega(M)$ , we have

$$\pi_{\mathcal{H}}(\eta) = \int^{M_1} H \wedge \text{pr}_1^* \eta.$$

(b) The form  $H$  is closed and Poincaré dual to  $\Delta \subset M \times M$ .

(c) The following symmetry condition is satisfied:

$$\tau^* H = (-1)^n H. \quad (4.10)$$

*Proof.* (a) For the purpose of the proof, we denote  $\mathcal{H}(\eta) := \int^{M_1} H \wedge \text{pr}_1^* \eta$ . For every  $k = 1, \dots, m$ , we use the projection formula to compute

$$\begin{aligned} \mathcal{H}(\nu_k) &= \sum_{i=1}^m (-1)^{\nu_i n + \nu_i \nu_k} \int^{M_1} \text{pr}_1^*(\nu_i \wedge \nu_k) \wedge \text{pr}_2^*(\nu_i) \\ &= \sum_{i=1}^m (-1)^{\nu_i(n+\nu_k) + \nu_k(n+\nu_i)} \left( \int_M \nu_k \wedge \nu_i \right) \nu_i \\ &= \nu_k. \end{aligned}$$

It is easy to see that  $\mathcal{H}(\eta) \in \mathcal{H}(M)$  for all  $\eta \in \Omega(M)$ . Therefore,  $\mathcal{H}$  is a projection to  $\mathcal{H}(M)$ . Relations  $\mathcal{H}(d\eta) = \mathcal{H}(d^*\eta) = 0$  for all  $\eta \in \Omega(M)$  follow from the second line of the computation above with  $\nu_k$  replaced by  $d\eta$  and  $d^*\eta$  using that  $\text{im } d^* \oplus \text{im } d$  is  $L^2$ -orthogonal to  $\mathcal{H}(M)$ . We see that  $\mathcal{H} = \pi_{\mathcal{H}}$ .

(b) Using  $d \circ \mathcal{H} = \mathcal{H} \circ d = 0$  and Stokes' theorem, we get

$$\int^{M_1} dH \wedge \text{pr}_1^* \eta = (-1)^n d\mathcal{H}(\eta) - \mathcal{H}(d\eta) = 0 \quad \text{for all } \eta \in \Omega(M).$$

It follows that  $dH = 0$ . Using the Künneth formula, we can write a given  $\kappa \in \Omega(M \times M)$

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with  $d\kappa = 0$  as  $\kappa = \text{pr}_1^* \eta_1 \wedge \text{pr}_2^* \eta_2 + d\eta$  for some  $\eta_1, \eta_2 \in \mathcal{H}(M)$  and  $\eta \in \Omega(M)$ . Then

$$\begin{aligned} \int_{M \times M} H \wedge \kappa &= \int_{M \times M} H \wedge \text{pr}_1^* \eta_1 \wedge \text{pr}_2^* \eta_2 \\ &= \int_M \mathcal{H}(\eta_1) \wedge \eta_2 \\ &= \int_M \eta_1 \wedge \eta_2 = \int_{\Delta} \kappa. \end{aligned}$$

This shows that  $H$  is Poincaré dual to  $\Delta$ .

(c) It follows from the Hodge decomposition that

$$(\pi_{\mathcal{H}}(\eta_1), \eta_2) = (\pi_{\mathcal{H}}(\eta_1), \pi_{\mathcal{H}}(\eta_2)) = (\eta_1, \pi_{\mathcal{H}}(\eta_2)) \quad \text{for all } \eta_1, \eta_2 \in \Omega(M). \quad (4.11)$$

As in (iii) of Remark 4.2.5, one shows that this is equivalent to (4.10).  $\square$

**Proposition 4.2.8** (Differential condition). *Let  $M$  be an oriented closed  $n$ -dimensional Riemannian manifold. For  $P \in \Omega^{n-1}(\text{Bl}_{\Delta}(M \times M))$ , the following claims are equivalent:*

(1) *The operator  $\mathcal{P} : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$  defined by  $\mathcal{P}(\eta) := \int^{\text{Bl}_{*}M_1} P \wedge \widetilde{\text{pr}}_1^* \eta$  for  $\eta \in \Omega(M)$  is a Hodge homotopy.*

(2) *It holds*

$$dP = (-1)^n H \quad \text{on } (M \times M) \setminus \Delta. \quad (4.12)$$

*Proof.* Before we begin, note that (4.12) is equivalent to the equation  $d\tilde{P} = (-1)^n \pi^* H$  on  $\text{Bl}_{\Delta}(M \times M)$  for the extension  $\tilde{P}$  of  $P$ ; we denote  $\tilde{P}$  by  $P$  and  $\pi^* H$  by  $H$  by uniqueness.

We will first prove  $2) \implies 1)$ . Using Stokes' formula, we get for every  $\eta \in \Omega(M)$  the following:

$$\begin{aligned} d\mathcal{P}(\eta) &= d \int^{\text{Bl}_{*}M_1} P \wedge \widetilde{\text{pr}}_1^* \eta \\ &= (-1)^n \left( \int^{\text{Bl}_{*}M_1} d(P \wedge \widetilde{\text{pr}}_1^* \eta) - \int^{\partial \text{Bl}_{*}M_1} P \wedge \widetilde{\text{pr}}_1^* \eta \right) \\ &= \pi_{\mathcal{H}}(\eta) - \mathcal{P}(d\eta) + \int^{\partial \text{Bl}_{*}M_1} (-1)^{n+1} P \wedge \widetilde{\text{pr}}_1^* \eta. \end{aligned}$$

Since  $\widetilde{\text{pr}}_1 = \widetilde{\text{pr}}_2$  on  $\partial \text{Bl}_{\Delta}(M \times M)$ , we get with the help of the projection formula the following:

$$\int^{\partial \text{Bl}_{*}M_1} P \wedge \widetilde{\text{pr}}_1^* \eta = \int^{\partial \text{Bl}_{*}M_1} P \wedge \widetilde{\text{pr}}_2^* \eta = \left( \int^{\partial \text{Bl}_{*}M_1} P \right) \wedge \eta.$$

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We will show that the 0-form  $\int^{\partial \text{Bl}_* M_1} P$  is constant  $(-1)^n$ . Stokes' formula implies

$$\int^{\partial \text{Bl}_* M_1} P = \int^{\text{Bl}_* M_1} dP = (-1)^n \int^{\text{Bl}_* M_1} H.$$

Using that  $H$  is Poincaré dual to  $\Delta$ , we get for every  $\eta \in \Omega^n(M)$  the following:

$$\begin{aligned} \int_M \left( \int^{\text{Bl}_* M_1} H \right) \wedge \eta &= \int_{\text{Bl}_\Delta(M \times M)} H \wedge \widetilde{\text{pr}}_2^* \eta \\ &= \int_{M \times M} H \wedge \text{pr}_2^* \eta \\ &= \int_\Delta \text{pr}_2^* \eta = \int_M 1 \wedge \eta. \end{aligned}$$

The implication follows.

We will now prove  $1) \implies 2)$ . Assume that (4.6) holds and that  $P$  extends smoothly to the blow-up. Denote

$$K := (-1)^n dP - H \quad \text{and} \quad L := -1 + \int^{\partial \text{Bl}_*(M_1)} (-1)^n P.$$

Notice that  $L$  is a function on  $M$ . From the previous computations, we deduce that

$$\int^{\text{Bl}_*(M_1)} K \wedge \widetilde{\text{pr}}_1^* \eta = L\eta \quad \text{for all } \eta \in \Omega(M),$$

and hence

$$\int_{\text{Bl}_\Delta(M \times M)} K \wedge \widetilde{\text{pr}}_1^*(\eta_1) \wedge \widetilde{\text{pr}}_2^*(\eta_2) = \int_M L\eta_1 \wedge \eta_2 \quad \text{for all } \eta_1, \eta_2 \in \Omega(M).$$

If  $K(x, y) \neq 0$  for some  $(x, y) \in (M \times M) \setminus \Delta$ , we can choose  $\eta_1, \eta_2$  with disjoint supports such that the left-hand side is non-zero. This is a contradiction. Consequently, we have  $K \equiv 0$ .  $\square$

In general, the *Schwartz kernel* of a linear operator  $\mathcal{P} : \Omega(M) \rightarrow \Omega(M)$  is a distributional form  $P$  on  $M \times M$  which satisfies<sup>1</sup>

$$\mathcal{P}(\eta)(x) = \int_{y \in M_1} P(y, x) \eta(y) \quad \text{for all } \eta \in \Omega(M) \text{ and } x \in M_2.$$

We consider the following conditions on  $\mathcal{P}$  and  $P$ :

---

<sup>1</sup>We may consider such class of  $\mathcal{P}$ 's, e.g., pseudo-differential operators, such that  $P$  exists and is unique (c.f., the well-known Schwartz kernel theorem).

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- (G1) The Schwartz kernel  $P$  of  $\mathcal{P}$  is a restriction of a smooth form on  $\text{Bl}_\Delta(M \times M)$ .
- (G2)  $d \circ \mathcal{P} + \mathcal{P} \circ d = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1}$ .
- (G3)  $(\mathcal{P}(\eta_1), \eta_2) = (-1)^n (\eta_1, \mathcal{P}(\eta_2))$  for all  $\eta_1, \eta_2 \in \Omega(M)$ .
- (G4)  $\mathcal{P} \circ \iota_{\mathcal{H}} = 0, \pi_{\mathcal{H}} \circ \mathcal{P} = 0$ .
- (G5)  $\mathcal{P} \circ \mathcal{P} = 0$ .

Clearly, (P1)–(P3) are equivalent to  $P$  being a Hodge propagator from Definition 4.2.4. Conditions (P4) and (P5) play a crucial role in the vanishing results for the Chern-Simons Maurer-Cartan element  $\mathfrak{n}$  in Section 4.4 — the more conditions are satisfied, the more vanishing we get.

The following lemma will be used in the proof of the upcoming proposition.

**Lemma 4.2.9.** *Let  $\mathcal{P}_1, \mathcal{P}_2$  be two linear operators  $\Omega(M) \rightarrow \Omega(M)$  with Schwartz kernels  $P_1, P_2 \in \Omega(\text{Bl}_\Delta(M \times M))$ . Then  $\mathcal{P} := \mathcal{P}_1 \circ \mathcal{P}_2$  is a smoothing operator, i.e., its Schwartz kernel  $P$  is a smooth form on  $M \times M$ .*

*Proof.* It holds  $P(x_1, x_2) = \pm \int_x P_2(x_1, x) P_1(x, x_2)$ . The lemma follows from properties of convolution.  $\square$

A version of the following proposition can be found in [CM10].

**Proposition 4.2.10** (Existence of special Hodge propagator). *Every oriented closed Riemannian manifold  $M$  admits a special Hodge propagator.*

*Proof of Proposition 4.2.10.* Because  $H$  is Poincaré dual to  $\Delta$ , we have for any closed  $\kappa \in \Omega_c((M \times M) \setminus \Delta)$  the following:

$$\int_{(M \times M) \setminus \Delta} H \wedge \kappa = \int_{M \times M} H \wedge \kappa = \int_{\Delta} \kappa = 0.$$

Poincaré duality for non-compact oriented manifolds (see [BT82]) implies that  $H$  is exact on  $(M \times M) \setminus \Delta$ . Because a manifold with boundary is homotopy equivalent to its interior, the restriction of the blow-down map induces an isomorphism  $\pi^* : H_{\text{dR}}((M \times M) \setminus \Delta) \rightarrow H_{\text{dR}}(\text{Bl}_\Delta(M \times M))$ . It follows that  $(-1)^n \pi^* H$  admits a primitive  $P \in \Omega(\text{Bl}_\Delta(M \times M))$ . According to Proposition 4.2.8, the corresponding  $\mathcal{P}$  satisfies (P1) and (P2).

If we define

$$\tilde{P} := \frac{1}{2}(P + (-1)^n \tau^* P) \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M)),$$

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then  $\tilde{P}$  satisfies  $\tau^*\tilde{P} = (-1)^n\tilde{P}$  and is still a primitive to  $(-1)^n\pi^*H$ . Proposition 4.2.8 and (4.8) imply that the corresponding  $\mathcal{P}$  satisfies (P1)–(P3).

Given  $\mathcal{P}$  satisfying (P1)–(P3), we will now show that we can arrange (P4). Let us define

$$\tilde{\mathcal{P}} := (\mathbb{1} - \pi_{\mathcal{H}}) \circ \mathcal{P} \circ (\mathbb{1} - \pi_{\mathcal{H}}),$$

where we view  $\pi_{\mathcal{H}}$  as a map  $\pi_{\mathcal{H}} : \Omega \rightarrow \mathcal{H} \subset \Omega$ . Then  $\tilde{\mathcal{P}}$  is a Hodge homotopy because

$$d \circ \tilde{\mathcal{P}} + \tilde{\mathcal{P}} \circ d = (\mathbb{1} - \pi_{\mathcal{H}}) \circ (d \circ \mathcal{P} + \mathcal{P} \circ d) \circ (\mathbb{1} - \pi_{\mathcal{H}}) = \mathbb{1} - \pi_{\mathcal{H}}.$$

Using (4.11) and (4.8), we see that  $\tilde{\mathcal{P}}$  satisfies (P3). Using the intersection pairing and Proposition 4.2.7, we can write

$$\pi_{\mathcal{H}}(\eta) = \sum_{i=1}^m (-1)^{(n+\eta)\nu_i} (*\nu_i, \eta) \nu_i \quad \text{for all } \eta \in \Omega(M),$$

and hence we have for all  $\eta_1, \eta_2 \in \Omega(M)$  the following:

$$\begin{aligned} (\mathcal{P}(\pi_{\mathcal{H}}(\eta_1)), \eta_2) &= \sum_{i=1}^m (-1)^{(n+\eta_1)\nu_i} (*\nu_i, \eta_1) (\mathcal{P}(\nu_i), \eta_2) \\ &= \sum_{i=1}^m (-1)^{(n+1)\nu_i} \int_{M \times M} \text{pr}_1^>(*\nu_i) \wedge \text{pr}_2^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_1^*(\eta_1) \wedge \text{pr}_2^*(\eta_2). \end{aligned}$$

It follows that the Schwartz kernel of  $\mathcal{P} \circ \pi_{\mathcal{H}}$  is the smooth form

$$\mathcal{K}_{\mathcal{P} \circ \pi_{\mathcal{H}}} := \sum_{i=1}^m (-1)^{(n+1)\nu_i} \text{pr}_1^>(*\nu_i) \wedge \text{pr}_2^*(\mathcal{P}(\nu_i)).$$

Moreover, if we replace  $\mathcal{P}$  with  $\pi_{\mathcal{H}} \circ \mathcal{P}$ , we get the smooth Schwartz kernel  $\mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}}$  of  $(\pi_{\mathcal{H}} \circ \mathcal{P}) \circ \pi_{\mathcal{H}}$ . In the same way, but now using in addition (4.8), we can write

$$\begin{aligned} (\pi_{\mathcal{H}}(\mathcal{P}(\eta_1)), \eta_2) &= (-1)^{\eta_1} (\eta_1, \mathcal{P}(\pi_{\mathcal{H}}(\eta_2))) = (-1)^{\eta_1 \eta_2} (\mathcal{P}(\pi_{\mathcal{H}}(\eta_2)), \eta_1) \\ &= \sum_{i=1}^m (-1)^{\eta_1 \eta_2 + (n+1)\nu_i} \int_{M \times M} \text{pr}_1^>(*\nu_i) \wedge \text{pr}_2^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_1^*(\eta_2) \wedge \text{pr}_2^*(\eta_1) \\ &= \sum_{i=1}^m (-1)^{(n+1)\nu_i + n} \int_{M \times M} \text{pr}_2^>(*\nu_i) \wedge \text{pr}_1^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_1^*(\eta_1) \wedge \text{pr}_2^*(\eta_2), \end{aligned}$$

where in the last equality we pulled back the integral along the twist map. It follows

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that the Schwartz kernel of  $\pi_{\mathcal{H}} \circ \mathcal{P}$  is the smooth form

$$\mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P}} := \sum_{i=1}^m (-1)^{\nu_i} \text{pr}_1^*(\mathcal{P}(\nu_i)) \wedge \text{pr}_2^*(\nu_i).$$

The Schwartz kernel of  $\tilde{\mathcal{P}} = \mathcal{P} - \pi_{\mathcal{H}} \circ \mathcal{P} - \mathcal{P} \circ \pi_{\mathcal{H}} + \pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}$  is then

$$\tilde{\mathcal{P}} = \mathcal{P} - \pi^* \mathcal{K}_{\mathcal{P} \circ \pi_{\mathcal{H}}} - \pi^* \mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P}} + \pi^* \mathcal{K}_{\pi_{\mathcal{H}} \circ \mathcal{P} \circ \pi_{\mathcal{H}}},$$

which is a smooth form on  $\text{Bl}_{\Delta}(M \times M)$ . Therefore,  $\tilde{\mathcal{P}}$  satisfies (P1)–(P4).

Given  $\mathcal{P}$  satisfying (P1)–(P4), we will show that we can arrange (P5). The trick from [CM10] is to define

$$\tilde{\mathcal{P}} = \mathcal{P} d\mathcal{P}.$$

Applying (P1) and (P2) repeatedly, we compute

$$\begin{aligned} d\mathcal{P}\mathcal{P}\mathcal{P}d &= d\mathcal{P}\mathcal{P} - d\mathcal{P}\mathcal{P}d\mathcal{P} = d\mathcal{P}\mathcal{P} - d\mathcal{P}\mathcal{P} + d\mathcal{P}d\mathcal{P}\mathcal{P} \\ &= d\mathcal{P}\mathcal{P} - dd\mathcal{P}\mathcal{P} = d\mathcal{P}\mathcal{P} = \mathcal{P} - \mathcal{P}d\mathcal{P}, \end{aligned} \tag{4.13}$$

and hence

$$\tilde{\mathcal{P}} = \mathcal{P} - d\mathcal{P}\mathcal{P}\mathcal{P}d.$$

Clearly,  $\tilde{\mathcal{P}}$  satisfies (P1) and (P2). As for (P3), we compute

$$(\eta_1, \tilde{\mathcal{P}}\eta_2) = (-1)^{\eta_1} (\mathcal{P}\eta_1, d\mathcal{P}\eta_2) = (d\mathcal{P}\eta_1, \mathcal{P}\eta_2) = (-1)^{\eta_1} (\tilde{\mathcal{P}}\eta_1, \eta_2).$$

As for (P5), we have

$$\begin{aligned} \tilde{\mathcal{P}}\tilde{\mathcal{P}} &= \mathcal{P}d\mathcal{P}(\mathcal{P}d)\mathcal{P} = \mathcal{P}d\mathcal{P}\mathcal{P} - \mathcal{P}d(\mathcal{P}d)\mathcal{P}\mathcal{P} \\ &= \mathcal{P}d\mathcal{P}\mathcal{P} - \mathcal{P}d\mathcal{P}\mathcal{P} + \mathcal{P}dd\mathcal{P}\mathcal{P} = 0. \end{aligned} \tag{4.14}$$

In order to show (P4), we have to compute the Schwartz kernel of  $d\mathcal{P}\mathcal{P}\mathcal{P}d$ . By Lemma 4.2.9, the Schwartz kernel  $T$  of  $T := \mathcal{P}\mathcal{P}\mathcal{P}$  is a smooth form on  $M \times M$ . Therefore, Stokes' formula without the boundary term applies, and we get

$$(dTd)\eta = d \int^{M_1} T \wedge d\pi_1^*(\eta) = \int^{M_1} dT \wedge d\pi_1^*(\eta) = (-1)^T \int^{M_1} d_1 dT \wedge \pi_1^*(\eta).$$



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Here  $d_1 : \Omega(M \times M) \rightarrow \Omega(M \times M)$  is the operator defined in local coordinates by

$$d_1(f(x, y) dx^I dy^J) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x, y) dx^i dx^I dy^J.$$

It follows that the Schwartz kernel  $\tilde{P}$  of  $\tilde{\mathcal{P}}$  satisfies

$$\tilde{P} = P + (-1)^n d_1 dT$$

and is a smooth  $(n-1)$ -form on  $\text{Bl}_\Delta(M \times M)$ . Conditions (P1)–(P5) are satisfied.  $\square$

*Remark 4.2.11* (Property (P5) in dimensions 1 and 2). In dimension 1, every operator of degree  $-1$  satisfies (P5) from degree reasons. In dimension 2, every operator satisfying (P1) and (P2) satisfies (P5) as well, which follows from (4.13) and (4.14).  $\triangleleft$

*Remark 4.2.12* (The standard Hodge propagator). Consider the Hodge-de Rham Laplacian  $\Delta = d \circ d^* + d^* \circ d : \Omega(M) \rightarrow \Omega(M)$  and its “Green operator”  $\mathcal{G}$  of degree 0 (see (v) of Remark 4.2.5 for the collision of terminology) which was defined in [War83, Definition 6.9] by

$$\mathcal{G} := (\Delta|_{\mathcal{H}(M)^\perp})^{-1} \circ \pi_{\mathcal{H}(M)^\perp},$$

where  $\perp$  denotes the  $L^2$ -orthogonal complement. We introduce the *standard Hodge homotopy* by

$$\mathcal{P}_{\text{std}} := -d^* \circ \mathcal{P}_\Delta. \quad (4.15)$$

Using the properties of  $\mathcal{G}$ ,  $d$  and  $d^*$ , one can show that  $\mathcal{P}_{\text{std}}$  satisfies (P2)–(P5).

As for (P1), the author was able to show it for flat manifolds ( $:=$  locally isometric to  $\mathbb{R}^n$ ) by transforming the following formula inspired by [Har04] to blow-up coordinates and explicitly computing the integral and limit:

$$P_{\text{std}} = -\lim_{t \rightarrow 0} \int_t^\infty \frac{1}{2} d^* K_\tau d\tau,$$

where  $K_t(x, y) = \sum_i (-1)^{ne_i} e^{-\lambda_i t} (*e_i)(x) \wedge e_i(y)$  is the heat kernel of  $\Delta$  and  $e_i$  the  $L^2$ -orthonormal eigenbasis of  $\Delta$  with eigenvalues  $\lambda_i$  (the signs come from our convention for fiberwise integration, c.f., (4.9)).  $\triangleleft$

### 4.3. Chern-Simons Maurer-Cartan element

We first recall ribbon graphs and their labelings based on [CFL15].

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**Definition 4.3.1** (Ribbon graph). A graph  $\Gamma$  is a quadruple  $(V, H, \mathcal{V}, \mathcal{E})$ , where  $V$  is a finite set of vertices,  $H$  a finite set of half-edges,  $\mathcal{V} : H \rightarrow V$  the “vertex map” and  $\mathcal{E} : H \rightarrow H$  with  $\mathcal{E} \circ \mathcal{E} = \text{id}$  and without fixed points the “edge map”. The preimage  $\mathcal{E}^{-1}(h_1) = \{h_1, h_2\}$  for some  $h_1, h_2 \in H$  is called an edge; the set of edges is denoted by  $E$ . We assume that the graphs are connected, i.e., that for any  $v_1, v_2 \in V$  there exists a path in  $E$  connecting  $v_1$  to  $v_2$ .

A ribbon graph is a graph  $\Gamma$  together with a free transitive action  $\mathbb{Z}_{d(v)} \curvearrowright \mathcal{V}^{-1}(v)$  for every  $v \in V$ , where

$$d(v) := |\mathcal{V}^{-1}(v)|$$

is the valency of  $v$ . We denote by  $\mathcal{N} : H \rightarrow H$  the bijection induced by  $1 \in \mathbb{Z}_{d(v)}$  for every  $v \in V$ .

For a ribbon graph  $\Gamma$ , consider the set of sequences  $(h_n)_{n \in \mathbb{Z}} \subset H$  such that the following conditions holds:

$$\forall n \in \mathbb{Z} : \quad h_{n+1} = \begin{cases} \mathcal{E}(h_n) & n \text{ even,} \\ \mathcal{N}(h_n) & n \text{ odd.} \end{cases}$$

Two such sequences  $(h_n)_{n \in \mathbb{Z}}$  and  $(h'_n)_{n \in \mathbb{Z}}$  are equivalent if and only if there exist  $n_0, n'_0 \in \mathbb{Z}$  both even or both odd such that  $h_{n_0} = h'_{n'_0}$ . An equivalence class  $[(h_n)_{n \in \mathbb{Z}}]$  is called a boundary (or a boundary component) of  $\Gamma$ . The set of boundaries of  $\Gamma$  is denoted by  $\partial\Gamma$ .

An IE ribbon graph is a ribbon graph  $\Gamma$  together with the decomposition  $V = V_{\text{int}} \sqcup V_{\text{ext}}$  into internal and external vertices  $V_{\text{int}}$  and  $V_{\text{ext}}$ , respectively, such that  $d(v) = 1$  for all  $v \in V_{\text{ext}}$ . This decomposition induces the decomposition  $E = E_{\text{int}} \sqcup E_{\text{ext}}$ , where an edge  $e$  is internal if it connects two internal vertices and is external otherwise. We allow only graphs with at least one internal vertex. We often identify an external vertex with its unique adjacent half-edge or the unique adjacent external edge; we call either of these an external leg. For any  $b \in \partial\Gamma$ , we define the valency of  $b$  by

$$s(b) := |\mathcal{V}(b) \cap V_{\text{ext}}|,$$

where  $\mathcal{V}(b) = \{\mathcal{V}(h_n) \mid n \in \mathbb{Z}\}$ . We also have the free transitive  $\mathbb{Z}_{s(b)}$ -action on  $\mathcal{V}(b) \cap V_{\text{ext}}$  mapping  $v \in \mathcal{V}(b) \cap V_{\text{ext}}$  to the next external vertex in the sequence  $(\mathcal{V}(h_n))_{n \in \mathbb{Z}}$ . We will denote this action by  $\mathcal{N}$  again.

We say that an IE ribbon graph  $\Gamma$  is reduced if  $s(b) \geq 1$  for all  $b \in \partial\Gamma$ .

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The following letters will be used to denote the numerical invariants of a graph:

$k$	$\dots$	the number of internal vertices,
$s$	$\dots$	$\text{---}^{\#}\text{---}$ external vertices.
$l$	$\dots$	$\text{---}^{\#}\text{---}$ boundary components,
$e$	$\dots$	$\text{---}^{\#}\text{---}$ internal edges.

Moreover, we define the genus  $g \in \mathbb{N}_0$  so that the following Euler formula holds:

$$k - e + l = 2 - 2g. \quad (4.16)$$

We denote by  $\text{RG}_{klg}$  the set of isomorphism classes of connected IE ribbon graphs with fixed  $k, l, g$ . We let  $\overline{\text{RG}}_{klg} \subset \text{RG}_{klg}$  be the subset of reduced graphs. For  $m \in \mathbb{N}_0$ , we denote by  $\text{RG}_{klg}^{(m)} \subset \text{RG}_{klg}$  the set of isomorphism classes of connected IE ribbon graphs with all internal vertices  $m$ -valent, i.e., with

$$d(v) = m \quad \text{for all } v \in V_{\text{int}}.$$

The notation  $\Gamma \in \text{RG}_{klg}$  means that  $\Gamma$  is a representative of an equivalence class  $[\Gamma] \in \text{RG}_{klg}$ .

**Remark 4.3.2** (On ribbon graphs). (i) An  $m$ -valent ribbon graph with  $m \geq 2$  has a unique decomposition  $V = V_{\text{int}} \sqcup V_{\text{ext}}$ , and hence we can omit writing IE.

(ii) In this text, we will use only reduced ribbon graphs. Non-reduced ribbon graphs may play a role in the extension of the theory of  $\text{dIBL}^n(C(\mathcal{H}))$  to non-reduced cyclic cochains or in the weak  $\text{IBL}_{\infty}$ -theory (see Remarks 3.2.10 and 3.3.2).  $\triangleleft$

**Definition 4.3.3** (Labeling). A labeling of an IE ribbon graph  $\Gamma$  is the triple  $L = (L_1, L_2, L_3)$ , where  $L_i$  have the following meanings:

- The symbol  $L_1$  represents an ordering of internal vertices ( $=: L_1^v$ ), and of boundary components ( $=: L_1^b$ ). Given  $L_1$ , we write  $V_{\text{ext}} = \{v_1, \dots, v_k\}$ ,  $\partial\Gamma = \{b_1, \dots, b_l\}$  and denote

$$d_i := d(v_i) \quad \text{and} \quad s_j := s(b_j).$$

- The symbol  $L_2$  represents an ordering and orientation of internal edges. Given  $L_2$ , we write  $E_{\text{int}} = \{e_1, \dots, e_e\}$  and  $e_i = \{h_{i,1}, h_{i,2}\}$  for  $h_{i,1}, h_{i,2} \in H$ .
- The symbol  $L_3$  represents an ordering of half-edges at every internal vertex ( $=: L_3^v$ ) and of external vertices at every boundary component ( $=: L_3^b$ ), both compatible

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with the ribbon structure ( $\text{:= the } \mathbb{Z}_m\text{-actions}$ ). Given  $L_3$ , we write  $\mathcal{V}^{-1}(\mathbf{v}) = \{h_{\mathbf{v},1}, \dots, h_{\mathbf{v},d(\mathbf{v})}\}$  and  $\mathcal{V}(\mathbf{b}) \cap V_{\text{ext}} = \{v_{\mathbf{b},1}, \dots, v_{\mathbf{b},s(\mathbf{b})}\}$  with  $\mathcal{N}(h_{\mathbf{v},i}) = h_{\mathbf{v},i+1}$  and  $\mathcal{N}(v_{\mathbf{b},j}) = v_{\mathbf{b},j+1}$  for all  $i, j$ , respectively.

We sometimes call  $L_i$  partial labelings and  $L$  a full labeling. A ribbon graph  $\Gamma$  together with a labeling  $L$  is called a labeled ribbon graph.

Given a ribbon graph  $\Gamma$ , one can construct an oriented surface with boundary  $\Sigma_\Gamma$  — the thickening of  $\Gamma$  — in the obvious way and a closed oriented surface  $\hat{\Sigma}_\Gamma$  by gluing oriented disks to the oriented boundaries of  $\Sigma_\Gamma$ . If partial labelings  $L_1$  and  $L_2$  are given, we obtain the following chain complex with oriented chain groups (vector spaces over  $\mathbb{R}$ ):

$$C_2 := \langle b_1, \dots, b_l \rangle \xrightarrow{\partial_2} C_1 := \langle e_1, \dots, e_e \rangle \xrightarrow{\partial_1} C_0 := \langle v_k, \dots, v_1 \rangle. \quad (4.17)$$

Here  $b_i$  stands for the oriented disc glued to the  $i$ -th boundary component of  $\Sigma_\Gamma$  and now being mapped into  $\hat{\Sigma}_\Gamma$ ,  $e_i$  stands for the 1-simplex in  $\hat{\Sigma}_\Gamma$  corresponding to the  $i$ -th internal edge,  $v_i$  stands for the 0-simplex in  $\hat{\Sigma}_\Gamma$  corresponding to the  $i$ -th internal vertex, and the boundary map  $\partial$  is the “geometric” boundary operator. The homology of this chain complex is isomorphic to the singular homology  $H(\hat{\Sigma}) := H(\hat{\Sigma}_\Gamma; \mathbb{R})$ .

The orientation of  $C_i$  ( $\text{:= the order of generators in (4.17)}$ ) induces naturally an orientation of  $H(\hat{\Sigma}_\Gamma)$ . The construction from [CFL15, Appendix A] is as follows. We pick complements  $H_i$  of  $\text{im}(\partial_{i+1})$  in  $\ker(\partial_i)$  and complements  $V_i$  of  $\ker(\partial_i)$  in  $C_i$  and write

$$C_2 = V_2 \oplus H_2 \xrightarrow{\partial_2} C_1 = V_1 \oplus H_1 \oplus \text{im}(\partial_2) \xrightarrow{\partial_1} C_0 = \text{im}(\partial_1) \oplus H_0.$$

We orient  $V_i$  arbitrarily and transfer the orientation to  $\text{im}(\partial_i)$  via  $\partial_i : V_i \xrightarrow{\sim} \text{im}(\partial_i)$ . Then, assuming the direct sum orientation, orienting  $H_i$  is equivalent to orienting  $C_i$ , and we obtain the orientation of  $H_i(\hat{\Sigma}_\Gamma)$  via the canonical projection  $\pi : H_i \xrightarrow{\sim} H_i(\hat{\Sigma}_\Gamma) = \ker(\partial_i) / \text{im}(\partial_{i+1})$ . This construction does not depend on the choices of complements and orientations of  $V_i$ .

**Definition 4.3.4** (Compatibility of  $L_1$  and  $L_2$ ). *Given a ribbon graph  $\Gamma$  with partial labelings  $L_1$  and  $L_2$ , we say that  $L_2$  is compatible with  $L_1$  if the orientation on  $H(\hat{\Sigma}_\Gamma)$  induced by (4.17) agrees with the canonical orientation*

$$H(\hat{\Sigma}_\Gamma) = \langle v_1 + \dots + v_k \rangle \oplus H_1(\hat{\Sigma}_\Gamma) \oplus \langle b_1 + \dots + b_l \rangle,$$

where  $H_1(\hat{\Sigma}_\Gamma)$  is oriented using the canonical symplectic intersection form.

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Given a labeled IE ribbon graph  $\Gamma$ , the set of half-edges adjacent to internal vertices  $\mathcal{V}^{-1}(V_{\text{int}})$  can be ordered in two ways corresponding to writing

$$2e + (s_1 + \cdots + s_l) = d_1 + \cdots + d_k.$$

This leads to the following definition.

**Definition 4.3.5** (Edge order and vertex order). *For a labeled IE ribbon graph  $\Gamma$ , we define the following two orders on the set of half-edges  $H$ :*

- Edge order: *The first  $2e$  half-edges  $h_{i,j}^e$  are the ones from internal edges; they are ordered according to  $L_2$ . They are followed by blocks of  $s_1, \dots, s_l$  half-edges  $h_{i,j}^b$  which come from the boundary components  $i = 1, \dots, l$ , respectively, and which are ordered according to  $L_3^b$  inside the blocks. Schematically, we have*

$$(h_{1,1}^e h_{1,2}^e) \cdots (h_{e,1}^e h_{e,2}^e) (h_{1,1}^b \cdots h_{1,s_1}^b) \cdots (h_{l,1}^b \cdots h_{l,s_l}^b).$$

- Vertex order: *It consists of blocks of  $d_1, \dots, d_k$  half-edges  $h_{i,j}^v$  which come from internal vertices  $1, \dots, k$ , and which are ordered according to  $L_3^v$  inside the blocks. Schematically, we have*

$$(h_{1,1}^v \cdots h_{1,d_1}^v) \cdots (h_{k,1}^v \cdots h_{k,d_k}^v).$$

We denote by  $\sigma_L \in \mathbb{S}_{|H|}$  the permutation from the edge to the vertex order which is constructed such that the  $i$ -th half-edge in the edge order is the same as the  $\sigma_L(i)$ -th half-edge in the vertex order.

From now on, we will consider only reduced trivalent ribbon graphs  $\overline{\text{RG}}_{klg}^{(3)}$  with  $k, l \geq 1, g \geq 0$ . We will often use the equation

$$2e + s = 3k. \tag{4.18}$$

**Definition 4.3.6** (Chern-Simons Maurer-Cartan element). *Let  $M$  be an oriented closed Riemannian manifold, and let  $P \in \Omega^{n-1}(\text{Bl}_\Delta(M \times M))$  be an admissible Hodge propagator from Definition 4.2.4. The Chern-Simons Maurer-Cartan element  $\mathfrak{n}$ , or formal pushforward Maurer-Cartan element, is the collection of*

$$\mathfrak{n}_{lg} \in \hat{E}_l C(\mathcal{H}(M)) \quad \text{for all } l \geq 1, g \geq 0$$

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defined on generating words  $\omega_i = \alpha_{i1} \dots \alpha_{is_i} \in B^{\text{cyc}}\mathcal{H}(M)$ , where  $\alpha_{ij} = \theta\eta_{ij}$  with  $\eta_{ij} \in \mathcal{H}(M)$  for  $s_i \geq 1$  and  $i = 1, \dots, l$ , by the formula

$$\begin{aligned} \mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \omega_l) \\ := \frac{1}{l!} \sum_{[\Gamma] \in \overline{\text{RG}}_{klg}^{(3)}} \frac{1}{|\text{Aut}(\Gamma)|} (-1)^{s(k,l)+P(\omega)} \sum_{L_1, L_3^b} (-1)^{\sigma_L} I(\sigma_L), \end{aligned} \quad (4.19)$$

which we explain as follows:

- The second sum is over all partial labelings  $L_1$  and  $L_3^b$  of a representative  $\Gamma$  of  $[\Gamma]$ . In every summand, we complete  $L_1$  and  $L_3^b$  to a full labeling  $L = (L_1, L_2, L_3)$  by picking an arbitrary  $L_3^v$  and an arbitrary  $L_2$  compatible with  $L_1$ .
- Suppose that  $\Gamma$  and  $L_1$  are admissible with respect to the input  $\omega_1, \dots, \omega_l$ ; this means that  $\Gamma$  has  $l$  boundary components and that the  $i$ -th boundary component has valency  $s_i$  for every  $i = 1, \dots, l$ . In this case, denoting  $\sigma = \sigma_L$ , we define

$$\begin{aligned} I(\sigma_L) := \int_{x_1, \dots, x_k} P(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots P(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) \\ \eta_{11}(x_{\xi(\sigma_{2e+1})}) \cdots \eta_{ls_l}(x_{\xi(\sigma_{2e+s})}), \end{aligned} \quad (4.20)$$

where  $\xi : \{1, \dots, 3k\} \rightarrow \{1, \dots, k\}$  is the function defined by

$$\xi(3j-2) = \xi(3j-1) = \xi(3j) := j$$

for all  $j = 1, \dots, k$ ,  $s = s_1 + \dots + s_l$ ,  $\eta(x_i)$  denotes the pullback of  $\eta$  along the canonical projection  $\pi_i : M^{\times k} \rightarrow M$  to the  $i$ -th component  $M_i$ ,  $P(x_i, x_j)$  denotes the pullback of  $P$  along  $\pi_i \times \pi_j : M^{\times k} \rightarrow M_i \times M_j$ , and  $\int_{x_1, \dots, x_k}$  denotes the integral of an  $nk$ -form over  $k$  copies of  $M$ .

If  $\Gamma$  and  $L_1$  are not admissible, then we set  $I(\sigma_L) := 0$ .

- $s(k, l) := k + kl(n-1) + \frac{1}{2}k(k-1)n \pmod{2}$ .
- $P(\omega) := \sum_{i=1}^l \sum_{j=1}^{s_i} (s - s_1 - \dots - s_{i-1} - j) \eta_{ij} \pmod{2}$ .

In order to show that  $\mathbf{n}_{lg}$  is well-defined and that the collection  $(\mathbf{n}_{lg})$  satisfies Definition 3.2.6 for  $\text{dIBL}(C(\mathcal{H}(M)))$ , there are several things to check:

- (1) The integral  $I(\sigma_L)$  converges.
- (2) The sums are finite.

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- (3) The product  $(-1)^{\sigma_L} I(\sigma_L)$  is independent of the choice of  $L_3^v$  and  $L_2$  compatible with  $L_1$ .
- (4) The sum over labelings is independent of the chosen representative  $\Gamma$  in an isomorphism class from  $\overline{\text{RG}}_{klg}^{(3)}$ .
- (5) The map  $\mathfrak{n}_{lg} : \text{B}^{\text{cyc}} \mathcal{H}(M)[3-n]^{\otimes l} \rightarrow \mathbb{R}$  is graded symmetric on permutations of its inputs  $s\omega_i$ .
- (6) The map  $\mathfrak{n}_{lg}$  is graded symmetric on cyclic permutations of the components  $\alpha_{ij}$  of each  $\omega_i$ .
- (7) The degree condition 1) from Definition 3.2.6 holds with  $d = n - 3$ .
- (8) The filtration-degree condition 2) from Definition 3.2.6 holds with  $\gamma = 2$ .
- (9) The Maurer-Cartan equation (3.20) holds.

Conditions 1) and 9) will be proven in [CVona] using the theory of iterated blow-ups. In this text, we will take 1) and 9) for granted.

**Lemma 4.3.7.** *Assuming 1), the conditions 2) – 8) hold.*

*Proof.* As for 2), the fixed input  $\omega_1, \dots, \omega_l$  fixes the number  $s$  of external vertices of  $\Gamma$  by admissibility. Expressing  $e$  from (4.16) and plugging it in (4.18) gives

$$k = s + 2l + 4g - 4. \quad (4.21)$$

We see that all parameters are fixed. Now, there is only finitely many elements with fixed  $s$  in  $\overline{\text{RG}}_{klg}^{(3)}$ , and each of them has only finitely many labelings. Therefore, the sums are finite.

As for 3), we have to consider the orientation of the complex (4.17). Clearly, if two  $L_2$ 's are compatible with  $L_1$ , then they differ by an even number of the following operations: a transposition of two internal edges or a change of the orientation of an internal edge. The former operation introduces no sign in  $(-1)^{\sigma_L}$  but generates the sign  $(-1)^{n-1}$  in  $I(\sigma_L)$  from swapping the corresponding P's. The latter operation induces the sign  $-1$  in  $(-1)^{\sigma_L}$  and the sign  $(-1)^n$  in  $I(\sigma_L)$  from the symmetry  $P(x, y) = (-1)^n P(y, x)$ . Because the overall signs in  $(-1)^{\sigma_L} I(\sigma_L)$  are the same, an even number of these operations preserves  $(-1)^{\sigma_L} I(\sigma_L)$ . This implies the independence of an  $L_2$  compatible with  $L_1$ . A change in  $L_3^v$  produces no sign in  $(-1)^{\sigma_L}$  because every internal vertex is trivalent and a cyclic permutation of an odd number of elements is even. The integral  $I(\sigma_L)$

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remains unchanged because the change in  $\sigma_L$  is compensated by the composition with  $\xi$ . Independence of the choice of  $L_3^v$  follows.

As for 4), every isomorphism of ribbon graphs  $\Gamma \rightarrow \Gamma'$  induces the bijection  $L \mapsto L'$  of compatible labelings such that  $\sigma_L = \sigma_{L'}$  ( $L'$  is the “pushforward” labeling). The independence of the choice of a representative of  $[\Gamma]$  follows.

As for 5), let  $\mu \in \mathbb{S}_l$  be a permutation of the inputs  $s\omega_1, \dots, s\omega_l$ . The set of graphs which admit an admissible labeling is the same for both  $\mathbf{n}_{lg}(s^l\omega_1 \otimes \dots \otimes \omega_l)$  and  $\mathbf{n}_{lg}(s^l\omega_{\sigma_1^{-1}} \otimes \dots \otimes \omega_{\sigma_l^{-1}})$ ; we will pick one such  $\Gamma$  and study the admissible labelings  $L$  and  $L'$ , respectively. We write  $\eta_i = \eta_{i1} \dots \eta_{is_i}$  and  $\Omega_i = s\omega_i$  for all  $i, j$ , and denote by  $I'(\sigma_{L'})$  the integral in the definition of  $\mathbf{n}_{lg}(s^l\omega_{\mu_1^{-1}} \otimes \dots \otimes \omega_{\mu_l^{-1}})$ . Let  $\tilde{\mu} \in \mathbb{S}_{3k}$  be the permutation which acts as the identity on  $1, \dots, 2e$  and as the block permutation determined by  $\mu$  on  $2e+1, \dots, 2e+s$  divided into  $l$  blocks of lengths  $s_1, \dots, s_l$ . For any  $\sigma \in \mathbb{S}_{3k}$ , we have

$$\begin{aligned} I'(\sigma) &= \int_{x_1, \dots, x_k} P(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots P(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) \\ &\quad \eta_{\mu_1^{-1}1}(x_{\xi(\sigma_{2e+1})}) \cdots \eta_{\mu_l^{-1}s_{\mu_l^{-1}}}(x_{\xi(\sigma_{2e+s})}) \\ &= \varepsilon(\mu, \eta) \int_{x_1, \dots, x_k} P(x_{\xi((\sigma \circ \tilde{\mu})_1)}, x_{\xi((\sigma \circ \tilde{\mu})_2)}) \cdots P(x_{\xi((\sigma \circ \tilde{\mu})_{2e-1})}, x_{\xi((\sigma \circ \tilde{\mu})_{2e})}) \\ &\quad \eta_{11}(x_{\xi((\sigma \circ \tilde{\mu})_{2e+1})}) \cdots \eta_{ls_l}(x_{\xi((\sigma \circ \tilde{\mu})_{2e+s})}) \\ &= \varepsilon(\mu, \eta) I(\sigma \circ \tilde{\mu}). \end{aligned}$$

The precomposition with  $\tilde{\mu}$  corresponds to a bijection  $(L_1, L_3^b) \mapsto (L'_1, L_3^{b'})$  of partial labelings for  $\mathbf{n}_{lg}(s^l\omega_1 \dots \omega_l)$  and  $\mathbf{n}_{lg}(s^l\omega_{\mu_1^{-1}} \otimes \dots \otimes \omega_{\mu_l^{-1}})$ , respectively. However, if  $L_2$  is compatible with  $L_1$ , then in order to get an  $L'_2$  compatible with  $L'_1$ , the labeling  $L_2$  has to be altered by as many operations of switching two internal edges or changing the orientation of an internal edge as there are transpositions in  $\mu$ . We explained in the proof of 3) that this produces the sign  $(-1)^{(n-1)\mu}$  in  $(-1)^{\sigma_{L'}} I(\sigma_{L'})$ . Therefore, after the choice of compatible  $L_2$  and  $L'_2$ , we have

$$(-1)^{\sigma_{L'}} I'(\sigma_{L'}) = (-1)^{(n-1)\mu} (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta) (-1)^{\sigma_L} I(\sigma_L).$$

If we view  $\eta$  as  $\eta_{11} \dots \eta_{ls_l}$ , we can understand  $(-1)^{P(\omega)}$  as the Koszul sign  $\varepsilon(\theta, \eta)$ . Similarly, we write  $(-1)^{P(\mu(\omega))} = \varepsilon(\theta, \mu(\eta))$ , where we first view  $\eta$  as  $\eta_1 \otimes \dots \otimes \eta_l$  to apply  $\mu$  and then as the list of components  $\eta_{ij}$  to compute the Koszul sign (this is a little ambiguity in our notation). If we denote by  $\bar{\mu}$  the permutation of  $1, \dots, s$  permuting the  $l$  blocks of lengths  $s_1, \dots, s_l$  according to  $\mu$ , then  $\bar{\mu}$  has the same sign as  $\tilde{\mu}$ , and the



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decomposition of  $\varepsilon(\theta, \mu(\eta))$  into the moves

$$\begin{aligned} \theta_1 \dots \theta_s \eta_{\mu_1^{-1}1} \dots \eta_{\mu_l^{-1}s_{\mu_l^{-1}}} &\xrightarrow{(1)} \theta_{\bar{\mu}_1} \dots \theta_{\bar{\mu}_s} \eta_{11} \dots \eta_{ls_l} \xrightarrow{(2)} \theta_{\bar{\mu}_1} \eta_{11} \dots \theta_{\bar{\mu}_s} \eta_{ls_l} \\ &\xrightarrow{(3)} \theta_1 \eta_{\mu_1^{-1}1} \dots \theta_s \eta_{\mu_l^{-1}s_l} \end{aligned}$$

shows that

$$(-1)^{P(\mu(\omega))} = \underbrace{(-1)^{\tilde{\mu}} \varepsilon(\mu, \eta)}_{(1)} \underbrace{(-1)^{P(\omega)}}_{(2)} \underbrace{\varepsilon(\mu, \omega)}_{(3)}.$$

Using this, we write

$$(-1)^{P(\mu(\omega))} (-1)^{\sigma_{L'}} I'(\sigma_{L'}) = \varepsilon(\mu, \omega) (-1)^{(n-1)\mu} (-1)^{P(\omega)} (-1)^{\sigma_L} I(\sigma_L),$$

and compute

$$\begin{aligned} \mathbf{n}_{lg}(\Omega_{\mu_1^{-1}} \otimes \dots \otimes \Omega_{\mu_l^{-1}}) &= \varepsilon(\mu(s), \mu(\omega)) \mathbf{n}_{lg}(s^l \omega_{\mu_1^{-1}} \otimes \dots \otimes \omega_{\mu_l^{-1}}) \\ &= \varepsilon(\mu(s), \mu(\omega)) (-1)^{|s|\mu} \varepsilon(\mu, \omega) \mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \omega_l) \\ &= \underbrace{\varepsilon(\mu(s), \mu(\omega))}_{(1)} \underbrace{(-1)^{|s|\mu} \varepsilon(\mu, \omega)}_{(2)} \underbrace{\varepsilon(s, \omega)}_{(3)} \mathbf{n}_{lg}(s\omega_1 \otimes \dots \otimes s\omega_l) \\ &= \varepsilon(\mu, \Omega) \mathbf{n}_{lg}(\Omega_1 \otimes \dots \otimes \Omega_l). \end{aligned}$$

We used  $|s| = n - 1 \bmod 2$ , and the last equality follows from the decomposition of  $\varepsilon(\mu, \Omega)$  into the moves

$$\begin{aligned} s_1 \omega_1 \dots s_l \omega_l &\xrightarrow{(3)} s_1 \dots s_l \omega_1 \dots \omega_l \xrightarrow{(2)} s_{\mu_1^{-1}} \dots s_{\mu_l^{-1}} \omega_{\mu_1^{-1}} \dots \omega_{\mu_l^{-1}} \\ &\xrightarrow{(1)} s_{\mu_1^{-1}} \omega_{\mu_1^{-1}} \dots s_{\mu_l^{-1}} \omega_{\mu_l^{-1}}. \end{aligned}$$

This proves the symmetry of  $\mathbf{n}_{lg}$ .

As for 6), fix an  $i = 1, \dots, l$  and let  $\mu \in \mathbb{S}_{s_i}$  be a cyclic permutation permuting the components of  $\omega_i = \alpha_{i1} \dots \alpha_{is_i}$ . Similarly to the previous case, we denote by  $\tilde{\mu}$  the corresponding permutation of  $1, \dots, 3k$  and get a bijection  $(L_1, L_3^b) \mapsto (L'_1 = L_1, L_3^{b'})$  of admissible labelings of a given graph  $\Gamma$  for  $\mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \alpha_{i1} \dots \alpha_{is_i} \otimes \dots \otimes \omega_l)$  and  $\mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \alpha_{i\mu_1^{-1}} \dots \alpha_{i\mu_{s_i}^{-1}} \otimes \dots \otimes \omega_l)$ , respectively. This time, there is no change in

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$L_1$ , and thus we can take  $L'_2 = L_2$ , producing no sign. Therefore, we have

$$(-1)^{\sigma_{L'}} I'(\sigma_{L'}) = (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta_i) (-1)^{\sigma_L} I(\sigma_L),$$

where  $\varepsilon(\mu, \eta_i)$  comes from permuting the forms in  $I'(\sigma_{L'})$ . Further, we deduce

$$(-1)^{P(\mu(\omega))} = (-1)^{\tilde{\mu}} \varepsilon(\mu, \eta_i) (-1)^{P(\omega)} \varepsilon(\mu, \omega_i),$$

and hence

$$\begin{aligned} \mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \alpha_{i\mu_1^{-1}} \cdots \alpha_{i\mu_{s_i}^{-1}} \otimes \cdots \otimes \omega_l) \\ = \varepsilon(\mu, \omega_i) \mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \alpha_{i1} \cdots \alpha_{is_i} \otimes \cdots \otimes \omega_l). \end{aligned}$$

This shows the symmetry of  $\mathbf{n}_{lg}$  on cyclic permutations of the components of  $\omega_i$ .

As for 7), suppose that  $\mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \omega_l) \neq 0$ , and let  $D$  denote the total form-degree of the input  $\eta_{11}, \dots, \eta_{ls_l} \in \mathcal{H}(M)$ ; i.e., we define

$$D := \deg(\eta_{11}) + \cdots + \deg(\eta_{1s_1}) + \cdots + \deg(\eta_{l1}) + \cdots + \deg(\eta_{ls_l}).$$

Clearly, we must have

$$nk = (n-1)e + D, \tag{4.22}$$

where the left-hand side is the dimension of  $M^{\times k}$  and the right-hand side the form-degree of the integrand of  $I(\sigma_L)$ . If we plug in  $e$  from (4.16) and  $k$  from (4.21), we get

$$\begin{aligned} D &= nk - (n-1)e \\ &= nk - (n-1)(k+l+2g-2) \\ &= k - (n-1)(l+2g-2) \\ &= s + 2l + 4g - 4 - (n-1)(l+2g-2) \\ &= s - (n-3)(l+2g-2). \end{aligned}$$

It follows that

$$|\mathbf{n}_{lg}| = |s^l| + |\omega_1| + \cdots + |\omega_l| = l(n-3) + D - s = -2(n-3)(g-1).$$

This is exactly the degree from Definition 3.2.6.

As for 8), if  $\mathbf{n}_{lg}(s^l \omega_1 \otimes \cdots \otimes \omega_l) \neq 0$ , then

$$s = k - 2l - 4g - 4 \geq 1 + 2(2 - 2g - l) = 1 + 2\chi_{0lg},$$

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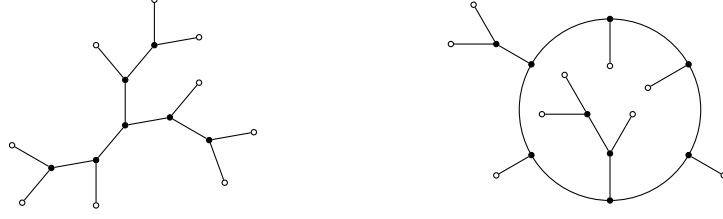


Figure 4.1.: A tree and a circular graph. Internal vertices are denoted with a full dot and external vertices with an empty dot.



Figure 4.2.: The Y-graph and an  $O_6$ -graph.

and hence  $\mathbf{n}_{lg} \in \mathcal{F}^{1+2\chi_{0lg}} \hat{\mathbf{E}}_l C$  for the filtration induced from the dual of the filtration of  $\mathbf{B}^{\text{cyc}} \mathcal{H}$  by weights. Therefore, we get

$$\|\mathbf{n}_{lg}\| \geq 1 + 2\chi_{0lg} > 2\chi_{0lg} \quad \text{for all } l \geq 1, g \geq 0.$$

This finishes the proof. □

Superfluous signs: Add here a remark that we do not have to sum over  $L_1^{v!}$ .

**Definition 4.3.8** (Vertices of types A, B, C and some special graphs). *Let  $\Gamma \in \text{RG}_{klg}^{(3)}$  be a trivalent ribbon graph and  $v$  its internal vertex. We say that  $v$  is of type A, B or C if it is connected to precisely 1, 2 or 3 internal vertices, respectively (see Figure 4.3). The graph  $\Gamma$  is called (see Figures 4.1 and 4.2):*

- a tree if  $[\Gamma] \in \text{RG}_{k10}$  for some  $k \geq 1$ ;
- circular if  $[\Gamma] \in \text{RG}_{k20}$  for some  $k \geq 1$ ;
- the Y-graph is the unique tree with  $k = 1$ ;
- an  $O_k$ -graph if  $\Gamma$  is circular with  $k$  internal vertices and no A-vertex.

We denote the Y-graph simply by  $Y$ .

*Remark 4.3.9* (On A, B, C vertices and special graphs). We observe the following:

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(i) A trivalent graph  $\Gamma \neq Y$  has each internal vertex of type  $A$ ,  $B$  or  $C$ .

(ii) The term  $\mathbf{n}_{10}$  is a sum over trees, and the term  $\mathbf{m}_{10}$  is the contribution of the  $Y$ -graph to  $\mathbf{n}_{10}$  (see Proposition 4.3.10 below). The term  $\mathbf{n}_{20}$  is a sum over circular graphs.  $\triangleleft$

We will also denote by  $A$ ,  $B$ ,  $C$  the numbers of internal vertices of the corresponding type. Under the change of variables

$$\begin{aligned} s &= 2A + B, \\ e &= B + \frac{1}{2}A + \frac{3}{2}C, \\ k &= A + B + C, \end{aligned} \tag{4.23}$$

the trivalent formula (4.18) becomes trivial and the Euler formula (4.16) becomes

$$C - A = 2l - 4 + 4g. \tag{4.24}$$

**Proposition 4.3.10** (Chern-Simons Maurer-Cartan element). *The collection  $\mathbf{n} = (\mathbf{n}_{lg})$  from Definition 4.3.6 is a Maurer-Cartan element for  $\mathrm{dIBL}(\mathcal{H}(M))$  which is compatible with  $\mathbf{m}$ . In particular, the  $A_\infty$ -algebra  $\mathcal{H}(M)_{\mathbf{n}}$  is homologically unital and augmented.*

*Proof.* The fact that  $\mathbf{n}$  is a Maurer-Cartan element follows from Lemma 4.3.7 assuming 1) and 9) from [CVona].

As for the compatibility with  $\mathbf{m}$ , the only graph contributing to  $\mathbf{n}_{10}(s\alpha_1\alpha_2\alpha_3)$  is the  $Y$ -graph with  $k = 1$ . The group  $\mathrm{Aut}(Y)$  consists of three rotations, and there is only one possible  $L_1$ , no  $L_2$  and three  $L_3^b$ . In Definition 4.3.6, we get  $s(1, 1) = n - 2$ ,  $(-1)^{\sigma_L} = 1$  because a cyclic permutation of an odd number of elements is even, and also  $P(\alpha_1\alpha_2\alpha_3) = \eta_2$ . Finally, we compute

$$\begin{aligned} \mathbf{n}_{10}(s\alpha_1\alpha_2\alpha_3) &= \frac{1}{3}(-1)^{n-2+\eta_2} \sum_{L_3^b} \int_x \alpha_1(x_{\xi(\sigma_1)})\alpha_2(x_{\xi(\sigma_2)})\alpha_3(x_{\xi(\sigma_3)}) \\ &= (-1)^{n-2+\eta_2} \int_M \eta_1 \wedge \eta_2 \wedge \eta_3 \\ &= \mathbf{m}_{10}(s\alpha_1\alpha_2\alpha_3). \end{aligned}$$

DONE Redundant check of vanishing for  $w \leq 2$ : Change it here because the definition of compatible changed!! In particular, the vanishing on words of length 1 and 2 follows from the filtration degree condition. Do not have to show it here.

□

**Definition 4.3.11** (Contributions of  $A$ ,  $B$ ,  $C$  vertices). *Consider an internal vertex of type  $A$ ,  $B$  or  $C$  as in Figure 4.3. We define the following smooth forms on  $M$ ,  $M^{\times 2}$  and*

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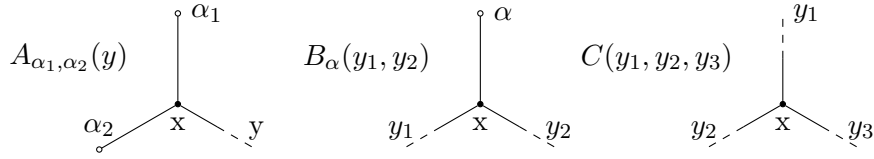


Figure 4.3.: Trivalent vertices of types  $A$ ,  $B$  and  $C$  with the corresponding forms  $A_{\alpha_1, \alpha_2}$ ,  $B_\alpha$  and  $C$ , respectively.

$M^{\times 3}$ , respectively:<sup>2</sup>

$$\begin{aligned} A_{\alpha_1, \alpha_2}(y) &:= \int_x P(y, x) \eta_1(x) \eta_2(x), \\ B_\alpha(y_1, y_2) &:= \int_x P(y_1, x) P(x, y_2) \eta(x), \\ C(y_1, y_2, y_3) &:= \int_x P(x, y_1) P(x, y_2) P(x, y_3). \end{aligned}$$

#### 4.4. Results about vanishing of Chern-Simons Maurer-Cartan element

Do I really need that  $\pi_{\mathcal{H}} \mathcal{P} = 0$ ? What is precisely the definition?

In the situation of Definition 4.3.6, let  $\Gamma \in \overline{\text{RG}}_{klg}^{(3)}$  be a reduced trivalent ribbon graph,  $L = (L_1, L_2, L_3)$  its labeling,  $x_i$  the integration variable associated to the  $i$ -th internal vertex,  $P(x_i, x_j)$  an admissible Hodge propagator on the oriented internal edge between  $x_i$  and  $x_j$ , and  $\alpha_{ij} \in \mathcal{H}(M)[1]$  the harmonic form on the  $j$ -th external vertex on the  $i$ -th boundary component. Recall that we denote by  $\omega_i = s\alpha_{i1} \dots \alpha_{is_i}$  the  $i$ -th input of  $\mathbf{n}_{lg}$  and by  $D$  the total form-degree of all inputs.

By saying “a graph vanishes” we mean that  $I(\sigma_L) = 0$  in the given context.

**Proposition 4.4.1** (Vanishing of graphs with 1). *In the setting of Definition 4.3.6, suppose that the following condition is satisfied:*

**(V<sub>1</sub>)** Every graph  $\Gamma \in \overline{\text{RG}}_{klg}^{(3)}$ ,  $\Gamma \neq Y$  which has  $1 = \theta 1 \in \mathcal{H}(M)[1]$  at an external vertex vanishes.

Then  $\mathbf{n}$  is strictly reduced, and the following holds depending on the dimension  $n$ :

- (a) For  $n > 3$ : All graphs which are not trees or circular vanish. Therefore,  $\mathbf{n}_{lg} = 0$  for all  $(l, g) \neq (1, 0), (2, 0)$ , and it follows that all higher operations  $\mathbf{q}_{1lg}^{\mathbf{n}}$  vanish on the chain level.

<sup>2</sup>The definitions can be made precise in local coordinates. Smoothness of  $A_{\alpha_1, \alpha_2}$  is clear, smoothness of  $B_\alpha$  follows from Lemma 4.2.9, and smoothness of  $C$  can be shown by a similar argument.

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(b) For  $n = 3$ : A tree vanishes unless all  $\eta_1, \dots, \eta_s$  are one-forms. Therefore,  $\mathbf{n}_{10}(\mathbf{s}\alpha_1 \dots \alpha_s) \neq 0$  implies  $\deg(\eta_i) = 1$  for all  $i$ .

(c) For  $n < 3$ : All trees except for  $Y$  vanish. Therefore, we have  $\mathbf{n}_{10} = \mathbf{m}_{10}$ , and consequently  $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110}^{\mathbf{m}}$ .

Moreover, we have

(d) A circular graph vanishes unless all  $\eta_{11}, \dots, \eta_{2s_2}$  are one-forms. Therefore,  $\mathbf{n}_{20}(\mathbf{s}^2\alpha_{11} \dots \alpha_{1s_1} \otimes \alpha_{21} \dots \alpha_{2s_2}) \neq 0$  implies  $\deg(\eta_{ij}) = 1$  for all  $i, j$ .

In addition to  $(V_1)$ , suppose that  $H_{\text{dR}}^1(M) = 0$ . Then:

(e) All circular graphs vanish. Therefore, we have  $\mathbf{n}_{20} = 0$ , and consequently  $\mathbf{q}_{120}^{\mathbf{n}} = \mathbf{q}_{120}$ .

(f) For  $n \leq 6$ : All trees except for  $Y$  vanish. Therefore, we have  $\mathbf{n}_{10} = \mathbf{m}_{10}$ , and consequently  $\mathbf{q}_{110}^{\mathbf{n}} = \mathbf{q}_{110}^{\mathbf{m}}$ .

*Proof.* The proof is just combinatorics with  $D$ . Suppose that a trivalent ribbon graph  $\Gamma \neq Y$  does not vanish on the input  $\omega_1, \dots, \omega_l$ . Because all external vertices of  $\Gamma$  are adjacent to an  $A$ -vertex or a  $B$ -vertex, the assumption  $(V_1)$  implies  $D \geq s$ , where  $s$  is the total number of external vertices. A combination of (4.22) and (4.18) yields

$$nk - (n-1)e = D \geq s = 3k - 2e \iff (n-3)k \geq (n-3)e.$$

(a) For  $n > 3$ , we get  $k \geq e$ , which implies that  $\Gamma$  is either a tree or a circular graph.

(b) If  $\Gamma$  is a tree, then  $s = k + 2$  and  $e = k - 1$ . From (4.22) we get

$$D = nk - (n-1)(k-1) = k + n - 1. \quad (4.25)$$

Now  $D$  is the sum of  $s = k + 2$  form-degrees  $\deg(\eta_{ij}) > 0$ , and hence (4.25) for  $n = 3$  implies that  $\deg(\eta_{ij}) = 1$  for all  $i, j$ .

(c) For  $n < 3$ , we get  $e \geq k$ , which implies that  $\Gamma$  is not a tree.

(d) If  $\Gamma$  is a circular graph, then  $e = k = s$ , and we get using (4.22) that

$$D = nk - (n-1)k = k.$$

Here  $D$  is the sum of  $s = k$  form-degrees  $\deg(\eta_{ij}) > 0$ , and hence  $\deg(\eta_{ij}) = 1$  for all  $i, j$ .

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We will now assume, in addition, that  $\mathcal{H}^1(M) \simeq H_{\text{dR}}^1(M) = 0$ .

DONE Add in addition: This is now assumed in addition to (1)!

(e) We must have  $D \geq 2s$ , which is in contradiction with  $D = s$  for a circular graph. Therefore,  $\mathbf{n}_{20} = 0$ .

(f) Finally, for a tree  $\Gamma \neq Y$ , we have

$$k + n - 1 = D \geq 2s = 2(k + 2) \iff n - 5 \geq k.$$

This finishes the proof of the proposition.  $\square$

**Proposition 4.4.2** (Special Hodge propagator). *In the setting of Definition 4.3.6, suppose that the Hodge propagator  $P$  is special. Then the condition  $(V_1)$ , and hence Proposition 4.4.1 holds.*

*Proof.* It is easy to see that  $A_{\alpha_1, \alpha_2} = \mathcal{P}(\eta_1 \wedge \eta_2)$  for all  $\alpha_1, \alpha_2 \in \mathcal{H}(M)[1]$ , and that  $-B_1$  is the Schwartz kernel of  $\mathcal{P} \circ \mathcal{P}$ . Therefore, (P4) and (P5) imply  $A_{\alpha_1, 1} = 0$  and  $B_1 = 0$ , respectively.

As for the integral  $I(\sigma_L)$ , one has to apply the Fubini theorem in order to integrate out single vertices  $A_{\alpha_1, 1}$  and  $B_1$ . This step relies on  $L^1$ -integrability of the integrand which follows from [CVona] (the integrand comes from a smooth form on a compact manifold with corners).  $\square$

**Proposition 4.4.3** (Vanishing of  $A$ -vertices). *In the setting of Definition 4.3.6, suppose that the following condition is satisfied:*

**( $V_A$ )** *Every graph with an  $A$ -vertex vanishes.*

*Then we have  $\mathbf{n}_{10} = \mathbf{m}_{10}$ , and the only contribution to  $\mathbf{n}_{20}(s^2 \alpha_{11} \dots \alpha_{1s_1} \otimes \alpha_{21} \dots \alpha_{2s_2})$  comes from  $O_k$ -graphs with  $k = s_1 + s_2 = D$ .*

*Proof.* The only trees and circular graphs which are not excluded by the assumption are the  $Y$ -graph and  $O_k$ -graphs, respectively (the external branches contract). The condition on form-degrees is obtained as in the proof of Proposition 4.4.1.

To argue that  $I(\sigma_L) = 0$ , we again need  $L^1$ -integrability as in the proof of Proposition 4.4.2.  $\square$

**Remark 4.4.4** (Integrability for trees). Given a tree, we can start at a leaf and write  $I(\sigma_L)$  as an iterative integral of contributions  $A_{\alpha_1, \alpha_2}$  for  $\alpha_1, \alpha_2 \in \Omega(M)$ . These are smooth forms, and hence integrability is guaranteed. Therefore, the result  $\mathbf{n}_{10} = \mathbf{m}_{10}$  is independent of the convergence results from [CVona].  $\triangleleft$

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**Proposition 4.4.5** (1-connected geometrically formal manifolds). *Let  $M$  be a geometrically formal  $n$ -manifold and  $P$  a special Hodge propagator (it exists by Proposition 4.2.10). If  $H_{\text{dR}}^1(M) = 0$ , then the following holds:*

( $n \neq 2$ ) All  $Y \neq \Gamma \in \overline{\text{RG}}_{klg}$  with  $k, l \geq 1, g \geq 0$  vanish, and hence  $\mathbf{n} = \mathbf{m}$ .

( $n = 2$ ) All  $Y \neq \Gamma \in \overline{\text{RG}}_{kl0}$  with  $k, l \geq 1$  vanish, and hence  $\mathbf{n}_{l0} = \mathbf{m}_{l0}$  for all  $l \geq 1$ .

*Proof.* Given  $\eta_1, \eta_2 \in \mathcal{H}$ , geometric formality implies  $\eta_1 \wedge \eta_2 \in \mathcal{H}$ , and hence  $A_{\alpha_1, \alpha_2} = \mathcal{P}(\eta_1 \wedge \eta_2) = 0$ . We see that  $(V_1)$  and  $(V_A)$  are satisfied, and hence the implications of Propositions 4.4.1 and 4.4.3 hold. The claim for  $n > 3$  follows.

As for  $n = 3$ , Poincaré duality implies  $H_{\text{dR}}^2(M; \mathbb{R}) = 0$ . Therefore, the total form-degree  $D$  satisfies  $D = nB$ , where  $B$  is the number of  $B$ -vertices. We see using (4.23) that (4.22) is equivalent to

$$B + \frac{1}{2}(3 - n)C = D = nB \iff (n - 1)B = \frac{1}{2}(3 - n)C. \quad (4.26)$$

It follows that  $B = 0$ , and hence all reduced graphs vanish.

As for  $n = 2$ , we get from (4.26) and (4.24) that  $B \geq l$  is equivalent to  $g \geq 1$ .  $\square$

*Remark 4.4.6* ( $A_\infty$ -homotopy transfer). In [CVona], it will be shown that the  $A_\infty$ -algebra  $\mathcal{H}(M)_\mathbf{n} = (\mathcal{H}(M), (\mu_k))$  induced by  $\mathbf{n}_{10}$  agrees with the  $A_\infty$ -algebra obtained by the  $A_\infty$ -homotopy transfer

$$\left( \begin{array}{c} \Omega(M) \\ m_1, m_2 \end{array} \right) \rightsquigarrow \left( \begin{array}{c} \mathcal{H}(M) \\ \mu_1 \equiv 0, \mu_2 = \pi_{\mathcal{H}} m_2(\iota_{\mathcal{H}}, \iota_{\mathcal{H}}), \mu_3, \dots \end{array} \right)$$

using the homotopy retract (see [Val12])

$$\mathcal{P} \left( \begin{array}{c} \curvearrowright \end{array} \right) (\Omega(M), m_1) \xrightleftharpoons[\iota_{\mathcal{H}}]{\pi_{\mathcal{H}}} (\mathcal{H}(M), m_1 \equiv 0).$$

The operation  $\mu_k$  of the transferred  $A_\infty$ -structure is computed as a sum over planar trees with a root and  $k$  leaves decorated by  $\iota_{\mathcal{H}}$  at the leaves,  $\pi_{\mathcal{H}}$  at the root and  $\mathcal{P}$  at the internal edges (see [Aka07]). The result of [CVona] is plausible because the part of  $\mathbf{n}_{10}$  contributing to  $\mu_k$  is a sum over trivalent ribbon trees with  $k + 1$  leaves.

In [CVona], they will also show that  $\iota_1 := \iota_{\mathcal{H}} : \mathcal{H} \rightarrow \Omega$  extends to an  $A_\infty$ -quasi-isomorphism  $(\iota_k)_{k \geq 1}$  from  $(\mathcal{H}, (\mu_k))$  to  $(\Omega, m_1, m_2)$ . The induced chain map on the

DONE Wrong reference: Here is not Eq:GenusFormula but the relation of A B C vertices to graph variables



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dual cyclic bar complexes is then the map  $f_{110}^m$  coming from the  $\text{IBL}_\infty$ -theory in the Overview.  $\triangleleft$

**Proposition 4.4.7** (Twisted boundary operator for formal manifolds). *In the setting of Definition 4.3.6, suppose that  $M$  is formal in the sense of rational homotopy theory. Then there is a quasi-isomorphism*

$$\mathfrak{h}_{110} : (\hat{B}_{\text{cyc}}^* H_{\text{dR}}(M)[3-n], \mathfrak{q}_{110}^m) \longrightarrow (\hat{B}_{\text{cyc}}^* \mathcal{H}(M)[3-n], \mathfrak{q}_{110}^n).$$

*Proof.* Formality of  $M$  is equivalent to the existence of a zig-zag of quasi-isomorphisms of dga's (see [Val12])

$$(H_{\text{dR}}(M), m_1 \equiv 0, m_2) \rightsquigarrow \bullet \quad \cdots \quad \bullet \leftarrow (\Omega(M), m_1, m_2).$$

Because a dga-quasi-isomorphism has a homotopy inverse in the category of  $A_\infty$ -algebras, we get a direct  $A_\infty$ -quasi-isomorphism

$$(g_k) : (\Omega(M), m_1, m_2) \rightsquigarrow (H_{\text{dR}}(M), m_1 \equiv 0, m_2).$$

Precomposing with  $(\iota_k)$  from Remark 4.4.6, we get the  $A_\infty$ -isomorphism

$$(h_k) : (\mathcal{H}(M), (\mu_k)) \rightsquigarrow (H_{\text{dR}}(M), m_1 \equiv 0, m_2).$$

This induces the quasi-isomorphism  $\mathfrak{h}_{110}$  of the corresponding cyclic cochain complexes.  $\square$

*Remark 4.4.8* (On formality). Geometrically formal manifolds include  $\mathbb{S}^n$ ,  $\mathbb{C}P^n$  and Lie groups (see [Kot01]). Any geometrically formal manifold is formal. Every simply-connected manifold of dimension at most 6 is formal (see [Mil79]).  $\triangleleft$

### 4.5. Conjectured relation to string topology

DONE Simplify string top. and cyc. hom. comparison: Add somewhere that the map from  $(\hat{B}_{\text{cyc}}^* H_{\text{dR}}(M), b^*)$  to  $(C(L_{\mathbb{S}^1} M), \theta)$  with no degree shifts does the job (after removing all degree shifts from definitions).

Given a smooth connected oriented  $n$ -dimensional manifold  $M$ , we consider the equivariant homology of the free loop space  $LM := \{\gamma : \mathbb{S}^1 \rightarrow M \text{ continuous}\}$  with respect to the reparametrization action of  $\mathbb{S}^1$ . It is defined as the singular homology of the Borel construction

$$L_{\mathbb{S}^1} M := E\mathbb{S}^1 \times_{\mathbb{S}^1} LM := (E\mathbb{S}^1 \times LM)/\mathbb{S}^1,$$

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where  $E\mathbb{S}^1 = \mathbb{S}^\infty \rightarrow B\mathbb{S}^1 = \mathbb{CP}^\infty$  is a model for the universal bundle for  $\mathbb{S}^1$ , and we quotient out the diagonal action. We denote this homology by

$$H^{\mathbb{S}^1}(LM) := H(L_{\mathbb{S}^1}M).$$

Recall that  $(E\mathbb{S}^1 \times LM)/\mathbb{S}^1$  is the homotopically correct quotient replacing  $LM/\mathbb{S}^1$ , which we prefer to use because the diagonal action of  $\mathbb{S}^1$  on  $E\mathbb{S}^1$  is free, and hence  $E\mathbb{S}^1 \times LM \rightarrow (E\mathbb{S}^1 \times LM)/\mathbb{S}^1$  is a principal  $\mathbb{S}^1$ -bundle (in contrast to the pathological map  $LM \rightarrow LM/\mathbb{S}^1$ ). Recall also that  $E\mathbb{S}^1$  is contractible.

The “geometric versions” of the homologies were defined in [CS99] as the degree shifts

$$\mathbb{H}(LM) := H(LM)[n] \quad \text{and} \quad \mathcal{H}(LM) := H^{\mathbb{S}^1}(LM)[n].$$

There is the *loop product*  $\bullet : \mathbb{H}(LM)^{\otimes 2} \rightarrow \mathbb{H}(LM)$  of degree 0 which makes  $\mathbb{H}(LM)$  into a graded commutative associative algebra. There is also the *loop coproduct*  $\tau : \bar{\mathbb{H}}(LM) \rightarrow \bar{\mathbb{H}}(LM)^{\otimes 2}$  of degree  $1 - 2n$  which is graded cocommutative and coassociative and is a derivation of  $\bullet$ . The geometric construction of  $\bullet$  and  $\tau$  on transverse smooth chains in  $LM$  was described in [CS99] and [Bas11], respectively. Here, the symbol  $\bar{\mathbb{H}}(LM)$  stands for the degree shifted relative homology

$$\bar{\mathbb{H}}(LM) := H(LM, M)[n]$$

with respect to constant loops  $M \hookrightarrow LM$ . The geometric construction of  $\tau$  does not work on the whole  $\mathbb{H}(LM)$  because of the phenomenon of “vanishing of small loops” depicted in [CL09, Figure 4, p. 13].

The projection  $E\mathbb{S}^1 \times LM \rightarrow L_{\mathbb{S}^1}M$  is an  $\mathbb{S}^1$ -principal bundle and thus induces a Gysin sequence. This sequence written using the geometric versions reads

$$\dots \longrightarrow \mathbb{H}_i \xrightarrow{\mathcal{E}} \mathcal{H}_i \xrightarrow{\cap c} \mathcal{H}_{i-2} \xrightarrow{\mathcal{M}} \mathbb{H}_{i-1} \longrightarrow \dots, \quad (4.27)$$

where the map  $\mathcal{M}$  adds a marked point in each string in a family in all possible positions, the map  $\mathcal{E}$  erases the marked point of each string in a family,  $c \in H_{\mathbb{S}^1}^2(LM)$  is the Euler class of the circle bundle and  $\cap$  the cap product.

The *string bracket*  $\tilde{m}_2 : \mathcal{H}(LM)^{\otimes 2} \rightarrow \mathcal{H}(LM)$  and the *string cobracket*  $\tilde{c}_2 : \bar{\mathcal{H}}(LM) \rightarrow \bar{\mathcal{H}}(LM)^{\otimes 2}$  are defined by

$$\tilde{m}_2 := \mathcal{E} \circ \bullet \circ \mathcal{M}^{\otimes 2} \quad \text{and} \quad \tilde{c}_2 := \mathcal{E}^{\otimes 2} \circ \nu \circ \mathcal{M}.$$

DONE Reduced loop product: Is it really true? What about the example of torus.

#### 4. Chern-Simons Maurer-Cartan element and string topology

Here, the symbol  $\bar{\mathcal{H}}(LM)$  stands for the degree shifted relative  $\mathbb{S}^1$ -equivariant homology

$$\bar{\mathcal{H}}(LM) := \underbrace{H^{\mathbb{S}^1}(E\mathbb{S}^1 \times_{\mathbb{S}^1} LM, E\mathbb{S}^1 \times_{\mathbb{S}^1} M)}_{=: \bar{H}^{\mathbb{S}^1}(LM)}[n].$$

Because  $|\mathcal{M}| = 1$  and  $|\mathcal{E}| = 0$ , we have for all  $\xi \in \bar{\mathcal{H}}(LM)$  and  $\xi_1, \xi_2 \in \mathcal{H}$  the relations

DONE Typo:  $\xi$  in the exponent should be  $|\xi_1|$

$$\begin{aligned} \tilde{\mathfrak{m}}_2(\xi_1, \xi_2) &= (-1)^{|\xi_1|} \mathcal{E}(\mathcal{M}(\xi_1) \bullet \mathcal{M}(\xi_2)), \\ \tilde{\mathfrak{c}}_2(\xi) &= \sum \mathcal{E}(\nu^1) \otimes \mathcal{E}(\nu^2), \end{aligned} \tag{4.28}$$

where we write  $\nu(\mathcal{M}(\xi)) = \sum \nu^1 \otimes \nu^2$ . The operations  $\tilde{\mathfrak{m}}_2$  and  $\tilde{\mathfrak{c}}_2$  have degrees 2 and  $2-2n$  with respect to the grading on  $\mathcal{H}(LM)$ , respectively. In fact, we will consider  $\tilde{\mathfrak{m}}_2$  and  $\tilde{\mathfrak{c}}_2$  given by (4.28) as operations on the even degree shift  $H^{\mathbb{S}^1}(LM)[2-n] = \mathcal{H}(LM)[2-2n]$ , which have degrees  $2(2-n)$  and 0, respectively. The symbols  $\mathfrak{m}_2$  and  $\mathfrak{c}_2$  will denote their degree shifts to  $H^{\mathbb{S}^1}(LM)[3-n]$ , which have degrees of an IBL-algebra from Definition 3.2.4.

In work in progress [CVonb], they consider the map

$$I_{\lambda,*} : H_{-\bullet-1}^{\lambda}(\Omega(M)) \longrightarrow H_{\mathbb{S}^1}^{\bullet}(LM; \mathbb{R})$$

defined on the chain level as a cyclic version of Chen's iterated integrals  $I_{\lambda}$ . Recall that  $H_{-\bullet-1}^{\lambda}(\Omega) = H_{\bullet}(\text{B}^{\text{cyc}}\Omega, \mathfrak{b})$ , where  $\mathfrak{b} : B\Omega = \bigoplus_{k \geq 1} \Omega[1]^{\otimes k} \rightarrow B\Omega$  is the Hochschild differential of the de Rham dga  $(\Omega, m_1, m_2)$ , and the grading on  $H^{\lambda}(\Omega)$  was chosen such that  $H_{\bullet}^{\lambda}(\Omega) \simeq H_{\bullet}^{\lambda, \text{cl}}(\Omega)$  for the classical cyclic homology of a dga (see Section 3.3). They prove in [CVonb] that if  $M$  is simply-connected, then the map  $I_{\lambda,*}$  induces an isomorphism  $H_{-\bullet-1}^{\lambda, \text{red}}(\Omega(M)) \simeq H_{\mathbb{S}^1, \text{red}}^{\bullet}(LM)$ , where

DONE Homology exchanged with cohomology: DONE The homology of  $\mathfrak{q}_{110}$  should be the cyclic cohomology

$$H_{\mathbb{S}^1, \text{red}}(LM) := H_{\mathbb{S}^1}(E\mathbb{S}^1 \times_{\mathbb{S}^1} LM, E\mathbb{S}^1 \times_{\mathbb{S}^1} \{x_0\})$$

is the *reduced  $\mathbb{S}^1$ -equivariant cohomology* with respect to a base point  $x_0 \in M$  (the constant loop at  $x_0$ ). Dualizing their map, we obtain the isomorphism

$$H_{\lambda, \text{red}}^{-\bullet-1}(\Omega(M)) \simeq H_{\bullet}^{\mathbb{S}^1, \text{red}}(LM; \mathbb{R}). \tag{4.29}$$

Suppose from now on that  $M$  is closed. Pick a Riemannian metric and an admissible Hodge propagator  $P \in \Omega^{n-1}(\text{Bl}_{\Delta}(M \times M))$ . We will assume that  $P$  is special, i.e., that it satisfies (P1)–(P5) from Section 4.2, so that the Chern-Simons Maurer-Cartan element  $\mathfrak{n}$  is strictly reduced, and hence the twisted reduced IBL $_{\infty}$ -algebra  $\text{dIBL}^{\mathfrak{n}}(C_{\text{red}}(\mathcal{H}))$  and

#### 4. Chern-Simons Maurer-Cartan element and string topology

the induced IBL-algebra  $\text{IBL}(\mathbb{H}^{\mathfrak{n},\text{red}}(C(\mathcal{H})))$  are well-defined. Recall that  $\mathbb{H}^{\mathfrak{n}}_{\bullet}(C(\mathcal{H})) = H^{\lambda}_{n-3-\bullet}(\mathcal{H}_{\mathfrak{n}})$ , where  $\mathcal{H}_{\mathfrak{n}}$  is the  $A_{\infty}$ -algebra on  $\mathcal{H}$  twisted by  $\mathfrak{n}_{10}$ . From [CVona], we have

$$H_{\lambda}(\mathcal{H}(M)_{\mathfrak{n}}) \simeq H_{\lambda}(\Omega(M)). \quad (4.30)$$

A combination of (4.29) and (4.30) gives the following version of the string topology conjecture from the Overview.

**Conjecture 4.5.1** (String topology conjecture for simply-connected manifold). *Let  $M$  be an oriented closed manifold of dimension  $n$ . There is a chain map*

$$(C^{\text{sing}}(\text{L}_{\mathbb{S}^1}M; \mathbb{R}), \partial) \longrightarrow (\hat{\text{B}}_{\text{cyc}}^* \mathcal{H}(M), \mathfrak{q}_{110}^{\mathfrak{n}}),$$

where  $C^{\text{sing}}$  denotes the (smooth) singular chain complex and  $\partial$  the standard boundary operator, which, if  $M$  is simply-connected, satisfies the following:

- It induces an isomorphism  $H^{\mathbb{S}^1, \text{red}}(\text{LM}; \mathbb{R})[2-n] \simeq \mathbb{H}^{\mathfrak{n}, \text{red}}(C(\mathcal{H}(M)))$ .
- It intertwines  $\mathfrak{m}_2$  on  $H^{\mathbb{S}^1}(\text{LM}; \mathbb{R})$  and  $\mathfrak{q}_{210}$ .
- The pullback of  $\mathfrak{q}_{120}^{\mathfrak{n}}$  to  $H^{\mathbb{S}^1, \text{red}}(\text{LM}; \mathbb{R})$  is compatible with  $\mathfrak{c}_2$  on  $\bar{H}^{\mathbb{S}^1}(\text{LM}; \mathbb{R})$  under the morphism induced by the inclusion  $(\text{LM}, x_0) \rightarrow (\text{LM}, M)$ .

*Remark 4.5.2* (On string topology conjecture). (i) The conjecture can be interpreted as follows. There is an IBL-structure on  $H^{\mathbb{S}^1, \text{red}}(\text{LM}; \mathbb{R})$  compatible with Chas-Sullivan operations, and the  $\text{IBL}_{\infty}$ -algebra  $\text{dIBL}^{\mathfrak{n}}(C_{\text{red}}(\mathcal{H}(M)))$  is its chain model.

(ii) The loop coproduct  $\tau$  is geometrically defined only on  $\bar{H}^{\mathbb{S}^1}(\text{LM})$ ; the conjecture thus provides an extension of  $\mathfrak{c}_2$  to  $H^{\mathbb{S}^1, \text{red}}(\text{LM})$ . In [Bas11], it is shown that the geometric definition of  $\tau$  can be extended to  $H(\text{LM})$  for manifolds with zero Euler characteristic, i.e.,  $\chi(M) = 0$ . This extension depends on the choice of a non-vanishing vector field on  $M$ . By homotopy invariance (see (v) below), our extension of  $\mathfrak{c}_2$  should not depend on the admissible Hodge propagator  $P$ .

(iii) The loop product  $\bullet$  is geometrically defined on  $H(\text{LM})$ ; however, it does not always induce an associative product on  $H^{\text{red}}(\text{LM}) = H(\text{LM}, x_0)$ . Indeed, the examples of  $\mathbb{T}^2$  (see [Bas11]) and  $\mathbb{S}^3$  (see [CS99]) show that  $H(x_0; \mathbb{R}) \subset H(\text{LM}; \mathbb{R})$  is not an ideal with respect to  $\bullet$ . By [Tam10], this does not happen when  $\chi(M) \neq 0$ , and hence, in this case,  $\bullet$  restricts to  $H(\text{LM}, x_0; \mathbb{R})$ .

#### 4. Chern-Simons Maurer-Cartan element and string topology

(iv) The computation for  $\mathbb{S}^n$  with  $n \geq 2$  and the computation for  $\mathbb{CP}^n$  in Section 5 support the conjecture. The computation for  $\mathbb{S}^1$  in Section 5.3 provides a counterexample for non-simply-connected  $M$ . Surfaces of genus  $g \geq 1$  should be considered.

(v) We expect that if  $M_1$  and  $M_2$  are homotopy equivalent, then the  $\mathrm{IBL}_\infty$ -algebras  $\mathrm{dIBL}^{\mathfrak{n}_1}(C(\mathrm{H}_{\mathrm{dR}}(M_1)))$  and  $\mathrm{dIBL}^{\mathfrak{n}_2}(C(\mathrm{H}_{\mathrm{dR}}(M_2)))$  are  $\mathrm{IBL}_\infty$ -homotopy equivalent.  $\triangleleft$

## 5. Explicit computations

In Section 5.1, we solve the differential equation for the Hodge propagator  $P$  for  $\mathbb{S}^n$  (Proposition 5.1.2) using the Relative Poincaré Lemma (Lemma 5.1.1). In the rest of the section, we will be showing that  $P$  is admissible (Proposition 5.1.10); the most work is to show that  $P$  extends smoothly to the blow-up (Proposition 5.1.9). Another Hodge propagator for  $\mathbb{S}^1$  can be obtained in an alternative simple way by writing  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ , and there are nice geometric formulas for  $P$  for  $\mathbb{S}^2$  (Example 5.1.3).

In Section 5.2, we use  $P$  from Section 5.1 to compute the Chern-Simons Maurer-Cartan element  $\mathbf{n}$  for  $\mathbb{S}^n$  (Proposition 5.2.10). We first prove that the condition  $(V_1)$  from Proposition 4.4.1 is satisfied (Lemma 5.2.1) and then perform combinatorics with degrees to show vanishing of some more integrals (Proposition 5.2.2). In fact, all the integrals vanish for  $\mathbb{S}^n$  with  $n \geq 3$ , and the only non-vanishing integrals for  $\mathbb{S}^1$  are the  $O_k$ -graphs with even  $k$ . We compute these integrals explicitly together with all signs and combinatorial coefficients required to obtain  $\mathbf{n}_{20}$  (Lemmas 5.2.5, 5.2.6, 5.2.7 and 5.2.8). There might be some non-vanishing integrals associated to reduced graphs for  $\mathbb{S}^2$  as well as some non-vanishing integrals associated to graphs without external vertices for  $\mathbb{S}^3$ ; however, the simplest examples vanish (Remarks 5.2.3 and 5.2.4).

In the remaining Sections 5.3 and 5.4, we compute  $\text{IBL}(\mathbb{H}^n(C(\mathcal{H}(M))))$  and the higher operations  $\mathbf{q}_{1lg}^{\mathbf{n}}$  on  $\mathbb{H}^n$  for  $M = \mathbb{S}^n, \mathbb{CP}^n$ . As soon as we argue that  $\mathbf{n}_{10} = \mathbf{m}_{10}$  due to geometric formality, the computation of  $\mathbb{H}^m(C(\mathcal{H}(\mathbb{S}^n)))$  and  $\mathbb{H}^m(C(\mathcal{H}(\mathbb{CP}^n)))$  is an easy exercise in cyclic homology. The operations for  $\mathbb{S}^{2m}$  and  $\mathbb{CP}^n$  vanish for degree reasons (Remark 5.3.1). Therefore, the integrals from Section 5.2 help only in the case of  $\mathbb{S}^{2m-1}$ . We compare our results to Chas-Sullivan string topology from [Bas11] and confirm Conjecture 4.5.1 for  $\mathbb{S}^n$  with  $n \geq 2$  and for  $\mathbb{CP}^n$ .

## 5.1. Construction of a Hodge propagator for spheres

The standard Riemannian volume form on the round sphere  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$  is the restriction of the following closed form on  $\mathbb{R}^{n+1} \setminus \{0\}$ :

$$\text{Vol}(x) := \frac{1}{|x|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}.$$

Here  $\widehat{dx_i}$  means that  $dx_i$  is omitted. We denote the Riemannian volume of  $\mathbb{S}^n$  by

$$V := \int_{\mathbb{S}^n} \text{Vol}.$$

The  $n$ -form  $H$  from Proposition 4.2.7 reads

$$H = \frac{1}{V} (\text{pr}_1^* \text{Vol} + (-1)^n \text{pr}_2^* \text{Vol}).$$

According to Proposition 4.2.8, the equation which we want to solve reads

$$dP = \frac{1}{V} ((-1)^n \text{pr}_1^* \text{Vol} + \text{pr}_2^* \text{Vol}). \quad (5.1)$$

We denote

$$\tilde{G} := VP \quad \text{and} \quad \tilde{H} := VH.$$

The following lemma will be used to construct a solution to (5.1).

**Lemma 5.1.1** (Relative Poincaré Lemma). *Let  $M$  be a smooth oriented manifold and  $\psi : [0, 1] \times M \rightarrow M$  a smooth map. Consider the operator  $T : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$  defined by*

$$T(\eta) := \int^{[0,1]} \psi^* \eta \quad \text{for all } \eta \in \Omega(M),$$

*where we integrate along the fiber of the oriented fiber bundle  $\text{pr}_2 : [0, 1] \times M \rightarrow M$ . Then we have*

$$d \circ T + T \circ d = \psi_1^* - \psi_0^*.$$

*Proof.* Stokes' formula from Proposition 4.2.6 gives

$$d \int^{[0,1]} \psi^* \eta = - \left( \int^{[0,1]} d\psi^* \eta - \int^{\partial[0,1]} \psi^* \eta \right) = - \int^{[0,1]} \psi^* d\eta + \psi_1^* \eta - \psi_0^* \eta$$

for all  $\eta \in \Omega(M)$ . □

## 5. Explicit computations

**Proposition 5.1.2** (Solution to (5.1)). *For all  $(x, y) \in (\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ , let*

$$P(x, y) := (-1)^n \sum_{k=0}^{n-1} g_k(x, y) \omega_k(x, y), \quad (5.2)$$

where

$$g_k(x, y) := \int_0^1 \frac{t^k (t-1)^{n-1-k}}{(2t(t-1)(1+x \cdot y) + 1)^{\frac{n+1}{2}}} dt \quad (5.3)$$

and

$$\omega_k(x, y) := \frac{1}{k!} \frac{1}{(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n+1}} (-1)^\sigma x^{\sigma_1} y^{\sigma_2} dx^{\sigma_3} \dots dx^{\sigma_{2+k}} dy^{\sigma_{3+k}} \dots dy^{\sigma_{n+1}}. \quad (5.4)$$

The form (5.2) is a smooth solution to (5.1) on  $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ .

*Proof.* Define the set

$$N := (\mathbb{R}_{\neq 0}^{n+1} \times \mathbb{R}_{\neq 0}^{n+1}) \setminus \{(x, ax) \mid x \in \mathbb{R}^{n+1}, a > 0\}.$$

It is an open thickening of  $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$  in  $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta$ . Consider the smooth deformation retraction

$$\begin{aligned} \psi : [0, 1] \times N &\longrightarrow N \\ (t, x, y) &\longmapsto \psi_t(x, y) := (x, (1-t)y - tx) \end{aligned}$$

with

$$\psi_0(x, y) = (x, y) \quad \text{and} \quad \psi_1(x, y) = (x, -x) \quad \text{for all } (x, y) \in N.$$

The retraction is depicted in Figure 5.1. Denote by  $A : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ ,  $x \mapsto -x$  the antipodal map. It is easy to see that

$$A^* \text{Vol} = (-1)^{n+1} \text{Vol},$$

and hence

$$\psi_1^* \tilde{H} = \psi_1^* \text{pr}_1^* \text{Vol} + (-1)^n \psi_1^* \text{pr}_2^* \text{Vol} = \text{pr}_1^* \text{Vol} + (-1)^n \text{pr}_1^* A^* \text{Vol} = 0.$$

Define

$$P := (-1)^{n+1} \int_{[0,1]} \psi^* H. \quad (5.5)$$

Let  $T : \Omega^*(N) \rightarrow \Omega^{*-1}(N)$  be the cochain homotopy from Lemma 5.1.1 associated to  $\psi$ .



## 5. Explicit computations

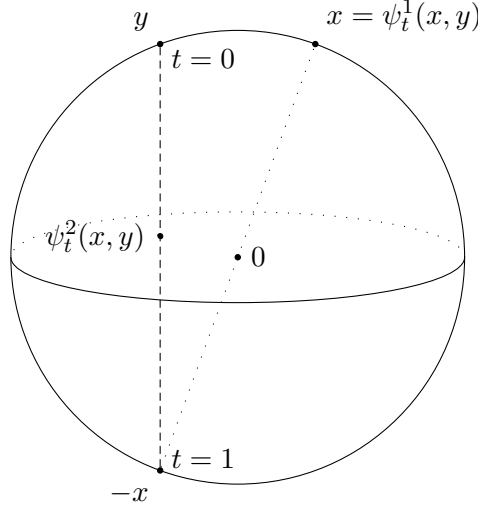


Figure 5.1.: Retraction  $\psi_t = (\psi_t^1, \psi_t^2)$ . A point of  $\mathbb{S}^n \times \mathbb{S}^n$  is visualized as a pair of points on  $\mathbb{S}^n$ .

Because  $dH = 0$ , we get

$$dP = (-1)^{n+1} dT(H) = (-1)^{n+1} (dT + Td)H = (-1)^{n+1} (\psi_1^* - \psi_0^*)H = (-1)^n H.$$

For every  $i = 1, \dots, n+1$ , we have

$$\psi^*(dx^i) = dx^i \quad \text{and} \quad \psi^*(dy^i) = (1-t) dy^i - t dx^i - (y^i + x^i) dt.$$

We compute

$$\begin{aligned} (-1)^{n+1} \int_{[0,1]} \psi^* \tilde{H} &= - \int_{[0,1]} \psi^* \text{pr}_2^* \text{Vol} \\ &= \int_{[0,1]} \sum_{i=1}^{n+1} (-1)^i \frac{((1-t)y^i - tx^i)}{|(1-t)y - tx|^{n+1}} \psi^*(dy^1 \dots \widehat{dy^i} \dots dy^{n+1}) \\ &= \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} (x^i y^j - y^i x^j) \int_{[0,1]} \frac{dt \psi^*(dy^1 \dots \widehat{dy^i} \dots \widehat{dy^j} \dots dy^{n+1})}{|(1-t)y - tx|^{n+1}} \\ &= (-1)^n \sum_{k=0}^{n-1} \left( \int_0^1 \frac{t^k (t-1)^{n-1-k}}{|(1-t)y - tx|^{n+1}} dt \right) \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} (x^i y^j - y^i x^j) \\ &\quad \sum_{\substack{\sigma: \{1, \dots, n-1\} \rightarrow \{1, \dots, \hat{i}, \dots, \hat{j}, \dots, n+1\} \\ \sigma_1 < \dots < \sigma_k \\ \sigma_{k+1} < \dots < \sigma_{n-1}}} (-1)^\sigma dx^{\sigma_1} \dots dx^{\sigma_k} dy^{\sigma_{k+1}} \dots dy^{\sigma_{n-1}}. \end{aligned}$$

## 5. Explicit computations

The formulas (5.3) and (5.4) are obtained from this by writing

$$|(1-t)y - tx|^2 = 2t(t-1)(1+x \cdot y) + 1$$

in the denominator of the integrand and by simple combinatorics in the form part, respectively. Smoothness of  $P$  on  $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$  follows from the expression (5.5).  $\square$

Note that  $g_k$  are smooth functions on  $(\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta$ .

*Example 5.1.3* ( for  $\mathbb{S}^1$  and  $\mathbb{S}^2$ ). (a) Let

$$\alpha : (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta \rightarrow (0, 2\pi)$$

be the smooth function assigning to a pair  $(x, y) \in (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta$  the counterclockwise angle from  $x$  to  $y$ . Let  $\alpha_1, \alpha_2 \in [0, 2\pi)$  be such that  $x = \cos(\alpha_1)e_1 + \sin(\alpha_1)e_2$  and  $y = \cos(\alpha_2)e_1 + \sin(\alpha_2)e_2$  for the standard Euclidean basis  $e_1, e_2$  of  $\mathbb{R}^2$ . It is easy to see that

$$\alpha(x, y) = \begin{cases} \alpha_2 - \alpha_1 & \text{if } \alpha_1 < \alpha_2, \\ \alpha_2 - \alpha_1 + 2\pi & \text{if } \alpha_1 > \alpha_2. \end{cases}$$

Therefore, we get

$$d\alpha = d\alpha_2 - d\alpha_1 = -2\pi H \quad \text{on } (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta.$$

On the other hand, we can compute  $P$  from (5.2) as follows. Using the substitution  $u = 2t - 1$ , we get for all  $x, y \in \mathbb{S}^1$  with  $x \neq \pm y$  the following:

$$\begin{aligned} g_0(x, y) &= \int_0^1 \frac{dt}{2t(t-1)(1+x \cdot y) + 1} = \frac{1}{1-x \cdot y} \int_{-1}^1 \frac{du}{\frac{1+x \cdot y}{1-x \cdot y} u^2 + 1} \\ &= \frac{2}{\sqrt{1-(x \cdot y)^2}} \arctan\left(\sqrt{\frac{1+x \cdot y}{1-x \cdot y}}\right) \\ &= \frac{\pi - \arccos(x \cdot y)}{\sqrt{1-(x \cdot y)^2}} = \frac{\pi - \arccos(x \cdot y)}{|x^1 y^2 - x^2 y^1|} = \frac{\pi - \alpha(x, y)}{x^1 y^2 - x^2 y^1}. \end{aligned}$$

The third from last equality can be obtained by trigonometric considerations and the second from last equality by an algebraic manipulation with the denominator. We will explain the last equality. Consider the matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

## 5. Explicit computations

representing the counterclockwise rotation by  $\frac{\pi}{2}$ . The function  $\arccos : (-1, 1) \rightarrow (0, \pi)$  satisfies

$$\arccos(x \cdot y) = \begin{cases} \alpha(x, y) & \text{if } y \cdot Rx > 0, \\ 2\pi - \alpha(x, y) & \text{if } y \cdot Rx < 0. \end{cases}$$

The last equality becomes clear when we notice that  $x^1y^2 - x^2y^1 = y \cdot Rx$ .

Finally, we have  $\omega_0(x, y) = x^1y^2 - x^2y^1$ , and hence

$$2\pi G(x, y) = -g_0(x, y)\omega_0(x, y) = \alpha(x, y) - \pi = \pi - \alpha(y, x).$$

(b) For  $n = 2$ , we get the formulas

$$\begin{aligned} g_0(x, y) &= -g_1(x, y) = \frac{1}{x \cdot y - 1} \quad \text{and} \\ \omega_0(x, y) &= (x^2y^3 - x^3y^2) dy^1 + (x^3y^1 - x^1y^3) dy^2 + (x^1y^2 - x^2y^1) dy^3 \\ &= \sum_{i=1}^3 (x \times y)^i dy^i. \end{aligned}$$

The formula for  $\omega_1(x, y)$  is obtained from the formula for  $\omega_0(x, y)$  by replacing  $dy$  with  $dx$ .  $\triangleleft$

Consider the diagonal action of the orthogonal group  $O(n+1)$  on  $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$  by matrix multiplication.

**Proposition 5.1.4** (Symmetries of P). *Consider P from Proposition 5.1.2. For all  $R \in O(n+1)$ , we have*

$$R^*P = (-1)^R P,$$

where  $(-1)^R = \det(R)$ . Moreover, if  $\tau$  denotes the twist map, then

$$\tau^*P = (-1)^n P.$$

*Proof.* We will use the thickening  $N$ , the antipodal map  $A$  and the expression (5.5) for  $P$  from the proof of Proposition 5.1.2.

It is easy to check that both  $\tau$  and  $R$  preserve  $N$ . Let  $\tilde{\tau}$  and  $\tilde{R}$  be the isomorphisms of the fiber bundle  $\text{pr}_2 : [0, 1] \times N \rightarrow N$  given by

$$\tilde{\tau}(t, x, y) := (1 - t, y, x) \quad \text{and} \quad \tilde{R}(t, x, y) := (t, Rx, Ry)$$

for all  $(t, x, y) \in [0, 1] \times N$ . Then  $\tilde{\tau}$  covers  $\tau$  and  $\tilde{R}$  covers  $R$ . A simple computation

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directly from Definition 4.2.1 shows that the fiberwise integration commutes with the pullback along a bundle morphism if the bundle map and the base map are both either orientation preserving or reversing. In our case, we have

$$(-1)^{\tau+\tilde{\tau}} = -1 \quad \text{and} \quad (-1)^{R+\tilde{R}} = 1.$$

Using this and the equation

$$\text{pr}_2 \circ \psi \circ \tilde{\tau} = A \circ \text{pr}_2 \circ \psi,$$

we get firstly

$$\begin{aligned} \tau^* \int^{[0,1]} \psi^* \tilde{H} &= - \int^{[0,1]} \tilde{\tau}^* \psi^* \text{pr}_2^* \text{Vol} \\ &= - \int^{[0,1]} \psi^* \text{pr}_2^* A^* \text{Vol} \\ &= (-1)^n \int^{[0,1]} \psi^* \text{pr}_2^* \text{Vol} \\ &= (-1)^n \int^{[0,1]} \psi^* \tilde{H} \end{aligned}$$

and secondly

$$R^* \int^{[0,1]} \psi^* H = \int^{[0,1]} \tilde{R}^* \psi^* H = \int^{[0,1]} \psi^* R^* H = (-1)^{n+1} \int^{[0,1]} \psi^* H.$$

This proves the proposition. □

Both diffeomorphisms  $R$  and  $\tau$  preserve  $\Delta$ , and hence they extend to diffeomorphisms of  $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ . If also  $P$  extends, then the statement of Proposition 5.1.4 holds for  $P$  on  $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ .

In the rest of the section, we will be proving that  $P$  extends smoothly to  $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ . This is a local problem at the boundary, where we introduce the following radial coordinates. Define the set

$$X := \{(r, \eta, x) \in [0, \infty) \times \mathbb{S}^n \times \mathbb{S}^n \mid \eta \cdot x = 0\},$$

## 5. Explicit computations

and let  $\kappa : X \longrightarrow \text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$  be the map defined by

$$\kappa(r, \eta, x) := \begin{cases} \left( x, \frac{x + r\eta}{|x + r\eta|} \right) \in (\mathbb{S}^n \times \mathbb{S}^n) \setminus \Delta & \text{for } r > 0, \\ [(-\eta, \eta)] \in P^+ N_{(x,x)} \Delta & \text{for } r = 0. \end{cases}$$

For the upcoming computations, it is convention to define the map  $\gamma : \mathbb{R} \rightarrow (-1, 1)$  by

$$\gamma(r) := \frac{r}{\sqrt{1 + r^2} + 1} \quad \text{for all } r \in \mathbb{R}.$$

It is a diffeomorphism with inverse  $r = \frac{2\gamma}{1-\gamma^2}$ .

**Lemma 5.1.5** (Parametrization of the collar neighborhood). *The subset  $X \subset \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$  is a submanifold with boundary, and the map  $\kappa : X \longrightarrow \text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$  is an embedding onto a neighborhood of  $\partial \text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ .*

*Proof.* The set  $X$  is a Cartesian product of  $[0, \infty)$  and a regular level set; therefore, it is a submanifold with boundary. The inclusion  $\mathbb{S}^n \times \mathbb{S}^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$  induces an embedding of manifolds with boundary  $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n) \subset \text{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ . Consider the global chart  $\tilde{\mathbb{I}} : \text{Bl}_\Delta(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \rightarrow [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^{n+1}$  from (4.4) induced by the identity. We have

$$\begin{aligned} Y &:= \tilde{\mathbb{I}}(\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)) \\ &= \{(\tilde{r}, w, u) \in [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^{n+1} \mid |u|^2 + \tilde{r}^2 = 1, w \cdot u = 0\}, \end{aligned}$$

where we denote  $r$  on  $Y$  by  $\tilde{r}$  in order to distinguish it from  $r$  on  $X$ . It suffices to prove the claim for the map  $\mu := \tilde{\mathbb{I}} \circ \kappa : X \rightarrow Y$ . For  $(r, \eta, x) \in X$ , we compute

$$\mu(r, \eta, x) = \left( \frac{\gamma}{\sqrt{1 + \gamma^2}}, \frac{1}{\sqrt{1 + \gamma^2}}(\gamma x - \eta), \frac{1}{1 + \gamma^2}(x + \gamma\eta) \right).$$

This formula defines a smooth map of  $\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ . It is a local diffeomorphism because its Jacobian is non-vanishing:

$$|\text{D}\mu| = \frac{\partial \tilde{r}}{\partial r} \left( \frac{\partial w}{\partial \eta} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} \frac{\partial u}{\partial \eta} \right)^{n+1} = (-1)^{n+1} (1 + \gamma^2)^{-\frac{n+4}{2}} \frac{\partial \gamma}{\partial r}.$$

Moreover, the map  $\mu$  is injective, maps  $X$  into  $Y$  and  $\partial X$  onto  $\partial Y$ . The claim follows.  $\square$

## 5. Explicit computations

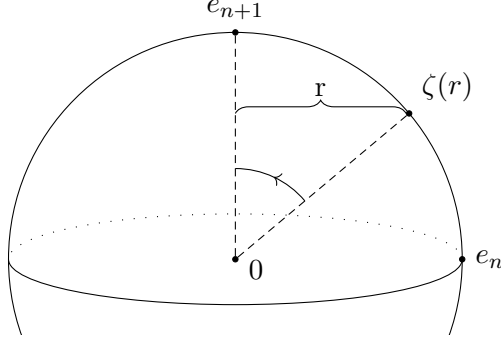


Figure 5.2.: The curve  $\zeta := \kappa \circ \zeta'$  is given by  $\zeta(r) = (e_{n+1}, \frac{e_{n+1} + r e_n}{|e_{n+1} + r e_n|})$  for  $r > 0$ .

Consider the action of  $O(n+1)$  on  $X$  defined by

$$R \cdot (r, \eta, x) := (r, R\eta, Rx) \quad \text{for all } (r, \eta, x) \in X \text{ and } R \in O(n+1).$$

Via  $\kappa$ , this agrees with the diagonal action of  $O(n+1)$  on  $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ . Denote

$$P' := \kappa^* P \in \Omega^{n-1}(\text{Int}(X)).$$

From Proposition 5.1.4 we get

$$R^* P' = (-1)^R P' \quad \text{for all } R \in O(n+1). \quad (5.6)$$

Consider the smooth curve (see Figure 5.2)

$$\begin{aligned} \zeta' : [0, \infty) &\longrightarrow X \\ r &\longmapsto (r, e_n, e_{n+1}). \end{aligned}$$

We have the following lemma.

**Lemma 5.1.6** (Smooth extension along the curve). *The form  $P'$  extends smoothly to  $X$  if and only if the map  $P' \circ \zeta' : (0, \infty) \rightarrow \Lambda^{n-1} T^* X$  extends smoothly to the interval  $[0, \infty)$ .*

*Proof.* As for the non-trivial implication, let  $(0, \eta_0, x_0) \in X$  be a boundary point. Pick vectors  $v_1, \dots, v_{n-1} \in \mathbb{R}^{n+1}$  so that the vectors  $v_1, \dots, v_{n-1}, \eta_0, x_0$  are linearly independent, and define the set

$$U := \{(r, \eta, x) \in X \mid v_1, \dots, v_{n-1}, \eta, x \text{ are linearly independent}\}.$$

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It is an open neighborhood of  $(0, \eta_0, x_0)$  in  $X$ . Applying the Gram-Schmidt orthogonalization to  $v_1, \dots, v_{n-1}, \eta, x$ , we find a smooth map  $R : U \rightarrow O(n+1)$  such that

$$R(r, \eta, x) \cdot (r, \eta, x) = (r, e_n, e_{n+1}) \quad \text{for all } (r, \eta, x) \in U.$$

The equation (5.6) implies

$$P'(r, \eta, x) = (-1)^R R(r, \eta, x)^* (P'(r, e_n, e_{n+1})) \quad \text{for all } (r, \eta, x) \in \text{Int}(U),$$

where  $R(r, \eta, x)^* : \Lambda^* T^* X \rightarrow \Lambda^* T^* X$  is the smooth cotangential map which is induced by the diffeomorphism  $R(r, \eta, x) : X \rightarrow X$ , and which maps the fiber over  $z \in X$  to the fiber over  $R(r, \eta, x)^{-1}z$ . By the assumption, all maps in the composition are smooth in their arguments. The lemma follows.  $\square$

**Lemma 5.1.7** (Local expression at the boundary). *On the interval  $(0, \infty)$ , we have*

$$\tilde{P}' \circ \zeta' = (-1)^{n+1} (1 + \gamma^2)^{-\frac{n-1}{2}} \sum_{k=0}^{n-1} \gamma^{n-k} (h_k \circ \gamma) (\nu_k \circ \zeta'),$$

where the functions  $h_k : (0, 1) \rightarrow \mathbb{R}$  are defined by

$$h_k(\gamma) := \int_{-1}^1 \frac{(u + \gamma^2)^k (u - 1)^{n-1-k}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du \quad \text{for all } \gamma \in (0, 1)$$

and the forms  $\nu_k \in \Omega(X)$  are defined by

$$\nu_k(r, x, \eta) := \frac{1}{k!(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n-1}} (-1)^\sigma dx^{\sigma_1} \dots dx^{\sigma_k} d\eta^{\sigma_{k+1}} \dots d\eta^{\sigma_{n-1}}.$$

*Proof.* We start with the following formula from the proof of Proposition 5.1.2:

$$\tilde{P} = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} (x^i y^j - y^i x^j) \int_{[0,1]} \frac{dt \psi^*(dy^1 \dots \widehat{dy^i} \dots \widehat{dy^j} \dots dy^{n+1})}{|(1-t)y - tx|^{n+1}}.$$

We restrict to the points  $(x, y) = \kappa(r, e_n, e_{n+1})$  with  $r > 0$ . There, we have

$$\begin{aligned} x^1 &= \dots = x^n = 0, \quad x^{n+1} = 1, \\ y^1 &= \dots = y^{n-1} = 0, \quad y^n = \frac{2\gamma}{1 + \gamma^2}, \quad y^{n+1} = \frac{1 - \gamma^2}{1 + \gamma^2}. \end{aligned}$$

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Under the substitution  $u = 2t - 1$ , we get

$$|(1-t)y - tx|^2 = \frac{4t(t-1)}{1+\gamma^2} + 1 = \frac{u^2 + \gamma^2}{1+\gamma^2}.$$

We make the following preliminary computations:

$$\begin{aligned} x^i y^j - y^i x^j &= 0 \quad \text{for } 1 \leq i \leq n-1 \text{ and } i < j \leq n+1, \\ x^n y^{n+1} - y^n x^{n+1} &= -\frac{2\gamma}{1+\gamma^2}, \\ \kappa^*(dy^i) &= \frac{1}{1+\gamma^2} ((1-\gamma^2) dx^i + 2\gamma d\eta^i) \quad \text{for } 1 \leq i \leq n-1. \end{aligned}$$

We plug these in the formula for  $\tilde{G}$  and get

$$\begin{aligned} \tilde{P}'(\zeta'(r)) &= 2\gamma(1+\gamma^2)^{\frac{n-1}{2}} \int_{[0,1]} dt \frac{\prod_{i=1}^{n-1} ((1-t)\kappa^*(dy^i) - t dx^i)}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} \\ &= (-1)^{n+1} \gamma(1+\gamma^2)^{-\frac{n-1}{2}} \int_{[-1,1]} du \frac{\prod_{i=1}^{n-1} ((u+\gamma^2) dx^i + \gamma(u-1) d\eta^i)}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} \\ &= (-1)^{n+1} (1+\gamma^2)^{-\frac{n-1}{2}} \sum_{k=0}^{n-1} \gamma^{n-k} \left( \int_{-1}^1 \frac{(u+\gamma^2)^k (u-1)^{n-1-k}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du \right) \nu_k. \end{aligned}$$

The lemma follows. □

**Lemma 5.1.8** (Integrals depending on parameter). *Let  $n \in \mathbb{N}$ , and let  $l = 0, 1, \dots, n-1$ . The function  $F_{n,l} : (0, \infty) \rightarrow \mathbb{R}$  defined by*

$$F_{n,l}(t) := \int_{-1}^1 \frac{t^{n-l} u^l}{(u^2 + t^2)^{\frac{n+1}{2}}} du \quad \text{for all } t \in (0, \infty) \quad (5.7)$$

*extends smoothly to  $\mathbb{R}$ .*

*Proof.* We have

$$F_{1,0}(t) = 2 \arctan\left(\frac{1}{t}\right) = \pi - 2 \arctan(t) \quad \text{for all } t \in (0, \infty).$$

The right-hand side is a smooth function on  $\mathbb{R}$ .

For  $n \geq 2$ , we deduce the recursive formula

$$F_{n,0}(t) = \frac{1}{n-1} \left( (n-2) F_{n-2,0}(t) + \frac{2t^{n-2}}{(1+t^2)^{\frac{n-1}{2}}} \right).$$



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If  $l$  is odd, then  $F_{n,l} \equiv 0$  for all  $n$  because the integrand of (5.7) is odd as a function of  $u$ .

For  $n \geq 3$  and even  $2 \leq l \leq n-1$ , we deduce yet another recursive formula

$$F_{n,l}(t) = \frac{1}{n-l} \left( (l-1)F_{n,l-2}(t) - \frac{2t^{n-l}}{(1+t^2)^{\frac{n-1}{2}}} \right).$$

The claim for all  $F_{n,l}$  follows by induction.  $\square$

**Proposition 5.1.9** (Smooth extension to the boundary). *The form  $P$  from (5.2) extends smoothly to  $\text{Bl}_\Delta(\mathbb{S}^n \times \mathbb{S}^n)$ .*

*Proof.* According to Lemmas 5.1.5 and 5.1.6, it suffices to show that the curve  $P' \circ \zeta' : (0, \infty) \rightarrow \Lambda^{n-1}T^*X$  extends smoothly to  $[0, \infty)$ . Lemma 5.1.7 gives an expression for  $P' \circ \zeta'$  as a linear combination of smooth forms  $\nu_k \in \Omega^{n-1}(X)$  with coefficients  $\gamma^{n-k}(h_k \circ \gamma)$  for  $k = 0, \dots, n-1$  multiplied by the overall coefficient  $(-1)^n(1+\gamma^2)^{-\frac{n-1}{2}}$ . We expand

$$\gamma^{n-k}(h_k \circ \gamma) = \sum_{a=0}^k \sum_{b=0}^{n-1-k} (-1)^{n-1-k-b} \binom{k}{a} \binom{n-1-k}{b} \int_{-1}^1 \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du$$

and notice that we can write

$$\int_{-1}^1 \frac{\gamma^{n+k-2a} u^{a+b}}{(u^2 + \gamma^2)^{\frac{n+1}{2}}} du = \gamma^{k-a+b} (F_{n,a+b} \circ \gamma)$$

for the function  $F_{n,l}$  from (5.7) with  $l := a+b$ . Because  $0 \leq l \leq n-1$ , Lemma 5.1.8 asserts that  $F_{n,l}$  extends smoothly to  $[0, \infty)$ . Because  $k-a+b \geq 0$ , the entire coefficient at  $\nu_k$  extends smoothly to  $[0, \infty)$  for every  $k = 0, \dots, n-1$ . The lemma follows.  $\square$

We summarize our results in the following proposition:

**Proposition 5.1.10** (for  $\mathbb{S}^n$ ). *The form  $P$  from (5.2) defines a form for  $\mathbb{S}^n$  satisfying Definition 4.2.4. Moreover, we have the symmetries*

$$\begin{aligned} R^*P &= (-1)^R P \quad \text{for all } R \in O(n+1) \text{ and} \\ \tau^*P &= (-1)^n P. \end{aligned}$$

*Proof.* The proposition is a summary of Propositions 5.1.2, 5.1.4 and 5.1.9.  $\square$

*Remark 5.1.11* (Better notation due to R. Bryant, see [Bry18]). Pick an oriented basis  $e_1, \dots, e_{n+1}$  of  $\mathbb{R}^{n+1}$  as generators of the exterior algebra  $\Lambda^*(\mathbb{R}^{n+1})$ , and view  $x, y, dx$ ,

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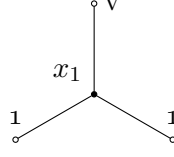


Figure 5.3.: The Y-graph for  $\mathbb{S}^n$ .

$dy$  as  $\Lambda^*(\mathbb{R}^{n+1})$ -valued forms on  $\mathbb{R}^{n+1}$ . For example, we view  $x$  as the map  $x \in \mathbb{R}^{n+1} \mapsto \sum_{i=1}^{n+1} x^i e_i \in \Lambda^1(\mathbb{R}^{n+1})$  and  $dx$  as the map  $x \in \mathbb{R}^{n+1} \mapsto \sum_{i=1}^{n+1} (dx_i)_x e_i \in \Lambda^1(\mathbb{R}^{n+1})$ . There is a natural wedge product on the space of  $\Lambda^*(\mathbb{R}^{n+1})$ -valued forms. If  $\omega$  is a top-form, we denote by  $[\omega]$  the coefficient of  $\omega$  at  $e_1 \wedge \cdots \wedge e_{n+1}$ . Then it holds

$$\omega_k(x, y) = \frac{1}{k!} \frac{1}{(n-1-k)!} [x \wedge y \wedge (dx)^k \wedge (dy)^{n-1-k}].$$

Note that if we view  $e_i$  as odd variables, then  $[\cdot]$  corresponds to the odd integration  $\int De(\cdot)$ . It would be interesting to know whether this notation simplifies some proofs, especially if Lemma 5.2.1 can be deduced from abstract algebraic facts or rules valid for odd integration.  $\triangleleft$

## 5.2. Computation of Chern-Simons Maurer-Cartan element for spheres

We recall from Definition 4.3.6 that the Chern-Simons Maurer-Cartan element  $\mathbf{n}$  is computed as a sum over trivalent ribbon graphs decorated with the admissible Hodge Propagator  $P$  at internal edges, integration variables  $x_i$  at internal vertices and, in the case of  $\mathbb{S}^n$ , with  $1$  or  $v$  at external vertices.

The canonical Maurer-Cartan element  $\mathbf{m}$  is the contribution of the Y-graph (see Figure 5.3), and it is easy to see that

$$\mathbf{m}_{10}(\mathbf{sv}11) = (-1)^n \mathbf{m}_{10}(\mathbf{s}1\mathbf{v}1) = \mathbf{m}_{10}(\mathbf{s}11\mathbf{v}) = (-1)^{n-2}.$$

Throughout this section, we will be in the setting of Definition 4.3.6. In particular,  $\Gamma \in \text{RG}_{klg}^{(3)}$  is a ribbon graph,  $L$  its compatible labeling admissible with respect to an input  $\omega_1, \dots, \omega_l$  and  $I(\sigma_L)$  the corresponding integral.

**Lemma 5.2.1** (Condition  $(V_1)$  holds). *For the Hodge Propagator  $P$  from (5.2) for  $\mathbb{S}^n$ , every graph  $\Gamma \neq Y$  with  $1$  at an external vertex vanishes.*

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*Proof.* The only contribution of an  $A$ -vertex which does not vanish for degree reasons is

$$A_{v,1}(y) = \int_x P(x, y) \text{Vol}(x).$$

From the symmetry of  $P$  and  $\text{Vol}$  under the action of  $O(n+1)$ , we get

$$R^* A_{v,1} = (-1)^R A_{v,1} \quad \text{for all } R \in O(n+1).$$

Therefore, it suffices to check that  $A_{v,1}(e_1) = 0$ , where  $e_1, \dots, e_{n+1}$  denotes the standard basis of  $\mathbb{R}^{n+1}$ . Evaluation of (5.4) at  $(x, e_1)$  gives

$$\omega_0(x, e_1) = \frac{1}{(n-1)!} \sum_{\substack{\sigma \in \mathbb{S}_{n+1} \\ \sigma_2=1}} (-1)^\sigma x^{\sigma_1} dy^{\sigma_3} \dots dy^{\sigma_{n+1}}.$$

Therefore, we get

$$\begin{aligned} A_{v,1}(e_1) &= (-1)^n \int_x g_0(x \cdot e_1) \omega_0(x, e_1) \text{Vol}(x) \\ &= \sum_{j=2}^{n+1} (-1)^{n+j+1} \left( \int_x g_0(x^1) x^j \text{Vol}(x) \right) dy^2 \dots \widehat{dy^j} \dots dy^{n+1}, \end{aligned}$$

where we view  $g_0$  as a function of  $x \cdot y$ . For every  $j = 2, \dots, n+1$ , consider the orientation reversing diffeomorphism

$$\begin{aligned} I_j : \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ (x^1, \dots, x^{n+1}) &\longmapsto (x^1, \dots, -x^j, \dots, x^{n+1}). \end{aligned}$$

Then we have

$$\int_x g_0(x^1) x^j \text{Vol}(x) = - \int_x I_j^* (g_0(x^1) x^j \text{Vol}(x)) = - \int_x g_0(x^1) (-x^j) (-\text{Vol}(x)),$$

and it follows that  $A_{v,1}(e_1) = 0$ .

Let us now consider the contribution of a  $B$ -vertex with 1:

$$B_1(y, z) = \int_x P(y, x) P(x, z) = (-1)^n \int_x P(y, x) P(x, z).$$

For  $n = 1$ , the degree of  $P(y, x) P(x, z)$  is 0, and hence  $B_1 = 0$  trivially. Suppose that

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$n \geq 2$ . As in the case of  $A_{v,1}$ , we get that

$$R^* B_1 = (-1)^R B_1 \quad \text{for all } R \in O(n+1).$$

Therefore, it suffices to check that  $B_1(e_1, c_1 e_1 + c_2 e_2) = 0$  for all  $(c_1, c_2) \in \mathbb{S}^1$ . We have

$$B_1(e_1, c_1 e_1 + c_2 e_2) = (-1)^n \sum_{a=1}^{n-1} \int_x g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) \omega_a(x, e_1) \omega_{n-a}(x, c_1 e_1 + c_2 e_2).$$

We will show that for every  $a = 1, \dots, n-1$  we can write

$$\mu_a(x) := \omega_a(x, e_1) \omega_{n-a}(x, c_1 e_1 + c_2 e_2) = \left( \sum_{i=3}^{n+1} \pm x^i \text{Vol}(x) \right) \eta_a(y, z) \quad (5.8)$$

for some form  $\eta_a(y, z)$ . Then, using the same argument as for  $A_{v,1}$ , we will have

$$\begin{aligned} & \int_x g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) x^i \text{Vol}(x) \\ &= - \int_x I_i^*(g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) x^i \text{Vol}(x)) \\ &= - \int_x g_a(x^1) g_{n-a}(c_1 x^1 + c_2 x^2) (-x^i) (-\text{Vol}(x)) \end{aligned}$$

for all  $3 \leq i \leq n+1$ , and hence  $B_1(e_1, c_1 e_1 + c_2 e_2) = 0$ .

In order to show (5.8), we have to study the product of  $\omega_i$ 's. From (5.4) we get

$$\begin{aligned} & \omega_a(x, y) \omega_{n-a}(x, z) \\ &= \frac{1}{a!(n-1-a)!(n-a)!(a-1)!} \sum_{\sigma, \mu \in \mathbb{S}_{n+1}} (-1)^{\sigma+\mu} x^{\sigma_1} x^{\mu_1} y^{\sigma_2} z^{\mu_2} \\ & \quad dx^{\sigma_3} \dots dx^{\sigma_{2+a}} dx^{\mu_3} \dots dx^{\mu_{2+n-a}} dy^{\sigma_{3+a}} \dots dy^{\sigma_{n+1}} \\ & \quad dz^{\mu_{3+n-a}} \dots dz^{\mu_{n+1}}. \end{aligned} \quad (5.9)$$

In order to simplify this expression, we decompose  $\sigma \in \mathbb{S}_{n+1}$  as

$$\sigma = \sigma^5 \circ \sigma^4 \circ \sigma^3 \circ \sigma^2 \circ \sigma^1, \quad (5.10)$$

where  $\sigma^1, \dots, \sigma^5 \in \mathbb{S}_{n+1}$  are permutations defined as follows:

- The permutation  $\sigma^1$  is a shuffle permutation  $\sigma^1 \in \mathbb{S}_{2+a, n-a-1}$  such that its first block denoted by  $\sigma^1(1) = (\sigma_1^1, \dots, \sigma_{2+a}^1)$  is equal to the ordered set  $\{\sigma_1, \dots, \sigma_{2+a}\}$ .

## 5. Explicit computations

The second block  $\sigma^1(2)$  is then the ordered set  $\{\sigma_{3+a}, \dots, \sigma_{n+1}\}$ , which will be denoted by  $J_\sigma$ .

- The permutation  $\sigma^2$  acts on the block  $\sigma^1(1)$  by moving  $\sigma_2$  in front. We denote the new block  $\sigma^1(1) \setminus \{\sigma_2\}$  by  $I_\sigma$ , so that we can write  $\sigma^2 : \sigma^1(1) \mapsto (\sigma_2, I_\sigma)$ .
- The permutation  $\sigma^3$  acts on the block  $I_\sigma$  by moving  $\sigma_1$  in front. Together with the previous step we get  $\sigma^1(1) \mapsto (\sigma_2, \sigma_1, I_\sigma \setminus \{\sigma_1\})$ .
- The permutation  $\sigma^4$  is a transposition of  $\sigma_1$  and  $\sigma_2$ .
- The permutation  $\sigma^5$  is determined by the pair  $(\sigma^{51}, \sigma^{52}) \in \mathbb{S}_a \times \mathbb{S}_{n-1-a}$  of permutations  $\sigma^{51}$  and  $\sigma^{52}$  acting on blocks  $I_\sigma \setminus \{\sigma_1\}$  and  $J_\sigma$  to get  $(\sigma_3, \dots, \sigma_{2+a})$  and  $(\sigma_{3+a}, \dots, \sigma_{n+1})$ , respectively.

We define the decomposition  $\mu^1, \dots, \mu^5$  for  $\mu \in \mathbb{S}_{n+1}$  from (5.9) analogously with  $a$  replaced by  $n - a$ . Using (5.10), the product (5.9) can be written as

$$\begin{aligned} & \frac{1}{a!(n-1-a)!(n-a)!(a-1)!} \sum_{\substack{\sigma^1, \dots, \sigma^5 \\ \mu^1, \dots, \mu^5}} (-1)^{\sigma^1 + \dots + \sigma^5 + \mu^1 + \dots + \mu^5} x^{\sigma^1} x^{\mu^1} y^{\sigma^2} z^{\mu^2} \\ & \quad dx^{\sigma^{51}(I_\sigma \setminus \{\sigma_1\})} dx^{\mu^{51}(I_\mu \setminus \{\mu_1\})} dy^{\sigma^{52}(J_\sigma)} dz^{\mu^{52}(J_\mu)} \\ & = - \sum_{\sigma^1, \mu^1} (-1)^{\sigma^1 + \mu^1} \left( \sum_{\sigma^2, \mu^2} (-1)^{\sigma^2 + \mu^2} \sum_{\sigma^3, \mu^3} (-1)^{\sigma^3 + \mu^3} x^{\sigma^1} x^{\mu^1} y^{\sigma^2} z^{\mu^2} \right. \\ & \quad \left. dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}} \right) dy^{J_\sigma} dz^{J_\mu}, \end{aligned}$$

where  $-1$  comes from  $(-1)^{\sigma^4}$  and  $\sigma^5$  is compensated by permutations of forms. For fixed  $\sigma^1$  and  $\mu^1$ , consider the coefficient at  $dy^{J_\sigma} dz^{J_\mu}$  in the brackets. If we evaluate it at  $y = e_1, z = c_1 e_1 + c_2 e_2$ , we get

$$\begin{aligned} & \overbrace{c_1 \sum_{\substack{\sigma^3, \mu^3 \\ \sigma_2=1 \\ \mu_2=1}} (-1)^{\sigma^3 + \mu^3} x^{\sigma^1} x^{\mu^1} dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}}} =: \text{I} \\ & + (-1)^{\mu^2} c_2 \overbrace{\sum_{\substack{\sigma^3, \mu^3 \\ \sigma_2=1 \\ \mu_2=2}} (-1)^{\sigma^3 + \mu^3} x^{\sigma^1} x^{\mu^1} dx^{I_\sigma \setminus \{\sigma_1\}} dx^{I_\mu \setminus \{\mu_1\}}} =: \text{II}, \end{aligned}$$

where  $(-1)^{\mu^2} = -1$  if and only if  $1 \in I_\mu$ .

## 5. Explicit computations

More generally, for multiindices  $I_1, I_2 \subset \{1, \dots, n+1\}$  of lengths  $a+1$  and  $n-a+1$ , respectively, consider the sum

$$S(I_1, I_2) := \sum_{\substack{i_1 \in I_1 \\ i_2 \in I_2}} (-1)^{(i_1, I_1) + (i_2, I_2)} x^{i_1} x^{i_2} dx^{I_1 \setminus \{i_1\}} dx^{I_2 \setminus \{i_2\}}, \quad (5.11)$$

where  $(i_j, I_j)$  is the number of transpositions required to move  $i_j$  in front of  $I_j$ . The following implication holds:

$$S(I_1, I_2) \neq 0 \implies 1 \leq |I_1 \cap I_2| \leq 2.$$

We distinguish the two cases left:

*Case  $I_1 \cap I_2 = \{i, j\}$  with  $i < j$ :* We get

$$\begin{aligned} S(I_1, I_2) &= (-1)^{(i, I_1) + (j, I_2)} x^i x^j dx^{I_1 \setminus \{i\}} dx^{I_2 \setminus \{j\}} \\ &\quad + (-1)^{(j, I_1) + (i, I_2)} x^j x^i dx^{I_1 \setminus \{j\}} dx^{I_2 \setminus \{i\}} \\ &= (-1)^{(i, I_1) + (j, I_2) + (j, I_1) + 1 + (i, I_2)} x^i x^j dx^j dx^{I_1 \setminus \{i, j\}} \\ &\quad dx^i dx^{I_2 \setminus \{i, j\}} + (-1)^{(j, I_1) + (i, I_2) + (i, I_1) + (j, I_2) + 1} x^i x^j \\ &\quad dx^i dx^{I_1 \setminus \{i, j\}} dx^j dx^{I_2 \setminus \{i, j\}} \\ &= \pm(-1+1)x^i x^j dx^i dx^{I_1 \setminus \{i, j\}} dx^j dx^{I_2 \setminus \{i, j\}}, \end{aligned}$$

where in the last step we switched  $dx^i \leftrightarrow dx^j$  in the first summand. Therefore, it holds  $S(I_1, I_2) = 0$ .

*Case  $I_1 \cap I_2 = \{i\}$ :* We must have  $I_1 \cup I_2 = \{1, \dots, n+1\}$ . A non-zero summand in (5.11) has either  $i_1 = i$  and  $i_2 \in I_2$ , in which case

$$I_1 \setminus \{i_1\} \cup I_2 \setminus \{i_2\} = \{1, \dots, \widehat{i_2}, \dots, n+1\},$$

or  $i_2 = i$  and  $i_1 \in I_1$  with  $i_1 \neq i$ , in which case

$$I_1 \setminus \{i_1\} \cup I_2 \setminus \{i_2\} = \{1, \dots, \widehat{i_1}, \dots, n+1\}.$$

Indices  $i_2$  from the first case and  $i_1$  from the second case constitute  $\{1, \dots, n+1\}$ . Therefore, for some signs  $\pm$ , we can write

$$S(I_1, I_2) = x^i \sum_{j=1}^{n+1} \pm x^j dx^1 \dots \widehat{dx^j} \dots dx^{n+1}.$$

## 5. Explicit computations

We will prove that the signs alternate, and hence  $S(I_1, I_2) = \pm x^i \text{Vol}(x)$ . Suppose that  $j, j+1 \in I_1$  for some  $j \in \{1, \dots, n\}$ . The two summands in (5.11) with  $(i_1, i_2) = (j, i)$  and  $(i_1, i_2) = (j+1, i)$ , respectively, give

$$\begin{aligned} & (-1)^{(j, I_1) + (i, I_2)} x^j x^i \, dx^{I_1 \setminus \{j\}} \, dx^{I_2 \setminus \{i\}} + (-1)^{(j+1, I_1) + (i, I_2)} x^{j+1} x^i \\ & \quad dx^{I_1 \setminus \{j+1\}} \, dx^{I_2 \setminus \{i\}} \\ &= (-1)^{(i, I_2)} x^j x^i \, dx^{j+1} \, dx^{I_1 \setminus \{j, j+1\}} \, dx^{I_2 \setminus \{i\}} \\ & \quad + (-1)^{1 + (i, I_2)} x^{j+1} x^i \, dx^j \, dx^{I_1 \setminus \{j, j+1\}} \, dx^{I_2 \setminus \{i\}} \\ &= (-1)^{(i, I_2)} x^i (x^j \, dx^{j+1} - x^{j+1} \, dx^j) \, dx^{I_1 \setminus \{j, j+1\}} \, dx^{I_2 \setminus \{i\}}. \end{aligned}$$

The signs clearly alternate. A symmetric argument holds when  $j, j+1 \in I_2$ . Now assume that  $j \in I_1$  and  $j+1 \in I_2$ . The two summands in (5.11) which have  $(i_1, i_2) = (j, i)$  and  $(i_1, i_2) = (i, j+1)$ , respectively, give

$$\begin{aligned} & (-1)^{(j, I_1) + (i, I_2)} x^j x^i \, dx^{I_1 \setminus \{j\}} \, dx^{I_2 \setminus \{i\}} + (-1)^{(i, I_1) + (j+1, I_2)} x^i x^{j+1} \\ & \quad dx^{I_1 \setminus \{i\}} \, dx^{I_2 \setminus \{j+1\}} \\ &= (-1)^{(j, I_1) + (i, I_1 \setminus \{j\}) + (a+1)} x^j x^i \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2} \\ & \quad + (-1)^{(i, I_1) + (j+1, I_2) + (j, I_1 \setminus \{i\})} x^i x^{j+1} \, dx^j \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}} \\ &= (-1)^{(j, I_1) + (i, I_1 \setminus \{j\}) + (j+1, I_2)} x^i x^j \, dx^{j+1} \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}} \\ & \quad + (-1)^{1 + (j, I_1) + (i, I_1 \setminus \{j\}) + (j+1, I_2)} x^i x^{j+1} \, dx^j \, dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}} \\ &= (-1)^{(j, I_1) + (i, I_1 \setminus \{j\}) + (j+1, I_2)} x^i (x^j \, dx^{j+1} - x^{j+1} \, dx^j) \\ & \quad dx^{I_1 \setminus \{i, j\}} \, dx^{I_2 \setminus \{j+1\}}. \end{aligned}$$

The signs alternate again. A symmetric argument holds for  $j \in I_2$  and  $j+1 \in I_1$ .

Back to the original problem, we have  $I = S(I_\sigma, I_\mu)$  with  $I_\sigma, I_\mu \subset \{2, \dots, n+1\}$ . It follows that the first case applies, and hence  $I = 0$ . We have  $II = S(I_\sigma, I_\mu)$  with  $I_\sigma \subset \{2, \dots, n+1\}$  and  $I_\mu \subset \{1, \widehat{2}, \dots, n+1\}$ . It follows that either the first case or the second case with  $i \geq 3$  applies. This proves (5.8). Consequently, we get  $B_1 = 0$  also for  $n \geq 2$ .

The last paragraph of the proof of Proposition 4.4.2 finishes the proof.  $\square$

We summarize the consequences in the following proposition. The main argument is the same as in the proof of Proposition (4.4.5).

**Proposition 5.2.2** (Vanishing of graphs for  $\mathbb{S}^n$ ). *Consider  $\mathbb{S}^n$  with the Hodge Propagator (5.2). Only the following trivalent ribbon graphs  $\Gamma \neq Y$  do not necessarily vanish:*

## 5. Explicit computations

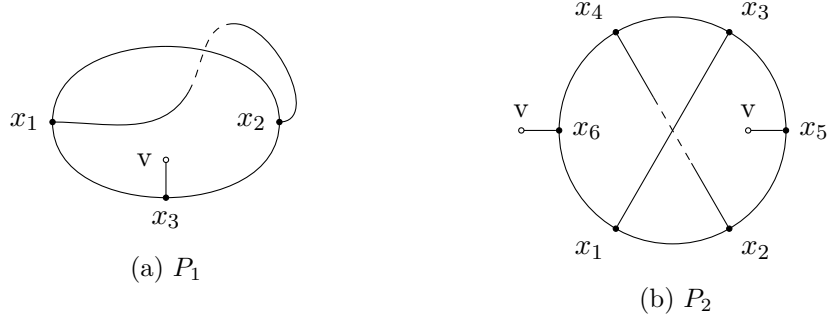


Figure 5.4.: Vanishing graphs  $P_1$  and  $P_2$  for  $\mathbb{S}^2$ .

( $n = 1$ ): The  $O_k$ -graph with  $k \in 2\mathbb{N}$  internal vertices of type  $B$  with  $v$  at the external vertex (see Figure 5.7).

( $n = 2$ ): It must hold  $A = 0$ ,  $C = 2B$  and all  $B$  vertices must have  $v$  at the external vertex. Moreover, if  $\Gamma$  is reduced, it must have  $g \geq 1$ .

( $n = 3$ ): There is no external vertex and  $4 \mid C$  holds.

( $n > 3$ ): All graphs vanish.

*Proof.* Lemma 5.2.1 implies that  $A = 0$  and that the total form-degree  $D$  satisfies  $D = nB$ . Therefore, we get from (4.26) the following: for  $n > 3$  there is neither a  $B$ -vertex nor a  $C$ -vertex; for  $n = 3$ , there is no  $B$ -vertex; for  $n = 2$ , we have  $C = 2B$ ; and for  $n = 1$ , there is no  $C$ -vertex.

Consider the pullback of  $I(\sigma_L)$  along the (multi)diagonal action of an  $R \in O(n+1)$  with  $\det(R) = -1$  on  $(\mathbb{S}^n)^{\times k}$ . We get schematically

$$\int_{(\mathbb{S}^n)^{\times k}} P^e \text{Vol}^s = (-1)^{k+e+s} \int_{(\mathbb{S}^n)^{\times k}} G^e \text{Vol}^s.$$

Therefore,  $k + e + s$  has to be even. If we plug-in from (4.23), we get

$$k + e + s = \begin{cases} 3B & \text{for } n = 1, \\ 8B & \text{for } n = 2, \\ \frac{5}{2}C & \text{for } n = 3. \end{cases}$$

A non-vanishing reduced graph must have  $B \geq l$ . For  $n = 2$ , so that  $C = 2B$ , the formula (4.24) gives  $g \geq 1$ . □



## 5. Explicit computations

*Remark 5.2.3* (Graphs for  $\mathbb{S}^2$ ). The simplest possibly non-vanishing graph for  $\mathbb{S}^2$  has  $A = 0$ ,  $B = 1$ ,  $C = 2$ . If it is reduced, we must have  $l = g = 1$ , and hence it will contribute to  $\mathbf{n}_{11}$ . Up to an isomorphism, there is only one such graph, which we denote by  $P_1$  (see Figure 5.4). However, we see that the pair of internal vertices  $x_1$  and  $x_2$  is connected by two edges, which implies that  $P_1 = 0$ . Indeed,  $P(x, y)$  has odd degree, and hence we have<sup>1</sup>

$$P(x, y)P(y, x) = P(x, y)^2 = 0$$

by the symmetry on the pullback along the twist map. It follows that  $\mathbf{n}_{11} = 0$ .

The second simplest possibly non-vanishing reduced graph is the graph  $P_2$  from Figure 5.4. Let

$$\begin{aligned} \eta(x_1, x_2, x_3, x_4, x_5) &:= P(x_1, x_2)P(x_1, x_3)P(x_4, x_2)P(x_4, x_3)P(x_3, x_5) \\ &\quad P(x_2, x_5)\text{Vol}(x_5) \end{aligned}$$

denote the form in the integrand coming from the part of the graph on the right-hand side of the vertical axis going through  $x_1, x_4$ . If  $\tau_{1,4}$  denotes the exchange of  $x_1$  and  $x_4$ , then clearly  $\tau_{1,4}^*\eta = \eta$  because the graph is symmetric with respect to the horizontal axis going through  $x_5, x_6$ . Using this, we compute

$$\begin{aligned} &\int_{x_1, x_2, x_3, x_4, x_5, x_6} v(x_6)P(x_1, x_6)P(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5) \\ &= \int_{\tau_{1,4}(x_1, x_2, x_3, x_4, x_5, x_6)} \tau_{1,4}^*(v(x_6)P(x_1, x_6)P(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5)) \\ &= \int_{x_4, x_2, x_3, x_1, x_5, x_6} v(x_6)P(x_4, x_6)P(x_1, x_6)\eta(x_4, x_2, x_3, x_1, x_5) \\ &= - \int_{x_1, x_2, x_3, x_4, x_5, x_6} v(x_6)P(x_1, x_6)P(x_4, x_6)\eta(x_1, x_2, x_3, x_4, x_5), \end{aligned}$$

where the minus sign comes from switching the first two P's. We see that  $P_2$  vanishes. The other variants with  $x_5$  moved on the edge  $x_3, x_4$  and  $x_2, x_4$  vanish by a similar argument using the compositions  $\tau_{1,3} \circ \tau_{5,6}$  and  $\tau_{1,2} \circ \tau_{5,6}$ , respectively. We conclude that  $\mathbf{n}_{21} = 0$ , and hence  $\mathbf{q}_{121}^{\mathbf{n}} = 0$ .

We sum up some general observations about the integrals for  $\mathbb{S}^2$ :

- We have  $B_1 \neq 0$  and  $C \neq 0$  for the corresponding forms.

---

<sup>1</sup>We recall from Section 4.3 that the notation  $P(x_i, x_j)$  means  $(\pi_i \times \pi_j)^*P$  and not just the evaluation at  $(x_i, x_j)$ .

## 5. Explicit computations

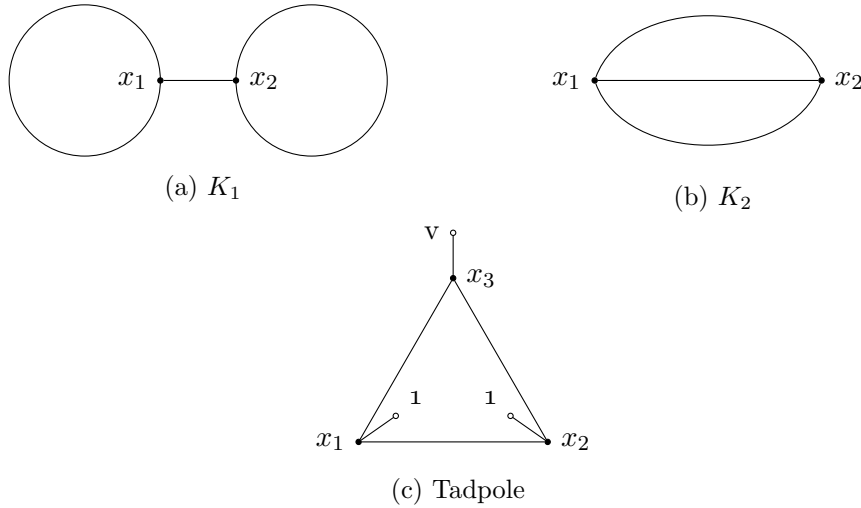


Figure 5.5.: Graphs  $K_1$  and  $K_2$  from the Chern-Simons theory and the tadpole graph with  $(l, g) = (2, 0)$  for  $n = 3$ .

- We have the multiplication formula (c.f., Example 5.1.3)

$$\omega_1(x, y)\omega_1(x, z) = x \cdot (y \times z)\text{Vol}(x).$$

- The number  $(-1)^{\sigma_L} I(\sigma_L)$  does not depend on the choice of  $L_1$  provided a compatible  $L_2$  is chosen.
- It holds  $\sum_{L_3^b} (-1)^{\sigma_L} I(\sigma_L) = 0$  whenever there is a boundary component with even number of  $v$ 's.
- If there is a  $B$ -vertex  $x$  such that the underlying graph (after forgetting the ribbon structure) is symmetric on the reflection along an axis going through  $x$ , then  $I(\sigma_L) = 0$ . ◁

*Remark 5.2.4* (Graphs for  $\mathbb{S}^3$ ). For  $\mathbb{S}^3$ , we consider the non-reduced graphs  $K_1$  and  $K_2$  and the tadpole graph from Figure 5.5. The graphs  $K_1$  and  $K_2$  appear in the definition of the Chern-Simons topological invariant in [Koh02] (with a gauge group). The corresponding integrals from our theory vanish “algebraically”, i.e., at the level of wedge products of  $\omega_i$ . Indeed, every summand in  $K_1$  contains

$$\omega_a(x_1, x_1) = 0 \quad \text{for some } a = 0, 1, 2,$$

DONE Non-sense about lower bound:  $g \geq 1$  is equivalent to  $B \geq l$  and moreover we must have  $C = 2B$ . What else was I thinking here?

## 5. Explicit computations

and, for degree reasons, the form part of  $K_2$  can contain only

$$\omega_1(x_1, x_2)^3 = 0 \quad \text{or} \quad \omega_0(x_1, x_2)\omega_1(x_1, x_2)\omega_2(x_1, x_2) = 0.$$

The tadpole graph contains only

$$\omega_2(x_1, x_3)\omega_1(x_1, x_2)\omega_2(x_2, x_3) = 0. \quad \triangleleft$$

Equations in Remarks 5.2.3 and 5.2.4 were checked by the computer. The program for Wolfram Mathematica 10.4 will be made available at [Háj18a].

We will now compute  $\mathfrak{n}_{20}$  for  $\mathbb{S}^1$ , which according to Proposition 5.2.2 consists only of contributions from the  $O_k$ -graphs with  $k$  even. By analogy with the finite dimensional case (see Appendix A), we expect that the number  $(-1)^{\sigma_L} I(\sigma_L)$  does not depend on  $L$ . All inputs are namely the same and the degrees even, i.e.,  $|m_2^+| = -2$ ,  $|\theta^2 P| = -2$  and  $|v| = 0$ .

We fix  $s_1, s_2 \geq 1$  such that  $k = s_1 + s_2$  is even and make the ansatz

$$n_{20}(\mathfrak{sv}^{s_1} \otimes \mathfrak{sv}^{s_2}) := \varepsilon(s_1, s_2) C(s_1, s_2) I(k),$$

where  $I(k)$  is the integral

$$\frac{1}{V^k} \int_{x_1, \dots, x_k} G(x_1, x_2) \cdots G(x_{k-1}, x_k) G(x_k, x_1) \text{Vol}(x_1) \cdots \text{Vol}(x_k), \quad (5.12)$$

$\varepsilon(s_1, s_2)$  a sign and  $C(s_1, s_2)$  a combinatorial coefficient to be determined.

We fix a circle in the plane with  $k$  points (= internal vertices) and denote by  $O(s_1, s_2)$  the set of ribbon graphs constructed by attaching external legs from which  $s_1$  points in the interior and  $s_2$  in the exterior, or the other way round, so that  $O(s_1, s_2) = O(s_2, s_1)$  (see Figure 5.7). Recall that the ribbon structure is induced from the counterclockwise orientation of the plane. It is easy to see that all graphs in  $O(s_1, s_2)$  admit a labeling which is admissible with respect to  $\mathfrak{sv}^{s_1} \otimes \mathfrak{sv}^{s_2}$ , and that  $O(s_1, s_2)$  contains a representative of every such  $O_k$ -graph.

DONE Sign not necessary: There is no need to write  $(-1)^{k+1}$  when we know that  $k$  is even!!! Who cares? Let's keep it so that it is clear where the sign comes from.

**Lemma 5.2.5** (Integral for the  $O_k$ -graph for  $\mathbb{S}^1$ ). *For every even  $k \geq 2$ , the integral*

## 5. Explicit computations

$I(k)$  is equal to

$$(-1)^{k+1} \frac{1}{2^k} \sum_{i=2,4,\dots,k} \frac{i}{(i+1)!} \sum_{\substack{i_1+\dots+i_r=k-i \\ i_1,\dots,i_r \in 2\mathbb{N}, r \in \mathbb{N}}} (-1)^r \frac{1}{(i_1+1)! \cdots (i_r+1)!}. \quad (5.13)$$

*Proof.* Denote  $\bar{P}(x, y) := -2\pi P$ . For all  $k, l \geq 1$ , we consider the more general integral

$$I(k, l) := \int_{x_1, \dots, x_k} \bar{P}(x_1, x_2) \cdots \bar{P}(x_{k-1}, x_k) \bar{P}(x_k, x_1)^l \text{Vol}(x_1) \cdots \text{Vol}(x_k).$$

Taking the pullback along  $(x_1, x_2, \dots, x_{k-1}, x_k) \mapsto (x_k, x_{k-1}, \dots, x_2, x_1)$  and using the antisymmetry of  $\bar{P}(x, y)$ , we get  $I(k, l) = 0$  whenever  $k + l$  is even. We will compute  $I(k, 1)$  for  $k \in 2\mathbb{N}$  from a recursive relation which arises from successive integration.

For the recursion step, we need to evaluate the integral

$$\int_y \bar{P}(x, y) \bar{P}(y, z)^l \text{Vol}(y)$$

for fixed  $(x, z) \in (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta$ . Pick the chart  $g : \mathbb{S}^1 \setminus \{z\} \rightarrow (-\pi, \pi)$  defined by

$$g(y) = \bar{P}(y, z) = \pi - \alpha(y, z) \quad \text{for } y \in \mathbb{S}^1 \setminus \{z\},$$

where the angle  $\alpha$  was defined in Example 5.1.3. It holds  $dg(y) = \text{Vol}(y)$  and

$$\bar{P}(x, y) = \begin{cases} \bar{P}(x, z) - g(y) - \pi & \text{for } -\pi < g(y) < \bar{P}(x, z), \\ \bar{P}(x, z) - g(y) + \pi & \text{for } \bar{P}(x, z) < g(y) < \pi. \end{cases}$$

We compute

$$\begin{aligned} \int_y \bar{P}(x, y) \bar{P}(y, z)^l \text{Vol}(y) &= \int_{-\pi}^{\pi} (\bar{P}(x, z) - g) g^l dg - \pi \int_{-\pi}^{\bar{P}(x, z)} g^l dg \\ &\quad + \pi \int_{\bar{P}(x, z)}^{\pi} g^l dg \\ &= \frac{2\pi}{l+1} \begin{cases} \pi^l \bar{P}(x, z) - \bar{P}(x, z)^{l+1} & \text{for } l \text{ even,} \\ \frac{\pi^{l+1}}{l+2} - \bar{P}(x, z)^{l+1} & \text{for } l \text{ odd.} \end{cases} \end{aligned}$$

From now on,  $\int$  will stand for the Riemannian integral, i.e.,  $\int f := \int f \text{Vol}$  for a

## 5. Explicit computations

function  $f$ . We compute

$$\begin{aligned} I(2, l) &= \int_{x_1, x_2} \bar{P}(x_1, x_2) \bar{P}(x_2, x_1)^l = - \int_{yz} \bar{P}(y, z)^{l+1} = -2\pi \int_{-\pi}^{\pi} g^{l+1} dg \\ &= \begin{cases} 0 & \text{for } l \text{ even,} \\ -\frac{4\pi^{l+3}}{l+2} & \text{for } l \text{ odd.} \end{cases} \end{aligned}$$

For  $k \geq 4$  even and  $l$  odd, we compute

$$\begin{aligned} I(k, l) &= \frac{2\pi}{l+1} \int_{x_1, \dots, x_{k-1}} \bar{P}(x_1, x_2) \cdots \bar{P}(x_{k-2}, x_{k-1}) \\ &\quad \left( \frac{\pi^{l+1}}{l+2} - \bar{P}(x_{k-1}, x_1)^{l+1} \right) \\ &= \frac{-4\pi^2}{(l+1)(l+2)} \int_{x_1, \dots, x_{k-2}} \bar{P}(x_1, x_2) \cdots \bar{P}(x_{k-3}, x_{k-2}) \\ &\quad \left( \pi^{l+1} \bar{P}(x_{k-2}, x_1) - \bar{P}(x_{k-2}, x_1)^{l+2} \right) \\ &= \frac{4\pi^2}{(l+1)(l+2)} (-\pi^{l+1} I(k-2, 1) + I(k-2, l+2)). \end{aligned}$$

For the second equality, we used  $\int_{x_1} \bar{P}(x_1, x_2) = 0$  to show that the term multiplied by  $\frac{\pi^{l+1}}{l+2}$  vanishes. It follows that

$$\begin{aligned} I(k, 1) &= \frac{(2\pi)^{k-2}}{(k-1)!} I(2, k-1) - \sum_{l=2,4,\dots,k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l, 1) \\ &= -\frac{k(2\pi^2)^k}{(k+1)!} - \sum_{l=2,4,\dots,k-2} \frac{(2\pi^2)^{k-l}}{(k-l+1)!} I(l, 1) \quad \text{for all } k = 2, 4, \dots \end{aligned}$$

This is a recursive equation of the form  $a_k = c_k + \sum_{l=1}^{k-1} d_{k-l} a_l$ . Its solution is  $a_k = \sum_{i=1}^k c_i D_{k-i}$  with  $D_0 := 1$  and  $D_i = \sum d_{i_1} \cdots d_{i_r}$ , where we sum over all  $r = 1, \dots, i$  and  $i_1, \dots, i_r \in \mathbb{N}$  such that  $i_1 + \cdots + i_r = i$ . Therefore, we get

$$I(k, 1) = -(2\pi^2)^k \sum_{i=2,4,\dots,k} \frac{i}{(i+1)!} \sum_{\substack{i_1+\dots+i_r=k-i \\ i_1,\dots,i_r \in 2\mathbb{N}, r \in \mathbb{N}}} (-1)^r \frac{1}{(i_1+1)! \cdots (i_r+1)!}.$$

The result has to be multiplied by  $(-1)^k (2\pi)^{-2k}$  in order to get  $I(k)$ . □

**Lemma 5.2.6** (Independence of labeling). *The summand  $(-1)^{\sigma_L} I(\sigma_L)$  in the definition of  $\mathfrak{n}_{20}(\mathfrak{sv}^{s_1} \otimes \mathfrak{sv}^{s_2})$  for  $\mathbb{S}^1$  is independent of the choice of  $\Gamma \in O(s_1, s_2)$  and its labeling  $L$*

## 5. Explicit computations

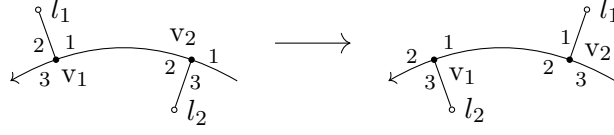


Figure 5.6.: Swapping adjacent legs.

which is compatible and admissible with respect to the input.

*Proof.* Pick  $\Gamma \in O(s_1, s_2)$  and its admissible labeling  $L$ . Let  $L'$  be an other admissible labeling of  $\Gamma$ . We distinguish the following situations:

- Suppose that  $L$  and  $L'$  differ by a permutation  $\mu$  in  $L_3^b$ . A similar argument as in the proof of Lemma 4.3.7 shows that  $(-1)^{\sigma_{L'}} = (-1)^\mu (-1)^{\sigma_L}$  and  $I(\sigma_{L'}) = (-1)^\mu I(\sigma_L)$ , where the sign in the integral comes from the permutation of Vol's, which have form-degree 1. Hence  $(-1)^{\sigma_{L'}} I(\sigma_{L'}) = (-1)^{\sigma_L} I(\sigma_L)$ .
- Suppose that the boundaries are permuted, i.e., that  $L$  and  $L'$  differ in  $L_1^b$ . Notice that  $s_1 = s_2$  because otherwise one of  $L$  or  $L'$  would not be admissible. The sign from changing  $L_1^b$  cancels as in the previous case.
- Suppose that  $L$  and  $L'$  differ in  $L_2$ . It was explained in the proof of Lemma 4.3.7 that a single change of  $L_2$  induces the sign  $(-1)^{n-1} = 1$  in  $(-1)^{\sigma_L} I(\sigma_L)$ .
- A cyclic permutation in  $L_3^v$  induces a sign neither in  $(-1)^{\sigma_L}$  nor in  $I(\sigma_L)$ .
- A permutation  $\mu$  in  $L_1^v$  induces  $(-1)^\mu$  in  $(-1)^{\sigma_L}$  and a change in  $I(\sigma_L)$ , which can be realized by taking the pullback along  $\mu : (x_1, \dots, x_k) \mapsto (x_{\mu_1}, \dots, x_{\mu_k})$ . However, the sign of the Jacobian is  $(-1)^\mu$ , which cancels the sign from  $(-1)^{\sigma_L}$ .

Next, we prove the independence of  $\Gamma \in O(s_1, s_2)$ . Let  $L$  be an admissible and compatible labeling of  $\Gamma$ . Pick two adjacent internal vertices with external legs pointing to different regions, i.e., one to the interior of the circle and the other to the exterior. Suppose that the vertices are labeled by  $v_1$  and  $v_2$  and the legs by  $l_1$  and  $l_2$ , respectively. Let  $\Gamma' \in O(s_1, s_2)$  be the graph with the two legs turned inside out (see Figure 5.6). We can construct an admissible and compatible labeling  $L'$  of  $\Gamma'$  by making the following changes to  $L$ : The new leg at  $v_1$  will be labeled by  $l_2$  and the new leg at  $v_2$  by  $l_1$ . The cyclic orderings at  $v_1$  and  $v_2$ , respectively, have to be modified by a transposition in order to get compatibility with the new ribbon structure. All other labelings can be copied from  $L$ . In total, we get

$$(-1)^{\sigma_L - \sigma_{L'}} = -1.$$

## 5. Explicit computations

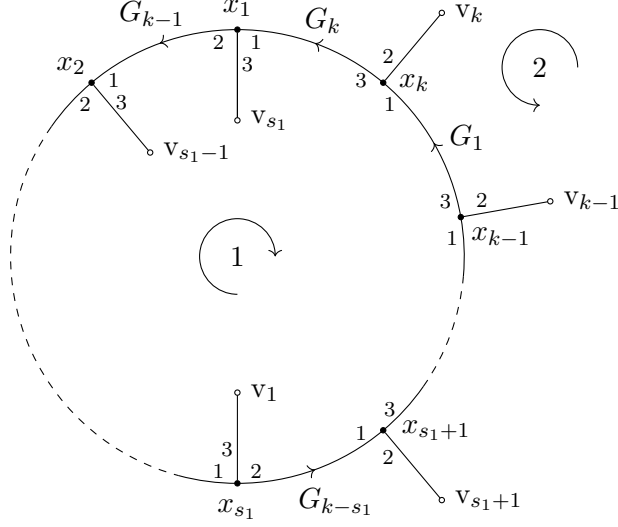


Figure 5.7.: The graph  $\Gamma^*$  with the labeling  $L^*$ . It can be checked that  $L_1$  and  $L_2$  are compatible.

This sign is compensated by swapping the one-forms in  $I(\sigma_L)$ :

$$\text{Vol}(x_{v_1}) \dots \text{Vol}(x_{v_2}) \longleftrightarrow \text{Vol}(x_{v_2}) \dots \text{Vol}(x_{v_1}).$$

The independence of  $\Gamma \in O(s_1, s_2)$  follows from the fact that we can span the entire  $O(s_1, s_2)$  by repeating the swap-of-legs operation.  $\square$

**Lemma 5.2.7** (Sign). *We have*

$$\varepsilon(s_1, s_2) = (-1)^{s_1+1}.$$

*Proof.* By Lemma 5.2.6, in order to compute  $(-1)^{\sigma_L} I(\sigma_L)$ , we can pick  $\Gamma^* \in O(s_1, s_2)$  and its admissible and compatible labeling  $L^*$  from Figure 5.7. We abbreviate  $\sigma_0 = \sigma_{L^*}$ . The corresponding integral (4.20) reads

$$I(\sigma_0) = \frac{1}{V^k} \int_{x_1, \dots, x_k} G(x_{k-1}, x_k) \cdots G(x_1, x_2) G(x_k, x_1) \text{Vol}(x_{s_1}) \cdots \text{Vol}(x_1) \\ \text{Vol}(x_{s_1+1}) \cdots \text{Vol}(x_k).$$

It differs from  $I(k)$  from (5.12) in the order of P's and Vol's. A reordering produces the sign

$$(-1)^{\frac{1}{2}s_1(s_1-1)}.$$

## 5. Explicit computations

We will compute  $(-1)^{\sigma_0}$  by ordering half-edges from the edge order back to the vertex order while looking at Figure 5.7. The steps are as follows:

- Transpose half-edges at internal vertices so that the first half-edge goes inside the vertex and the third outside with respect to the counterclockwise orientation. This gives  $(-1)^a$ .
- Permute external legs so that  $v_i$  is at  $x_i$  for all  $i = 1, \dots, k$ . This gives

$$(-1)^{\frac{1}{2}s_1(s_1-1)}.$$

- Permute internal edges so that  $P_i$  starts at the third half-edge of  $x_i$  and ends at the first half-edge of  $x_{i+1}$  for all  $i = 1, \dots, k-1$ . This does not produce any sign as swapping of two  $P$ 's requires two transpositions.
- At this point, we have the permutation

$$\begin{pmatrix} 1 & 2 & \dots & 2(e-1)-1 & 2(e-1) & 2e-1 & 2e & 2e+1 & \dots & 3k \\ 3 & 4 & \dots & 3k-3 & 3k-2 & 3k & 1 & 2 & \dots & 3k-1 \end{pmatrix}.$$

We interpret the last line as  $P_1 \dots P_k v_1 \dots v_k$  and permute it to the sequence  $v_1 P_1 v_2 P_2 \dots v_k P_k$ , which does not produce any sign. We end up with

$$\sigma'_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & 3k-1 & 3k \\ 2 & 3 & 4 & \dots & 3k & 1 \end{pmatrix}.$$

It is now easy to see that

$$(-1)^{\sigma'_0} = (-1)^{3k-1}.$$

In total, we get

$$(-1)^{\sigma_0} = (-1)^{s_1 + \frac{1}{2}s_1(s_1-1) + k+1}.$$

As for the other signs in Definition 4.3.6, we have  $s(k, l) = k + \frac{1}{2}k(k-1)$  and  $P(v^k) = \frac{1}{2}k(k-1)$ . There is no sign from  $s^2 v^{s_1} \otimes v^{s_2} = s v^{s_1} \otimes s v^{s_2}$  since  $|s| = -2$ . Multiplying everything together, we get  $\varepsilon(s_1, s_2)$ .  $\square$

**Lemma 5.2.8** (Combinatorial coefficient). *It holds*

$$C(s_1, s_2) = \frac{1}{2}ak! \binom{k-1}{s_1}.$$



## 5. Explicit computations

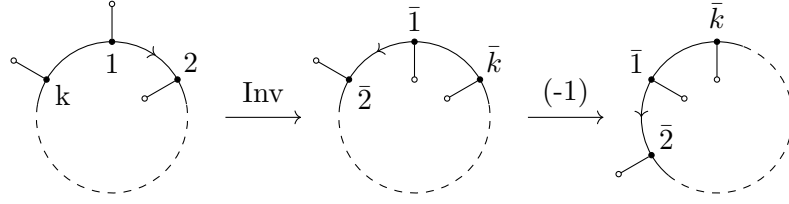


Figure 5.8.: The mirror isomorphism  $M : 1 \dots k \mapsto \bar{k} \dots \bar{1}$  is a composition of the inversion and the counterclockwise rotation by one place.

*Proof.* Every isomorphism of ribbon graphs  $\Gamma$  and  $\Gamma'$  from  $O(s_1, s_2)$  is a composition of the clockwise rotation  $(r)$  for  $r \in \mathbb{Z}$  and the mirror operation  $M$  defined as follows: If  $1, \dots, k$  label internal vertices in the clockwise direction starting from the north-pole, then the result of  $M$  is  $\bar{k}, \dots, \bar{1}$ , where  $\bar{i}$  means that the external leg is reversed (see Figure 5.8). These operations satisfy

$$(r + k) = (r), (r)(-r) = \mathbb{1}, M^2 = \mathbb{1}, (r)M = M(-r),$$

and hence generate a group  $G$  which is isomorphic to the dihedral group  $\mathbb{Z}_k \rtimes \mathbb{Z}_2$ . The orbit space  $O(s_1, s_2)/G$  is in 1 : 1 correspondence with isomorphism classes of admissible  $O_k$ -graphs and  $\text{Aut}(\Gamma)$  is in 1 : 1 correspondence with  $\text{Stab}(\Gamma)$ . From the orbit-stabilizer formula, we get

$$\begin{aligned} \sum_{\substack{[\Gamma] \text{ admiss.} \\ O_k\text{-graph}}} \frac{1}{|\text{Aut}(\Gamma)|} &= \sum_{[\Gamma] \in O(s_1, s_2)/G} \frac{1}{|\text{Stab}(\Gamma)|} = \sum_{\Gamma \in O(s_1, s_2)} \frac{1}{|\text{Orb}(\Gamma)| |\text{Stab}(\Gamma)|} \\ &= \frac{|O(s_1, s_2)|}{|G|} = \frac{1}{2k} \binom{k}{s_1} \times \begin{cases} 1 & \text{for } s_1 = s_2, \\ 2 & \text{for } s_1 \neq s_2. \end{cases} \end{aligned}$$

The two cases are compensated in the sum over labelings: For  $s_1 = s_2$ , both labelings  $L_1^b$  are admissible, and hence we get the factor 2.

Next, we multiply by  $k!s_1(k - s_1)$ , which is the number of  $L_1^v$  and  $L_3^b$ . There is also the factor  $\frac{1}{l!} = \frac{1}{2}$ . Multiplying everything together, we get  $C(s_1, s_2)$ .  $\square$

Before we summarize the results of our computations (see Proposition 5.2.10 below), we show directly that  $\mathbf{n}$  is a Maurer-Cartan element.

**Lemma 5.2.9** (Maurer-Cartan equation for  $\mathbb{S}^n$ ). *Consider  $\mathbb{S}^n$  with the Hodge Propagator from (5.2). The collection  $(\mathbf{n}_g)$  satisfies the Maurer-Cartan equation (3.20) for  $\text{dIBL}(C(\mathcal{H}(\mathbb{S}^n)))$ .*

## 5. Explicit computations

*Proof.* We will show that for every  $l \geq 1, g \geq 0$  all summands in the relation corresponding to  $(l, g)$  vanish. The summands for  $(l, g) = (1, 0)$  are  $\mathbf{q}_{110}(\mathbf{n}_{10})$  and  $\frac{1}{2}\mathbf{q}_{210}(\mathbf{n}_{10}, \mathbf{n}_{10})$ , and the summand for  $(l, g) = (2, 0)$  is  $\mathbf{q}_{120}(\mathbf{n}_{10})$ . The first term vanishes trivially as  $\mathbf{q}_{110} = 0$ , while the other two terms vanish by [CFL15, Proposition 12.5] because  $\mathbf{n}_{10} = \mathbf{m}_{10}$  is the canonical Maurer-Cartan element. For  $(l, g) \neq (1, 0)$ , we have the following four situations:

$\mathbf{q}_{210} \circ_2 \mathbf{n}_{lg}$ ,  $l \geq 2$ : Let  $\Psi = \Psi_1 \cdots \Psi_l \in E_l C$  be a summand of  $\mathbf{n}_{lg}$ . From Proposition 5.2.2 it follows that the summands can be chosen such that  $\Psi_1, \dots, \Psi_l \in B_{\text{cyc,red}}^* \mathcal{H}(\mathbb{S}^n)[3-n]$ , i.e., such that  $\Psi_i$  evaluates to 0 whenever 1 is a part of its argument. From Definition 3.2.2, we compute

$$\mathbf{q}_{210} \circ_2 (\Psi_1 \cdots \Psi_l) = \sum_{\sigma \in \mathbb{S}_{2,l-2}} \varepsilon(\sigma, \Psi) \mathbf{q}_{210}(\Psi_{\sigma_1^{-1}} \cdot \Psi_{\sigma_2^{-1}}) \cdot \Psi_{\sigma_3^{-1}} \cdots \Psi_{\sigma_l^{-l}}.$$

We clearly have  $\mathbf{q}_{210}(\Psi_{\sigma_1^{-1}} \cdot \Psi_{\sigma_2^{-1}}) = 0$  because  $\mathbf{q}_{210}$  feeds 1 into one of its inputs. It follows that  $\mathbf{q}_{210} \circ_2 \mathbf{n}_{lg} = 0$ .

$\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{l_1 g_1} \odot \mathbf{n}_{l_2 g_2})$ ,  $(l_i, g_i) \neq (1, 0)$ : A similar argument as above.

$\mathbf{q}_{120} \circ_1 \mathbf{n}_{lg}$ ,  $(l, g) \neq (1, 0)$ : A similar argument as above using that  $\mathbf{q}_{120}$  also feeds 1 into its input.

$\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{10} \odot \mathbf{n}_{lg})$ ,  $(l, g) \neq (1, 0)$ : As in the case of  $\mathbf{q}_{210} \circ_2 \mathbf{n}_{lg}$ , suppose that  $\Psi_1, \dots, \Psi_l \in B_{\text{cyc,red}}^* \mathcal{H}(\mathbb{S}^n)[3-n]$ . Recall that we write  $\Omega_i = s\omega_i \in B^{\text{cyc}} \mathcal{H}(\mathbb{S}^n)[3-n]$  and  $\Omega = \Omega_1 \otimes \cdots \otimes \Omega_l$ . From Definition 3.2.2, we compute

$$\begin{aligned} & [\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{10} \odot \Psi)](\Omega_1 \otimes \cdots \otimes \Omega_l) \\ &= \left[ \sum_{i=1}^l (-1)^{|\Psi_i|(|\Psi_1| + \cdots + |\Psi_{i-1}|)} \mathbf{q}_{210}(\mathbf{n}_{10} \Psi_i) \Psi_1 \cdots \widehat{\Psi_i} \cdots \Psi_l \right] \\ & \quad (\Omega_1 \otimes \cdots \otimes \Omega_l) \\ &= \sum_{\substack{\mu \in \mathbb{S}_l \\ i=1, \dots, l}} \frac{1}{l!} (-1)^{|\Psi_i|(|\Psi_1| + \cdots + |\Psi_{i-1}|)} \varepsilon(\mu, \Omega) (\mathbf{q}_{210}(\mathbf{n}_{10} \cdot \Psi_i) \otimes \Psi_1 \otimes \cdots \\ & \quad \widehat{\Psi_i} \cdots \otimes \Psi_l) (\Omega_{\mu_1^{-1}} \otimes \cdots \otimes \Omega_{\mu_l^{-1}}). \end{aligned}$$

For every  $i = 1, \dots, l$ , we have

$$\mathbf{q}_{210}(\mathbf{n}_{10} \cdot \Psi_i)(\Omega) = \mathbf{q}_{210}(\mathbf{n}_{10} \otimes \Psi_i)(\Omega)$$

## 5. Explicit computations

$$\begin{aligned}
&= - \sum \varepsilon(\omega \mapsto \omega^1 \omega^2) [(-1)^{(n-1)|\omega^1|} \mathbf{n}_{10}(\mathbf{s}1\omega^1) \Psi_i(\mathbf{sv}\omega^2) \\
&\quad + (-1)^{|\omega^1|} \mathbf{n}_{10}(\mathbf{sv}\omega^1) \Psi_i(\mathbf{s}1\omega^2)] \\
&= - \sum \varepsilon(\omega \mapsto \omega^1 \omega^2) (-1)^{(n-1)|\omega^1|} \mathbf{n}_{10}(\mathbf{s}1\omega^1) \Psi_i(\mathbf{sv}\omega^2).
\end{aligned}$$

This can be non-zero only if  $\omega = 1\mathbf{v}^{s-1}$  for some  $s \geq 2$  (up to a cyclic permutation). For this input, we get

$$\begin{aligned}
&\mathbf{q}_{210}(\mathbf{n}_{10} \cdot \Psi_i)(\mathbf{s}1\mathbf{v}^{s-1}) \\
&= - [\varepsilon(1\mathbf{v}^{s-1} \mapsto 1\mathbf{v}^{s-1}) \mathbf{n}_{10}(\mathbf{s}11\mathbf{v}) \Psi_i(\mathbf{sv}^{s-1}) \\
&\quad + \varepsilon(1\mathbf{v}^{s-1} \mapsto \mathbf{v}1\mathbf{v}^{s-2}) \mathbf{n}_{10}(\mathbf{s}1\mathbf{v}1) \Psi_i(\mathbf{sv}^{s-1})] \\
&= (-1)^{n-3} [1 + (-1)^{ns+s-1}] \Psi_i(\mathbf{sv}^{s-1}).
\end{aligned}$$

The prefactor in brackets is 0 for  $n$  odd or  $s$  even, whereas  $\mathbf{v}^{s-1} = 0$  for  $n$  even and  $s$  odd. Therefore, we have  $\mathbf{q}_{210} \circ_{1,1} (\mathbf{n}_{10} \odot \mathbf{n}_g) = 0$ .  $\square$

**Proposition 5.2.10** (Chern-Simons Maurer-Cartan element for  $\mathbb{S}^n$ ). *Consider the round sphere  $\mathbb{S}^n$  with the Hodge Propagator (5.2). The Chern-Simons Maurer-Cartan element  $\mathbf{n}$  is a strictly reduced Maurer-Cartan element for  $\text{dIBL}(C(\mathcal{H}(\mathbb{S}^n)))$  which satisfies*

$$\mathbf{n}_{10} = \mathbf{m}_{10} \quad \text{for all } n \in \mathbb{N}$$

plus the following properties depending on the dimension:

$(n = 1)$ : It holds  $\mathbf{n}_{lg} = 0$  for all  $l \geq 1, g \geq 0$  such that  $(l, g) \neq (1, 0), (2, 0)$ ; the only non-trivial relation for  $\mathbf{n}_{20}$  is

$$\mathbf{n}_{20}(\mathbf{sv}^{s_1} \otimes \mathbf{sv}^{s_2}) = (-1)^{s_1+1} \frac{1}{2} s_1 k! \binom{k-1}{s_1} I(k), \quad (5.14)$$

where  $s_1, s_2 \geq 1$  are such that  $k = s_1 + s_2$  is even, and  $I(k)$  is given by (5.13).

$(n = 2)$ : It holds  $\mathbf{n}_{10} = 0$  for all  $l \geq 2$ . We also have  $\mathbf{n}_{11} = 0$ .

$(n \geq 3)$ : It holds  $\mathbf{n}_{lg} = 0$  for all  $l \geq 1, g \geq 0$  such that  $(l, g) \neq (1, 0)$ .

Notice that  $\mathbf{n}_{20} \notin E_2 C(\mathcal{H}(\mathbb{S}^1))$ , i.e.  $\mathbf{n}_{20}$  is a long cochain, because it is non-zero in infinitely many weights.

### 5.3. Twisted IBL-infinity-structure for spheres

Let  $e_0, e_1$  be the basis of  $\mathcal{H}(\mathbb{S}^n)[1]$  defined by

$$e_0 := \mathbf{1} := \theta \mathbf{1}, \quad e_1 := \mathbf{v} := \frac{1}{V} \theta \text{Vol}.$$

The degrees satisfy

$$|\mathbf{1}| = -1, \quad |\mathbf{v}| = n - 1.$$

The matrix of the pairing  $\Pi$  with respect to the basis  $e_0, e_1$  reads

$$\Pi = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

The dual basis  $e^0, e^1$  to  $e_0, e_1$  with respect to  $\Pi$  is thus

$$e^0 = \mathbf{v}, \quad e^1 = (-1)^n \mathbf{1}.$$

It follows that the matrix  $(T^{ij})$  from (3.32) satisfies

$$(T^{ij}) = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We clearly have

$$\hat{\mathbf{B}}_{\text{cyc,red}}^* \mathcal{H}(\mathbb{S}^1) = \left\{ \sum_{k=1}^{\infty} c_k \mathbf{v}^{k*} \mid c_k \in \mathbb{R} \right\},$$

where  $\mathbf{v}^{k*}$  is the dual to the cyclic word  $\mathbf{v}^k = \mathbf{v} \dots \mathbf{v}$  of length  $k$ . Observe that the cyclic symmetry gives

$$\mathbf{v}^i = (-1)^{(n-1)(i-1)} \mathbf{v}^i \quad \text{for all } i \geq 1.$$

Therefore,  $\mathbf{v}^{i*} = 0$  holds if both  $n$  and  $i$  are even.

For  $n \geq 2$ , the vector space  $\mathcal{H}(\mathbb{S}^n)$  is connected and simply-connected, and Proposition 3.3.15 implies that there are no long reduced cyclic cochains (i.e., we have only finite sums of  $\mathbf{v}^{k*}$ 's).

The product  $\mu_2 : \mathcal{H}[1]^{\otimes 2} \rightarrow \mathcal{H}[1]$  from (4.2) has the following matrix with respect to the basis  $\mathbf{1}, \mathbf{v}$ :

$$\mu_2 = \begin{pmatrix} \mathbf{1} & \mathbf{v} \\ (-1)^n \mathbf{v} & 0 \end{pmatrix}.$$

## 5. Explicit computations

Because  $\mu_2(v, v) = 0$ , we get

$$\mathbb{H}^m(C_{\text{red}}(\mathcal{H}(\mathbb{S}^n)))[1] = \begin{cases} \langle sv^{i*} \mid i \geq 1 \rangle & \text{for } n \geq 3 \text{ odd,} \\ \langle sv^{2i-1*} \mid i \geq 1 \rangle & \text{for } n \text{ even,} \\ \{s \sum_{k=1}^{\infty} c_k v^{k*} \mid c_k \in \mathbb{R}\} & \text{for } n = 1. \end{cases}$$

Because we are in the strictly unital and strictly augmented case, we obtain

$$\mathbb{H}^m(C)[1] = \begin{cases} \langle sv^{i*}, s1^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \geq 3 \text{ odd,} \\ \langle sv^{2i-1*}, s1^{2j-1*} \mid i, j \geq 1 \rangle & \text{for } n \text{ even,} \\ \langle s \sum_{k=1}^{\infty} c_k v^{k*}, s1^{2j-1*} \mid c_k \in \mathbb{R}, j \geq 1 \rangle & \text{for } n = 1. \end{cases} \quad (5.15)$$

The canonical IBL-operations can be written as

$$\begin{aligned} q_{210}(s^2\psi_1 \otimes \psi_2)(s\omega) &= - \sum \varepsilon(\omega \mapsto \omega^1\omega^2)[(-1)^{(n-1)|\omega^1|}\psi_1(e_0\omega^1) \\ &\quad \psi_2(e_1\omega^2) + (-1)^{|\omega^1|}\psi_1(e_1\omega^1)\psi_2(e_0\omega^2)], \\ q_{120}(s\psi)(s^2\omega_1 \otimes \omega_2) &= - \frac{1}{2} \sum \varepsilon(\omega_1 \mapsto \omega_1^1)\varepsilon(\omega_2 \mapsto \omega_2^1)[(-1)^{(n-1)|\omega_1^1|} \\ &\quad \psi(e_0\omega_1^1e_1\omega_2^1) + (-1)^{|\omega_1^1|}\psi(e_1\omega_1^1e_0\omega_2^1)] \end{aligned}$$

for all  $\psi, \psi_1, \psi_2 \in \hat{B}_{\text{cyc}}^* \mathcal{H}$  and generating words  $\omega, \omega_1, \omega_2 \in B^{\text{cyc}} \mathcal{H}$ . For all  $k, k_1, k_2 \geq 1$ , we have

$$q_{210}((sv^{k_1*}) \cdot (sv^{k_2*})) = 0 \quad \text{and} \quad q_{120}(sv^{k*}) = 0$$

because both  $q_{210}$  and  $q_{120}$  feed 1 into their inputs. For the *canonically twisted reduced IBL-algebra*, this implies the following:

$$\text{IBL}(\mathbb{H}^m(C_{\text{red}})) = (\mathbb{H}^m(C_{\text{red}}), q_{210} \equiv 0, q_{120} \equiv 0) \quad \text{for all } n \in \mathbb{N}.$$

By Proposition 3.4.10, the only possibly non-zero relation of  $\text{IBL}(\mathbb{H}^m(C))$  is

$$\begin{aligned} q_{210}(s1^* \otimes sv^{k*}) &= (-1)^{n-2} s(v^{k*} \circ \iota_v) \\ &= (-1)^{n-2} \left( \sum_{i=1}^{k-1} (-1)^{i|v|} \right) sv^{k-1*} = \begin{cases} -(k-1)sv^{k-1*} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

The reason for 0 for even  $n$  is that either  $k$  is odd, in which case  $\sum_{i=1}^{k-1} (-1)^i = 0$ , or  $k$  is

## 5. Explicit computations

even, in which case  $v^{k*} = 0$ . Therefore, for the *canonically twisted IBL-algebra*, we have

$$\text{IBL}(\mathbb{H}^m(C)) = (\mathbb{H}^m(C), \mathfrak{q}_{210}, \mathfrak{q}_{120} \equiv 0) \quad \text{for all } n \in \mathbb{N},$$

where  $\mathbb{H}^m(C)$  is given by (5.15) and  $\mathfrak{q}_{210}$  satisfies the following:

( $n$  even):  $\mathfrak{q}_{210} \equiv 0$ .

( $n \geq 3$  odd): The non-trivial relations are

$$\mathfrak{q}_{210}(s_1^* \otimes sv^{k*}) = \mathfrak{q}_{210}(sv^{k*} \otimes s_1^*) = -(k-1)v^{k-1*} \quad \text{for } k \geq 2.$$

( $n = 1$ ): The non-trivial relations are

$$\mathfrak{q}_{210}\left(s_1^* \otimes s \sum_{k=1}^{\infty} c_k v^{k*}\right) = -s \sum_{k=1}^{\infty} k c_{k+1} v^{k*} \quad \text{for } c_k \in \mathbb{R}.$$

Recall that the twist by  $\mathfrak{m}$  does not produce any higher operation  $\mathfrak{q}_{1lg}^m$ .

We will now consider  $\text{dIBL}^n(C(\mathcal{H}(\mathbb{S}^n)))$ . Recall that  $\mathfrak{q}_{110}^n = \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{10}$ ,  $\mathfrak{q}_{210}^n = \mathfrak{q}_{210}$  and  $\mathfrak{q}_{120}^n = \mathfrak{q}_{120} + \mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20}$ . By Proposition 5.2.10, we have  $\mathfrak{n}_{10} = \mathfrak{m}_{10}$  for all  $n \in \mathbb{N}$  and  $\mathfrak{n}_{20} = 0$  for all  $n \geq 2$ . It follows that  $\mathfrak{q}_{110}^n = \mathfrak{q}_{110}^m$  for all  $n \in \mathbb{N}$  and that the only non-trivial twist may occur in  $\mathfrak{q}_{120}^n$  for  $\mathbb{S}^1$ . Using (3.36), we get for all  $\psi \in \hat{B}_{\text{cyc}}^* \mathcal{H}(\mathbb{S}^n)$  and generating words  $\omega_1, \omega_2 \in B^{\text{cyc}} \mathcal{H}(\mathbb{S}^n)$  the following:

$$\begin{aligned} & (\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s\psi)(s\omega_1 \otimes s\omega_2) \\ &= (-1)^{n-2} \left[ \sum \varepsilon(\omega_1 \mapsto \omega_1^1 \omega_1^2) \psi(1\omega_1^1) \mathfrak{n}_{20}(sv\omega_1^2 \otimes s\omega_2) \right. \\ & \quad \left. + (-1)^{(n-3+|\omega_1|)(n-3+|\omega_2|)} \sum \varepsilon(\omega_2 \mapsto \omega_2^1 \omega_2^2) \psi(1\omega_2^1) \right. \\ & \quad \left. \mathfrak{n}_{20}(sv\omega_2^2 \otimes s\omega_1) \right]. \end{aligned} \tag{5.16}$$

In this paragraph, we suppose that  $n = 1$  and compute  $\mathfrak{q}_{120}^n$ . Clearly,  $(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(sv^{k*}) = 0$  for all  $k \geq 1$  since  $1$  is fed into  $v^{k*}$ . A non-zero evaluation of  $(\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s_1^{k*})$  for some  $k \geq 1$  odd is possible only on  $s_1^{k-1} v^{k_1} \otimes sv^{k_2}$  for  $k_1, k_2 \geq 0$  (up to a transposition of arguments and their cyclic permutation). If  $k > 1$ , only the first

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summand of (5.16) contributes, and we get

$$\begin{aligned}
& (\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s_1^{k*})(s_1^{k-1}v^{k_1} \otimes sv^{k_2}) \\
&= (-1)^{n-2} \sum \varepsilon(1^{k-1}v^{k_1} \mapsto \omega_1\omega_2)1^{k*}(1\omega_1)\mathfrak{n}_{20}(sv\omega_2 \otimes sv^{k_2}) \\
&= (-1)^{n-2}1^{k*}(11^{k-1})\mathfrak{n}_{20}(svv^{k_1} \otimes sv^{k_2}) \\
&= -\mathfrak{n}_{20}(sv^{k_1+1} \otimes sv^{k_2}).
\end{aligned}$$

According to Proposition 5.2.10, this is non-zero if and only if  $k_1 + k_2$  is odd. It follows that

$$\mathfrak{q}_{120}^n \neq \mathfrak{q}_{120}^m = \mathfrak{q}_{120} \quad \text{on the chain level for } \mathbb{S}^1.$$

However, the chains  $s_1^{k-1}v^{k_1} \otimes sv^{k_2}$  for  $k > 1$  do not survive to the homology (c.f., (5.15)). The only possibility is thus  $k = 1$ . In this case, both summands of (5.16) contribute, and using (5.14), we get for all  $k_1, k_2 \geq 1$  the following:

$$\begin{aligned}
& (\mathfrak{q}_{210} \circ_1 \mathfrak{n}_{20})(s_1^*)(sv^{k_1} \otimes sv^{k_2}) \\
&= (-1)^{n-2} \left[ \sum \varepsilon(v^{k_1} \mapsto v^0v^{k_1})1^*(1)\mathfrak{n}_{20}(sv^{k_1+1} \otimes sv^{k_2}) \right. \\
&\quad \left. + (-1)^{(n-3+k_1(n-1))(n-3+k_2(n-1))} \sum \varepsilon(v^{k_2} \mapsto v^0v^{k_2})1^*(1) \right. \\
&\quad \left. \mathfrak{n}_{20}(sv^{k_2+1} \otimes sv^{k_1}) \right] \\
&= -k_1\mathfrak{n}_{20}(sv^{k_1+1} \otimes sv^{k_2}) - k_2\mathfrak{n}_{20}(sv^{k_2+1} \otimes sv^{k_1}) \\
&= -\frac{1}{2}(k_1 + k_2 + 1)!I(k_1 + k_2 + 1) \left[ (-1)^{k_1}k_1(k_1 + 1) \binom{k_1 + k_2}{k_1 + 1} \right. \\
&\quad \left. + (-1)^{k_2}k_2(k_2 + 1) \binom{k_1 + k_2}{k_2 + 1} \right] \\
&= -\frac{1}{2}(k_1 + k_2 + 1)!k_1k_2 \binom{k_1 + k_2}{k_1} \underbrace{I(k_1 + k_2 + 1)[(-1)^{k_1} + (-1)^{k_2}]}_{=:(*)}.
\end{aligned}$$

Denoting  $k := k_1 + k_2 + 1$ , we have that  $(-1)^{k_1} + (-1)^{k_2} = 0$  for  $k$  even and  $I(k) = 0$  for  $k$  odd. Therefore,  $(*) = 0$  for any  $k_1, k_2 \geq 1$ . This implies that

$$\mathfrak{q}_{120}^n = \mathfrak{q}_{120}^m = \mathfrak{q}_{120} \quad \text{on the homology for } \mathbb{S}^1.$$

We conclude that the *twisted IBL-algebra* satisfies

$$\text{IBL}(\mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^n)))) = \text{IBL}(\mathbb{H}^m(C(\mathcal{H}(\mathbb{S}^n)))) \quad \text{for all } n \in \mathbb{N}.$$

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As for the *higher twisted operations*, combining Propositions 3.2.11 and 5.2.10, we see that for  $\mathbb{S}^n$  with  $n \in \mathbb{N} \setminus \{2\}$  all higher operations  $\mathfrak{q}_{1lg}^n$  vanish already on the chain level. For  $n = 2$ , we have that  $\mathfrak{q}_{1l0}^n = 0$  for all  $l \geq 3$  and  $\mathfrak{q}_{111}^n = 0$  on the chain level. However, we did not prove that all higher operations vanish on the chain level. As for the operations induced on the homology, the graded vector space  $\mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^2)))$  is concentrated in even degrees and  $\mathfrak{q}_{1lg}^n$  are odd (see Definition 3.2.4). Therefore, all higher operations vanish also on  $\mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^2)))$ .

The string topology  $H^{\mathbb{S}^1}(\mathbb{S}^n)$  and the string operations  $\mathfrak{m}_2$  and  $\mathfrak{c}_2$  were computed in [Bas11] for all  $n \in \mathbb{N}$ . We review their results and basic ideas below:

We will consider *even spheres* first. The minimal model for the Borel construction  $L_{\mathbb{S}^1}\mathbb{S}^{2m}$  for  $m \in \mathbb{N}$  is denoted by  $\Lambda^{\mathbb{S}^1}(2, m)$  — it is the free graded commutative dga ( $=:\text{cdga}$ ) over  $\mathbb{R}$  generated by homogenous vectors  $x_1, y_1, x_2, y_2, u$  of degrees

$$|x_1| = 2m, \quad |y_1| = 2m - 1, \quad |x_2| = 4m - 1, \quad |y_2| = 2(2m - 1), \quad |u| = 2,$$

whose differential  $d$  satisfies

$$dy_1 = 0, \quad dx_1 = y_1 u, \quad dy_2 = -2x_1 y_1, \quad dx_2 = x_1^2 + y_2 u.$$

The minimal model for the loop space  $L\mathbb{S}^{2m}$  is the dga  $\Lambda(2, m)$  which is obtained from  $\Lambda^{\mathbb{S}^1}(2, m)$  by setting  $u = 0$ . A computation (see [Bas11, Theorem 3.6]) gives the following for all  $m \in \mathbb{N}$ :

$$\begin{aligned} H^*(L\mathbb{S}^{2m}; \mathbb{R}) &\simeq H(\Lambda(2, m), d) = \langle y_2^i x_1 - 2iy_1 x_2 y_2^{i-1}, y_1 y_2^j, 1 \mid i, j \in \mathbb{N}_0 \rangle, \\ H_{\mathbb{S}^1}^*(L\mathbb{S}^{2m}; \mathbb{R}) &\simeq H(\Lambda^{\mathbb{S}^1}(2, m), d) = \langle y_1 y_2^i, u^j \mid i, j \in \mathbb{N}_0 \rangle, \end{aligned} \tag{5.17}$$

where  $y_2^0 := u^0 := 1$  is the unit in  $\Lambda^{\mathbb{S}^1}(2, m)$  and  $\langle \cdot \rangle$  denotes the linear span over  $\mathbb{R}$ . Clearly, the cohomology groups are degree-wise finite-dimensional, and hence, using the universal coefficient theorem, they are isomorphic to the corresponding homology groups. We can thus identify  $H(L\mathbb{S}^{2m}; \mathbb{R})$  and  $H^{\mathbb{S}^1}(L\mathbb{S}^{2m}; \mathbb{R})$  with the vector spaces on the right hand side of (5.17). We have  $H_{2k}^{\mathbb{S}^1} = \langle u^k \rangle$  for all  $k \in \mathbb{N}_0$ , and hence the multiplication with  $u$  induces an isomorphism  $H_{2k}^{\mathbb{S}^1} \simeq H_{2k+2}^{\mathbb{S}^1}$ . This corresponds to the cap product with the Euler class in (4.27), and exactness of the sequence implies  $\mathcal{M}(H_{2k}^{\mathbb{S}^1}) = \mathcal{E}(H_{2k}^{\mathbb{S}^1}) = 0$ . Using this and degree considerations, we get  $\mathfrak{m}_2 = \mathfrak{c}_2 = 0$ .

We will now consider *odd spheres* with  $n \geq 3$ . The minimal model for  $L_{\mathbb{S}^1}\mathbb{S}^{2m+1}$  for  $m \in \mathbb{N}$  is denoted simply by  $\Lambda(x, y, u)$  — it is the free cdga on homogenous vectors  $x$ ,



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$y, u$  of degrees

$$|x| = 2m + 1, \quad |y| = 2m, \quad |u| = 2,$$

such that

$$dx = yu, \quad dy = du = 0.$$

We get immediately

$$\begin{aligned} H^*(LS^{2m+1}; \mathbb{R}) &\simeq \langle x^i, y^j \mid i, j \in \mathbb{N}_0 \rangle, \\ H_{\mathbb{S}^1}^*(LS^{2m+1}; \mathbb{R}) &\simeq \langle y^i, u^j \mid i, j \in \mathbb{N}_0 \rangle, \end{aligned}$$

and we can again identify  $H$  and  $H^{\mathbb{S}^1}$  with the vector spaces on the right hand side. Clearly,  $H_{2k-1}^{\mathbb{S}^1} = 0$  for all  $k \in \mathbb{N}$ , and hence  $\mathfrak{m}_2 = \mathfrak{c}_2 = 0$  for degree reasons (the operations are odd).

We will now consider *the circle*  $\mathbb{S}^1$ . For every  $i \in \mathbb{Z}$ , let  $\alpha_i : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  and  $\theta_i : \mathbb{S}^1 \rightarrow LS^1$  be the maps defined by

$$\alpha_i(z) := z^i \quad \text{and} \quad \theta_i(w) := w\alpha_i \quad \text{for all } w, z \in \mathbb{S}^1 \subset \mathbb{C}.$$

By examining the equivariant homology of connected components of  $LS^1$  containing  $\alpha_i$  separately as in [Bas11, Section 2.1.4], we get

$$\begin{aligned} H(LS^1; \mathbb{R}) &= \langle \alpha_i, \theta_j \mid i, j \in \mathbb{Z} \rangle, \\ H^{\mathbb{S}^1}(LS^1; \mathbb{R}) &= \langle u^i, \theta_0 u^j, \alpha_k \mid i, j \in \mathbb{N}_0, k \in \mathbb{Z} \setminus \{0\} \rangle, \end{aligned}$$

where  $u$  corresponds to the Euler class and

$$|u| = 2, \quad |\theta_i| = 1, \quad |\alpha_i| = 0$$

are the degrees in the singular chain complex. On [Bas11, p. 21] they show that the string cobracket  $\mathfrak{c}_2$  is 0 and that all non-trivial relations for the string bracket  $\mathfrak{m}_2 : H^{\mathbb{S}^1}(LS^1)[2]^{\otimes 2} \rightarrow H^{\mathbb{S}^1}(LS^1)[2]$  are the following:

$$\mathfrak{m}_2(s\alpha_k, s\alpha_{-k}) = k^2 s\theta_0 \quad \forall k \in \mathbb{N}.$$

We will now compare the reduced IBL-structures motivated by Conjecture 4.5.1. The point-reduced versions  $H^{\mathbb{S}^1, \text{red}}(LS^n)$  for  $n \geq 2$  are obtained from  $H^{\mathbb{S}^1}(LS^n)$  by deleting

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$u^i$ . We have the following isomorphisms of graded vector spaces:

$$\begin{aligned} \mathbb{H}^n(C_{\text{red}}(\mathcal{H}(\mathbb{S}^n)))[1] &\longrightarrow \mathbb{H}^{\mathbb{S}^1, \text{red}}(\mathbb{L}\mathbb{S}^n)[3-n] \\ \text{sv}^i &\longmapsto \text{sy}^i && \text{for } n > 1 \text{ odd,} \\ \text{sv}^{2i+1} &\longmapsto \text{sy}_1 y_2^i && \text{for } n \text{ even.} \end{aligned}$$

Because all operations are trivial, it induces the isomorphism

$$\text{IBL}(\mathbb{H}^n(C_{\text{red}}(\mathcal{H}(\mathbb{S}^n)))) \simeq \text{IBL}(\mathbb{H}^{\mathbb{S}^1, \text{red}}(\mathbb{L}\mathbb{S}^n)[2-n]) \quad \text{for } n \geq 2.$$

For  $n = 1$ , the reduced homology is seemingly different.

*Remark 5.3.1* (Triviality for degree reasons). The graded vector spaces

$$\mathbb{H}^{\mathbb{S}^1}(\mathbb{L}\mathbb{S}^{2m-1})[3-n] \quad \text{and} \quad \mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^{2m}))) [1]$$

are concentrated in even degrees, and so any  $\text{IBL}_\infty$ -structure must be trivial for degree reasons. On the other hand, the graded vector spaces

$$\mathbb{H}^{\mathbb{S}^1}(\mathbb{L}\mathbb{S}^{2m}) \quad \text{and} \quad \mathbb{H}^n(C(\mathcal{H}(\mathbb{S}^{2m-1}))) [1]$$

have both even and odd degrees, and hence an additional argument is needed to prove vanishing of the IBL-structure. This is not the case of the reduced homology, which is again concentrated in even degree.  $\triangleleft$

DONE Too dense text: Add paragraphs here — too dense. Ans also add  $\mathbb{S}^1$ !!

### 5.4. Homology of twisted IBL-infinity structure for complex projective space

Let  $K \in \Omega^2(\mathbb{CP}^n)$  be the Fubini–Study Kähler form on  $\mathbb{CP}^n$  (see [Huy05, Examples 3.1.9]). The powers of  $K$  are harmonic,<sup>2</sup> and we get easily

$$\mathcal{H}(\mathbb{CP}^n) = \langle 1, K, \dots, K^n \rangle.$$

DONE Non-trivial degree: Add here the computation of which relations on homology are not implied automatically by degree reasons.

<sup>2</sup>This follows by induction on the power of  $K$  using the fact that, on a general Kähler manifold  $M$ , the Lefschetz operator  $L : \Omega(M) \rightarrow \Omega(M)$  defined by  $L(\eta) := \eta \wedge K$  for all  $\eta \in \Omega(M)$  commutes with the Hodge–de Rham Laplacian  $\Delta$  (see [Huy05, Chapter 3]).

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We denote the Riemannian volume of  $\mathbb{CP}^n$  by

$$V := \int_{\mathbb{CP}^n} \frac{1}{n!} K^n.$$

Consider the basis  $e_0, \dots, e_n$  of  $\mathcal{H}(\mathbb{CP}^n)[1]$  defined for all  $i = 0, \dots, n$  by

$$e_i := \frac{k^i}{(n!V)^{\frac{i}{n}}}, \quad \text{where } k^i := \theta K^i.$$

The matrix of the pairing  $\Pi$  from (2.3) with respect to the basis  $e_0, \dots, e_n$  reads:

$$(\Pi^{ij}) = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

The basis  $e^0, \dots, e^n$  dual to  $e_0, \dots, e_n$  with respect to  $\Pi$  thus satisfies

$$e^i = e_{n-i} \quad \text{for all } i = 0, \dots, n.$$

Therefore, the following holds for the matrix  $(T^{ij})$  from (3.32):

$$(T^{ij}) = -(\Pi^{ij}).$$

For all  $1 \leq i, j, k \leq n$ , we have

$$\mu_2(e_i, e_j) = e_{i+j} \quad \text{and} \quad \mathfrak{m}_{10}(se_i e_j e_k) = \delta_{i+j+k, n}.$$

For  $\psi, \psi_1, \psi_2 \in \hat{\mathcal{B}}_{\text{cyc}}^* \mathcal{H}$  and generating words  $\omega, \omega_1, \omega_2 \in \mathcal{B}^{\text{cyc}} \mathcal{H}$ , we have

$$\begin{aligned} \mathfrak{q}_{210}(s^2 \psi_1 \otimes \psi_2)(s\omega) &= - \sum_{i=0}^n \sum \varepsilon(\omega \mapsto \omega^1 \omega^2) (-1)^{|\omega^1|} \psi_1(e_i \omega^1) \psi_2(e_{n-i} \omega^2), \\ \mathfrak{q}_{120}(s\psi)(s^2 \omega_1 \otimes \omega_2) &= - \sum_{i=0}^n \sum \varepsilon(\omega_1 \mapsto \omega_1^1) \varepsilon(\omega_2 \mapsto \omega_2^1) (-1)^{|\omega_1^1|} \psi(e_i \omega_1^1 e_{n-i} \omega_2^1). \end{aligned}$$

The cyclic homology of  $\mathcal{H}(\mathbb{CP}^n)$  is that of the truncated polynomial algebra

$$A := \mathbb{R}[x]/(x^{n+1}) \quad \text{with } \deg(x) = 2.$$

The computation of  $H^{\lambda, \text{cl}}(A)$  for  $|x| = 0$  over a field is the goal of [Lod92, Exercise

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4.1.8.] or [Wei94, Exercise 9.1.1]. The case of  $|x| = d$  can be solved by taking suitable degree shifts in the proposed projective resolution which is used to compute  $\mathrm{HH}(A)$ . Unfortunately, using a non-canonical projective resolution, we lose the concrete form of the cyclic cycles and obtain just the following result:

For all  $i = 1, \dots, n$  and  $k \in \mathbb{N}_0$ , there are cycles  $\tilde{t}_{2k+1,i} \in \tilde{D}_q(A)$  of weights  $2k+1$  and degrees  $d(i + (n+1)k)$  which form a basis of  $H^{\lambda, \mathrm{cl}}(A)$ . We apply the degree shift  $U : \tilde{D}(A) \rightarrow D(A)$  from Proposition 3.3.14 to get the generators

$$t_{w,i} := U(\tilde{t}_{w,i}) \in D^\lambda(\mathcal{H}(\mathbb{CP}^n))$$

of weights  $w$  and degrees  $2i + (w-1)n - 1$ , so that

$$H^\lambda(\mathcal{H}(\mathbb{CP}^n)) = \langle t_{w,i}, 1^w \mid w \in \mathbb{N} \text{ odd}, i = 1, \dots, n \rangle.$$

By the universal coefficient theorem we have  $H_\lambda^* = (H^\lambda)^{*g}$  with respect to the grading by the degree. Given  $d \in \mathbb{Z}$ , the equation  $d = 2i + (w-1)n - 1$  has only finitely many solution  $(w, i) \in \mathbb{N} \times \{1, \dots, n\}$ , and hence we get

$$\mathbb{H}^m(C(\mathcal{H}(\mathbb{CP}^n))) = \langle st_{w,i}^*, s1^{w*} \mid w \in \mathbb{N} \text{ odd}, i = 1, \dots, n \rangle, \quad (5.18)$$

where  $t_{w,i}^*$  and  $1^{w*} \in B_{\mathrm{cyc}}^* \mathcal{H}$  are the duals to  $t_{w,i}$  and  $1^w$ , respectively (see Remark 3.3.16). Notice that both  $|st_{w,i}^*|$  and  $|s1^{w*}|$  are even since  $|s| = 2n - 3$ .

Because  $\mathbb{CP}^n$  is geometrically formal, Proposition 4.4.5 implies that  $\mathfrak{n}_{10} = \mathfrak{m}_{10}$ . Because  $\mathbb{H}^m(C)$  is concentrated in even degrees and because a general  $\mathrm{IBL}_\infty$ -operation  $\mathfrak{q}_{klg}$  is odd, all operations vanish on the homology. Therefore, for the *twisted IBL-algebras* we have

$$\mathrm{IBL}(\mathbb{H}^n(C)) = \mathrm{IBL}(\mathbb{H}^m(C)) = (\mathbb{H}^m(C), \mathfrak{q}_{210} \equiv 0, \mathfrak{q}_{120} \equiv 0),$$

where  $\mathbb{H}^m(C)$  is given by (5.18).

According to [Bas11, Section 3.1.2], the minimal model for the Borel construction  $L_{\mathbb{S}^1} \mathbb{CP}^n$  is the cdga  $\Lambda^{\mathbb{S}^1}(n+1, 1)$ , which is freely generated (over  $\mathbb{R}$ ) by the homogenous vectors  $x_1, x_2, y_1, y_2, u$  of degrees

$$|x_1| = 2, \quad |x_2| = 2n+1, \quad |y_1| = 1, \quad |y_2| = 2n, \quad |u| = 2,$$

and whose differential  $d$  satisfies

$$dy_1 = 0, \quad dx_1 = y_1 u, \quad dy_2 = -(n+1)x_1^n y_1, \quad dx_2 = x_1^{n+1} + y_2 u.$$

## 5. Explicit computations

By [Bas11, Theorem 3.6], the string cohomology  $H_{\mathbb{S}^1}^*(\mathrm{LCP}^n; \mathbb{R}) \simeq H(\Lambda^{\mathbb{S}^1}(n+1, 1), d)$  satisfies for all  $m \in \mathbb{N}_0$  the following:

$$H_{\mathbb{S}^1}^m(\mathrm{LCP}^n; \mathbb{R}) = \begin{cases} \langle u^j \rangle & \text{if } m = 2j, \\ \langle y_1 y_2^p x_1^q \mid 0 \leq q \leq n-1, p \geq 0; q + np = j \rangle & \text{if } m = 2j + 1. \end{cases}$$

The right-hand side can be identified with  $H^{\mathbb{S}^1}(\mathrm{LCP}^n; \mathbb{R})$  by the universal coefficient theorem. According to [Bas11, Proposition 3.7], we have  $\mathfrak{m}_2 = 0$  and  $\mathfrak{c}_2 = 0$ . We conclude that the map

$$\begin{aligned} \mathbb{H}^n(C_{\mathrm{red}}(\mathcal{H}(\mathbb{CP}^n)))[1] &\longrightarrow H^{\mathbb{S}^1, \mathrm{red}}(\mathrm{LCP}^n; \mathbb{R})[3-n] \\ st_{2k+1, l}^* &\longmapsto sy_1 y_2^k x_1^{l-1} \quad \text{for } k \geq 0 \text{ and } l = 1, \dots, n \end{aligned}$$

induces an isomorphism of IBL-algebras

$$\mathrm{IBL}(\mathbb{H}^n(C_{\mathrm{red}}(\mathcal{H}(\mathbb{CP}^n)))) \simeq \mathrm{IBL}(H^{\mathbb{S}^1, \mathrm{red}}(\mathrm{LCP}^n; \mathbb{R})[3-n]).$$

## **Part II.**

### **Follow-up topics**

## 6. IBL-infinity formality and Poincaré duality models

In this chapter, we study the following question:

**Question 6.0.1.** *Let  $M$  be a connected oriented closed  $n$ -manifold which is formal in the sense of rational homotopy theory. Let  $\mathfrak{m}$  be the canonical and  $\mathfrak{n}$  the formal pushforward (or Chern-Simons) Maurer-Cartan element for  $\mathrm{dIBL}(C(H_{\mathrm{dR}}(M)))$ . Are  $\mathrm{dIBL}^{\mathfrak{m}}(C(H_{\mathrm{dR}}(M)))$  and  $\mathrm{dIBL}^{\mathfrak{n}}(C(H_{\mathrm{dR}}(M)))$  homotopy equivalent as  $\mathrm{IBL}_{\infty}$ -algebras?*

We remind that Theorem B states that for a geometrically formal manifold  $M$  with  $H_{\mathrm{dR}}^1(M) = 0$ , one can pick a special Hodge propagator  $P_{\mathrm{std}}$  such that  $\mathfrak{n} = \mathfrak{m}$  at least for  $n \neq 2$ .<sup>1</sup> Question 6.0.1 asks for a generalization of Theorem B. In the upcoming sections, we will propose a strategy to answer Question 6.0.1 in the case of  $H_{\mathrm{dR}}^1(M) = 0$ , when the expected answer is “yes”. Our main tool is the theory of Poincaré duality models from [LS08]. Nevertheless, we also study DGA’s of Hodge type and their small subalgebras from [Fio+19], which, as we think, naturally fit in the picture.

In Section 6.1, we consider DGA’s and define the notions of an orientation and cyclic structure (Definition 6.1.1), non-degeneracy and Poincaré duality (Definition 6.1.6), degenerate subspace and non-degenerate quotient (Definition 6.1.8), Hodge decomposition and Hodge pair (Definition 6.1.9) and Hodge and small subalgebra from [Fio+19] (Definition 6.1.14). We prove that in some cases orientations and cyclic structures are in one-to-one correspondence (Proposition 6.1.4) and that being of Hodge type is equivalent to acyclicity of the degenerate subspace (Proposition 6.1.13). We describe the small subalgebra of [Fio+19] in terms of Kontsevich-Soibelman-like evaluations of rooted binary trees with internal edges labeled with the identity or the standard Hodge homotopy (Proposition 6.1.15). We study what happens when we take small subalgebras and non-degenerate quotients iteratively (Proposition 6.1.17). We give examples when small subalgebras and their non-degenerate quotients differ for different Hodge decompositions of the same DGA (Examples 6.1.21 and 6.1.22). We prove an important lemma allowing

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<sup>1</sup>We did not prove that the homotopy class of  $\mathrm{dIBL}^{\mathfrak{n}}(C(H_{\mathrm{dR}}(M)))$  does not depend on  $P_{\mathrm{std}}$ , but we expect so.

## 6. IBL-infinity formality and Poincaré duality models

to extend an oriented DGA to a DGA of Hodge type later (Lemma 6.1.23). We also prove several little lemmas (Lemmas 6.1.12, 6.1.18, 6.1.19 and 6.1.20) and make several remarks (Remarks 6.1.2, 6.1.5, 6.1.7, 6.1.11 and 6.1.16) which might be useful for future investigations. We finish with some open questions (Questions 6.1.24)

In Section 6.2, we define differential Poincaré duality algebras<sup>2</sup> and Poincaré DGA's (Definition 6.2.1), i.e., DGA's whose homology is a Poincaré duality algebra. We consider Poincaré duality models (Definition 6.2.3). We argue that a Poincaré DGA is formal if and only if it is formal as a DGA (Proposition 6.2.5). We prove that an oriented DGA extends under some conditions to a DGA of Hodge type (Proposition 6.2.6) and use it to prove the existence and uniqueness of Poincaré duality models in some cases (Propositions 6.2.8 and 6.2.9). We discuss minimal models (Remark 6.2.12). We give examples of manifolds whose de Rham algebras do not admit a Poincaré duality model with just one arrow (Example 6.2.13). We finish with some open questions (Questions 6.2.14).

In Section 6.3, we prove functoriality of the canonical  $\mathrm{dIBL}^m$  construction on differential Poincaré duality algebras up to  $\mathrm{IBL}_\infty$ -homotopy (Proposition 6.3.1) and the uniqueness up to  $\mathrm{IBL}_\infty$ -homotopy in a weak homotopy equivalence class of PDGA's in some cases (Proposition 6.3.2). We define  $\mathrm{IBL}_\infty$ -formality of a PDGA (Definition 6.3.4) and conjecture that  $\mathrm{IBL}_\infty$ -formality is implied by DGA-formality (Conjecture 6.3.5); in fact, this conjecture is implied by the previous in some cases. We discuss another relevant notions of formality (Remark 6.3.6). For the de Rham complex of a smooth manifold, we conjecture that the  $\mathrm{IBL}_\infty$ -algebra on cyclic cochains of de Rham cohomology twisted by the Chern-Simons Maurer-Cartan element is homotopy equivalent to the  $\mathrm{dIBL}$ -algebra on cyclic cochains of the non-degenerate quotient of the small subalgebra of the de Rham complex twisted by the canonical Maurer-Cartan element (Conjecture 6.3.7). Assuming that the conjectures hold, we obtain a positive answer to Question 6.0.1 when  $H_{\mathrm{dR}}^1(M) = 0$  (Conjecture 6.3.8).

### 6.1. Orientation, Poincaré duality, Hodge decomposition

In this section, a DGA  $V$  is a triple  $(V, d, \wedge)$ , where

- $V = \bigoplus_{n \in \mathbb{Z}} V^n$  is a  $\mathbb{Z}$ -graded vector space,
- $d : V \rightarrow V$  a differential of degree 1, and

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<sup>2</sup>The same as cyclic DGA's for unital commutative DGA's up to a degree shift and the correspondence of orientations and cyclic structures



## 6. IBL-infinity formality and Poincaré duality models

- $\wedge : V \otimes V \rightarrow V$  an associative product of degree 0 such that  $d$  and  $\wedge$  satisfy the Leibnitz identity  $d(v_1 \wedge v_2) = dv_1 \wedge v_2 + (-1)^{v_1} v_1 \wedge dv_2$  for all homogenous  $v_1, v_2 \in V$ .

In other words, we consider general, possibly non-unital and non-commutative,  $\mathbb{Z}$ -graded DGA's. We denote by  $\deg(v)$  the degree of a homogenous element  $v \in V$  and write  $(-1)^v$  in the exponent.

**Definition 6.1.1** (Orientations and cyclic structures). *Let  $(V, d)$  be a  $\mathbb{Z}$ -graded cochain complex. We define the following:*

- An orientation in degree  $n$  is a linear function  $\circ : V^n \rightarrow \mathbb{R}$  such that

$$\circ \neq 0 \quad \text{and} \quad \circ \circ d = 0.$$

*In other words, it is a surjective chain map  $\circ : (V, d) \rightarrow (\mathbb{R}[-n], 0)$ .*

- A cyclic structure of degree  $n$  is a homogenous bilinear form  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  of degree  $-n$  as a map such that for all homogenous  $v_1, v_2 \in V$ , we have

$$\langle dv_1, v_2 \rangle = (-1)^{1+v_1 v_2} \langle dv_2, v_1 \rangle. \quad (6.1)$$

*A cyclic structure on a DGA  $(V, d, \wedge)$  is additionally required to satisfy*

$$\langle v_1 \wedge v_2, v_3 \rangle = (-1)^{v_3(v_1+v_2)} \langle v_3 \wedge v_1, v_2 \rangle \quad (6.2)$$

*for all homogenous  $v_1, v_2, v_3 \in V$ .*

Analogously, one can define a cyclic structure on an  $A_\infty$ -algebra.

*Remark 6.1.2* (Cyclic DGA). A cyclic DGA  $(V, \Pi, \mu_1, \mu_2)$  of degree  $n$  from Definition 3.3.7 is the same as a DGA equipped with a cyclic structure  $\langle \cdot, \cdot \rangle$  of degree  $n$  which is symmetric, i.e.,

$$\langle v_1, v_2 \rangle = (-1)^{v_1 v_2} \langle v_2, v_1 \rangle$$

for all homogenous  $v_1, v_2 \in V$ , and non-degenerate (see Definition 6.1.6). The correspondence is via the degree shift

$$\begin{aligned} \mu_1(\theta v) &= \theta d(v), \\ \mu_2(\theta v_1, \theta v_2) &= (-1)^{v_1} \theta(v_1 \wedge v_2), \\ \Pi(\theta v_1, \theta v_2) &= (-1)^{v_1} \langle v_1, v_2 \rangle, \end{aligned}$$

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where  $\theta$  is a formal symbol of degree  $-1$ . From this reason, we sometimes call a DGA equipped with a non-degenerate symmetric cyclic structure a cyclic DGA, although there are degree shifts involved.  $\triangleleft$

The homology  $H(V) := H(V, d)$  of a DGA  $(V, d, \wedge)$  is also a DGA with the induced product  $\wedge$  and with zero differential. Given an orientation  $o$  or a cyclic structure  $\langle \cdot, \cdot \rangle$  on  $V$ , we define the maps  $o^H : H(V) \rightarrow \mathbb{R}$  and  $\langle \cdot, \cdot \rangle^H : H(V) \otimes H(V) \rightarrow \mathbb{R}$  for all closed  $h_1, h_2 \in V$  by

$$\begin{aligned} o^H([h_1]) &:= o(h_1) \text{ and} \\ \langle [h_1], [h_2] \rangle^H &:= \langle h_1, h_2 \rangle, \end{aligned} \tag{6.3}$$

respectively, where  $[\cdot]$  denotes the cohomology class. It is easy to see that  $\langle \cdot, \cdot \rangle^H$  is a cyclic structure on  $H(V)$  and that  $o^H$  is an orientation on  $H(V)$  provided that  $o|_{\ker d} \neq 0$ .

**Proposition 6.1.3** (Orientation on homology). *We have the following:*

- (a) *Let  $(V, d)$  be a  $\mathbb{Z}$ -graded cochain complex, and let  $n \in \mathbb{Z}$ . Given an orientation  $\tilde{o} : H^n(V) \rightarrow \mathbb{R}$ , there is an orientation  $o^V : V^n \rightarrow \mathbb{R}$  such that  $\tilde{o} = o^H$ . If  $dV^n = 0$ , then we have the correspondence*

$$\text{Orientations on } V \text{ in degree } n \xrightarrow{1:1} \text{Orientations on } H(V) \text{ in degree } n.$$

- (b) *If  $(V_1, d_1, o_1)$  and  $(V_2, d_2, o_2)$  are  $\mathbb{Z}$ -graded cochain complexes oriented in degree  $n$  with  $dV_1^n = 0 = dV_2^n$ , then a chain map  $f : V_1 \rightarrow V_2$  preserves the orientation if and only if the induced map  $f_* : H(V_1) \rightarrow H(V_2)$  preserves the induced orientation.*

*Proof.* (a) We define  $o^V(v) := \tilde{o}([v])$  for all closed  $v \in V$  and extend it by 0 to a complement of  $\ker d$  in  $V$ . It is obvious that  $o^H = \tilde{o}$ . If  $dV^n = 0$ , any complement is trivial and  $o^V$  is uniquely determined by  $o^H$ .

- (b) Let  $v \in V_1^n$ . Because  $dv = 0$ , we have

$$o_2(f(v)) = o_2^H([f(v)]) = o_2^H(f_*[v]) = o_1^H([v]) = o_1(v).$$

This finishes the proof.  $\square$

Suppose that 1 is a unit for  $(V, d, \wedge)$ , i.e.,  $1 \in V^0$ ,  $d1 = 0$  and  $1 \wedge v = v \wedge 1 = v$  for all

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$v \in V$ , and let  $\langle \cdot, \cdot \rangle$  be a cyclic structure on  $V$ . For all homogenous  $v_1, v_2 \in V$ , we have

$$\begin{aligned}\langle v_1, v_2 \rangle &= \langle v_1 \wedge 1, v_2 \rangle \\ &= (-1)^{v_1 v_2} \langle 1 \wedge v_2, v_1 \rangle \\ &= (-1)^{v_1 v_2} \langle v_2, v_1 \rangle,\end{aligned}$$

and hence  $\langle \cdot, \cdot \rangle$  is automatically symmetric.

Recall that a DGA is called commutative if  $v_1 \wedge v_2 = (-1)^{v_1 v_2} v_2 \wedge v_1$  for all homogenous  $v_1, v_2 \in V$ . Commutativity of a DGA and symmetry of a general cyclic structure on it seem to be unrelated.

**Proposition 6.1.4** (Correspondence of orientations and cyclic structures on DGA's). *Let  $(V, d, \wedge)$  be a DGA. Then the following holds:*

- (a) *If  $\wedge$  is commutative, then any orientation  $\circ$  in degree  $n$  induces a cyclic structure  $\langle \cdot, \cdot \rangle$  of degree  $n$  which is given for all homogenous  $v_1, v_2 \in V$  by*

$$\langle v_1, v_2 \rangle := \begin{cases} \circ(v_1 \wedge v_2) & \text{if } \deg(v_1) + \deg(v_2) = n, \\ 0 & \text{otherwise.} \end{cases} \quad (6.4)$$

- (b) *If  $1$  is a unit, then any non-zero cyclic structure  $\langle \cdot, \cdot \rangle$  of degree  $n$  induces an orientation  $\circ$  in degree  $n$  by defining*

$$\circ(v) := \begin{cases} \langle v, 1 \rangle & \text{for } v \in V^n, \\ 0 & \text{otherwise.} \end{cases} \quad (6.5)$$

- (c) *For a unital commutative DGA  $(V, d, \wedge, 1)$ , formulas (6.4) and (6.5) define the correspondence*

$$\text{Orientations in degree } n \stackrel{1:1}{\cong} \text{Non-zero cyclic structures of degree } n.$$

*Proof.* (a) Using the Leibnitz identity, properties of an orientation and commutativity,

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we check that

$$\begin{aligned}
\langle dv_1, v_2 \rangle &= o(dv_1 \wedge v_2) \\
&= o(d(v_1 \wedge v_2) - (-1)^{v_1} v_1 \wedge dv_2) \\
&= (-1)^{1+v_1} o(v_1 \wedge dv_2) \\
&= (-1)^{1+v_1 v_2} o(dv_2 \wedge v_1) \\
&= (-1)^{1+v_1 v_2} \langle dv_2, v_1 \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle v_1 \wedge v_2, v_3 \rangle &= o(v_1 \wedge v_2 \wedge v_3) \\
&= (-1)^{v_3(v_1+v_2)} o(v_3 \wedge v_1 \wedge v_2) \\
&= (-1)^{v_3(v_1+v_2)} \langle v_3 \wedge v_1, v_2 \rangle
\end{aligned}$$

for all homogenous  $v_1, v_2, v_3 \in V$ .

(b) For all  $v \in V$ , we have

$$o(dv) = \langle dv, 1 \rangle = -\langle d1, v \rangle = 0.$$

From  $\langle \cdot, \cdot \rangle \neq 0$  it follows that there are homogenous  $v_1, v_2 \in V$  with  $\deg(v_1) + \deg(v_2) = n$  such that  $\langle v_1, v_2 \rangle \neq 0$ . Then  $v_1 \wedge v_2 \in V^n$  and

$$\begin{aligned}
o(v_1 \wedge v_2) &= \langle v_1 \wedge v_2, 1 \rangle \\
&= \langle 1 \wedge v_1, v_2 \rangle \\
&= \langle v_1, v_2 \rangle \neq 0.
\end{aligned}$$

Therefore,  $o$  is an orientation on  $V$ .

(c) This is a combination of (a) and (c) plus the uniqueness, which is easy to check.  $\square$

*Remark 6.1.5* (Volume forms). If  $H^n(V) \simeq \mathbb{R}$ , then orientations  $o : H^n(V) \rightarrow \mathbb{R}$  and elements  $0 \neq [\text{Vol}] \in H^n(V)$  called *volume forms* are in one-to-one correspondence via

$$o([\text{Vol}]) = 1.$$

A consequence is the following: Suppose that  $(V_1, d_1, o_1)$  and  $(V_2, d_2, o_2)$  are cochain

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complexes oriented in degree  $n$  which satisfy

$$H^n(V_1) \simeq H^n(V_2) \simeq \mathbb{R} \quad \text{and} \quad d_1 V_1^n \simeq d_2 V_2^n = 0, \quad (6.6)$$

so that Proposition 6.1.3 applies. Then a chain map  $f : V_1 \rightarrow V_2$  preserves orientation if and only if the induced map  $f_* : H^n(V_1) \rightarrow H^n(V_2)$  maps  $[\text{Vol}_1]$  to  $[\text{Vol}_2]$ . In the category of unital commutative DGA's satisfying (6.6), so that also Proposition 6.1.4 holds, if the orientations come from cyclic structures, then  $f$  preserves cyclic structure if and only if  $f_*$  preserves volume form.  $\triangleleft$

**Definition 6.1.6** (Non-degeneracy and Poincaré duality). *Given a graded vector space  $V$ , a homogenous bilinear form ( $\equiv$ : pairing)  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  of degree  $-n$  as a map which is graded symmetric is called non-degenerate if for every  $v \in V$ , the following implication holds:*

$$\langle v, w \rangle = 0 \quad \text{for all } w \in V \quad \implies \quad v = 0.$$

We say that  $\langle \cdot, \cdot \rangle$  satisfies Poincaré duality if the map  $\flat : V \rightarrow V^{*\text{g}}$  (graded dual) defined by

$$\flat(v)(w) := \langle v, w \rangle \quad \text{for all } v, w \in V$$

is a graded isomorphism (it has degree  $-n$  as a map) of graded vector spaces.

*Remark 6.1.7* (On non-degeneracy and Poincaré duality). (i) Clearly, Poincaré duality implies non-degeneracy. If the degree  $k$  component  $V^k$  of  $V$  is finite-dimensional for every  $k \in \mathbb{Z}$  — we say that  $V$  is of *finite type* — then the opposite is true as well. If  $n = 0$ , then Poincaré duality implies that  $V$  is of finite type.

(ii) If  $V$  is non-negatively graded, then non-degeneracy of  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  implies  $V = V^0 \oplus \dots \oplus V^n$ . Therefore, non-negatively graded vector spaces of finite type which admit a non-degenerate homogenous bilinear form are finite-dimensional.

(iii) The de Rham complex  $(\Omega(M), d, \wedge)$  of an oriented closed  $n$ -manifold  $M$  with the orientation  $\int : \Omega^n(M) \rightarrow \mathbb{R}$  is an oriented DGA whose cyclic structure is non-degenerate but does not satisfy Poincaré duality. On the other hand, the induced structure on homology  $H(\Omega(M))$  satisfies Poincaré duality.  $\triangleleft$

An analog of the following definition is used in [Fio+19] and also in [LS08] (under the name “set of orphans”).

**Definition 6.1.8** (Degenerate subspace and non-degenerate quotient). *Given a symmetric pairing  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  on a graded vector space  $V$ , we define the degenerate*

In the augmented unital case, if the pairing is non-deg, one defines  $\text{Vol}$  uniquely by requiring that  $\text{Vol} \perp \bar{V}$  and  $\circ(\text{Vol}) = 1$ . Let us not write it.

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subspace  $V^\perp \subset V$  by

$$V^\perp := \{v \in V \mid \langle w, v \rangle = 0 \text{ for all } w \in V\}.$$

If  $(V, d, \wedge)$  is a DGA and  $\langle \cdot, \cdot \rangle$  a cyclic structure, then (6.1) implies that  $V^\perp$  is a differential graded ideal in  $V$ , and thus we obtain the short exact sequence of DGA's

$$0 \longrightarrow V^\perp \xrightarrow{\iota} V \xrightarrow{\pi^\mathcal{Q}} \mathcal{Q}(V) := V/V^\perp \longrightarrow 0, \quad (6.7)$$

where  $\iota$  is the inclusion and  $\pi^\mathcal{Q}$  the canonical projection. We call the DGA  $\mathcal{Q}(V)$  together with the induced non-degenerate cyclic structure  $\langle \cdot, \cdot \rangle^\mathcal{Q}$  such that  $\langle \pi^\mathcal{Q}(\cdot), \pi^\mathcal{Q}(\cdot) \rangle^\mathcal{Q} = \langle \cdot, \cdot \rangle$  the non-degenerate quotient.

It was observed in [Fio+19] that the question whether  $(V^\perp, d)$  is acyclic, and hence  $\pi^\mathcal{Q}$  is a quasi-isomorphism, turns out to be related to the existence of Hodge decomposition.

**Definition 6.1.9** (Hodge decomposition). *A cochain complex  $(V, d)$  with a symmetric cyclic structure  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  is of Hodge type if there exist subspaces  $\mathcal{H} \subset \ker d$  and  $C \subset V$  such that*

$$V = \ker d \oplus C, \quad \ker d = \operatorname{im} d \oplus \mathcal{H} \quad \text{and} \quad C \perp \mathcal{H} \oplus C, \quad (6.8)$$

where  $\perp$  denotes the relation of being perpendicular with respect to  $\langle \cdot, \cdot \rangle$ . Such decomposition is called a Hodge decomposition. We call  $\mathcal{H}$  the harmonic subspace and  $C$  the coexact part.

Given a Hodge decomposition, we define the standard Hodge homotopy  $\mathcal{P}_{\text{std}} : V \rightarrow V$  by

$$\mathcal{P}_{\text{std}} := \begin{cases} -(d|_C)^{-1} & \text{on } \operatorname{im} d, \\ 0 & \text{on } \mathcal{H} \oplus C. \end{cases}$$

Then we have  $d\mathcal{P}_{\text{std}} = -\pi_{\operatorname{im} d}$ ,  $\mathcal{P}_{\text{std}}d = -\pi_C$ , and hence

$$[d, \mathcal{P}_{\text{std}}] = d\mathcal{P}_{\text{std}} + \mathcal{P}_{\text{std}}d = \pi_{\mathcal{H}} - \mathbb{1}.$$

We call  $(\mathcal{H}, \mathcal{P}_{\text{std}})$  the Hodge pair associated to the Hodge decomposition (6.8).

**Proposition 6.1.10** (Non-deg., fin. type implies Hodge type). *Any cochain complex of finite type with a non-degenerate symmetric cyclic structure is of Hodge type.*

*Proof.* This is [CFL15, Lemma 11.1], and the proof uses formal Hodge theory.  $\square$

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*Remark 6.1.11* (Harmonic subspaces). In the situation of Proposition 6.1.10, it was shown in [Fio+19, Remark 2.6] that for any complement  $\mathcal{H}$  of  $\text{im } d$  in  $\ker d$  (in other words, the image of a section  $H(V) \rightarrow \ker d$ ) there is a coexact part  $C$  such that  $V = \mathcal{H} \oplus \text{im } d \oplus C$  is a Hodge decomposition. From this reason, we call any complement of  $\text{im } d$  in  $\ker d$  a *harmonic subspace*.<sup>3</sup>  $\triangleleft$

The following lemma will be used in the proof of Proposition 6.1.13.

**Lemma 6.1.12** (Complement of acyclic subcomplex over  $\mathbb{R}$ ). *Let  $f : V_1 \rightarrow V_2$  be an injective chain map of cochain complexes  $(V_1, d_1)$  and  $(V_2, d_2)$  over  $\mathbb{R}$  such that  $(V_1, d_1)$  is acyclic. Then there is a chain map  $g : V_2 \rightarrow V_1$  such that  $g \circ f = \mathbb{1}$ .<sup>4</sup>*

*Proof.* For every  $i \in \mathbb{Z}$ , consider the diagram

$$\begin{array}{ccc} \ker d_1^i \oplus C_1^i & \xrightarrow{f^i} & \ker d_2^i \oplus C_2^i \\ \downarrow d_1^i & & \downarrow d_2^i \\ \ker d_1^{i+1} \oplus C_1^{i+1} & \xrightarrow{f^{i+1}} & \ker d_2^{i+1} \oplus C_2^{i+1}, \end{array}$$

where  $C_1^i$  is a complement of  $\ker d_1^i$  in  $V_1^i$  and  $C_2^i$  is a complement of  $\ker d_2^i$  in  $V_2^i$ . With respect to this decomposition, we write

$$\begin{aligned} f^i &= \begin{pmatrix} f_{11}^i & f_{12}^i \\ f_{21}^i & f_{22}^i \end{pmatrix}, & g^i &= \begin{pmatrix} g_{11}^i & g_{12}^i \\ g_{21}^i & g_{22}^i \end{pmatrix}, \\ d_1^i &= \begin{pmatrix} 0 & d_1^i \\ 0 & 0 \end{pmatrix}, & d_2^i &= \begin{pmatrix} 0 & d_2^i \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The assumption  $H(V_1) = 0$  implies that  $d_1^i$  is an isomorphism. The fact that  $f$  is a chain map translates to

$$d_2^i f_{21}^i = 0, \quad f_{21}^{i+1} d_1^i = 0, \quad f_{11}^{i+1} d_1^i = d_2^i f_{22}^i. \quad (6.9)$$

From the second relation and surjectivity of  $d_1^i$ , we get that  $f_{21}^{i+1} = 0$ . Now,  $f_{11}^{i+1}$  has to be injective because it is the only possibly non-zero part of  $f$  on  $\ker d_1^{i+1}$ . From the third relation of (6.9) and the fact that  $d_1^i$  is injective, we get that  $f_{22}^i$  is injective as well. Relations (6.9) hold also for  $g$  with  $d_1$  and  $d_2$  switched. In particular, we have

<sup>3</sup>Given a Hodge decomposition  $V = \mathcal{H} \oplus \text{im } d \oplus C$  and a harmonic subspace  $\mathcal{H}'$ , then it holds  $\mathcal{H}' = \text{gr}(\alpha : \mathcal{H} \rightarrow dV)$  because  $\mathcal{H} \oplus dV = \mathcal{H}' \oplus dV$ , and one can take  $C' = \text{gr}(-\alpha^\dagger - \frac{1}{2}\alpha\alpha^\dagger : C \rightarrow \mathcal{H} \oplus dV)$ .

<sup>4</sup>This lemma can be used to prove that over  $\mathbb{R}$ , every surjective quasi-isomorphism is a deformation retraction and every injective quasi-isomorphism is a section of a deformation retraction.

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$g_{21}^i = 0$ . The relation  $g \circ f = \mathbb{1}$  translates using  $f_{21}^i = g_{21}^i = 0$  to

$$g_{11}^i f_{11}^i = \mathbb{1}, \quad g_{11}^i f_{12}^i + g_{12}^i f_{22}^i = 0, \quad g_{22}^i f_{22}^i = \mathbb{1}. \quad (6.10)$$

Because  $d_1^i$  is an isomorphism, the last equation is equivalent to

$$\mathbb{1} = d_1^i g_{22}^i f_{22}^i (d_1^i)^{-1} = g_{11}^{i+1} d_2^i f_{22}^i (d_1^i)^{-1} = g_{11}^{i+1} f_{11}^{i+1} d_1^i (d_1^i)^{-1} = g_{11}^{i+1} f_{11}^{i+1}.$$

We see that  $g$  can be constructed as follows. For all  $i \in \mathbb{Z}$ , let  $g_i^{11}$  be an arbitrary left inverse of  $f_i^{11}$ . Set  $g_{22}^i := (d_1^i)^{-1} g_{11}^{i+1} d_2^i$  and  $g_i^{21} := 0$ . Finally,  $g_{12}^i$  has to be chosen such that the second equation of (6.10) is satisfied. This is possible since we can first define  $g_{12}^i$  on  $\text{im } f_{22}^i$  because  $f_{22}^i$  injective and then extend it by 0 to a complement.  $\square$

Claim (b) of the following proposition corresponds to [Fio+19, Lemma 2.8]. Claim (c) was suggested by Prof. Hồng Vân Lê via e-mail correspondence.

**Proposition 6.1.13** (Hodge decomposition and acyclicity of  $V^\perp$ ). *Let  $(V, d)$  be a cochain complex with a symmetric cyclic structure  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$ . If  $V$  is of Hodge type, then the following implications hold:*

(a) *If  $\langle \cdot, \cdot \rangle$  is non-degenerate, then  $\langle \cdot, \cdot \rangle^H$  is non-degenerate.*

(b) *If  $\langle \cdot, \cdot \rangle^H$  is non-degenerate, then  $(V^\perp, d)$  is acyclic.*

Moreover, the following reverse implication holds:

(c) *If  $V$  is of finite type and  $(V^\perp, d)$  is acyclic, then  $V$  is of Hodge type.*

*Proof.* (a) Let  $V = \text{im } d \oplus \mathcal{H} \oplus C$  be a Hodge decomposition. Then  $\text{im } d \oplus C \subset \mathcal{H}^\perp$ , and hence

$$\langle d\eta + b + c, b' \rangle = \langle b, b' \rangle \quad \text{for all } \eta \in V, c \in C \text{ and } b, b' \in \mathcal{H}.$$

The claim follows easily. Notice that having a Hodge decomposition, it holds

$$\langle \cdot, \cdot \rangle|_{\mathcal{H} \otimes \mathcal{H}} \text{ non-degenerate} \iff \text{im } d \oplus C = \mathcal{H}^\perp.$$

(b) Consider a Hodge decomposition as above, and let  $v \in V^\perp \cap \ker d$  be a non-zero vector. Suppose that  $v \notin \text{im } d$ . Then  $[v] \neq 0$  in  $H(V)$ , and hence there is a  $b \in \mathcal{H}$  such that  $\langle v, b \rangle \neq 0$  by non-degeneracy of  $\langle \cdot, \cdot \rangle^H$ . This is a contradiction with  $v \in V^\perp$ . Therefore, it holds  $V^\perp \cap \ker d = V^\perp \cap \text{im } d$ . In particular, there is an  $\eta \in C$  such that



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$v = d\eta$ . Now, for any  $\eta' \in C$ ,  $b \in \mathcal{H}$  and  $c \in C$ , we have using  $C \perp \mathcal{H} \oplus C$  and  $v \in V^\perp$  the following:

$$\begin{aligned} \langle d\eta' + b + c, \eta \rangle &= \langle d\eta', \eta \rangle \\ &= (-1)^{1+\eta'} \langle d\eta, \eta' \rangle \\ &= (-1)^{1+\eta'} \langle v, \eta' \rangle \\ &= 0. \end{aligned}$$

Therefore, it holds  $\eta \in V^\perp$ , and we have shown that  $V^\perp \cap \text{im } d = dV^\perp$ . The claim follows.

(c) Because  $V^\perp \subset V$  is an acyclic subcomplex and we work over  $\mathbb{R}$ , there is a complementary subcomplex  $Z \subset V$ ; i.e.,  $dZ \subset Z$  and  $V = V^\perp \oplus Z$ . This follows from Lemma 6.1.12 by setting  $V^\perp = \text{im } f$  and  $Z = \ker g$ . Now, the restriction of  $\langle \cdot, \cdot \rangle$  to  $Z$  is non-degenerate, and Proposition 6.1.10 provides its Hodge decomposition  $Z = dZ \oplus \mathcal{H} \oplus D$ . Let  $E \subset V^\perp$  be a graded vector space which is complementary to  $dV^\perp$  in  $V^\perp$ ; i.e.,  $V^\perp = dV^\perp \oplus E$ . It is easy to check that  $V = \text{im } d \oplus \mathcal{H} \oplus C$  with  $C := D \oplus E$  is a Hodge decomposition.  $\square$

Deformation retract over  $\mathbb{R}$   
add: Add here deformation retract somewhere.

The following notions were taken from [Fio+19].

**Definition 6.1.14** (Hodge subalgebra and small subalgebra). *Consider a DGA  $(V, d, \wedge)$  with a symmetric cyclic structure  $\langle \cdot, \cdot \rangle$ . Suppose that it admits a Hodge decomposition with the Hodge pair  $(\mathcal{H}, \mathcal{P}_{\text{std}})$ . A Hodge subalgebra is a differential graded subalgebra  $W \subset V$  which satisfies*

$$\mathcal{H} \subset W \quad \text{and} \quad \mathcal{P}_{\text{std}} W \subset W.$$

*We denote the smallest Hodge subalgebra of  $V$  by  $\mathcal{S}(V)$  and call it the small subalgebra. We stress that the definition of  $\mathcal{S}(V)$  depends on  $(\mathcal{H}, \mathcal{P}_{\text{std}})$ !*

The following is a version of [Fio+19, Proposition 3.3] which generalizes to the non-simply-connected case (see (iii) of Remark 6.1.16 below for the comparison).

**Proposition 6.1.15** (Description of small subalgebra). *Consider the situation of Definition 6.1.14. The small subalgebra  $\mathcal{S}(V)$  is generated as a graded vector space by Kontsevich-Soibelman-like evaluations of rooted binary trees with  $k \geq 1$  leaves labeled with homogenous elements of  $\mathcal{H}$ , interior vertices labeled with  $\wedge$  and interior edges labeled either with  $\mathcal{P}_{\text{std}}$  or with  $\mathbb{1}$ .*

Does the following hold without assuming finite type? I.e. does Poincaré duality imply Hodge type?

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*Proof.* We abbreviate  $\mathcal{S} := \mathcal{S}(V)$ , denote the set of labeled trees by  $\mathcal{T}$  and denote the vector space generated by evaluations of elements of  $\mathcal{T}$  by  $\langle \mathcal{T} \rangle$ .

Clearly, we have  $\langle \mathcal{T} \rangle \subset \mathcal{S}$ . For “=”, it suffices to check that for any  $T, T_1, T_2 \in \mathcal{T}$ , it holds  $dT, \mathcal{P}_{\text{std}}T, T_1 \wedge T_2 \in \langle \mathcal{T} \rangle$ , i.e., that  $\langle \mathcal{T} \rangle$  is a Hodge subalgebra.

As for  $dT$ , we imagine  $d$  propagating from the root to the leaves. When it encounters  $L \wedge R$ , where  $L$  stands for the left and  $R$  for the right sub-branch, it duplicates the tree and continues propagating in  $L$  and  $R$ , respectively (we take the sum of the two copies in the end). This is justified by the Leibnitz identity  $d(L \wedge R) = dL \wedge R + (-1)^L L \wedge dR$ . When it encounters  $\mathcal{P}_{\text{std}}$ , it triples the tree and either goes past  $\mathcal{P}_{\text{std}}$  and continues propagating, or exchanges  $\mathcal{P}_{\text{std}}$  for  $\mathbb{1}$  and stops, or exchanges  $\mathcal{P}_{\text{std}}$  for  $\pi_{\mathcal{H}}$  and stops. This is justified by  $d\mathcal{P}_{\text{std}} = -\mathcal{P}_{\text{std}}d - \mathbb{1} + \pi_{\mathcal{H}}$ . If it encounters  $\mathbb{1}$ , nothing happens and it keeps propagating. If it reaches a leaf with  $h \in \mathcal{H}$ , then the corresponding tree evaluates to 0. We see that we always obtain an element of  $\langle \mathcal{T} \rangle$ .

As for  $\mathcal{P}_{\text{std}}T$ , we have either  $\mathcal{P}_{\text{std}}T = 0$  if the interior edge adjacent to the root (= the root edge) is labeled with  $\mathcal{P}_{\text{std}}$ , or  $\mathcal{P}_{\text{std}}T$  is a new tree in  $\mathcal{T}$  which arises from  $T$  by replacing  $\mathbb{1}$  by  $\mathcal{P}_{\text{std}}$  on the root edge.

As for  $T_1 \wedge T_2$ , using  $\mathbb{1} = \pi_{\mathcal{H}} - d\mathcal{P}_{\text{std}} - \mathcal{P}_{\text{std}}d$  and the Leibnitz identity, we get

$$\begin{aligned} T_1 \wedge T_2 &= \pi_{\mathcal{H}}(T_1 \wedge T_2) - (d\mathcal{P}_{\text{std}})(T_1 \wedge T_2) - (\mathcal{P}_{\text{std}}d)(T_1 \wedge T_2) \\ &= \pi_{\mathcal{H}}(T_1 \wedge T_2) - d(\mathcal{P}_{\text{std}}(T_1 \wedge T_2)) - \mathcal{P}_{\text{std}}(dT_1 \wedge T_2) - (-1)^{T_1} \mathcal{P}_{\text{std}}(T_1 \wedge dT_2). \end{aligned}$$

We see that  $T_1 \wedge T_2 \in \langle \mathcal{T} \rangle$ . □

*Remark 6.1.16* (On Hodge subalgebra and small subalgebra). (i) Any Hodge subalgebra  $W$  inherits the Hodge decomposition

$$W = \mathcal{H} \oplus dW \oplus \mathcal{P}_{\text{std}}W \tag{6.11}$$

with the Hodge pair  $(\mathcal{H}, \mathcal{P}_{\text{std}}|_W)$ .

(ii) We will be in the situation of Definition 6.1.14 and use the notation of the proof of Proposition 6.1.15. In addition, we suppose that  $V$  is non-negatively graded. Having established  $\mathcal{S} = \langle \mathcal{T} \rangle$ , consider the Hodge decomposition (6.11) for  $W = \mathcal{S}$ . Let  $\bar{\mathcal{S}} = \bigoplus_{k \geq 1} \mathcal{S}^k$  denote the reduced part of  $\mathcal{S}$ , and let  $\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle$  denote the graded vector space

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generated by products  $v_1 \wedge v_2$  for  $v_1, v_2 \in \bar{\mathcal{S}}$ . We have

$$\begin{aligned}\mathcal{P}_{\text{std}}\langle \mathcal{T} \rangle &= \langle \{T \in \mathcal{T} \text{ with } \mathcal{P}_{\text{std}} \text{ on the root edge}\} \rangle \\ &= \mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle\end{aligned}$$

and

$$\begin{aligned}d\langle \mathcal{T} \rangle &= d(\pi_{\mathcal{H}} - d\mathcal{P}_{\text{std}} - \mathcal{P}_{\text{std}}d)\langle \mathcal{T} \rangle \\ &= d\mathcal{P}_{\text{std}}d\langle \mathcal{T} \rangle \\ &\subset d\mathcal{P}_{\text{std}}\langle \mathcal{T} \rangle \\ &\subset d\mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle.\end{aligned}$$

It holds even  $d\langle \mathcal{T} \rangle = d\mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle$  due to (6.11). We can now write (6.11) as

$$\mathcal{S}^k = \mathcal{H}^k \oplus d\mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle^k \oplus \mathcal{P}_{\text{std}}\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle^{k+1}. \quad (6.12)$$

If  $H^1 = 0$ , this agrees with the formula from [Fio+19, Proposition 3.3]. In this case, it holds  $\mathcal{S}^1 = 0$ , and hence  $\langle \bar{\mathcal{S}} \wedge \bar{\mathcal{S}} \rangle^{k+1}$  depends only on  $\mathcal{S}^i$  for  $i < k$ . Therefore, we can compute  $\mathcal{S}^k$  from (6.12) inductively starting with  $\mathcal{S}^0 = \langle 1 \rangle$ .

(iii) The previous remark implies the following: Let  $(V, d, \wedge)$  be a non-negatively graded DGA with cyclic structure  $\langle \cdot, \cdot \rangle$  of Hodge type. Suppose that  $H^1(V) = 0$ . If  $H(V)$  is of finite type, then so is  $\mathcal{S}(V)$ .

(iv) The advantage of  $\mathcal{S}(V)$  is that we have the diagram of pairing preserving DGA-quasi-isomorphisms

$$V \longleftarrow \mathcal{S}(V) \longrightarrow \mathcal{Q}(\mathcal{S}(V)).$$

Moreover, in the case of (iii), the non-degenerate quotient is finite-dimensional, and hence a Poincaré duality model of  $V$  (see the next section).

(v) Notice that if there is a quasi-isomorphism  $f : H(V) \rightarrow V$ , then  $\mathcal{S}(V) = V$  for any Hodge decomposition with harmonic subspace  $\text{im } f$ .  $\triangleleft$

**Proposition 6.1.17** (Properties of  $\mathcal{S}$  and  $\mathcal{Q}$ ). *Let  $(V, d, \wedge)$  be a DGA with a symmetric cyclic structure  $\langle \cdot, \cdot \rangle$ . Suppose that it admits a Hodge decomposition with Hodge pair  $(\mathcal{H}, \mathcal{P}_{\text{std}})$ . Then  $\mathcal{S}(V)$  and  $\mathcal{Q}(V)$  admit Hodge decompositions with the Hodge pairs  $(\mathcal{H}, \mathcal{P}_{\text{std}}^{\mathcal{S}} := \mathcal{P}_{\text{std}}|_{\mathcal{S}(V)})$  and  $(\pi^{\mathcal{Q}}(\mathcal{H}), \mathcal{P}_{\text{std}}^{\mathcal{Q}})$ , respectively, where  $\pi^{\mathcal{Q}} : V \rightarrow \mathcal{Q}(V)$  is the canonical projection and  $\mathcal{P}_{\text{std}}^{\mathcal{Q}}$  the unique map on  $\mathcal{Q}(V)$  satisfying  $\mathcal{P}_{\text{std}}^{\mathcal{Q}} \circ \pi^{\mathcal{Q}} = \pi^{\mathcal{Q}} \circ \mathcal{P}_{\text{std}}$ .*

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Furthermore, with respect to the induced Hodge decompositions, we have

$$\mathcal{S}(\mathcal{S}(V)) = \mathcal{S}(V), \quad \mathcal{Q}(\mathcal{Q}(V)) = \mathcal{Q}(V) \quad \text{and} \quad \mathcal{S}(\mathcal{Q}(\mathcal{S}(V))) = \mathcal{Q}(\mathcal{S}(V)). \quad (6.13)$$

*Proof.* The fact that  $\mathcal{S} := \mathcal{S}(V)$  admits a Hodge decomposition with Hodge pair  $(\mathcal{H}, \mathcal{P}_{\text{std}}^{\mathcal{S}})$  was stated in (i) of Remark 6.1.16 and is easy to check.

We prove that  $\mathcal{Q} := \mathcal{Q}(V)$  has a Hodge decomposition with Hodge pair  $(\pi^{\mathcal{Q}}(\mathcal{H}), \mathcal{P}_{\text{std}}^{\mathcal{Q}})$ . Because  $\pi^{\mathcal{Q}}$  is a quasi-isomorphism,  $\pi^{\mathcal{Q}}(\mathcal{H})$  is a harmonic subspace of  $\mathcal{Q}$ , and we have  $\ker d^{\mathcal{Q}} = \pi^{\mathcal{Q}}(\mathcal{H}) \oplus \text{im } d^{\mathcal{Q}}$ . Let  $c \in C$  with  $dc \in V^{\perp}$  such that  $\pi^{\mathcal{Q}}(c) \in \ker d^{\mathcal{Q}} \cap \pi^{\mathcal{Q}}(C)$ . From the cyclicity of  $\langle \cdot, \cdot \rangle$  with respect to  $d$  and from  $C \perp \mathcal{H} \oplus C$ , it follows that  $c \in V^{\perp}$ , and thus  $\pi^{\mathcal{Q}}(c) = 0$ . Together with surjectivity of  $\pi^{\mathcal{Q}}$  this implies that  $\mathcal{Q} = \ker d^{\mathcal{Q}} \oplus \pi^{\mathcal{Q}}(C)$ . From  $\langle \pi^{\mathcal{Q}}(\cdot), \pi^{\mathcal{Q}}(\cdot) \rangle^{\mathcal{Q}}$ , we see that  $\pi^{\mathcal{Q}}(C) \perp \pi^{\mathcal{Q}}(\mathcal{H}) \oplus \pi^{\mathcal{Q}}(C)$ . Therefore,  $\mathcal{Q} = \pi^{\mathcal{Q}}(\mathcal{H}) \oplus \text{im } d^{\mathcal{Q}} \oplus \pi^{\mathcal{Q}}(C)$  is a Hodge decomposition, and it is easy to see that its standard Hodge homotopy  $\mathcal{P}_{\text{std}}^{\mathcal{Q}}$  satisfies  $\mathcal{P}_{\text{std}}^{\mathcal{Q}} \circ \pi^{\mathcal{Q}} = \pi^{\mathcal{Q}} \circ \mathcal{P}_{\text{std}}$ . This defines  $\mathcal{P}_{\text{std}}^{\mathcal{Q}}$  uniquely because  $\pi^{\mathcal{Q}}$  is surjective.

As for the relations (6.13), the first two are clear. The third can be seen as follows. Since  $\pi^{\mathcal{Q}} : V \rightarrow \mathcal{Q}$  is a DGA-morphism mapping the harmonic subspaces isomorphically onto each other and commuting with  $d$  and  $\mathcal{P}_{\text{std}}$ , the assignments  $Y \subset V \mapsto \pi^{\mathcal{Q}}(Y) \subset \mathcal{Q}$  and  $Z \subset \mathcal{Q} \mapsto (\pi^{\mathcal{Q}})^{-1}(Z) \subset V$  preserve Hodge subalgebras. Therefore, if  $Z \subset \mathcal{Q}(\mathcal{S})$  is a Hodge subalgebra, then  $(\pi^{\mathcal{Q}})^{-1}(Z) \subset \mathcal{S}$  is a Hodge subalgebra. It holds even  $(\pi^{\mathcal{Q}})^{-1}(Z) = \mathcal{S}$  by minimality of  $\mathcal{S}$ , and hence  $Z = \mathcal{Q}(\mathcal{S})$  by surjectivity of  $\pi^{\mathcal{Q}}$ .  $\square$

A natural question is, how does  $\mathcal{Q}(\mathcal{S}(V))$  depend on the chosen Hodge pair and how does it behave under quasi-isomorphisms? The following lemmas might be useful.

**Lemma 6.1.18** (Kernel of pairing-preserving morphism). *Let  $V_1$  and  $V_2$  be vector spaces with symmetric bilinear forms  $\langle \cdot, \cdot \rangle_1 : V_1 \otimes V_1 \rightarrow \mathbb{R}$  and  $\langle \cdot, \cdot \rangle_2 : V_2 \otimes V_2 \rightarrow \mathbb{R}$ , respectively. Let  $f : V_1 \rightarrow V_2$  be a linear map such that*

$$\langle v_1, v_2 \rangle_1 = \langle f(v_1), f(v_2) \rangle_2 \quad \text{for all } v_1, v_2 \in V_1. \quad (6.14)$$

*Then it holds*

$$\ker f \subset V_1^{\perp} \quad \text{and} \quad f(V_1^{\perp}) \subset f(V_1)^{\perp}.$$

*In particular, the following statements are true:*

- (a) *If  $\langle \cdot, \cdot \rangle_1$  is non-degenerate, then  $f$  is injective.*
- (b) *If  $\langle \cdot, \cdot \rangle_2$  is non-degenerate and  $f$  is surjective, then  $\ker f = V_1^{\perp}$ .*

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*Proof.* Clear. □

**Lemma 6.1.19** (Injectivity on domain with non-degenerate orientation). *Let  $(V_1, d_1, \wedge_1)$  be a non-negatively graded commutative DGA with an orientation  $\mathfrak{o}_1$  in degree  $n$  such that the induced cyclic structure is non-degenerate. Let  $(V_2, d_2, \wedge_2)$  be any DGA, and let  $f : V_1 \rightarrow V_2$  be a morphism of DGA's. Then injectivity of  $f_* : H^n(V_1) \rightarrow H^n(V_2)$  implies injectivity of  $f : V_1 \rightarrow V_2$ .*

*Proof.* Firstly, injectivity of a homogenous map is equivalent to degree-wise injectivity. Secondly, because  $V_1$  is non-negatively graded and the cyclic structure of degree  $n$  is non-degenerate, we have  $V_1 = V_1^0 \oplus \cdots \oplus V_1^n$ . Now, suppose that  $v \in V_1^k$  for some  $k = 0, \dots, n$  satisfies  $f(v) = 0$ . For any  $w \in V^{n-k}$ , the product  $v \wedge w$  lies in  $V^n$ , and thus  $d(v \wedge w) = 0$ . We compute

$$f_*[v \wedge w] = [f(v \wedge w)] = [f(v) \wedge f(w)] = 0,$$

and hence  $v \wedge w = d\eta$  for some  $\eta \in V_1^{n-1}$  by injectivity of  $f_*$ . Consequently, we have

$$\mathfrak{o}_1(v \wedge w) = \mathfrak{o}_1(d\eta) = 0,$$

and hence  $v = 0$  by non-degeneracy of  $\mathfrak{o}_1$ . □

**Lemma 6.1.20** (Small subalgebra of cyclic DGA and quasi-iso.). *Let  $(V_1, d_1, \wedge_1, \langle \cdot, \cdot \rangle_1)$  and  $(V_2, d_2, \wedge_2, \langle \cdot, \cdot \rangle_2)$  be non-negatively graded unital commutative DGA's of finite type with non-degenerate cyclic structures of degree  $n$  (hence finite-dimensional). Let  $f : V_1 \rightarrow V_2$  be a DGA-morphism such that  $f_* : (H(V_1), \mathfrak{o}_1^H) \rightarrow (H(V_2), \mathfrak{o}_2^H)$  is an isomorphism. Then a Hodge decomposition of  $V_1$  with Hodge pair  $(\mathcal{H}_1, \mathcal{P}_{\text{std}}^1)$  induces a Hodge decomposition of  $V_2$  with Hodge pair  $(f(\mathcal{H}_1), \mathcal{P}_{\text{std}}^2)$ , where  $\mathcal{P}_{\text{std}}^2$  satisfies  $\mathcal{P}_{\text{std}}^2 \circ f = f \circ \mathcal{P}_{\text{std}}^1$ . Consequently,  $f$  induces an isomorphism  $\mathcal{S}(V_1) \simeq \mathcal{S}(V_2)$ .*

*Proof.* Because  $f_*$  preserves orientation and it holds  $V_1^{n+1} = 0 = V_2^{n+1}$ , Proposition 6.1.3 implies that  $f$  preserves cyclic structure and hence is injective by Lemma 6.1.18. Let  $V_1 = \mathcal{H}_1 \oplus \text{im } d_1 \oplus C_1$  be a Hodge decomposition. From the injectivity of  $f$ , it follows that

$$f(V_1) = f(\mathcal{H}_1) \oplus df(V_1) \oplus f(C_1)$$

and that the restriction of  $\langle \cdot, \cdot \rangle_2$  to  $f(V_1) \otimes f(V_1)$  is non-degenerate. Because  $V_1$  is of finite type,  $f(V_1)$  is of finite type, and so non-degeneracy implies Poincaré duality

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$f(V_1)^{*g} \simeq f(V_1)$ . It follows that

$$V_2 = f(V_1) \oplus f(V_1)^\perp.$$

Cyclicity of  $d_2$  with respect to  $\langle \cdot, \cdot \rangle_2$  implies that  $f(V_1)^\perp \subset V_2$  is a subcomplex. Because  $f$  is a quasi-isomorphism, we have  $H(V_1) \simeq H(f(V_1)) \simeq H(V_2)$ ; because the homology is additive, we have  $H(V_2) \simeq H(f(V_1)) \oplus H(f(V_1)^\perp) \simeq H(V_2) \oplus H(f(V_1)^\perp)$ ; finally, because  $H(V_2)$  is of finite type, we have  $H(f(V_1)^\perp) = 0$ . Because  $V_2 = f(V_1) \oplus f(V_1)^\perp$  and  $\langle \cdot, \cdot \rangle_2$  is non-degenerate, its restriction to  $f(V_1)^\perp$  is non-degenerate too. As  $V_2$  and hence  $f(V_1)^\perp$  is of finite type, Proposition 6.1.10 gives a Hodge decomposition  $f(V_1)^\perp = df(V_1)^\perp \oplus C'_2$ . It is easy to check that

$$V_2 = f(\mathcal{H}_1) \oplus \underbrace{(df(V_1) \oplus df(V_1)^\perp)}_{= \text{im } d_2} \oplus \underbrace{(f(C_1) \oplus C'_2)}_{=: C_2}$$

is a Hodge decomposition of  $V_2$ . The corresponding standard Hodge homotopy clearly satisfies  $\mathcal{P}_{\text{std}}^2 \circ f = f \circ \mathcal{P}_{\text{std}}^1$ , and it holds  $\mathcal{P}_{\text{std}}^2(f(V_1)) \subset f(V_1)$ , so that  $f(V_1)$  is a Hodge subalgebra. By injectivity, it follows that  $\mathcal{S}(V_1) \simeq \mathcal{S}(V_2)$ .  $\square$

Since  $\mathcal{S} = \mathcal{S}(V)$  is “small”, a question whether it can be fit inside the image of a Sullivan’s minimal model arose. This would imply, under some additional assumptions, that for any two Hodge decompositions of  $V$ , the non-degenerate quotients  $\mathcal{Q}_1$ , resp.  $\mathcal{Q}_2$  of the corresponding small subalgebras  $\mathcal{S}_1$ , resp.  $\mathcal{S}_2$  would be isomorphic as Poincaré duality algebras. Let us sketch the idea of this construction assuming that  $f_1 : \Lambda U_1 \rightarrow \mathcal{S}_1$  and  $f_2 : \Lambda U_2 \rightarrow \mathcal{S}_2$  are surjective Sullivan’s minimal models. Uniqueness from [FOT08, Theorem 2.24] gives an isomorphism  $\Lambda U := \Lambda U_1 \simeq \Lambda U_2$  such that the following diagram commutes up to homotopy of DGA’s:

$$\begin{array}{ccccc}
 & & f_1 & \twoheadrightarrow & \mathcal{S}_1 & \longrightarrow & \mathcal{Q}_1 \\
 & \nearrow & & & \nwarrow & & \\
 \Lambda U & & & & V & & \\
 & \searrow & & & \nearrow & & \\
 & & f_2 & \twoheadrightarrow & \mathcal{S}_2 & \longrightarrow & \mathcal{Q}_2.
 \end{array} \tag{6.15}$$

Now,  $\Lambda U \xrightarrow{f_1} \mathcal{S}_1 \hookrightarrow V$  and  $\Lambda U \xrightarrow{f_2} \mathcal{S}_2 \hookrightarrow V$  induce the same isomorphism on homology; this can be used to pullback the orientation  $\text{o}^H$  on  $H(V)$  to an orientation  $\tilde{\text{o}}$  on  $H(\Lambda U)$ , so that all maps in (6.15) will preserve the orientation on homology. Under the assumptions

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$dV^n = 0$  and  $d(\Lambda U)^n = 0$ , Proposition 6.1.3 applies, and we obtain a cyclic structure  $\langle \cdot, \cdot \rangle^{\Lambda U}$  on  $\Lambda U$  which is preserved by both  $f_1$  and  $f_2$  on the chain level. We denote  $f_i^{\mathcal{Q}} := \pi_i^{\mathcal{Q}} \circ f_i$  and write down the following diagram with pairing-preserving DGA-quasi-isomorphisms:

$$\begin{array}{ccc}
 & (\Lambda U, \langle \cdot, \cdot \rangle^{\Lambda U}) & \\
 f_1^{\mathcal{Q}} \swarrow & & \searrow f_2^{\mathcal{Q}} \\
 (\mathcal{Q}_1, \langle \cdot, \cdot \rangle_1^{\mathcal{Q}}) & & (\mathcal{Q}_2, \langle \cdot, \cdot \rangle_2^{\mathcal{Q}})
 \end{array} \tag{6.16}$$

Claim (b) of Lemma 6.1.18 implies that  $\ker f_1^{\mathcal{Q}} = \ker f_2^{\mathcal{Q}} = (\Lambda U)^{\perp}$ , and hence

$$\mathcal{Q}_1 \simeq \Lambda U / (\Lambda U)^{\perp} \simeq \mathcal{Q}_2.$$

Unfortunately, the next two examples show that one can not, in general, expect  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , or even  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  to be isomorphic and  $\mathcal{S}$  to fit inside the image of the minimal model.<sup>5</sup>

*Example 6.1.21* (Small algebras are, in general, not unique and not contained in images of Sullivan minimal models). Consider  $M = \mathbb{CP}^2$ , and let  $K$  be the Fubini–Study form on  $M$ . For  $\alpha \in \Omega^1(M)$ , set  $K_{\alpha} := K + d\alpha$ . Then  $K_{\alpha} \wedge K_{\alpha} = K \wedge K + d(2\alpha \wedge K + \alpha \wedge d\alpha)$ . We can choose  $\alpha$  such that  $d(2\alpha \wedge K + \alpha \wedge d\alpha) \neq 0$ . Consider the Riemannian Hodge decomposition of  $\Omega(M)$  with  $\mathcal{H} = \langle 1, K, K \wedge K \rangle$ . According to Remark 6.1.11, there is also a “twisted” Hodge decomposition with the harmonic subspace  $\mathcal{H}_{\alpha} := \langle 1, K_{\alpha}, K \wedge K \rangle$ .

The small subalgebra  $\mathcal{S}$  for the Riemann Hodge decomposition is  $\mathcal{S} = \langle 1, K, K \wedge K \rangle$ . The small subalgebra  $\mathcal{S}_{\alpha}$  for the twisted Hodge decomposition must contain 1,  $K_{\alpha}$  and both  $K_{\alpha} \wedge K_{\alpha}$  and  $K \wedge K$  (we require  $\mathcal{H}_{\alpha} \subset \mathcal{S}_{\alpha}$  by definition). Therefore, it contains

$$K_{\alpha} \wedge K_{\alpha} - K \wedge K = d(2\alpha \wedge K + \alpha \wedge d\alpha)$$

and also

$$P_{\text{std}}^{\alpha} d(2\alpha \wedge K + \alpha \wedge d\alpha) = \pi_{C_{\alpha}}(2\alpha \wedge K + \alpha \wedge d\alpha).$$

Proposition 6.1.15 asserts that these vectors, together with 1 and  $K_{\alpha}$ , generate  $\mathcal{S}_{\alpha}$  as a

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<sup>5</sup>Note that it is always possible to construct a surjective (non-minimal) Sullivan model  $f : \Lambda U \twoheadrightarrow V$  by taking the minimal Sullivan model and inductively adding generators  $\xi_i$  and  $\mu_i$  with  $d\mu_i = 0$  and  $d\xi_i = \mu_i$  (the minimality condition on a Sullivan’s algebra would require  $d\xi_i$  to be decomposable), and mapping them to  $v \in V$  and  $dv$ , respectively. Nevertheless, the uniqueness property of non-minimal Sullivan models is much weaker, see [FOT08, Lemma 2.20].

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graded vector space. Clearly,  $\mathcal{S}_\alpha$  is not isomorphic to  $\mathcal{S}$  for generic  $\alpha$ , but

$$\mathcal{Q}(\mathcal{S}_\alpha) = \langle 1, K_\alpha, K_\alpha \wedge K_\alpha \rangle \simeq \langle 1, K, K \wedge K \rangle = \mathcal{Q}(\mathcal{S}).$$

In fact, the previous argument works in general and implies that for  $\mathbb{CP}^2$ , the non-degenerate quotients of two small subalgebras are isomorphic as Poincaré duality algebras (this does not hold for any  $M$ , see Example 6.1.22).

The Sullivan minimal model of  $M$  is the free DGA  $\mathcal{M} := \Lambda(\eta, \mu)$  with  $|\eta| = 2$ ,  $|\mu| = 5$ ,  $d\eta = 0$  and  $d\mu = \eta \wedge \eta$ . A DGA-quasi-isomorphism  $f : \mathcal{M} \rightarrow \Omega(M)$  is specified by its values on  $\eta$  and  $\mu$ ; for example,  $f(\eta) := K$  and  $f(\mu) := 0$ . We see that neither  $\mathcal{S}_\alpha$  nor  $\mathcal{H}_\alpha$  can lie in  $\text{im } f$  because  $\dim(\text{im } f)^4 = 1$  for any  $f$ .  $\triangleleft$

*Example 6.1.22* (Non-degenerate quotients of small subalgebras for different Hodge decompositions are, in general, not isomorphic). Consider  $M = \text{SU}(6)$ . It is a compact simply-connected Lie group of dimension 35 ( $= n^2 - 1$  for  $n = 6$ ) whose cohomology ring is freely generated by single elements in degrees 3, 5,  $\dots$ , 11; see [MT91, Corollary 3.11]. There is a biinvariant Riemannian metric and there are biinvariant differential forms  $x_3, x_5, \dots, x_{11}$  in the corresponding degrees such that

$$\mathcal{H} := \Lambda(x_3, \dots, x_{11}) \subset \Omega(M)$$

is the algebra of harmonic forms, see [FOT08, Chapter 1].

For  $\eta_6 \in \Omega^6(M)$  and  $\eta_8 \in \Omega^8(M)$ , which are going to be specified later, set

$$\tilde{x}_7 := x_7 + d\eta_6 \quad \text{and} \quad \tilde{x}_9 := x_9 + d\eta_8,$$

and let  $\mathcal{H}'$  denote the graded vector space obtained from  $\mathcal{H}$  by replacing the vectors  $x_7$  and  $x_9$  with  $\tilde{x}_7$  and  $\tilde{x}_9$ , respectively. We emphasize that we are replacing just the vectors, not the products; e.g.,  $x_7 \wedge x_9$  is an element of  $\mathcal{H}'$  but  $\tilde{x}_7 \wedge \tilde{x}_9$  might not be. Let  $\mathcal{S} = \mathcal{H}$  be the small subalgebra corresponding to the Riemannian Hodge decomposition, and let  $\mathcal{S}'$  be the small subalgebra corresponding to a Hodge decomposition based on  $\mathcal{H}'$  (such always exists by Remark 6.1.11). The following elements in degrees 15, resp. 20



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must be contained in  $\mathcal{S}'$ :

$$\begin{aligned} y &:= P'_{\text{std}}(\tilde{x}_7 \wedge x_9 - x_7 \wedge x_9) \\ &= P'_{\text{std}}d(\eta_6 \wedge x_9), \\ z &:= \tilde{x}_9 \wedge x_{11} - x_9 \wedge x_{11} \\ &= d(\eta_8 \wedge x_{11}). \end{aligned}$$

Using Stokes theorem and  $d \circ P'_{\text{std}} = \pi_{\text{im}d}$ , we get

$$\begin{aligned} \langle y, z \rangle &= \pm \int_M P'_{\text{std}}d(\eta_6 \wedge x_9) \wedge d(\eta_8 \wedge x_{11}) \\ &= \pm \int_M d\eta_6 \wedge \eta_8 \wedge x_9 \wedge x_{11}. \end{aligned}$$

We claim that the integral can be made non-zero by a choice of  $\eta_6$  and  $\eta_8$ . Indeed, because  $x_9 \wedge x_{11}$  generates non-zero homology, there is an  $m \in M$  such that  $x_9(m) \wedge x_{11}(m) \neq 0$ . Pick local coordinates  $(x^i)$  centered at  $m$ , and let  $\alpha^I dx^I$  be a non-zero coefficient in  $x_9(m) \wedge x_{11}(m)$ . Consider the complement  $J = I^C$  and decompose  $J = J_1 \cup J_2$  into two parts with  $|J_1| = 7$  and  $|J_2| = 8$ . For some  $0 \neq c \in \mathbb{R}$ , set locally

$$\eta_6 := ((x^{J_{11}} + c) dx^{J_1 \setminus \{J_{11}\}}) \quad \text{and} \quad \eta_8 = dx^{J_2},$$

and extend them to the whole of  $M$  by multiplying with a bump function which is constant non-zero in a neighborhood of  $m$ . We have achieved that the integrand  $\omega := d\eta_6 \wedge \eta_8 \wedge x_9 \wedge x_{11}$  is non-zero around  $m$ . Because the intersection pairing is non-degenerate and because  $\omega \neq 0$ , there is a function  $f \in C^\infty(M)$  such that  $\int_M f\omega \neq 0$ . We can now just rescale  $\eta_8$  by  $f$ .

We have shown that  $z$  induces a non-zero element  $\pi'_Q(z) \in \mathcal{Q}(\mathcal{S}')^{20}$ , where  $\pi'_Q : \mathcal{S}' \rightarrow \mathcal{S}'/\mathcal{S}'^\perp$  is the canonical projection. Now,  $\pi'_Q$  is a DGA-morphism, and hence  $\pi'_Q(z)$  is exact. Because  $\mathcal{S}'$  is of Hodge type,  $\pi'_Q$  is also a quasi-isomorphism, and hence  $\pi'_Q(x_9 \wedge x_{11})$  generates non-trivial homology. It follows that  $\pi'_Q(z)$  is not a multiple of  $\pi'_Q(x_9 \wedge x_{11})$ , and thus  $\dim \mathcal{Q}(\mathcal{S}')^{20} \geq 2$ . However, we have  $\mathcal{Q}(\mathcal{S})^{20} = \mathcal{H}^{20} = \langle x_9 \wedge x_{11} \rangle$ . This shows that  $\mathcal{Q}(\mathcal{S})$  and  $\mathcal{Q}(\mathcal{S}')$  can not be isomorphic as vector spaces.  $\triangleleft$

It is not hard to come up with artificial examples of oriented DGA's which are not of Hodge type.

The following lemma and proposition show that in some cases it is possible to extend a DGA to a DGA of Hodge type.

Not of Hodge type: Example of an oriented dga without a Hodge decomposition.

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**Lemma 6.1.23** (Giving partners to non-degenerates). *Let  $(V, d, \wedge, o)$  be a unital commutative DGA which is non-negatively graded and oriented in degree  $n$ . For  $k = \lceil \frac{n}{2} \rceil, \dots, n$ , consider the following property  $(P_k)$  of a direct sum decomposition*

$$V = \mathcal{H} \oplus dV \oplus C, \quad (6.17)$$

where  $\mathcal{H}$  is a harmonic subspace and  $C$  a complement of  $\ker d$  in  $V$  perpendicular to  $\mathcal{H}$  with respect to the induced cyclic structure  $\langle \cdot, \cdot \rangle$ :

$(P_k)$  There is a complement  $E$  of

$$C^\perp := \{c^\perp \in C \mid \langle c^\perp, c \rangle = 0 \text{ for all } c \in C\}$$

in  $C$  and a homogenous linear map

$$\rho : E^{\lceil n/2 \rceil} \oplus \dots \oplus E^k \longrightarrow dV$$

such that for all  $e' \in E$  and  $c^\perp \in C^\perp$ , the following holds:

$$\langle e', \rho(e) \rangle = \langle e', e \rangle, \quad (6.18)$$

$$\langle c^\perp, \rho(e) \rangle = 0. \quad (6.19)$$

Suppose that  $V$  is non-negatively graded, of finite type and satisfies  $V^0 = \text{span}\{1\}$  and  $V^1 = 0$ . Suppose that  $n \geq 5$  and that  $(H(V), \wedge, o^H)$  is a Poincaré duality algebra. Given  $\lceil \frac{n}{2} \rceil \leq l \leq n$ , suppose that  $V$  admits a decomposition of type (6.17) such that either  $l = \lceil \frac{n}{2} \rceil$  or  $l > \lceil \frac{n}{2} \rceil$  and  $(P_{l-1})$  holds. Then there is an  $m \in \mathbb{N}_0$  and a Sullivan DGA

$$\Lambda := \Lambda(w_1, \dots, w_m, z_1, \dots, z_m) \quad (6.20)$$

specified by  $\deg w_i = l - 1$ ,  $\deg z_i = l$ ,  $dz_i = 0$  and  $dw_i = z_i$  for all  $i = 1, \dots, m$  such that the tensor product DGA

$$\hat{V} := V \otimes \Lambda$$

admits an orientation  $\hat{o} : \hat{V} \rightarrow \mathbb{R}$  which extends  $o : V \rightarrow \mathbb{R}$  on the canonical inclusion  $V \hookrightarrow \hat{V}$ , and there is a decomposition

$$\hat{V} = \hat{\mathcal{H}} \oplus d\hat{V} \oplus \hat{C} \quad (6.21)$$

of type (6.17) for which  $(P_l)$  holds.

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*Proof.* The proof consists of a construction of  $\hat{o} : \hat{V} \rightarrow \mathbb{R}$ , a construction of a harmonic subspace  $\hat{\mathcal{H}}$  in  $\hat{V}$ , a construction of a complement  $\hat{C}$  of  $\ker d$  in  $\hat{V}$ , a degree-wise description of  $\hat{C}$  and  $\hat{C}^\perp$ , a degree-wise construction of a complement  $\hat{E}$  of  $\hat{C}^\perp$  in  $\hat{C}$  and a proof of the property  $(P_l)$  for the constructed decomposition.

*Construction of  $\hat{o}$ :* Consider the decomposition (6.17). Because  $d : C \rightarrow dV$  is an isomorphism, we can write

$$V = \mathcal{H} \oplus \underbrace{dE \oplus dC^\perp}_{dV} \oplus \underbrace{E \oplus C^\perp}_C. \quad (6.22)$$

The restriction of  $\langle \cdot, \cdot \rangle$  to  $E$  is non-degenerate, and  $V$  is of finite type by assumption; hence,  $E$  is finite-dimensional. Set

$$m := \dim E^l.$$

Let  $\xi_1, \dots, \xi_m$  be a basis of  $E^l$  and  $\xi^1, \dots, \xi^m$  its dual basis in  $E^{n-l}$ . The Sullivan algebra  $\Lambda$  can be written as a direct sum

$$\Lambda = \bigoplus_{k=0}^{\infty} \Lambda_k \quad \text{with} \quad \Lambda_k = \bigoplus_{\substack{r, m \geq 0 \\ r+m=k}} \Lambda_r(w) \otimes \Lambda_m(z), \quad (6.23)$$

where  $\Lambda_r(w)$  and  $\Lambda_m(z)$  are the graded vector spaces generated by monomials  $w_I = w_{i_1} \dots w_{i_r}$  and  $z_J = z_{j_1} \dots z_{j_m}$  for all multiindices  $I = \{i_1, \dots, i_r\}$  and  $J = \{j_1, \dots, j_m\}$ , respectively. The direct sum decompositions (6.22) and (6.23) induce a direct sum decomposition of  $\hat{V} = V \otimes \Lambda$  via the distributivity of  $\otimes$  and  $\oplus$ . We denote

$$\hat{V}_k := V \otimes \Lambda_k \quad \text{for } k \geq 0.$$

Let  $\hat{o} : \hat{V} \rightarrow \mathbb{R}$  be the linear map satisfying

$$\hat{o}(v) := o(v) \quad \text{for all } v \in V, \quad (6.24)$$

$$\hat{o}(\xi^i \wedge z_j) := o(\xi^i \wedge \xi_j) \quad \text{and} \quad (6.25)$$

$$\hat{o}(d\xi^i \wedge w_j) := (-1)^{\deg \xi^i + 1} o(\xi^i \wedge \xi_j) \quad \text{for all } i, j = 1, \dots, m, \quad (6.26)$$

and which is zero on  $(\mathcal{H} \oplus dV \oplus C^\perp \oplus \bigoplus_{i \geq 0, i \neq n-l} E^i) \otimes \Lambda_1(z)$ , on  $(\mathcal{H} \oplus C \oplus dC^\perp \oplus \bigoplus_{i \geq 0, i \neq n-l} dE^i) \otimes \Lambda_1(w)$  and on  $\hat{V}_k$  for  $k \geq 2$ .

In order to show that  $\hat{o}$  is an orientation, we must check that  $\hat{o} \neq 0$  and  $\hat{o} \circ d = 0$ . The

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first condition is clear from  $\hat{o}|_V = o \neq 0$ . As for the second condition,  $\hat{V}$  is generated by elements  $v \wedge w_I \wedge z_J$  for  $v \in V$  and multiindices  $I, J$ . It holds  $d\hat{V}_k \subset \hat{V}_k$  for all  $k \geq 0$ , and hence  $d\hat{V} = \bigoplus_{k=0}^{\infty} d\hat{V}_k$ . From the definition of  $\hat{o}$ , we have immediately  $d\hat{V}_0 = dV \subset \ker o \subset \ker \hat{o}$  and  $\bigoplus_{k=2}^{\infty} d\hat{V}_k \subset \ker \hat{o}$ . As for  $d\hat{V}_1$ , we write  $\hat{V}_1 = \text{span}\{v \wedge w_j, v \wedge z_j \mid v \in V, j = 1, \dots, m\}$  as a graded vector space and compute

$$\begin{aligned} d\hat{V}_1 &= \text{span}\{d(v \wedge w_j), d(v \wedge z_j) \mid v \in V, j = 1, \dots, m\} \\ &= \text{span}\{dv \wedge w_j + (-1)^{\deg v} v \wedge z_j \mid v \in V, j = 1, \dots, m\}. \end{aligned} \tag{6.27}$$

Write  $v \in V^{n-l}$  as  $v = h + dc + c^\perp + \sum_{i=1}^m \alpha_i \xi^i$  for  $h \in \mathcal{H}^{n-l}$ ,  $c \in C^{n-l-1}$ ,  $c^\perp \in C^{\perp n-l}$  and  $\alpha_i \in \mathbb{R}$ , and compute for every  $j = 1, \dots, m$  the following:

$$\begin{aligned} \hat{o}(dv \wedge w_j) &= \hat{o}\left(dc^\perp \wedge w_j + \sum_{i=1}^m \alpha_i d\xi^i \wedge w_j\right) \\ &= \sum_{i=1}^m \alpha_i \hat{o}(d\xi^i \wedge w_j) \\ &= \sum_{i=1}^m (-1)^{\deg \xi^i + 1} \alpha_i \hat{o}(\xi^i \wedge \xi_j) \\ &= (-1)^{n-l+1} \sum_{i=1}^m \alpha_i \hat{o}(\xi^i \wedge z_j) \\ &= (-1)^{n-l+1} \hat{o}\left((h + dc + c^\perp) \wedge z_j + \sum_{i=1}^m \alpha_i \xi^i \wedge z_j\right) \\ &= (-1)^{\deg v + 1} \hat{o}(v \wedge z_j). \end{aligned}$$

Consequently,  $d\hat{V}_1 \subset \ker \hat{o}$ . This shows  $\hat{o} \circ d = 0$ .

The inclusion  $V \hookrightarrow \hat{V}$  is clearly orientation preserving.

*Construction of  $\hat{\mathcal{H}}$ :* It holds  $\bar{H}(\Lambda) = 0$  for the reduced homology, and hence  $H(\hat{V}) \simeq H(V) \otimes H(\Lambda) = H(V)$  by Künneth's formula. Because  $\mathcal{H} \subset \ker d$ ,  $\mathcal{H} \cap \text{im } d = 0$  and  $\dim(\mathcal{H}) = \dim H(\hat{V})$ ,  $\mathcal{H}$  is a complement of  $d\hat{V}$  in  $\ker d$ . Therefore,

$$\hat{\mathcal{H}} := \mathcal{H}$$

is a harmonic subspace of  $\hat{V}$ . Also note that  $H(\hat{V}) = \bigoplus_{k=0}^{\infty} H(\hat{V}_k)$  and  $\dim H(\hat{V}) = \dim H(\hat{V}_0)$ , and hence  $H(\hat{V}_k) = 0$  for all  $k \geq 1$ .

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*Construction of  $\hat{C}$ :* We construct  $\hat{C}$  as a direct sum  $\hat{C} = \bigoplus_{k=0}^{\infty} \hat{C}_k$ . Set

$$\hat{C}_0 := C.$$

For  $k = 1$ , define

$$\begin{aligned}\tilde{C}_1 &:= \text{span}\{v \wedge w_i \mid v \in V, i = 1, \dots, m\} \subset \hat{V}_1, \\ \hat{C}_1 &:= \{\tilde{c} - \pi(\tilde{c}) \mid \tilde{c} \in \tilde{C}_1\} \subset \hat{V}_0 \oplus \hat{V}_1,\end{aligned}\tag{6.28}$$

where  $\pi : \hat{V} \rightarrow \hat{\mathcal{H}}$  is the orthogonal projection. For all  $k \geq 2$ , let  $\hat{C}_k \subset \hat{V}_k$  be an arbitrary complement of  $\ker d$  in  $\hat{V}_k$  as a graded vector space.

We show first that  $\hat{C}_i$  for  $i \geq 0$  are disjoint. Clearly,  $\hat{C}_j \cap \sum_{i=0, i \neq j}^{\infty} \hat{C}_i = 0$  for all  $j \geq 2$ . Because  $(\mathcal{H} + C) \cap \tilde{C}_1 = 0$  and  $\mathcal{H} \cap C = 0$ , it holds  $\hat{C}_1 \cap \hat{C}_0 = 0$ , and  $(\hat{C}_0 + \hat{C}_1) \cap \sum_{i \geq 2} \hat{C}_i = 0$  implies that  $\hat{C}_j \cap \sum_{i=0, i \neq j}^{\infty} \hat{C}_i = 0$  holds also for  $j = 0, 1$ .

We show that  $\hat{C} = \bigoplus_{k \geq 0} \hat{C}_k$  is a complement of  $\ker d$  in  $\hat{V}$ . For  $k \geq 2$ ,  $\hat{C}_k$  are complements of  $\ker d$  in  $\hat{V}_k$  by construction. For  $k = 0$ ,  $\hat{C}_0 \oplus \ker d \cap \hat{V}_0 = \hat{V}_0$  follows from (6.17). For  $k = 1$ , we compare (6.27) and (6.28) to see that  $\tilde{C}_1 \oplus d\hat{V}_1 = \hat{V}_1$ . Because  $\mathcal{H} \subset \ker d$ , it is easy to see that  $\hat{C}_0 \oplus \hat{C}_1$  is a complement of  $\ker d$  in  $\hat{V}_0 \oplus \hat{V}_1$ .

Finally,  $\hat{C}_0 \perp \hat{\mathcal{H}}$  holds by (6.17),  $\hat{C}_1 \perp \mathcal{H}$  holds by the construction of  $\hat{C}_1$  from  $\tilde{C}_1$  and  $\hat{C}_k \perp \mathcal{H}$  follows from the definition of  $\hat{o}$ .

*Degree-wise description of  $\hat{C}$  and  $\hat{C}^\perp$ :* We are interested in  $\hat{C}^i$  for  $0 \leq i \leq l$ . For all  $k \geq 2$ , the graded vector space  $\hat{C}_k$  is concentrated in degrees  $i \geq 2(l-1)$ . But  $2(l-1) > l$  due to  $n \geq 5$ . Therefore,  $\hat{C}_k^i = 0$  for  $k \geq 2$  and  $0 \leq i \leq l$ . We denote  $\bar{V} := \bigoplus_{i=1}^{\infty} V^i$  and write  $\tilde{C}_1 = \Lambda_1(w) \oplus (\bar{V} \otimes \Lambda_1(w))$ . Because  $V^1 = 0$ , the graded vector space  $\bar{V} \otimes \Lambda_1(w)$  is concentrated in degrees  $i \geq 2 + (l-1) = l+1$ . We obtain

$$\hat{C}^i = (\hat{C}_0 \oplus \hat{C}_1)^i = \begin{cases} C^i & \text{for } 0 \leq i \leq l-1, \\ C^i + \text{span}\{w_j \mid j = 1, \dots, m\} & \text{for } i = l-1, \\ C^i & \text{for } i = l. \end{cases}\tag{6.29}$$

Here we used that  $\hat{C}_1^{l-1} = \tilde{C}_1^{l-1} = \text{span}\{w_i \mid i = 1, \dots, m\}$ , which is true because  $\Lambda_1(w) \perp \mathcal{H}$  from the definition of  $\hat{o}$ .

We are now interested in  $\hat{C}^{\perp i}$  for  $n-l \leq i \leq l$ . Note that  $n-l \leq i \leq l$  is equivalent to  $n-l \leq n-i \leq l$ . By definition,  $\hat{o}$  vanishes on  $C \wedge \Lambda_1(w) = C \otimes \Lambda_1(w)$  and

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$\Lambda_1(w) \wedge \Lambda_1(w) \subset \Lambda_2(w)$ . Looking at (6.29), we see the following:

$$\hat{C}^{\perp i} = \begin{cases} C^{\perp i} & \text{for } n-l \leq i \leq l-2, \\ C^{\perp i} + \text{span}\{w_i \mid i = 1, \dots, m\} & \text{for } i = l-1, \\ C^{\perp i} & \text{for } i = l. \end{cases} \quad (6.30)$$

Because  $\hat{V}$  is non-negatively graded, it holds  $\hat{C}^{\perp i} = \hat{C}^i$  for  $i > n$ . Notice that  $\hat{C}^{\perp i}$  might be smaller than  $C^i$  for  $0 \leq i \leq n-l-1$ . The reason for this is a possible existence of  $v_1, v_2 \in V$  such that  $v_1 \wedge v_2$  has a non-trivial  $dE$ -component and  $\langle v_1, v_2 \wedge w_i \rangle = \hat{o}((v_1 \wedge v_2) \wedge w_i) \neq 0$ .

*Construction of  $\hat{E}$ :* Because  $\hat{V}_i \wedge \hat{V}_j \subset \hat{V}_{i+j}$  and  $\hat{V}_k \subset \ker \hat{o}$  for  $k \geq 2$ , we have  $\hat{C}_k \subset \hat{C}^{\perp}$  for all  $k \geq 2$ . It follows that

$$\hat{C}^{\perp} = (\hat{C}_0 \oplus \hat{C}_1) \cap \hat{C}^{\perp} \oplus \bigoplus_{k \geq 2} \hat{C}_k,$$

and hence it is enough to construct  $\hat{E} \subset \hat{C}_0 \oplus \hat{C}_1$  such that  $\hat{E} \oplus (\hat{C}_0 \oplus \hat{C}_1) \cap \hat{C}^{\perp} = \hat{C}_0 \oplus \hat{C}_1$ . Because  $E \cap \hat{C}^{\perp} \subset E \cap C^{\perp} = 0$ , we can get  $\hat{E}$  by extending  $E$ . For  $0 \leq i \leq n$ , we define

$$\hat{E}^i := \begin{cases} E^i \oplus \text{span}\{c_1^i, \dots, c_{k_i}^i\} & \text{for } c_1^i, \dots, c_{k_i}^i \in C^i & \text{for } 0 \leq i \leq n-l-1, \\ E^i & & \text{for } n-l \leq i \leq l, \\ E^i \oplus \text{span}\{\hat{v}_1, \dots, \hat{v}_k\} & \text{for } \hat{v}_1, \dots, \hat{v}_k \in (\hat{C}_0 \oplus \hat{C}_1)^i & \text{for } l+1 \leq i \leq n, \end{cases} \quad (6.31)$$

where the existence of  $c_j^i$  and  $\hat{v}_j^i$  and the fact that  $\hat{C} = \hat{E} \oplus \hat{C}^{\perp}$  are justified by (6.29), (6.30) and (6.19).

*Property ( $P_l$ ):* We define  $\hat{\rho} : \hat{E}^{\lceil \frac{n}{2} \rceil} \oplus \dots \oplus \hat{E}^l \rightarrow d\hat{V}$  by  $\hat{\rho} := \rho$  on  $\hat{E}^{\lceil \frac{n}{2} \rceil} \oplus \dots \oplus \hat{E}^{l-1} = E^{\lceil \frac{n}{2} \rceil} \oplus \dots \oplus E^{l-1}$  and by

$$\hat{\rho}(\xi_i) := z_i \quad \text{for all } i = 1, \dots, m.$$

Conditions (6.18) and (6.19) are checked easily using  $(P_{l-1})$ , (6.29), (6.30), (6.31) and the definition of  $\hat{o}$ .  $\square$

*Questions 6.1.24.* (i) Given a cochain complex  $(V, d)$  with a symmetric pairing  $\langle \cdot, \cdot \rangle$ , which conditions have to be satisfied by the maps  $\pi_{\mathcal{H}}, \mathcal{P}_{\text{std}} : V \rightarrow V$  so that  $V = \ker \pi_{\mathcal{H}} \oplus \text{im } d \oplus \text{im } \mathcal{P}_{\text{std}}$  is a Hodge decomposition with Hodge pair  $(\ker \pi_{\mathcal{H}}, \mathcal{P}_{\text{std}})$ ? This would characterize Hodge decompositions in terms of Hodge pairs.

(ii) It should be possible to prove Lemma 6.1.23 also for  $n \leq 4$  by hand. Check that!

## 6.2. Poincaré DGA's and Poincaré duality models

In this section, we restrict to non-negatively graded unital commutative DGA's, which we denote by  $\text{CDGA}_1^{\geq 0}$ . In this case, the notions of orientation and of cyclic structure agree by Proposition 6.1.4.

We modify and combine definitions from [Fio+19] and [LS08] as follows.

**Definition 6.2.1** (Dif. Poincaré duality algebra, PDGA and formality). *A differential Poincaré duality algebra of degree  $n$  is a  $\text{CDGA}_1^{\geq 0}(V, d, \wedge)$  of finite type with orientation  $\circ$  in degree  $n$  such that the induced pairing on  $V$  satisfies Poincaré duality. If  $d = 0$ , we call it just Poincaré duality algebra.*

*A Poincaré DGA (shortly PDGA) of degree  $n$  is a  $\text{CDGA}_1^{\geq 0}(V, d, \wedge)$  together with an orientation  $\circ^H : H(V) \rightarrow \mathbb{R}$  of degree  $n$  which makes  $H(V)$  into a Poincaré duality algebra.*

*A morphism of PDGA's  $(V_1, d_1, \wedge_1, \circ_1^H)$  and  $(V_2, d_2, \wedge_2, \circ_2^H)$  is a DGA-morphism  $f : V_1 \rightarrow V_2$  such that the induced map  $f_* : H(V_1) \rightarrow H(V_2)$  preserves orientation, i.e., it holds  $\circ_2^H \circ f_* = \circ_1^H$ .*

*A quasi-isomorphism (or weak equivalence) of PDGA's is a morphism of PDGA's  $f : V_1 \rightarrow V_2$  such that  $f_* : H(V_1) \rightarrow H(V_2)$  is an isomorphism of oriented DGA's.*

*Two PDGA's are weakly homotopy equivalent (or isomorphic in the homotopy category) if they are connected by a zig-zag of PDGA-quasi-isomorphisms.*

*A PDGA is formal if it is weakly homotopy equivalent (as a PDGA) to its homology.*

It follows from Proposition 6.1.4 that a differential Poincaré duality algebra according to Definition 6.2.1 is precisely a cyclic DGA from Part I. In particular, it is finite dimensional. However, if we relax finite type, unitality or commutativity, we obtain a different notion.

*Remark 6.2.2* (Frobenius algebra). A differential Poincaré duality algebra, resp. a cyclic DGA, is precisely a finite-dimensional symmetric dg-Frobenius algebra from [Val12, p. 13] or [CHV06, Theorem 1.1].  $\triangleleft$

**Definition 6.2.3** (Poincaré duality model). *A Poincaré duality model of a PDGA  $(V, d, \wedge, \circ^H)$  is a differential Poincaré duality algebra  $(\mathcal{M}, d^{\mathcal{M}}, \wedge^{\mathcal{M}}, \circ^{\mathcal{M}})$  which is weakly homotopy equivalent to  $V$  as a PDGA.*

DONE Hom  
to weak Hom:  
Homotopy to  
weak homotopy

DONE Add ori-  
entation to the  
data of PDGA:  
Add orientation  
to the data of  
PDGA

DONE Change  
integral to or:  
Change integral  
to or

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We call a Poincaré duality model small if  $\mathcal{Q}(\mathcal{S}(\mathcal{M})) \simeq \mathcal{M}$  for every Hodge decomposition.

*Remark 6.2.4* (On Poincaré duality models). (i) The definition of a Poincaré duality model in [LS08] requires only weak homotopy equivalence of DGA's, i.e., it does not require quasi-isomorphisms to preserve orientation on homology.

(ii) In general, a Sullivan minimal model  $\Lambda U$  fails easily to be a Poincaré duality model because it often has non-zero elements in degree  $> n$ , e.g., powers of even generators; see Example 6.2.7 for  $\mathbb{S}^2$ .

For a compact connected Lie group  $G$ , the subalgebra of harmonic forms  $\mathcal{H}$  for any biinvariant Riemannian metric is isomorphic to a free algebra on odd generators, see [FOT08, Chapter 1]. Therefore,  $\mathcal{H}$  with zero differential and the induced cyclic structure is at the same time the Sullivan minimal model and a Poincaré duality model for  $\Omega(G)$ .

(iii) Poincaré duality models are not “strongly unique” in the sense that two Poincaré duality models of the same algebra must not be isomorphic; see Example 6.1.22 for  $\mathrm{SU}(6)$ . However, there is a “weak uniqueness” statement in Proposition 6.2.9 below. The situation is similar to the situation with Sullivan and minimal Sullivan models; see [FOT08]. We introduced “smallness” in Definition 6.2.3 as a candidate for a minimality condition on a Poincaré duality model which might imply its “strong uniqueness”; see Question 6.2.14.  $\triangleleft$

Consider the functor from PDGA's to DGA's which forgets the orientation on homology. We have the following trivial yet somewhat surprising observation.

**Proposition 6.2.5** (PDGA-formality is the same as DGA-formality). *A Poincaré DGA  $(V, d, \wedge, \circ^H)$  is formal (as a PDGA) if and only if it is formal as a DGA.*

*Proof.* The “only if” part is clear.

As for the “if” part, let

$$V \longleftarrow \bullet \cdots \bullet \longrightarrow H(V) \tag{6.32}$$

be a weak homotopy equivalence of DGA's. Denote by  $f : H(V) \rightarrow H(V)$  the isomorphism on homology induced by (6.32) from the left to the right. We adjoin  $H(V)$  to the right of (6.32) to obtain the homotopy

$$V \longleftarrow \bullet \cdots \bullet \longrightarrow H(V) \xrightarrow{f^{-1}} H(V) \tag{6.33}$$

whose induced map on homology from the left to the right is the identity. Therefore, we



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can orient homologies of the inner nodes of (6.33) so that all maps preserve orientation on homology.  $\square$

The next proposition will be used to show the existence of Poincaré duality models.

**Proposition 6.2.6** (Extension of Hodge type). *Let  $V$  be a PDGA of degree  $n \geq 5$  which is of finite type and satisfies  $V^0 = \text{span}\{1\}$  and  $V^1 = 0$ . Then it is a retract of an oriented DGA  $E(V)$  of Hodge type in the category of PDGA's.*

*Proof.* Pick an arbitrary harmonic subspace  $\mathcal{H}$  and an arbitrary complement  $C$  of  $\ker d$  in  $V$ . If  $C$  is not perpendicular to  $\mathcal{H}$ , replace it with  $\{c - \pi(c) \mid c \in C\}$ , where  $\pi : V \rightarrow \mathcal{H}$  is the orthogonal projection. We start with  $l = \lceil \frac{n}{2} \rceil$  and apply Lemma 6.1.23 inductively to get an extension  $\hat{V} = V \otimes \Lambda$  which admits a decomposition

$$\hat{V} = \hat{\mathcal{H}} \oplus d\hat{V} \oplus \hat{C} \quad (6.34)$$

of type (6.17) such that there is a complement  $\hat{E}$  of  $\hat{C}^\perp$  in  $\hat{C}$  and a linear map  $\hat{\rho} : \hat{E}^{\lceil n/2 \rceil} \oplus \dots \oplus \hat{E}^n \rightarrow d\hat{V}$  such that (6.18) and (6.19) hold. We consider a linear map

$$\kappa : \hat{C} \longrightarrow d\hat{V}$$

such that

$$\kappa(e) = \begin{cases} 0 & \text{for } e \in \hat{E}^i \text{ with } i < \lceil \frac{n}{2} \rceil, \\ \hat{\rho}(e) & \text{for } e \in \hat{E}^i \text{ with } i > \lceil \frac{n}{2} \rceil, \end{cases} \quad \text{and} \quad \kappa(c^\perp) = 0 \quad \text{for } c^\perp \in \hat{C}^\perp.$$

The case  $n = 2k$  and  $e \in \hat{E}^k$  is specified as follows. If  $k$  is even, then  $\langle \cdot, \cdot \rangle : \hat{E}^k \otimes \hat{E}^k \rightarrow \mathbb{R}$  is an inner product, and there is an orthonormal basis  $\eta_1, \dots, \eta_m$  for some  $m \in \mathbb{N}$ . We require

$$\kappa(\eta_i) = \frac{1}{2} \hat{\rho}(\eta_i) \quad \text{for all } i = 1, \dots, m. \quad (6.35)$$

If  $k$  is odd, then  $\langle \cdot, \cdot \rangle : \hat{E}^k \otimes \hat{E}^k \rightarrow \mathbb{R}$  is a symplectic form, and there is a symplectic basis  $\eta_1, \theta_1, \dots, \eta_m, \theta_m$  for some  $m \in \mathbb{N}$ . We use the convention  $\langle \theta_i, \eta_j \rangle = \delta_{ij}$  for  $i, j = 1, \dots, m$ . We require

$$\kappa(\eta_i) = \hat{\rho}(\eta_i) \quad \text{and} \quad \kappa(\theta_i) = 0 \quad \text{for } i = 1, \dots, m. \quad (6.36)$$

Let

$$\hat{C}' := \{c - \kappa(c) \mid c \in \hat{C}\}.$$

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This is a complement of  $\ker d$  in  $\hat{V}$  perpendicular to  $\hat{\mathcal{H}}$  because  $\hat{C}$  is and  $\text{im } \kappa \subset d\hat{V}$ . Given homogenous  $c_1, c_2 \in \hat{C}$  with  $\deg c_1 + \deg c_2 = n$  and  $\deg c_1 \leq \deg c_2$ , write  $c_1 = c_1^\perp + e_1$  and  $c_2 = c_2^\perp + e_2$  for  $c_1^\perp, c_2^\perp \in \hat{C}^\perp$  and  $e_1, e_2 \in \hat{E}$ , and compute

$$\begin{aligned} \langle c_1 - \kappa(c_1), c_2 - \kappa(c_2) \rangle &= \langle c_1, c_2 \rangle - \langle \kappa(c_1), c_2 \rangle - \langle c_1, \kappa(c_2) \rangle \\ &= \underbrace{\langle e_1, e_2 \rangle - \langle \kappa(e_1), e_2 \rangle - \langle e_1, \kappa(e_2) \rangle}_{=:(*)} \\ &\quad - \underbrace{\langle \kappa(e_1), c_2^\perp \rangle - \langle c_1^\perp, \kappa(e_2) \rangle}_{=:(**)}. \end{aligned}$$

Now,  $(**) = 0$  because of (6.19). As for  $(*)$ , if  $\deg c_1 < \deg c_2$ , then

$$\begin{aligned} (**) &= \langle e_1, e_2 \rangle - \langle e_1, \kappa(e_2) \rangle \\ &= \langle e_1, e_2 \rangle - \langle e_1, \hat{\rho}(e_2) \rangle \\ &= 0 \end{aligned}$$

because of (6.18). If  $\deg c_1 = \deg c_2 = k$  and  $k$  is even, we plug in the orthonormal basis and get using (6.35) that

$$\begin{aligned} e_1 = \eta_i, e_2 = \eta_j : \quad (**) &= \langle \eta_i, \eta_j \rangle - \langle \kappa(\eta_i), \eta_j \rangle - \langle \eta_i, \kappa(\eta_j) \rangle \\ &= \langle \eta_i, \eta_j \rangle - \langle \eta_j, \kappa(\eta_i) \rangle - \langle \eta_i, \kappa(\eta_j) \rangle \\ &= \langle \eta_i, \eta_j \rangle - \frac{1}{2} \langle \eta_j, \hat{\rho}(\eta_i) \rangle - \frac{1}{2} \langle \eta_i, \hat{\rho}(\eta_j) \rangle \\ &= \langle \eta_i, \eta_j \rangle - \frac{1}{2} \langle \eta_j, \eta_i \rangle - \frac{1}{2} \langle \eta_i, \eta_j \rangle \\ &= 0. \end{aligned}$$

If  $k$  is odd, we plug in the symplectic basis and get using (6.36) that

$$\begin{aligned} e_1 = \eta_i, e_2 = \eta_j : \quad (**) &= \langle \eta_j, \kappa(\eta_i) \rangle - \langle \eta_i, \kappa(\eta_j) \rangle \\ &= \langle \eta_j, \hat{\rho}(\eta_j) \rangle - \langle \eta_i, \hat{\rho}(\eta_j) \rangle \\ &= 0, \\ e_1 = \theta_i, e_2 = \eta_j : \quad (**) &= \langle \theta_i, \eta_j \rangle - \langle \theta_i, \kappa(\eta_j) \rangle \\ &= \langle \theta_i, \eta_j \rangle - \langle \theta_i, \hat{\rho}(\eta_j) \rangle \\ &= 0, \\ e_1 = \theta_i, e_2 = \theta_j : \quad (**) &= 0. \end{aligned}$$

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This shows that  $\hat{C} \perp \hat{C}$ , and hence (6.34) is a Hodge decomposition.

Finally, because  $\hat{V} = V \otimes \Lambda$  as a DGA, both the inclusion  $\iota : V \rightarrow \hat{V}$  of  $V$  into  $\hat{V}_0$  and the projection  $\pi : \hat{V} \rightarrow V$  from  $\hat{V}_0$  onto  $V$  are DGA morphisms. Because  $\pi \circ \iota = \mathbb{1}$  and because  $\iota_*$  is an orientation preserving isomorphism,  $\pi_*$  is an orientation preserving isomorphism as well. Therefore,  $V$  is a retract of  $E(V) := \hat{V}$  in the category of PDGA's.  $\square$

The next example shows that the Sullivan minimal model is sometimes of Hodge type.

*Example 6.2.7* (Sullivan minimal model of  $\mathbb{S}^2$  is of Hodge type). The Sullivan minimal model of  $\mathbb{S}^2$  is the free graded commutative algebra  $\mathcal{M} := \Lambda(\eta_2, \eta_3)$  with  $|\eta_2| = 2$ ,  $|\eta_3| = 3$ ,  $d\eta_2 = 0$  and  $d\eta_3 = \eta_2 \wedge \eta_2$ . From degree reasons, it holds  $\mathcal{M} = \text{span}\{\eta_2^k \eta_3^l \mid k \geq 0, l \in \{0, 1\}\}$  as a graded vector space. We have a canonical decomposition  $\mathcal{M} = \mathcal{H} \oplus d\mathcal{M} \oplus C$ , where  $\mathcal{H} = \text{span}\{\eta_2\}$ ,  $d\mathcal{M} = \text{span}\{\eta_2^k \mid k \geq 2\}$  and  $C = \text{span}\{\eta_2^k \eta_3 \mid k \geq 0\}$ . We define an orientation  $\circ : \mathcal{M} \rightarrow \mathbb{R}$  in degree 2 by  $\circ(\eta_2) := 1$  on  $\mathcal{H}$  and by 0 on  $d\mathcal{M}$  and  $C$ . It is easy to see that  $C \perp \mathcal{H}$  and  $C \perp C$  with respect to the induced cyclic structure  $\langle \cdot, \cdot \rangle$ . Consider the DGA-quasi-isomorphism  $f : \mathcal{M} \rightarrow \Omega(\mathbb{S}^2)$  defined by  $f(\eta_2) := \text{Vol}$  and  $f(\eta_3) := 0$ . Clearly, it is orientation preserving. We have  $\mathcal{M}/\mathcal{M}^\perp \simeq \Lambda(\eta_2)$   $\triangleleft$

The following proposition about the existence of a Poincaré duality model of a Poincaré DGA  $V$  with  $H^1(V) = 0$  was originally proven in [LS08, Theorem 1.1]. It was formulated in the category of DGA's, i.e., not checking whether the arrows are orientation preserving on homology. The idea was to construct an extension  $E(\Lambda U)$  of the Sullivan minimal model  $\Lambda U$  of  $V$  and an orientation on it such that the degenerate subspace is acyclic; the Poincaré duality model is then obtained by taking the quotient. The extension is constructed by adding elements which kill the so called orphans.

By Proposition 6.1.13, we know that  $V^\perp$  is acyclic if and only if  $V$  is of Hodge type ( $V$  needs to be of finite type for the direct implication). Based on this, we give a new construction of  $E(\Lambda U)$  using Lemma 6.1.23, i.e., by adding exact partners to non-degenerates. Our construction works for  $n \geq 5$ , whereas the assumption of [LS08] is  $n \geq 7$ . We also do not need  $d(\Lambda U)^2 = 0$ , although it follows from  $H^1(V) = 0$ . It is also clear from our construction that the arrows preserve orientation on homology. However, this can be checked for the construction of [LS08] as well.

**Proposition 6.2.8** (Existence of Poincaré duality model for  $H^1 = 0$ ). *A Poincaré DGA  $V$  with  $H^0(V) = \text{span}\{1\}$  and  $H^1(V) = 0$  admits a Poincaré duality model  $\mathcal{M}$ .*

*Proof.* If  $\circ^H$  comes from a pairing on  $V$  which is of Hodge type, then we can take

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$\mathcal{M} = \mathcal{Q}(\mathcal{S}(V))$ . The weak homotopy equivalence of PDGA's looks like

$$\begin{array}{ccc} & \mathcal{S}(V) & \\ \swarrow & & \searrow \\ \mathcal{Q}(\mathcal{S}(V)) & & V. \end{array} \quad (6.37)$$

If  $V$  is not of Hodge type, we proceed as follows. Let  $n$  be the degree of the orientation on  $H(V)$ . If  $n \leq 6$ , then  $V$  is formal as a DGA by [Mil79], and we can take  $\mathcal{M} = H(V)$  by Proposition 6.2.5. The weak homotopy equivalence of PDGA's looks like

$$\begin{array}{ccc} & \Lambda U & \\ \swarrow & & \searrow \\ H(V) & & V, \end{array} \quad (6.38)$$

where  $\Lambda U$  is the Sullivan minimal model of  $V$ . The Sullivan minimal model  $W := \Lambda U$  is of finite type and satisfies  $W^0 = \mathbb{R}$ ,  $W^1 = 0$  and  $dW^2 = 0$ . Suppose that  $n \geq 7$ . Let  $E(W)$  be the extension of  $W$  of Hodge type either from Proposition 6.2.6 or from [LS08, Section 4]. This extension is of finite type, the inclusion  $W \hookrightarrow E(W)$  is a quasi-isomorphism of DGA's and  $E(W)^\perp$  is acyclic. Moreover,  $W \hookrightarrow E(W)$  preserves orientation on homology. We take  $\mathcal{M} = \mathcal{Q}(E(\Lambda U))$  and obtain the following weak homotopy equivalence of PDGA's:

$$\begin{array}{ccc} & \Lambda U & \\ \swarrow & & \searrow \\ \mathcal{Q}(E(\Lambda U)) & & V. \end{array} \quad (6.39)$$

This proves the proposition. □

The following is mostly [LS08, Theorem 7.1]. In addition, we check that the orientation on homology is preserved. Also, by using our extension of Hodge type, we can improve from  $n \geq 7$  to  $n \geq 5$ .

**Proposition 6.2.9** (“Weak uniqueness” of Poincaré duality model). *Let  $V_1$  and  $V_2$  be differential Poincaré duality algebras of degree  $n$  which are weakly homotopy equivalent as PDGA's. Suppose that  $H^0(V_1) = H^0(V_2) = \text{span}\{1\}$  and  $H^1(V_1) = H^1(V_2) = 0$ . In addition, suppose that  $H^2(V_1) = H^2(V_2) = 0$ ,  $V_1^1 = V_2^1 = 0$  and  $n \geq 5$ . Then there is a*

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differential Poincaré duality algebra  $V_3$  and PDGA-quasi-isomorphisms<sup>6</sup>

$$\begin{array}{ccc} & V_3 & \\ \nearrow & & \nwarrow \\ V_1 & & V_2. \end{array} \quad (6.40)$$

*Proof.* By the assumption, there is  $k \geq 1$  and a zig-zag of PDGA-quasi-isomorphisms

$$V_1 \longleftarrow Z_1 \longrightarrow Z_2 \longleftarrow Z_3 \longrightarrow Z_4 \longleftarrow \cdots \longleftarrow Z_k \longrightarrow V_2. \quad (6.41)$$

Consider the Sullivan minimal model  $\Lambda U \rightarrow Z_2$  and use the Lifting Lemma [FOT08, Lemma 2.15] to construct DGA-quasi-isomorphisms  $\Lambda U \rightarrow Z_1$  and  $\Lambda U \rightarrow Z_3$  such that the diagram

$$\begin{array}{ccccc} & \Lambda U & & & \\ & \swarrow \downarrow \searrow & & & \\ Z_1 & \longrightarrow & Z_2 & \longleftarrow & Z_3 \end{array}$$

commutes up to homotopy of DGA's. It is easy to see that there is an orientation on  $H(\Lambda U)$  such that all morphisms preserve orientation on homology. Therefore, we can replace the segment  $V_1 \longleftarrow Z_1 \longrightarrow Z_2 \longleftarrow Z_3 \longrightarrow Z_4$  in (6.41) by  $V_1 \longleftarrow \Lambda U \longrightarrow Z_4$ . Repeating this process, we can shorten (6.41) to

$$\begin{array}{ccc} & \Lambda U & \\ f_1 \swarrow & & \searrow f_2 \\ V_1 & & V_2, \end{array} \quad (6.42)$$

where  $f_1$  and  $f_2$  are PDGA-quasi-isomorphisms. In order to take  $\mathcal{Q}(E(\Lambda U))$  and obtain a zig-zag with three terms, we have to revert the arrows in (6.42). The trick from [LS08] is the following:

Consider the relative minimal model of the multiplication  $\mu : \Lambda U \otimes \Lambda U \rightarrow \Lambda U$ ; from [FOT08, Example 2.48], it is given by

$$\begin{array}{ccc} \Lambda U \otimes \Lambda U & \xrightarrow{\mu} & \Lambda U \\ & \searrow i & \uparrow p \\ & M(\mu) := \Lambda U \otimes \Lambda U \otimes \Lambda(U[1]), & \end{array} \quad (6.43)$$

where  $i$  is the inclusion into the first two factors, which is a cofibration, and  $p$  is a

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<sup>6</sup>These are automatically injective and orientation preserving.

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surjective quasi-isomorphism. Let  $\iota_i : \Lambda U \rightarrow \Lambda U \otimes \Lambda U$  for  $i = 1, 2$  be the inclusions to the first and the second factor, respectively. Because  $\mu \circ \iota_i = \mathbb{1}$  and  $p$  is a quasi-isomorphism, the maps  $i \circ \iota_i : \Lambda U \rightarrow M(\mu)$  for  $i = 1, 2$  are quasi-isomorphisms. Moreover, it is easy to see that  $H(M(\mu))$  inherits an orientation such that  $p_*$  and  $(i \circ \iota_i)_*$  are orientation preserving. To transfer this situation to  $V_i$ , we use the diagram

$$\begin{array}{ccc}
 \Lambda U & \xrightarrow{f_i} & V_1 \\
 \downarrow \iota_i & & \downarrow \iota_i^V \\
 \Lambda U \otimes \Lambda U & \xrightarrow{f_1 \otimes f_2} & V_1 \otimes V_2 \\
 \downarrow i & & \downarrow g_2 \\
 M(\mu) & \xrightarrow{g_1} & \tilde{V}_3 := M(\mu) \otimes_{\Lambda U \otimes \Lambda U} (V_1 \otimes V_2),
 \end{array} \tag{6.44}$$

where  $\iota_i^V : V_i \rightarrow V_1 \otimes V_2$  for  $i = 1, 2$  are inclusions. The lower square, i.e., the maps  $g_1, g_2$  and the DGA  $\tilde{V}_3$ , is a pushout diagram (see [Men15, Example 1.4]). According to [Law15], the model category of  $\text{CDGA}_1^{\geq 0}$  is proper, and hence pushouts along cofibrations preserve quasi-isomorphisms. Therefore,  $f_1 \otimes f_2$  being a quasi-isomorphism implies that  $g_1$  is a quasi-isomorphism. We push the orientation to  $H(\tilde{V}_3)$  via  $g_{1*}$ . Since  $i \circ \iota_i, g_1$  and  $f_i$  are quasi-isomorphisms preserving orientation on homology, it follows that  $h_i := g_2 \circ \iota_i^V : V_i \rightarrow V_3$  are quasi-isomorphisms preserving orientation of homology as well. It holds

$$\tilde{V}_3 \simeq \Lambda(U[1]) \otimes V_1 \otimes V_2.$$

Clearly,  $\tilde{V}_3$  is of finite type. Using  $H^1(V_i) = H^2(V_i) = 0$ , we have  $U^1 = U^2 = 0$ , and hence  $(\Lambda(U[1]))^1 = 0$ . This together with  $V_i^1 = 0$  implies that  $\tilde{V}_3^1 = 0$ . Therefore, all conditions for an application of the Hodge extension  $E$  from Proposition 6.2.6 are satisfied, and we can set

$$V_3 := \mathcal{Q}(E(\tilde{V}_3)).$$

This finishes the proof. □

**Conjecture 6.2.10.** *The additional assumptions of Proposition 6.2.9 can be dropped.*

*Remark 6.2.11* (Weak uniqueness in the case of  $H^1(V) = 0$  and  $n \leq 3$ ). For  $n = 1$ , there is no differential Poincaré duality algebra  $V$  with  $H^1(V) = 0$ .

For  $n = 2$ , a general differential Poincaré duality algebra can be written in terms of

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its Hodge decomposition as

$$\begin{aligned} V^2 &= \text{span}\{\text{Vol}\} \oplus dC^1 \\ V^1 &= dC^0 \oplus C^1 \\ V^0 &= \text{span}\{1\} \oplus C^0. \end{aligned}$$

Now,  $\mathcal{H} = \text{span}\{1\} \oplus \text{span}\{\text{Vol}\}$  is a dg-subalgebra which is itself a Poincaré duality algebra. Therefore, two differential Poincaré duality algebras with  $H(V_1) \simeq H(V_2)$  and  $H^1(V_i) = 0$  are connected via the zig-zag

$$\begin{array}{ccc} & \mathcal{H} & \\ \swarrow & & \searrow \\ V_1 & & V_2. \end{array}$$

For  $n = 3$ , we have

$$\begin{aligned} V^3 &= \text{span}\{\text{Vol}\} \oplus dC^2 \\ V^2 &= dC^1 \oplus C^2 \\ V^1 &= dC^0 \oplus C^1 \\ V^0 &= \text{span}\{1\} \oplus C^0, \end{aligned}$$

and the same situation as for  $n = 2$  occurs.

For  $n = 4$  and  $V^1 = 0$ , we have  $V \simeq \mathcal{H}$ .  $\triangleleft$

We would like to define a “minimal Poincaré duality model”. We motivate this notion in the following remark.

*Remark 6.2.12* (Model and minimal model). We shall understand models and minimal models in terms of model categories and their homotopy categories.

Let us illustrate this on Sullivan models. A Sullivan DGA is a free graded commutative algebra  $\Lambda U$  over a positively graded vector space  $U$  which admits a well-ordered homogenous basis  $(v_\alpha)$  such that  $dv_\alpha \in \Lambda(v_\beta \mid \beta < \alpha)$  ( $:=$  the subalgebra of  $\Lambda U$  generated by the  $v_\beta$ ’s) for all  $\alpha$ . A Sullivan DGA is called minimal if  $\text{im } d \subset \Lambda_{\geq 2} U$  ( $:=$  the set of decomposable elements).

According to [BG76, Theorem 4.3], the category  $\text{CDGA}_{\mathbb{I}}^{\geq 0}$  is a model category with weak equivalences being DGA-quasi-isomorphisms, fibrations being degree-wise surjective DGA-morphisms and cofibrations being retracts of relative Sullivan algebras (see [FOT08, Proposition 2.22 and Proposition 2.28]). Cofibrant objects are then precisely

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Sullivan algebras.

So, the homotopy extension property holds already for Sullivan DGA's. To see the role of minimality, we shall descend to the homotopy category. The homotopy category is constructed from a model category by localizing morphisms at weak equivalences. An isomorphism in the homotopy category, called weak homotopy equivalence, corresponds to a zig-zag of weak equivalences. If  $V_1$  is weakly homotopy equivalent to  $V_2$ , we say that  $V_2$  is a model of  $V_1$ . If there is a weak equivalence  $V_2 \rightarrow V_1$ , we say that  $V_2$  is a resolution of  $V_1$ . We understand minimality as a condition which is in each weak homotopy equivalence class satisfied by at most one object up to isomorphism in the model category. If minimal models exist, they form a skeleton of the homotopy category. This is precisely the case of  $\text{CDGA}_1^{\geq 0}$  and minimal Sullivan algebras. Indeed, by [FOT08, Theorem 2.24], every connected  $\text{CDGA}_1^{\geq 0}$  is resolved by a minimal Sullivan algebra. Next, by [FOT08, Proposition 2.26], a DGA-morphism lifts to resolutions by Sullivan algebras, and by [FOT08, Corollary 2.13], quasi-isomorphic minimal Sullivan DGA's are isomorphic. Finally, this implies that weakly homotopy equivalent minimal Sullivan algebras are isomorphic, and hence are minimal in the sense above.

As another example, for an operad (or properad)  $\mathcal{O}$ , one wants to construct a dg-operad  $\mathcal{O}_\infty$  which is a quasi-free resolution of  $\mathcal{O}$  (see [Val12]). Quasi-free means that after forgetting the differential, the operad  $\mathcal{O}_\infty$  is free over  $\mathbb{S}$ -bimodules. This is similar to Sullivan models which are free over vector spaces. For quadratic operads,  $\mathcal{O}_\infty$  is often constructed as the cobar construction  $\Omega$  of the Koszul dual cooperad  $\mathcal{O}^i$ . This is the case of  $A_\infty$ ,  $L_\infty$  or of the properad  $\text{IBL}_\infty$ . The differential on  $\Omega\mathcal{O}^i$  is the extension of the decomposition on  $\mathcal{O}^i$  to a derivative, and hence it has decomposable image (c.f., the explicit formula [Pek18, Formula (2)]). This is similar to minimal Sullivan models. It would be interesting to know whether  $\mathcal{O}_\infty$  can be constructed using the same inductive method of “killing” and “adding” generators of homology as Sullivan minimal models.

Finally, let us note that in [CR19], they use the inductive method to construct (minimal) models of  $\mathcal{O}$ -algebras for a wide class of operads  $\mathcal{O}$ . Note that cyclic  $A_\infty$ -algebras can be formulated in the language of cyclic operads. However, in the case of PDGA's, we have the non-degenerate pairing on homology and an operadic description is not clear.  $\triangleleft$

The following example shows that the zig-zag of a Poincaré duality model can not be shortened to one arrow.

*Example 6.2.13* (No Poincaré duality model with one arrow). (a) The closed genus 2 surface  $\Sigma_2$  does not admit a Poincaré duality model  $A$  with just one arrow  $A \rightarrow \Omega(\Sigma_2)$ . Suppose the contrary. We consider the quasi-isomorphism  $f : A \rightarrow \Omega(\Sigma_2)$  and compute



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for homogenous  $v_1, v_2 \in A$  the following:

$$\begin{aligned}
\langle f(v_1), f(v_2) \rangle &= \pm \langle f(v_1) \wedge f(v_2), 1 \rangle \\
&= \pm \langle f(v_1 \wedge v_2), 1 \rangle \\
&= \pm \langle [f(v_1 \wedge v_2)], [1] \rangle \quad (*) \\
&= \pm \langle f_*[v_1 \wedge v_2], f^*[1] \rangle \\
&= \pm \langle [v_1 \wedge v_2], [1] \rangle \\
&= \pm \langle v_1 \wedge v_2, 1 \rangle \quad (*) \\
&= \langle v_1, v_2 \rangle.
\end{aligned}$$

Stars hold because the only non-zero case is when  $\deg(v_1) + \deg(v_2) = \deg(\langle \cdot, \cdot \rangle)$ , and hence  $d(v_1 \wedge v_2) = 0$  because non-degeneracy implies vanishing of higher degrees. We see that  $f$  preserves the pairing on the chain-level and is injective by non-degeneracy. We can thus assume that  $A \subset \Omega(\Sigma_2)$  is a dg-subalgebra equipped with the restriction of the intersection pairing. Since  $\dim(A) < \infty$ , there is a Hodge decomposition

$$\begin{aligned}
A^2 &= \mathcal{H}^2 \oplus dC^1 \\
A^1 &= \mathcal{H}^1 \oplus dC^0 \oplus C^1 \\
A^0 &= \mathcal{H}^0 \oplus C^0.
\end{aligned}$$

If  $1 \neq f \in C^\infty(\Sigma_2)$ , then  $f^k$ ,  $k \in \mathbb{N}_0$  are linearly independent over  $\mathbb{R}$ . Therefore, it must hold  $\mathcal{H}^0 = \text{span}\{1\}$  and  $C^0 = 0$ . Duality implies  $dC^1 = 0$ , and hence  $\mathcal{H}^2$  is spanned by  $\omega \in \Omega^2(\Sigma_2)$  with  $\int_{\Sigma_2} \omega = 1$ . It holds even  $C^1 = 0$  as  $\ker d \cap C = 0$  in a Hodge decomposition. Since  $A \simeq H(\Sigma_2)$  as a DGA, there are closed  $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \Omega^1(\Sigma_2)$  such that  $\mathcal{H}^1 = \text{span}\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$  and such that for all  $x \in \Sigma_2$  the following holds:

$$\begin{aligned}
\alpha_1(x) \wedge \alpha_2(x) &= \alpha_1(x) \wedge \beta_2(x) = 0 \\
\alpha_1(x) \wedge \beta_1(x) &= \alpha_2(x) \wedge \beta_2(x) = \omega(x).
\end{aligned}$$

Taking an  $x \in \Sigma_2$  with  $\omega(x) \neq 0$  gives a contradiction.

(b) The simply-connected 4-manifold  $\mathbb{CP}^{2\#7}$ , where  $\#$  denotes the connected sum, does not admit a Poincaré duality model  $A$  with just one arrow  $A \rightarrow \Omega(M)$ . Similarly as in the proof for  $\Sigma_2$ , we can restrict to the case  $A \subset \Omega(\mathbb{CP}^{2\#7})$ . We obtain  $A^4 = \mathcal{H}^4 = \langle \omega \rangle$  for a somewhere non-vanishing 4-form  $\omega$  and  $\mathcal{H}^2 = \langle K_0, \dots, K_6 \rangle$  such that for all  $x \in \mathbb{CP}^{2\#7}$

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the following holds:

$$K_i(x) \wedge K_i(x) = \pm \omega(x)$$

$$K_i(x) \wedge K_j(x) = 0.$$

We now view  $K_i(x)$  as vectors in  $\mathbb{R}^6$  so that  $K_0(x) \wedge K_j(x) = 0$  corresponds to taking the scalar product. If  $\sum_{i=0}^6 \lambda_i K_i(x) = 0$  for some  $\lambda_i \in \mathbb{R}$  with  $\lambda_{i_0} \neq 0$ , then

$$0 = K_{i_0}(x) \wedge \left( \sum_{i=0}^6 \lambda_i K_i(x) \right) = \pm \lambda_{i_0} \omega(x).$$

Therefore,  $\omega(x) \neq 0$  implies that  $K_i(x)$  are linearly independent. Hence, in this case,  $K_0(x) \wedge K_i(x) = 0$  for all  $i = 1, \dots, 6$  implies  $K_0(x) = 0$ , which is a contradiction with  $K_0(x) \wedge K_0(x) = \omega(x)$ .  $\triangleleft$

*Questions 6.2.14.* (i) Give an example of a Sullivan minimal model  $f : \Lambda U \rightarrow V$  of a PDGA  $V$  which is not of Hodge type for any orientation such that  $f_* : H(\Lambda U) \rightarrow H(V)$  is orientation preserving.

(ii) Can some of the additional assumptions on  $V$  in Lemma 6.1.23 and Proposition 6.2.6 be dropped?

(iii) Can we describe a model category structure on Poincaré DGA's ?

(iv) Is smallness or minimal dimension the correct notion of minimality of a Poincaré duality model?

### 6.3. Consequences for IBL-infinity theory

Consider the construction  $\mathrm{dIBL}^{\mathrm{m}}(C(\cdot))$  on cyclic DGA's (= differential Poincaré duality algebras) from Section 3.4. We first need the following technical result.

**Proposition 6.3.1** (Functoriality of  $\mathrm{dIBL}^{\mathrm{m}}$ -construction up to homotopy). *Suppose that  $V_1$  and  $V_2$  are differential Poincaré duality algebras, and let  $f : V_1 \rightarrow V_2$  be a PDGA-quasi-isomorphism. Then there is an  $\mathrm{IBL}_{\infty}$ -quasi-isomorphism  $\mathfrak{f} : \mathrm{dIBL}(C(V_2)) \rightarrow \mathrm{dIBL}(C(V_1))$  which is also an  $\mathrm{IBL}_{\infty}$ -quasi-isomorphism  $\mathfrak{f} : \mathrm{dIBL}^{\mathrm{m}}(C(V_2)) \rightarrow \mathrm{dIBL}^{\mathrm{m}}(C(V_1))$ , and it holds  $\mathfrak{f}_{110} = f^*$ .* ■

*Proof.* Lemma 6.1.18 implies that  $f$  is injective, and hence we can consider the case when  $V_1 \subset V_2$  is a dg-subalgebra and  $f = \iota : V_1 \hookrightarrow V_2$  is the inclusion. In the proof of

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Lemma 6.1.20 it was argued that there are Hodge decompositions  $V_1 = \mathcal{H} \oplus dC_1 \oplus C_1$  and  $V_1^\perp = dC_2 \oplus C_2$  such that  $V_2 = \mathcal{H} \oplus (dC_1 \oplus dC_2) \oplus (C_1 \oplus C_2)$  is a Hodge decomposition of  $V$ . Define  $P : V_2 \rightarrow V_2$  by

$$P(dc_2) := -c_2 \quad \text{for all } c_2 \in C_2$$

and by 0 on other direct summands. Let  $\pi : V_2 \rightarrow V_1$  be the canonical projection induced by  $V_2 = V_1 \oplus V_1^\perp$ . For  $v, v' \in V_2$ , write  $v = h + dc_{11} + c_{12} + dc_{21} + c_{22}$ ,  $v' = h' + dc'_{11} + c'_{12} + dc'_{21} + c'_{22}$  for  $h, h' \in \mathcal{H}$ ,  $c_{11}, c'_{11}, c_{12}, c'_{12} \in C_1$ ,  $c_{21}, c'_{21}, c_{22}, c'_{22} \in C_2$  and compute

$$\begin{aligned} (P \circ d + d \circ P)(v) &= P(dc_{21}) + dP(dc_{22}) \\ &= -c_{21} - dc_{22} \\ &= (\pi - \mathbb{1})(v) \end{aligned}$$

and

$$\begin{aligned} \langle Pv, v' \rangle &= \langle -c_{21}, dc'_{22} \rangle \\ &= (-1)^{\deg c_{21}} \langle dc_{21}, c'_{22} \rangle \\ &= (-1)^{\deg c_{21} + 1} \langle dc_{21}, Pv' \rangle \\ &= (-1)^{\deg v} \langle v, Pv' \rangle. \end{aligned}$$

These are conditions (11.2), resp. (11.3) from the beginning of [CFL15, Section 11]. We can now apply [CFL15, Theorem 11.3] to obtain an  $\text{IBL}_\infty$ -quasi-isomorphism  $f : \text{dIBL}(C(V_1)) \rightarrow \text{dIBL}(C(V_2))$  with  $f_{110} = \iota^*$ . Because  $\iota$  is a morphism of algebras,  $\iota^*$  commutes also with the Hochschild differential, i.e., with  $q_{110}^m$ . The claim follows because  $\text{dIBL} = (q_{110}, q_{210}, q_{120})$  and  $\text{dIBL}^m = (q_{110}^m, q_{210}, q_{120})$ .  $\square$

For any PDGA  $(V, d, \wedge, o^H)$  with  $H^0(V) = \mathbb{R}$  and  $H^1(V) = 0$ , we can pick its Poincaré duality model  $\mathcal{M}(V)$  and associate to it the  $\text{dIBL}$ -algebra  $\text{dIBL}^m(C(\mathcal{M}(V)))$ . This extends the domain of  $\text{dIBL}^m(C(\cdot))$  to PDGA's. However, removing choices requires us to relax the codomain to  $\text{IBL}_\infty$ -homotopy equivalence classes.

**Proposition 6.3.2** (Uniqueness of  $\text{dIBL}^m$  for PDGA up to homotopy). *Let  $V$  be a PDGA of degree  $n$  with  $H^0(V) = \text{span}\{1\}$  and  $H^1(V) = 0$ . Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be two Poincaré duality models of  $V$ . In addition, suppose that  $\mathcal{M}_1^1 = \mathcal{M}_2^1 = 0$  if  $n \geq 4$  and that  $H^2(V) = 0$  if  $n \geq 5$ . Then  $\text{dIBL}^m(C(\mathcal{M}_1))$  and  $\text{dIBL}^m(C(\mathcal{M}_2))$  are  $\text{IBL}_\infty$ -quasi-*

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*isomorphic.*

*Proof.* By Proposition 6.2.9, there is a differential Poincaré duality algebra  $V_3$  and PDGA-quasi-isomorphisms  $f_1 : \mathcal{M}_1 \rightarrow V_3$  and  $f_2 : \mathcal{M}_2 \rightarrow V_3$ . Proposition 6.3.1 gives  $\text{IBL}_\infty$ -quasi-isomorphisms  $f_1 : \text{dIBL}^m(C(\mathcal{M}_1)) \rightarrow \text{dIBL}^m(C(V_3))$  and  $f_2 : \text{dIBL}^m(C(\mathcal{M}_2)) \rightarrow \text{dIBL}^m(C(V_3))$  extending  $f_1^*$  and  $f_2^*$ , respectively. Application of [CFL15, Theorem 1.2] (existence of  $\text{IBL}_\infty$ -homotopy inverses for  $\text{IBL}_\infty$ -quasi-isomorphisms) finishes the proof.  $\square$

**Conjecture 6.3.3.** *The additional assumptions of Proposition 6.3.2 can be removed.*

If we had the notion of a minimal Poincaré duality model  $\mathcal{M}_0(V)$  which is unique in every weak homotopy class of PDGA's (c.f., Question 6.2.14), we could associate to a weak homotopy equivalence class of a PDGA  $V$  a canonical  $\text{dIBL}$ -algebra  $\text{dIBL}^m(C(\mathcal{M}_0(V)))$  up to an  $\text{IBL}_\infty$ -isomorphism.<sup>7</sup>

**Definition 6.3.4** ( $\text{IBL}_\infty$ -formality). *We say that a differential Poincaré duality algebra  $V$  is  $\text{IBL}_\infty$ -formal if  $\text{dIBL}^m(C(V))$  and  $\text{dIBL}^m(C(H(V)))$  are weakly  $\text{IBL}_\infty$ -homotopy equivalent.*

Due to the existence of homotopy inverses of quasi-isomorphisms in the  $\text{IBL}_\infty$ -category, being weakly  $\text{IBL}_\infty$ -homotopy equivalent (the existence of zig-zag of quasi-isomorphisms) is equivalent to being  $\text{IBL}_\infty$ -quasi-isomorphic (the existence of a direct quasi-isomorphism), which is equivalent to being  $\text{IBL}_\infty$ -homotopy equivalent (the existence of a quasi-isomorphism with a homotopy inverse).

We formulate the following statement as a conjecture although a special case holds by Proposition 6.3.2. The proof might be easier than the proof of Conjecture 6.3.3.

**Conjecture 6.3.5** (Algebraic formality conjecture). *Let  $V$  be a PDGA with  $H^0(V) = \text{span}\{1\}$  and  $H^1(V) = 0$  which is formal as a DGA. Then any its Poincaré duality model  $\mathcal{M}(V)$  is  $\text{IBL}_\infty$ -formal.*

*Proof.* By Proposition 6.2.5, DGA-formality is equivalent to PDGA-formality. Therefore,  $H(V)$  is a Poincaré duality model of  $V$ . The rest follows from Conjecture 6.3.3.  $\square$

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<sup>7</sup>In order to think about functors, we would first have to find a canonical construction of an  $\text{IBL}_\infty$ -morphism induced by a PDGA-morphism of cyclic DGA's. The construction of [CFL15, Section 11] applies only to quasi-isomorphisms, and even if we pick the standard Hodge propagator  $P_{\text{std}}$ , it still depends on the choice of the Hodge decomposition. Therefore, the best we can hope for is an assignment of PDGA-morphisms to  $\text{IBL}_\infty$ -homotopy classes of  $\text{IBL}_\infty$ -morphisms.

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*Remark 6.3.6* (IBL<sub>∞</sub>-formality of  $V$  and  $C(V)$ ). Let  $\mathcal{O}$  be a dg-operad. If  $V$  is an  $\mathcal{O}$ -algebra, then  $H(V)$  is an  $\mathcal{O}$ -algebra with zero differential. We say that  $V$  is  $\mathcal{O}$ -formal if  $V$  and  $H(V)$  are weakly homotopy equivalent as  $\mathcal{O}$ -algebras; i.e., if there is a zig-zag of  $\mathcal{O}$ -quasi-isomorphisms between  $V$  and  $H(V)$ .

Consider DGA and  $A_\infty$ . If a DGA  $V$  is formal, then it is also  $A_\infty$ -formal because DGA is a subcategory of  $A_\infty$ .<sup>8</sup> The converse is also true by the rectification procedure (see [Idr18]). Therefore, DGA-formality and  $A_\infty$ -formality of a DGA  $V$  are equivalent. This should be true for any suitable (pro)perad  $\mathcal{O}$  and its quasi-free resolution  $\mathcal{O}_\infty$ .

In the spirit above, we can speak about dIBL-, resp. IBL<sub>∞</sub>-formality of the dIBL-algebra  $\text{dIBL}^m(C(V))$  on  $C(V)$ , which is about the existence of zig-zags of dIBL-, resp. IBL<sub>∞</sub>-quasi-isomorphisms between  $C(V)$  and  $H(C(V), \mathfrak{q}_{110}^m)$ . These notions are most likely equivalent as in the case of DGA- and  $A_\infty$ -formality.

On the other hand, IBL<sub>∞</sub>-formality of a differential Poincaré duality algebra  $V$  from Definition 6.3.4 is a different notion because  $V$  is not an IBL<sub>∞</sub>-algebra.

We conclude that the following three types of “formalities” might be interesting:

- 1) DGA-formality of a PDGA  $V$ ,
- 2) IBL<sub>∞</sub>-formality of a PDGA  $V$  (resp. of its Poincaré duality model),
- 3) IBL<sub>∞</sub>-formality of  $C(V)$  as the dIBL-algebra  $\text{dIBL}^m(C(V))$ .

Except for the fact that (1) implies (2), we do not have any other guesses. Notions (1)–(3) should also be interpreted geometrically in the context of the String Topology Conjecture.  $\triangleleft$

Let  $M$  be a connected oriented closed  $n$ -manifold with  $H_{\text{dR}}^1(M) = 0$ . We know that  $\Omega(M)$  is of Hodge type and that a Riemannian metric on  $M$  induces a canonical Hodge decomposition  $\Omega(M) = \mathcal{H} \oplus \text{im } d \oplus \text{im } d^*$ , where  $\mathcal{H}$  is the subspace of harmonic forms and  $d^*$  the codifferential. So, in this case, we have a canonical Poincaré duality model

$$\mathcal{M}(\Omega(M)) = \mathcal{Q}(\mathcal{S}(\Omega(M))).$$

Due to  $H_{\text{dR}}^1(M) = 0$ , it is of finite type and hence finite-dimensional. Therefore,  $\text{dIBL}^m(C(\mathcal{M}(\Omega(M))))$  is well-defined. Recall that  $(\Omega, d, \wedge, f)$  is not of finite type and  $\text{dIBL}^m(C(\cdot))$  is not well-defined; that was the reason to define the Chern-Simons, aka

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<sup>8</sup>Note that  $A_\infty$ , similarly as IBL<sub>∞</sub>, admits homotopy inverses of quasi-isomorphisms, and hence  $A_\infty$ -formality is equivalent to the existence of a direct  $A_\infty$ -quasi-isomorphism  $V \rightarrow H(V)$ .

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formal pushforward, Maurer-Cartan element  $\mathfrak{n}$  for  $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}(M)))$  directly as a summation over Feynman integrals. The idea was that  $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$  is “homotopy equivalent” to “ $\mathrm{dIBL}^{\mathfrak{m}}(C(\Omega(M)))$ ”.

We conjecture the following:

**Conjecture 6.3.7** (Equivalence of algebraic and geometric approach). *Let  $M$  be a connected oriented closed Riemannian  $n$ -manifold with  $\mathrm{H}_{\mathrm{dR}}^1(M) = 0$ . Then there is an admissible Hodge propagator  $P$  such that if we consider the Chern-Simons Maurer-Cartan element  $\mathfrak{n}$ , then the  $\mathrm{IBL}_{\infty}$ -algebras*

$$\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M))) \quad \text{and} \quad \mathrm{dIBL}^{\mathfrak{m}}(C(\mathcal{Q}(\mathcal{S}(\Omega(M))))))$$

*are  $\mathrm{IBL}_{\infty}$ -homotopy equivalent.*

*Comment on proof.* By the vanishing result from Theorem B, we can find  $P$  such that the only non-zero component of  $(\mathfrak{n}_{l_g})$  is  $\mathfrak{n}_{10}$ . It satisfies  $\mathfrak{n}_{10} = \sum \pm m_k^+$ , where  $m_k : \mathcal{H}^{\otimes k} \rightarrow \mathcal{H}$  are the  $A_{\infty}$ -operations obtained by homotopy transfer from  $\Omega$  (we use the canonical Hodge isomorphism  $\mathrm{H}_{\mathrm{dR}} \simeq \mathcal{H}$ ). By lifting the zig-zag of PDGA’s to  $A_{\infty}$ , we obtain an  $\mathrm{IBL}_{\infty}$ -quasi-isomorphism  $\mathfrak{g} : (\mathcal{Q}(\mathcal{S}(\Omega)), d, \wedge) \rightarrow (\mathrm{H}_{\mathrm{dR}}, (m_k))$ . The idea is to show that  $\mathfrak{g}_*\mathfrak{m}$  and  $\mathfrak{n}$ , where  $\mathfrak{m}$  is the canonical Maurer-Cartan element, are gauge equivalent Maurer-Cartan elements (see [CFL15, Definition 9.7]). For this, it might be useful to describe a path object of the IBL-algebra  $\mathrm{dIBL}(C(\mathrm{H}_{\mathrm{dR}}))$  (it is a  $\mathrm{dIBL}$ -algebra), how it arises from the path object of a DGA, and how an overlying Maurer-Cartan element can be constructed from the  $A_{\infty}$ -morphism.  $\square$

The following is a consequence of Conjecture 6.3.7. However, we formulate it separately as it might be easier to prove.

**Conjecture 6.3.8** (Geometric formality conjecture). *In the situation of Conjecture 6.3.7, suppose that  $M$  is formal. Then  $\mathrm{dIBL}^{\mathfrak{n}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$  is  $\mathrm{IBL}_{\infty}$ -homotopy equivalent to  $\mathrm{dIBL}^{\mathfrak{m}}(C(\mathrm{H}_{\mathrm{dR}}(M)))$ .*

*Comment on proof.* One has to prove the following. If there is an  $A_{\infty}$ -isomorphism  $(f_i)$  of  $(\mathrm{H}_{\mathrm{dR}}, \wedge)$  and  $(\mathrm{H}_{\mathrm{dR}}, \wedge, m_3, \dots)$  with  $f_1 = \mathbb{1}$ , then  $\mathfrak{m}_{10}$  and  $\mathfrak{n}_{10} = \sum \pm \mathfrak{m}_k^+$  are gauge equivalent.  $\square$

Together with the String Topology Conjecture 4.5.1, this gives a canonical chain model for the equivariant Chas-Sullivan string topology for formal  $M$  with  $\mathrm{H}_{\mathrm{dR}}^1(M) = 0$ .

## 7. Standard Hodge propagator

In this chapter, we tackle the question whether the standard Hodge propagator  $P_{\text{std}}$  extends smoothly to the blow-up.

In Section 7.1, we recall the Schwartz kernel theorem (Proposition 7.1.1) and define the Schwartz form of an operator between sections of the exterior bundle of the cotangent bundle (Proposition 7.1.2 and Definition 7.1.3). We mention that the Schwartz form of a pseudo-differential operator is smooth outside of the diagonal (Proposition 7.1.4). On the example of the identity (Example 7.1.5) we illustrate that the smooth part does not always determine the pseudo-differential operator. We recall basic facts about regularity of the Laplace Green kernel and standard Hodge propagator (Proposition 7.1.6). We formulate the problem of smooth extension to the blow-up (Definition 7.1.7) and ask whether it is satisfied by  $G$  and  $P_{\text{std}}$ .

In Section 7.2, we study uniqueness of the Hodge homotopy (Proposition 7.2.1). We would like to characterize the standard Hodge propagator as a unique primitive to the harmonic kernel satisfying certain properties. We give some ideas how to do it. We also illustrate that one has to be careful with blow-ups (Proposition 7.2.2).

In Section 7.3, we consider the heat form approximation. We define the heat form and sum up its computational properties (Proposition 7.3.1). We write down formulas for the heat form approximations (Proposition 7.3). We postulate that the standard Hodge propagator is the codifferential of the Laplacian Green kernel with respect to one variable (Proposition 7.3.3).

In Section 7.4, we study the standard Hodge propagator on  $\mathbb{R}^n$ . We recall the well-known formulas for the heat form and the Green form on  $\mathbb{R}^n$  (Proposition 7.4.1). We compute the standard Hodge propagator in two ways — as a coderivative of the Green form and as an integral using heat form approximation (Proposition 7.4.2 for  $n \geq 2$  and Example 7.4.3 for  $n = 1$ ).

In Section 7.5, we study the standard Hodge propagator on  $\mathbb{S}^n$ . We compute it explicitly for  $\mathbb{S}^1$  (Example 7.5.1). Next, we study the structure of the space of Hodge propagators on  $\mathbb{S}^n$  (Proposition 7.5.2). We show that the Hodge propagator for  $\mathbb{S}^n$  constructed in Part I is coexact (Proposition 7.5.3). Finally, using the previous results, we

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prove that the standard Hodge propagator for  $\mathbb{S}^2$  extends smoothly to the blow-up and give an explicit formula with two unknown constants (Proposition 7.5.4).

### 7.1. Schwartz form and smooth extension to blow-up

For a smooth vector bundle  $\pi : E \rightarrow M$ , let  $\mathcal{D}(E)$  be the space of “test sections of  $E$ ”, and let  $\mathcal{D}'(E^*)$  be the space of distributions. Let  $\langle \cdot, \cdot \rangle : \mathcal{D}'(E^*) \otimes \mathcal{D}(E) \rightarrow \mathbb{R}$  be the natural pairing. Proper definitions can be formulated easily based on [Hör03, Section 6].

**Proposition 7.1.1** (Schwartz kernel theorem, see [Hör03, Theorem 5.2.1] for the local version). *Let  $E_1$  and  $E_2$  be smooth vector bundles over smooth manifolds  $M_1$  and  $M_2$ , respectively. Then there is a one-to-one correspondence between continuous operators  $T : \mathcal{D}(E_1) \rightarrow \mathcal{D}'(E_2)$  and elements  $K_T \in \mathcal{D}'(E_1 \boxtimes E_2^*)$  which is given by the equation*

$$\langle T s_1, \psi_2 \rangle = \langle K_T, s_1 \boxtimes \psi_2 \rangle \quad \text{for all } s_1 \in \mathcal{D}(E_1) \text{ and } \psi_2 \in \mathcal{D}(E_2^*).$$

The distributional section  $K_T$  is called the Schwartz kernel of  $T$ .

An  $L^1_{\text{loc}}$ -integrable function  $k : M \times M \rightarrow \text{Hom}(E_1, E_2) = E_1^* \boxtimes E_2$  on an oriented Riemannian manifold  $M$  defines the distribution  $K \in \mathcal{D}'(E_1 \boxtimes E_2^*)$  by

$$\langle K, s_1 \boxtimes \psi_2 \rangle = \int_{x_1, x_2} \langle k(x_1, x_2) s_1(x_1), \psi_2(x_2) \rangle \text{Vol}(x_1) \text{Vol}(x_2)$$

In the case of exterior bundles, we introduce an equivalent notion of a Schwartz form. This name was proposed by Dr. A. Hermann in a discussion in Potsdam.

**Proposition 7.1.2** (Schwartz kernel and Schwartz form). *In the setting of Proposition 7.1.1, suppose that  $M_1 = M_2 = M$  is a smooth oriented Riemannian manifold and  $E_1 = E_2 = \Lambda T^* M$ . We consider the isomorphism of vector bundles  $\Psi : \Lambda T^* M \boxtimes (\Lambda T^* M)^* \rightarrow \Lambda T^*(M \times M)$  which is for every  $x_1, x_2 \in M$  given by*

$$\begin{aligned} \Psi : \Lambda T^*_{x_1} M \otimes (\Lambda T^*_{x_2} M)^* &\longrightarrow \Lambda T^*_{(x_1, x_2)}(M \times M) \\ \omega_1 \otimes \xi_2 &\longmapsto \omega_1 \wedge \sharp \xi_2. \end{aligned}$$

Here,  $\sharp : \Lambda T^* M \rightarrow (\Lambda T^* M)^*$  denotes the musical isomorphism with respect to the natural



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pointwise inner product on  $\Lambda T^*M$ . We obtain the isomorphism

$$\begin{aligned}\Psi_* : \Omega'(M \times M) &\longrightarrow \mathcal{D}'(E_1^* \boxtimes E_2) \\ \Omega_T &\longmapsto K_T,\end{aligned}$$

where  $\Omega'$  denotes the space of de Rham currents.

If  $T$  is homogenous of degree  $|T|$ , then the degree of  $\Omega_T$  satisfies

$$\deg(\Omega_T) = \dim(M) + |T|.$$

If  $K_T$  is represented by an  $L^1_{\text{loc}}$ -integrable section of  $\text{Hom}(E_1, E_2) \simeq E_1^* \boxtimes E_2$ , then  $\Omega_T$  is represented by an  $L^1_{\text{loc}}$ -integrable form, and the other way round. In this case, for all  $\omega_1 \in \Omega(M)$ ,  $\omega_2 \in \Omega_c(M)$ , we have

$$\begin{aligned}\int_{x_2} ((T\omega_1)(x_2))[\omega_2(x_2)] &= \int_{x_1, x_2} K_T(x_1, x_2)[\omega_1(x_1)] \wedge \text{Vol}(x_1) \wedge \omega_2(x_2) \\ &= \int_{x_1, x_2} \Omega_T(x_1, x_2) \wedge \omega_1(x_1) \wedge \omega_2(x_2).\end{aligned}\tag{7.1}$$

*Proof.* Straightforward computations similar to Example 7.1.5 below.  $\square$

**Definition 7.1.3** (Schwartz form). *The current  $\Omega_T \in \Omega'(M \times M)$  from Proposition 7.1.2 is called the Schwartz form of  $T$ .*

We will consider pseudo-differential operators  $T : \Omega(M) \rightarrow \Omega(M)$  on a Riemannian manifold  $M$ ; this class of operators generalizes differential operators and contains generalized inverses of elliptic operators (see [Hör07] for thorough treatment).

**Proposition 7.1.4** (Schwartz form of pseudo-differential operators). *Let  $T : \Omega(M) \rightarrow \Omega(M)$  be a pseudo-differential operator on a smooth oriented Riemannian manifold  $M$ . Then the Schwartz form  $\Omega_T$  restricts to a smooth form on  $M \times M \setminus \Delta$ .*

*Proof.* Use methods of [Hör07].  $\square$

*Example 7.1.5* (Schwartz form of  $\mathbb{1}$ ). The Schwartz form of the identity  $\mathbb{1} : \Omega(\mathbb{R}^n) \rightarrow \Omega(\mathbb{R}^n)$  reads

$$\Omega_{\mathbb{1}}(x, y) = \delta(x - y)(dx^1 - dy^1) \cdots (dx^n - dy^n),\tag{7.2}$$

where  $\delta$  denotes the Dirac delta function on  $\mathbb{R}^n$  centered at 0. In order to prove this, we

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start by rewriting

$$\begin{aligned} (dx^1 - dy^1) \cdots (dx^n - dy^n) &= \sum_I (-1)^{|I|} \varepsilon(I^c I \mapsto [n]) dx^{I^c} \wedge dy^I \\ &= \sum_I (-1)^{n|I|} (* dx^I) \wedge dy^I. \end{aligned}$$

Here, the sum is over all multiindices  $I \subset \{1, \dots, n\}$ , and we use that  $*(dx^I) = \varepsilon(I, I^c) dx^{I^c}$ , where  $\varepsilon(I, I^c)$  denotes the sign to order  $II^c$  to  $\{1, \dots, n\}$ . Now, for any  $\omega \in \Omega_c(\mathbb{R}^n)$ , which we write as  $\omega(x) = \sum_K \omega_K(x) dx^K$ , we compute using  $dx^K \wedge *(dx^I) = (dx^K, dx^I) \text{Vol}(x) = \delta^{KI} \text{Vol}(x)$  the following:

$$\begin{aligned} \int_x \delta(x - y) (dx^1 - dy^1) \cdots (dx^n - dy^n) \omega(x) \\ &= \sum_{I, K} \int_x \delta(x - y) (-1)^{n|I|} \omega_K(x) (* dx^I) dy^I dx^K \\ &= \sum_{I, K} \int_x \delta(x - y) (-1)^{n|I| + n|K|} \delta(x - y) \omega_K(x) dx^K *(dx^I) dy^I \\ &= \sum_I \int_x \delta(x - y) \omega_I(x) \text{Vol}(x) dy^I \\ &= \omega(y). \end{aligned}$$

This shows (7.1), and (7.2) follows. Notice that

$$\Omega_1|_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta} = 0,$$

and thus the smooth part of  $\Omega_1$  does not recover the data of the operator  $\mathbb{1}$ .  $\triangleleft$

We consider the Green operator  $\mathcal{G}$  for the Laplacian  $\Delta$  (see [War83]) and the standard Hodge propagator  $\mathcal{P}_{\text{std}}$ . They are both pseudo-differential operators, and it holds  $\mathcal{P}_{\text{std}} = \pm d^* \mathcal{G}$ . We will study their Schwartz forms  $G$  and  $P_{\text{std}}$ , which are called the Green kernel and the standard Hodge propagator, respectively.

**Proposition 7.1.6** (Basic facts about  $G$  and  $P_{\text{std}}$ ). *The Green kernel  $G$  represents an  $L^2$ -integrable form. The standard Hodge propagator  $P_{\text{std}}$  represent an  $L^1$ -integrable form.*

*Proof.* The fact that  $G$  is  $L^2$  is an exercise in [War83]. The fact that  $P_{\text{std}}$  defines an  $L^1$ -integrable form on  $M \times M$  for a compact manifold  $M$  was proved in [Har04] using the heat kernel approximation (see the next section).  $\square$

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A consequence of Proposition 7.1.6 is that the Schwartz forms  $G$  and  $P_{\text{std}}$  are determined by their smooth restrictions to  $M \times M \setminus \Delta$ . This follows from (7.1) because the integral does not depend on sets of zero measure. Therefore, we will write  $G, P_{\text{std}} \in \Omega^{n-1}(M \times M \setminus \Delta)$ .

**Definition 7.1.7** (Smooth extension to the blow-up). *Let  $M$  be a smooth manifold, and let  $\pi : \text{Bl}_\Delta(M \times M) \rightarrow M \times M$  be the Melrose blow-up of  $M \times M$  at the diagonal  $\Delta$  (this is another name for the spherical blow-up). Consider the blow-up diagram*

$$\begin{array}{ccc} & & \text{Bl}_\Delta(M \times M) \\ & \nearrow \tilde{\iota} & \downarrow \pi \\ M \times M \setminus \Delta & \xhookrightarrow{\iota} & M \times M, \end{array}$$

where  $\iota$  is the inclusion and  $\tilde{\iota}$  its unique smooth lift — the embedding of the interior. We say that a smooth form  $\omega \in \Omega(M \times M \setminus \Delta)$  extends smoothly to the blow-up if there is a smooth form  $\tilde{\omega} \in \Omega(\text{Bl}_\Delta(M \times M))$  such that

$$\tilde{\iota}^* \tilde{\omega} = \omega.$$

Note that the extension, if it exists, is necessarily unique.

**Question 7.1.8** (Smooth extension of standard propagator). *Do  $G$  and  $P_{\text{std}}$  extend smoothly to the blow-up?*

## 7.2. Uniqueness of Hodge propagator

**Proposition 7.2.1** (Uniqueness of Hodge homotopy). *Let  $M$  be a closed oriented manifold, and let  $\Omega(M) = \text{im } d \oplus \text{im } d^* \oplus \mathcal{H}$  be its Hodge decomposition. Any linear map  $\mathcal{P} : \Omega(M) \rightarrow \Omega(M)$  satisfying*

$$d \circ \mathcal{P} + \mathcal{P} \circ d = \iota_{\mathcal{H}} \circ \pi_{\mathcal{H}} - \mathbb{1} \tag{7.3}$$

can be expressed as

$$\mathcal{P} = \begin{cases} -(d|_{\text{im } d^*})^{-1} + R_1 & \text{on } \text{im } d, \\ (d|_{\text{im } d^*})^{-1} R_1 d + R_2 & \text{on } \text{im } d^*, \\ R_3 & \text{on } \mathcal{H}, \end{cases} \tag{7.4}$$

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where  $d|_{\text{im } d^*} : \text{im } d^* \rightarrow \text{im } d$  is an isomorphism, for some linear maps  $R_1 : \text{im } d \rightarrow \text{im } d$ ,  $R_2 : \text{im } d^* \rightarrow \ker d$  and  $R_3 : \mathcal{H} \rightarrow \ker d$ . Moreover, the following facts are equivalent for an operator  $\mathcal{P}$  satisfying (7.3):

- (1)  $\mathcal{P} = \mathcal{P}_{\text{std}}$  is the standard Hodge homotopy,
- (2)  $R_1 = R_2 = R_3 = 0$ ,
- (3)  $\text{im } \mathcal{P} \subset \text{im } d^*$ ,

*Proof.* Suppose first that  $\omega \in \text{im } d$ . Plugging it in (7.3), we get that  $\mathcal{P}\omega$  has to satisfy

$$d\mathcal{P}\omega = \omega.$$

We see that

$$\mathcal{P}\omega \in \eta + \ker d,$$

where  $\eta$  is the unique coexact form with  $d\eta = \omega$ . In other words,  $\eta$  is the preimage of  $\omega$  under the isomorphism  $d : \text{im } d^* \rightarrow \text{im } d$ , and we can write

$$\mathcal{P}\omega = (d|_{\text{im } d^*})^{-1}\omega + R_1\omega, \tag{7.5}$$

where  $R_1 : \text{im } d \rightarrow \ker d$ .

Suppose now that  $\omega \in \text{im } d^*$ . Using (7.3) and (7.5), we obtain

$$d\mathcal{P}\omega = \omega - \underbrace{(d|_{\text{im } d^*})^{-1}d\omega}_{=0} - R_1d\omega,$$

where the two terms cancel because  $\omega$  is the unique coexact primitive to  $d\omega$ . Notice that this equation restricts  $R_1$  to  $R_1 : \text{im } d \rightarrow \text{im } d$ . Similarly as in the first case, we obtain

$$\mathcal{P}\omega = -(d|_{\text{im } d^*})^{-1}R_1d\omega + R_2\omega,$$

where  $R_2 : \text{im } d^* \rightarrow \ker d$ .

If  $\omega \in \mathcal{H}$ , then (7.3) gives

$$d\mathcal{P}\omega = 0,$$

and hence  $\mathcal{P}\omega = R_3\omega$  for some  $R_3 : \mathcal{H} \rightarrow \ker d$ .

Therefore, (7.3) implies (7.4). The other direction clearly holds as well.

As for the equivalent facts, because  $\ker d \perp \text{im } d^*$  (with respect to the  $L^2$ -inner product), it is clear from (7.4) that  $\text{im}(\mathcal{P}) \subset \text{im}(d^*)$  is equivalent to  $R_1 = R_2 = R_3 = 0$ .

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Therefore, if there is a  $\mathcal{P}$  satisfying (7.3) and (3), then it is unique; it must be  $\mathcal{P}_{\text{std}} = -d^* \mathcal{G}$ .  $\square$

We see that the standard Hodge homotopy  $\mathcal{P}_{\text{std}}$  can be characterized as the unique Hodge homotopy with coexact image.

We would like to use the equation

$$dP_{\text{std}} = \pm H \quad \text{on } (M \times M) \setminus \Delta \quad (7.6)$$

and characterize  $P_{\text{std}}$  as its unique coexact solution. This is probably not enough and additional assumptions on the asymptotic behavior near  $\Delta$  are needed.

Another idea is to lift the equation (7.6) to the blow-up and study primitives to  $\pi^*H$  on  $\text{Bl}_\Delta(M \times M)$ . The advantage is that  $\text{Bl}_\Delta(M \times M)$  is a compact manifold with boundary, and hence for a given Riemannian metric, we have the Hodge decomposition with boundary conditions. However, the pull-back metric  $g$  along  $\pi : \text{Bl}_\Delta(M \times M) \rightarrow M \times M$  is singular at the boundary. We would have to approximate it with a family of metrics  $g_t$  in a correct way, use Hodge theory to find a  $d^*$ -primitive  $\eta_t$  to a unique coexact  $d$ -primitive  $P_t$  to  $\pi^*H$  with certain boundary conditions, and finally prove that  $P_t$  converges to a solution of (7.6) on the interior and  $\eta_t$  to its  $d^*$ -primitive.

Natural differential operators on  $\Omega(M \times M)$  do not always pull-back to differential operators on  $\Omega(\text{Bl}_\Delta(M \times M))$  via the blow-down map. The total differential  $d$  does, but the operators  $\mathbb{1} \otimes d_y$  and  $d_x \otimes \mathbb{1}$  do not. We suppose that none of  $d^*$ ,  $\mathbb{1} \otimes d_y^*$  and  $d_x^* \otimes \mathbb{1}$  does. We illustrate the consequences on the following seemingly pathological example.

**Proposition 7.2.2** (Pathological example). *For any  $n \in \mathbb{N}$ , there exists a smooth form  $\eta \in \Omega(\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n))$  with compact vertical support with respect to the fiber bundle  $\tilde{\pi}_2 = \text{pr}_2 \circ \pi : \text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow \mathbb{R}$  such that  $d_y \eta = 0$  on  $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta$  but  $d\tilde{\pi}_{2*}\omega \neq 0$ .*

*If  $\eta \in \Omega(\text{Bl}_\Delta(\mathbb{R} \times \mathbb{R}))$  is as above and satisfies in addition  $\tau^* \eta = \pm \eta$ , then  $d_y \eta = 0$  on  $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$  implies  $d\tilde{\pi}_{2*}\eta = 0$ .*

*Proof.* Let us start with  $n = 1$ . Let  $f : \mathbb{R} \times \mathbb{R} \setminus \Delta \rightarrow \mathbb{R}$  be a function such that

1.  $f : \mathbb{R} \times \mathbb{R} \setminus \Delta \rightarrow \mathbb{R}$  is smooth.
2. It holds  $\frac{\partial f}{\partial y}(x, y) = 0$  for all  $(x, y) \in \mathbb{R} \times \mathbb{R} \setminus \Delta$ .
3. For every  $y \in \mathbb{R}$ , the function  $f(\cdot, y)$  has compact support in  $\mathbb{R}$ .

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Clearly, (1) and (2) implies

$$f(x, y) = \begin{cases} f^-(x) & \text{for } x < y, \\ f^+(x) & \text{for } x > y, \end{cases} \quad \text{for all } (x, y) \in \mathbb{R} \times \mathbb{R} \setminus \Delta, \quad (7.7)$$

where  $f^+, f^- : \mathbb{R} \rightarrow \mathbb{R}$  are smooth functions. It is easy to see that (3) implies the existence of  $x^+, x^- \in \mathbb{R}$  such that

$$f^+(x) = 0 \quad \text{for all } x > x^+ \quad \text{and} \quad f^-(x) = 0 \quad \text{for all } x < x^-.$$

We consider the form

$$\eta(x, y) = f(x, y) \, dx \quad \text{on } \mathbb{R} \times \mathbb{R} \setminus \Delta. \quad (7.8)$$

Recall that in general, a form  $\eta(x, y)$  on  $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$  is restriction of a smooth form on  $\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$  if and only if the form  $(\Phi^*\eta)(r, \omega, u)$  for the diffeomorphism  $\Phi : (0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$  given by  $\Phi(r, \omega, u) = (u + r\omega, u)$  is restriction of a smooth form on  $[0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$ . In our case, we have

$$(\Phi^*\eta)(r, u) = \begin{cases} f^+(u + r)(du + dr) & \text{on } D^+ := [0, \infty) \times \{1\} \times \mathbb{R}, \\ f^-(u - r)(du - dr) & \text{on } D^- := [0, \infty) \times \{-1\} \times \mathbb{R}, \end{cases}$$

where we used the fact that  $\mathbb{S}^0 = \{\pm 1\}$  and splitted the domain in two connected components. We see that  $\Phi^*\eta$  extends smoothly to  $[0, \infty) \times \mathbb{S}^0 \times \mathbb{R}$  in the obvious way. We denote the extension by  $\widetilde{\Phi^*\eta}$  and the induced extension of  $\eta$  to  $\text{Bl}_\Delta(\mathbb{R} \times \mathbb{R})$  by  $\tilde{\eta}$ . It holds  $\tilde{\Phi}^*\tilde{\eta} = \widetilde{\Phi^*\eta}$  under the extended diffeomorphism  $\tilde{\Phi} : [0, \infty) \times \mathbb{S}^0 \times \mathbb{R} \rightarrow \text{Bl}_\Delta(\mathbb{R} \times \mathbb{R})$ . The fiberwise integral along the smooth oriented fiber bundle  $\tilde{\pi}_2 : \text{Bl}_\Delta(\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{R}$  transforms under  $\tilde{\Phi}$  to the fiberwise integral along  $\tilde{p}_3 : [0, \infty) \times \mathbb{S}^0 \times \mathbb{R} \rightarrow \mathbb{R}$ , and we get for all  $y = u \in \mathbb{R}$  by the definition

$$(\tilde{\pi}_{3*}\tilde{\eta})(y) = \tilde{p}_{3*}(\widetilde{\Phi^*\eta})(u) = \int_0^\infty (f^+(u + r) + f^-(u - r)) \, dr.$$

The algebraic sign of the second term was canceled by the geometric sign coming from different orientations of  $[0, \infty)$  in  $[0, \infty) \times \{0\}$  and in  $[0, \infty) \times \{1\}$ . We think of  $\tilde{\Phi}$  as of an isomorphism of fiber bundles  $\tilde{p}_3$  and  $\tilde{\pi}_2$  covering the identity, which explains the notation  $y = u$ . The fiberwise integration along  $\tilde{p}_3 : [0, \infty) \times \mathbb{S}^0 \times \mathbb{R} \rightarrow \mathbb{R}$  reduces to Lebesgue integration of a smooth function  $g$  on  $[0, \infty) \times \mathbb{S}^0 \times \mathbb{R}$  with respect to  $(r, \omega)$ .

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Clearly, we can permute any differential operator acting on  $u$  with the integral. In our case, we use  $d$  and obtain

$$\begin{aligned}
(d\tilde{\pi}_{3*}\tilde{\eta})(y) &= (d\tilde{p}_{3*}\widetilde{\Phi^*\eta})(u) \\
&= \frac{\partial}{\partial u} \left( \int_0^\infty (f^+(u+r) + f^-(u-r)) dr \right) du \\
&= \left( \int_0^\infty (f^+)'(u+r) + (f^-)'(u-r) dr \right) du \\
&= \left( \int_u^\infty (f^+)'(z) dz + \int_{-\infty}^u (f^-)'(z) dz \right) du \\
&= (f^-(u) - f^+(u)) du
\end{aligned}$$

This is zero precisely when  $f$  extends continuously to  $\mathbb{R} \times \mathbb{R}$ .

A general form on  $\mathbb{R} \times \mathbb{R} \setminus \Delta$  is

$$\eta(x, y) = f_0(x, y) + f_1(x, y) dx + f_2(x, y) dy + f_3(x, y) dx dy.$$

Imposing the condition  $d_y \eta = 0$ , we get (7.7) for both  $f_0$  and  $f_1$ . Imposing  $\tau^* \omega = -\omega$ , we get  $f_3(x, y) = f_3(y, x)$ ,  $f_2(x, y) = -f_1(y, x)$  and  $f_0(x, y) = -f_0(y, x)$ . Now, requiring that  $\eta$  has compact vertical support in  $x$  implies  $f_0 = f_1 \equiv 0$  because of (7.7), and hence also  $f_2 \equiv 0$ . However, the form  $f_3(x, y) dx dy$  satisfies  $d_y \eta = 0$  and  $d\tilde{\pi}_{2*} = 0$  trivially.

For any  $n \geq 1$  and  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ , consider

$$\eta(x, y) := \frac{\lambda(x)}{|x - y|^n} \sum_{i=1}^n (-1)^{i+1} (x^i - y^i) dx^1 \cdots dx^n dy^1 \cdots \widehat{dy^i} \cdots dy^n,$$

where  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth bump function. Then  $\eta$  extends smoothly to  $\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$ , it holds  $d_y \eta(x, y) = 0$ , but

$$d \int_x \eta(x, y) = \text{Vol}(\mathbb{S}^n) \lambda(y) dy^1 \cdots dy^n.$$

This finishes the proof. □

### 7.3. Approximation using heat form

Given an oriented Riemannian manifold  $(M, g)$ , consider the *heat form*

$$K_t(x, y) = \sum (-1)^{kn} e^{-\lambda_i t} (\star e_i)(x) \wedge e_i(y), \quad (7.9)$$

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where  $(e_i)$  are eigenvectors of  $\Delta$  and  $\lambda_i$  the corresponding eigenvalues. It is equivalently a solution of the equation  $\Delta K_t(x, y) = -\frac{\partial}{\partial t} K_t(x, y)$  with certain boundary condition.

**Proposition 7.3.1** (Properties of the heat kernel). *Let  $M$  be an oriented Riemannian manifold. The heat form  $K_t(x, y)$  is smooth on  $M \times M \times (0, \infty)$  and satisfies*

$$dK_t = 0, \quad \tau^* K_t = (-1)^n K_t \quad \text{and} \quad \frac{1}{2} \Delta K_t = \Delta_x K_t = -\frac{\partial}{\partial t} K_t.$$

*Proof.* Straightforward computation. □

**Proposition 7.3.2** (Approximation using heat form). *Let  $M$  be an oriented Riemannian manifold and  $K_t(x, y)$  the heat form. For all  $(t, x, y) \in ([0, \infty) \times M \times M) \setminus \{0\} \times \Delta =: D(K)$ , we define*

$$\begin{aligned} G_t(x, y) &:= \int_t^\infty K_\tau(x, y) d\tau \quad \text{and} \\ P_t(x, y) &:= (-1)^{n+1} \int_t^\infty (\mathbb{1} \hat{\otimes} d_y^*) K_\tau(x, y) d\tau. \end{aligned} \tag{7.10}$$

*Then:*

- (a) *The forms  $G_t$  and  $P_t$  are smooth on  $D(K)$ , the point-wise limits  $G'$  and  $P'$  exist, and it holds  $G_t \xrightarrow{t} G'$  and  $P_t \xrightarrow{t} P'$  in  $C_{loc}^\infty(M \times M \setminus \Delta)$ .*
- (b) *On  $D(K)$ , the following relations hold:*

$$\begin{aligned} dG_t &= 0 & P_t &= (-1)^{n+1} \frac{1}{2} d^* G_t \\ \Delta G_t &= K_t - H & dP_t &= (-1)^n (H - K_t) \\ \tau^* G_t &= (-1)^n G_t & \tau^* P_t &= (-1)^n P_t. \end{aligned}$$

*It follows that  $G' = G$  is the (Laplace) Green form and  $P' = P_{\text{std}}$  the standard Hodge propagator.*

*Proof.* The formal computation is clear. An honest proof uses the standard heat kernel estimates. □

**Proposition 7.3.3** ( $P_{\text{std}}$  is codifferential of  $G$ ). *Let  $M$  be an oriented Riemannian manifold, and let  $G \in \Omega^n(M \times M \setminus \Delta)$  be the Green form. Then the standard Hodge propagator  $P_{\text{std}}$  satisfies*

$$P_{\text{std}}(x, y) = (-1)^{n+1} (\mathbb{1} \otimes d_y^*) G(x, y),$$



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where  $\mathbb{1} \otimes d_y^* : \Omega^\bullet(M \times M) \rightarrow \Omega^{\bullet-1}(M \times M)$  is the differential operator defined in local coordinates by commuting  $d^*$  over the first factor with the Koszul sign and applying it to the second factor.

*Proof.* As for the signs,  $(-1)^n$  comes from  $T\mathcal{G}\omega(y) = (-1)^{nT} \int_x (\mathbb{1} \otimes T_y)G(x, y)\omega(x)$  with  $T = d^*$  and  $-1$  from  $\mathcal{P}_{\text{std}} = -d^*\mathcal{G}$ . The rest can be proven using the heat kernel approximation and standard heat kernel estimates. There is an other method using the asymptotic expansion of  $G$ , which was shown to the author by Prof. Christian Bär.  $\square$

### 7.4. Standard Hodge propagator for Euclidean space

We will use Propositions 7.3.3 and 7.3.2 to prove in two ways that  $\mathcal{P}_{\text{std}}$  for  $\mathbb{R}^n$  extends smoothly to the blow-up. We will start with the following well known formulas.

**Proposition 7.4.1** (Green form and heat kernel for  $\mathbb{R}^n$ ). *The Green form for  $\mathbb{R}^n$ , which can be equivalently characterized as the unique solution  $G \in \Omega^n(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta)$  of*

$$\Delta_y G(x, y) = \delta(x - y)(dx^1 - dy^1) \cdots (dx^n - dy^n), \quad (7.11)$$

where  $\delta(x - y)$  is the Dirac delta function, satisfies

$$G(x, y) = \begin{cases} \frac{1}{(n-2)\text{Vol}(\mathbb{S}^{n-1})} \frac{1}{|x - y|^{n-2}} (dx^1 - dy^1) \cdots (dx^n - dy^n) & \text{for } n \geq 3, \\ -\frac{1}{2\pi} \ln |x - y| (dx^1 - dy^1)(dx^2 - dy^2) & \text{for } n = 2. \end{cases}$$

The heat kernel for  $\mathbb{R}^n$ , i.e., the solution of  $\Delta_y Q_t(x, y) = -\frac{\partial}{\partial t} Q_t(x, y)$ , is given by

$$Q_t(x, y) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x - y|^2}{4t}\right) (dx^1 - dy^1) \cdots (dx^n - dy^n). \quad (7.12)$$

*Proof.* See [BGV04]  $\square$

**Proposition 7.4.2** (Standard Hodge propagator for  $\mathbb{R}^n$ ). *The standard Hodge propagator for  $\mathbb{R}^n$  with  $n \geq 2$  satisfies*

$$\mathcal{P}_{\text{std}}(x, y) = \frac{(-1)^{n+1}}{\text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n (-1)^{i-1} \frac{x^i - y^i}{|x - y|^n} (dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n)$$

and extends smoothly to  $\text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n)$ .

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*Proof using the Green form.* Consider the formulas from Proposition 7.4.1. Recall the formula for the codifferential

$$d^* f(x) dx^I = - \sum_{i \in I} \varepsilon(i, I) \frac{\partial f(x)}{\partial x^i} dx^{I \setminus \{i\}}. \quad (7.13)$$

For  $n \geq 3$ , we compute

$$\begin{aligned} & (\mathbb{1} \otimes d_y^*) G(x, y) \\ &= \frac{1}{(n-2)\text{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|} (\mathbb{1} \otimes d_y^*) \left( \frac{1}{|x-y|^{n-2}} (*dx^I) \wedge dy^I \right) \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|+|I|+n+1} \sum_{i \in I} \frac{x^i - y^i}{|x-y|^n} \varepsilon(i, I) (*dx^I) \wedge dy^{I \setminus \{i\}} \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_I (-1)^{n|I|+|I|+n+1} \sum_{i \in I} \frac{x^i - y^i}{|x-y|^n} \varepsilon(i, I) \varepsilon(I, I^c) dx^{I^c} \wedge dy^{I \setminus \{i\}} \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n \frac{x^i - y^i}{|x-y|^n} \sum_{J \subset \{1, \dots, \hat{i}, \dots, n\}} (-1)^{n|J|+|J|} \varepsilon(i, J) \varepsilon(J \cup \{i\}, J^c \setminus \{i\}) \\ & \quad \quad \quad dx^{J^c \setminus \{i\}} \wedge dy^J \\ &= \frac{1}{\text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n (-1)^{i-1} \frac{x^i - y^i}{|x-y|^n} (dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n). \end{aligned}$$

In the last step, we used that

$$\varepsilon(i, J) \varepsilon(J \cup \{i\}, J^c \setminus \{i\}) = (-1)^{i-1} \varepsilon(J, J^c \setminus \{i\})$$

and

$$\begin{aligned} & dx^{J^c \setminus \{i\}} \wedge dy^J \\ &= (-1)^{|J|} \varepsilon(J^c \setminus \{i\}, J) [(dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n)]_{J^c \setminus \{i\}, J} \\ &= (-1)^{n|J|+|J|} [(dx^1 - dy^1) \cdots \widehat{(dx^i - dy^i)} \cdots (dx^n - dy^n)]_{J^c \setminus \{i\}, J}, \end{aligned}$$

where  $[\cdot]_{I_1, I_2}$  denotes the part of the product which picks the first variable at positions  $I_1$  and the second at positions  $I_2$ . The computation for  $n = 2$  gives the same result, and the formula for  $P_{\text{std}}$  is justified by Proposition 7.3.3.

We will now study whether  $P_{\text{std}}$  extends smoothly to the blow-up. Consider the polar

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coordinates in one variable

$$\begin{aligned}\phi : [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \\ (r, \omega, u) &\longmapsto (u + r\omega, u).\end{aligned}$$

There is a unique smooth map  $\tilde{\phi}$  which fits in the blow-up diagram

$$\begin{array}{ccc} & & \text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n) \\ & \nearrow \tilde{\phi} & \downarrow \pi \\ [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R}^n \times \mathbb{R}^n,\end{array}$$

and it is a diffeomorphism of manifolds with boundary. We denote by

$$\phi_0 := \phi|_{(0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n} : (0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$$

the restriction of  $\phi$  to the interior. The commutative diagram

$$\begin{array}{ccc} [0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n & \xrightarrow{\tilde{\phi}} & \text{Bl}_\Delta(\mathbb{R}^n \times \mathbb{R}^n) \\ \uparrow & & \uparrow \tilde{\iota} \\ (0, \infty) \times \mathbb{S}^n \times \mathbb{R}^n & \xrightarrow{\phi_0} & \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \end{array}$$

shows that we can equivalently study whether the form

$$\mathbf{P}_{\text{pol}} := \phi_0^* \mathbf{P}_{\text{std}}$$

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admits a smooth extension  $\widetilde{P}_{\text{pol}}$  to  $[0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$ . We have

$$\begin{aligned}
P_{\text{pol}}(r, \omega, u) &= \phi_0^*(d_y^* G)(r, \omega, u) \\
&= \frac{(-1)^{n+1}}{r^{n-1} \text{Vol}(\mathbb{S}^{n-1})} \sum_{i=1}^n (-1)^{i-1} \omega^i (r d\omega^1 + \omega^1 dr) \cdots \widehat{(r d\omega^i + \omega^i dr)} \cdots \\
&\quad (r d\omega^n + \omega^n dr) \\
&= \frac{(-1)^{n+1}}{r^{n-1} \text{Vol}(\mathbb{S}^{n-1})} \left( r^{n-1} \sum_{1 \leq i \leq n} (-1)^{i-1} \omega^i d\omega^1 \cdots \widehat{d\omega^i} \cdots d\omega^n \right. \\
&\quad \left. - r^{n-2} \sum_{1 \leq i < j \leq n} (-1)^{i+j} \omega^i \omega^j dr d\omega^1 \cdots \widehat{d\omega^i} \cdots \widehat{d\omega^j} \cdots d\omega^n \right. \\
&\quad \left. + r^{n-2} \sum_{1 \leq j < i \leq n} (-1)^{i+j} \omega^j \omega^i dr d\omega^1 \cdots \widehat{d\omega^j} \cdots \widehat{d\omega^i} \cdots d\omega^n \right) \\
&= (-1)^{n+1} \frac{\text{Vol}_{\mathbb{S}^{n-1}}(\omega)}{\text{Vol}(\mathbb{S}^{n-1})},
\end{aligned}$$

where the last two sums canceled. The result extends smoothly beyond  $r = 0$  because it does not depend on  $r$  at all.  $\square$

*Proof using the heat kernel.* Consider the polar coordinates with respect to the diagonal

$$\begin{aligned}
\varphi : [0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \\
(r, \omega, u) &\longmapsto (x, y) = (u + r\omega, u - r\omega).
\end{aligned}$$

There is a unique smooth map  $\tilde{\varphi}$  which fits in the blow-up diagram

$$\begin{array}{ccc}
& & \text{Bl}_{\Delta}(\mathbb{R}^n \times \mathbb{R}^n) \\
& \nearrow \tilde{\varphi} & \downarrow \pi \\
[0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n & \xrightarrow{\varphi} & \mathbb{R}^n \times \mathbb{R}^n,
\end{array}$$

and it is a diffeomorphism of manifolds with boundary. The heat kernel (7.12) transforms under  $\varphi$  as

$$\begin{aligned}
Q_t^{\text{pol}}(r, \omega, u) &:= (\varphi^* Q_t)(r, \omega, u) \\
&= (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) 2^n r^{n-1} dr \text{Vol}(\omega) \\
&= (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) r^{n-1} dr \text{Vol}(\omega).
\end{aligned}$$

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It follows from Proposition 7.3.2 that for  $(r, \omega, u) \in (0, \infty) \times \mathbb{S}^{n-1} \times \mathbb{R}^n$  we have

$$P_{\text{pol}}(r, \omega, u) = (-1)^{n+1} \frac{1}{2} \lim_{t \rightarrow 0} \int_t^\infty d^{*\text{pol}} Q_\tau^{\text{pol}}(r, \omega, u) d\tau, \quad (7.14)$$

where  $d^{*\text{pol}}$  is the codifferential on  $\mathbb{R}^n \times \mathbb{R}^n$  computed with respect to the pullback of the product Riemannian metric. If  $g : \mathbb{R}^n \otimes \mathbb{R}^n \rightarrow \mathbb{R}$  is an inner product, then the pullback Riemannian metric satisfies

$$g^{\text{pol}} := \varphi^*(g \oplus g)(r, \omega, u) = 2 \begin{pmatrix} g(\omega, \omega) & 0 & 0 \\ 0 & r^2 g & 0 \\ 0 & 0 & g \end{pmatrix}; \quad (7.15)$$

it is degenerate at the boundary  $r = 0$ . Next, it is easy to check that the conformal transformation  $g \mapsto \lambda g$  on vectors induces the conformal transformation  $g \mapsto \frac{1}{\lambda^k} g$  on  $k$ -forms. Thus, if  $\|\cdot\|^{\text{pol}}$  denotes the point-wise norm with respect to  $g^{\text{pol}}$  for the standard Euclidean metric, then

$$\|\text{dr}\|^{\text{pol}} = 2^{-\frac{1}{2}}, \quad \|\text{Vol}(u)\|^{\text{pol}} = 2^{-\frac{n}{2}}, \quad \|\text{Vol}(\omega)\|^{\text{pol}} = 2^{-\frac{n-1}{2}} r^{-(n-1)},$$

and hence

$$\text{Vol}^{\text{pol}}(r, \omega, u) = 2^n r^{n-1} \text{dr Vol}(\omega) \text{Vol}(u).$$

In order to get the Hodge star  $*^{\text{pol}}$ , we compute

$$\begin{aligned} \text{dr Vol}(\omega) \wedge *^{\text{pol}}(\text{dr Vol}(\omega)) &\stackrel{!}{=} \|\text{dr Vol}(\omega)\|^{\text{pol}^2} \text{Vol}^{\text{pol}}(r, \omega, u) \\ &= r^{-(n-1)} \text{dr Vol}(\omega) \text{Vol}(u), \\ \text{dr Vol}(u) \wedge *^{\text{pol}}(\text{dr Vol}(u)) &\stackrel{!}{=} \|\text{dr Vol}(u)\|^{\text{pol}^2} \text{Vol}^{\text{pol}}(r, \omega, u) \\ &= \frac{1}{2} r^{n-1} \text{dr Vol}(\omega) \text{Vol}(u). \end{aligned}$$

Using the product structure (7.15), it follows that

$$\begin{aligned} *^{\text{pol}}(\text{dr Vol}(\omega)) &= r^{-(n-1)} \text{Vol}(u) \quad \text{and} \\ *^{\text{pol}}(\text{dr Vol}(u)) &= \frac{1}{2} r^{n-1} \text{Vol}(\omega). \end{aligned}$$

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Recalling the definition of the codifferential  $d^*\alpha = (-1)^{n(k-1)+1} * d * \alpha$ , we compute

$$\begin{aligned} (*^{\text{pol}} d *^{\text{pol}})(Q_t^{\text{pol}}(r, \omega, u)) &= *^{\text{pol}} d \left( (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) \text{Vol}(u) \right) \\ &= *^{\text{pol}} \left( \frac{-2r}{t} (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) dr \text{Vol}(u) \right) \\ &= -\frac{1}{t} (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) r^n \text{Vol}(\omega), \end{aligned}$$

and because the product dimension is even, we obtain

$$d^{\text{pol}} Q_t^{\text{pol}}(r, \omega, u) = \frac{1}{t} (\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{t}\right) r^n \text{Vol}(\omega).$$

In order to integrate this according to (7.14), we will make use of the  $\Gamma$ -function  $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ . Then, we have

$$\begin{aligned} (-1)^{n+1} P_{\text{pol}}(r, \omega, u) &= \frac{1}{2} \lim_{t \rightarrow 0} \int_t^\infty d^{\text{pol}} Q_\tau^{\text{pol}}(r, \omega, u) d\tau \\ &= \frac{1}{2} \pi^{-\frac{n}{2}} r^n \left( \lim_{t \rightarrow 0} \int_t^\infty \tau^{-\frac{n}{2}-1} \exp\left(-\frac{r^2}{\tau}\right) d\tau \right) \text{Vol}(\omega) \\ &= \frac{1}{2} \pi^{-\frac{n}{2}} r^n \left( \lim_{t \rightarrow 0} \int_{\frac{r^2}{t}}^0 \left(\frac{r^2}{z}\right)^{-\frac{n}{2}-1} \exp(-z) (-z^{-2} r^2 dz) \right) \text{Vol}(\omega) \\ &= \frac{1}{2} \pi^{-\frac{n}{2}} \left( \lim_{t \rightarrow 0} \int_0^{\frac{r^2}{t}} z^{\frac{n}{2}-1} \exp(-z) dz \right) \text{Vol}(\omega) \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \text{Vol}(\omega) \\ &= \frac{\text{Vol}(\omega)}{\text{Vol}(\mathbb{S}^{n-1})}, \end{aligned}$$

which recovers the formula in the proof of Proposition 7.4.2. □

*Example 7.4.3* (The case  $n = 1$ ). It is easy to check directly that

$$P(x, y) := \theta(x - y),$$

where  $\theta$  is the Heavyside step function, is the Schwartz form of a Hodge homotopy  $\mathcal{P}$ . Indeed, for any smooth  $f : \mathbb{R} \rightarrow \mathbb{R}$  with compact support, we have

$$\mathcal{P}(f dx)(y) = \int_x P(x, y) f(x) dx = \int_y^\infty f(x) dx,$$

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and hence  $d \circ \mathcal{P} = -\mathbb{1}$ . It is also easy to check that the following is the Laplace Green form:

$$G(x, y) = -\frac{1}{2}|x - y|(dx - dy).$$

Indeed, using  $\text{sgn}(x) = 2\theta(x) - 1$ , we get

$$\frac{\partial^2}{\partial y^2} \frac{1}{2}|x - y| = \frac{\partial}{\partial y} \frac{1}{2} \text{sgn}(y - x) = \delta(y - x) = \delta(x - y),$$

which shows (7.11). We carefully compute

$$\begin{aligned} (\mathbb{1} \otimes d_y^*)G(x, y) &= d_y^* \left( \frac{1}{2}|x - y| dy \right) = -\frac{1}{2} \frac{\partial}{\partial y} |x - y| = -\frac{1}{2}(2\theta(y - x) - 1) \\ &= -\theta(y - x) + \frac{1}{2} = \theta(x - y) - \frac{1}{2} \end{aligned}$$

and apply Proposition 7.3.3 to get

$$P_{\text{std}}(x, y) = \theta(x - y) - \frac{1}{2}.$$

In particular, we see that the sign agrees with the direct computation of the Hodge propagator above. Smoothness on the blow-up is for  $n = 1$  equivalent to smoothness on the closures of connected components of  $\mathbb{R} \times \mathbb{R} \setminus \Delta$ ; this is readily satisfied.  $\triangleleft$

## 7.5. Standard Hodge propagator for 1- and 2-sphere

In this section, we denote the Hodge propagator for  $\mathbb{S}^n$  constructed in Part I by  $P_{\text{art}}$ . We would like to use  $P_{\text{art}}$  to study the standard Hodge propagator  $P_{\text{std}}$  for  $\mathbb{S}^n$ .

For  $\mathbb{S}^1$ , we can compute  $K_t$  and hence  $P_{\text{std}}$  explicitly.

*Example 7.5.1* ( $P_{\text{std}}$  for  $\mathbb{S}^1$ ). We write  $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$  and use the coordinate  $x \in [0, 2\pi)$ . Because  $\mathbb{S}^1$  is flat, we have

$$\Delta = -\frac{\partial^2}{\partial x^2}.$$

Solving the eigenvalue problem  $\Delta \omega = \lambda \omega$  for  $\omega \in \Omega(\mathbb{S}^1)$  and  $\lambda \in \mathbb{R}$ , we get  $\lambda \in \{0, n^2 \mid n \in \mathbb{N}\}$  and the corresponding eigenvectors

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \cos(nx) dx, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \sin(nx) dx \mid n \in \mathbb{N} \right\},$$

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which we normalized in the  $L^2$ -norm. Plugging in (7.9), we get

$$\begin{aligned} K_t(x_1, x_2) &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} (\cos(nx_1) \cos(nx_2) + \sin(nx_1) \sin(nx_2)) (dx_1 - dx_2) \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \cos(nx_1 - nx_2) (dx_1 - dx_2). \end{aligned}$$

Applying the product codifferential, we get

$$\begin{aligned} d^* K_t(x_1, x_2) &= \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \left[ -\frac{\partial}{\partial x_1} \cos(nx_1 - nx_2) + \frac{\partial}{\partial x_2} \cos(nx_1 - nx_2) \right] \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} n \sin(nx_1 - nx_2). \end{aligned}$$

Finally, the integration gives

$$\begin{aligned} P_{\text{std}}(x_1, x_2) &= \frac{1}{2} \int_0^{\infty} d^* K_t(x_1, x_2) dt \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \left( \int_0^{\infty} e^{-n^2 t} dt \right) n \sin(nx_1 - nx_2) \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n(x_1 - x_2))}{n} \\ &= \frac{1}{2\pi} \begin{cases} \pi - (x_1 - x_2) & x_1 > x_2, \\ -\pi - (x_1 - x_2) & x_1 < x_2. \end{cases} \\ &= \frac{1}{2\pi} (\alpha(x_1, x_2) - \pi). \end{aligned}$$

This is precisely  $P_{\text{art}}$  from Part I.  $\triangleleft$

For  $n \geq 2$ , an explicit formula for any of  $P_{\text{std}}$ ,  $G$  or  $K_t$  seems to be unknown. For  $\mathbb{S}^2$ , the formula for  $P_{\text{std}}$  on functions was derived by A. Hermann.

Our idea to study  $P_{\text{std}}$  via  $P_{\text{art}}$  is to examine the uniqueness of Hodge propagators. Consider the Schwartz form  $H(x, y) = \frac{1}{V} (\text{Vol}(x) + (-1)^n \text{Vol}(y))$  of the harmonic projection for  $\mathbb{S}^n$ . Let  $C_2(\mathbb{S}^n) := \mathbb{S}^n \times \mathbb{S}^n \setminus \Delta$  denote the configuration space, and let

$$\mathcal{S}_n := \{G \in \Omega^{n-1}(C_2(\mathbb{S}^n)) \mid dG = (-1)^n H\} \quad (7.16)$$

be the space of primitives to  $(-1)^n H$ . We know that  $P_{\text{std}} \in \mathcal{S}_n$ . The following holds.



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**Proposition 7.5.2** (The space of primitives to  $H$  for  $\mathbb{S}^n$ ). *Let*

$$V_n := \begin{cases} \Omega^{n-2}(C_2(\mathbb{S}^n))/d\Omega^{n-3}(C_2(\mathbb{S}^n)) & \text{for } n \geq 3, \\ \Omega^0(C_2(\mathbb{S}^n))/\mathbb{R} & \text{for } n = 2, \end{cases}$$

where  $\mathbb{R} \subset \Omega^0(C_2(\mathbb{S}^n))$  denotes the constants. The action  $\rho : V_n \times \mathcal{S}_n \rightarrow \mathcal{S}_n$ ,  $(\lambda, G) \mapsto G + d\lambda$  of the additive group  $V_n$  on  $\mathcal{S}_n$  defines the structure of an affine space on  $\mathcal{S}_n$  for  $n \geq 2$ . If we require  $SO(n+1)$  or  $(-1)^n \tau^*$ -invariance, then the same holds with  $\Omega$  replaced by the correspondingly invariant forms.

*Proof.* We have to check that the action  $\rho$  is free and transitive. For  $G_1, G_2 \in \mathcal{S}_n$ , the difference  $\eta := G_1 - G_2$  is a closed  $(n-1)$ -form; it is exact because  $C_2(\mathbb{S}^n)$  is homotopy equivalent to  $\mathbb{S}^n$ . A primitive  $\lambda_1$  is an  $n-2$  form. If  $\lambda_2$  is another primitive, then  $\lambda_1 - \lambda_2$  is closed, and hence it is a constant for  $n = 2$  and an exact form for  $n \geq 3$ . Therefore,  $\mathcal{S}_n$  is an affine space over  $V_n$ . As for the invariance, we can average a primitive of an invariant form over  $SO(n+1)$  or take  $\frac{1}{2}(\mathbb{1} + (-1)^n \tau^*)$ .  $\square$

Note that  $\mathcal{S}_1 \simeq \mathbb{R}$  by adding the constant and that all functions on  $C_2(\mathbb{S}^1)$  are coexact. Therefore,  $P_{\text{std}}$  for  $\mathbb{S}^1$  can not be characterized as a unique coexact solution of the differential equation for the Hodge propagator.

**Proposition 7.5.3** (Coexactness of artificial Hodge propagator). *The Hodge propagator  $P_{\text{art}} \in \Omega^{n-1}(\mathbb{S}^n \times \mathbb{S}^n \setminus \Delta)$  constructed in Part I is coexact for every  $n \in \mathbb{N}$ .*

*Proof.* First of all, we rewrite

$$\begin{aligned} \omega_k(x, y) &= \frac{1}{k!(n-1-k)!} \sum_{\sigma \in \mathbb{S}_{n+1}} x^{\sigma_1} y^{\sigma_1} dx^{\sigma_3} \dots dx^{\sigma_{2+k}} dy^{\sigma_{3+k}} \dots dy^{\sigma_{n+1}} \\ &= (-1)^k \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} \iota_x(dx^I) \wedge \underbrace{\iota_y *^{\mathbb{R}^{n+1}}}_{*^{\mathbb{S}^n}}(dy^I). \end{aligned}$$

Recall the formulas  $d^* \alpha = (-1)^{d(k-1)+1} * d * \alpha$  and  $** \alpha = (-1)^{k(n-k)} \alpha$  for  $\alpha \in \Omega^k(M)$ ,

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where  $d = \dim(M)$ . For all  $(x, y) \in \mathbb{S}^n \times \mathbb{S}^n \setminus \Delta$ , we compute

$$\begin{aligned}
d_y^* G(x, y) &= d_y^* \left( \sum_{k=0}^{n-1} (-1)^k g_k(x \cdot y) \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} (\iota_x dx^I) \wedge *^{\mathbb{S}^n} (dy^I) \right) \\
&= \sum_{k=0}^{n-1} \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} (\iota_x dx^I) \wedge d_y^* (g_k(x \cdot y) *^{\mathbb{S}^n} (dy^I)) \\
&= \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} (\iota_x dx^I) \wedge *^{\mathbb{S}^n} d_y (g_k(x \cdot y) dy^I) \\
&\quad \uparrow \\
&\quad d_y^* = (-1)^{n(n-k)+1} *^{\mathbb{S}^n} d *^{\mathbb{S}^n} \\
&\quad *^{\mathbb{S}^n} *^{\mathbb{S}^n} = (-1)^{(k+1)(n-k-1)} \mathbb{1} \\
&\quad \text{tot. sign} = (-1)^k \\
&= \sum_{k=0}^{n-1} (-1)^k g'_k(x \cdot y) \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=k+1}} \sum_{i \in I} \sum_{j \in \{1, \dots, n+1\} \setminus I} \varepsilon(i, I) \varepsilon(j, I) x^i x^j \\
&\quad \underbrace{dx^{I \setminus \{i\}} \wedge *^{\mathbb{S}^n} (dy^{I \cup \{j\}})}_{=0} \\
&= 0.
\end{aligned}$$

The cancellation occurs because the summand  $(I, i, j)$  contains the same terms as the summand  $(I' = I \setminus \{i\} \cup j, j, i)$ , and the signs satisfy

$$\varepsilon(j, I') \varepsilon(i, I') = -\varepsilon(i, I) \varepsilon(j, I).$$

We have

$$H_{n-1}(\Omega(\mathbb{S}^n \times \mathbb{S}^n \setminus \Delta), d^*) \simeq H_{\text{dR}}^{n+1}(\mathbb{S}^n \times \mathbb{S}^n \setminus \Delta) = 0,$$

and hence any coclosed  $(n-1)$ -form is coexact. □

**Proposition 7.5.4** (Smooth extension to the blow-up for  $\mathbb{S}^2$ ). *The standard Hodge propagator for  $\mathbb{S}^2$  extends smoothly to the blow-up.*

*Proof.* Let  $G_1, G_2 \in \mathcal{S}_2$  be two  $\text{SO}(3)$ -symmetric solutions. Proposition 7.5.2 asserts that there is a smooth  $\text{SO}(3)$ -symmetric function  $\alpha : C_2(\mathbb{S}^2) \rightarrow \mathbb{R}$  such that  $G_1 - G_2 = d\lambda$ . Because  $\text{SO}(3)$  acts on  $\mathbb{S}^2$  transitively, there is a smooth function  $f : [-1, 1) \rightarrow \mathbb{R}$  such that

$$\lambda(x, y) = f(x \cdot y) \quad \text{for all } (x, y) \in C_2(\mathbb{S}^2).$$

## 7. Standard Hodge propagator

Note that one can let  $f$  explode at 1 and obtain Hodge propagators which do not extend smoothly to the blow-up. Let us assume, in addition, that  $G_1 - G_2$  is coexact. We obtain

$$0 = d^*(G_1 - G_2) = d^*d\lambda = \Delta\lambda.$$

Therefore,  $\lambda$  is a harmonic function on  $C_2(\mathbb{S}^2)$ . Denoting  $B(x, y) := x \cdot y$ , we can write  $\lambda = f \circ B$ , which implies

$$\Delta(f \circ B) = f''\|\nabla B\|^2 + f' \Delta B.$$

The computation of  $\|\nabla B\|$  and  $\Delta B$  is straightforward and we will do it for any  $n \in \mathbb{N}$ . If  $\tilde{f} : \mathbb{S}^n \rightarrow \mathbb{R}$  is a smooth function and  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  is defined by

$$f(x) := \tilde{f}\left(\frac{x}{|x|}\right) \quad \text{for all } x \in \mathbb{R}^{n+1} \setminus \{0\},$$

then

$$\Delta^{\mathbb{S}^n} \tilde{f} = (\Delta^{\mathbb{R}^{n+1}} f)|_{\mathbb{S}^n} \quad \text{and} \quad \nabla^{\mathbb{S}^n} \tilde{f} = (\nabla^{\mathbb{R}^{n+1}} f)|_{\mathbb{S}^n}.$$

We compute

$$\begin{aligned} \Delta_x^{\mathbb{R}^{n+1}} \left( \frac{x}{|x|} \cdot y \right) &= \sum_{i=1}^{n+1} -\frac{\partial}{\partial x^i} \left( \frac{y^i}{|x|} - \frac{x^i}{|x|^3} x \cdot y \right) \\ &= \sum_{i=1}^{n+1} \frac{x^i y^i}{|x|^3} - 3 \frac{x^i x^i}{|x|^5} x \cdot y + \frac{x \cdot y}{|x|^3} + \frac{x^i y^i}{|x|^3} \\ &= 0, \\ \nabla_x^{\mathbb{R}^{n+1}} \left( \frac{x}{|x|} \cdot y \right) &= \sum_{i=1}^{n+1} \left( \frac{y^i}{|x|} - \frac{x^i}{|x|^3} x \cdot y \right) \frac{\partial}{\partial x^i} \quad \text{and} \\ \|\nabla(x \cdot y)\|^2 &= \left\| \sum_{i=1}^{n+1} (y^i - (x \cdot y)x^i) \frac{\partial}{\partial x^i} + \sum_{i=1}^{n+1} (x^i - (x \cdot y)y^i) \frac{\partial}{\partial y^i} \right\|^2 \\ &= \sum_{i=1}^{n+1} (y^i - (x \cdot y)x^i)^2 + (x^i - (x \cdot y)y^i)^2 \\ &= 1 - 2(x \cdot y)^2 + (x \cdot y)^2 + 1 - 2(x \cdot y)^2 + (x \cdot y)^2 \\ &= 2(1 - (x \cdot y)^2). \end{aligned}$$

Therefore,  $\Delta B = 0$ ,  $\|\nabla B(x_1, x_2)\|^2 = 2(1 - (x_1 \cdot x_2)^2)$ , and we arrive to the equation

$$2(1 - u^2)f''(u) = 0 \quad \text{for all } u \in [-1, 1)$$

### 7. *Standard Hodge propagator*

and a smooth function  $f : [-1, 1) \rightarrow \mathbb{R}$ . The only solution is a linear function, and it must hold

$$\lambda(x_1, x_2) = A(x_1 \cdot x_2) + B \quad \text{for some } A, B \in \mathbb{R}.$$

Now,  $B$  comes from a smooth function on  $\mathbb{S}^n \times \mathbb{S}^n$ , and hence  $\lambda$  extends smoothly to the blow-up.  $\square$

If we determine the constants  $A$  and  $B$  in the proof of Proposition 7.5.4, then we get a formula for  $\mathcal{P}_{\text{std}}$  for  $\mathbb{S}^2$  with  $\mathcal{P}_{\text{art}}$ .

## 8. BV-formalism for IBL-infinity theory on cyclic cochains

In this chapter, we formulate the theory of dIBL-algebras on cyclic cochains of an odd symplectic vector space and their twisting with Maurer-Cartan elements in BV-formalism for quantum field theories. The fields are cyclic Hochschild chains and the BV-operator comes from the algebraic string bracket and cobracket. This point of view will hopefully help to understand conceptually the relation between string topology, Chern-Simons theory, symplectic field theory and string field theory.

In Section 8.1, we summarize relevant  $\text{IBL}_\infty$ -theory from [CFL15].

In Section 8.2, we define the canonical string BV-operator on the space of observables and the total action; it consists of the free part and of the interaction part which corresponds to the Maurer-Cartan element. We argue that the quantum master equation is satisfied for the canonical (Chern-Simons) Maurer-Cartan element and that the string BV-operator twisted by the action corresponds to the twisted dIBL-algebra (Proposition 8.2.1). In general, we show that the action satisfies the quantum master equation if and only if its interaction part encodes a Maurer-Cartan element and that the twistings in both formalisms agree (Proposition 8.2.2).

In Section 8.3, we sketch how to apply ideas from [DJP19] for quantum  $L_\infty$ -algebras to  $\text{IBL}_\infty$ -algebras. We ask whether their formulas for the effective action and the path integral which come from the Homological Perturbation Lemma (and Wick's Theorem) agree with the formulas from [CFL15] with summations over ribbon graphs (Question 8.3.1).

### 8.1. Summary of IBL-infinity theory for cyclic DGA

Let  $(V, \Pi, m_1, m_2)$  be a cyclic DGA of degree  $n$  of finite type. We have:

- Cyclic cochains  $C(V) = \hat{B}_{\text{cyc}}^* V[2 - n]$ ,
- Canonical dIBL-algebra  $\mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}$  on  $C(V)$  denoted by  $\text{dIBL}(C(V))$ ,

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- Canonical Maurer-Cartan element  $\mathbf{m} = (\mathbf{m}_{10})$ ,
- Twisted dIBL-algebra  $\mathbf{q}_{110}^{\mathbf{m}}, \mathbf{q}_{210}, \mathbf{q}_{120}$  on  $C(V)$  denoted by  $\text{dIBL}^{\mathbf{m}}(C(V))$ .

Recall from Appendix C that, up to signs and permutation of variables,  $\mathbf{q}_{210}$  is the cyclization  $^+$  of the Gerstenhaber bracket  $[\cdot, \cdot]$  and  $\mathbf{q}_{120}$  fits in  $\Delta_{\text{cyc}} = \mu_{\text{cyc}} \circ \cdot$  where  $\Delta_{\text{cyc}}$  is an extension of the Schwarz's BV-operator on polynomial functions on  $V$  to  $C(V)$  and  $\mu_{\text{cyc}}$  is the cyclic shuffle coproduct.

Let  $(V', \Pi', m'_1, m'_2)$  be another cyclic DGA of degree  $n$  of finite type, and let

$$\mathcal{P} \begin{array}{c} \longrightarrow \\ \circlearrowleft \end{array} (V, m_1) \xrightleftharpoons[\iota]{\pi} (V', m'_1) \quad (8.1)$$

be a deformation retraction. We have the following:

- IBL $_{\infty}$ -morphism  $\mathbf{f} : \text{dIBL}(C(V)) \rightarrow \text{dIBL}(C(V'))$  with components  $\mathbf{f}_{klg} : \hat{\mathbf{E}}_k C(V) \rightarrow \hat{\mathbf{E}}_l C(V')$  such that

$$\mathbf{f}_{110} = \iota^* : (C(V), \hat{\mathbf{q}}_{110}) \longrightarrow (C(V'), \hat{\mathbf{q}}'_{110})$$

is a quasi-isomorphism. The number  $\mathbf{f}_{klg}(s^k \psi_1 \otimes \dots \otimes \psi_k)(s^l \omega_1 \otimes \dots \otimes \omega_l)$  for  $\psi_1, \dots, \psi_k \in \hat{\mathbf{B}}_{\text{cyc}}^* V$  and  $\omega_1 = \omega_{11} \dots \omega_{1s_1}, \dots, \omega_l = \omega_{l1} \dots \omega_{ls_l} \in \mathbf{B}^{\text{cyc}} V'$ , where  $s$  is the formal symbol of degree  $n - 3$ , is computed via summation over ribbon graphs with interior vertices decorated with  $\psi_1, \dots, \psi_k$ , interior edges decorated with the propagator — the Schwartz kernel  $\mathbf{P}$  of  $\mathcal{P}$  — and exterior vertices decorated with  $\omega_{ij}$  at the  $i$ -th boundary component and with  $j$  respecting the orientation. We will refer to such graphs shortly as *Feynman graphs*.

- Pushforward Maurer-Cartan element  $\mathbf{n} := \mathbf{f}_* \mathbf{m}$  with components  $\mathbf{n}_{lg} \in \hat{\mathbf{E}}_l C(V)$ . The number  $\mathbf{n}_{lg}(s^l \omega_1 \otimes \dots \otimes \omega_l)$  is computed via summation over Feynman graphs as above with trivalent internal vertices  $\mathbf{m}_{10}$ .
- Twisted IBL $_{\infty}$ -morphism  $\mathbf{f}^{\mathbf{m}} : \text{dIBL}^{\mathbf{m}}(C(V)) \rightarrow \text{dIBL}^{\mathbf{n}}(C(V'))$  with components  $\mathbf{f}_{klg}^{\mathbf{m}} : \hat{\mathbf{E}}_k C(V) \rightarrow \hat{\mathbf{E}}_l C(V')$  such that

$$\begin{aligned} \mathbf{f}_{110}^{\mathbf{m}} : (C(V), \hat{\mathbf{q}}_{110}^{\mathbf{m}}) &\longrightarrow (C(V'), \hat{\mathbf{q}}'_{110}^{\mathbf{n}}) \\ &= \iota^* + \mathbf{f}_{210} \circ_1 \mathbf{m}_{10} + \frac{1}{2!} \mathbf{f}_{310} \circ_{1,1} (\mathbf{m}_{10}, \mathbf{m}_{10}) + \dots \end{aligned}$$

is a quasi-isomorphism. The number  $\mathbf{f}_{klg}(s^k \psi_1 \otimes \dots \otimes \psi_k)(s^l \omega_1 \otimes \dots \otimes \omega_l)$  is computed via summation over Feynmann graphs as above with interior vertices  $\psi_1, \dots, \psi_k$

and  $\mathfrak{m}_{10}$ .

Recall that an  $\text{IBL}_\infty$ -quasi-isomorphism is automatically an  $\text{IBL}_\infty$ -homotopy equivalence, which has been proven in [CFL15, Theorem 1.2] via obstruction theory. Obstruction theory also gives the existence of a minimal model of any  $\text{IBL}_\infty$ -algebra, which is the content of [CFL15, Theorem 1.3]. What is not proven yet but what we suspect is true is the following:

- (1) For a different choice of the deformation retraction (8.1), the corresponding  $\text{IBL}_\infty$ -morphisms  $\mathfrak{f}$  and  $\mathfrak{f}'$  are  $\text{IBL}_\infty$ -homotopic.
- (2) If  $\iota$  is in addition a morphism of Poincaré DGA's (perhaps simply-connected), then  $\text{dIBL}^{\mathfrak{m}}(C(V))$  and  $\text{dIBL}^{\mathfrak{m}'}(C(V'))$  are  $\text{IBL}_\infty$ -homotopy equivalent (perhaps even the Maurer-Cartan elements  $\mathfrak{m}'$  and  $\mathfrak{f}_*\mathfrak{m}$  are gauge equivalent).

## 8.2. BV-formulation for dIBL-algebra on cyclic cochains

In the setting of Section 8.1, we define

$$B(V) := B^{\text{cyc}}V[3-n] \quad \text{and} \quad \mathcal{F}(B(V)) := \hat{\text{EC}}(V)((\hbar)) = \text{EC}(V) \hat{\otimes} \mathbb{K}((\hbar)).$$

We call  $\mathcal{F}(B(V))$  the *space of functions on  $B(V)$* . This makes sense because  $C(V) \subset (B^{\text{cyc}}V[2-n])^{*\mathfrak{g}}$ ,  $\text{EC}(V) = \text{S}(C(V)[1])$  and the symmetric algebra of the dual can be viewed as polynomial functions. In contrast to [DJP19], we do not have any canonical odd symplectic form on  $B(V)$ , and hence there is no Schwarz's BV-operator on  $\mathcal{F}(B(V))$ . However, we have the following BV-operators from [CFL15]:

$$\begin{aligned} \Delta_0 &:= \hat{\mathfrak{q}}_{120} + \hbar \hat{\mathfrak{q}}_{210}, \\ \Delta &:= \hat{\mathfrak{q}}_{110} + \Delta_0, \\ \Delta^{\mathfrak{m}} &:= \widehat{\mathfrak{q}_{210} \circ_1 \mathfrak{m}_{10}} + \Delta. \end{aligned}$$

Graded symplectic form: Define and use graded symplectic form and dg-symplectic vector space.

The first BV-operator is canonical for an odd symplectic vector space  $(V, \Pi)$  and we call it the *string BV-operator*. The second BV-operator is canonical for a cyclic cochain complex  $(V, \Pi, m_1)$  and the third for a cyclic DGA  $(V, \Pi, m_1, m_2)$  (in both cases,  $(V, \Pi)$  is an odd symplectic vector space).

For  $\Delta$ , we consider the associated Gerstenhaber bracket  $\{\cdot, \cdot\} : \mathcal{F}(B(V))^{\otimes 2} \rightarrow \mathcal{F}(B(V))$ , which is for all  $\Psi_1, \Psi_2 \in \mathcal{F}(B(V))$  given by

$$\{\Psi_1, \Psi_2\} := (-1)^{|\Psi_1|} (\Delta(\Psi_1 \Psi_2) - \Delta(\Psi_1) \Psi_2 - (-1)^{|\Psi_1|} \Psi_1 \Delta(\Psi_2)).$$

## 8. BV-formalism for IBL-infinity theory on cyclic cochains

It is easy to see that

$$\{\Psi_1, \Psi_2\} = (-1)^{1+(|\Psi_1|+1)(|\Psi_2|+1)} \{\Psi_2, \Psi_1\}.$$

If  $\{\cdot, \cdot\}_0$  and  $\{\cdot, \cdot\}^m$  are the Gerstenhaber brackets for  $\Delta_0$  and  $\Delta^m$ , respectively, then it holds

$$\{\cdot, \cdot\}_0 = \{\cdot, \cdot\}^m = \{\cdot, \cdot\}$$

because  $\Delta$ ,  $\Delta_0$  and  $\Delta^m$  differ only by differential operators of order  $\leq 1$ .

Consider the cyclizations  $m_1^+, m_2^+ : B^{\text{cyc}}V \rightarrow \mathbb{R}$  defined for all  $v_1, v_2, v_3 \in V[1]$  by

$$m_1^+(v_1, v_2) := \Pi(m_1(v_1), v_2) \quad \text{and} \quad m_2^+(v_1, v_2, v_3) := \Pi(m_2(v_1, v_2), v_3).$$

They have degree  $3 - n$  as maps, thus degree  $n - 3$  as elements of  $B_{\text{cyc}}^*V$  with the cohomological grading, and thus  $sm_1^+, sm_2^+ \in \mathcal{F}(B(V))$  have degree  $2(n - 3)$ . They are “linear functions” in the sense that  $sm_1^+, sm_2^+ \in \hat{E}_1C(V)$ . We define the *total action*

$$S := \underbrace{(-1)^{n-2}(sm_1^+)\hbar^{-1}}_{=: S_{\text{free}}} + \underbrace{(-1)^{n-2}(sm_2^+)\hbar^{-1}}_{=: S_{\text{int}}} \in \mathcal{F}(B(V)) \quad (8.2)$$

and call  $S_{\text{free}}$  the *free action* and  $S_{\text{int}}$  the *interaction*. The total action is linear and has degree 0. As a function, we can write

$$S(sb) = (-1)^{n-2}m_1^+(b)\hbar^{-1} + (-1)^{n-2}m_2^+(b)\hbar^{-1} \quad \text{for } b \in B^{\text{cyc}}V.$$

Recall that  $\mathbf{m}_{10} = (-1)^{n-2}sm_2^+$  is the canonical Maurer-Cartan element.

**Proposition 8.2.1** (BV-action for canonical Maurer-Cartan element). *Let  $(V, \Pi, m_1, m_2)$  be a cyclic DGA of degree  $n$  of finite type, and let  $\Delta_0 = \hat{\mathbf{q}}_{120} + \hbar\hat{\mathbf{q}}_{210}$  be the string BV-operator on  $\mathcal{F}(B(V))$ . The total action  $S \in \mathcal{F}(B(V))$  from (8.2) satisfies the quantum master equation*

$$\Delta_0 S + \frac{1}{2}\{S, S\}_0 = 0, \quad (8.3)$$

and it holds

$$\Delta^m(f) = \Delta_0(f) + \{S, f\}_0 \quad \text{for all } f \in \mathcal{F}(C(V)), \quad (8.4)$$

where  $\Delta^m = \hat{\mathbf{q}}_{110}^m + \hat{\mathbf{q}}_{120} + \hbar\hat{\mathbf{q}}_{210}$  is the BV-operator associated to the twisted dIBL-algebra  $\text{dIBL}^m(C(V))$ .

*Proof.* Let  $f \in \mathcal{F}(B(V))$ , and consider the left multiplication  $L_f : \mathcal{F}(B(V)) \rightarrow \mathcal{F}(B(V))$



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given by  $g \mapsto fg$ . Using formulas from Appendix D, we compute

$$\begin{aligned}\hat{\mathbf{q}}_{210} \circ L_f &= \hat{\mathbf{q}}_{210}(f \odot \mathbb{1}) \\ &= \mathbf{q}_{210} \circ_{0,2} (f, \mathbb{1}) + \mathbf{q}_{210} \circ_{1,1} (f, \mathbb{1}) + \mathbf{q}_{210} \circ_{2,0} (f, \mathbb{1}) \\ &= (-1)^{|f|} L_f \circ \hat{\mathbf{q}}_{210} + \widehat{\mathbf{q}_{210} \circ_1 f} + L_{\hat{\mathbf{q}}_{210}(f)},\end{aligned}$$

where we view  $f$  as a linear map  $\mathcal{F}(B(V)) \rightarrow \mathcal{F}(B(V))$  which maps the constant 1 to  $f$  and vanishes on the reduced part. Therefore, it holds

$$\begin{aligned}\{f, \cdot\}_0 &= \hbar(\hat{\mathbf{q}}_{210} \circ L_f - L_{\hat{\mathbf{q}}_{210}(f)} - (-1)^{|f|} L_f \circ \hat{\mathbf{q}}_{210}) \\ &= \hbar \widehat{\mathbf{q}_{210} \circ_1 f}.\end{aligned}$$

We compute

$$\begin{aligned}(\mathbf{q}_{210} \circ_1 (sm_1^+))(s\psi)(\omega) &= \mathbf{q}_{210}(sm_1^+, s\psi)(\omega) \\ &= (-1)^{|s|} \mathbf{q}_{210}(s^2 m_1^+, \psi) \\ &= (-1)^{|s|} \sum \varepsilon(\omega \rightarrow \omega^1 \omega^2) (-1)^{|e_j||\omega^1|} T^{ij} m_1^+(e_i \omega^1) \psi(e_j \omega^2) \\ &\quad \uparrow \\ &\quad T^{ij} = (-1)^{|s|+|e_i||e_j|} T^{ji} \\ &\quad T^{ji} = (-1)^{|e_j|} \Pi(e^j, e^i) \\ \Pi(e^j, e^i) \neq 0 &\implies |e_j| + |e_i| = |s| + 1 \\ &= \sum \varepsilon(\omega \rightarrow \omega^1 \omega^2) (-1)^{|e_j|(|s|+1+|\omega^1|)+|e^j||\omega^1|} \Pi(m_1(\omega^1), e^j) \psi(e_j \omega^2) \\ &\quad \uparrow \\ &\quad \sum_i \Pi(v, e^i) e_i = v \\ m_1^+(e^j \omega^1) &= (-1)^{|e^j||\omega^1|} m_1^+(\omega^1 e^j) \\ &= (-1)^{|s|+1} \sum \varepsilon(\omega^1 \omega^2) \psi(m_1(\omega^1) \omega^2) \\ &\quad \uparrow \\ \Pi(m_1(\omega^1), e^j) \neq 0 &\implies 1 + |\omega^1| + |e^j| = |s| + 1 \\ &\quad |e^j| + |e_j| = |s| + 1 \\ &= (-1)^{|s|+1} \sum_{i=1}^k (-1)^{|\omega_1|+\dots+|\omega_{i-1}|} \psi(\omega_1 \dots m_1(\omega_i) \dots \omega_k) \\ &= (-1)^{|s|+1} \mathbf{q}_{110}(s\psi)(s\omega).\end{aligned}\tag{8.5}$$

Therefore, we have

$$\{S_{\text{free}}, \cdot\}_0 = \hat{\mathbf{q}}_{110}.\tag{8.6}$$

Since  $m_1 \circ m_1 = 0$  and since  $m_1^+$  is cyclically symmetric, it holds

$$\{S_{\text{free}}, S_{\text{free}}\}_0 = (-1)^{n-2} \hbar^{-1} \mathbf{q}_{110}(sm_1^+) = 0.$$

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Because  $m_1^+ \in B_{\text{cyc}}^* V$  has weight 2, i.e., it vanishes on words of length 3 and more,  $\mathbf{q}_{120}$  decreases the weight by 2, and  $B_{\text{cyc}}^* V$  is reduced, i.e., its weight-0 part is 0, we have

$$\Delta_0 S_{\text{free}} = (-1)^{n-2} \mathbf{q}_{120}(sm_1^+) = 0.$$

We now compute

$$\begin{aligned} \{S, S\}_0 &= \{S_{\text{free}}, S_{\text{free}}\}_0 + 2\{S_{\text{free}}, S_{\text{int}}\}_0 + \{S_{\text{int}}, S_{\text{int}}\}_0 \\ &= 2\{S_{\text{free}}, S_{\text{int}}\}_0 + \{S_{\text{int}}, S_{\text{int}}\}_0 \\ &= 2\hbar^{-1} \left( \mathbf{q}_{110}(\mathbf{m}_{10}) + \frac{1}{2} \mathbf{q}_{210}(\mathbf{m}_{10}, \mathbf{m}_{10}) \right) \end{aligned} \tag{8.7}$$

and

$$\Delta_0 S = \hbar^{-1} \mathbf{q}_{120}(\mathbf{m}_{10}). \tag{8.8}$$

The right-hand side of (8.7) vanishes due to [CFL15, Proposition 12.3], and the right-hand side of (8.8) vanishes by the discussion preceding [CFL15, Proposition 12.5] (for degree reasons). Therefore,  $S$  satisfies the quantum master equation (8.3). The twisting equation (8.4) is also clear.  $\square$

The inclusion of  $\hbar^{-1}$  into  $S$  is the convention from [CFL15]. After the transformation  $\Delta_0 \rightarrow \Delta'_0 = \Delta_0$ ,  $S \rightarrow S' = \hbar S$ , we get the well-known equations

$$\hbar \Delta'_0 S' + \frac{1}{2} \{S', S'\}'_0 = 0 \quad \text{and} \quad \hbar \Delta'^m = \hbar \Delta'_0 + \{S', \cdot\}'_0.$$

**Proposition 8.2.2** (BV-action for general Maurer-Cartan element). *Let  $(V, \Pi, m_1, m_2)$  be a cyclic DGA of degree  $n$  of finite type, and let  $\Delta_0 = \hat{\mathbf{q}}_{120} + \hbar \hat{\mathbf{q}}_{210}$  be the string BV-operator on  $\mathcal{F}(B(V))$ . Let  $\mathbf{m}_{lg} \in \hat{\mathbf{E}}_l C(V)$  for all  $l, g \geq 0$  satisfy the degree and filtration degree conditions of a Maurer-Cartan element for an  $\text{IBL}_\infty$ -algebra of bidegree  $(n-3, 2)$ . Then  $\mathbf{m} = (\mathbf{m}_{lg})$  satisfies the Maurer-Cartan equation for  $\text{dIBL}(C(V))$  if and only if the total action*

$$S := S_{\text{free}} + \sum_{l, g \geq 0} \mathbf{m}_{lg} \hbar^{g-1} \in \mathcal{F}(B(V)) \tag{8.9}$$

*satisfies the quantum master equation (8.3). The  $\text{BV}_\infty$ -operator  $\Delta^m$  of the twisted  $\text{IBL}_\infty$ -algebra  $\text{dIBL}^m(C(V))$  is given by (8.4). The elements  $\mathbf{m}_{0g}$  appear neither in (8.3) nor in (8.4).*

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*Proof.* Using  $\Delta_0 S_{\text{free}} = \{S_{\text{free}}, S_{\text{free}}\}_0 = 0$ , we compute

$$\begin{aligned}
\Delta_0 S + \frac{1}{2}\{S, S\}_0 &= \Delta_0 S_{\text{int}} + \{S_{\text{free}}, S_{\text{int}}\}_0 + \frac{1}{2}\{S_{\text{int}}, S_{\text{int}}\}_0 \\
&= \sum_{l,g \geq 0} (\hat{q}_{120}(\mathbf{m}_{lg})\hbar^{g-1} + \hat{q}_{210}(\mathbf{m}_{lg})\hbar^g + \hat{q}_{110}(\mathbf{m}_{lg})\hbar^{g-1}) \\
&\quad + \frac{1}{2} \sum_{l_1, l_2, g_1, g_2 \geq 0} \widehat{q_{210} \circ_1 \mathbf{m}_{l_1 g_1}(\mathbf{m}_{l_2 g_2})} \hbar^{g_1 + g_2 - 1} \\
&= \sum_{l,g \geq 0} (\iota_l \pi_l) \left( q_{120} \circ_1 \mathbf{m}_{l-1,g} + q_{210} \circ_2 \mathbf{m}_{l+1,g-1} + q_{110} \circ_1 \mathbf{m}_{lg} \right. \\
&\quad \left. + \frac{1}{2} \sum_{\substack{l_1, l_2, g_1, g_2 \geq 0 \\ l_1 + l_2 - 1 = l \\ g_1 + g_2 = g}} q_{210} \circ_{1,1}(\mathbf{m}_{l_1 g_1}, \mathbf{m}_{l_2 g_2}) \right) \hbar^{g-1}.
\end{aligned}$$

Comparing to Proposition 3.2.11, we see that the  $(l, g)$ -components for  $l \geq 1, g \geq 0$  give precisely the Maurer-Cartan equation in  $\text{dIBL}(C(V))$ . We compute

$$\begin{aligned}
\Delta_0 + \{S, \cdot\}_0 &= \hat{q}_{120} + \hbar \hat{q}_{210} + \hat{q}_{110} + \sum_{l,g \geq 0} \widehat{q_{210} \circ_1 \mathbf{m}_{lg}} \hbar^g \\
&= (\hat{q}_{110} + \widehat{q_{210} \circ_1 \mathbf{m}_{10}}) + (\hat{q}_{120} + \widehat{q_{210} \circ_1 \mathbf{m}_{20}}) + (\widehat{q_{210} \circ_1 \mathbf{m}_{30}}) + \cdots \\
&\quad + [(\hat{q}_{210}) + (\widehat{q_{210} \circ_1 \mathbf{m}_{11}}) + (\widehat{q_{210} \circ_1 \mathbf{m}_{21}}) + \cdots] \hbar \\
&\quad + \sum_{l \geq 0, g \geq 2} (\widehat{q_{210} \circ_1 \mathbf{m}_{lg}}) \hbar^g \\
&= (\hat{q}_{110}^{\mathbf{m}}) + (\hat{q}_{120}^{\mathbf{m}}) + (\hat{q}_{130}^{\mathbf{m}}) + \cdots \\
&\quad + [(\hat{q}_{210}^{\mathbf{m}}) + (\hat{q}_{111}^{\mathbf{m}}) + (\hat{q}_{121}^{\mathbf{m}}) + \cdots] \hbar \\
&\quad + \sum_{l \geq 0, g \geq 2} (\hat{q}_{1lg}^{\mathbf{m}}) \hbar^g
\end{aligned}$$

and see that this is indeed the  $\text{BV}_\infty$ -operator  $\Delta^{\mathbf{m}}$  of  $\text{dIBL}^{\mathbf{m}}(C(V))$ . □

### 8.3. Homotopy transfer and effective action

In [DJP19], they consider multilinear operations  $l_{lg} : V^{\otimes l} \rightarrow \mathbb{R}$  for  $l \geq 1, g \geq 0$  on an odd symplectic vector space  $V$  and write down an action  $S \in \mathcal{F}(V)$  in the form of (8.9) with  $\mathbf{m}_{lg}$  replaced by  $l_{lg}^+$ . We remark that their “vertices” are  $l_{lg}^+(v, \dots, v)$  for a “field”  $v \in V$ , whereas ours are  $\mathbf{m}_{lg}(v_1, \dots, v_l)$  for a “string of fields”  $v_1 \cdots v_l \in B(V)$ . They consider the Schwarz’s canonical BV-operator on  $\mathcal{F}(V)$  from [Sch93] and show that  $S$  satisfies the quantum master equation if and only if  $(l_{lg})$  satisfy the relations of a quantum

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$L_\infty$ -algebra.<sup>1</sup> On the other hand, we have the string BV-operator on  $\mathcal{F}(B(V))$  and our action (8.9) satisfies the quantum master equation if and only if  $\mathfrak{m} = (\mathfrak{m}_g)$  is a Maurer-Cartan element for  $\mathrm{dIBL}(C(V))$ . Next, they consider a deformation retract ( $=: \mathrm{DR}$ ) as in (8.1) and obtain explicit formulas for the homotopy transferred quantum  $L_\infty$ -algebra on  $V'$  together with all maps and homotopies via the Homological Perturbation Lemma ( $=: \mathrm{HPL}$ ). In what follows, we will sketch how to apply their construction to  $\mathrm{IBL}_\infty$ -algebras. We stress that the details have NOT been done yet!

A  $\mathrm{DR}$  (8.1) induces a  $\mathrm{DR}$

$$K_C \left( \begin{array}{c} \curvearrowright \end{array} \right) (C(V), \mathfrak{q}_{110}) \xrightleftharpoons[I_C]{P_C} (C(V'), \mathfrak{q}'_{110}),$$

which further induces a  $\mathrm{DR}$

$$K \left( \begin{array}{c} \curvearrowright \end{array} \right) (\mathcal{F}(B(V)), \hat{\mathfrak{q}}_{110}) \xrightleftharpoons[I]{P} (\mathcal{F}(B(V')), \hat{\mathfrak{q}}'_{110}). \quad (8.10)$$

In [DJP19, Remark 3], they write down a formula for  $K$  on  $\mathcal{F}(V)$  given  $\mathcal{P}$  using a “tensor trick” due to Eilenberg Mac-Lane; the same method may apply to get  $K_C$  from  $\mathcal{P}$  on  $C(V)$  and  $K$  on  $\mathcal{F}(B(V))$  from  $K_C$  in our case. With such  $K$ ’s, one can take  $P_C$  and  $P$ , resp.  $I_C$  and  $I$  to be the natural extensions of  $\iota^*$ , resp.  $\pi^*$ . Note that any surjective quasi-isomorphism over  $\mathbb{R}$  is a deformation retraction, but their formula is explicit and preserves special  $\mathrm{DR}$ ’s, i.e.,  $\mathrm{DR}$ ’s satisfying  $\mathcal{P}^2 = \mathcal{P}\iota = \pi\mathcal{P} = 0$  ( $=: \mathrm{SDR}$ ). The crucial idea of [DJP19] translated to our situation is to view  $\Delta$  and  $\Delta^{\mathfrak{m}}$  as perturbations of  $\{S_{\mathrm{free}}, \cdot\} = \hat{\mathfrak{q}}_{110}$ , which we denote by  $\delta^{(1)}$  and  $\delta^{(2)}$ , respectively, and apply the HPL from [Cra04]:

injectiv qi: Are injective quasi-isomorphisms sections of deformation re-tractions?

For  $i = 1, 2$ , suppose that  $\delta^{(i)}$  is “small”, i.e., that  $(\mathbb{1} - \delta^{(i)}K)$  is invertible, and consider the maps

$$\begin{aligned} \Delta^{(i)} &:= \hat{\mathfrak{q}}'_{110} + P(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}I = \hat{\mathfrak{q}}'_{110} + P\delta^{(i)}I + P\delta^{(i)}K\delta^{(i)}I + \dots, \\ I^{(i)} &:= I + K(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}I = I + K\delta^{(i)}I + K\delta^{(i)}K\delta^{(i)}I + \dots, \\ P^{(i)} &:= P + P(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}K = P + P\delta^{(i)}K + P\delta^{(i)}K\delta^{(i)}K + \dots, \\ K^{(i)} &:= K + K(\mathbb{1} - \delta^{(i)}K)^{-1}\delta^{(i)}K = K + K\delta^{(i)}K + K\delta^{(i)}K\delta^{(i)}K + \dots. \end{aligned}$$

<sup>1</sup>This is equivalent to the notion of a loop homotopy algebra from [Mar01] and to string brackets in closed string field theory from [Zwi93].

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The HPL asserts that if (8.10) is an SDR, then the following are SDR's as well:

$$\begin{array}{ccc}
 K \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & (\mathcal{F}(B(V)), \hat{\mathbf{q}}_{110}) \xrightleftharpoons[I]{P} & (\mathcal{F}(B(V')), \hat{\mathbf{q}}'_{110}) \quad \delta^{(1)} = \Delta_0 \\
 K^{(1)} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & (\mathcal{F}(B(V)), \Delta) \xrightleftharpoons[I^{(1)}]{P^{(1)}} & (\mathcal{F}(B(V')), \Delta^{(1)}) \quad \delta^{(1)} = \Delta_0 \\
 K^{(2)} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & (\mathcal{F}(B(V)), \Delta^{\mathfrak{m}}) \xrightleftharpoons[I^{(2)}]{P^{(2)}} & (\mathcal{F}(B(V')), \Delta^{(2)}) \quad \delta^{(2)} = \Delta_0 + \{S_{\text{int}}, \cdot\}
 \end{array}$$

One defines the *effective action*

$$W := \log(P^{(1)}(e^{S_{\text{int}}})) \in \mathcal{F}(B(V'))$$

and the *path integral*

$$Z := L_{e^{-W}} \circ P^{(1)} \circ L_{e^{S_{\text{int}}}} : \mathcal{F}(B(V)) \rightarrow \mathcal{F}(B(V')).$$

Under certain circumstances, the following formulas hold:

$$\Delta^{(1)} = P \Delta_0 I, \quad \Delta^{(2)} = \Delta^{(1)} + \{W, \cdot\}^{(1)} \quad \text{and} \quad P^{(2)} = Z. \quad (8.11)$$

This is proven in [DJP19] on  $\mathcal{F}(V)$  when  $V' = \mathcal{H}$  is a “harmonic” subspace in a Hodge decomposition  $V = \mathcal{H} \oplus C$  into odd symplectic subspaces,  $\pi$  and  $\iota$  are the canonical projection and inclusion, respectively, the homotopy  $\mathcal{P}$  is such that  $(\pi, \iota, \mathcal{P})$  is an SDR, and the homotopy  $K$  was constructed from  $\mathcal{P}$  via the tensor trick. Since we deal with the string BV-operator on  $\mathcal{F}(B(V))$ , we can not talk about this “symplectic compatibility” and the proof of (8.11) might be based on another arguments.

**Question 8.3.1** (HPL and  $\text{IBL}_\infty$ ). *If one picks an SDR of  $V$  onto a harmonic subspace  $\mathcal{H}$  as above (basically equivalent to the setting of [CFL15, Section 11]) and constructs  $K_C$  and  $K$  in a particular way, can one achieve that (8.11) and the following identities hold?*

$$\Delta^{(1)} = \Delta'_0, \quad \Delta^{(2)} = \Delta^{\mathfrak{m}}, \quad W = S_{\mathfrak{f}_* \mathfrak{m}}, \quad P^{(1)} = e^{\mathfrak{f}}, \quad P^{(2)} = e^{\mathfrak{f}^{\mathfrak{m}}}.$$

Here,  $S_{\mathfrak{f}_* \mathfrak{m}}$  denotes the action (8.9) for the Maurer-Cartan element  $\mathfrak{f}_* \mathfrak{m}$ .

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*Remark 8.3.2* (On BV-formalism for  $\text{IBL}_\infty$ ). (i) As summarized in [DJP19, Section 5], ■ given a BV-action  $S$ , there are various approaches to obtain  $W$  and  $Z$  as summations over Feynman graphs (see [Mne17] for the stationary phase formula approach).

(ii) The appearance of Feynman graphs can be explained from the proof of [DJP19, Theorem 2], where they show that in the special setting above, it holds

$$P^{(1)} = P e^{D_P}$$

for an order  $\leq 2$  differential operator  $D_P$  which “connects” two legs with the propagator  $P$  (obtained by “rising one index” of  $\mathcal{P}$  using the odd symplectic form). This is reminiscent of the Wick’s Theorem for the (formal) perturbative expansion of the path integral (the classical approach to quantum field theories).

(iii) Having a BV-formulation of the  $\text{IBL}_\infty$ -theory, it is intriguing to compare it to [MS13], ■ where certain  $\text{IBL}_\infty$ -structures are considered in the context of open-closed string field theory. ◁

**Part III.**

## **Appendices**

## A. Evaluation of labeled ribbon graphs

In this appendix, we consider an algebraic propagator  $P$  and the graph pairing  $\langle \cdot, \cdot \rangle_\Gamma^P$  (Definition A.1.1), which encapsulates the contribution of a ribbon graph  $\Gamma$  to the map  $f_{klg} : (B_{\text{cyc}}^* V)^{\otimes k} \rightarrow (B_{\text{cyc}}^* V)^{\otimes l}$  defined as a sum of contributions of ribbon graphs (Proposition A.1.2). Such maps were already defined in [CFL15, Section 11] using coordinates; here we use an invariant framework inspired by [Mne17]. As an example, we work out in details expressions for the canonical dIBL-operations  $q_{210}$  and  $q_{120}$  (Example A.1.5). We also explain the technicality of identifying symmetric maps with maps on symmetric powers (Remark A.1.3).

DONE Propagator: Change propagator to admissible propagator or symmetric propagator. Let's leave it as a propagator. It is the propagator in our theory, hence, we can have some definitions.

Next, we show that the matrix  $(T^{ij})$  from Definition 3.4.1 corresponds to the algebraic Schwartz kernel (Definition A.1.4) of the identity  $\mathbb{1}$  up to a sign. Assuming that the Hodge propagator  $P$  from Definition 4.2.4 is algebraic, we deduce natural candidates for the signs in Definition 4.3.6 based on the formula from [CFL15, Remark 12.10] for the genuine pushforward Maurer-Cartan element  $\mathfrak{n}$  in the finite-dimensional case. Establishing the formal analogy between the de Rham case and the finite-dimensional case is our main application of the invariant framework. Finally, we sketch how to obtain signs for the Fréchet dIBL-structure on  $\Omega(M)$  (Remark A.2.2).

Throughout this appendix, we will use Notation 3.3.3 without further remarks.

### A.1. Finite dimensional case

**Definition A.1.1** (Propagator & graph pairing). *Let  $V$  be a graded vector space. We will call a tensor  $P \in V[1]^{\otimes 2}$  an algebraic propagator. We call it admissible if the symmetry condition*

$$\tau(P) = (-1)^{|P|} P \quad (\text{A.1})$$

*is satisfied, where  $\tau : V[1]^{\otimes 2} \rightarrow V[1]^{\otimes 2}$  is the twist map defined by  $\tau(v_1 \otimes v_2) = (-1)^{|v_1||v_2|} v_2 \otimes v_1$  for all  $v_1, v_2 \in V[1]$ .*

*For a ribbon graph  $\Gamma \in \overline{\text{RG}}_{klg}$  and its labeling  $L$ , consider the permutation  $\sigma_L$  from Definition 4.3.5. It acts on tensor powers according to Definition 3.1.6 and thus defines*



## A. Evaluation of labeled ribbon graphs

the map

$$\sigma_L : (V[1]^{\otimes 2})^{\otimes e} \otimes V[1]^{\otimes s_1} \otimes \cdots \otimes V[1]^{\otimes s_l} \longrightarrow V[1]^{\otimes d_1} \otimes \cdots \otimes V[1]^{\otimes d_k},$$

where  $d_i$  and  $s_i$  are the valencies of internal vertices  $1, \dots, k$  and boundary components  $1, \dots, l$ , respectively, and  $e$  is the number of internal edges. We extend  $\sigma_L$  by 0 to other combinations of tensor powers.

The graph pairing

$$\langle \cdot, \cdot \rangle_{\Gamma}^P : (B_{\text{cyc}}^* V)^{\otimes k} \otimes (B^{\text{cyc}} V)^{\otimes l} \longrightarrow \mathbb{R}$$

is defined for all  $\psi_1, \dots, \psi_k \in B_{\text{cyc}}^* V$  and generating words  $w_i = v_{i1} \dots v_{im_i}$  with  $v_{ij} \in V[1]$  for  $m_i \in \mathbb{N}$  and  $i = 1, \dots, l$  by the following formula:

$$\begin{aligned} & \langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^P \\ &:= \sum_{L_1, L_3^b} (\psi_1 \otimes \cdots \otimes \psi_k) (\sigma_L(P^{\otimes e} \otimes (v_{11} \otimes \cdots \otimes v_{1m_1}) \otimes \cdots \\ & \quad \otimes (v_{l1} \otimes \cdots \otimes v_{lm_l}))), \end{aligned}$$

where we use the pairing from Definition 3.3.4 and in every summand an  $L_2$  compatible with  $L_1$  and an  $L_3^b$  are chosen arbitrarily to get a full labeling  $L$  of  $\Gamma$ . The graph pairing extends to  $\langle \cdot, \cdot \rangle_{\Gamma}^P : \bar{TB}_{\text{cyc}}^* V \otimes \bar{TB}^{\text{cyc}} V \rightarrow \mathbb{R}$ , where  $\bar{TW} = \bigoplus_{k=1}^{\infty} W^{\otimes k}$  is the reduced tensor product.

**Proposition A.1.2.** *In the setting of Definition A.1.1, we denote  $w = w_1 \otimes \cdots \otimes w_l$  and  $\psi = \psi_1 \otimes \cdots \otimes \psi_k$ . If  $P$  is admissible, then the following holds:*

- (a) *The number  $\psi(\sigma_L(P^{\otimes e} \otimes w))$  does not depend on the choice of  $L_3^b$  and an  $L_2$  compatible with  $L_1$ . Moreover,  $\langle \cdot, \cdot \rangle_{\Gamma}^P$  does not depend on the representative of  $[\Gamma] \in \overline{\text{RG}}_{klg}$ .*
- (b) *If  $V$  is finite-dimensional, then for every  $k, l \geq 1, g \geq 0$  there is a unique linear map*

$$f_{klg} : (B_{\text{cyc}}^* V)^{\otimes k} \longrightarrow (B_{\text{cyc}}^* V)^{\otimes l}$$

such that

$$\begin{aligned} & f_{klg}(\psi_1 \otimes \cdots \otimes \psi_k)(w_1 \otimes \cdots \otimes w_l) \\ &= \frac{1}{l!} \sum_{[\Gamma] \in \overline{\text{RG}}_{klg}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi_1 \otimes \cdots \otimes \psi_k, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^P. \end{aligned}$$

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(c) The following holds for the map  $f_{klg}$  from (b):

- It is homogenous of degree

$$|f_{klg}| = -|P|(k + l - 2 + 2g). \quad (\text{A.2})$$

- The filtration degree satisfies

$$\|f_{klg}\| \geq -2(k + l - 2 + 2g). \quad (\text{A.3})$$

- For all  $\eta \in \mathbb{S}_l$  and  $\mu \in \mathbb{S}_k$ , we have

$$\eta \circ f_{klg} \circ \mu = (-1)^{|P|(\eta+\mu)} f_{klg}. \quad (\text{A.4})$$

*Proof.* (a) Let us denote by  $\bar{i}$  and  $ij$  the operations on  $L_2$  given by  $e_i \mapsto -e_i$  and  $e_i \leftrightarrow e_j$ , respectively. An even number of these operations does not change the orientation of the complex (4.17). Their effect in  $\sigma_L$  acting on  $P^{\otimes e} \otimes w$  is

$$\bar{i} : P_i \mapsto \tau(P_i) = (-1)^{|P|} P_i \quad \text{and} \quad ij : P_i \dots P_j \mapsto (-1)^{|P|} P_j \dots P_i.$$

Therefore, an even number of them does not change  $\sigma_L(P^{\otimes e} \otimes w)$ . This proves the independence of the choice of a compatible  $L_2$ . The independence of the choice of  $L_3^v$  is clear since  $\psi_i$  are cyclic symmetric.

An isomorphism of ribbon graphs  $\eta : \Gamma \rightarrow \Gamma'$  induces the map of compatible labelings  $L \mapsto L' = \eta_* L$  such that  $\sigma_L = \sigma_{L'}$ . The independence of the choice of a representative of  $[\Gamma]$  follows.

(b) Suppose that  $\psi = \psi_1 \otimes \dots \otimes \psi_k$  with  $\psi_i \in (B_{\text{cyc}}^* V)_{r_i}^{c_i}$ , where  $r_i \in \mathbb{N}$  and  $c_i \in \mathbb{Z}$  for  $i = 1, \dots, k$ . A general element of  $(B_{\text{cyc}}^* V)^{\otimes k}$  is then a finite linear combination of such  $\psi$ 's.

First of all, let us argue that the sum  $\sum_{\overline{\text{RG}}_{klg}}$  is finite. The number of internal edges  $e$  is fixed from the Euler formula (4.16). Therefore, the number of contributing graphs  $(V_{\text{int}}, E_{\text{int}})$  is finite. In order to bound the number of external vertices, we notice that  $d_1 = r_1, \dots, d_k = r_k$  must hold for  $\psi(\sigma_L(P^{\otimes e} \otimes w))$  to be non-zero. Therefore, the sum is finite.

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We now have the linear functional

$$f_{klg}(\psi) := \frac{1}{l!} \sum_{[\Gamma] \in \overline{\text{RG}}_{klg}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi \mid \cdot \rangle_{\Gamma}^{\text{P}} : (\text{B}^{\text{cyc}} V)^{\otimes l} \longrightarrow \mathbb{R}$$

and need to show that  $f_{klg}(\psi) \in (\text{B}_{\text{cyc}}^* V)^{\otimes l} \subset (\text{B}^{\text{cyc}} V)^{\otimes l*}$ . Because  $V$  has finite dimension, the weight-filtration of  $\text{B}^{\text{cyc}} V$  satisfies (WG1) & (WG2) (see (3.11) and Proposition 3.3.6), and hence we have

$$(\text{B}_{\text{cyc}}^* V)^{\otimes l} = (\text{B}^{\text{cyc}} V)^{*_{\text{wg}} \otimes l} = ((\text{B}^{\text{cyc}} V)^{\otimes l})^{*_{\text{wg}}}$$

for the weight-graded duals. Therefore, it suffices to show that  $f_{klg}(\psi)$  vanishes on all but finitely many degrees and weights of  $(\text{B}^{\text{cyc}} V)^{\otimes k}$ . However, the relation  $f_{klg}(\psi)(w) \neq 0$  for a generating word  $w \in (\text{B}^{\text{cyc}} V)^{\otimes k}$  implies

$$\begin{aligned} |w| &= |\psi| - e|\text{P}| \quad \text{and} \\ k(w) &= k(\psi) - 2e, \end{aligned} \tag{A.5}$$

where  $k$  denotes the weight, and hence  $f_{klg}(\psi) \in (\text{B}_{\text{cyc}}^* V)^{\otimes l}$  indeed holds.

(c) The formulas (A.2) and (A.3) follow from (A.5) and from (4.16).

As for the symmetry (A.4), suppose that  $L$  and  $L'$  are compatible labelings of the same graph  $\Gamma$  such that  $L'_1$  differs from  $L_1$  by a permutation  $\mu \in \mathbb{S}_k$  of internal vertices and a permutation  $\eta \in \mathbb{S}_l$  of boundary components. Viewing  $\mu$  and  $\eta$  as block permutations in the vertex and edge order, respectively, we get

$$\sigma_{L'}(\text{P}^{\otimes e} \otimes w) = (-1)^{|\text{P}|(\eta+\mu)} \mu(\sigma_L(\text{P}^{\otimes e} \otimes \eta(w))).$$

The sign comes from the difference of  $L_2$  and  $L'_2$  which compensates the change of the orientation of (4.17) caused by  $\mu$  and  $\eta$ .  $\square$

Given  $\mu \in \mathbb{S}_k$  and  $\psi = \psi_1 \otimes \cdots \otimes \psi_k \in (\text{B}_{\text{cyc}}^* V)^{\otimes k}$ , it is easy to see that

$$\varepsilon(\mu, \Psi) = \varepsilon(\mu(s), \mu(\psi)) \varepsilon(\mu, s) \varepsilon(s, \psi) \varepsilon(\mu, \psi),$$

where  $\Psi = (s\psi_1) \otimes \cdots \otimes (s\psi_k) \in (\text{B}_{\text{cyc}}^* V[A])^{\otimes k}$  and  $\varepsilon(\mu, s) = (-1)^{|s|\mu}$ . If  $A = -|\text{P}|$ , then we get from (A.4) that the degree shift  $f_{klg} : (\text{B}_{\text{cyc}}^* V[A])^{\otimes k} \rightarrow (\text{B}_{\text{cyc}}^* V[A])^{\otimes l}$  has the

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following symmetries:

$$\forall \mu \in \mathbb{S}_k, \eta \in \mathbb{S}_l : \quad \eta \circ \mathfrak{f}_{klg} \circ \mu = \mathfrak{f}_{klg}. \quad (\text{A.6})$$

Note that the degrees satisfy

$$|\mathfrak{f}_{klg}| = |f_{klg}| + (k - l)A. \quad (\text{A.7})$$

*Remark A.1.3* (Symmetric maps versus maps on symmetric powers). In the situation above, we define  $\tilde{\mathfrak{f}}_{klg}$  as the unique map such that the solid lines of the following diagram commute:

$$\begin{array}{ccc} (\mathbf{B}_{\text{cyc}}^* V[A])^{\otimes k} & \xrightarrow{\mathfrak{f}_{klg}} & (\mathbf{B}_{\text{cyc}}^* V[A])^{\otimes l} \\ \downarrow \scriptstyle \begin{array}{c} \swarrow \pi \\ \searrow \iota \end{array} & & \downarrow \scriptstyle \begin{array}{c} \swarrow \pi \\ \searrow \iota \end{array} \\ \mathbf{S}_k \mathbf{B}_{\text{cyc}}^* V[A] & \xrightarrow{\tilde{\mathfrak{f}}_{klg}} & \mathbf{S}_l \mathbf{B}_{\text{cyc}}^* V[A]. \end{array}$$

The symmetry condition (A.6) provides the existence of  $\tilde{\mathfrak{f}}_{klg}$  and implies commutativity of the dotted diagram as well. Moreover, for all  $\psi_1, \dots, \psi_k \in \mathbf{B}_{\text{cyc}}^* V$  and  $w_1, \dots, w_l \in \mathbf{B}_{\text{cyc}}^* V$ , we have

$$\tilde{\mathfrak{f}}_{klg}(s^k \psi_1 \cdots \psi_k)(s^l w_1 \cdots w_l) = \mathfrak{f}_{klg}(s^k \psi_1 \otimes \cdots \otimes \psi_k)(s^l w_1 \otimes \cdots \otimes w_l),$$

where we use the pairing from Definition 3.3.4. We denote  $\tilde{\mathfrak{f}}_{klg}$  again by  $\mathfrak{f}_{klg}$ .  $\triangleleft$

**Definition A.1.4** (Algebraic Schwartz kernel). *Let  $V$  be a graded vector space and  $\Pi : V \otimes V \rightarrow \mathbb{R}$  a non-degenerate pairing on  $V$ . We extend  $\Pi$  to a non-degenerate pairing  $\Pi : V^{\otimes k} \otimes V^{\otimes k} \rightarrow \mathbb{R}$  for  $k \geq 1$  by setting*

$$\Pi(v_{11} \otimes \cdots \otimes v_{1k}, v_{21} \otimes \cdots \otimes v_{2k}) := \varepsilon(v_1, v_2) \Pi(v_{11}, v_{21}) \cdots \Pi(v_{1k}, v_{2k})$$

for all  $v_{11}, \dots, v_{1k}, v_{21}, \dots, v_{2k} \in V$ , where  $\varepsilon$  is the Koszul sign (see Definition 3.1.2). For  $k = 0$ , we let  $\Pi : \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R}$  be the multiplication on  $\mathbb{R}$ .

For  $k, l \geq 0$ , we say that  $\mathcal{K}_L \in V^{\otimes k+l}$  is the algebraic Schwartz kernel of a linear operator  $L : V^{\otimes k} \rightarrow V^{\otimes l}$  if the following is satisfied:

$$\forall w_1 \in V^{\otimes k}, w_2 \in V^{\otimes l} : \quad \Pi(L(w_1), w_2) = \Pi(\mathcal{K}_L, w_1 \otimes w_2). \quad (\text{A.8})$$

We usually omit writing “algebraic” if it is clear from the context (i.e., if we do not consider any “extensions” of  $V^{\otimes k}$ ).

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In the situation of Definition A.1.4, let  $(e_i) \subset V$  be a basis and  $(e^i)$  its dual basis such that  $\Pi(e_i, e^j) = \delta_{ij}$ . We define the coordinates  $K_L^{ij} \in \mathbb{R}$  and  $L^{ij} \in \mathbb{R}$  by

$$\mathcal{K}_L = \sum_{i,j} K_L^{ij} e_i \otimes e_j \quad \text{and} \quad L^{ij} := \Pi(L(e^i), e^j).$$

From (A.8) we have

$$\mathcal{K}_L^{ij} = (-1)^{(|L|+1)(|\Pi|+|e_i|)} L^{ij} \quad \text{for all } i, j. \quad (\text{A.9})$$

From now on, we will be in the situation of (A) and (B) in the Overview; in particular, we put  $V[1]$  in place of  $V$  in Definition A.1.4. Let  $\mathcal{K}_1 \in V[1]^{\otimes 2}$  be the Schwartz kernel of the identity  $\mathbb{1} : V[1] \rightarrow V[1]$  and  $\mathcal{K}_{\mathcal{P}} \in V[1]^{\otimes 2}$  the Schwartz kernel of the cochain homotopy  $\mathcal{P} : V[1] \rightarrow V[1]$ . From (A.9), we get

$$\mathcal{K}_{\mathcal{P}}^{ij} = \mathcal{P}^{ij} \quad \text{and} \quad \mathcal{K}_1^{ij} = (-1)^{|e_i|+|\Pi|} \Pi(e^i, e^j) \quad \text{for all } i, j.$$

We see that the tensor  $T = \sum_{i,j} T^{ij} e_i \otimes e_j$  from (3.32) can be expressed as

$$T = (-1)^{n-2} \mathcal{K}_1.$$

This is the invariant meaning of  $T$ . Note that the degrees satisfy

$$|T| = n - 2 \quad \text{and} \quad |\mathcal{K}_{\mathcal{P}}| = n - 3.$$

The assumption (2.1) on  $\mathcal{P}$  is equivalent to graded antisymmetry of the bilinear form  $\mathcal{P}^+ := \Pi \circ (\mathcal{P} \otimes \mathbb{1}) : V[1]^{\otimes 2} \rightarrow \mathbb{R}$ . This is further equivalent to

$$\tau(\mathcal{K}_{\mathcal{P}}) = (-1)^{|\mathcal{K}_{\mathcal{P}}|} \mathcal{K}_{\mathcal{P}}.$$

Therefore,  $\mathcal{K}_{\mathcal{P}}$  satisfies (A.1), and hence it defines an admissible propagator for the construction of  $f_{klg}$  for every  $k, l \geq 1, g \geq 0$ . We have from (A.4) that the degree shift  $\mathfrak{f}_{klg} : (B_{\text{cyc}}^* V[3-n])^{\otimes k} \rightarrow (B_{\text{cyc}}^* V[3-n])^{\otimes l}$  is symmetric. Moreover, using (A.2), (A.3) and (A.7), we obtain

$$\begin{aligned} |\mathfrak{f}_{klg}| &= -2d(k+g-1), \\ \|\mathfrak{f}_{klg}\| &\geq \gamma(2-2g-k-l), \end{aligned}$$

where  $(d, \gamma) = (n-3, 2)$ . These are the degree and filtration conditions on an  $\text{IBL}_{\infty}$ -morphism from [CFL15, Definition 2.8 and (8.3)]. In fact, our  $\mathfrak{f} = (\mathfrak{f}_{klg})_{k,l \geq 1, g \geq 0}$  is

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precisely the  $\text{IBL}_\infty$ -homotopy from [CFL15, Theorem 11.3].

Graded antisymmetry of  $\Pi$  is equivalent to

$$\tau(T) = (-1)^{|T|+1}T.$$

Visibly,  $T$  does not satisfy (A.1), and hence is not an admissible propagator. Nevertheless, we can still use it to define  $f_{210}$  and  $f_{120}$  since the corresponding graphs  $\Gamma$  (see Figure A.1) have only one internal edge  $e$ , and, for a given  $L_1$ , there is a unique compatible  $L_2$  determined by the orientation of  $e$  (see Example A.1.5 for the compatibility condition). As for the symmetry of the resulting maps, a transposition of internal vertices or boundary components in (4.17) can be compensated only by  $e \mapsto -e$ , which produces  $(-1)^{|T|+1}$  (c.f. the proof of Proposition A.1.2 (a)). Therefore, if we shift the degrees by  $A = -|T| + 1 = n - 3$ , we obtain symmetric maps  $\mathbf{q}_{210} : (\mathbf{B}_{\text{cyc}}^* V[A])^{\otimes 2} \rightarrow \mathbf{B}_{\text{cyc}}^* V[A]$  and  $\mathbf{q}_{120} : \mathbf{B}_{\text{cyc}}^* V[A] \rightarrow (\mathbf{B}_{\text{cyc}}^* V[A])^{\otimes 2}$ . We show in Example A.1.5 below that these operations agree with those defined in Definition 3.4.1.

*Example A.1.5* (The canonical dIBL-operations). We have

$$\begin{aligned} f_{210}(\psi_1 \otimes \psi_2)(w) &= \frac{1}{1!} \sum_{[\Gamma] \in \overline{\text{RG}}_{210}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi_1 \otimes \psi_2 \mid w \rangle_\Gamma^P \quad \text{and} \\ f_{120}(\psi)(w_1 \otimes w_2) &= \frac{1}{2!} \sum_{[\Gamma] \in \overline{\text{RG}}_{120}} \frac{1}{|\text{Aut}(\Gamma)|} \langle \psi \mid w_1 \otimes w_2 \rangle_\Gamma^P. \end{aligned} \tag{A.10}$$

We parametrize  $\text{RG}_{210}$  by the ribbon graphs  $\Gamma_{k_1, k_2}$  with  $1 \leq k_1 \leq k_2$  and  $\text{RG}_{120}$  by the ribbon graphs  $\Gamma^{s_1, s_2}$  with  $0 \leq s_1 \leq s_2$ ; these graphs are depicted in Figure A.1. We have  $\overline{\text{RG}}_{210} = \text{RG}_{210} \setminus \{[\Gamma_{1,1}]\}$  and  $\overline{\text{RG}}_{120} = \text{RG}_{120} \setminus \{[\Gamma^{0,0}], [\Gamma^{0,1}]\}$ . We also have

$$|\text{Aut}(\Gamma_{k_1, k_2})| = \begin{cases} 1 & \text{if } k_1 \neq k_2, \\ 2 & \text{if } k_1 = k_2, \end{cases}$$

and likewise for  $\Gamma^{s_1, s_2}$ . We fix labelings  $L_3^v$  and parametrize  $L_3^b$  by  $c = 1, \dots, k_1 + k_2 - 2$  for  $\Gamma_{k_1, k_2}$  and by  $c_1 = 1, \dots, s_1$  and  $c_2 = 1, \dots, s_2$  for  $\Gamma^{s_1, s_2}$  as it is indicated in Figure A.1.

There are two possible labelings  $L_1^v$  for  $\Gamma_{k_1, k_2}$  and two possible labelings  $L_1^b$  for  $\Gamma^{s_1, s_2}$ ; this is the only freedom in choosing a full labeling  $L$  because  $L_3$  is fixed and  $L_2$  is just the orientation of the single internal edge, which is uniquely determined by  $L_1$ . For both  $\Gamma_{k_1, k_2}$  and  $\Gamma^{s_1, s_2}$ , we will denote the two possible full labelings by  $L^1$  and  $L^2$ . They can be depicted as follows:

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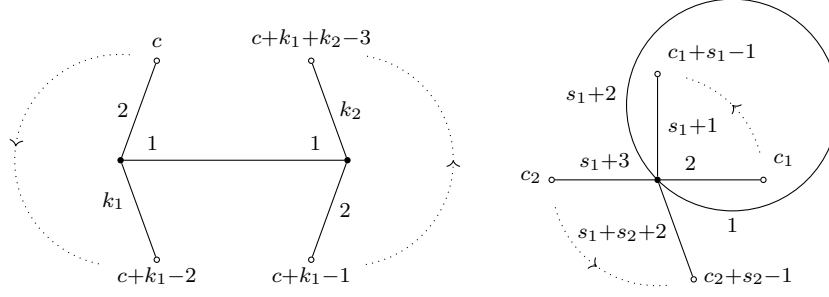


Figure A.1.: Graphs  $\Gamma_{k_1, k_2}$  and  $\Gamma^{s_1, s_2}$  with fixed labelings  $L_3$ .

	$\Gamma_{k_1, k_2}$	$\Gamma^{s_1, s_2}$
$L^1$	$1 \rightarrow 2$	$2 \leftarrow 1$
$L^2$	$\begin{array}{c} \curvearrowright \\ \textcircled{1} \end{array} 2$	$\begin{array}{c} \curvearrowright \\ \textcircled{2} \end{array} 1$

(A.11)

Let us check that the indicated  $L_1$  and  $L_2$  are compatible. For the complexes  $C_2 \rightarrow C_1 \rightarrow C_0$  from (4.17), we have the following:

$$\begin{aligned} \Gamma_{k_1, k_2} : \quad & \langle b \rangle \xrightarrow{\partial_2=0} \langle e \rangle \xrightarrow{\partial_1} \langle v_2 - v_1 \rangle \oplus \langle v_1 + v_2 \rangle, \\ \Gamma^{s_1, s_2} : \quad & \langle b_1 - b_2 \rangle \oplus \langle b_1 + b_2 \rangle \xrightarrow{\partial_2} \langle e \rangle \xrightarrow{\partial_1=0} \langle v \rangle. \end{aligned}$$

As for  $\Gamma_{k_1, k_2}$ , the basis  $v_2 - v_1, v_1 + v_2$  of  $C_0$  is positively oriented with respect to the basis  $v_2, v_1$ . Therefore,  $e$  has to be oriented such that  $\partial_1 e = v_2 - v_1$ ; i.e., it is a path from  $v_1$  to  $v_2$ . As for  $\Gamma^{s_1, s_2}$ , the basis  $b_1 - b_2, b_1 + b_2$  of  $C_2$  is positively oriented with respect to  $b_1, b_2$ . Therefore,  $e$  has to be oriented such that  $e = \partial_2(b_1 - b_2)$ . Recall that we orient the boundary of a 2-simplex by the “outer normal first” convention. We conclude that the labelings from (A.11) are indeed compatible.

As for  $f_{210}$ , the permutations  $\sigma_1 := \sigma_{L^1}$  and  $\sigma_2 := \sigma_{L^2}$  corresponding to the labelings  $L^1$  and  $L^2$ , respectively, read

$$\begin{aligned} \sigma_1 &= \left( \begin{array}{cc|ccc} 1 & 2 & \dots & c+2 & \dots \\ 1 & k_1+1 & \dots & 2 & \dots \end{array} \right) \quad \text{and} \\ &\quad \underbrace{\hspace{10em}}_{k_1+k_1-2} \\ \sigma_2 &= \left( \begin{array}{cc|ccc} 1 & 2 & \dots & c+2 & \dots \\ 1 & k_2+1 & \dots & k_2+2 & \dots \end{array} \right). \\ &\quad \underbrace{\hspace{10em}}_{k_1+k_2-2} \end{aligned}$$

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The underbracket marks the block which represents a cyclic permutation of the remaining indices. We see that

$$\begin{aligned}\sigma_1 : V^{\otimes 2} \otimes V^{\otimes s} &\longrightarrow V^{\otimes k_1} \otimes V^{\otimes k_2}, & e_i e_j w &\longmapsto e_i w^1 e_j w^2, \\ \sigma_2 : V^{\otimes 2} \otimes V^{\otimes s} &\longrightarrow V^{\otimes k_2} \otimes V^{\otimes k_1}, & e_i e_j w &\longmapsto e_i w^2 e_j w^1,\end{aligned}$$

where  $w^1 = w_c \dots w_{c+k_1-2}$ ,  $w^2 = w_{c+k_1-1} \dots w_{c+k_1+k_2-3}$  and  $s := k_1 + k_2 - 2$ . Defining  $\tilde{w}^1 := w^2$  and  $\tilde{w}^2 := w^1$ , The Koszul sign of  $\sigma_2$  can be written as

$$\varepsilon(w \mapsto w^1 w^2) (-1)^{|w^2||e_j|+|w^1||w^2|} = \varepsilon(w \mapsto \tilde{w}^1 \tilde{w}^2) (-1)^{|\tilde{w}^1||e_j|}.$$

We use these facts to rewrite (A.10) as follows:

$$\begin{aligned}f_{210}(\psi_1 \otimes \psi_2)(w) &= \sum_{1 \leq k_1 < k_2} \sum_{i,j} T^{ij} \left( \sum_{k(w^1)=k_1-1} \varepsilon(w \mapsto w^1 w^2) (-1)^{|w^1||e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2) \right. \\ &\quad + \sum_{k(w^1)=k_2-1} \varepsilon(w \mapsto w^1 w^2) (-1)^{|w^2||e_j|+|w^1||w^2|} \psi_1(e_i w^2) \psi_2(e_j w^1) \Big) \\ &\quad + \sum_{1 < k_1 = k_2} \frac{1}{2} \left( \sum_{k(w^1)=k_1-1} \varepsilon(w \mapsto w^1 w^2) (-1)^{|w^1||e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2) \right. \\ &\quad + \sum_{k(w^1)=k_2-1} \varepsilon(w \mapsto w^1 w^2) (-1)^{|w^2||e_j|+|w^1||w^2|} \psi_1(e_i w^2) \psi_2(e_j w^1) \Big) \\ &= \sum_{\substack{k_1, k_2 \geq 1 \\ k_1 + k_2 > 2}} \sum_{\substack{k(w^1)=k_1-1 \\ k(w^2)=k_2-1}} T^{ij} \varepsilon(w \mapsto w^1 w^2) (-1)^{|w^1||e_j|} \psi_1(e_i w^1) \psi_2(e_j w^2).\end{aligned}$$

This coincides with the formula from Definition 3.4.1.

As for  $f_{120}$ , the permutations  $\sigma_1 := \sigma_{L^1}$  and  $\sigma_2 := \sigma_{L^2}$  corresponding to the labelings  $L^1$  and  $L^2$ , respectively, read

$$\begin{aligned}\sigma_1 &= \left( \begin{array}{cc|ccc} 1 & 2 & \dots & c_1 + 2 & \dots \\ 1 & s_1 + 2 & \dots & 2 & \dots \end{array} \middle| \underbrace{\dots}_{s_1} \underbrace{\dots}_{s_2} \right) \text{ and} \\ \sigma_2 &= \left( \begin{array}{cc|ccc} 1 & 2 & \dots & c_2 + 2 & \dots \\ s_1 + 2 & 1 & \dots & s_1 + 3 & \dots \end{array} \middle| \underbrace{\dots}_{s_2} \underbrace{\dots}_{s_1} \right),\end{aligned}$$

where the underbracketed blocks denote cyclic permutations of consecutive indices on



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the corresponding boundary component. We see that

$$\begin{aligned}\sigma_1 : V^{\otimes 2} \otimes V^{\otimes s_1} \otimes V^{\otimes s_2} &\longrightarrow V^{\otimes k}, & e_i e_j w_1 w_2 &\longmapsto e_i w_1^1 e_j w_2^1, \\ \sigma_2 : V^{\otimes 2} \otimes V^{\otimes s_2} \otimes V^{\otimes s_1} &\longrightarrow V^{\otimes k}, & e_i e_j w_1 w_2 &\longmapsto e_j w_2^1 e_i w_1^1,\end{aligned}$$

where  $w_i^1$  denotes a cyclic permutation and  $k := s_1 + s_2 + 2$ . The Koszul sign of  $\sigma_2$  can be written as

$$\begin{aligned}(-1)^{|e_i||e_j|+|w_1||w_2|+|e_i||w_2|}\varepsilon(w_1 \mapsto w_1^1)\varepsilon(w_2 \mapsto w_2^1) \\ = (-1)^{(|e_i|+|w_1|)(|e_j|+|w_2|)+|w_1||e_j|}\varepsilon(w_1 \mapsto w_1^1)\varepsilon(w_2 \mapsto w_2^1).\end{aligned}$$

We use this fact and the cyclic symmetry of  $\psi$  to rewrite (A.10) as follows:

$$\begin{aligned}f_{120}(\psi)(w_1 \otimes w_2) \\ = \sum_{0 \leq s_1 < s_2} \left( \delta_{\substack{k(w_1)=s_1 \\ k(w_2)=s_2}} \sum \mathsf{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|w_1||e_j|} \right. \\ \quad \left. \psi(e_i w_1^1 e_j w_2^1) + \delta_{\substack{k(w_1)=s_2 \\ k(w_2)=s_1}} \sum \mathsf{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \right. \\ \quad \left. (-1)^{|e_i||e_j|+|w_2||w_1|+|e_i||w_2|} \psi(e_j w_2^1 e_i w_1^1) \right) \\ + \sum_{0 < s_1 = s_2} \delta_{\substack{k(w_1)=k(w_2)=s_1 \\ k(w_2)=k(w_2)=s_2}} \frac{1}{2} \left( \sum \mathsf{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \right. \\ \quad \left. (-1)^{|w_1||e_j|} \psi(e_i w_1^1 e_j w_2^1) + \sum \mathsf{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) \right. \\ \quad \left. (-1)^{|e_i||e_j|+|w_2||w_1|+|e_i||w_2|} \psi(e_j w_2^1 e_i w_1^1) \right) \\ = \sum_{\substack{s_1, s_2 \geq 0 \\ s_1 + s_2 > 0}} \delta_{\substack{k(w_1)=s_1 \\ k(w_2)=s_2}} \sum \mathsf{T}^{ij} \varepsilon(w_1 \mapsto w_1^1) \varepsilon(w_2 \mapsto w_2^1) (-1)^{|w_1||e_j|} \\ \quad \psi(e_i w_1^1 e_j w_2^1).\end{aligned}$$

This coincides with the formula from Definition 3.4.1.  $\triangleleft$

## A.2. De Rham case

We will now establish a formal analogy between the finite-dimensional and the de Rham case, which will explain the signs in Definition 4.3.6.

**The finite-dimensional case.** Consider the situation of (A) – (D) in the Overview. To recall briefly, we have a finite-dimensional cyclic dga  $(V, \Pi, m_1, m_2)$  and a subcomplex  $\mathcal{H} \subset V$  such that there is a projection  $\pi : V[1] \rightarrow \mathcal{H}[1]$  chain homotopic to  $\mathbb{1}$  via a chain homotopy  $\mathcal{P} : V[1] \rightarrow V[1]$ . Using  $m_2$ , one constructs the canonical Maurer-Cartan element  $\mathfrak{m}$  for  $\mathrm{dIBL}(C(V))$ . The algebraic Schwartz kernel  $\mathcal{K}_{\mathcal{P}}$  of  $\mathcal{P}$  is an admissible

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propagator used to construct the  $\text{IBL}_\infty$ -quasi-isomorphism  $\mathfrak{f} = (\mathfrak{f}_{klg}) : \text{dIBL}(C(V)) \rightarrow \text{dIBL}(C(\mathcal{H}))$ . The Maurer-Cartan element  $\mathfrak{m}$  is then pushed forward along  $\mathfrak{f}$  to obtain the Maurer-Cartan element  $\mathfrak{n} := \mathfrak{f}_* \mathfrak{m}$  for  $\text{dIBL}(C(\mathcal{H}))$  (see [CFL15, Lemma 9.5]). The formula for  $\mathfrak{n}$  given in [CFL15, Remark 12.10] reads

$$\begin{aligned} & \mathfrak{n}_{lg}(s^l w_1 \otimes \cdots \otimes w_l) \\ &= \frac{1}{l!} \sum_{[\Gamma] \in \text{RG}_{klg}^{(3)}} \frac{1}{|\text{Aut}(\Gamma)|} (-1)^{k(n-2)} \langle (m_2^+)^{\otimes k}, w_1 \otimes \cdots \otimes w_l \rangle_{\Gamma}^{\mathcal{K}_{\mathcal{P}}}. \end{aligned} \quad (\text{A.12})$$

Here the artificial sign  $(-1)^{k(n-2)}$  is added because our sign conventions for  $m_2^+$  differ (see Remark 3.3.8).

**The de Rham case.** We are in the setting of Definition 4.3.6. To recall briefly, we have the cyclic dga  $(\Omega(M), \Pi, m_1, m_2)$ , the subspace of harmonic forms  $\mathcal{H} \subset \Omega$ , the harmonic projection  $\pi_{\mathcal{H}} : \Omega \rightarrow \mathcal{H}$  and a Hodge propagator  $P \in \Omega(\text{Bl}_{\Delta}(M \times M))$ , which is the Schwartz kernel of a chain homotopy  $\mathcal{P} : \Omega \rightarrow \Omega$  between  $\pi_{\mathcal{H}}$  and  $\mathbb{1}$ . In analogy with the finite-dimensional case, the canonical Maurer-Cartan element (4.3) for  $\text{dIBL}(\mathcal{H})$  satisfies  $\mathfrak{m}_{10} = (-1)^{n-2} m_2^+$  with  $m_2^+ = \Pi(m_2 \otimes \mathbb{1})$ . Because  $\dim(\Omega)$  is not of finite type, Definition 3.4.1 does not give the canonical  $\text{dIBL}$ -structure on  $C(\Omega)$ , and hence we have neither  $\mathfrak{f}$  nor  $\mathfrak{n}$  in the standard sense.

In order to deduce the formal analogy, we embed  $\Omega(M)^{\otimes 2}$  into  $\Omega(\text{Bl}_{\Delta}(M \times M))$  using the external wedge product  $(\eta_1, \eta_2) \mapsto \tilde{\pi}_1^* \eta_1 \wedge \tilde{\pi}_2^* \eta_2$  and suppose that the Hodge propagator  $P$  satisfies  $P \in \Omega^{\otimes 2}$ . This never happens, so what follows is just a formal computation whose purpose is to deduce candidates for signs.

**Proposition A.2.1.** *In the de Rham case, suppose that  $P \in \Omega(M)^{\otimes 2}$ . Then (A.12) reduces to (4.19).*

*Proof.* Consider the intersection pairing  $\tilde{\Pi}$  and its degree shift  $\Pi$  (see Proposition 4.1.2). According to Definition A.1.4, they extend to pairings on  $\Omega(M)^{\otimes k}$  and  $\Omega(M)[1]^{\otimes k}$  for all  $k \geq 1$ , respectively. For all  $\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22} \in \Omega(M)$ , we have:

$$\begin{aligned} & \Pi(\theta^2 \eta_{11} \otimes \eta_{12}, \theta^2 \eta_{21} \otimes \eta_{22}) \\ &= (-1)^{\eta_{11} + \eta_{21}} \Pi(\theta \eta_{11} \otimes \theta \eta_{12}, \theta \eta_{21} \otimes \theta \eta_{22}) \\ &= (-1)^{\eta_{11} + \eta_{21} + (1 + \eta_{12})(1 + \eta_{21})} \Pi(\theta \eta_{11}, \theta \eta_{21}) \Pi(\theta \eta_{12}, \theta \eta_{22}) \\ &= (-1)^{1 + \eta_{12} \eta_{21}} \tilde{P}(\eta_{11}, \eta_{21}) \tilde{\Pi}(\eta_{12}, \eta_{22}) \\ &= -\tilde{\Pi}(\eta_{11} \otimes \eta_{12}, \eta_{21} \otimes \eta_{22}). \end{aligned} \quad (\text{A.13})$$

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One can also check that

$$\tilde{\Pi}(\eta_{11} \otimes \eta_{12}, \eta_{21} \otimes \eta_{22}) = \int_{x,y} \eta_{11}(x) \eta_{12}(y) \eta_{21}(x) \eta_{22}(y).$$

For the Hodge homotopy  $\mathcal{P} : \Omega(M) \rightarrow \Omega(M)$  and its Hodge propagator  $P \in \Omega(M)^{\otimes 2}$ , we have the following:

$$\forall \eta_1, \eta_2 \in \Omega(M) : \quad \tilde{\Pi}(\mathcal{P}(\eta_1), \eta_2) = \int_{x,y} P(x, y) \eta_1(x) \eta_2(y) = \tilde{\Pi}(P, \eta_1 \otimes \eta_2).$$

From this and (A.13), we obtain

$$\begin{aligned} \Pi(\mathcal{P}(\theta\eta_1), \eta_2) &= \Pi(\theta\mathcal{P}(\eta_1), \theta\eta_2) = (-1)^{1+\eta_1} \tilde{\Pi}(\mathcal{P}(\eta_1), \eta_2) \\ &= (-1)^{1+\eta_1} \tilde{\Pi}(P, \eta_1 \otimes \eta_2) = (-1)^{\eta_1} \Pi(\theta^2 P, \theta^2 \eta_1 \otimes \eta_2) \\ &= \Pi(\theta^2 P, \theta\eta_1 \otimes s\eta_2). \end{aligned}$$

Therefore, the element  $\theta^2 P \in V[1]^{\otimes 2}$  corresponds to the Schwartz kernel  $\mathcal{K}_{\mathcal{P}}$  of  $\mathcal{P} : V[1] \rightarrow V[1]$ . We write this correspondence as

$$\mathcal{K}_{\mathcal{P}} \in V[1]^{\otimes 2} \sim \theta^2 P \in \text{Bl}_{\Delta}(M \times M)[2].$$

Let us check that  $\theta^2 P$  satisfies (A.1). First of all, if we embed  $\Omega(M)^{\otimes k}$  into  $\Omega(M^{\times k})$  using the external wedge product  $\eta_1 \otimes \cdots \otimes \eta_k \mapsto \pi_1^* \eta_1 \wedge \cdots \wedge \pi_k^* \eta_k =: \eta_1(x_1) \wedge \cdots \wedge \eta_k(x_k)$ , then for all  $\eta_1, \dots, \eta_k \in \Omega(M)$  we have

$$\sigma(\eta_1 \otimes \cdots \otimes \eta_k)(x_1, \dots, x_k) = \eta_1(x_{\sigma_1}) \wedge \cdots \wedge \eta_k(x_{\sigma_k}),$$

where the action on the left-hand side is given by (3.8). Now, the symmetry property (4.7) implies

$$\tau(\theta^2 P) = -\theta^2 \tau^*(P) = (-1)^{n+1} \theta^2 P = (-1)^{|\theta^2 P|} \theta^2 P.$$

Therefore, the symmetry condition (A.1) is indeed satisfied.

Let  $\Gamma \in \overline{\text{RG}}_{klg}^{(3)}$ , and let  $L$  be a labeling of  $\Gamma$ . We abbreviate  $\sigma := \sigma_L \in \mathbb{S}_{3k}$ . Given  $\eta_{ij} \in \Omega(M)$  for  $j = 1, \dots, s_i$  and  $i = 1, \dots, l$ , where  $s_i$  is the valency of the  $i$ -th boundary component, we set  $\eta_i = \eta_{i1} \otimes \cdots \otimes \eta_{is_i}$ ,  $\eta = \eta_1 \otimes \cdots \otimes \eta_l$ ,  $\alpha_{ij} = \theta\eta_{ij}$ ,  $\omega_i = \alpha_{i1} \otimes \cdots \otimes \alpha_{is_i}$  and  $\omega = \omega_1 \otimes \cdots \otimes \omega_l$ . We denote  $s := s_1 + \cdots + s_l$ , so that  $3k = 2e + s$ , where  $e$  is the

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number of internal edges. We have

$$\begin{aligned} (m_2^+)^{\otimes k}(\sigma((\theta^2 \mathbf{P})^{\otimes e} \otimes \omega)) &= \varepsilon(\theta, \eta)(m_2^+)^{\otimes k}(\sigma((\theta^2 \mathbf{P})^{\otimes e} \otimes \theta^s \eta)) \\ &= (-1)^{se(n-1)} \varepsilon(\theta, \eta)(m_2^+)^{\otimes k}(\sigma(\theta^{2e+s} \mathbf{P}^{\otimes e} \otimes \eta)) \\ &= \underbrace{(-1)^{\sigma+se(n-1)}}_{=:\varepsilon_1} \varepsilon(\theta, \eta)(m_2^+)^{\otimes k}(\theta^{2e+s} \sigma(\mathbf{P}^{\otimes e} \otimes \eta)), \end{aligned}$$

where  $\varepsilon(\theta, \eta)$  is the Koszul sign to order  $\theta^s \eta_{11} \dots \eta_{l_{s_l}} \mapsto \theta \eta_{11} \dots \theta \eta_{l_{s_l}}$  and the operation  $m_2^+ : \Omega(M)[1]^{\otimes 3} \rightarrow \mathbb{R}$  is given by  $m_2^+ = \Pi(m_2 \otimes \mathbb{1})$ . We denote  $\kappa := P^{\otimes e} \otimes \eta = \kappa_1 \otimes \dots \otimes \kappa_{3k}$ ,  $\kappa_i \in \Omega(M)[1]$  and compute

$$\begin{aligned}
& (m_2^+)^{\otimes k} (\theta^{3k} \sigma(\kappa)) \\
&= \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes k} (\theta^{3k} \kappa_{\sigma_1^{-1}} \otimes \cdots \otimes \kappa_{\sigma_{3k}^{-1}}) \\
&\quad \uparrow \\
&|m_2^+|=3-n \\
&= (-1)^{\frac{1}{2}k(k-1)n} \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes k} (\theta^3 (\kappa_{\sigma_1^{-1}} \otimes \kappa_{\sigma_2^{-1}} \otimes \kappa_{\sigma_3^{-1}}) \otimes \cdots \\
&\qquad \qquad \qquad \otimes \theta^3 (\kappa_{\sigma_{3k-2}^{-1}} \otimes \kappa_{\sigma_{3k-1}^{-1}} \otimes \kappa_{\sigma_{3k}^{-1}})) \\
&\quad \underbrace{\hspace{10em}}_{=: \varepsilon_2} \\
&= (-1)^{\frac{1}{2}k(k-1)n + \kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \varepsilon(\sigma, \kappa) (m_2^+)^{\otimes k} ((\theta \kappa_{\sigma_1^{-1}} \otimes \theta \kappa_{\sigma_2^{-1}} \\
&\qquad \qquad \qquad \otimes \theta \kappa_{\sigma_3^{-1}}) \otimes \cdots \otimes (\theta \kappa_{\sigma_{3k-2}^{-1}} \otimes \theta \kappa_{\sigma_{3k-1}^{-1}} \otimes \theta \kappa_{\sigma_{3k}^{-1}})).
\end{aligned}$$

Next, using the formula (2) for  $m_2^+$ , we get

$$\begin{aligned}
& (m_2^+)^{\otimes k} ((\theta \kappa_{\sigma_1^{-1}} \otimes \theta \kappa_{\sigma_2^{-1}} \otimes \theta \kappa_{\sigma_3^{-1}}) \otimes \cdots \otimes (\theta \kappa_{\sigma_{3k-2}^{-1}} \otimes \theta \kappa_{\sigma_{3k-1}^{-1}} \otimes \theta \kappa_{\sigma_{3k}^{-1}})) \\
&= (-1)^{\kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \left( \int_{x_1} \kappa_{\sigma_1^{-1}}(x_1) \kappa_{\sigma_2^{-1}}(x_1) \kappa_{\sigma_3^{-1}}(x_1) \right) \cdots \\
&\quad \left( \int_{x_k} \kappa_{\sigma_{3k-2}^{-1}}(x_k) \kappa_{\sigma_{3k-1}^{-1}}(x_k) \kappa_{\sigma_{3k}^{-1}}(x_k) \right) \\
&= (-1)^{\kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \int_{x_1, \dots, x_k} \kappa_{\sigma_1^{-1}}(x_1) \kappa_{\sigma_2^{-1}}(x_1) \kappa_{\sigma_3^{-1}}(x_1) \cdots \\
&\quad \kappa_{\sigma_{3k-2}^{-1}}(x_k) \kappa_{\sigma_{3k-1}^{-1}}(x_k) \kappa_{\sigma_{3k}^{-1}}(x_k) \\
&= \overbrace{(-1)^{\kappa_{\sigma_2^{-1}} + \cdots + \kappa_{\sigma_{3k-1}^{-1}}} \varepsilon(\sigma, \kappa)}^{=: \varepsilon_3} \int_{x_1, \dots, x_k} \kappa_1(x_{\xi(\sigma_1)}) \kappa_2(x_{\xi(\sigma_2)}) \kappa_3(x_{\xi(\sigma_3)}) \cdots \\
&\quad \kappa_{3k-2}(x_{\xi(\sigma_{3k-2})}) \kappa_{3k-1}(x_{\xi(\sigma_{3k-1})}) \kappa_{3k}(x_{\xi(\sigma_{3k})}),
\end{aligned}$$

where  $\xi(3j-2) = \xi(3j-1) = \xi(3j) = j$  for  $j = 1, \dots, k$  (see Definition 4.3.6). In total,

## A. Evaluation of labeled ribbon graphs

we have

$$\begin{aligned} & (m_2^+)^{\otimes k}(\sigma((\theta^2 P)^{\otimes e} \otimes \omega)) \\ &= \varepsilon_1 \varepsilon_2 \varepsilon_3 \int_{x_1, \dots, x_k} P(x_{\xi(\sigma_1)}, x_{\xi(\sigma_2)}) \cdots P(x_{\xi(\sigma_{2e-1})}, x_{\xi(\sigma_{2e})}) \\ & \quad \alpha_{11}(x_{\xi(\sigma_{2e+1})}) \cdots \alpha_{l_{s_l}}(x_{\xi(\sigma_{2e+s})}), \end{aligned}$$

where

$$\varepsilon_1 \varepsilon_2 \varepsilon_3 = (-1)^{\sigma + se(n-1) + \frac{1}{2}k(k-1)n} \varepsilon(\theta, \eta).$$

Using (4.16) and (4.18) and  $\varepsilon(\theta, \eta) = (-1)^{P(\omega)}$ , we get the total sign

$$(-1)^{k(n-2)} \varepsilon_1 \varepsilon_2 \varepsilon_3 = (-1)^{s(k,l) + \sigma + P(\omega)},$$

where  $(-1)^{k(n-2)}$  is the artificial sign from (A.12). This proves the proposition.  $\square$

*Remark A.2.2* (Signs for the Fréchet dIBL-structure on  $\Omega(M)$ ). In [CFL15, Section 13], they consider the weight-graded nuclear Fréchet space  $B_{\text{cyc}}^* \Omega(M)_\infty \subset B_{\text{cyc}}^* \Omega(M)$  generated by  $\varphi \in B_{\text{cyc}}^* \Omega(M)$  which have a smooth Schwartz kernel  $k_\varphi \in \Omega(M^{\times k})$ ; they showed that there is a canonical Fréchet dIBL-structure on  $B_{\text{cyc}}^* \Omega(M)_\infty[2-n]$ . In order to deduce the signs, we can consider the subspace  $B_{\text{cyc}}^* \Omega(M)_{\text{alg}} \subset B_{\text{cyc}}^* \Omega(M)_\infty$  generated by  $\varphi \in B_{\text{cyc}}^* \Omega(M)$  with an algebraic Schwartz kernel  $\mathcal{K}_\varphi \in \Omega(M)[1]^{\otimes k}$ , rewrite (A.10) in terms of  $\mathcal{K}_\varphi$  and extend the obtained formulas to  $B_{\text{cyc}}^* \Omega(M)_\infty$ .  $\triangleleft$

Other de Rham formulas: What are the de Rham formulas for  $f_{klg}$ ,  $f_{klg}^m$ , and so on? What kind of Feynman graphs? See Page 99 in Diary I.

## B. Reduced cyclic homology of $A_\infty$ -algebras

In this appendix, we prove Proposition 3.3.13. The idea from [Lod92] is to resolve cyclic (co)invariants degree-wise and obtain certain bicomplexes with better properties.

In Section B.1, given a strictly unital  $A_\infty$ -algebra on a graded vector space  $V$ , we define the normalized and reduced Hochschild (co)chain complexes (Definition B.1.1) and prove the computational prerequisites CP1–CP4 (Lemmas B.1.4, B.1.5, B.1.6 and B.1.7); they are necessary for the development of the cyclic homology theory in the upcoming section. These prerequisites, like squaring to zero of the Hochschild differential, seem to be much harder computationally for  $A_\infty$ -algebras than for DGA's. Proofs of some of these relations in different formalisms appeared already in [Mes16] and [Laz03].

In Section B.2, we define Loday's and Connes' cyclic half-plane bicomplexes for  $A_\infty$ -algebras together with their normalized and reduced versions (Definition B.2.3). We then summarize some convergence results for spectral sequences associated to horizontal, vertical and diagonal filtrations (Proposition B.2.1). In a series of lemmas (Lemmas B.2.5, B.2.7, B.2.8 and B.2.9), we prove that some of the (co)homologies are isomorphic. These lemmas copy results for DGA's from [Lod92]; we just do them carefully for half-plane bicomplexes and more explicitly. There is a new phenomenon of long chains coming from completing the direct sum total complex; these long chains seem to disappear in homology if the degrees of  $V$  are bounded (Lemma B.2.6). Additionally, we point out some differences between first-quadrant and half-plane bicomplexes (Remark B.2.2) and mention the relation to mixed complexes (Remark B.2.4).

In Section B.3, we obtain short exact sequences for reduced Connes' bicomplexes (Lemma B.3.1), which replace, up to quasi-isomorphisms, the non-exact sequences for reduced cyclic Hochschild (co)chains. We summarize the isomorphisms of (co)homologies from Section B.2 (Figure B.3), finish the proof of Proposition 3.3.13 and formulate a few open question (Remark B.3.2).

## B.1. Computational prerequisites

The heart of cyclic (co)homology theory, following [Lod92], are the following five *computational prerequisites* (CP):

**CP0** (horizontal relations):  $\ker N = \text{im}(\mathbb{1} - t)$ ,  $\ker(\mathbb{1} - t) = \text{im } N$ ,

**CP1** (vertical relations):  $b \circ b = 0$ ,  $b' \circ b' = 0$ ,

**CP2** (vertical-horizontal relations):  $b' \circ N = N \circ b$ ,  $(\mathbb{1} - t) \circ b' = b \circ (\mathbb{1} - t)$ ,

and in the strictly unital case

**CP3** (null-homotopy of the bar resolution):  $b' \circ \iota_1 + \iota_1 \circ b' = \mathbb{1}$  and

**CP4** (contraction onto normalized chains):  $\bar{p} : DV \rightarrow \bar{D}V$  is a quasi-isomorphisms.

The definitions of  $t$  (cyclic permutation),  $b$  ( $A_\infty$ -Hochschild differential),  $b'$  (acyclic  $A_\infty$ -Hochschild differential) and  $DV$  ((reduced) bar complex with reversed grading shifted by one) can be found in Section 3.3; in particular, consult Definition 3.3.9. The new players are the *counting operator*

$$N := \sum_{k=1}^{\infty} \underbrace{\sum_{i=0}^{k-1} t_k^i}_{=: N_k} : DV \longrightarrow DV$$

and the projection  $\bar{p} : DV \rightarrow \bar{D}V$  to normalized Hochschild chains — this we define below.

**Definition B.1.1** (Normalized and reduced Hochschild complex). *Let  $(V, (\mu_j), \mathbb{1})$  be a strictly unital  $A_\infty$ -algebra. Let  $\bar{V}[1] := V[1]/\langle \mathbb{1} \rangle$ . We define the normalized Hochschild chain complex by*

$$\bar{D}V := \bigoplus_{l=0}^{\infty} V[1] \otimes \bar{V}[1]^{\otimes l}.$$

We consider the canonical projection  $\bar{p} : V[1] \rightarrow \bar{V}[1]$  and define  $\bar{p} : DV \rightarrow \bar{D}V$  by

$$\bar{p}|_{V[1]^{\otimes l}} := \mathbb{1} \otimes \underbrace{\bar{p} \otimes \cdots \otimes \bar{p}}_{l\text{-times}}.$$

For every  $l \geq 1$ , we define the operator  $\iota_l : DV \rightarrow DV$  by inserting  $\mathbb{1}$  at the  $l$ -th position of a tensor product, where the position  $l = 1$  is in front; i.e., we have

$$\iota_1(v_1 \otimes \cdots \otimes v_i) = \mathbb{1} \otimes v_1 \otimes \cdots \otimes v_i \quad \text{for all } v_j \in V[1] \text{ and } i \geq j \geq 1.$$

DONE Wrong def of normalized: The normalized complex is not properly defined!! The projection does not work.

## B. Reduced cyclic homology of A-infinity-algebras

We define the normalized Hochschild cochain complex by

$$\bar{D}^*V := \{\varphi \in D^*V \mid \varphi \circ \iota_l = 0 \text{ for all } l \geq 2\}.$$

If  $u : D\mathbb{R} \rightarrow DV$  is the unit map (in the strictly unital case,  $u|_{\mathbb{R}[1]^{\otimes k}} := u^{\otimes k}$  where  $u : \mathbb{R}[1] \rightarrow V[1]$  is the canonical injection; see also Definition 3.3.10 and the discussion below) and  $\bar{\iota} : \bar{D}^*V \rightarrow D^*V$  the inclusion, we define the reduced Hochschild chain and cochain complexes  $D^{\text{red}}V$  and  $D_{\text{red}}^*V$  by

$$D^{\text{red}}V := \text{coker}(\bar{p} \circ u) \quad \text{and} \quad D_{\text{red}}^*V := \ker(u^* \circ \bar{\iota}), \quad \text{respectively.}$$

We denote by  $p^{\text{red}} : DV \rightarrow D^{\text{red}}V$  and  $\iota_{\text{red}} : D_{\text{red}}^*V \rightarrow D^*V$  the canonical projection and inclusion, respectively.

All chain complexes above are graded by degree and equipped with a boundary operator induced naturally from  $b$  (see the remark below).

*Remark B.1.2* (Some details on normalized and reduced complexes). Since  $1$  is a unit for  $\mu_2$ , we have

$$\begin{aligned} 0 &= (-1)^{|v_1|+\dots+|v_{k-1}|} (v_1 \cdots \mu_2(v_k, 1) + (-1)^{|v_k|} \mu_2(1, v_1) \cdots v_k) \quad \text{and} \\ 0 &= (-1)^{|v_1|+\dots+|v_{i-2}|} (v_1 \cdots \mu_2(v_{i-1}, 1) v_i \cdots v_k + (-1)^{|v_{i-1}|} v_1 \cdots v_{i-1} \mu_2(1, v_i) \cdots v_k) \end{aligned}$$

for all  $i = 2, \dots, k$ . This fact and strict unitality implies

$$b\left(\sum_{i \geq 2} \text{im } \iota_i\right) \subset \sum_{i \geq 2} \text{im } \iota_i = \ker \bar{p}.$$

Therefore,  $b$  induces a differential on  $\bar{D}V$ . Since  $\bar{D}^*V = \{\varphi \in D^*V \mid \varphi(\sum_{i \geq 2} \text{im } \iota_i) = 0\}$ , the dual  $b^*$  restricts to  $\bar{D}^*V$ . Clearly, both  $\bar{p}$  and  $\bar{\iota}$  are chain maps, and they are compatible under the dualization from Definition 3.3.4; i.e.,  $\bar{\iota} \simeq \bar{p}^*$  under  $\bar{D}^*V \simeq (\bar{D}V)^{*g}$  and  $D^*V \simeq (DV)^{*g}$ , where  $^{*g}$  denotes the graded dual.

As for the reduced complexes,  $u$  is a chain map, and thus  $\ker$  and  $\text{coker}$  are chain complexes. Again, it holds  $D_{\text{red}}^*V \simeq (D^{\text{red}}V)^{*g}$  and  $\iota_{\text{red}} \simeq p^{\text{red},*}$  under the dualization.  $\triangleleft$

We will now prove CP1, CP2, CP3 and CP4 for strictly unital  $A_\infty$ -algebras. We do not prove CP0 because it is a standard fact which does not depend on the algebra we work with (see [Lod92]). A proof of CP1 in a slightly different notation and in a more general setting (coefficients in a bimodule) can also be found in [Mes16]. The proofs



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of CP2 and CP3 work in the same way as the proofs for DGA's from [Lod92]. The computation is just a little longer. As for CP4, we can not use the proof for DGA's from [Lod92, Proposition 1.6.5] anymore because we do not have a simplicial module; instead of this, we consider an explicit homotopy inspired by [Laz03], where CP4 is also proven in a slightly different notation.

We first introduce some notation which simplifies computations:

**Definition B.1.3** (Notation). *For the cyclic permutation  $t_k^i$  ( $:= i$ -times  $t_k$ ), we define  $c := k - i + 1$  and write*

$$v_c \cdots v_{c-1} := t_k^i(v_1 \otimes \cdots \otimes v_k).$$

*We compute indices modulo  $k$  and often omit writing the tensor product.*

*For every  $i = 1, \dots, k$  and  $j = 1, \dots, k - i + 1$ , we define the closed bracket by*

$$\begin{aligned} v_1 \cdots \overline{v_i \cdots v_{i+j-1}} \cdots v_k \\ := (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \otimes \cdots \otimes \mu_j(v_i \otimes \cdots \otimes v_{i+j-1}) \otimes \cdots \otimes v_k. \end{aligned} \tag{B.1}$$

*If we apply the closed bracket two-times, we write the first application as an underbracket and the second as an overbracket; for instance, we have*

$$\begin{aligned} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \underbrace{v_{i_3} \cdots v_{i_4}} \cdots v_k \\ = (-1)^{|v_{i_1}| + \cdots + |v_{i_3-1}|} v_1 \otimes \cdots \otimes \mu_{j_2}(v_{i_1} \otimes \cdots \otimes v_{i_2}) \otimes \cdots \\ \otimes \mu_{j_1}(v_{i_3} \otimes \cdots \otimes v_{i_4}) \otimes \cdots \otimes v_k, \end{aligned}$$

*where  $j_1 = i_4 - i_3 + 1$ ,  $j_2 = i_2 - i_1 + 1$ . Clearly, the difference is only in the sign. We denote*

$$v_1 \cdots v_{i-1} \bar{1} v_i \cdots v_k := \iota_i(v_1 \cdots v_k) = (-1)^{|v_1| + \cdots + |v_{i-1}|} v_1 \cdots v_{i-1} 1 v_i \cdots v_k.$$

*If  $\iota_i$  is composed with an other operation, we write  $\underline{1}$  if the corresponding 1 was inserted first and  $\bar{1}$  if it was inserted second. For example, we have*

$$v_1 \underline{1} v_2 v_3 \bar{1} v_4 = \iota_5(\iota_2(v_1 v_2 v_3 v_4)).$$

*For  $j \geq 1$  and  $1 \leq i_1 \leq i_2 \leq k$  with  $i_2 - i_1 \geq j$ , we define the open bracket as follows:*

$$v_1 \cdots \overbrace{v_{i_1} \cdots v_{i_2}}^j \cdots v_k := \sum_{\substack{i_1 \leq i_3 \leq i_4 \leq i_2 \\ i_4 - i_3 = j}} v_1 \cdots v_{i_1} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_{i_2} \cdots v_k.$$

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Using the notation above, it holds

$$\begin{aligned} b'(v_1 \cdots v_k) &= \sum_{1 \leq i_1 \leq i_2 \leq k} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots v_k = \sum_{j=1}^k \frac{j}{v_1 \cdots v_k}, \\ R(v_1 \cdots v_k) &= \sum_{2 \leq c \leq k} \overline{v_c \cdots v_1} \cdots v_{c-1}, \end{aligned}$$

and the  $A_\infty$ -relations simplify to

$$\sum_{1 \leq i_1 \leq i_2 \leq k} \overline{v_1 \cdots v_{i_1} \cdots v_{i_2} \cdots v_k} = \sum_{j=1}^k \frac{j}{v_1 \cdots v_k} = 0. \quad (\text{B.2})$$

Because all signs are, in fact, Koszul signs for the symbols  $\mu_{j_1}, \mu_{j_2}, v_1, \dots, v_k$ , and because  $\mu$ 's have odd degree, we have for every  $1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq k$  the following relation:

$$v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k + v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots v_{i_3} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k = 0. \quad (\text{B.3})$$

**Lemma B.1.4** (CP1). *For an  $A_\infty$ -algebra  $(V, (\mu_j))$ , it holds*

$$b' \circ b' = 0 \quad \text{and} \quad b \circ b = 0.$$

*Proof.* We write

$$b \circ b = (b' + R) \circ (b' + R) = b' \circ b' + b' \circ R + R \circ b' + R \circ R$$

and evaluate it on a tensor  $v_1 \cdots v_k \in DV$ . We claim that a summand of  $b(b(v_1 \cdots v_k))$  coming from the subsequent application of the operations can be uniquely determined by the following data:

- the information whether it comes from  $b' \circ b'$ ,  $b' \circ R$ ,  $R \circ b'$  or  $R \circ R$ ;
- a cyclic permutation  $c$  of  $v_1, \dots, v_k$ ;
- positions of the under- and upperbracket.

The reason for this is that both  $b'$  and  $R$  produce only Koszul signs, and hence the total sign of a summand in  $b(b(v_1 \cdots v_k))$  is the Koszul sign for the symbols  $\mu_{j_1}, \mu_{j_2}, v_1, \dots, v_k$ , which depends only on the start and final position of the symbols; this is precisely encoded in the data above.

## B. Reduced cyclic homology of A-infinity-algebras

For  $c = 1$ , only  $b' \circ b'$  contributes; however, using (B.2) and (B.3), we get

$$\begin{aligned}
 (b' \circ b')(v_1 \cdots v_k) &= \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq k} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2} \cdots v_{i_3} \cdots v_{i_4}} \cdots v_k \\
 &+ \sum_{\substack{1 \leq i_1 \leq i_2 \leq k-1 \\ i_2+1 \leq i_3 \leq i_4 \leq k}} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k \\
 &+ \sum_{\substack{2 \leq i_3 \leq i_4 \leq k \\ 1 \leq i_1 \leq i_2 \leq i_3-1}} v_1 \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_k \\
 &= 0.
 \end{aligned}$$

Let  $c \geq 2$  and consider the summands from  $b' \circ R$ ,  $R \circ b'$  and  $R \circ R$  based on  $v_c \cdots v_{c-1}$ . The contribution of  $R \circ b'$  consists of the following three parts:

$$\begin{aligned}
 \text{I.} \quad & \sum_{\substack{c \leq i_1 \leq i_2 \leq k \\ 1 \leq i_4 \leq c-1}} \overline{v_c \cdots v_{i_1} \cdots v_{i_2} \cdots v_k v_1 \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}, \\
 \text{II.} \quad & \sum_{1 \leq i_1 \leq i_2 \leq i_4 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_1} \cdots v_{i_2} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}, \\
 \text{III.} \quad & \sum_{1 \leq i_4 < i_1 \leq i_2 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_4}} v_{i_4+1} \cdots \overline{v_{i_1} \cdots v_{i_2}} \cdots v_{c-1}.
 \end{aligned}$$

The contribution of  $b' \circ R$  consists of the following two parts:

$$\begin{aligned}
 \text{IV.} \quad & \sum_{1 \leq i_2 \leq i_4 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_2} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}, \\
 \text{V.} \quad & \sum_{1 \leq i_2 < i_3 \leq i_4 \leq c-1} \overline{v_c \cdots v_k v_1 \cdots v_{i_2}} v_{i_2+1} \cdots \overline{v_{i_3} \cdots v_{i_4}} \cdots v_{c-1}.
 \end{aligned}$$

The contribution of  $R \circ R$  is:

$$\text{VI.} \quad \sum_{\substack{c < i_1 \leq k \\ 1 \leq i_2 \leq i_4 \leq c-1}} \overline{v_c \cdots v_{i_1} \cdots v_k v_1 \cdots v_{i_2} \cdots v_{i_4}} v_{i_4+1} \cdots v_{c-1}.$$

Using (B.3), it is easy to see that III cancels with V. The sum of the other terms I, II, IV, VI vanishes for fixed  $1 \leq i_4 \leq c-1$  due to (B.2).  $\square$

**Lemma B.1.5** (CP2). *For an  $A_\infty$ -algebra  $(V, (\mu_j))$ , the following relations hold:*

- (a)  $b' \circ N = N \circ b$ ,
- (b)  $(\mathbb{1} - t) \circ b' = b \circ (\mathbb{1} - t)$ .

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*Proof.* (a) We denote  $z_j := \mu_j \otimes \mathbb{1}^{k-j}$  and omit writing the composition  $\circ$ . We consider the components

$$b'_k{}^j := \sum_{i=0}^{k-j} t_{k-j+1}^i z_j t_k^{-i} \quad \text{and} \quad R_k^j := \sum_{i=1}^{j-1} z_j t_k^i.$$

It holds

$$\begin{aligned} b'_k{}^j N_k &= \sum_{l=0}^{k-1} \sum_{i=0}^{k-j} t_{k-j+1}^i z_j t_k^{-i+l} \\ &= \sum_{l=0}^{k-1} \sum_{i=0}^{k-j} t_{k-j+1}^l t_{k-j+1}^{i-l} z_j t_k^{-(i-l)} \\ &= \sum_{u=1-k}^{k-j} \left( \sum_{l \in L_u} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u}, \end{aligned}$$

where  $u := i - l$  and

$$L_u := \{l \in \{0, \dots, k-1\} \mid \exists i \in \{0, \dots, k-j\} : u = i - l\}.$$

We distinguish the cases

$$L_u = \begin{cases} \{0, \dots, k-j-u\} & \text{for } 0 \leq u \leq k-j, \\ \{-u, \dots, k-j-u\} & \text{for } 1-j \leq u \leq -1 \\ \{-u, \dots, k-1\} & \text{for } 1-k \leq u \leq -j \end{cases} \quad \text{and}$$

and denote the corresponding sums by I, II and III, respectively. It holds

$$\text{I} = \sum_{u=0}^{k-j} \left( \sum_{l=0}^{k-j-u} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u}$$

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and

$$\begin{aligned}
 \text{III} &= \sum_{u=1-k}^{-j} \sum_{l=-u}^{k-1} t_{k-j+1}^l t_{k-j+1}^u z_j t_k^{-u} \\
 &= \sum_{u=1-k}^{-j} \sum_{l=-u}^{k-1} t_{k-j+1}^l t_{k-j+1}^{u+k-j+1} z_j t_k^{-u-k} \\
 &= \sum_{u=1}^{k-j} \sum_{l=k-u}^{k-1} t_{k-j+1}^{l-j+1} t_{k-j+1}^u z_j t_k^{-u} \\
 &= \sum_{u=1}^{k-j} \left( \sum_{l=k-j-u+1}^{k-j} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u}.
 \end{aligned}$$

Therefore, we have

$$\text{I} + \text{III} = \sum_{u=0}^{k-j} \left( \sum_{l=0}^{k-j} t_{k-j+1}^l \right) t_{k-j+1}^u z_j t_k^{-u} = N_{k-j+1} b_k^j$$

Next, we have

$$\text{II} = \sum_{u=1-j}^{-1} \sum_{l=-u}^{k-j-u} t_{k-j+1}^l t_{k-j+1}^u z_j t_k^{-u} = \sum_{u=1}^{j-1} \left( \sum_{l=0}^{k-j} t_{k-j+1}^l \right) z_j t_k^u = N_{k-j+1} R_k^j.$$

We conclude that

$$b_k^j N_k = N_{k-j+1} b_k^j + N_{k-j+1} R_k^j = N_{k-j+1} b_k.$$

This proves the claim.

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(b) For every  $k \geq 1$  and  $1 \leq j \leq k$  we compute

$$\begin{aligned}
 (\mathbb{1} - t)b_k^j &= b_k^j - \sum_{i=0}^{k-j} t_{k-j+1}^{i+1} z_j t_k^{-(i+1)} t_k \\
 &= b_k^j - \sum_{i=1}^{k-j+1} t_{k-j+1}^i z_j t_k^{-i} t_k \\
 &= b_k^j - \sum_{i=0}^{k-j} t_{k-j+1}^i z_j t_k^{-i} t_k + t_{k-j+1}^0 z_j t_k^{-0} t_k - t_{k-j+1}^{k-j+1} z_j t_k^{-k+j-1} t_k \\
 &= b_k^j (\mathbb{1} - t_k) + z_j t_k - z_j t_k^j \\
 &= b_k^j (\mathbb{1} - t_k) + \sum_{i=1}^{j-1} z_j t_k^i (\mathbb{1} - t_k) \\
 &= (b_k^j + R_k^j) (\mathbb{1} - t_k).
 \end{aligned}$$

This proves the claim.  $\square$

**Lemma B.1.6** (CP3). *For a strictly unital  $A_\infty$ -algebra  $(V, (\mu_k), \mathbb{1})$ , it holds*

$$b' \circ \iota_1 + \iota_1 \circ b' = \mathbb{1}.$$

*Proof.* For any  $k \geq 1$  and  $v_1, \dots, v_k \in V[1]$ , we compute

$$\begin{aligned}
 b' \iota_1(v_1 \dots v_k) + \iota_1 b'(v_1 \dots v_k) &= \overline{\mathbb{1} v_1} v_2 \dots v_k + \overline{\mathbb{1} v_1 \dots v_k} + \overline{\mathbb{1} v_1 \dots v_k} \\
 &= v_1 \dots v_k.
 \end{aligned}$$

This proves the claim.  $\square$

**Lemma B.1.7** (CP4). *Let  $\mathcal{A} = (V, (\mu_k), \mathbb{1})$  be a strictly unital  $A_\infty$ -algebra. For all  $k \geq 2$ , we define  $h_k : DV \rightarrow DV$  by*

$$h_k := \iota_k \circ b + b \circ \iota_k + \mathbb{1}.$$

*Then the formulas*

$$\begin{aligned}
 s^* &:= s_2^* + s_3^* \circ h_2^* + s_4^* \circ h_3^* \circ h_2^* + \dots \quad \text{and} \\
 h^* &:= \dots \circ h_k^* \circ \dots \circ h_2^*
 \end{aligned}$$

*define homogenous linear maps  $s^*$  and  $h^* : D^*V \rightarrow D^*V$  of degrees 1 and 0, respectively.*

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The map  $h^*$  is a projection onto  $\bar{D}^*V$ , and the following homotopy relation holds:

$$s^* \circ b^* + b^* \circ s^* = h^* - \mathbb{1}. \quad (\text{B.4})$$

It implies that  $\bar{\iota}$  and  $\bar{p}$  are quasi-isomorphisms.

*Proof.* We set  $\bar{D}_{(1)}^*V := D^*V$ , and for all  $k \geq 2$ , we define

$$\bar{D}_{(k)}^*V := \{\psi \in D^*V \mid \psi \circ \iota_i = 0 \text{ for all } i = 2, \dots, k\}.$$

We will show first that  $h_k^*$  restricts to a projection  $\bar{D}_{(k-1)}^*V \rightarrow \bar{D}_{(k)}^*V$ . Let  $i \geq 1$  and  $v_1, \dots, v_i \in V[1]$ . We make the following computations:

(1) For  $i < k - 1$ , we have

$$(\iota_i b + b \iota_i)(v_1 \dots v_i) = 0$$

by the definition of  $\iota_i k$  and by the fact that  $b$  does not increase weights.

(2) For  $i = k - 1$ , we have

$$\begin{aligned} & (\iota_i b + b \iota_i)(v_1 \dots v_i) \\ &= \frac{v_1 \dots v_i \bar{1}}{1} + \frac{1}{v_1 \dots v_i \underline{1}} + \sum_{j=2}^i \frac{j}{v_1 \dots v_i \underline{1}} + v_1 \dots \overline{v_i \underline{1}} + \underline{1} v_1 \dots v_i \\ &= \sum_{j=2}^i \frac{j}{v_1 \dots v_i \underline{1}}. \end{aligned}$$

Notice that  $\bar{1}$  in the result is at positions  $< k$ .

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(3) For  $i > k - 1$ , we have

$$\begin{aligned}
& (\iota_i b + b \iota_i)(v_1 \dots v_i) \\
&= \sum_{j=1}^{i-k+2} \frac{v_1 \dots v_{k+j-2} \bar{1} v_{k+j-1} \dots v_i}{j} + \sum_{j=1}^{i-k+1} \frac{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i}{j} \\
&+ \sum_{m=1}^{i-k+1} \sum_{c=m+k-1}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2} \bar{1} v_{m+k-1} \dots v_{c-1} \\
&+ \sum_{j=1}^{k-1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i} + \sum_{j=1}^{i-k+1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i} \\
&+ v_1 \dots \overline{v_{k-1} \bar{1} v_k} \dots v_i + v_1 \dots v_{k-1} \bar{1} v_k \dots v_i \\
&+ \sum_{m=1}^{k-1} \sum_{c=k}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1} \bar{1} v_k \dots v_{c-1} \\
&= \sum_{j=1}^{i-k+2} \frac{v_1 \dots v_{k+j-2} \bar{1} v_{k+j-1} \dots v_i}{j} + \sum_{j=1}^{k-1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i} \\
&+ \sum_{m=1}^{i-k+1} \sum_{c=m+k-1}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2} \bar{1} v_{m+k-1} \dots v_{c-1} \\
&+ \sum_{m=1}^{k-1} \sum_{c=k}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1} \bar{1} v_k \dots v_{c-1} \\
&= \overbrace{\sum_{j=2}^{i-k+2} \frac{v_1 \dots v_{k+j-2} \bar{1} v_{k+j-1} \dots v_i}{j}}^{=:I} + \overbrace{\sum_{j=2}^{k-1} \frac{j}{v_1 \dots v_{k-1} \bar{1} v_k \dots v_i}}^{=:II} \\
&+ \overbrace{\sum_{m=2}^{i-k+1} \sum_{c=m+k-1}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{m+k-2} \bar{1} v_{m+k-1} \dots v_{c-1}}^{=:III} \\
&+ \overbrace{\sum_{m=2}^{k-1} \sum_{c=k}^i \overline{v_c \dots v_i v_1 \dots v_m} v_{m+1} \dots v_{k-1} \bar{1} v_k \dots v_{c-1}}^{=:IV}
\end{aligned}$$

Notice that  $\bar{1}$  is at the  $k$ -th position in I and III, whereas at positions  $< k$  in II and IV.

Let  $k \geq 2$ , and let  $\psi \in \bar{D}_{(k-1)}^* V$ . In order to show that  $\iota_j^* h_k^* \psi = 0$  for  $2 \leq j \leq k$ , let  $i \geq j$ , and let  $v_1, \dots, v_i \in V[1]$  be such that  $v_j = \bar{1}$ . Clearly,  $\psi(\text{II}) = \psi(\text{IV}) = 0$  for any  $v$ 's. As for III, the vector  $v_j = \bar{1}$  lies either inside  $\mu_j$  with  $j \geq 3$  or at a position



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$< k$ . It follows that  $\psi(\text{III}) = 0$ . As for I, we write

$$\text{I} = \underbrace{\frac{v_1 \cdots v_k \bar{1} v_{k+1} \cdots v_i}{2}}_{\text{Ia}} + \underbrace{\sum_{j \geq 3}^{i-k+2} \frac{v_1 \cdots v_{k+j-2} \bar{1} v_{k+j-1} \cdots v_i}{j}}_{\text{Ib}}.$$

It holds  $\psi(\text{Ib}) = 0$ . For  $2 \leq j < k$ , it holds

$$\begin{aligned} \psi(\text{Ia}) &= \psi(v_1 \cdots \underbrace{v_{j-1} \bar{1}}_{j-1} v_{j+1} \cdots v_k \bar{1} v_{k+1} \cdots v_i) \\ &\quad + \psi(v_1 \cdots v_{j-1} \bar{1} \underbrace{v_{j+1}}_{j+1} \cdots v_k \bar{1} v_{k+1} \cdots v_i) \\ &= 0, \end{aligned}$$

whereas for  $j = k$ , we have

$$\psi(v_1 \cdots \underbrace{v_{k-1} \bar{1}}_{k-1} v_{k+1} \cdots v_i) = -\psi(v_1 \cdots v_i).$$

It follows that

$$h_k^* \psi(v_1 \cdots v_i) = \psi(v_1 \cdots v_i) + \psi((\iota_i b + b \iota_i)(v_1 \cdots v_i)) = 0. \quad (\text{B.5})$$

Therefore, we have  $h_k^* \psi \in \bar{D}_{(k)}^* V$ . If  $\psi \in \bar{D}_{(k)}^* V$ , then clearly  $h_k^*(\psi) = \psi$ . Consequently,  $h_k^*$  is a projection  $h_k^* : \bar{D}_{(k-1)}^* V \rightarrow \bar{D}_{(k)}^* V$ .

For  $k \geq 2$ , we define

$$\begin{aligned} {}^k s^* &:= \iota_2^* + \iota_3^* \circ h_2^* + \cdots + \iota_k^* \circ h_{k-1}^* \circ \cdots \circ h_2^*, \\ {}^k h^* &:= h_k^* \circ \cdots \circ h_2^*. \end{aligned}$$

Let  $\mathcal{F}_{\mathbf{w}}^n DV = \bigoplus_{k=1}^n D_k V$  be the filtration of  $DV$  by weights. For all  $k \geq n+1$  it holds

$${}^k h^* \psi \Big|_{\mathcal{F}_{\mathbf{w}}^n DV} = {}^{n+1} h^* \psi \quad \text{and} \quad {}^k s^* \psi \Big|_{\mathcal{F}_{\mathbf{w}}^n DV} = {}^{n+1} s^* \psi.$$

It follows that  $h^*, s^* : D^* V \rightarrow D^* V$  are well-defined and that  $h^*$  is a projection onto  $\bar{D}^* V$ . Also, it suffices to prove (B.4) with  ${}^k s^*$  and  ${}^k h^*$  instead of  $s^*$  and  $h^*$  for each  $k \geq 2$ . For  $k = 2$ , it holds by definition. Suppose that (B.4) holds for some  $k \geq 2$ .

Then

$$\begin{aligned}
 & b^* \circ ({}^{k+1}s^*) + ({}^{k+1}s^*) \circ \mathbb{1} \\
 &= b^* \circ ({}^k s^*) + ({}^k s^*) \circ b^* + b^* \circ \iota_{k+1}^* \circ h_k^* \circ \cdots \circ h_2^* + \iota_{k+1}^* \circ h_k^* \circ \cdots \circ h_2^* \circ b^* \\
 &= {}^k h^* - \mathbb{1} + (b^* \circ \iota_{k+1}^* + \iota_{k+1}^* \circ b^*) \circ h_k^* \circ \cdots \circ h_2^* \\
 &= {}^k h^* - \mathbb{1} + (h_{k+1}^* - \mathbb{1}) \circ h_k^* \circ \cdots \circ h_2^* \\
 &= {}^{k+1} h^* - \mathbb{1}.
 \end{aligned}$$

The lemma is finally proven.  $\square$

## B.2. Homological algebra of bicomplexes

We will consider homological and cohomological half-plane bicomplexes, which we depict as

$$\begin{array}{ccccc}
 & \downarrow & & \downarrow & & \downarrow \\
 & B_{q,p+2} & \longleftarrow & B_{q+1,p+2} & \longleftarrow & B_{q+2,p+2} & \longleftarrow \\
 & \downarrow & & \downarrow & & \downarrow \\
 B : & B_{q,p+1} & \longleftarrow & B_{q+1,p+1} & \longleftarrow & B_{q+2,p+1} & \longleftarrow \\
 & \downarrow & & \downarrow & & \downarrow \\
 & B_{q,p} & \longleftarrow & B_{q+1,p} & \longleftarrow & B_{q+2,p} & \longleftarrow \\
 & \downarrow & & \downarrow & & \downarrow
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow \\
 & B^{q,p+2} & \longrightarrow & B^{q+1,p+2} & \longrightarrow & B^{q+2,p+2} & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow \\
 B^* : & B^{q,p+1} & \longrightarrow & B^{q+1,p+1} & \longrightarrow & B^{q+2,p+1} & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow \\
 & B^{q,p} & \longrightarrow & B^{q+1,p} & \longrightarrow & B^{q+2,p} & \longrightarrow \\
 & \uparrow & & \uparrow & & \uparrow
 \end{array} ,$$

respectively. The standard convention is that the squares anticommute (see [Lod92])!

We consider the *total complexes*  $(\text{Tot}_I(B), \partial)$  and  $(\text{Tot}_{II}(B), \partial)$ , where for all  $q \in \mathbb{Z}$ ,

DONE Switch p q: Switch p and q! There is more mistakes. It should be just  $B_{q,p} \mapsto B_{q,p}^*$  in the picture, not a change of numbering

DONE Why completion arises for half-plane: Give here the example with the snake of  $\mathbb{R}$ 's.

DONE Do bicomplexes commute or anticommute?: According to our convention, the squares anti-commute. Boardmann says they should anti-commute!!!

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the chain groups are defined by

$$(\mathrm{Tot}_I B)_q := \bigoplus_{i+j=q} B_{i,j}, \quad \text{and} \quad (\mathrm{Tot}_{II} B)_q := \prod_{i+j=q} B_{i,j},$$

respectively, and where  $\partial = \partial_v + \partial_h$  is the total boundary operator consisting of the vertical and horizontal boundary operators  $\partial_v$  and  $\partial_h$ , respectively. The homologies of  $\mathrm{Tot}_I$  and  $\mathrm{Tot}_{II}$  are denoted by  $HB$  and  $H\hat{B}$ , respectively. We proceed similarly in cohomology.

For each  $B$  and  $B^*$ , we consider the vertical and horizontal filtrations which are defined in such a way that they are preserved by all the arrows and that the  $k$ -th group contains the  $k$ -th column and the  $k$ -th row, respectively. More precisely, the vertical filtration  $\mathcal{F}_v^k B$  of  $B$  consists of the columns  $0, \dots, k$ , whereas the vertical filtration  $\mathcal{F}_v^k B^*$  of  $B^*$  consists of the columns  $k+1, k+2, \dots$ ; the horizontal filtration  $\mathcal{F}_h^k B$  of  $B$  consists of the rows  $k, k-1, \dots$ , whereas the horizontal filtration  $\mathcal{F}_h^k B^*$  of  $B^*$  consists of the rows  $k, k+1, \dots$ , and so on.

In order to check that morphisms of bicomplexes induce quasi-isomorphisms of total complexes, we will use the techniques of spectral sequences. Because we work with half-plane bicomplexes, our spectral sequences do not lie in the first quadrant, as in [Wei94], and the notion of conditional convergence from [Boa99] comes in handy. In the following, we recall some basic theory and formulate a proposition about the convergence of some unbounded spectral sequences.

A *cohomological spectral sequence* is a collection  $E_r$  of bigraded vector spaces and differentials  $d_r : E_r^{\bullet, \bullet} \rightarrow E_r^{\bullet+r, \bullet-r+1}$  for  $r \in \mathbb{N}$  such that  $E_{r+1} = H(E_r, d_r)$ . Let  $(C^*, d)$  be a cochain complex with a decreasing filtration  $\mathcal{F}^s C^*$  (it has to be graded and preserved by  $d$ ). For every  $s \in \mathbb{Z}$ , the short exact sequence

$$0 \longrightarrow \mathcal{F}^{s+1} C^* \xrightarrow{i} \mathcal{F}^s C^* \xrightarrow{j} \mathrm{gr}_s(C^*) := \mathcal{F}^s C^* / \mathcal{F}^{s+1} C^* \longrightarrow 0$$

induces the long exact sequence

$$\dots \longrightarrow H^\bullet(\mathcal{F}^{s+1} C^*) \xrightarrow{i} H^\bullet(\mathcal{F}^s C^*) \xrightarrow{j} H^\bullet(\mathrm{gr}_s C^*) \xrightarrow{\delta} H^{\bullet+1}(\mathcal{F}^{s+1} C^*) \longrightarrow \dots,$$

DONE Remark about half-plane bi: Make a remark why half-plane happen in A-infinity, example of the snake where the spectral sequence does not compute the homology properly, make a remark about dualizations. These are the main problems.

DONE Labels on arrows: Add labels  $i, j, \delta$  to arrows!

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which wraps into the exact couple of bigraded vector spaces

$$\begin{array}{ccc}
 A_1 := \bigoplus_{s \in \mathbb{Z}} H(\mathcal{F}^s C^*)[s] & \xrightarrow{\quad i \quad} & A_1 \\
 & \nwarrow \delta \quad \nearrow j & \\
 & E_1 := \bigoplus_{s \in \mathbb{Z}} H(\mathrm{gr}_s C^*)[s]. &
 \end{array}$$

This is the so called *geometric grading* convention.<sup>1</sup> We also define  $A_1^s := H(\mathcal{F}^s C^*)$  and  $E_1^s := H(\mathrm{gr}_s C^*)$ . By deriving this triangle (see, e.g., [Cie13]), one obtains a spectral sequence  $E_r$  associated to the filtration. One defines the  $E_\infty$  page (see [Boa99]) and studies the convergence of  $E_r$  to a filtered group  $G$ . In order to formulate this, one considers the limit  $A^\infty := \lim_s A^s$ , the colimit  $A^{-\infty} := \mathrm{colim}_s A^s$  and the right derived module for the limit  $RA^\infty := \lim_s^1 A^s$ . We will use the following notions of convergence:

1. *Strong convergence to a filtered group  $G$*  :  $\iff$  For each  $s \in \mathbb{Z}$ , we have  $\mathrm{gr}_s G \simeq E_\infty^s / E_\infty^{s+1}$  and the filtration on  $G$  is exhaustive, Hausdorff and complete (i.e.,  $G^\infty = 0$ ,  $G^{-\infty} = G$  and  $RG^\infty = 0$  for  $G^s := \mathcal{F}^s G$ ).
2. *Conditional convergence to the colimit  $G := A^{-\infty}$*  :  $\iff$  It holds  $A^\infty = 0$  and  $RA^\infty = 0$ .

Note that neither notion implies, in general, the other (see (b) of Remark B.2.2).

**Proposition B.2.1** (Convergence of certain unbounded spectral sequences). *The following statements about convergence of spectral sequences hold:*

- (a) *For any  $\mathbb{Z}$ -graded cochain complex  $(C^*, d)$  with the canonical filtration  $\mathcal{F}_{\mathrm{can}}^k C^* := \bigoplus_{i \geq k} C^i$ , the associated spectral sequence converges strongly and conditionally to the colimit  $H(C^*)$ .*
- (b) *The spectral sequence associated to the total complex  $\mathrm{Tot}_I B^*$  of a cohomological half-plane bicomplex  $B^*$  with the filtration induced from the horizontal filtration  $\mathcal{F}_h^k B^* = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \geq k} B^{ij}$  converges strongly to the colimit  $H(\mathrm{Tot}_I B^*)$ .*
- (c) *The spectral sequence associated to the total complex  $\mathrm{Tot}_I B^*$  of a cohomological half-plane bicomplex  $B^*$  with the filtration induced from the diagonal filtration*

$$\mathcal{F}^k B^* = \bigoplus_{j-i > k} B^{ij} \oplus \bigoplus_{i \in \mathbb{N}_0} Z^{i, k+i},$$

---

<sup>1</sup>It is chosen such that  $E_1^{sd} = H(B^{sd}, d_v)$  for the vertical filtration.

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where  $Z^{ij} \subset B^{ij}$  is such that  $d_h Z^{ij} = 0$  and  $d_h B^{i-1j} \subset Z^{ij}$ , converges strongly to the colimit  $H(\text{Tot}_I B^*)$ .

- (d) The spectral sequence associated the total complex  $\text{Tot}_I B^*$  of a cohomological half-plane bicomplex  $B^*$  with the filtration induced from the vertical filtration  $\mathcal{F}_v^k B^* = \bigoplus_{i \geq k} \bigoplus_{j \in \mathbb{Z}} B^{ij}$  converges conditionally to the colimit  $H(\text{Tot}_I B^*)$ .

The following statements about morphisms of spectral sequences hold:

- (e) Let  $f$  be a morphism of filtered complexes of types (a), (b) or (c). If it induces an isomorphism of  $E_r$  for some  $r$ , then it induces an isomorphism of the target groups.
- (f) Let  $f$  be a morphism of filtered complexes of type (d). If it induces an isomorphism of  $E_r$  for some  $r$ , then it induces an isomorphism of the target groups.

*Proof.* (a) Strong convergence follows from (b) by embedding  $C^*$  as the first column of  $B^*$ . Also, the computation there show that  $A^{-\infty} = H(C^*)$  and  $A^\infty = 0$ . The condition  $RA^\infty = 0$  is equivalent to the degreewise completeness of the filtration, which is true as the filtration is degreewise trivial. This shows the conditional convergence.

- (b) Let us compute the first page with the geometrical bigrading:

$$\begin{aligned} E_1^{sd} &= H(\text{gr}_s \text{Tot}_I B^*, d_h)[s]^d \\ &= H^{d+s}(\text{Tot}_I(s\text{-th row of } B^*), d_h) \\ &= H(B^{d,s}, d_h). \end{aligned}$$

We want to use the following theorem:

*Theorem B.2.1* ([Boa99, Theorem 6.1]). Suppose that  $E^s = 0$  for all  $s > 0$  and  $A^{-\infty} = 0$ . Then the spectral sequence converges strongly to the colimit  $A^\infty$ .

The proof can be done degreewise (see [Rog19]), and it can be shown that Theorem B.2.1 generalizes appropriately under the weaker assumption of “exiting differentials”. This means that the pages occupy a half-plane and for any fixed  $(s, d)$ , all but finitely many differentials  $d_r : E_r^{sd} \rightarrow E_r^{s+r, d-r+1}$  leave the half-plane (and thus vanish). In our case,  $E_r^{sd}$  occupy the half-plane  $\{(s, d) \mid d \geq 0\}$ , and because  $d - r + 1 \rightarrow -\infty$  as  $r \rightarrow \infty$ , the condition of exiting differentials is satisfied.

We still have to check that  $A^\infty = 0$  and compute  $A^{-\infty}$ . Because the colimit is an

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exact functor, it commutes with  $H$ , and we have

$$A^{-\infty} = \operatorname{colim}_s H(\mathcal{F}^s \operatorname{Tot}_I B^*) = H(\operatorname{colim}_s \mathcal{F}^s \operatorname{Tot}_I B^*) = H\left(\bigcup_s \mathcal{F}^s \operatorname{Tot}_I B^*\right) = H(\operatorname{Tot}_I B^*).$$

We used here that  $\mathcal{F}$  is exhaustive. The limit  $A^\infty$  can be represented as

$$A^\infty \simeq \left\{ ([a_s]) \in \prod_{s \in \mathbb{Z}} A^s \mid [a_{s+1}] \mapsto [a_s] \right\},$$

where  $H(\mathcal{F}^{s+1} \operatorname{Tot}_I B^*) \rightarrow H(\mathcal{F}^s \operatorname{Tot}_I B^*)$  is induced by the inclusion  $\mathcal{F}^{s+1} \operatorname{Tot}_I B^* \hookrightarrow \mathcal{F}^s \operatorname{Tot}_I B^*$ . Pick  $s_0 \in \mathbb{Z}$  and consider  $[a_{s_0}] \in A^{s_0}$  with a fixed representative  $a_{s_0} \in \mathcal{F}^{s_0} B^*$ . Because the cohomological degrees of  $a_{s_0}$  are bounded, let's say that  $d_0 \in \mathbb{Z}$  is an upper bound, and the filtration is degreewise bounded from below, there is an  $s_1 \geq s_0$  such that  $\mathcal{F}^s \operatorname{Tot}_I B^* \cap (\operatorname{Tot}_I B^*)^d = 0$  for all  $s \geq s_1$  and  $d \leq d_0$ . Now, we have  $[a_{s_1}] \mapsto [a_{s_0}]$ , and hence  $[a_{s_0}] = 0$ . Because  $s_0$  was arbitrary, we get  $([a_s]) = 0$ . Therefore, it holds  $A^\infty = 0$ .

Alternatively, a direct proof of (b) can be found in [Cen98].

(c) The first page reads

$$E_1^{sd} = H^{s+d}(\operatorname{gr}_s \operatorname{Tot}_I B^*) = H(B^{\lfloor \frac{d}{2} \rfloor, s + \lceil \frac{d}{2} \rceil}, d'),$$

where  $d'$  is the differential on  $\operatorname{gr} \operatorname{Tot}_I B^*$ . We see that  $E_r^{sd}$  occupy the half-plane  $\{(s, d) \mid d \geq 0\}$ , and hence the condition of exiting differentials is satisfied. The groups  $A^\infty$  and  $A^{-\infty}$  are computed as in (b). The strong convergence is again implied by a generalization of Theorem B.2.1.

Alternatively, one can modify the direct proof for the horizontal filtration from [Cen98].

(d) We want to use the following theorem:

*Theorem B.2.2* ([Boa99, Theorem 9.2]). Let  $C^*$  be a cochain complex filtered by an exhaustive, Hausdorff and complete filtration. Then the spectral sequence converges conditionally to  $H(C^*)$ .

We have

$$\mathcal{F}_v^k(\operatorname{Tot}_I B^*)^d = \bigoplus_{i \geq k} B^{i, d-i},$$

and hence  $\operatorname{Tot}_{II} B^*$  is the completion of  $\operatorname{Tot}_I B^*$  with respect to  $\mathcal{F}_v$ . Hence, it is complete. Clearly, it is also Hausdorff and exhaustive, and we can apply Theorem B.2.2.

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(e) This follows from (a), (b), (c) and the following theorem:

*Theorem B.2.3* ([Boa99, Theorem 5.3]). Let  $f : C^* \rightarrow \bar{C}^*$  be a morphism of filtered cochain complexes. Suppose that  $E_r$  converges strongly to a filtered group  $G$  and that  $\bar{E}_r$  converges (strongly) to a filtered group  $\bar{G}$ . If  $f$  induces an isomorphism  $E_r \simeq \bar{E}_r$  for some  $r$ , then it induces an isomorphism  $G \simeq \bar{G}$ .

(f) We want to use the following theorem:

*Theorem B.2.4* ([Boa99, Theorem 7.2]). Let  $f : C^* \rightarrow \bar{C}^*$  be a morphism of filtered cochain complexes. Suppose that  $E^s = \bar{E}^s = 0$  for all  $s < 0$  and that the spectral sequences converge conditionally to the colimits  $A^{-\infty}$  and  $\bar{A}^{-\infty}$ , respectively. If  $f$  induces isomorphisms  $E_\infty \simeq \bar{E}_\infty$  and  $RE_\infty \simeq R\bar{E}_\infty$ , then it induces an isomorphism  $A^\infty \simeq \bar{A}^\infty$ .

The first page for the vertical filtration reads:

$$\begin{aligned} E_1^{sd} &= H(\mathrm{gr}_s \mathrm{Tot}_{\mathrm{II}} B^*, d_v)[s]^d \\ &= H^{d+s}(\mathrm{Tot}_{\mathrm{II}}(s\text{-th column of } B^*), d_v) \\ &= H(B^{sd}, d_v). \end{aligned}$$

Therefore, the condition  $E^s = \bar{E}^s = 0$  for all  $s < 0$  is satisfied.<sup>2</sup> By (d), conditional convergence is guaranteed. Since both  $E_\infty$  and  $RE_\infty$  depend only on  $E_r$  for  $r \geq r_0$  and any  $r_0$  (see [Boa99, p. 16]), the rest of the assumptions of Theorem B.2.4 is fulfilled.  $\square$

*Remark B.2.2* (Differences to first-quadrant bicomplexes). (i) Given a half-plane bi-complex  $B$ , we have

$$(\mathrm{Tot}_{\mathrm{I}} B)^{*g} \simeq \mathrm{Tot}_{\mathrm{II}} B^*,$$

where  $B^* = \bigoplus_{i,j} B_{ij}^*$  is the “pointwise dual” to  $B$ . This is why we have to consider homology and cohomology separately and can not just dualize the results.

(ii) The vertical filtration of  $B^*$  might not converge strongly to  $H(\mathrm{Tot}_{\mathrm{I}} B^*)$ . Indeed, let

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<sup>2</sup>It is again possible to relax this assumption and prove B.2.4 when the condition of “entering differentials” is satisfied. This means that the pages occupy a half-plane and for any fixed  $(s, d)$ , all but finitely many differentials  $d_r : E_r^{s-r, d+r-1} \rightarrow E_r^{s, d}$  start outside of the half-plane (and thus vanish). See [Rog19].

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$$\begin{array}{ccccccc}
 & & \mathbb{R} & & 0 & & 0 \\
 & & \uparrow \mathbb{1} & & & & \\
 B^* : & \mathbb{R} & \xrightarrow{\mathbb{1}} & \mathbb{R} & & 0 & \cdot \\
 & & & \uparrow \mathbb{1} & & & \\
 & 0 & & \mathbb{R} & \xrightarrow{\mathbb{1}} & \dots & 
 \end{array}$$

Then  $H(\text{Tot}_I B^*) = \mathbb{R}$  (the  $\mathbb{R}$  in the first column), but  $E_1 = 0$  because every column is exact. Notice that  $H(\text{Tot}_{II} B^*) = 0$ . Taking 0's instead of  $\mathbb{1}$ 's in the definition of  $B^*$ , we see that the horizontal filtration does not converge conditionally to  $H(\text{Tot}_I B^*)$  because its filtration by  $A^s$  is incomplete.  $\triangleleft$

We will work with the following bicomplexes.

**Definition B.2.3** (Bicomplexes for cyclic (co)homology). *Let  $\mathcal{A} = (V, (\mu_k))$  be an  $A_\infty$ -algebra. Loday's cyclic bicomplexes are defined by*

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & D_2 V & \xleftarrow{\mathbb{1}-t} & D_2 V & \xleftarrow{N} & D_2 V & \xleftarrow{\mathbb{1}-t} & D_2 V & \xleftarrow{N} \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' \\
 CC(\mathcal{A}) : & D_1 V & \xleftarrow{\mathbb{1}-t} & D_1 V & \xleftarrow{N} & D_1 V & \xleftarrow{\mathbb{1}-t} & D_1 V & \xleftarrow{N} \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' \\
 & D_0 V & \xleftarrow{\mathbb{1}-t} & D_0 V & \xleftarrow{N} & D_0 V & \xleftarrow{\mathbb{1}-t} & D_0 V & \xleftarrow{N} \\
 & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b'
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & b^* \uparrow & & -b'^* \uparrow & & b^* \uparrow & & -b'^* \uparrow \\
 & D^2 V & \xrightarrow{\mathbb{1}-t^*} & D^2 V & \xrightarrow{N^*} & D^2 V & \xrightarrow{\mathbb{1}-t^*} & D^2 V & \xrightarrow{N^*} \\
 & b^* \uparrow & & -b'^* \uparrow & & b^* \uparrow & & -b'^* \uparrow \\
 CC^*(\mathcal{A}) : & D^1 V & \xrightarrow{\mathbb{1}-t^*} & D^1 V & \xrightarrow{N^*} & D^1 V & \xrightarrow{\mathbb{1}-t^*} & D^1 V & \xrightarrow{N^*} \\
 & b^* \uparrow & & -b'^* \uparrow & & b^* \uparrow & & -b'^* \uparrow \\
 & D^0 V & \xrightarrow{\mathbb{1}-t^*} & D^0 V & \xrightarrow{N^*} & D^0 V & \xrightarrow{\mathbb{1}-t^*} & D^0 V & \xrightarrow{N^*} \\
 & b^* \uparrow & & -b'^* \uparrow & & b^* \uparrow & & -b'^* \uparrow
 \end{array}$$

Clearly,  $CC^*$  is the “pointwise” graded dual to  $CC$  and analogously for other bicomplexes we are going to define.

Let  $\mathbb{1}$  be a strict unit for  $\mathcal{A}$ . We define the Connes' operator  $B : DV \rightarrow DV$  by

$$B := (\mathbb{1} - t) \circ \iota_1 \circ N. \quad (\text{B.6})$$

Connes' cyclic bicomplexes are defined by



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$$\begin{array}{ccccccc}
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 & D_2V & \xleftarrow{\quad \mathfrak{B} \quad} & D_1V & \xleftarrow{\quad \mathfrak{B} \quad} & D_0V & \xleftarrow{\quad \mathfrak{B} \quad} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 BC(\mathcal{A}) : & D_1V & \xleftarrow{\quad \mathfrak{B} \quad} & D_0V & \xleftarrow{\quad \mathfrak{B} \quad} & D_{-1}V & \xleftarrow{\quad \mathfrak{B} \quad} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 & D_0V & \xleftarrow{\quad \mathfrak{B} \quad} & D_{-1}V & \xleftarrow{\quad \mathfrak{B} \quad} & D_{-2}V & \xleftarrow{\quad \mathfrak{B} \quad} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} &
 \end{array}$$

and

$$\begin{array}{ccccccc}
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 & D^2V & \xrightarrow{\quad \mathfrak{B}^* \quad} & D^1V & \xrightarrow{\quad \mathfrak{B}^* \quad} & D^0V & \xrightarrow{\quad \mathfrak{B}^* \quad} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 BC^*(\mathcal{A}) : & D^1V & \xrightarrow{\quad \mathfrak{B}^* \quad} & D^0V & \xrightarrow{\quad \mathfrak{B}^* \quad} & D^{-1}V & \xrightarrow{\quad \mathfrak{B}^* \quad} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & \\
 & D^0V & \xrightarrow{\quad \mathfrak{B}^* \quad} & D^{-1}V & \xrightarrow{\quad \mathfrak{B}^* \quad} & D^{-2}V & \xrightarrow{\quad \mathfrak{B}^* \quad} \\
 & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow & & \mathfrak{b}^* \uparrow &
 \end{array}$$

We define the normalized Connes' operator  $\bar{\mathfrak{B}} : \bar{D}V \rightarrow \bar{D}V$  by<sup>3</sup>

$$\bar{\mathfrak{B}} := \bar{p} \circ \mathfrak{B} = \bar{p} \circ \iota_1 \circ N. \quad (\text{B.7})$$

The normalized Connes' cyclic bicomplexes are defined by

$$\begin{array}{ccccccc}
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 & \bar{D}_2V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} & \bar{D}_1V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} & \bar{D}_0V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 \overline{BC}(\mathcal{A}) : & \bar{D}_1V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} & \bar{D}_0V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} & \bar{D}_{-1}V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & \\
 & \bar{D}_0V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} & \bar{D}_{-1}V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} & \bar{D}_{-2}V & \xleftarrow{\quad \bar{\mathfrak{B}} \quad} \\
 & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} & & \downarrow \mathfrak{b} &
 \end{array}$$

and

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<sup>3</sup>The definition does not depend on the chosen section of  $\bar{p} : DV \rightarrow \bar{D}V$ .

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$$\overline{BC}^*(\mathcal{A}) : \begin{array}{ccccccc} & \begin{array}{c} b^* \uparrow \\ \bar{D}^2 V \end{array} & \xrightarrow{\bar{B}^*} & \begin{array}{c} b^* \uparrow \\ \bar{D}^1 V \end{array} & \xrightarrow{\bar{B}^*} & \begin{array}{c} b^* \uparrow \\ \bar{D}^0 V \end{array} & \xrightarrow{\bar{B}^*} \\ \begin{array}{c} b^* \uparrow \\ \bar{D}^1 V \end{array} & \xrightarrow{\bar{B}^*} & \begin{array}{c} b^* \uparrow \\ \bar{D}^0 V \end{array} & \xrightarrow{\bar{B}^*} & \begin{array}{c} b^* \uparrow \\ \bar{D}^{-1} V \end{array} & \xrightarrow{\bar{B}^*} & \cdot \\ \begin{array}{c} b^* \uparrow \\ \bar{D}^0 V \end{array} & \xrightarrow{\bar{B}^*} & \begin{array}{c} b^* \uparrow \\ \bar{D}^{-1} V \end{array} & \xrightarrow{\bar{B}^*} & \begin{array}{c} b^* \uparrow \\ \bar{D}^{-2} V \end{array} & \xrightarrow{\bar{B}^*} & \\ & \begin{array}{c} b^* \uparrow \\ \bar{D}^{-1} V \end{array} & & \begin{array}{c} b^* \uparrow \\ \bar{D}^{-2} V \end{array} & & \begin{array}{c} b^* \uparrow \\ \bar{D}^{-3} V \end{array} & \end{array}$$

The reduced Connes' cyclic bicomplexes  $BC^{\text{red}}$  and  $BC_{\text{red}}^*$  are defined by replacing  $\bar{D}V$  and  $\bar{D}^*V$  by  $D^{\text{red}}V$  and  $D_{\text{red}}^*V$ , respectively.

The coordinate  $(0,0)$  in the bicomplexes above always correspond to  $D^0V$  in the first column (bottom-left position in the figures).

Note that another convention of drawing homological bicomplexes in the left half-plane and cohomological bicomplexes in the right half-plane might be more natural.

*Remark B.2.4* (Mixed complexes). One can equivalently encode the data of a cohomological Connes bicomplex into that of a mixed complex  $(D^*, b^*, B^*)$ . In general, it is a graded vector space  $D^*$  with a differential  $b^*$ ,  $|b^*| = 1$  and a boundary operator  $B^*$ ,  $|B^*| = -1$  which anticommute. One introduces the formal symbol  $u$  of degree  $|u| = 2$  and considers the polynomial ring  $D^*[u]$  in  $u$  with values in  $D^*$  and the ring of power series  $D^*[[u]]$  with the differential  $b^* + B^*u$ . Clearly, the former is quasi-isomorphic to  $\text{Tot}_I(BC^*)$  and the latter to  $\text{Tot}_{II}(BC^*)$  (columns of  $BC^*$  are indexed with non-negative powers of  $u$ ). Altogether, there are seven versions  $[u]$ ,  $[u^{-1}]$ ,  $[u, u^{-1}]$ ,  $[[u^{-1}, u]$ ,  $[u^{-1}, u]$ ,  $[[u, u^{-1}]$  whose relation is studied in [CVonb]. Some of these are related to periodic and negative versions of cyclic homology (see [Lod92]).  $\triangleleft$

DONE Origin of bicomplexes: Define where the point  $(0,0)$  is for the bicomplexes.

In the following proofs, we might not need the full strength of Proposition B.2.1 since the spectral sequences for the bicomplexes from Definition B.2.3 mostly collapse already on the second page (see question (iii) of Remark B.3.2).

**Lemma B.2.5** (Loday's cyclic bicomplexes and cyclic homology). *Let  $\mathcal{A} = (V, (\mu_k))$  be an  $A_\infty$ -algebra. The projection  $p^\lambda : CC \rightarrow D^\lambda$  to the first column modulo  $\text{im}(\mathbb{1} - t)$  is a chain map and induces an isomorphism  $H(\widehat{CC}) \simeq H^\lambda(\mathcal{A})$ . The inclusion  $\iota_\lambda : D_\lambda^* \rightarrow CC^*$  into the first column is a chain map and induces an isomorphism  $H_\lambda^*(\mathcal{A}) \simeq H(CC^*)$ .*

*Proof.* The fact that  $p^\lambda$  and  $\iota_\lambda$  are chain maps is obvious. We consider the horizontal filtration  $\mathcal{F}^h$  of  $CC^*$ . Because of CP1, the rows are acyclic, and we see that the only

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non-zero terms of the first page of the corresponding spectral sequence are

$$E_1^{0d} = D^d V / \ker(\mathbb{1} - t^*).$$

The differential  $d_1$  is easy to check to be  $b^*$ , and the inclusion  $\iota_\lambda$  induces the isomorphism  $D_\lambda^d \simeq D^d V / \ker(\mathbb{1} - t^*)$ . Considering the canonical filtration on  $D_\lambda^*$ , claims (a), (b) and (e) of Proposition B.2.1 apply.

As for homology, we consider the degree reversed cochain complex  $r(\text{Tot}_{\text{II}} CC)$  and the reversed filtration  $r(\mathcal{F}^h)_s = \mathcal{F}_{-s}^h$ . For the corresponding cohomological spectral sequence  $\tilde{E}_r$ , we have

$$\begin{aligned} \tilde{E}_1^{sd} &= H^{s+d}(r(\mathcal{F})_s r(\text{Tot}_{\text{II}} CC) / r(\mathcal{F})_{s+1} r(\text{Tot}_{\text{II}} CC)) \\ &= H_{-s-d}(\text{Tot}_{\text{II}}(-s\text{-th row of } CC)) \\ &= H(B_{-d, -s}, \partial_h). \end{aligned}$$

Therefore, the only groups are  $\tilde{E}^{s0} = D_{-s} / \text{im}(\mathbb{1} - t)$ , and the spectral sequence converges conditionally to  $r(H(\widehat{CC}))$ . Clearly,  $\iota_\lambda$  induces an isomorphism of the first pages, where on  $r(D^\lambda)$  we consider the canonical filtration. Proposition B.2.1 and its proof finishes the argument.  $\square$

Claim (a) of the following is similar to [CVonb, Lemma 2.12].

**Lemma B.2.6** (No long chains in homology for bounded degrees). *Let  $\mathcal{A} = (V, (\mu_k), 1)$  be a strictly unital  $A_\infty$ -algebra. Suppose that  $V$  has bounded degrees. Then the canonical inclusion  $\text{Tot}_{\text{I}} \hookrightarrow \text{Tot}_{\text{II}}$  induces the following isomorphism:*

- (a)  $H(CC(\mathcal{A})) \simeq H(\widehat{CC}(\mathcal{A}))$ ,
- (b)  $H(\overline{BC}(\mathcal{A})) \simeq H(\widehat{\overline{BC}}(\mathcal{A}))$ .

*Proof.* (a) We will denote  $CC(\mathcal{A})$ ,  $\overline{BC}(\mathcal{A})$  and  $DV$  simply by  $CC$ ,  $\overline{BC}$  and  $D$ , respectively. Consider the (increasing) filtration  $\mathcal{F}_w$  of  $D$  by weights. We first prove the following subclaim:

*Subclaim* (Weight normalization). Let  $c = (c_i)_{i=0}^\infty \in \text{Tot}_{\text{II}k}(CC)$  be a closed chain of degree  $k$ ; i.e., for all  $i \in \mathbb{N}_0$ , we have  $c_i \in D_{k-i}$ , and the relations

$$bc_{2i} + (\mathbb{1} - t)c_{2i+1} = 0 \quad \text{and} \quad -b'c_{2i+1} + Nc_{2i+2} = 0$$

hold. Suppose that we are given  $j \geq 1$  and  $n_0 \in \mathbb{N}$  such that  $c_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j+1}$ . Then

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$$\begin{array}{ccc}
 \xleftarrow{\mathbb{1}-t} c_j, \tilde{c}_j \in D_{k-j} & \xleftarrow{N} & \tilde{z}_{j+1} \in D_{j-k} \\
 \downarrow -b' & & \downarrow b \\
 \xleftarrow{N} c_{j+1}, \tilde{c}_{j+1} \in D_{j-k-1} & & \\
 \downarrow b & & 
 \end{array}$$

Figure B.1.: Positions of element in  $CC$  for  $j$  odd.

we can construct  $\tilde{c}_j \in \mathcal{F}_w^{n_0} D_{k-j}$ ,  $\tilde{c}_{j+1} \in D_{k-j-1}$  and  $\tilde{z}_{j+1} \in D_{k-j}$  such that if we define

$$\tilde{c}_i := \begin{cases} \tilde{c}_j & \text{for } i = j, \\ \tilde{c}_{j+1} & \text{for } i = j+1, \\ c_i & \text{otherwise} \end{cases} \quad \text{and} \quad z_i := \begin{cases} \tilde{z}_{j+1} & \text{for } i = j+1, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{B.8})$$

then  $\tilde{c} := (\tilde{c}_i)_{i=0}^\infty$  is a closed chain and  $z := (z_i)_{i=0}^\infty$  satisfies  $\partial z = c - \tilde{c}$ . By repeating this procedure inductively, we obtain chains  $c' \in \text{Tot}_{\text{II}k}(CC)$  and  $z' \in \text{Tot}_{\text{II}k+1}(CC)$  such that  $c'_i \in \mathcal{F}_w^{n_0} D_{k-i}$  for all  $i \geq j$  and  $c - c' = \partial z'$ .

*Proof of the Subclaim.* We will assume that  $j$  is odd; the proof is analogous for  $j$  even with the roles of  $(\mathbb{1} - t)$  and  $N$ , resp.  $b$  and  $-b'$  switched. The situation is depicted in Figure B.1. Because  $c_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j+1}$  and  $b$  does not increase weights, we have  $bc_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j}$ . Since  $c$  is closed, we have  $(\mathbb{1} - t)c_j = -bc_{j-1} \in \mathcal{F}_w^{n_0} D_{k-j}$ . Therefore, there is a  $\tilde{c}_j \in \mathcal{F}_w^{n_0} D_{k-j}$  such that  $(\mathbb{1} - t)\tilde{c}_j = (\mathbb{1} - t)c_j$ . As  $c_j - \tilde{c}_j \in \ker(\mathbb{1} - t) = \text{im } N$ , we have  $c_j - \tilde{c}_j = N\tilde{z}_{j+1}$  for some  $\tilde{z}_{j+1} \in D_{k-j}$ . We define  $\tilde{c}_{j+1} := c_{j+1} - b\tilde{z}_{j+1}$ . The following relations hold:

- I.  $(\mathbb{1} - t)\tilde{c}_j = (\mathbb{1} - t)c_j$ ,
- II.  $-b'\tilde{c}_j + N\tilde{c}_{j+1} = -b'\tilde{c}_j + Nc_{j+1} - Nb\tilde{z}_{j+1}$ 

$$\begin{aligned}
 &= -b'\tilde{c}_j + Nc_{j+1} - b'N\tilde{z}_{j+1} \\
 &= -b'\tilde{c}_j + Nc_{j+1} - b'(c_j - \tilde{c}_j) \\
 &= -b'c_j + Nc_{j+1} \\
 &= 0,
 \end{aligned}$$
- III.  $b\tilde{c}_{j+1} = bc_{j+1}$ ,
- IV.  $N\tilde{z}_{j+1} = c_j - \tilde{c}_j$ ,
- V.  $b\tilde{z}_{j+1} = c_{j+1} - \tilde{c}_{j+1}$ .

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The relations I–III show that  $\tilde{c}$  is closed and the relations IV–V that  $\partial z = c - \tilde{c}$ .<sup>4</sup>

Starting with  $c^1 := \tilde{c}$  and  $z^1 := z$ , we repeat the construction above to produce the telescopic sequence of homotopies

$$\begin{aligned} c - c^1 &= \partial z^1 \\ c^1 - c^2 &= \partial z^2 \\ &\dots = \dots \end{aligned}$$

such that  $c_i^j \in \mathcal{F}_w^{n_0} D_{k-i}$  for all  $j \leq i \leq j+l$ . The limit chain  $c' := \sum_{k=0}^{\infty} c^k \in \text{Tot}_{\text{II}} CC$  has the property that  $c_i^j \in \mathcal{F}_w^{n_0} D_{k-i}$  for all  $i \geq j$ , and the limit homotopy  $z' := \sum_{k=1}^{\infty} z^k \in \text{Tot}_{\text{II}} CC$  converges and satisfies  $\partial z' = c - c'$ . (Subclaim)  $\square$

We will now prove surjectivity of the map on homology induced by the inclusion  $\text{Tot}_{\text{I}} \hookrightarrow \text{Tot}_{\text{II}}$ . Given  $[c] \in H_k(\widehat{CC})$ , using the Subclaim, we can assume that there is an  $n_0 \in \mathbb{N}$  such that  $c_i \in \mathcal{F}_w^{n_0} D_{k-i}$  for all  $i \in \mathbb{N}_0$ . However, we have  $|c_i| = -k + i - 1$  for the degrees in  $BV$ , and because the degrees of  $V$  are bounded,  $c_i$  eventually, as  $i \rightarrow \infty$ , reach degrees which can not be produced by  $n_0$  vectors. Therefore, there is an  $i_0 \in \mathbb{N}_0$  such that  $c_i = 0$  for all  $i \geq i_0$ ; this means that  $c \in \text{Tot}_{\text{I}} CC$ .

To show injectivity of the induced map on homology, suppose that  $c \in \text{Tot}_{\text{I}} CC$  satisfies  $c = \partial z$  for some  $z \in \text{Tot}_{\text{II}} CC$ . Let  $i_0 \in \mathbb{N}$  be such that  $c_i = 0$  for all  $i \geq i_0$ . We use the Subclaim to alter  $z$  and obtain a chain  $\tilde{z} \in \text{Tot}_{\text{I}} CC$  such that  $\tilde{z}_i = z_i$  for  $i \leq i_0$  and  $\partial \tilde{z} = c$ . This shows injectivity.

(b) We will prove an analogy of the Subclaim from (a):

Given a closed  $c = (c_i)_{i=0}^{\infty} \in \text{Tot}_{\text{II}k} \overline{BC}$ , every  $c_i \in \bar{D}_{k-2i} V$  can be written as

$$c_i = \tilde{c}_i + 1\hat{c}_i$$

for unique  $\tilde{c}_i \in D_{q-2i} \bar{V}$  and  $\hat{c}_i \in D_{q-2i-1} \bar{V}$ . Using strict unitality, we have  $b(1\hat{c}_i) = (\mathbb{1} - t)\hat{c}_i - 1b'\hat{c}_i$ , and hence

$$\begin{aligned} bc_i &= b\tilde{c}_i + (\mathbb{1} - t)\hat{c}_i - 1b'\hat{c}_i, \\ \bar{B}c_i &= 1N\tilde{c}_i. \end{aligned}$$

For the second equality, recall the definition (B.7) and note that the  $\bar{p}$  in front “kills” any input of  $\bar{B}$  containing at least one 1. We see that  $\partial c = 0$  is equivalent to  $bc_i = -\bar{B}c_{i+1}$

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<sup>4</sup>In fact,  $\partial \tilde{c} = 0$  follows from  $\partial c = 0$  and  $\partial z = c - \tilde{c}$ .

## B. Reduced cyclic homology of A-infinity-algebras

$$\begin{array}{ccccc}
 c_{j-1} \in \bar{D}_{k-2j+2} & \xleftarrow{\bar{B}} & z_j \in \bar{D}_{k-2j+1} & & \\
 \downarrow \text{b} & & \downarrow \text{b} & & \\
 & \xleftarrow{\bar{B}} & c_j, \tilde{c}_j \in \bar{D}_{k-2j} & \xleftarrow{\bar{B}} & z_{j+1} \in \bar{D}_{k-2j-1} \\
 & & \downarrow \text{b} & & \downarrow \text{b} \\
 & & & \xleftarrow{\bar{B}} & c_{j+1}, \tilde{c}_{j+1} \in \bar{D}_{k-2j-2}
 \end{array}$$

Figure B.2.: Positions of element in  $BC$  for  $j$  odd.

which is equivalent to

$$\begin{aligned}
 \text{b}\bar{c}_i + (\mathbb{1} - t)\hat{c}_i &= 0 \quad \text{and} \\
 \text{b}'\hat{c}_i &= N\bar{c}_{i+1}
 \end{aligned}$$

for all  $i \in \mathbb{N}_0$ .

Suppose that  $c_{j-1} \in \mathcal{F}_w^{n_0} \bar{D}_{k-2j+2}$  for some  $j \geq 1$  and  $n_0 \in \mathbb{N}_0$ . Then  $\bar{c}_{j-1} \in \mathcal{F}_w^{n_0} \bar{D}_{k-2j+2}$  and  $\hat{c}_{j-1} \in \mathcal{F}_w^{n_0-1} \bar{D}_{k-2j+1}$ . Because  $\text{b}'\hat{c}_{j-1} = N\bar{c}_j$ , we can find  $\bar{d}_j \in \mathcal{F}_w^{n_0-1} D_{k-2j} \bar{V}$  such that  $N\bar{d}_j = \text{b}'\hat{c}_{j-1}$ . Because  $\bar{d}_j - \bar{c}_j \in \ker N = \text{im}(\mathbb{1} - t)$ , we can find  $\hat{z}_j \in \bar{D}_{k-2j} V$  such that  $(\mathbb{1} - t)\hat{z}_j = \bar{d}_j - \bar{c}_j$ . We compute

$$\begin{aligned}
 \text{b}\bar{d}_j &= \text{b}(\bar{c}_j + (\mathbb{1} - t)\hat{z}_j) \\
 &= -(\mathbb{1} - t)\hat{c}_j + \text{b}(\mathbb{1} - t)\hat{z}_j \\
 &= -(\mathbb{1} - t)\hat{c}_j + (\mathbb{1} - t)\text{b}'\hat{z}_j \\
 &= -(\mathbb{1} - t)(\hat{c}_j - \text{b}'\hat{z}_j).
 \end{aligned}$$

Because  $\bar{d}_j \in \mathcal{F}_w^{n_0-1} D_{k-2j} \bar{V}$  and  $\text{b}$  does not increase the filtration, we can find  $\hat{d}_j \in \mathcal{F}_w^{n_0-1} D_{k-2j-1} \bar{V}$  such that  $(\mathbb{1} - t)\hat{d}_j = -\text{b}\bar{d}_j$ . Now,  $\hat{d}_j - (\hat{c}_j - \text{b}'\hat{z}_j) \in \ker(\mathbb{1} - t) = \text{im } N$ , and hence there is a  $\bar{z}_{j+1} \in \bar{D}_{k-2j-1}$  such that  $N\bar{z}_{j+1} = \hat{d}_j - (\hat{c}_j - \text{b}'\hat{z}_j)$ . We define the following elements:

DONE : in definition: Take care of alignment of  $\coloneqq$  and  $=$  in one column!

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$$\begin{aligned}
\tilde{c}_j &:= c_j + \bar{B}\bar{z}_{j+1} + b(1\hat{z}_j) \\
&= c_j + \bar{B}\bar{z}_{j+1} + (1-t)\hat{z}_j - 1b'\hat{z}_j \\
&= c_j + 1N\bar{z}_{j+1} + (1-t)\hat{z}_j - 1b'\hat{z}_j \\
&= c_j + 1\hat{d}_j - 1\hat{c}_j + 1b'\hat{z}_j + \bar{d}_j - \bar{c}_j - 1b'\hat{z}_j \\
&= \bar{d}_j + 1\hat{d}_j, \\
\tilde{c}_{j+1} &:= c_{j+1} + b\bar{z}_{j+1}, \\
\tilde{z}_j &:= 1\hat{z}_j, \\
\tilde{z}_{j+1} &:= \bar{z}_{j+1}.
\end{aligned}$$

The following relations hold:

- I.  $\bar{B}\tilde{c}_j = \bar{B}c_j$ ,
- II.  $b\tilde{c}_j = bc_j + b\bar{B}\bar{z}_{j+1} = -\bar{B}c_{j+1} - \bar{B}b\bar{z}_{j+1} = -\bar{B}\tilde{c}_{j+1}$ ,
- III.  $b\tilde{c}_{j+1} = bc_{j+1}$ ,
- IV.  $\bar{B}\tilde{z}_j = \bar{B}1\hat{z}_j = 0$ ,
- V.  $b\tilde{z}_j = \tilde{c}_j - c_j - \bar{B}\tilde{z}_{j+1}$ ,
- VI.  $b\tilde{z}_{j+1} = \tilde{c}_{j+1} - c_{j+1}$ .

Relations I–III show that  $\tilde{c}$  is closed, and relations IV–VI show that  $\partial z = \tilde{c} - c$ . Here  $\tilde{c}$  is defined as in (B.8) and  $z$  has one more term:

$$z_i := \begin{cases} \tilde{z}_j & \text{for } i = j, \\ \tilde{z}_{j+1} & \text{for } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\tilde{c}_j = \bar{d}_j + 1\hat{d}_j$ ,  $\bar{d}_j \in \mathcal{F}_w^{n_0-1}D_{k-2j}\bar{V}$  and  $\hat{d}_j \in \mathcal{F}_w^{n_0-1}D_{k-2j-1}\bar{V}$ , we have  $\tilde{c}_j \in \mathcal{F}_w^{n_0}\bar{D}_{k-2j}$ .

Having the recursive step, the rest can be done in the same way as in (a).  $\square$

**Lemma B.2.7** (Loday's and Connes' bicomplexes are quasi-isomorphic). *Let  $\mathcal{A} = (V, (\mu_k), 1)$  be a strictly unital  $A_\infty$ -algebra. The map*

$$\begin{aligned}
I : \text{Tot}_\Pi BC &\longrightarrow \text{Tot}_\Pi CC \\
(c_0, c_1, c_2, \dots) &\longmapsto (c_0, \iota_1 N c_1, c_1, \iota_1 N c_2, c_2, \dots)
\end{aligned}$$

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is a chain map inducing the isomorphisms

$$H(\widehat{BC}) \simeq H(\widehat{CC}) \quad \text{and} \quad H(BC) \simeq H(CC).$$

Analogously, the map

$$\begin{aligned} P : \text{Tot}_{\text{II}} CC^* &\longrightarrow \text{Tot}_{\text{II}} BC^* \\ (\psi_0, \psi_1, \psi_2, \dots) &\longmapsto (\psi_0, \psi_2 + N^* \iota_1^* \psi_1, \psi_4 + N^* \iota_1^* \psi_3, \dots) \end{aligned}$$

is a chain map inducing the isomorphisms

$$H(\widehat{BC}^*) \simeq H(\widehat{CC}^*) \quad \text{and} \quad H(BC^*) \simeq H(CC^*).$$

*Proof.* The following computation shows that  $\iota$  is a chain map:

$$\begin{aligned} \partial_{CC} I(c_0, c_1, c_2, \dots) &= (bc_0 + (\mathbb{1} - t)\iota_1 Nc_1, -b'\iota_1 Nc_1 + Nc_1, bc_1 + (\mathbb{1} - t)\iota_1 Nc_2, \dots) \\ &= (bc_0 + Bc_1, -Nc_1 + \iota_1 b' Nc_1 + Nc_1, bc_1 + Bc_2, \dots) \\ &= (bc_0 + Bc_1, \iota_1 Nbc_1, bc_1 + Bc_2, \dots) \\ &= I(bc_0 + Bc_1, bc_1 + Bc_2, \dots) \\ &= I\partial_{BC}(c_0, c_1, c_2, \dots). \end{aligned}$$

Clearly,  $I$  is injective, and hence it induces an isomorphisms of chain complexes

$$(\text{Tot}_{\text{II}} BC, \partial_{BC}) \simeq (\text{im}(I), \partial_{CC}|_{\text{im}(I)}) \subset (\text{Tot}_{\text{II}} CC, \partial_{CC}).$$

Consider the subcomplex  $(\text{Tot}_{\text{II}} CC_{\text{odd}, \bullet}, -b') \subset (\text{Tot}_{\text{II}} CC, \partial_{CC})$  which consists of odd columns of  $CC$ . It is a direct complement of  $\text{im}(I)$  in  $\text{Tot}_{\text{II}} CC$ . Indeed,  $(0, c_1, 0, c_2, \dots) \in \text{im}(I)$  implies  $c_i = 0$  for all  $i \in \mathbb{N}$ , which gives  $\text{Tot}_{\text{II}}(CC_{\text{odd}, \bullet}) \cap \text{im}(I) = 0$ ; also, for any  $(c_i) \in \text{Tot}_{\text{II}} CC$ , we have

$$(c_0, c_1, c_2, c_3, c_4, \dots) = I((c_0, c_2, c_4, \dots)) - (0, \iota_1 Nc_2 - c_1, 0, \iota_1 Nc_4 - c_3, 0, \dots),$$

which gives  $\text{im}(I) + \text{Tot}_{\text{II}}(CC_{\text{odd}, \bullet}) = \text{Tot}_{\text{II}} CC$ . Now,  $\text{Tot}_{\text{II}}(CC_{\text{odd}, \bullet})$  is contractible by CP3, and hence  $H(\text{Tot}_{\text{II}} BC) \simeq H(\text{Tot}_{\text{II}} CC)$  (using an argument with the long exact sequence in homology). Clearly,  $I$  restricts to short chains  $\text{Tot}_{\text{I}}$ , and thus  $H(\text{Tot}_{\text{I}} BC) \simeq H(\text{Tot}_{\text{I}} CC)$  holds too.

A similar discussion applies in cohomology. The following computation shows that  $P$



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is a chain map:

$$\begin{aligned}
& \text{Pd}_{CC}(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \dots) \\
&= P(\mathbf{b}^*\psi_0, (\mathbb{1} - t^*)\psi_0 - \mathbf{b}'\psi_1, N^*\psi_1 + \mathbf{b}^*\psi_2, (\mathbb{1} - t^*)\psi_2 - \mathbf{b}'^*\psi_3, N^*\psi_4 + \mathbf{b}^*\psi_3, \dots) \\
&= (\mathbf{b}^*\psi_0, N^*\psi_1 + \mathbf{b}^*\psi_2 + N^*\iota_1^*((\mathbb{1} - t^*)\psi_0 - \mathbf{b}'\psi_1), \\
&\quad N^*\psi_4 + \mathbf{b}^*\psi_3 + N^*\iota_1^*((\mathbb{1} - t^*)\psi_2 - \mathbf{b}'^*\psi_3), \dots) \\
&= (\mathbf{b}^*\psi_0, \mathbf{B}^*\psi_0 + \mathbf{b}^*\psi_2 + N^*(\psi_1 - \iota_1^*\mathbf{b}'^*\psi_1), \mathbf{B}^*\psi_2 + \mathbf{b}^*\psi_4 + N^*(\psi_3 - \iota_1^*\mathbf{b}'^*\psi_3), \dots) \\
&= (\mathbf{b}^*\psi_0, \mathbf{B}^*\psi_0 + \mathbf{b}^*(\psi_2 + N^*\iota_1^*\psi_1), \mathbf{B}^*\psi_2 + \mathbf{b}^*(\psi_4 + N^*\iota_1^*\psi_3), \dots) \\
&= d_{BC}(\psi_0, \psi_2 + N^*\iota_1^*\psi_1, \psi_2, \psi_4 + N^*\iota_1^*\psi_3, \dots) \\
&= d_{BC}P(\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \dots)
\end{aligned}$$

The fourth equality uses that  $\iota_1\mathbf{b}' + \mathbf{b}'\iota_1 = \mathbb{1}$  and  $\mathbf{b}'N = N\mathbf{b}$ . Because  $(\psi_0, \psi_1, \dots) = P(\psi_0, 0, \psi_1, 0, \dots)$ ,  $P$  is surjective, and hence it induces an isomorphism of cochain complexes  $\text{Tot}_{\Pi}CC^*/\ker(P) \simeq \text{Tot}_{\Pi}BC^*$ . It is easy to see that

$$\ker(P) = \{(0, \psi_1, -N^*\iota_1^*\psi_1, \psi_3, -N^*\iota_1^*\psi_3, \dots)\}$$

and that the map

$$\begin{aligned}
Z : \text{Tot}_{\Pi}(CC^{\text{odd}, \bullet}) &\longrightarrow \ker(p) \\
(\psi_1, \psi_3, \dots) &\longmapsto (0, \psi_1, -N^*\iota_1^*\psi_1, \psi_3, -N^*\iota_1^*\psi_3, \dots)
\end{aligned}$$

is an isomorphism of the cochain complexes

$$(\text{Tot}_{\Pi}(CC^{\text{odd}, \bullet}), -\mathbf{b}'^*) \simeq (\ker(P), \partial_{CC}|_{\ker(P)}) \subset (\text{Tot}_{\Pi}CC, \partial_{CC}).$$

Indeed, we have

$$\begin{aligned}
\partial_{CC}Z(\psi_1, \psi_3, \dots) &= (0, -\mathbf{b}'^*\psi_1, N^*\psi_1 - \mathbf{b}^*N^*\iota_1^*\psi_1, -(\mathbb{1} - t^*)N^*\iota_1^*\psi_1 - \mathbf{b}'^*\psi_3, \dots) \\
&= (0, -\mathbf{b}'^*\psi_1, -N^*\iota_1^*(-\mathbf{b}'^*\psi_1), -\mathbf{b}'^*\psi_3, \dots) \\
&= Z(-\mathbf{b}'^*)(\psi_1, \psi_3, \dots).
\end{aligned}$$

Therefore,  $\ker P$  is contractible, and the statement is implied by an argument with the long exact sequence in homology.  $\square$

**Lemma B.2.8** (Connes' cyclic bicomplexes are quasi-iso to their normalized versions).

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Let  $\mathcal{A} = (V, (\mu_k), 1)$  be a strictly unital  $A_\infty$ -algebra. The projection  $\bar{p}$  and the inclusion  $\bar{i}$  (see Definition B.2.3) induce the isomorphisms  $H(BC) \simeq H(\overline{BC})$  and  $H(\widehat{BC}^*) \simeq H(\overline{\widehat{BC}^*})$ , respectively.

*Proof.* It follows from CP4 using the spectral sequence associated to the vertical filtration. In cohomology, we use (d) and (f) of Proposition B.2.1.

In homology, we have  $\tilde{E}^{sd} = H(BC_{-s,-d}, \partial_v)$  for the reversed spectral sequence (see the proof of Lemma B.2.5), and hence strong convergence is implied by Theorem B.2.1 from the proof of Proposition B.2.1. Claim (e) of Proposition B.2.1 finishes the proof.  $\square$

The following is based on [Lod92, Proposition 2.2.14] and its proof.

**Lemma B.2.9** (Reduced Connes' cyclic bicomplexes and cyclic homology are quas-isomorphic). *Let  $\mathcal{A} = (V, (\mu_k), 1)$  be a strictly unital  $A_\infty$ -algebra. The projection  $p^\lambda : BC^{\text{red}} \rightarrow \bar{D}^\lambda$  to the first column modulo  $\text{im}(\mathbb{1} - t)$  is a chain map and induces an isomorphism  $H(\widehat{BC}^{\text{red}}) \simeq H(\bar{D}^\lambda) (=: H^{\lambda, \text{red}}(\mathcal{A}))$ . The inclusion  $\iota_\lambda : \bar{D}_\lambda^* \rightarrow BC_{\text{red}}^*$  into the first column of  $BC_{\text{red}}^*$  is a chain map and induces an isomorphism  $H(\bar{D}_\lambda^*) \simeq H(BC_{\text{red}}^*)$ .*

*Proof.* We start with the cohomology. It is easy to see that  $\iota_1$  induces  $\iota_1^* : D_{\text{red}}^* V \rightarrow D_{\text{red}}^* V$  and that for this map, we have

$$Z := \ker \iota_1^* = \operatorname{im} \iota_1^* \simeq D^* \bar{V}.$$

We consider the following diagonal filtration of  $BC_{\text{red}}^*$ :

$$\mathcal{F}^s BC_{\text{red}}^* :$$

$$\begin{array}{ccccccc} & & b^* \uparrow & & b^* \uparrow & & b^* \uparrow \\ & & D_{\text{red}}^{s+2} & \xrightarrow{\bar{B}^*} & D_{\text{red}}^{s+1} & \xrightarrow{\bar{B}^*} & Z^s & \xrightarrow{\bar{B}^*} \\ b^* \uparrow & & & & & & & \uparrow \\ \mathcal{F}^s BC_{\text{red}}^* : & & D_{\text{red}}^{s+1} & \xrightarrow{\bar{B}^*} & Z^s & \xrightarrow{\bar{B}^*} & 0 & \longrightarrow \\ b^* \uparrow & & & & \uparrow & & \uparrow & \\ & & Z^s & \xrightarrow{\bar{B}^*} & 0 & \longrightarrow & 0 & \longrightarrow \\ & \uparrow & & & \uparrow & & \uparrow & \end{array}$$

The first page of the corresponding spectral sequence consists of the columns

$$E_1^s = H(\mathrm{gr}_s \mathrm{Tot}_I) = H(Z^s \xrightarrow{b^*} D_{\mathrm{red}}^{s+1}/Z^{s+1} \xrightarrow{\bar{B}^*} Z^s \xrightarrow{b^*} \dots),$$

where the cochain complex starts in degree  $s$ . Because  $\iota_1^* b^* = -b'^* \iota_1^* + (\mathbb{1} - t^*)$  and

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$\bar{B}^* = N^* \iota_1^*$  on  $D_{\text{red}}$ , we have the commutative diagram

$$\begin{array}{ccccccc} Z^s & \xrightarrow{b^*} & D_{\text{red}}^{s+1}/Z^{s+1} & \xrightarrow{\bar{B}^*} & Z^s & \xrightarrow{b^*} & \dots \\ \downarrow \mathbb{1} & & \downarrow \iota_1^* & & \downarrow \mathbb{1} & & \\ Z^s & \xrightarrow{\mathbb{1}-t^*} & Z^s & \xrightarrow{N^*} & Z^s & \xrightarrow{\mathbb{1}-t^*} & \dots \end{array}$$

Therefore, the only non-zero terms of the first page are

$$E_1^{s,0} = Z_s / \ker(\mathbb{1} - t^*) = \bar{D}_\lambda^s$$

with the differential  $d_1 = b^*$ . This is precisely the first page of the canonical filtration of  $\bar{D}_\lambda^*$ . The map induced by  $\iota_\lambda$  on the first page is the identity, and Proposition B.2.1 implies the rest.

The situation in homology is analogous. We consider the restriction  $\iota_1 : D^{\text{red}} \rightarrow D^{\text{red}}$  and the subspace

$$B := \ker \iota_1 = \text{im } \iota_1.$$

The diagonal filtration is now

$$\begin{array}{ccccc} & \downarrow & \downarrow & \downarrow & \\ & 0 & \longleftarrow 0 & \xleftarrow{\bar{B}} B_s & \longleftarrow \\ & \downarrow & \downarrow & \downarrow b & \\ \mathcal{F}^s BC^{\text{red}} : & 0 & \xleftarrow{\bar{B}} B_s & \xleftarrow{\bar{B}} D_{s-1}^{\text{red}} & \longleftarrow \\ & \downarrow & \downarrow b & \downarrow b & \\ & B_s & \xleftarrow{\bar{B}} D_{s-1}^{\text{red}} & \xleftarrow{\bar{B}} D_{s-2}^{\text{red}} & \longleftarrow \\ & \downarrow & \downarrow & \downarrow & \end{array}$$

The reversed filtration  $r(\mathcal{F})_s = \mathcal{F}^{-s}$  of the degree reversed cochain complexes  $r(\text{Tot}_I)$  satisfies

$$\begin{aligned} \tilde{E}_1^s &= H(r(\mathcal{F})_s r(\text{Tot}_I) / r(\mathcal{F})_{s+1} r(\text{Tot}_I)) \\ &= H(D_{-s-1}^{\text{red}} / B_{-s-1} \xleftarrow{b} B_{-s} \xleftarrow{\bar{B}} D_{-s-1}^{\text{red}} / B_{-s-1} \xleftarrow{b} \dots) \end{aligned}$$

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where the first group has degree  $s$ . We have the commutative diagram

$$\begin{array}{ccccccc} D_{-s-1}^{\text{red}}/B_{-s-1} & \xleftarrow{\quad \text{b} \quad} & B_{-s} & \xleftarrow{\quad \bar{\text{B}} \quad} & D_{-s}^{\text{red}}/B_{-s} & \xleftarrow{\quad \text{b} \quad} & \cdots \\ \downarrow \iota_1 & & \downarrow \mathbb{1} & & \downarrow \iota_1 & & \\ B^{-s} & \xleftarrow{\quad \mathbb{1}-t \quad} & B^{-s} & \xleftarrow{\quad N \quad} & B^{-s} & \xleftarrow{\quad \mathbb{1}-t \quad} & \cdots, \end{array}$$

and thus  $\tilde{E}_1^{s,0} = B^{-s}/\text{im}(\mathbb{1}-t) = D_{-s}^{\lambda,\text{red}}$ . The rest is as in the case of cohomology.  $\square$

### B.3. Final argument and remarks

We are finally in position to prove Proposition 3.3.13. Precisely as in the proof sketch, we replace the unit-augmentation sequence (3.30), up to a quasi-isomorphism, by a short exact sequence of normalized Connes' cyclic bicomplexes.

**Lemma B.3.1** (Short exact sequence of normalized Connes' cyclic bicomplexes). *Let  $\mathcal{A} = (V, (\mu_j), 1, \varepsilon)$  be a strictly augmented strictly unital  $A_\infty$ -algebra. The short exact sequences of bicomplexes*

$$0 \longrightarrow \overline{BC}(\mathbb{R}) \xrightarrow{u} \overline{BC}(V) \xrightarrow{p^{\text{red}}} BC^{\text{red}} \longrightarrow 0$$

and

$$0 \longrightarrow BC_{\text{red}}^* \xrightarrow{\iota_{\text{red}}} \overline{BC}^*(V) \xrightarrow{u^*} \overline{BC}(\mathbb{R}) \longrightarrow 0$$

split. From this, we obtain the following isomorphisms:

$$\begin{aligned} H(\overline{BC}) &\simeq H(BC^{\text{red}}) \oplus H^\lambda(\mathbb{R}), & H(\widehat{\overline{BC}}) &\simeq H(\widehat{BC}^{\text{red}}) \oplus H^\lambda(\mathbb{R}), \\ H(\overline{BC}^*) &\simeq H(BC_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}), & H(\widehat{\overline{BC}}^*) &\simeq H(\widehat{BC}_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}). \end{aligned} \tag{B.9}$$

*Proof.* Because we have a strict unit and a strict augmentation, the maps  $u : D\mathbb{R} \rightarrow DV$  and  $\varepsilon : DV \rightarrow D\mathbb{R}$  satisfy  $\varepsilon \circ u = \mathbb{1}$ . It follows that  $DV = \ker \varepsilon \oplus \text{im } u$ . The same holds for the induced maps  $\bar{u} : \bar{D}\mathbb{R} \rightarrow \bar{D}V$  and  $\bar{\varepsilon} : \bar{D}V \rightarrow \bar{D}\mathbb{R}$ . We can now define the splitting  $r : \overline{BC}(V) \rightarrow \overline{BC}(\mathbb{R})$  of the homological short exact sequence by projecting to  $\ker \bar{\varepsilon}$  along  $\text{im } \bar{u}$ . It is easy to check that it is a morphism of bicomplexes. This dualizes “pointwisely” to cohomology.

Because

$$(\bar{D}\mathbb{R})_i = \begin{cases} 0 & \text{for } i \neq 0, \\ \mathbb{R} & \text{for } i = 0, \end{cases}$$

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the bicomplex  $\overline{BC}(\mathbb{R})$  is diagonal, and hence  $\text{Tot}_I$  and  $\text{Tot}_{II}$  are the same; both compute  $H^\lambda(\mathbb{R})$  (using results from the previous sections). The same is true in cohomology. This shows (B.9).  $\square$

We summarize our results in Figure B.3. Having  $H^\lambda(\mathcal{A}) \simeq H^{\lambda, \text{red}}(\mathcal{A}) \oplus H^\lambda(\mathbb{R})$ , Proposition 3.3.13 in Section 3.3 follows by dualization.

*Remark B.3.2* (Some questions). (i) Suppose that  $V$  has bounded degrees and look at Figure B.3. Does  $H(\widehat{CC}^*) \simeq H(CC^*)$  hold? Does  $H(BC^*) \simeq H(\overline{BC}^*)$  hold?

(ii) Does Proposition 3.3.13 hold for homological unital and homological augmented  $A_\infty$ -algebras? A strategy would be to construct a quasi-isomorphic strictly unital and strictly augmented  $A_\infty$ -algebra. Does it exist?

(iii) How is it with the conditional and strong convergence of spectral sequences associated to different filtrations of bicomplexes from Definition B.2.3? Because of the simple internal data, lots of them collapse.  $\triangleleft$

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$$\begin{array}{ccccc}
 H^\lambda(\mathcal{A}) & \xrightarrow{\text{Lem. B.2.5}} & H(\widehat{CC}) & \xrightarrow{\text{Lem. B.2.6}} & H(CC) \\
 & & \downarrow \text{Lem. B.2.7} & & \downarrow \text{Lem. B.2.7} \\
 & & H(\widehat{BC}) & & H(BC) \\
 & & & & \downarrow \text{Lem. B.2.8} \\
 & & H(\widehat{\overline{BC}}) & \xrightarrow{\text{Lem. B.2.6}} & H(\overline{BC}) \\
 & & \downarrow \text{Lem. B.3.1} & & \downarrow \text{Lem. B.3.1} \\
 & & H(\widehat{BC}^{\text{red}}) \oplus H^\lambda(\mathbb{R}) & & H(BC^{\text{red}}) \oplus H^\lambda(\mathbb{R}) \\
 & & \downarrow \text{Lem. B.2.9} & & \\
 & & H^{\lambda, \text{red}}(\mathcal{A}) \oplus H^\lambda(\mathbb{R}) & & 
 \end{array}$$
  

$$\begin{array}{ccc}
 H(\widehat{CC}^*) & & H(CC^*) \xrightarrow{\text{Lem. B.2.5}} H_\lambda^* \\
 \downarrow \text{Lem. B.2.7} & & \downarrow \text{Lem. B.2.7} \\
 H(\widehat{BC}^*) & & H(BC^*) \\
 \downarrow \text{Lem. B.2.8} & & \\
 H(\widehat{\overline{BC}}^*) & & H(\overline{BC}^*) \\
 \downarrow \text{Lem. B.3.1} & & \downarrow \text{Lem. B.3.1} \\
 H(\widehat{BC}_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}) & & H(BC_{\text{red}}^*) \oplus H_\lambda^*(\mathbb{R}) \\
 & & \downarrow \text{Lem. B.2.9} \\
 & & H_{\lambda, \text{red}}^*(\mathcal{A}) \oplus H_\lambda^*(\mathbb{R})
 \end{array}$$

Figure B.3.: Isomorphisms of (co)homologies for a strictly unital strictly augmented  $A_\infty$ -algebra  $\mathcal{A}$  on a graded vector space  $V$ . A solid line denotes an isomorphism which is always valid and a dashed line an isomorphism which is valid provided that the degrees of  $V$  are bounded. The dotted line denotes the isomorphism obtained by dualizing the corresponding isomorphism in homology under the assumptions that the degrees of  $V$  are bounded.

## C. Towards an invariant definition of canonical IBL-operations

The canonical IBL-operations  $\mathfrak{q}_{210}$  and  $\mathfrak{q}_{120}$  on cyclic cochains of an odd symplectic vector space  $V$  have been defined in coordinates in [CFL15]. In Part I, we used Definition 3.4.1; it takes and gives invariant objects but requires a choice of basis to describe the inner “trace” mechanism. Example A.1.5 shows that an invariant definition can be obtained from the invariant formalism for evaluation of ribbon graphs developed in Appendix A by plugging in the algebraic Schwarz kernel of the identity as a propagator. In this appendix, we ask the following question.

**Question C.0.1.** *Do  $\mathfrak{q}_{210}$  and  $\mathfrak{q}_{120}$  arise as a combination of some natural operations coming from the structure of Hochschild cochains on  $V$  or from odd symplectic geometry?*

In Section C.1, we define a canonical Lie bracket  $[\cdot, \cdot]$  on Hochschild cochains on  $V$  with values in  $V$  (Definition C.1.2 and Proposition C.1.3), which comes from a natural pre-Lie algebra structure which is visualized as grafting of trees (Equation (C.2), Figure C.1 and Lemma C.1.1). This is, in fact, the Gerstenhaber bracket from [Ger63]. Next, we use the symplectic form to define the operator  $+$  which “lowers indices” (Equation C.3) and show that it takes  $[\cdot, \cdot]$  to a Lie bracket on cyclic cochains on  $V$  with values in  $\mathbb{R}$  (Definition C.1.5 and Proposition C.1.6). By writing down everything in coordinates, we show that this Lie bracket is a degree shift of  $\mathfrak{q}_{210}$  (Proposition C.1.8). This answers Question C.0.1 for  $\mathfrak{q}_{210}$ .

In Section C.2, we relate the cobracket  $\mathfrak{q}_{120}$  to the canonical BV-operator  $\Delta_{\text{sym}}$  on “functions” on an odd symplectic vector space (Proposition C.2.4). We do it by rewriting  $\mathfrak{q}_{120}$  in terms of certain double derivative operators on the tensor algebra, which are associated to a given basis (Definition C.2.2). We prove that both definitions are equivalent (Proposition C.2.5). The BV-operator  $\Delta_{\text{sym}}$  can be defined geometrically (see [DJP19]), and the cobracket  $\mathfrak{q}_{120}$  is a factorization of an extension of  $\Delta_{\text{sym}}$  to cyclic invariants with respect to the cyclic shuffle product. We do not know whether this characterization determines  $\mathfrak{q}_{120}$  uniquely, and hence Question C.0.1 for  $\mathfrak{q}_{120}$  remains open

Change notation: Change the notation for the pre-Lie algebra product and for the  $D$ -space!!! I propose  $\triangle$  and

DONE Gerstenhaber bracket: Mention that it is, in fact, the Gerstenhaber bracket!

(see Remark C.2.6 for a list of related questions). This section was stimulated by a discussion with J. Pullman and L. Peksov at a winter school in Srn, 2019.

## C.1. Bracket and grafting of trees

Let  $V$  be a  $\mathbb{Z}$ -graded vector space. We define the weight-graded vector spaces

$$LV := \bigoplus_{k=0}^{\infty} \text{Hom}(V^{\otimes k}, V) \quad \text{and} \quad \bar{L}V := \bigoplus_{k=1}^{\infty} \text{Hom}(V^{\otimes k}, V), \quad (\text{C.1})$$

where  $\text{Hom}$  denotes the graded vector space generated by homogenous morphisms. The latter space is the weight-reduced version of the former (see Definition 3.1.1).

For homogenous  $\psi_1 \in \text{Hom}(V^{\otimes k_1}, V)$ ,  $\psi_2 \in \text{Hom}(V^{\otimes k_2}, V)$  with  $k_1, k_2 \in \mathbb{N}$  and vectors  $v_1, \dots, v_{k_1+k_2-1} \in V$ , we define<sup>1</sup>

$$\begin{aligned} & (\psi_1 * \psi_2)(v_1 \cdots v_{k_1+k_2-1}) \\ &:= \sum_{i=1}^{k_1} (-1)^{\psi_2(v_1+\cdots+v_{i-1})} \psi_1(v_1 \cdots v_{i-1} \psi_2(v_i \cdots v_{k_2+i-1}) v_{k_2+i} \cdots v_{k_1+k_2-1}). \end{aligned} \quad (\text{C.2})$$

Here and almost everywhere in this section, we suppress writing the tensor product. We recall that we use the same symbol  $v$  to denote both a homogenous vector and its degree in the exponent. From (C.2), we get a bilinear operation

$$*: LV \otimes LV \longrightarrow LV,$$

which preserves the degree and decreases the weight by 1. It can be visualized as grafting of trees (see Figure C.1).

**Lemma C.1.1** (Pre-Lie algebra). *Let  $V$  be a graded vector space. The pair  $(LV, *)$  is a graded pre-Lie algebra with respect to the degree, i.e., it holds*

$$(\psi_1 * \psi_2) * \psi_3 - \psi_1 * (\psi_2 * \psi_3) = (-1)^{\psi_2 \psi_3} [(\psi_1 * \psi_3) * \psi_2 - \psi_1 * (\psi_3 * \psi_2)]$$

for all homogenous  $\psi_1, \psi_2, \psi_3 \in LV$ .

*Proof.* In the following computations,  $U$  denotes a tensor product of homogenous vectors from the vector space  $V$  and  $U_i$  denotes a part of a decomposition of  $V$ , i.e.,  $U =$

<sup>1</sup>If  $k_1 = 0$  or  $k_2 = 0$ , we define  $*$  to be 0.

Non-reduced pre-Lie algebra.: Can one define  $*$  differently when there are no inputs and get something non-trivial?



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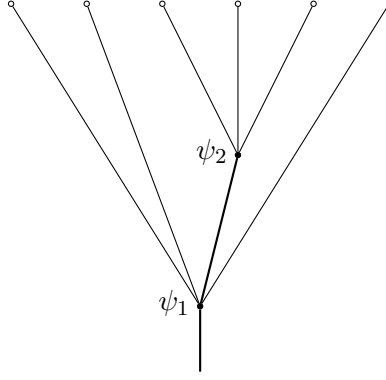


Figure C.1.: The Gerstenhaber bracket as grafting of trees.

$U_1 \cdots U_k$ . We compute schematically

$$\begin{aligned} ((\psi_1 * \psi_2) * \psi_3)(U) = & \sum (-1)^{\psi_3(U_1+U_2+U_3)+\psi_2 U_1} \psi_1(U_1 \psi_2(U_2) U_3 \psi_3(U_4) U_5) \\ & + \sum (-1)^{\psi_3 U_1 + \psi_2(U_1+U_2+U_3+\psi_3)} \psi_1(U_1 \psi_3(U_2) U_3 \psi_2(U_4) U_5) \\ & + \sum (-1)^{\psi_3(U_1+U_2)+\psi_2 U_1} \psi_1(U_1 \psi_2(U_2 \psi_3(U_3) U_4) U_5) \end{aligned}$$

and

$$(\psi_1 * (\psi_2 * \psi_3))(U) = \sum (-1)^{(\psi_2 - \psi_3)U_1 + \psi_3 U_2} \psi_1(U_1 \psi_2(U_2 \psi_3(U_3) U_4) U_5).$$

Therefore, we have

$$\begin{aligned} & [((\psi_1 * \psi_2) * \psi_3) - (\psi_1 * (\psi_2 * \psi_3))](U) \\ &= \sum (-1)^{\psi_3(U_1+U_2+U_3)+\psi_2 U_1} \psi_1(U_1 \psi_2(U_2) U_3 \psi_3(U_4) U_5) \\ & \quad + \sum (-1)^{\psi_3 U_1 + \psi_2(U_1+U_2+U_3+\psi_3)} \psi_1(U_1 \psi_3(U_2) U_3 \psi_2(U_4) U_5) \\ &= (-1)^{\psi_2 \psi_3} \left[ \sum (-1)^{\psi_2(U_1+U_2+U_3)+\psi_3 U_1} \psi_1(U_1 \psi_3(U_2) U_3 \psi_2(U_4) U_5) \right. \\ & \quad \left. + \sum (-1)^{\psi_2 U_1 + \psi_3(U_1+U_2+U_3+\psi_2)} \psi_1(U_1 \psi_2(U_2) U_3 \psi_3(U_4) U_5) \right] \\ &= (-1)^{\psi_2 \psi_3} [((\psi_1 * \psi_3) * \psi_2) - (\psi_1 * (\psi_3 * \psi_2))](U). \end{aligned}$$

This finishes the proof.  $\square$

**Definition C.1.2** (Canonical Lie algebra on Hochschild cochains). *Let  $V$  be a graded*

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vector space. For all  $\psi_1, \psi_2 \in LV$ , we define the bracket by

$$[\psi_1, \psi_2] := \psi_1 * \psi_2 - (-1)^{\psi_1 \psi_2} \psi_2 * \psi_1.$$

It is called the Gerstenhaber bracket.

**Proposition C.1.3** (Canonical Lie algebra on Hochschild cochains). *In the situation of Definition C.1.2, the pair  $(LV, [\cdot, \cdot])$  is a graded Lie algebra with bracket  $[\cdot, \cdot]$  of degree 0 and weight  $-1$ .*

*Proof.* The proof is standard. □

Let  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  be a homogenous bilinear form of degree  $-n$ . This means that

$$\langle v_1, v_2 \rangle \neq 0 \implies v_1 + v_2 = n.$$

To every  $\psi \in \text{Hom}(V^{\otimes k}, V)$ , we associate  $\psi^+ \in \text{Hom}(V^{\otimes k+1}, \mathbb{R})$  defined for all  $v_1, \dots, v_{k+1} \in V$  by the formula

$$\psi^+(v_1 \cdots v_{k+1}) := \langle \psi(v_1 \cdots v_k), v_{k+1} \rangle. \quad (\text{C.3})$$

We define the weight-graded vector spaces

$$L^+V := \bigoplus_{k=0}^{\infty} \text{Hom}(V^{\otimes k}, \mathbb{R}) \quad \text{and} \quad \bar{L}^+V := \bigoplus_{k=1}^{\infty} \text{Hom}(V^{\otimes k}, \mathbb{R}). \quad (\text{C.4})$$

We use here the cohomological grading convention; i.e.,  $\psi \in L^+V$  has degree  $d$  and weight  $k$  if and only if for all homogenous  $v_1, \dots, v_k \in V$ , the following implication holds:

$$\psi(v_1 \cdots v_k) \neq 0 \implies v_1 + \cdots + v_k = d.$$

We say that  $\varphi \in LV$  is cyclically symmetric if

$$\varphi(v_1 \cdots v_k) = \underbrace{(-1)^{v_k(v_1 + \cdots + v_{k-1})}}_{=: \varepsilon(v_1 \cdots v_k, v_k v_1 \cdots v_{k-1})} \varphi(v_k v_1 \cdots v_{k-1})$$

for all homogenous  $v_1, \dots, v_k \in V$ . We define the weight-graded vector spaces

$$\begin{aligned} L_{\text{cyc}}^+V &:= \{\varphi \in L^+V \mid \varphi \text{ is cyclically symmetric}\} \quad \text{and} \\ L_{\text{cyc}}V &:= \{\psi \in LV \mid \psi^+ \in L_{\text{cyc}}^+V\}, \end{aligned}$$

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and similarly for the reduced versions. Formula (C.3) defines the following linear maps of weight  $+1$ :

$$\begin{array}{ccc} + : & LV & \longrightarrow \bar{L}^+ V \\ & \uparrow & \uparrow \\ + := +|_{L_{\text{cyc}} V} : & L_{\text{cyc}} V & \longrightarrow \bar{L}_{\text{cyc}}^+ V \end{array} \quad (\text{C.5})$$

This might be related to Remark 3.3.2 about weight-reduced bar complexes.

Because  $L^+ V$  has the cohomological grading and  $LV$  the standard grading for homogenous maps,  $+$  is not homogenous; instead, it satisfies

$$|\psi^+| = n - |\psi|.$$

It is also useful to note that if  $\psi^+(v_1 \cdots v_{k+1}) \neq 0$ , then

$$\varepsilon(v_1 \cdots v_{k+1}, v_{k+1} v_1 \cdots v_k) = (-1)^{v_{k+1}(\psi^+ - 1)} = (-1)^{v_{k+1}(\psi - n - 1)}.$$

**Lemma C.1.4** (Restriction of Lie bracket to cyclic invariants). *Let  $V$  be a graded vector space and  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  a homogenous bilinear form of degree  $-n$  which is graded antisymmetric. This means that for all homogenous  $v_1, v_2 \in V$ , we have*

$$\langle v_1, v_2 \rangle = (-1)^{1+v_1 v_2} \langle v_2, v_1 \rangle.$$

*Then the Lie bracket  $[\cdot, \cdot]$  on  $LV$  satisfies  $[L_{\text{cyc}} V, L_{\text{cyc}} V] \subset L_{\text{cyc}} V$ , and thus it restricts to the Lie bracket (which we denote by the same symbol)*

$$[\cdot, \cdot] : L_{\text{cyc}} V \otimes L_{\text{cyc}} V \longrightarrow L_{\text{cyc}} V.$$

*Proof.* For any homogenous  $v_1, \dots, v_{k_1+k_2} \in V$ , we compute

$$\begin{aligned} & \langle [\psi_1, \psi_2](v_1, \dots, v_{k_1+k_2-1}), v_{k_1+k_2} \rangle \\ &= \sum_{i=1}^{k_1} (-1)^{\psi_2(v_1 + \cdots + v_{i-1})} \langle \psi_1(v_1 \cdots v_{i-1} \psi_2(v_i \cdots v_{i+k_2-1}) v_{i+k_2} \cdots \\ & \quad v_{k_1+k_2-1}), v_{k_1+k_2} \rangle - (-1)^{\psi_1 \psi_2} \sum_{j=1}^{k_2} (-1)^{\psi_1(v_1 + \cdots + v_{j-1})} \langle \psi_2(v_1 \cdots \\ & \quad v_{j-1} \psi_1(v_j \cdots v_{j+k_1-1}) v_{j+k_1} \cdots v_{k_1+k_2-1}), v_{k_1+k_2} \rangle \end{aligned}$$

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$$\begin{aligned}
&= (-1)^{v_{k_1+k_2}(v_1+\dots+v_{k_1+k_2-1})} \\
&\quad \left[ \sum_{i=1}^{k_1-1} (-1)^{\psi_2(v_{k_1+k_2}+v_1+\dots+v_{i-1})} \langle \psi_1(v_{k_1+k_2}v_1 \dots \psi_2(v_i \dots v_{i+k_2-1}) \right. \\
&\quad \left. v_{i+k_2} \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle + (-1)^{\psi_2(v_1+\dots+v_{k_1-1})+v_{k_1+k_2}\psi_2} \right. \\
&\quad \left. \langle \psi_1(v_{k_1+k_2}v_1 \dots v_{k_1-1}), \psi_2(v_{k_1} \dots v_{k_1+k_2-1}) \rangle \right. \\
&\quad \left. - (-1)^{\psi_1\psi_2} \left( \sum_{j=1}^{k_2-1} (-1)^{\psi_1(v_{k_1+k_2}+v_1+\dots+v_{j-1})} \langle \psi_2(v_1 \dots \psi_1(v_j \dots \right. \right. \\
&\quad \left. \left. v_{j+k_1-1})v_{j+k_1} \dots v_{k_1+k_2-1}), v_{k_1+k_2} \rangle + (-1)^{\psi_1(v_1+\dots+v_{k_2-1})+v_{k_1+k_2}\psi_1} \right. \right. \\
&\quad \left. \left. \langle \psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1}), \psi_1(v_{k_2}, \dots, v_{k_1+k_2-1}) \rangle \right) \right].
\end{aligned}$$

Using graded antisymmetry, we have for the last summand in square brackets

$$\begin{aligned}
&(-1)^{\psi_1(v_1+\dots+v_{k_2-1}+v_{k_1+k_2})} \langle \psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1}), \psi_1(v_{k_2}, \dots, v_{k_1+k_2-1}) \rangle \\
&= (-1)^{1+\psi_1\psi_2+(\psi_2+v_{k_1+k_2}+v_1+\dots+v_{k_2-1})(v_{k_2}+\dots+v_{k_1+k_2-1})} \\
&\quad \langle \psi_1(v_{k_2} \dots v_{k_1+k_2-1}), \psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1}) \rangle \\
&= (-1)^{1+\psi_1\psi_2} \langle \psi_1(\psi_2(v_{k_1+k_2}v_1 \dots v_{k_2-1})v_{k_2} \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle
\end{aligned}$$

and similarly for the second summand

$$\begin{aligned}
&(-1)^{\psi_2(v_1+\dots+v_{k_1-1}+v_{k_1+k_2})} \langle \psi_1(v_{k_1+k_2}v_1 \dots v_{k_1-1}), \psi_2(v_{k_1} \dots v_{k_1+k_2-1}) \rangle \\
&= (-1)^{1+\psi_1\psi_2} \langle \psi_2(\psi_1(v_{k_1+k_2}v_1 \dots v_{k_1-1})v_{k_1} \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle.
\end{aligned}$$

Therefore, the square bracket equals

$$\langle [\psi_1, \psi_2](v_{k_1+k_2}v_1 \dots v_{k_1+k_2-2}), v_{k_1+k_2-1} \rangle$$

and the lemma is proven.  $\square$

The horizontal arrows in (C.5) become isomorphisms provided that  $\langle \cdot, \cdot \rangle$  is non-degenerate (injectivity) and  $V$  is of finite type (surjectivity). Therefore, the following definition is possible.

**Definition C.1.5** (Candidate for IBL-product). *Let  $V$  be a graded vector space of finite type, and let  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  be a non-degenerate homogenous graded antisymmetric bilinear form of degree  $-n$ . Let  $\nu$  be a formal symbol of degree  $n$  realizing the degree shift*

DONE Nondeg, fintype and +: Be careful, I used to have here in addition degrees bounded from below and now I don't see why. NO PROBLEM, THE ISSUE DISCUSSED IN A FOOTNOTE LATER.

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by  $-n$ . We define the operation

$$\mathfrak{q}_{210}^* : (\bar{L}_{\text{cyc}}^+ V)[-n] \otimes (\bar{L}_{\text{cyc}}^+ V)[-n] \longrightarrow (\bar{L}_{\text{cyc}}^+ V)[-n]$$

for all  $\psi_1, \psi_2 \in L_{\text{cyc}} V$  by

$$\mathfrak{q}_{210}^*(\nu\psi_1^+ \otimes \nu\psi_2^+) := \nu[\psi_2, \psi_1]^+.$$

**Proposition C.1.6** (Candidate for IBL-product). *In the situation of Definition C.1.5, the pair  $((\bar{L}_{\text{cyc}}^+ V)[-n], \mathfrak{q}_{210}^*)$  is a graded Lie algebra with the bracket  $\mathfrak{q}_{210}^*$  of degree  $-2n$  and weight  $-2$ .*

*Proof.* As for the degree, we have

$$\begin{aligned} |\nu[\psi_2, \psi_1]^+| &= n + |[\psi_2, \psi_1]^+| \\ &= n - |[\psi_2, \psi_1]| + n \\ &= n - |\psi_1| - |\psi_2| + n \\ &= n + |\psi_1^+| + |\psi_2^+| - n \\ &= -n + |\nu\psi_1^+| + |\nu\psi_2^+| - n \\ &= |\nu\psi_1^+| + |\nu\psi_2^+| - 2n. \end{aligned}$$

The weights are clear. As for the graded anticommutativity, we have

$$\begin{aligned} \mathfrak{q}_{210}^*(\nu\psi_2^+ \otimes \nu\psi_1^+) &= \nu[\psi_1, \psi_2]^+ \\ &= -(-1)^{|\psi_1\psi_2|} \nu[\psi_2, \psi_1]^+ \\ &= -(-1)^{|\nu\psi_1^+||\nu\psi_2^+|} \mathfrak{q}_{210}^*(\nu\psi_1^+ \otimes \nu\psi_2^+), \end{aligned}$$

where we used that

$$|\nu\psi^+| = n + |\psi^+| = n + n - |\psi| = 2n - |\psi|.$$

From the same reason, the graded Jacobi identity for  $\mathfrak{q}_{210}^*$  is implied by the graded Jacobi identity for  $[\cdot, \cdot]$ .  $\square$

In the situation of Definition C.1.5, let  $(e_i)$  be a basis of  $V$ , and let  $(e^i)$  be the dual basis of  $V$  such that  $\langle e_i, e^j \rangle = \delta_i^j$ . To an index  $i$ , we will assign the degree of  $e_i$ , so that, e.g., we can write  $(-1)^i$  instead of  $(-1)^{e_i}$ . We will also use the Einstein summation convention.

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We define the coordinates

$$\begin{aligned} g_{ij} &= \langle e_i, e_j \rangle, \\ g^{ij} &= \langle e^i, e^j \rangle, \\ \psi(e_{i_1} \cdots e_{i_{k_1}}) &= \psi_{i_1 \cdots i_{k_1}}^i e_i, \\ \psi^+(e_{i_1} \cdots e_{i_{k_1+1}}) &= \psi_{i_1 \cdots i_{k_1+1}}, \end{aligned}$$

where  $\psi \in LV$  and  $\psi^+ \in L^+V$ . It holds

$$g_{ij} = (-1)^{ij+1} g_{ji}, \quad g^{ij} = (-1)^{ij+1} g^{ji} \quad \text{and} \quad g_{ik} g^{jk} = \delta_i^j.$$

Equation (C.3) is now equivalent to

$$\psi_{i_1 \cdots i_{k_1}}^i g_{ii_{k_1+1}} = \psi_{i_1 \cdots i_{k_1+1}}, \quad \text{or to} \quad \psi_{i_1 \cdots i_{k_1}}^i = \psi_{i_1 \cdots i_{k_1}} g^{ik}. \quad (\text{C.6})$$

We define the operation  $\mu : (\bar{L}_{\text{cyc}}^+ V)^{\otimes 2} \rightarrow \bar{L}_{\text{cyc}}^+ V$  for  $\varphi_1 \in \text{Hom}_{\text{cyc}}(V^{k_1+1}, \mathbb{R})$ ,  $\varphi_2 \in \text{Hom}_{\text{cyc}}(V^{\otimes k_2+1}, \mathbb{R})$  with  $k_1, k_2 \in \mathbb{N}_0$  such that  $k_1 + k_2 \geq 1$  in coordinates by the formula

$$\begin{aligned} \mu(\varphi_1 \otimes \varphi_2)_{i_1 \cdots i_{k_1+k_2}} &:= \sum_{i,j} \sum_{c=1}^{k_1+k_2} \varepsilon(i_1 \cdots i_{k_1+k_2}, i_c \cdots i_{c-1}) (-1)^{(\varphi_1-i)j} \\ &\quad (-1)^i g^{ij} (\varphi_1)_{ii_c \cdots i_{c+k_1-1}} (\varphi_2)_{ji_{c+k_1} \cdots i_{c-1}}. \end{aligned} \quad (\text{C.7})$$

*Remark C.1.7* (Comparison to [CFL15]). In order to compare  $\mu$  from (C.7) to  $\mu$  from [CFL15, Section 10] (and to the definition of  $\mathfrak{q}_{210}$  from Section 3.4), we must make the replacements

$$V \mapsto V[1], \quad n \mapsto n - 2 \quad \text{and} \quad \bar{L}_{\text{cyc}}^+ V \mapsto B_{\text{cyc}}^* V.$$

Then  $\mathfrak{q}_{210}$  and  $\mathfrak{q}_{210}^* : (B_{\text{cyc}}^* V)[2-n]^{\otimes 2} \rightarrow (B_{\text{cyc}}^* V)[2-n]$  clearly have the same degree  $2(2-n)$ .<sup>2</sup> Also,  $\varphi_1 j + i j$  is precisely the sign needed to move  $j$  over  $i_c \cdots i_{c+k_1-1}$ . We also recall the identification of  $B_{\text{cyc}}^* V$  with the weight-graded dual  $(B^{\text{cyc}} V)^{*_{\text{wg}}}$  from Remark 3.3.5 and remind that  $B^{\text{cyc}} V$  and  $B_{\text{cyc}}^* V$  were defined to be weight-reduced in order to ease the terminology and notation (see Definition 3.3.1).  $\triangleleft$

The equality  $\mathfrak{q}_{210}^* = \mathfrak{q}_{210}$  is obtained from the next proposition (c.f., Remark C.1.7).

**Proposition C.1.8** (Equivalence of definitions of IBL-product). *Let  $V$  be a graded vector space of finite type, and let  $\langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{R}$  be a non-degenerate homogenous graded*

<sup>2</sup>The degree of  $\mathfrak{q}_{210}$  on tensor powers of  $C = B_{\text{cyc}}^* V[2-n]$ , is obtained from the degree on  $EC = S(C[1])$  in Definition 3.4.1 by subtracting 1 (and by adding 1 for the cobracket).

DONE possibly wrong sign: there might be a sign mistake in  $(-1)^a$  because what is needed is probably  $(-1)^{e_a}$  which differs by  $n$ . IT IS CORRECT AS IT IS.

DONE finite type: One can replace finite dimensional by finite type.

DONE wrong degree: The degrees seem wrong.  $\mathfrak{q}_{210}$  has  $-2(n-3)-1$  and... NO PROBLEM.

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antisymmetric bilinear form of degree  $-n$ . Then the operation  $\mathfrak{q}_{210}^* : (\mathbb{L}_{\text{cyc}}^+ V)[-n]^{\otimes 2} \rightarrow (\mathbb{L}_{\text{cyc}}^+ V)[-n]$  defined invariantly in Definition C.1.5 is a degree shift of the operation  $\mu : (\bar{\mathbb{L}}_{\text{cyc}}^+ V)^{\otimes 2} \rightarrow \bar{\mathbb{L}}_{\text{cyc}}^+ V$  defined in coordinates above. More precisely, for all  $\varphi_1, \varphi_2 \in \bar{\mathbb{L}}_{\text{cyc}}^+ V$ , it holds

$$\mathfrak{q}_{210}^*(\nu^2 \varphi_1 \otimes \varphi_2) = \nu \mu(\varphi_1 \otimes \varphi_2) \quad \text{in } (\bar{\mathbb{L}}_{\text{cyc}}^+ V)[-n].$$

*Proof.* Consider  $\psi_1, \psi_2 \in \mathbb{L}_{\text{cyc}} V$  with  $k_1, k_2$  inputs, respectively. Writing the definition of  $[\cdot, \cdot]$  in coordinates, we get

$$\begin{aligned} & [\psi_1, \psi_2]_{i_1 \dots i_{k_1+k_2-1}}^i \\ &= \sum_{j=1}^{k_1} (-1)^{\psi_2(i_1+\dots+i_{j-1})} (\psi_1)_{i_1 \dots i_{j-1} k i_j+k_2 \dots i_{k_1+k_2-1}}^i (\psi_2)_{i_j \dots i_{j+k_2-1}}^k \\ &+ \sum_{j=1}^{k_2} (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1} (\psi_2)_{i_1 \dots i_{j-1} k i_j+k_1 \dots i_{k_1+k_2-1}}^i (\psi_1)_{i_j \dots i_{j+k_1-1}}^k. \end{aligned}$$

Using (C.6) and the cyclic symmetry of  $\psi_1^+$  and  $\psi_2^+$ , we compute

$$\begin{aligned} & [\psi_1, \psi_2]_{i_1 \dots i_{k_1+k_2}} \\ &= [\psi_1, \psi_2]_{i_1 \dots i_{k_1+k_2-1}}^i g_{i i_{k_1+k_2}} \\ &= \sum_{j=1}^{k_1} (-1)^{\psi_2(i_1+\dots+i_{j-1})} (\psi_1)_{i_1 \dots i_{j-1} k i_j+k_2 \dots i_{k_1+k_2}} (\psi_2)_{i_j \dots i_{j+k_2-1}}^k \\ &+ \sum_{j=1}^{k_2} (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1} (\psi_2)_{i_1 \dots i_{j-1} k i_j+k_1 \dots i_{k_1+k_2}} (\psi_1)_{i_j \dots i_{j+k_1-1}}^k \\ &= \sum_{j=1}^{k_1} (-1)^{\psi_2(i_1+\dots+i_{j-1})} g^{ab} (\psi_1)_{i_1 \dots i_{j-1} a i_j+k_2 \dots i_{k_1+k_2}} (\psi_2)_{i_j \dots i_{j+k_2-1}}^b \\ &+ \sum_{j=1}^{k_2} (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1} g^{ab} (\psi_2)_{i_1 \dots i_{j-1} a i_j+k_1 \dots i_{k_1+k_2}} (\psi_1)_{i_j \dots i_{j+k_1-1}}^b \\ &= \sum_{j=1}^{k_1} \underbrace{(-1)^{\psi_2(i_1+\dots+i_{j-1})+(i_1+\dots+i_{j-1})(\psi_1-n-1)+b(\psi_2-n-1)}}_{=:\varepsilon_1} \\ &\quad g^{ab} (\psi_1)_{a i_j+k_2 \dots i_{k_1+k_2}} i_1 \dots i_{j-1} (\psi_2)_{b i_j \dots i_{j+k_2-1}} \\ &+ \sum_{j=1}^{k_2} \underbrace{(-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+1+(i_1+\dots+i_{j-1})(\psi_2-n-1)+a(\psi_1-n-1)+ab+1}}_{=:\varepsilon_2} \\ &\quad g^{ab} (\psi_1)_{a i_j \dots i_{j+k_1-1}} (\psi_2)_{b i_j+k_1 \dots i_{k_1+k_2}} i_1 \dots i_{j-1}. \end{aligned}$$

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Notice that the lower indices  $i_1, \dots, i_{k_1+k_2}$  appear in cyclic permutations. The first sum consists of cyclic permutation starting with  $i_{k_2+1}$  and going up to  $i_{k_1+k_2+1}$ , and the second sum consists of cyclic permutations starting with  $i_1$  and going up to  $i_{k_2}$ ; hence, all cyclic permutations appear, and it just remains to check the signs. We see that

$$\begin{aligned}\varepsilon_1 &= (-1)^{\psi_2(i_1+\dots+i_{j-1})+(i_1+\dots+i_{j-1})(\psi_1-n-1)+b(\psi_2-n-1)} \\ &= \underbrace{(-1)^{(i_1+\dots+i_{j-1}+\psi_2-n-b)(\psi_1-n-a+\psi_2-n-b-1)}}_{=\varepsilon(i_1\dots i_{k_1+k_2}, i_{j+k_2}\dots i_{k_1+k_2} i_1\dots i_{j+k_2-1})} (-1)^{b(\psi_1-n-a)} (-1)^a \\ &\quad (-1)^{n(\psi_1-n)+\psi_2\psi_1}\end{aligned}$$

and

$$\begin{aligned}\varepsilon_2 &= (-1)^{\psi_1(i_1+\dots+i_{j-1}+\psi_2)+(i_1+\dots+i_{j-1})(\psi_2-n-1)+a(\psi_1-n-1)+ab} \\ &= \underbrace{(-1)^{(i_1+\dots+i_{j-1})(\psi_1-n-a+\psi_2-n-b-1)}}_{=\varepsilon(i_1\dots i_{k_1+k_2}, i_j\dots i_{k_1+k_2} i_1\dots i_{j-1})} (-1)^{b(\psi_1-n-a)} (-1)^a \\ &\quad (-1)^{n(\psi_1-n)+\psi_1\psi_2}.\end{aligned}$$

This finishes the proof.  $\square$

## C.2. Cobracket and odd symplectic geometry

We start with a remark about the naive dualization of the bracket on the dual.

*Remark C.2.1* (Naive dualization). If we ignore the fact that the dualization of spaces like  $LV$  is problematic and we might not get  $(LV^{*\mathfrak{g}})^{*}\mathfrak{g} \simeq LV$ ,<sup>3</sup> a naive way to obtain a cobracket  $\bar{\delta} : LV \rightarrow LV \otimes LV$  would be to dualize the canonical bracket  $[\cdot, \cdot]^* : LV^{*\mathfrak{g}} \otimes LV^{*\mathfrak{g}} \rightarrow LV^{*\mathfrak{g}}$  from the previous section (replacing  $V$  by  $V^{*\mathfrak{g}}$ ). Because  $w([\cdot, \cdot]^*) = -1$ , the dual would have  $w(\bar{\delta}) = 1$ , where  $w$  denotes the weights. However, the IBL-cobracket has  $w(\mathfrak{q}_{120}) = -2$  (see Definition 3.4.1), and hence

$$w(((+)^{-1} \otimes (+)^{-1}) \circ \mathfrak{q}_{120} \circ +) = w(\mathfrak{q}_{120}) - 1 = -3 \neq 1 = w(\bar{\delta}),$$

where the map  $^+ : LV \rightarrow L^+V$  of weight  $+1$  was defined in (C.5). It might be interesting to write down the coordinate expression for such  $\bar{\delta}$  which comes from  $\mathfrak{q}_{120}$  and study its origin and properties on  $LV$ .  $\triangleleft$

<sup>3</sup>It holds  $V \simeq (V^{*\mathfrak{g}})^{*}\mathfrak{g}$  provided that  $V$  is of finite type. The tensor product of graded vector spaces of finite type is also of finite type provided that the grading is non-negative. Moreover, one has to require that  $W^0 = W^1 = 0$  for  $W$  for which  $V = W[1]$  or take suitable completions with respect to weights in order to deal with arbitrary long tensor products in the same degree in the dual of  $LV$ .



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In the following definition, we will use the terms “cobracket” and “BV-operator” to denote some operations without actually knowing that they satisfy the corresponding relations (see Remark C.2.6 at the end of this section).

**Definition C.2.2** (Hochschild cobracket and BV-operator). *Let  $V$  be a graded vector space, let  $(e_i) \in V$  be its homogenous basis, and let  $(\eta^i) \subset V^{*\mathfrak{g}}$  be the dual basis, i.e., it holds  $\eta^i(e_j) = \delta_j^i$ . We denote by  $T(V^{*\mathfrak{g}})$  the tensor algebra over  $V^{*\mathfrak{g}}$  and consider its basis  $\eta^{\otimes I} := \eta^{i_1} \otimes \dots \otimes \eta^{i_k}$  for all multiindices  $I = (i_1, \dots, i_k)$ .*

*For each  $i, j$ , we define the cyclic double derivative  $\text{Der}_{ij} : T(V^{*\mathfrak{g}}) \longrightarrow T(V^{*\mathfrak{g}}) \otimes T(V^{*\mathfrak{g}})$  on the basis by the following formula, which we explain below:*

$$\text{Der}_{ij}(\eta^{\otimes I}) := \sum_{(i,j) \in I} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \frac{1}{k_1} \frac{1}{k_2} \varepsilon(I, ij I_1^{c_1} I_2^{c_2}) (\eta^{\otimes I_1^{c_1}} \otimes \eta^{\otimes I_2^{c_2}}) \quad (\text{C.8})$$

*The first sum is over all pairs of indices  $(i, j)$  at two distinct positions in  $I$ . We consider the cyclic order on positions in  $I$  (of length  $k$ ) starting from the left and define  $I_1$  and  $I_2$  as the substrings between  $i$  and  $j$  (of length  $k_1$ ) and between  $j$  and  $i$  (of length  $k_2$ ), respectively. We include neither  $i$  nor  $j$  in  $I_1$  or  $I_2$ , i.e.,  $k = k_1 + k_2 + 2$ . The second sum is an average over all cyclic permutations  $c_1$  and  $c_2$  of  $I_1$  and  $I_2$ , respectively. As usual,  $\varepsilon$  stands for the Koszul sign and  $i$  has the degree of  $e_i$ . If  $k_1$  or  $k_2$  vanishes, we replace the average over  $c_1$  or  $c_2$ , respectively, by 1 and set  $\eta^{\emptyset} = 1$ .*

*On  $T(V^{*\mathfrak{g}})$ , we consider the shuffle product  $\mu^{\text{sh}} : T(V^{*\mathfrak{g}}) \otimes T(V^{*\mathfrak{g}}) \rightarrow T(V^{*\mathfrak{g}})$  which is defined by*

$$\mu^{\text{sh}}(\eta^{\otimes I_1}, \eta^{\otimes I_2}) := \sum_{\mu \in \mathbb{S}_{k_1, k_2}} \frac{k_1! k_2!}{(k_1 + k_2)!} \varepsilon(I_1 I_2, \mu(I_1 I_2)) \eta^{\otimes \mu(I_1 I_2)},$$

*where  $k_1$  and  $k_2$  are the lengths of  $I_1$  and  $I_2$ , respectively, and  $\mathbb{S}_{k_1, k_2}$  denotes the shuffle permutations.*

*If  $V$  is of finite type and  $\langle \cdot, \cdot \rangle$  is a non-degenerate homogenous bilinear form on  $V$ , we consider the coordinates  $g^{ij} = \langle e^i, e^j \rangle$ , where  $(e^i) \subset V$  is the basis such that  $\langle e^i, e_j \rangle = \delta_j^i$ . We define the Hochschild cobracket  $\zeta_{\text{H}} : T(V^{*\mathfrak{g}}) \rightarrow T(V^{*\mathfrak{g}}) \otimes T(V^{*\mathfrak{g}})$  by*

$$\zeta_{\text{H}} = \frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} \text{Der}_{ij} \quad (\text{C.9})$$

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and the Hochschild BV-operator  $\Delta_H : T(V^{*\mathfrak{g}}) \rightarrow T(V^{*\mathfrak{g}})$  by

$$\Delta_H := \mu^{\text{sh}} \circ \zeta_H. \quad (\text{C.10})$$

It is easy to check that the definitions of  $\mu^{\text{sh}}$ ,  $\zeta_H$  and  $\Delta_H$  do not depend on the choice of the basis  $(e_i)$  of  $V$ .

We denote by  $T(V^{*\mathfrak{g}})_{\text{cyc}}$  the subspace of cyclic invariants of  $T(V^{*\mathfrak{g}})$  and by  $T(V^{*\mathfrak{g}})_{\text{sym}}$  the subspace of symmetric invariants. We consider the cyclization map  $\pi_{\text{cyc}} : T(V^{*\mathfrak{g}}) \rightarrow T(V^{*\mathfrak{g}})_{\text{cyc}}$  and the symmetrization map  $\pi_{\text{sym}} : T(V^{*\mathfrak{g}}) \rightarrow T(V^{*\mathfrak{g}})_{\text{sym}}$ ; they are defined by the averages

$$\begin{aligned} \pi_{\text{cyc}}(\eta^{\otimes I}) &= \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \frac{1}{k} \varepsilon(I, I^c) \eta^{\otimes I^c} \quad \text{and} \\ \pi_{\text{sym}}(\eta^{\otimes I}) &= \sum_{\sigma \in \mathbb{S}_k} \frac{1}{k!} \varepsilon(I, I^\sigma) \eta^{\otimes I^\sigma}, \end{aligned}$$

respectively, where  $k$  is the length of  $I$  (if  $k = 0$ , we have  $\eta^\emptyset = 1$  and  $\pi_{\text{cyc}}(1) := 1$ ). We denote by  $p_{\text{cyc}}$  the projection to cyclic coinvariants and by  $p_{\text{sym}}$  the projection to symmetric coinvariants.

**Definition C.2.3** (Cyclic and symmetric versions). *In the situation of Definition C.2.2, we define the cyclic and symmetric versions of the operations  $\text{Der}_{ij}$ ,  $\zeta_H$ ,  $\mu^{\text{sh}}$  and  $\Delta_H$  by precomposing with the inclusions  $\iota_{\text{cyc}} : T(V^{*\mathfrak{g}})_{\text{cyc}} \rightarrow T(V^{*\mathfrak{g}})$  and  $\iota_{\text{sym}} : T(V^{*\mathfrak{g}})_{\text{sym}} \rightarrow T(V^{*\mathfrak{g}})$  and postcomposing with the projections  $\pi_{\text{cyc}} : T(V^{*\mathfrak{g}}) \rightarrow T(V^{*\mathfrak{g}})_{\text{cyc}}$  and  $\pi_{\text{sym}} : T(V^{*\mathfrak{g}}) \rightarrow T(V^{*\mathfrak{g}})_{\text{sym}}$ , respectively. In formulas, we have*

$$\begin{aligned} D_{ij}^{\text{cyc}} &:= (\pi_{\text{cyc}} \otimes \pi_{\text{cyc}}) \circ \text{Der}_{ij} \circ \iota_{\text{cyc}}, & D_{ij}^{\text{sym}} &:= (\pi_{\text{sym}} \otimes \pi_{\text{sym}}) \circ \text{Der}_{ij} \circ \iota_{\text{sym}}, \\ \zeta_{\text{cyc}} &:= (\pi_{\text{cyc}} \otimes \pi_{\text{cyc}}) \circ \zeta_H \circ \iota_{\text{cyc}}, & \zeta_{\text{sym}} &:= (\pi_{\text{sym}} \otimes \pi_{\text{sym}}) \circ \zeta_H \circ \iota_{\text{sym}}, \\ \mu_{\text{cyc}} &:= \pi_{\text{cyc}} \circ \mu^{\text{sh}} \circ (\iota_{\text{cyc}} \otimes \iota_{\text{cyc}}), & \mu_{\text{sym}} &:= \pi_{\text{sym}} \circ \mu^{\text{sh}} \circ (\iota_{\text{sym}} \otimes \iota_{\text{sym}}), \\ \Delta_{\text{cyc}} &:= \pi_{\text{cyc}} \circ \Delta_H \circ \iota_{\text{cyc}}, & \Delta_{\text{sym}} &:= \pi_{\text{sym}} \circ \Delta_H \circ \iota_{\text{sym}}. \end{aligned} \quad (\text{C.11})$$

**Proposition C.2.4** (Hochschild, cyclic and symmetric operations). *In the situation of Definition C.2.3, we consider for every multiindex  $I$  the linear functional  $\alpha^I : TV \rightarrow \mathbb{R}$  given with respect to the basis  $e_{\otimes J}$  by*

$$\alpha^I(e_{\otimes J}) := \varepsilon(I, J).$$

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We set

$$\alpha_{\text{sym}}^I = \sum_{\sigma \in \mathbb{S}_k} \frac{1}{k!} \varepsilon(I, I^\sigma) \alpha^{I^\sigma} \quad \text{and} \quad \alpha_{\text{cyc}}^I = \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \frac{1}{k} \varepsilon(I, I^c) \alpha^{I^c}.$$

We define the monomorphisms  $I$ ,  $I_{\text{cyc}}$  and  $I_{\text{sym}}$  by

$$\begin{array}{ccc} TV^{*\mathfrak{g}} & \xhookrightarrow{I} & L^+V : & \eta^{\otimes I} & \longmapsto & \alpha^I, \\ \uparrow & & \uparrow & & & \\ (TV^{*\mathfrak{g}})_{\text{cyc}} & \xhookrightarrow{I_{\text{cyc}}} & L_{\text{cyc}}^+V : & \pi_{\text{cyc}}(\eta^{\otimes I}) & \longmapsto & \alpha_{\text{cyc}}^I, \\ \uparrow & & \uparrow & & & \\ (TV^{*\mathfrak{g}})_{\text{sym}} & \xhookrightarrow{I_{\text{sym}}} & L_{\text{sym}}^+V : & \pi_{\text{sym}}(\eta^{\otimes I}) & \longmapsto & \alpha_{\text{sym}}^I. \end{array} \quad (\text{C.12})$$

The vertical arrows are inclusions and the diagram commutes. Moreover,  $I$ ,  $I_{\text{cyc}}$  and  $I_{\text{sym}}$  become isomorphisms provided that  $V$  is of finite type with non-negative degrees. In this case, we transfer the operations and obtain the following commutative diagram:

$$\begin{array}{ccccc} \Delta_{\text{H}} : & L^+V & \xrightarrow{\zeta_{\text{H}}} & L^+V \otimes L^+V & \xrightarrow{\mu^{\text{sh}}} & L^+V \\ & \uparrow & & \uparrow & & \uparrow \\ \Delta_{\text{cyc}} : & L_{\text{cyc}}^+V & \xrightarrow{\zeta_{\text{cyc}}} & L_{\text{cyc}}^+V \otimes L_{\text{cyc}}^+V & \xrightarrow{\mu_{\text{cyc}}} & L_{\text{cyc}}^+V \\ & \uparrow & & \uparrow & & \uparrow \\ \Delta_{\text{sym}} : & L_{\text{sym}}^+V & \xrightarrow{\zeta_{\text{sym}}} & L_{\text{sym}}^+V \otimes L_{\text{sym}}^+V & \xrightarrow{\mu_{\text{sym}}} & L_{\text{sym}}^+V. \end{array} \quad (\text{C.13})$$

The following formulas hold:

$$\begin{aligned} \zeta_{\text{cyc}}(\alpha_{\text{cyc}}^I) &= \frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} \sum_{(i,j) \in I} \varepsilon(I, ijI_1I_2) (\alpha_{\text{cyc}}^{I_1} \otimes \alpha_{\text{cyc}}^{I_2}), \\ \mu_{\text{sym}}(\alpha_{\text{sym}}^{I_1}, \alpha_{\text{sym}}^{I_2}) &= \alpha_{\text{sym}}^{I_1I_2}, \\ \Delta_{\text{sym}}(\alpha_{\text{sym}}^I) &= \frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} \sum_{(i,j) \in I} \varepsilon(I, ijI_1I_2) \alpha_{\text{sym}}^{I_1I_2}. \end{aligned} \quad (\text{C.14})$$

*Proof.* The commutativity of (C.12) is clear from the definitions. The commutativity of (C.13) is equivalent to the commutativity of the same diagram with  $L^+V$ ,  $L_{\text{cyc}}^+V$ ,  $L_{\text{sym}}^+V$  replaced by  $T(V^{*\mathfrak{g}})$ ,  $T(V^{*\mathfrak{g}})_{\text{cyc}}$ ,  $T(V^{*\mathfrak{g}})_{\text{sym}}$ , respectively. This commutativity then follows from the defining relations (C.11) if we show the following inclusions:

$$(1) \quad \text{Der}_{ij}(T(V^{*\mathfrak{g}})_{\text{cyc}}) \subset T(V^{*\mathfrak{g}})_{\text{cyc}} \otimes T(V^{*\mathfrak{g}})_{\text{cyc}},$$

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$$(2) \text{ Der}_{ij}(\mathbf{T}(V^*_{\mathfrak{g}})_{\text{sym}}) \subset \mathbf{T}(V^*_{\mathfrak{g}})_{\text{sym}} \otimes \mathbf{T}(V^*_{\mathfrak{g}})_{\text{sym}},$$

$$(3) \mu^{\text{sh}}(\mathbf{T}(V^*_{\mathfrak{g}})_{\text{cyc}} \otimes \mathbf{T}(V^*_{\mathfrak{g}})_{\text{cyc}}) \subset \mathbf{T}(V^*_{\mathfrak{g}})_{\text{cyc}},$$

$$(4) \mu^{\text{sh}}(\mathbf{T}(V^*_{\mathfrak{g}})_{\text{sym}} \otimes \mathbf{T}(V^*_{\mathfrak{g}})_{\text{sym}}) \subset \mathbf{T}(V^*_{\mathfrak{g}})_{\text{sym}}.$$

Inclusion (1) is clear from (C.8). In fact, it holds even  $\text{Der}_{ij}(\mathbf{T}(V^*_{\mathfrak{g}})) \subset \mathbf{T}(V^*_{\mathfrak{g}})_{\text{cyc}} \otimes \mathbf{T}(V^*_{\mathfrak{g}})_{\text{cyc}}$ . As for inclusion (2), we have

$$\text{Der}_{ij}(\pi_{\text{sym}}(\eta^{\otimes I})) = \sum_{\sigma \in \mathbb{S}_k} \frac{1}{k!} \sum_{(i,j) \in I^\sigma} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \frac{1}{k_1} \frac{1}{k_2} \varepsilon(I, ij I_1^{\sigma c_1} I_2^{\sigma c_2}) \eta^{\otimes I_1^{\sigma c_1}} \otimes \eta^{\otimes I_2^{\sigma c_2}}.$$

A pair  $(\sigma_1, \sigma_2) \in \mathbb{S}_{k_1} \times \mathbb{S}_{k_2}$  induces a bijection of the domain of summation, which consists of the elements  $\sigma, (i, j), c_1, c_2$ , by keeping  $(i, j), c_1, c_2$  fixed and defining a new  $\tilde{\sigma}$  by altering  $\sigma$  to compensate for the effect of  $(\sigma_1, \sigma_2)$ . The signs fit since we only consider the Koszul sign. Therefore, we have

$$(\sigma_1 \otimes \sigma_2) \text{Der}_{ij}(\pi_{\text{sym}}(\eta^{\otimes I})) = \text{Der}_{ij}(\pi_{\text{sym}}(\eta^{\otimes I})),$$

and (2) follows. As for (3), we have

$$\begin{aligned} & \mu^{\text{sh}}(\pi_{\text{cyc}}(\eta^{\otimes I_1}), \pi_{\text{cyc}}(\eta^{\otimes I_2})) \\ &= \sum_{\mu \in \mathbb{S}_{k_1, k_2}} \frac{k_1! k_2!}{(k_1 + k_2)!} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \frac{1}{k_1} \frac{1}{k_2} \varepsilon(I_1 I_2, \mu(I_1^{c_1}, I_2^{c_2})) \eta^{\otimes \mu(I_1^{c_1}, I_2^{c_2})}. \end{aligned}$$

A permutation  $c \in \mathbb{S}_k$  again induces a bijection of the domain of summation, which consists of the elements  $\mu, c_1, c_2$ . A similar argument holds for (4).

We now derive (C.14). Because

$$\text{Der}_{ij}(\alpha_{\text{cyc}}^I) = \sum_{(i,j) \in I} \varepsilon(I, ij I_1 I_2) \alpha_{\text{cyc}}^{I_1} \otimes \alpha_{\text{cyc}}^{I_2},$$

the formula for  $\zeta_{\text{cyc}}$  is clear. As for  $\Delta_{\text{sym}}$ , because  $\mu_{\text{sym}} = \pi_{\text{sym}} \circ \mu_{\text{con}}$  for the concatenation product  $\mu_{\text{con}}(\eta^{\otimes I_1}, \eta^{\otimes I_2}) = \eta^{\otimes I_1 I_2}$ , we have

$$(\mu_{\text{sym}} \circ \text{Der}_{ij})(\pi_{\text{sym}}(\eta^{\otimes I})) = \sum_{\sigma \in \mathbb{S}_k} \frac{1}{k!} \sum_{(i,j) \in I^\sigma} \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \frac{1}{k_1} \frac{1}{k_2} \pi_{\text{sym}}(\varepsilon(I, ij I_1^{\sigma c_1} I_2^{\sigma c_2}) \eta^{\otimes I_1^{\sigma c_1}} I_2^{\sigma c_2})$$

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$$= \sum_{(i,j) \in I} \varepsilon(I, ijI_1I_2) \pi_{\text{sym}}(\eta^{\otimes I_1I_2}),$$

where we used that  $\pi_{\text{sym}} \circ \sigma = \pi_{\text{sym}}$  for every permutation  $\sigma$ . The formula for  $\Delta_{\text{sym}}$  follows. The formula for  $\mu_{\text{sym}}$  is also clear.  $\square$

In the formula (C.14) for  $\Delta_{\text{sym}}$ , we recognize the canonical BV-operator on  $SV^{*\mathfrak{g}}$  from [DJP19, Definition 4].<sup>4</sup>

We will now relate  $\zeta_{\text{cyc}}$  to  $\mathfrak{q}_{120}$ . The steps are similar to Section C.1. We define the coproduct  $\mathfrak{q}_{120}^* : (\bar{L}_{\text{cyc}}^+ V)[-n] \rightarrow (\bar{L}_{\text{cyc}}^+ V)[-n]^{\otimes 2}$ , where  $-n$  is the degree of  $\langle \cdot, \cdot \rangle$ , by

$$\mathfrak{q}_{120}^*(\nu\varphi) = \nu^2 \zeta_{\text{cyc}}(\varphi) \quad \text{for all } \varphi \in \bar{L}_{\text{cyc}}^+ V, \quad (\text{C.15})$$

where  $\nu$  is a formal symbol of degree  $n$ .

For  $\varphi \in \text{Hom}_{\text{cyc}}(V^{\otimes k}, \mathbb{R})$  with  $k \geq 4$ , we define  $(\delta\varphi) \in (\bar{L}_{\text{cyc}}^+ V)^{\otimes 2}$  using the coordinates

$$\varphi_I := \varphi(e_{\otimes I}) \quad \text{and} \quad (\delta\varphi)_{I_1; I_2} = (\delta\varphi)(e_{\otimes I_1} \otimes e_{\otimes I_2}) \quad (\text{C.16})$$

by

$$(\delta\varphi)_{I_1; I_2} := \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \sum_{i,j} (-1)^{e_i} g^{ij} \varepsilon(I_1, I_1^c) \varepsilon(I_2, I_2^c) (-1)^{e_j I_1} \varphi_{iI_1^{c_1} jI_2^{c_2}}. \quad (\text{C.17})$$

We set  $\delta\varphi = 0$  for  $k \leq 3$ . In order to obtain  $(\delta\varphi) \in (\bar{L}_{\text{cyc}}^+ V)^{\otimes 2}$ , we use the embedding  $(\bar{L}_{\text{cyc}}^+ V)^{\otimes k} \subset (\text{TV})^{\otimes k*}$  from Remark 3.3.5. Note that we can also use the canonical identification  $(\text{TV}/\text{cyc})^{*\text{wg}} \simeq \bar{L}_{\text{cyc}}^+ V$  and work with the cyclic words  $e_I := p_{\text{cyc}}(e_{\otimes I})$  instead. We see that (C.17) is precisely the defining equation [CFL15, Equation (10.6)] for the IBL-cobracket  $\delta : \bar{L}_{\text{cyc}}^+ V \rightarrow (\bar{L}_{\text{cyc}}^+ V)^{\otimes 2}$ , and  $\mathfrak{q}_{120} : (\bar{L}_{\text{cyc}}^+ V)[-n] \rightarrow (\bar{L}_{\text{cyc}}^+ V)[-n]^{\otimes 2}$  is the degree shift of  $\delta$ .

The equality  $\mathfrak{q}_{120}^* = \mathfrak{q}_{120}$  on the reduced part is obtained from the next proposition (c.f., Remark C.1.7 for the translation from the  $\bar{L}_{\text{cyc}}^+$ - to the  $B_{\text{cyc}}^*$ -notation).

**Proposition C.2.5** (Equivalence of definitions of IBL-cobracket). *Let  $V$  be a graded vector space of finite type, and let  $\langle \cdot, \cdot \rangle$  be a non-degenerate homogenous bilinear form of degree  $-n$ . For every  $\varphi \in \bar{L}_{\text{cyc}}^+ V$ , it holds*

$$\mathfrak{q}_{120}^*(\nu\varphi) = \nu^2 \delta\varphi \quad \text{in} \quad (\bar{L}_{\text{cyc}}^+ V)[-n]^{\otimes 2}.$$

<sup>4</sup>In fact, they work on a bigger “algebra of functions” on  $V$ , which is a completion of  $SV^{*\mathfrak{g}}$  with respect to a suitable filtration. Their statement is that the algebra of functions on an odd symplectic vector space carries a canonical BV-operator. It is given by the divergence of the Hamiltonian vector field.

DONE finite type: One can replace finite dimensional by finite type.

### C. Towards an invariant definition of canonical IBL-operations

*Proof.* We consider the basis  $(\alpha_{\text{cyc}}^I)$  of  $L_{\text{cyc}}^+ V$  from Proposition C.2.4. Given  $\varphi \in L_{\text{cyc}}^+ V$ , we can write  $\varphi = \sum_I \frac{1}{|I|} \varphi_I \alpha_{\text{cyc}}^I$ , where we sum over all multiindices  $I$  and  $\varphi_I \in \mathbb{R}$  are coefficients cyclically symmetric in  $I$ . For every  $I$  of length  $k$ , we have

$$\varphi(e_{\otimes I}) = \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \frac{1}{k} \varphi_{I^c} \alpha_{\text{cyc}}^{I^c}(e_{\otimes I}) = \sum_{c \in \mathbb{S}_k^{\text{cyc}}} \frac{1}{k} \varphi_{I^c} \varepsilon(I^c, I) = \varphi_I,$$

and hence  $\varphi_I$  correspond to the coefficients (C.16). Using (C.14), we compute

$$\begin{aligned} \zeta_{\text{cyc}}(\varphi) &= \frac{1}{2} \sum_I \sum_{(i,j) \in I} \frac{1}{|I|} \varphi_I (-1)^{e_i} g^{ij} \varepsilon(I, ij I_1 I_2) \alpha_{\text{cyc}}^{I_1} \otimes \alpha_{\text{cyc}}^{I_2} \\ &= \frac{1}{2} \sum_J \sum_{c \in \mathbb{S}_{|J|}^{\text{cyc}}} \sum_{j \in J \setminus \{j_1\}} \frac{1}{|J|} \varphi_{J^c} (-1)^{e_{j_1}} g^{j_1 j} (-1)^{j J_1} \varepsilon(J, J^c) \alpha_{\text{cyc}}^{J_1} \otimes \alpha_{\text{cyc}}^{J_2} \\ &= \frac{1}{2} \sum_J \sum_{j \in J \setminus \{j_1\}} \varphi_J (-1)^{e_{j_1}} g^{j_1 j} (-1)^{j J_1} \alpha_{\text{cyc}}^{J_1} \otimes \alpha_{\text{cyc}}^{J_2} \\ &= \sum_{J_1, J_2} \underbrace{\left( \frac{1}{2} \sum_{i,j} (-1)^{e_i} g^{ij} (-1)^{j J_1} \varphi_{i J_1 j J_2} \right)}_{=:(*)_{J_1; J_2}} \alpha_{\text{cyc}}^{J_1} \otimes \alpha_{\text{cyc}}^{J_2}. \end{aligned}$$

On the second line, we used the bijection of the summation domains consisting of  $I$ ,  $(i, j)$ , resp.  $J$ ,  $c$ ,  $b$ , which assigns to  $I$  its cyclic permutation  $J$  such that the position of  $i$  corresponds to the first position of  $J$ , i.e., it holds  $j_1 = i$ . On the third line, we used that  $\varphi_I$  are cyclic symmetric, and on the fourth line, we just rewrote the summation using  $J = i J_1 j J_2$ . We now relate  $(*)_{J_1; J_2}$  to  $(\delta\varphi)_{J_1; J_2}$ . We compute (notice that there is no Koszul sign from the application of the tensor product, c.f., Remark 3.3.5)

$$\begin{aligned} \zeta_{\text{cyc}}(\varphi)(e_{I_1} \otimes e_{I_2}) &= \left( \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} (*)_{I_1^{c_1}; I_2^{c_2}} \alpha_{\text{cyc}}^{I_1^{c_1}} \otimes \alpha_{\text{cyc}}^{I_2^{c_2}} \right) (e_{I_1} \otimes e_{I_2}) \\ &= \sum_{\substack{c_1 \in \mathbb{S}_{k_1}^{\text{cyc}} \\ c_2 \in \mathbb{S}_{k_2}^{\text{cyc}}}} \varepsilon(I_1^{c_1}, I_1) \varepsilon(I_2^{c_2}, I_2) (*)_{I_1^{c_1}; I_2^{c_2}} \\ &= (\delta\varphi)_{J_1; J_2}. \end{aligned}$$

This shows the proposition. □

*Remark C.2.6* (Some questions). (i) Are  $(L_{\text{cyc}}^+ V, \Delta_{\text{cyc}}, \mu_{\text{cyc}})$  and  $(L^+ V, \Delta_{\text{H}}, \mu^{\text{sh}})$  also BV-algebras like  $(L_{\text{sym}}^+ V, \Delta_{\text{sym}}, \mu_{\text{sym}})$ ? Recall the 2nd-derivative identity (or the "seven

### C. Towards an invariant definition of canonical IBL-operations

term identity”) which is used to define a BV-algebra:

$$\begin{aligned}\Delta(abc) = & \Delta(ab)c + (-1)^{a(b+c)} \Delta(bc)a + (-1)^{bc} \Delta(ac)b \\ & - \Delta(a)bc - (-1)^{ab} \Delta(b)ac - (-1)^{c(a+b)} \Delta(c)ab.\end{aligned}\tag{C.18}$$

It clearly uses associativity (and in this form also commutativity). It is a known fact that  $\mu^{\text{sh}}$  is both associative and commutative (see [Lod92]). Therefore, the question for  $\Delta_{\text{H}}$  makes sense. However,  $\mu_{\text{cyc}}$  might not be associative (is it?), and hence  $\Delta_{\text{cyc}}$  might not be a BV-operator in the classical sense. Is it possible to define a second order derivative, and hence the notion of a BV-operator, on a non-associative algebra?

(ii) Do the factorizations  $\Delta_{\text{cyc}} = \mu_{\text{cyc}} \circ \zeta_{\text{cyc}}$ , resp.  $\Delta_{\text{sym}} = \mu_{\text{sym}} \circ \zeta_{\text{sym}}$  define  $\zeta_{\text{cyc}}$ , resp.  $\zeta_{\text{sym}}$  uniquely? Do other BV-operators factorize like this? Is  $\zeta_{\text{sym}}$  a cobracket and is there a bracket on “symmetric Hochschild cochains” inducing a bialgebra structure on  $L_{\text{sym}}^+ V$ ?

(iii) Are the new operations chain maps with respect to  $b$  (Hochschild differential) and  $b + d$ ? How are our constructions related to the work of other authors, e.g., [Iri18]?

◁

DONE Kei Irie:  
Add Kei Irie  
reference to  
BV-structure  
on Hochschild.

## D. Filtered IBL-infinity-algebras as filtered MV-algebras

In this appendix, we want to understand the BV-formalism for  $\text{IBL}_\infty$ -algebras and the equivalence

$$\text{IBL}_\infty : \begin{array}{c} \text{Surface calculus} \\ (q_{klg}), (f_{klg}), \text{gluing relations} \end{array} \iff \begin{array}{c} \text{BV-formalismus} \\ \Delta, \mathfrak{f}, \text{relations } \Delta^2 = 0, e^{\mathfrak{f}} \Delta^+ = \Delta^- e^{\mathfrak{f}}. \end{array}$$

This was originally done in [CFL15] using filtrations and formal power series in  $\hbar$  as a “bookkeeping”. Nevertheless, the author thinks that some details were not fully addressed and was also curious about extending the MV-formalism from [MV17] to the filtered setting.

As a matter of fact, the main motivation for better understanding of these details was a brief discussion with Kyler Siegel at a workshop about BV-quantization in Stony Brook, 2019. It was pointed out that a twisted  $\infty$ -algebra is, in some sense, “isomorphic” to the untwisted one because the twisting is just conjugation with the exponential of the Maurer-Cartan element. The case of  $A_\infty$ -algebras from [Fuk+09] was mentioned. The author of this text was confused because, by Part I, the twisting with the Chern-Simons Maurer-Cartan element “adds” the information about string topology to the canonical structure and one expects to obtain a non-isomorphic structure. The precise notions of isomorphisms needed to be clarified.

In Section D.1, we start by dealing with filtrations and completions in terms of series in more details. We formulate the Resummation Lemma (Lemma D.1.1), define (complete) filtered algebras (Definition D.1.2) and comment on units and augmentations (Remark D.1.4). We then consider combinations of two filtrations (Equation (D.1)) and show that the symmetric bialgebra with the filtration which is the union of the induced filtration and the filtration by weights might not be a filtered bialgebra (Example D.1.5); it is, however, under a boundedness assumption on the filtration (Lemma D.1.6). We then consider the exponential and the logarithm on a complete filtered algebra (Lemma D.1.7) and generalize it for the convolution product of morphisms with codomain a complete fil-

BV and  $BV_{\text{infty}}$ :  
Change BV to  $BV_\infty$ .

DONE Degree of BV: Probably  $|\Delta| = -1$  in our formalism. Indeed, it is  $-1$ . Correct!

DONE Kyler's surname: Look up Kyler's surname.



tered algebra and domain a complete filtered coalgebra satisfying the limit conilpotency property (Proposition D.1.8). This is the case of completions of conilpotent coalgebras, e.g., the symmetric bialgebra (Lemma D.1.9). We remark that the exponential of a morphism might not be invertible as a map (Remark D.1.10).

In Section D.2, we recall MV-algebras from [MV17] and introduce their complete filtered version (Definition D.2.1). Based on this, in analogy to [MV17], we define the notion of a complete filtered  $\text{IBL}_\infty$ -algebra in MV-formalism and its morphisms (Definition D.2.3). We study the equivalent formulation in components via the surface calculus (Proposition D.2.6). We show that the notion of a filtered  $\text{IBL}_\infty$ -algebra in MV-formalism is equivalent to the notion of a filtered  $\text{IBL}_\infty$ -algebra from [CFL15] in some cases, e.g., for the dual cyclic bar complex from Part I (Proposition D.2.7). The formalism of [CFL15] is symmetric on exchanging inputs and outputs whereas the MV-formalism is not; we illustrate how bubblings at inputs and outputs are handled differently (Example D.2.9). Finally, we consider twisting with a Maurer-Cartan element (Proposition D.2.10).

In Section D.3, we define the BV-chain complexes associated to a complete filtered  $\text{IBL}_\infty$ -algebra in MV-formalism (Definition D.3.1) and observe that morphisms and the twisting with the Maurer-Cartan element induce chain maps of these chain complexes (Proposition D.3.3). We formulate some open questions (Remark D.3.4).

In Section D.4, we study the composition of maps which are convolutions of other maps (Lemma D.4.1). This leads to the definition of compositions controlling the number of “veins” between the individual maps (Definition D.4.2). This can be used to formulate the surface calculus algebraically (Proposition D.4.3). We give proofs of some formulas for partial compositions from Section 3.2 (Proposition D.4.4).

## **D.1. Filtered bialgebras and exponential in convolution product**

We use the definitions of a filtration, completion and the induced filtration on the completion from Section 3.1. We formulate the statements in the category of vector spaces, but similar statements hold in the category of graded vector spaces (writing “homogeneous” and “graded” everywhere). All filtrations are decreasing, i.e.,  $\mathcal{F}^{\lambda_1} \supseteq \mathcal{F}^{\lambda_2}$  whenever  $\lambda_1 \leq \lambda_2$ . We work over a field  $\mathbb{K}$  of characteristic zero.

We start with the following technical lemma, which shows that convergence in completion is very much like absolute convergence in analysis.<sup>1</sup>

---

<sup>1</sup>In fact, for complex numbers  $v_k \in \mathbb{C}$  for  $k \in \mathbb{N}_0$ , property (b) of Lemma D.1.1 is equivalent to the absolute convergence of  $\sum_{k=0}^{\infty} v_k$  in  $\mathbb{C}$ . Thanks to Jiří Zeman for noting that.

**Lemma D.1.1** (Resummation Lemma). *Let  $V$  be a vector space filtered by a decreasing filtration  $\mathcal{F}$ , and let  $\hat{V}$  be its completion.*

- (a) *Let  $v_{ij} \in V$  for all  $i, j \in \mathbb{N}_0$  be vectors such that  $\sum_{j=0}^{\infty} v_{ij}$  converges for every  $i \geq 0$  (i.e.,  $\|v_{ij}\| \rightarrow \infty$  as  $j \rightarrow \infty$ , so that  $\sum_{j=0}^{\infty} v_{ij} \in \hat{V}$ ) and  $\|\sum_{j=0}^{\infty} v_{ij}\| \rightarrow \infty$  as  $i \rightarrow \infty$ . Then for any bijection  $r : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ , the sum  $\sum_{i=0}^{\infty} v_{r(i)}$  converges and the limit does not depend on  $r$ ; we denote it by  $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_{ij} \in \hat{V}$ .<sup>2</sup>*
- (b) *Suppose that  $\sum_{k=0}^{\infty} v_k$  for  $v_k \in V$  converges, and let  $s : Z \subset \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$  be a bijection. If we define  $w_{ij} := v_{s(i,j)}$  for  $(i,j) \in Z$  and  $w_{ij} := 0$  otherwise, then  $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_{ij}$  converges (in the sense of (a)) and equals  $\sum_{k=0}^{\infty} v_k$ .*
- (c) *Let  $V$  and  $V'$  be vector spaces filtered by exhaustive filtrations, and let  $\sum_{i=0}^{\infty} v_i$  for  $v_i \in V$  and  $\sum_{i=0}^{\infty} v'_i$  for  $v'_i \in V'$  converge. Then for any bijection  $r : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ ,  $i \rightarrow (r_1(i), r_2(i))$ , the sum  $\sum_{i=0}^{\infty} v_{r_1(i)} \otimes v'_{r_2(i)}$  converges and the limit does not depend on  $r$ ; we denote it by  $\sum_{i=0}^{\infty} v_i \otimes \sum_{j=0}^{\infty} v'_j \in V \hat{\otimes} V'$ .*

Moreover, we have the following statements about maps of filtered vector spaces:

- (d) *If  $f : \hat{V} \rightarrow \hat{V}$  is a linear map of finite filtration degree, i.e.,  $\|f\| > -\infty$ , and  $\sum_{i=0}^{\infty} v_i$  with  $v_i \in V$  a convergent series, then  $\sum_{i=0}^{\infty} f(v_i)$  converges to  $f(\sum_{i=0}^{\infty} v_i)$ .*
- (e) *Given linear maps  $f_i : \hat{V} \rightarrow \hat{V}$  for  $i \in \mathbb{N}_0$ , we say that the sum  $\sum_{i=0}^{\infty} f_i$  converges if  $\sum_{i=0}^{\infty} f_i(v)$  converges for all  $v \in \hat{V}$ . In this case,  $f(v) := \sum_{i=0}^{\infty} f_i(v)$  defines a linear map  $\hat{V} \rightarrow \hat{V}$  which satisfies  $\|f\| \geq \inf_{i \in \mathbb{N}_0} \|f_i\|$ .*

*Proof.* (a) Let  $r : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$  be a bijection. We first prove the convergence of  $\sum_{i=0}^{\infty} v_{r(i)}$ . Given  $K > 0$ , let  $i_0 \in \mathbb{N}_0$  be such that  $\|v_{ij}\| \geq K$  for  $(i,j) \in \{i \geq i_0, j \geq 0\}$ . Such  $i_0$  exists by the assumption that  $\|\sum_{j=0}^{\infty} v_{ij}\| \rightarrow \infty$  as  $i \rightarrow \infty$ . Let  $j_0 \in \mathbb{N}_0$  be such that  $\|v_{ij}\| \geq K$  for  $(i,j) \in \{i \leq i_0, j \geq j_0\}$ . Such  $j_0$  exists from the convergence of  $\sum_{j=0}^{\infty} v_{0j}, \dots, \sum_{j=0}^{\infty} v_{i_0j}$ . We can now pick  $k_0 \in \mathbb{N}$  such that  $r(\{k < k_0\}) \supset \{i \leq i_0, j \leq j_0\}$ , and hence  $\|v_{r(k)}\| \geq K$  for all  $k \geq k_0$ . This implies the convergence of  $\sum_{i=0}^{\infty} v_{r(i)}$ .

Let now  $r' : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$  be another bijection. Given  $K > 0$ , we find  $i_0, j_0 \in \mathbb{N}_0$  so that  $\|v_{ij}\| \geq K$  for all  $(i,j) \notin \{i \leq i_0, j \leq j_0\}$  (like in the previous paragraph). There exists  $k_0 \in \mathbb{N}$  such that  $r(\{k < k_0\}), r'(\{k \leq k_0\}) \supset \{i \leq i_0, j \leq j_0\}$ . Hence, for any  $k \geq k_0$ , the contribution of  $v_{ij}$  with  $(i,j) \in \{i \leq i_0, j \leq j_0\}$  cancels in  $\Delta_k := \sum_{i=0}^k v_{r(i)} - \sum_{i=0}^k v_{r'(i)}$ , and we get  $\|\Delta_k\| \geq K$ . It follows that the sums are equal in the limit.

<sup>2</sup>It is an exercise to prove a similar statement for nested infinite sums  $\sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} v_{i_1, \dots, i_n}$ .

*D. Filtered IBL-infinity-algebras as filtered MV-algebras*

(b) It is easy to see that  $w_{ij}$  satisfy the assumptions of (a). The claim follows from (a) by defining  $r := s^{-1} \sqcup \mathbb{1} : \mathbb{N}_0 = \mathbb{N}_0 \sqcup (\mathbb{N}_0 \times \mathbb{N}_0 \setminus Z) \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ , where the notation  $\mathbb{N}_0 = \mathbb{N}_0 \sqcup (\mathbb{N}_0 \times \mathbb{N}_0 \setminus Z)$  means an infinite shuffle permutation. Clearly,  $\sum_{i=0}^{\infty} w_{r(i)}$  is precisely  $\sum_{i=0}^{\infty} v_i$  after crossing out possibly infinitely many zeros.

(c) We start with convergence. Let  $K > 0$ . Pick indices  $i_0 \in \mathbb{N}_0$  and  $i'_0 \in \mathbb{N}_0$  such that  $\|v_i\| \geq K - \inf_j \|v'_j\|$  and  $\|v'_i\| \geq K - \inf_j \|v_j\|$  for all  $i \geq i_0$  and  $i \geq i'_0$ , respectively. Note that the right-hand sides of the inequalities are finite because the filtrations are exhaustive. Pick  $k_0 \in \mathbb{N}$  such that  $r(\{k < k_0\}) \supset \{i \leq i_0, j \leq j_0\}$ . Then, for  $k \geq k_0$ , we have

$$\begin{aligned} \|v_{r_1(k)} \otimes v'_{r_2(k)}\| &\geq \|v_{r_1(k)}\| + \|v'_{r_2(k)}\| \\ &\geq \begin{cases} (K - \inf_j \|v'_j\|) + \inf_j \|v'_j\| = K & \text{if } r_1(k) \geq i_0, \\ \inf_j \|v_j\| + (K - \inf_j \|v_j\|) = K & \text{if } r_2(k) \geq i'_0. \end{cases} \end{aligned}$$

It follows that  $\sum_{i=0}^{\infty} v_{r_1(i)} \otimes v'_{r_2(i)}$  converges.

If  $r' : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$  is another bijection, we write  $r' = r \circ s$  for a bijection  $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  and apply (b) to  $V \otimes V'$  with the bijection  $s : Z := \mathbb{N}_0 \times \{0\} \simeq \mathbb{N}_0 \rightarrow \mathbb{N}_0$ . It follows that  $\sum_{i=0}^{\infty} v_{r(i)} = \sum_{i=0}^{\infty} v_{r'(i)}$ .

(d) This is clear because  $f(\sum_{i=0}^{\infty} v_i) = \sum_{i=0}^n f(v_i) + f(\sum_{i=n+1}^{\infty} v_i)$  for any  $n \in \mathbb{N}_0$  and  $\|f(\sum_{i=n+1}^{\infty} v_i)\| \geq \|f\| + \inf_{i>n} \|v_i\| \rightarrow \infty$  as  $n \rightarrow \infty$ .

(e) Linearity is clear. The inequality  $\|f\| \geq \inf_i \|f_i\|$  follows immediately from  $\|f(v)\| \geq \inf_i \|f_i(v)\| \geq \inf_i \|f_i\| + \|v\|$  for all  $v \in \hat{V}$ .  $\square$

Note that (a) and (c) of the Resummation Lemma are, in fact, reformulations of the facts that the canonical inclusions induce the isomorphisms  $\hat{V} \simeq \hat{\hat{V}}$  and  $V \hat{\otimes} V \simeq \hat{V} \hat{\otimes} \hat{V}$ , respectively.

We will work with (complete) filtered algebras, coalgebras and bialgebras, which we now define schematically. They are basically algebras over the corresponding (pr)operad in the category of (complete) filtered vector spaces.

**Definition D.1.2** ((Complete) filtered algebras, coalgebras, bialgebras). *A filtered algebra, coalgebra or bialgebra is a filtered vector space  $V$  with linear operations  $\mu_{klg} : V^{\otimes k} \rightarrow V^{\otimes l}$  of non-negative filtration degree for  $k, l, g \in \mathbb{N}_0$  such that  $(V, (\mu_{klg}))$  is an algebra, coalgebra or bialgebra, respectively.*

### D. Filtered IBL-infinity-algebras as filtered MV-algebras

A complete filtered algebra, coalgebra or bialgebra is a complete filtered vector space  $V$  with linear operations  $\mu_{klg} : V^{\hat{\otimes} k} \rightarrow V^{\hat{\otimes} l}$  of non-negative filtration degree for  $k, l, g \in \mathbb{N}_0$  such that  $\mu_{klg}$  satisfy the relations of an algebra, coalgebra or bialgebra with  $\otimes$  replaced by  $\hat{\otimes}$ , respectively.

A morphism of (complete) filtered algebras, coalgebras or bialgebras is a linear map  $f : V_1 \rightarrow V_2$  with non-negative filtration degree intertwining  $\mu_{klg}$ .

*Remark D.1.3* (On (complete) filtered algebras). (i) Clearly, completion is a functor from filtered algebras to complete filtered algebras. In deformation theory, we may deal with complete filtered algebras which are not in the image of this functor.

(ii) The  $\text{IBL}_\infty$ -algebra on  $C$  from Definition 3.2.4 for  $\gamma = 0$  is, in fact, a “complete filtered  $\text{IBL}_\infty$ -algebra on  $\hat{C}$ ” in the sense of Definition D.1.2 plus the data of the filtered vector space  $C$ . It might not be the completion of a “filtered  $\text{IBL}_\infty$ -algebra on  $C$ ”; in fact, we called an  $\text{IBL}_\infty$ -algebra “completion-free” if the operations on  $\hat{C}$  arose as continuous extensions of operations on  $C$ . Definition 3.2.4 seemed natural when we were trying to compute examples of the twisted structure, hoping that we do not encounter infinite sums. After dealing with completions, however, Definition D.1.2 is more logical as an abstract definition.

Notice that for  $\gamma > 0$ , the norm  $\|\mathbf{q}_{klg}\| \geq \gamma(2 - 2g - k - l)$  is allowed to be negative; in particular,  $\|\mathbf{q}_{210}\| \geq -\gamma$  and  $\|\mathbf{q}_{120}\| \geq -\gamma$ . Therefore, one has to allow finite filtration degree in Definition D.1.2 in order to accommodate it to  $\text{IBL}_\infty$ -algebras with  $\gamma \geq 0$ .  $\triangleleft$

*Remark D.1.4* (Units and augmentations). The ground field  $\mathbb{K}$  is filtered by the trivial complete filtration  $\mathcal{F}^{\lambda \leq 0} \mathbb{K} = \mathbb{K}$  and  $\mathcal{F}^{\lambda > 0} \mathbb{K} = 0$  (see also (3.9)). Suppose that  $\eta : \mathbb{K} \rightarrow V$  is a unit and  $\varepsilon : V \rightarrow \mathbb{K}$  an augmentation of a complete filtered algebra  $V$ ; in particular, we have  $\varepsilon \circ \eta = \mathbb{1}$  and  $\|\eta\|, \|\varepsilon\| \geq 0$ . From this, we deduce the following implications for  $v \in V$ :

$$\begin{aligned} \|v\| > 0 &\implies v \in \ker \varepsilon =: \bar{V}, \\ v \in \text{im } \eta &\implies \|v\| = 0. \end{aligned}$$

Recall that  $V = \text{im } \eta \oplus \bar{V}$ .  $\triangleleft$

Given a vector space  $U$  over  $\mathbb{K}$ , we will work with the *symmetric bialgebra*  $SU = \bigoplus_{k=0}^{\infty} S_k U$  from Definition 3.1.7. It has the concatenation product  $\mu : SU \otimes SU \rightarrow SU$  and the shuffle coproduct  $\delta : SU \rightarrow SU \otimes SU$  which are given for all  $u_{ij}, u_i \in U$  and  $k$ ,

### D. Filtered IBL-infinity-algebras as filtered MV-algebras

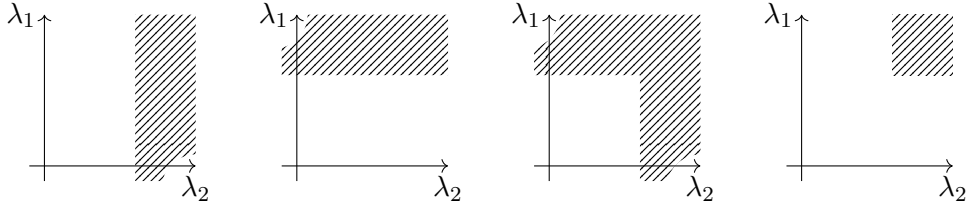


Figure D.1.: For a bigraded vector space  $V = \bigoplus_i \bigoplus_j V_{ij}$ , let  $\mathcal{F}_v^{\lambda_1} := \bigoplus_{i \geq \lambda_1} \bigoplus_j V_{ij}$  be the vertical and  $\mathcal{F}_h^{\lambda_2} := \bigoplus_i \bigoplus_{j \geq \lambda_2} V_{ij}$  the horizontal filtration. The illustrations above depict  $\mathcal{F}_v$ ,  $\mathcal{F}_h$ ,  $\mathcal{F}_v \cup \mathcal{F}_h$  and  $\mathcal{F}_v \cap \mathcal{F}_h$ , respectively. We imagine the component  $V_{ij}$  sitting at the position  $(i, j)$  in the plane. Given  $v_{ij} \in V_{ij}$ , the sum  $\sum_{i,j} v_{ij}$  converges with respect to the given filtration if and only if for every  $\lambda$ , only finitely many  $v_{ij}$ 's in the white region are non-zero.

$k' \in \mathbb{N}_0$  by

$$\begin{aligned} \mu(u_{11} \cdots u_{1k} \otimes u_{21} \cdots u_{2k'}) &= u_{11} \cdots u_{1k} u_{21} \cdots u_{2k'} \quad \text{and} \\ \delta(u_1 \cdots u_k) &= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = k}} \sum_{\sigma \in \mathbb{S}_{k_1, k_2}} \varepsilon(\sigma, u) u_{\sigma_1^{-1}}^{-1} \cdots u_{\sigma_{k_1}^{-1}}^{-1} \otimes u_{\sigma_{k_1+1}^{-1}}^{-1} \cdots u_{\sigma_{k_1+k_2}^{-1}}^{-1}, \end{aligned}$$

respectively. Here  $\mathbb{S}_{k_1, k_2}$  denotes the shuffle permutations and  $\varepsilon(\sigma, u)$  the Koszul sign. The unit  $\eta : \mathbb{K} \rightarrow SU$  is given by  $\eta(1) = 1 \in S_0 U = \mathbb{K}$ , and the augmentation  $\varepsilon : SU \rightarrow \mathbb{K}$  is determined by its reduced algebra  $\bar{S}U = \bigoplus_{k=1}^{\infty} S_k U$ . If  $U$  is filtered, then  $\mu$ ,  $\delta$ ,  $\eta$  and  $\varepsilon$  preserve the induced filtration on  $SU$  (see Definition 3.1.8), and hence  $SU$  is a filtered bialgebra. We will also use the canonical projection  $\pi_k : SU \rightarrow S_k U$  and the canonical inclusion  $\iota_l : S_l U \rightarrow SU$  for  $k \in \mathbb{N}_0$  and  $l \in \mathbb{N}_0$ . They too preserve the filtration.

The symmetric bialgebra of a filtered vector space has naturally two filtrations — the induced filtration and the filtration by weights  $\mathcal{F}_w^\lambda SU := \bigoplus_{k \geq \lambda} S_k U$ .<sup>3</sup> In general, having two filtrations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  on a vector space  $V$ , we define the filtrations

$$\mathcal{F}_\cup^\lambda := \mathcal{F}_1^\lambda + \mathcal{F}_2^\lambda \quad \text{and} \quad \mathcal{F}_\cap^\lambda := \mathcal{F}_1^\lambda \cap \mathcal{F}_2^\lambda \quad (\text{D.1})$$

and call them the *combined filtrations* — the union and the intersection. It is easy to see that

$$\|\cdot\|_\cup \geq \max(\|\cdot\|_1, \|\cdot\|_2) \quad \text{and} \quad \|\cdot\|_\cap \leq \min(\|\cdot\|_1, \|\cdot\|_2)$$

hold for the corresponding filtration degrees.

DONE Inequality for combined filtrations: There has to be an inequality!!

<sup>3</sup>Note that the filtration by weights can be also viewed as the induced filtration from the filtration on  $U$  defined by  $\mathcal{F}^{\leq 1} U := U$ ,  $\mathcal{F}^{> 1} U := 0$ .

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The following example shows that  $SU$  might not be a filtered bialgebra with respect to combined filtrations.

*Example D.1.5* (Combined filtrations of symmetric bialgebra  $SU$ ). Let  $U$  be a vector space filtered by a filtration  $\mathcal{F}$ . Consider the symmetric bialgebra  $SU$  with the induced filtration  $\mathcal{F}$  and the filtration by weights  $\mathcal{F}_w^\lambda$ . It is easy to see that the operations  $\mu, \delta, \eta$  and  $\varepsilon$  preserve both  $\mathcal{F}$  and  $\mathcal{F}_w$ . Consider the combined filtrations  $\mathcal{F}_\cup$  and  $\mathcal{F}_\cap$ . Because

$$\begin{aligned} \min(\lambda_1 + \lambda_2, k_1 + k_2) &\geq \min(\lambda_1, k_1) + \min(\lambda_2, k_2) \quad \text{and} \\ \max(\lambda_1 + \lambda_2, k_1 + k_2) &\leq \max(\lambda_1, k_1) + \max(\lambda_2, k_2), \end{aligned}$$

it holds  $\|\mu\|_\cap \geq 0$  and  $\|\delta\|_\cup \geq 0$ , respectively.

We now demonstrate that it might happen that  $\|\mu\|_\cup = -\infty$  and  $\|\delta\|_\cap = -\infty$ . Let  $U = TW$  be the tensor algebra over a non-zero vector space  $W$ . On  $U$ , consider the filtration by weights  $\mathcal{F}^\lambda U = \bigoplus_{k \geq \lambda} W^{\otimes k}$ . Let  $0 \neq w \in W$ , and define

$$v_k := \underbrace{w \otimes \cdots \otimes w}_{k\text{-times}} \in \mathcal{F}^k S_1 U \quad \text{for all } k \in \mathbb{N},$$

where  $\otimes$  denotes the tensor product on  $TW$ . For  $0 \neq u \in T_0 W = \mathbb{K}$ , define

$$v'_k := \underbrace{u \cdots u}_{k\text{-times}} \in \mathcal{F}^0 S_k U \quad \text{for all } k \in \mathbb{N},$$

where  $\cdot$  denotes the concatenation product on  $SU$ . It follows that

$$\|v_k\|_\cup = k, \quad \|v'_k\|_\cup = k \quad \text{and} \quad \|v_k v'_k\|_\cup = k + 1.$$

Therefore, for all  $k \in \mathbb{N}$ , it holds

$$\begin{aligned} \|\mu\|_\cup &\leq \|v_k v'_k\|_\cup - \|v_k \otimes v'_k\|_\cup \\ &\leq k + 1 - \|v_k\|_\cup - \|v'_k\|_\cup \\ &= 1 - k. \end{aligned}$$

Consequently  $\|\mu\|_\cup = -\infty$ . Next, it holds

$$\|v_k v'_k\|_\cap = k \quad \text{and} \quad \|v_k \otimes v'_k\|_\cap = 1,$$

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and we compute

$$\begin{aligned}\|\delta\|_{\cap} &\leq \|\delta(v_k v'_k)\|_{\cap} - \|v_k v'_k\|_{\cap} \\ &= 1 - k.\end{aligned}$$

We used here that the summands of  $\delta(v_k v'_k)$  are tensor products of  $v''_1 = u^i v_k \in \mathcal{F}^1 S_{i+1} U$  and  $v''_2 = u^{k-i} \in \mathcal{F}^0 S_{k-i} U$  for  $i = 0, \dots, k$ , so that  $\|v''_1 \otimes v''_2\|_{\cap} = 1$ . It follows that  $\|\delta\|_{\cap} = -\infty$ . A heuristic explanation of these phenomenons is that the tensor product of combined filtrations is not the combined filtration of the tensor product.

In the next paragraph, we will argue that it might still be possible to extend  $\mu$  to  $SU \hat{\otimes}_{\cup} SU$  — the completion of  $SU \otimes SU$  with respect to the tensor product of the union filtrations — even though  $\|\mu\| = -\infty$ . A similar discussion applies for  $\delta$ .

Suppose that the filtration on  $U$  is bounded from above. Let  $v_i \in \mathcal{F}^{\lambda_i} S_{k_i} U$  and  $v'_i \in \mathcal{F}^{\lambda'_i} S_{k'_i} U$  for all  $i \in \mathbb{N}_0$  be such that  $\max(\lambda_i, k_i) + \max(\lambda'_i, k'_i) \rightarrow \infty$  as  $i \rightarrow \infty$ , so that  $\sum_{i=0}^{\infty} v_i \otimes v'_i$  converges in  $SU \hat{\otimes}_{\cup} SU$  (in fact, any element of  $SU \hat{\otimes}_{\cup} SU$  can be written in this way). We have  $\mu(v_i, v'_i) \in \mathcal{F}^{\lambda_i + \lambda'_i} S_{k_i + k'_i} U$  for all  $i \in \mathbb{N}_0$ . Suppose that  $\sum_{i=0}^{\infty} v_i v'_i$  does not converge with respect to  $\mathcal{F}_{\cup}$ . Thus,  $\max(\lambda_i + \lambda'_i, k_i + k'_i)$  is bounded, hence  $\lambda_i + \lambda'_i$  and  $k_i + k'_i$  are bounded, and since  $k_i, k'_i \geq 0$ , also  $k_i$  and  $k'_i$  are bounded. Nevertheless, in order to comply with the assumption on the convergence of  $\sum_{i=0}^{\infty} v_i \otimes v'_i$ , we need one of  $\lambda_i$  or  $\lambda'_i$ , let's say  $\lambda_i$ , to diverge to  $\infty$ . Because  $\lambda_i + \lambda'_i$  is bounded,  $\lambda'_i$  has to diverge to  $-\infty$ . But this is not allowed by the boundedness assumption.  $\triangleleft$

It turns out that the union filtration is often identical with the induced filtration.

**Lemma D.1.6** (Combined filtration and boundedness condition). *Let  $U$  be a vector space filtered by a decreasing filtration  $\mathcal{F}$ . Suppose that there is  $\gamma > 0$  such that*

$$\mathcal{F}^{\gamma} U = U. \tag{D.2}$$

*Then the following holds:*

$$\begin{aligned}\gamma \geq 1 &\implies \forall \lambda \in \mathbb{R} : \mathcal{F}_{\cup}^{\lambda} SU = \mathcal{F}^{\lambda} SU, \\ 0 < \gamma < 1 &\implies \forall v \in SU : \|v\|_{\cup} \geq \|v\| \geq \gamma \|v\|_{\cup}.\end{aligned}$$

*Proof.* Under the assumption (D.2), we have for any  $k \in \mathbb{N}$  and  $\lambda \in \mathbb{R}$  the following:

$$\lambda \leq k\gamma \implies \mathcal{F}^{\lambda} S_k U = \sum_{\lambda_1 + \dots + \lambda_k = \lambda} \mathcal{F}^{\lambda_1} U \otimes \dots \otimes \mathcal{F}^{\lambda_k} U / S_k = S_k U. \tag{D.3}$$

We compute

$$\begin{aligned} \mathcal{F}_U^\lambda SU &= \bigoplus_{k < \lambda} \mathcal{F}^\lambda S_k U \oplus \bigoplus_{k \geq \lambda} S_k U \\ &= \begin{cases} \bigoplus_{k=0}^{\infty} \mathcal{F}^\lambda S_k U = \mathcal{F}^\lambda SU & \text{for } \gamma \geq 1, \\ \bigoplus_{k < \lambda \text{ or } k \geq \frac{\lambda}{\gamma}} \mathcal{F}^\lambda S_k U \oplus \bigoplus_{\lambda \leq k < \frac{\lambda}{\gamma}} S_k U \subset \mathcal{F}^{\lambda\gamma} SU & \text{for } 0 < \gamma < 1. \end{cases} \end{aligned}$$

The first case holds because if  $\gamma \geq 1$ , then  $k \geq \lambda$  implies  $\gamma k \geq \lambda$ , and thus  $S_k U = \mathcal{F}^\lambda S_k U$  by (D.3). The second case holds from the following reasons. If  $0 < \gamma < 1$  and  $k \geq \frac{\lambda}{\gamma}$ , so that  $S_k U = \mathcal{F}^\lambda S_k U$  by (D.3), then  $k > \lambda$ . Now,  $\mathcal{F}^\lambda S_k U \subset \mathcal{F}^{\lambda\gamma} S_k U$  because the filtration is decreasing and because  $\lambda \geq \lambda\gamma$ , and if  $v \in S_k U$  with  $\lambda \leq k < \frac{\lambda}{\gamma}$ , then  $k\gamma \geq \lambda\gamma$ , and hence  $S_k U = \mathcal{F}^{\lambda\gamma} S_k U$  by (D.3). The claim follows.  $\square$

We consider the exponential and the logarithm on filtered algebras.

**Lemma D.1.7** (Exponential and logarithm on filtered algebras). *Let  $V$  be a complete filtered associative algebra with unit 1. Given  $v \in V$  with  $\|v\| > 0$ , we define the exponential*

$$e^v := 1 + v + \frac{1}{2!}v^2 + \frac{1}{3!}v^3 + \cdots \in V.$$

*It holds  $e^v e^{-v} = 1$ . Given  $v \in V$  with  $\|v - 1\| > 0$ , we define the logarithm*

$$\log(v) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} (v - 1)^r \in V.$$

*It holds  $\log(e^v) = v$  for  $v \in V$  with  $\|v\| > 0$  and  $e^{\log v} = v$  for  $v \in V$  with  $\|v - 1\| > 0$ .*

*Proof.* For any  $v \in V$ , we have

$$\begin{aligned} e^{-v} e^v &= e^{-v} \left( \sum_{i=0}^{\infty} \frac{1}{i!} v^i \right) \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} e^{-v} v^i \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i! j!} (-1)^j v^{i+j} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^n (-1)^k \binom{n}{k}}_{=0 \text{ for } n > 0} v^n. \end{aligned}$$



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On the second line, we used that the multiplication with  $e^{-v}$  from the left has finite filtration degree, and hence it commutes with infinite sums; on the third line, we used that the multiplication with  $v^i$  from the right has finite filtration degree; on the fourth line, we used the Resummation Lemma (Lemma D.1.1). The facts about log are proven similarly, the equations reducing to known combinatorial identities.  $\square$

Let  $(V, \delta)$  be a coalgebra and  $(V', \mu')$  an algebra. The *convolution product*  $*$  on  $\mathbb{K}$ -linear maps  $f_1, f_2 : V \rightarrow V'$  is defined by

$$f_1 * f_2 := \mu' \circ (f_1 \otimes f_2) \circ \delta.$$

See also [LV12, Section 1.6]. If  $\varepsilon : V \rightarrow \mathbb{K}$  is a counit for  $V$  and  $\eta' : \mathbb{K} \rightarrow V'$  a unit for  $V'$ , then

$$e := \eta' \circ \varepsilon : V \rightarrow V' \tag{D.4}$$

is the unit for  $*$ . The convolution product is associative and commutative provided that the algebra and coalgebra are.

Let us recall the conilpotency property of a coaugmented counital coassociative coalgebra  $(V, \delta, \varepsilon, \eta)$ , where  $\varepsilon : V \rightarrow \mathbb{K}$  is the counit and  $\eta : \mathbb{K} \rightarrow V$  the coaugmentation. We write

$$\delta = \bar{\delta} + \delta_0,$$

where  $\delta_0 : V \rightarrow V \otimes V$  is defined by

$$\delta_0(v) := \begin{cases} 1 \otimes 1 & \text{if } v = 1, \\ 1 \otimes v + v \otimes 1 & \text{if } v \in \bar{V}, \end{cases} \tag{D.5}$$

with respect to the decomposition  $V = \langle 1 \rangle \oplus \bar{V}$ . The other map is then defined simply as the difference

$$\bar{\delta} := \delta - \delta_0 : V \longrightarrow V \otimes V.$$

We call  $\delta_0$  the *trivial coproduct* and  $\bar{\delta}$  the *reduced diagonal*. They are both coassociative, and  $\delta_0$  is even cocommutative. The *conilpotency property* reads: for every  $v \in V$ , there exists an  $n \in \mathbb{N}$  such that  $\bar{\delta}^{(n)}(v) = 0$ , where  $\bar{\delta}^{(n)} := (\bar{\delta} \otimes \mathbb{1}^{\otimes n-2}) \circ \dots \circ \bar{\delta}$  is the iterated reduced diagonal with  $n$  outputs.

Using the conilpotency property, it is possible to weaken the condition  $\|f\| > 0$  on the convergence of a power series in  $f \in \text{Hom}(V, V')$  in the convolution algebra  $(\text{Hom}(V, V'), *, e)$  to  $\|f\| \geq 0$  and  $\|f(1)\| > 0$ . The latter condition can not be weakened because  $e^f(1) = e^{f(1)}$  is the exponential in  $(V', \mu', \eta')$ .

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**Proposition D.1.8** (Power series in  $*$  with coefficients in  $\mathbb{K}$ ). *Let  $(V, \delta, \varepsilon, \eta)$  be a complete filtered coaugmented counital cocommutative coassociative coalgebra such that the following limit conilpotency property holds:*

$$\bar{\delta}^{(n)}(v) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for all } v \in V. \quad (\text{D.6})$$

*Let  $(V', \mu', \eta', \varepsilon')$  be a complete filtered augmented unital commutative associative algebra. Let  $R$  be a complete filtered  $\mathbb{K}$ -algebra. Then the following holds:*

- (a) *The convolution product  $*$  extends naturally to  $R$ -linear maps  $\mathfrak{f}_i : V \hat{\otimes} R \rightarrow V' \hat{\otimes} R$  ( $\hat{\otimes} = \hat{\otimes}_{\mathbb{K}}$ ) of finite filtration degrees, and for its iterations, we have*

$$\mathfrak{f}_1 * \cdots * \mathfrak{f}_n = \mu'^{(n)} \circ (\mathfrak{f}_1 \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n) \circ \delta^{(n)}, \quad (\text{D.7})$$

*where  $\mu'^{(n)}$  and  $\delta^{(n)}$  are the continuous  $R$ -linear extensions of the iterated product and coproduct, respectively. (Recall that the continuous extension of a linear map  $f$  of finite filtration degree on  $V$  to the completion  $\hat{V}$  is defined by  $f(\sum_{i=0}^{\infty} v_i) := \sum_{i=0}^{\infty} f(v_i)$  for all  $v_i \in V$  with  $\|v_i\| \rightarrow \infty$ .)*

- (b) *Let  $\mathfrak{f} : V \hat{\otimes} R \rightarrow V' \hat{\otimes} R$  be an  $R$ -linear map of finite filtration degree such that*

$$\|\mathfrak{f}\| \geq 0 \quad \text{and} \quad \|\mathfrak{f}(1)\| > 0. \quad (\text{D.8})$$

*Then any power series  $\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k}$  with coefficients  $\alpha_k \in \mathbb{K}$ , where  $\mathfrak{f}^{*0} := \mathfrak{e}$  is the unit of the convolution product, converges to an  $R$ -linear map  $V \hat{\otimes} R \rightarrow V' \hat{\otimes} R$  of non-negative filtration degree, and it holds*

$$\left( \sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k} \right) (1) = \sum_{k=0}^{\infty} \alpha_k \mathfrak{f}(1)^k.$$

*Proof.* (a) Given a  $\mathbb{K}$ -linear map  $u : V^{\hat{\otimes} k} \rightarrow \hat{V}^{\otimes l}$  of finite filtration degree for some  $k, l \geq 0$ , we consider its  $R$ -linear extension  $u \otimes \mathbb{1} : V^{\hat{\otimes} k} \otimes R \rightarrow V^{\hat{\otimes} l} \otimes R$  and extend it continuously to  $V^{\hat{\otimes} k} \hat{\otimes} R \rightarrow V^{\hat{\otimes} l} \hat{\otimes} R$ . Because  $V^{\hat{\otimes} n} \hat{\otimes} R \simeq (V \hat{\otimes} R)^{\hat{\otimes} n}$  via canonical maps (this can be proven explicitly using Proposition 3.1.11), we get the desired continuous  $R$ -linear extension  $(V \hat{\otimes} R)^{\hat{\otimes} k} \rightarrow (V \hat{\otimes} R)^{\hat{\otimes} l}$ . We apply this construction to  $u = \mu'^{(n)}$ ,  $\delta^{(n)}$  and define  $\mathfrak{f}_1 * \cdots * \mathfrak{f}_n$  using (D.7) for all  $n \geq 2$ . It is then easy to see that

$$\begin{aligned} & (\cdots ((\mathfrak{f}_1 * \mathfrak{f}_2) * \mathfrak{f}_3) * \cdots) * \mathfrak{f}_n \\ &= \mu'(\mu'(\cdots \mu'(\mathfrak{f}_1 \hat{\otimes}_R \mathfrak{f}_2) \delta \hat{\otimes}_R \mathfrak{f}_3) \delta \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n) \delta \end{aligned}$$

DONE Fin filtr degree: This holds for maps with finite filtration degree.

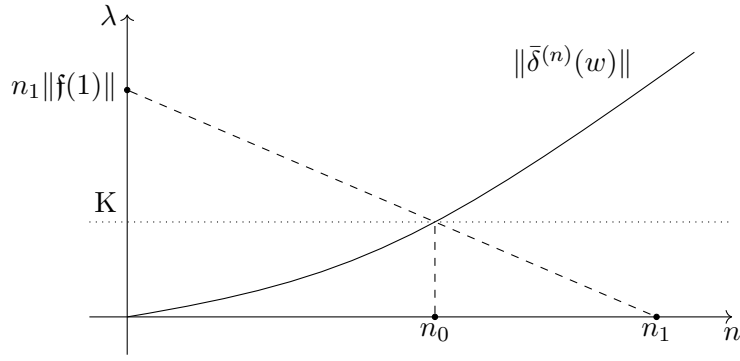


Figure D.2.: Given  $K > 0$ , pick  $n_0 \in \mathbb{N}$  such that  $\|\bar{\delta}^{(n)}(w)\| \geq K$  for  $n \geq n_0$ . Find  $n_1 \geq n_0$  such that  $n_1 \|\mathfrak{f}(1)\| \geq K + n_0 \|\mathfrak{f}(1)\|$ . Then  $(n-i)\|\mathfrak{f}(1)\| + \|\bar{\delta}^{(i)}(w)\| \geq K$  for  $i = 0, \dots, n$  for any  $n \geq n_1$ .

$$\begin{aligned}
 &= \underbrace{\mu'(\mu' \hat{\otimes}_R \mathbb{1}) \cdots (\mu' \hat{\otimes}_R \mathbb{1}^{n-2})}_{=\mu'^{(n)}} (\mathfrak{f}_1 \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}_n) \underbrace{(\delta \hat{\otimes}_R \mathbb{1}^{n-2}) \cdots (\delta \hat{\otimes}_R \mathbb{1}) \delta}_{=\delta^{(n)}} \\
 &= \mathfrak{f}_1 * \cdots * \mathfrak{f}_n.
 \end{aligned}$$

(b) Let  $w \in V \hat{\otimes} R$  and  $n \geq 2$ . For the map  $\delta^{(n)} : V \hat{\otimes} R \rightarrow (V \hat{\otimes} R)^{\hat{\otimes}_{R^n}}$ , we write

$$\delta^{(n)}(w) = \sum_{i=0}^n \sum_{(n-i) \times 1} \sum_{\delta^{(i)}} w_{(1)} \hat{\otimes}_R \mathbb{1} \hat{\otimes}_R \mathbb{1} \hat{\otimes}_R w_{(2)} \hat{\otimes}_R \mathbb{1} \hat{\otimes}_R \cdots \hat{\otimes}_R w_{(i)} \hat{\otimes}_R \mathbb{1}, \quad (\text{D.9})$$

where the second sum is over all insertions of  $n-i$  units  $\mathbb{1}$  into the tensor product and the third sum denotes the sum from the Sweedler's notation

$$\bar{\delta}^{(i)}(w) = \sum w_{(1)} \hat{\otimes}_R \cdots \hat{\otimes}_R w_{(i)}.$$

Formula (D.9) is easy to see from  $\delta = \delta_0 + \bar{\delta}$ .

Before we continue, let us check that  $\bar{\delta} : V \hat{\otimes} R \rightarrow (V \hat{\otimes} R)^{\hat{\otimes}_{R^2}}$  also satisfies the limit conilpotency property (the proof is similar to the proof of Lemma D.1.9 below). Every element of  $V \hat{\otimes} R$  can be written as  $\sum_{i=0}^{\infty} r_i v_i$  for  $r_i \in R$  and  $v_i \in V$  such that  $\|r_i v_i\| \rightarrow \infty$  as  $i \rightarrow \infty$ ; this follows from the Resummation Lemma (Lemma D.1.1). Given  $K > 0$ , we pick  $i_0 \in \mathbb{N}$  such that  $\|r_i v_i\| \geq K$  for all  $i \geq i_0$ . We pick  $n_0 \in \mathbb{N}$  such that  $\|\bar{\delta}^{(n)}(v_0)\| + \|r_0\|, \dots, \|\bar{\delta}^{(n)}(v_{i_0-1})\| + \|r_{i_0-1}\| \geq K$  for all  $n \geq n_0$ . According to the

construction in (a), we have

$$\bar{\delta}^{(n)}\left(\sum_{i=0}^{\infty} r_i v_i\right) = \sum_{i=0}^{\infty} r_i \bar{\delta}^{(n)}(v_i).$$

For  $n \geq n_0$ , it holds  $\|r_i \bar{\delta}^{(n)}(v_i)\| \geq \|r_i\| + \|\bar{\delta}^{(n)}(v_i)\| \geq K$  for  $i < i_0$  and  $\|r_i \bar{\delta}^{(n)}(v_i)\| \geq \|r_i v_i\| \geq K$  for  $i \geq i_0$ . It follows that  $\bar{\delta}^{(n)}\left(\sum_{i=0}^{\infty} r_i v_i\right) \rightarrow 0$  as  $n \rightarrow \infty$ .

Going on with the main proof, using (D.7) and (D.9), we have

$$\begin{aligned} \|\mathfrak{f}^{*n}(w)\| &= \|\mu^{(n)}(\mathfrak{f} \hat{\otimes}_R \cdots \hat{\otimes}_R \mathfrak{f}) \bar{\delta}^{(n)}(w)\| \\ &= \left\| \sum_{i=0}^n \sum_{(n-i) \times 1} \sum_{\bar{\delta}^{(i)}} \mathfrak{f}(w_{n,(1)}) \hat{\otimes}_R \mathfrak{f}(1) \hat{\otimes}_R \mathfrak{f}(1) \hat{\otimes}_R \mathfrak{f}(w_{n,(2)}) \hat{\otimes}_R \mathfrak{f}(1) \hat{\otimes}_R \cdots \right. \\ &\quad \left. \hat{\otimes}_R \mathfrak{f}(w_{n,(n-i)}) \hat{\otimes}_R \mathfrak{f}(1) \right\| \\ &\geq \min_{i=0, \dots, n} ((n-i)\|\mathfrak{f}(1)\| + i\|\mathfrak{f}\| + \|\bar{\delta}^{(i)}(w)\|) \\ &\geq \underbrace{\min_{i=0, \dots, n} ((n-i)\|\mathfrak{f}(1)\| + \|\bar{\delta}^{(i)}(w)\|)}_{=:(*)_n}. \end{aligned}$$

Let  $K > 0$  be arbitrary. Because  $\|\mathfrak{f}\| \geq 0$ ,  $\|\mathfrak{f}(1)\| > 0$  and  $\bar{\delta}^i(w) \rightarrow 0$  as  $i \rightarrow \infty$  by the assumptions, we obtain an  $n_1 \in \mathbb{N}$  such that  $(*)_n \geq K$  for all  $n \geq n_1$  (see Figure D.2). This implies the convergence of  $\sum_{n=0}^{\infty} \alpha_n \mathfrak{f}^{*n}(w)$  in  $V \hat{\otimes} R$ . Using (e) of the Resummation Lemma, we get a well-defined  $\mathbb{K}$ -linear map  $\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k} : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$  with  $\|\sum_{k=0}^{\infty} \alpha_k \mathfrak{f}^{*k}\| \geq \inf_{k=0, \dots, \infty} \|\mathfrak{f}^{*k}\| \geq 0$ . It is easy to see that it is  $R$ -linear as well.  $\square$

**Lemma D.1.9** (Completion of conilpotent coalgebra satisfies limit conilpotency). *Let  $(V, \delta, \varepsilon, \eta)$  be a filtered coaugmented counital coassociative coalgebra which is conilpotent. Then the completion  $(\hat{V}, \delta, \varepsilon, \eta)$  is a complete filtered coaugmented counital coassociative coalgebra which satisfies the limit conilpotency property (D.6).*

*Proof.* Let  $v \in \hat{V}$ , and write  $v = \sum_{i=0}^{\infty} v_i$  for  $v_i \in V$  with  $\|v_i\| \rightarrow \infty$ . Given  $K > 0$ , find  $i_0 \in \mathbb{N}$  such that  $\|v_i\| \geq K$  for all  $i \geq i_0$ . From the conilpotency, we can find  $n_0 \in \mathbb{N}$  such that  $\bar{\delta}^{(n)}(v_0) = \dots = \bar{\delta}^{(n)}(v_{i_0-1}) = 0$  for all  $n \geq n_0$ . Because  $\bar{\delta}$  has finite filtration degree,  $\bar{\delta}^{(n)}$  has finite filtration degree too, and so we can permute it with the infinite sum and write

$$\bar{\delta}^{(n)}(v) = \sum_{i=0}^{\infty} \bar{\delta}^{(n)}(v_i) = \sum_{i=i_0}^{\infty} \bar{\delta}^{(n)}(v_i).$$

Because  $\|\bar{\delta}\| \geq 0$ , we have  $\|\bar{\delta}^{(n)}\| \geq 0$ , and hence

$$\|\bar{\delta}^{(n)}(v)\| \geq \inf_{i=i_0, \dots, \infty} \|\bar{\delta}^{(n)}(v_i)\| \geq \inf_{i=i_0, \dots, \infty} \|v_i\| \geq K$$

for all  $n \geq n_0$ . This shows the limit conilpotency property.  $\square$

*Remark D.1.10* (Exponential and logarithm in  $*$ ). (i) It is obvious that Lemma D.1.7 holds for  $*$  with the weakened conditions (D.8). To sum up, the conditions for the existence of  $\exp(f)$  are  $\|f\| \geq 0$  and  $\|f(1)\| > 0$ , and the conditions for the existence of  $\log(f)$  are  $\|f\| \geq 0$  and  $\|f(1) - 1\| > 0$ .

As for  $\log(f)$ , we will explain how  $\|f\| \geq 0$  and  $\|f(1)\| > 0$  together imply  $\|f - e\| \geq 0$ . Let  $\lambda < 0$ ,  $\mu \in \mathbb{R}$  and  $v \in \mathcal{F}_\mu(V \hat{\otimes} R)$  be arbitrary. If  $v$  is not a multiple of 1, then  $(f - e)(v) = f(v) \in \mathcal{F}_{\mu+\lambda}$  as  $\lambda < \|f\|$ . If  $v = \tau$  for  $0 \neq \tau \in \mathbb{K}$ , then it must hold  $\mu \leq 0$ , and further because of  $\|f(1) - 1\| > 0$ , there is an  $\varepsilon > 0$  such that  $(f - e)(\tau) \in \mathcal{F}_\varepsilon \subset \mathcal{F}_{\mu+\lambda}$ . It follows that  $\|f - e\| \geq 0$ .

(ii) Given  $f : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$  satisfying (D.8), it holds  $e^{-f} * e^f = e$  by Lemma D.1.7 because  $e$  is the unit for the convolution product. However, there might not be any  $g : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$  such that  $g \circ e^f = 1$  (or  $e^f \circ g = 1$ ). To see this, let  $V = \hat{S}U$ ,  $R = \mathbb{K}$ , and let  $f_{110} : U \rightarrow U$  be a linear map with  $\|f_{110}\| \geq 0$  such that there is  $0 \neq v \in U$  with  $f_{110}(v) = 0$ . Consider the trivial extension  $f := f_{110} : \hat{S}U \rightarrow \hat{S}U$  (it equals  $f_{110}$  on  $\hat{S}_1U \rightarrow \hat{S}_1U$  and 0 otherwise). Because

$$f^{*n}(\underbrace{v \cdots v}_{k\text{-times}}) = \begin{cases} n! f_{110}(v)^n & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

for all  $k, n \in \mathbb{N}_0$ , it holds  $e^f(v) = f_{110}(v) = 0$ , and so  $e^f : \hat{S}U \rightarrow \hat{S}U$  is not invertible.

(iii) In the case of  $V = \hat{S}U$ , one can check that  $\log(1)$  is the trivial extension of  $1_{110} : \hat{U} \rightarrow \hat{U}$  (see also [MV17, Example 21] in the non-filtered setting).  $\triangleleft$

## D.2. Filtered IBL-infinity-algebras in filtered MV-formalism

In [MV17], MV-algebras were introduced; they are precisely the algebras governed by the BV-relations  $\Delta^2 = 0$  and  $e^f \Delta^+ = \Delta^- e^f$  (following [CFL15], we will denote by  $^+$  the source and by  $^-$  the target). Schematically, there is the following inclusion of categories

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(explanations will be given below):

$$\begin{array}{lll}
 \text{MV-algebras} & \supset & \text{BV}_\infty\text{-algebras}^* & \supset & \text{IBL}_\infty\text{-algebras}^{**} \\
 \text{Obj.: } (V, \mu, \delta, \eta, \varepsilon), R, & & R = \mathbb{K}[[\hbar]], & & V = SU, \\
 \Delta : V \hat{\otimes} R \rightarrow V \hat{\otimes} R, & & \Delta = \Delta_1 + \hbar \Delta_2 + \hbar^2 \Delta_3 + \cdots, & & S_{k>i}(U) \subset \ker \mathfrak{f}_i. \\
 \Delta^2 = 0, |\Delta| = 1, & & \Delta_i : V \rightarrow V \text{ differential} & & \\
 \Delta(1) = 0. & & \text{operator of order } \leq i, & & \\
 \text{Mor.: } \mathfrak{f} : V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R, & & \mathfrak{f} = \mathfrak{f}_1 + \hbar \mathfrak{f}_2 + \hbar^2 \mathfrak{f}_3 + \cdots. & & \\
 e^{\mathfrak{f}} \circ \Delta^+ = \Delta^- \circ e^{\mathfrak{f}}, & & & & \\
 \mathfrak{f}(1) \subset V^- \hat{\otimes} \mathfrak{m}^-. & & & & 
 \end{array} \tag{D.10}$$

Here,  $R$  is a complete local Noetherian ring with residue field  $\mathbb{K}$  of characteristic 0 and maximal ideal  $\mathfrak{m}$ ,  $(V, \mu, \eta, \varepsilon)$  is a graded commutative associative algebra over  $\mathbb{K}$  with unit  $\eta : \mathbb{K} \rightarrow V$  and augmentation  $\varepsilon : V \rightarrow \mathbb{K}$ ,  $(V, \delta, \varepsilon, \eta)$  is a *conilpotent* graded cocommutative coassociative coalgebra with counit  $\varepsilon$  and coaugmentation  $\eta$  (the same maps as  $\varepsilon$  and  $\eta$  for  $(V, \mu)$ ),  $V \hat{\otimes} R := \varprojlim_n (V \otimes R / V \otimes \mathfrak{m}^n)$ , where  $\mathfrak{m}^n = \mathfrak{m} \cdots \mathfrak{m}$ , is the completed tensor product, the operators  $\Delta$  and  $\mathfrak{f}$  are  $R$ -linear and have non-negative filtration degrees with respect to the filtration by  $V \hat{\otimes} \mathfrak{m}^n$ .

If  $R$  is a  $\mathbb{K}$ -algebra,<sup>4</sup> then this is precisely the same setting as that of Proposition D.1.8 for the filtrations  $\mathcal{F}^{\leq 0}V = V$ ,  $\mathcal{F}^{> 0}V = 0$  and  $\mathcal{F}^{\lambda \leq 0}R = R$ ,  $\mathcal{F}^{\lambda > 0}R = \mathfrak{m}^{[\lambda]}$ . Clearly,  $\mathfrak{f}(1) \subset V^- \hat{\otimes} \mathfrak{m}^-$  is equivalent to  $\|\mathfrak{f}(1)\| > 0$ . Therefore,  $e^{\mathfrak{f}}$  exists by Proposition D.1.8.

Recall from [MV17, p. 5] that a linear operator  $D : V \rightarrow V$  on a commutative associative algebra  $V$  with unit 1 is called a *differential operator* of order  $\leq k$  for  $k \in \mathbb{N}_0$  if it holds

$$\psi_{k+1}^D(v_1, \dots, v_{k+1}) := [[\cdots [D, L_{v_1}], L_{v_2}], \dots], L_{v_{k+1}}] = 0 \quad \text{for all } v_1, \dots, v_{k+1} \in V,$$

where

$$L_v(w) := vw \quad \text{for all } w \in V$$

is the left-multiplication with  $v \in V$  and  $[\cdot, \cdot]$  is the graded commutator.

The categories of  $\text{BV}_\infty$ - and  $\text{IBL}_\infty$ -algebras contained in the MV-formalism are not the most general ones; this is what  $*$  and  $**$  indicate. Before we explain this, let us

---

<sup>4</sup>The ring  $\mathbb{Z}_4$  is a local ring which is not a  $\mathbb{K}$ -algebra over its residue field  $\mathbb{K} = \mathbb{Z}_2$ . The ring of polynomials  $\mathbb{R}[x]$  in a single variable  $x$  is a  $\mathbb{K}$ -algebra over its residue field  $\mathbb{K} = \mathbb{R}$ , but it is not a local ring because both  $(x)$  and  $(x+1)$  are maximal ideals. In contrast to this, the ring of power series  $\mathbb{K}((\hbar))$  is both a local ring and a  $\mathbb{K}$ -algebra. Thanks to Thorsten Hertl for pointing this out.

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agree on calling  $(\mathfrak{q}_{klg})$  a *strict*  $\text{IBL}_\infty$ -algebra if  $\mathfrak{q}_{0lg} = \mathfrak{q}_{k0g} = 0$  for all  $k, l, g \in \mathbb{N}_0$ , an *input-strict*  $\text{IBL}_\infty$ -algebra if  $\mathfrak{q}_{0lg} = 0$  for all  $l, g \in \mathbb{N}_0$ , and a *weak*  $\text{IBL}_\infty$ -algebra if we want to emphasize that operations with no input and output are allowed. The same terminology will be used for morphisms.

\* It is the category of *augmented strictly commutative*  $\text{BV}_\infty$ -algebras (see [CL09, Section 5] for the definition of a commutative  $\text{BV}_\infty$ -algebra). The augmentation  $\varepsilon$  is an additional data to get an MV-algebra. The coproduct is defined by  $\delta := \delta_0$  (see (D.5)).

\*\* It is the category of *(non-filtered) input-strict*  $\text{IBL}_\infty$ -algebras  $(\mathfrak{q}_{klg})_{k \geq 1, l, g \geq 0}$  and input-strict morphisms  $(\mathfrak{f}_{klg})_{k \geq 1, l, g \geq 0}$  which satisfy the following *finiteness condition* for all  $k \geq 1$  and  $g \geq 0$ : for any  $v \in SU$ , we have

$$\mathfrak{q}_{klg}(v) = 0 \quad \text{and} \quad \mathfrak{f}_{klg}(v) = 0 \quad \text{for all but finitely many } l \in \mathbb{N}_0. \quad (\text{D.11})$$

See [CFL15, Remark (6), p. 14] for the same finiteness condition. This is precisely what symplectic field theory for exact cobordisms gives.

(The transformation formulas between  $\mathfrak{q}_{klg}$  and  $\Delta_i$  and  $\mathfrak{f}_{klg}$  and  $\mathfrak{f}_i$  were written down in Remark 3.2.9. We will repeat them in Proposition D.2.6 below.)

The following filtered version of MV-algebras will allow us to describe more general weak  $\text{IBL}_\infty$ -algebras. In particular, we will be able to remove the restriction (D.11).

**Definition D.2.1** (Complete filtered MV-algebra). *A complete filtered MV-algebra over a complete filtered graded algebra  $R$  over a field  $\mathbb{K}$  is a complete filtered graded vector space  $V$  over  $\mathbb{K}$ , a homogenous  $R$ -linear map  $\Delta : V \hat{\otimes} R \rightarrow V \hat{\otimes} R$  of finite filtration degree satisfying*

$$|\Delta| = -1, \quad \|\Delta\| \geq 0, \quad \|\Delta(1)\| > 0 \quad \text{and} \quad \Delta \circ \Delta = 0,$$

*and operations  $\mu : V \hat{\otimes} V \rightarrow V$ ,  $\delta : V \rightarrow V \hat{\otimes} V$ ,  $\eta : \mathbb{K} \rightarrow V$  and  $\varepsilon : V \rightarrow \mathbb{K}$  such that  $(V, \mu, \eta, \varepsilon)$  is a complete filtered augmented unital commutative associative algebra and  $(V, \delta, \varepsilon, \eta)$  is a complete filtered coaugmented counital cocommutative coassociative coalgebra satisfying the limit conilpotency property (D.6). We often denote a complete filtered MV-algebra simply by  $(V, \Delta)$ .*

*A morphism of complete filtered MV-algebras  $(V^+, \Delta^+)$  and  $(V^-, \Delta^-)$  over  $R$  is a homogenous  $R$ -linear map  $\mathfrak{f} : V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R$  of finite filtration degree such that*

DONE Add a reference: Remark (6) at the beginning of CFL15.

DONE Resolve strict and norm continuity: We choose to work with  $\|\cdot\|$  because it is more easier. However, it can be redone using  $\mathcal{F}_\lambda$ . There is some class of filtrations for which  $\|c\| = \lambda$  is equivalent to  $c \in \mathcal{F}_\lambda$  and  $c \notin \mathcal{F}_\mu$  for  $\mu > \lambda$ . Make a remark.

DONE Degree 0?: Add that  $\mathfrak{f}$  is of degree 0?

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$$\|\mathfrak{f}\| = 0, \quad \|\mathfrak{f}\| \geq 0, \quad \|\mathfrak{f}(1)\| > 0 \quad \text{and} \quad e^{\mathfrak{f}} \circ \Delta^+ = \Delta^- \circ e^{\mathfrak{f}}.$$

The composition  $\diamond$  of two composable morphisms  $\mathfrak{f}_1$  and  $\mathfrak{f}_2$  (i.e., the target of  $\mathfrak{f}_2$  is the source of  $\mathfrak{f}_1$ ) is defined by

$$\mathfrak{f}_1 \diamond \mathfrak{f}_2 := \log(e^{\mathfrak{f}_1} \circ e^{\mathfrak{f}_2}). \quad (\text{D.12})$$

(The exponential and logarithm are well-defined by Proposition D.1.8.)

**Remark D.2.2** (On complete filtered MV-algebras). (i) The condition  $\|\mathfrak{f}(1)\| > 0$  is required for the exponential to converge; i.e., it might be seen as a technical condition. On the other hand, the condition  $\|\Delta(1)\| > 0$  is optional, and we impose it in the manner of [MV17] and [CFL15].

(ii) We do not define “non-complete filtered MV-algebras” because it is not clear whether  $\mathfrak{f}_1 \diamond \mathfrak{f}_2 : V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R$  for  $V^+ \otimes R \xrightarrow{\mathfrak{f}_2} V^{+-} \otimes R \xrightarrow{\mathfrak{f}_1} V^- \otimes R$  restricts to a map  $V^+ \otimes R \rightarrow V^- \otimes R$ . This forces us to define morphisms as maps of completions  $V^+ \hat{\otimes} R \rightarrow V^- \hat{\otimes} R$ . In such category, two filtered MV-algebras would be isomorphic if and only if their completions were. This is not desired, and thus we define only “complete filtered MV-algebras”.  $\triangleleft$

Inspired by (D.10), we define the following version of weak  $\text{IBL}_\infty$ -algebras.

**Definition D.2.3** (Complete filtered  $\text{IBL}_\infty$ -algebra in MV-formalism). *Let  $d \in \mathbb{Z}$  and  $\gamma > 0$ . A complete filtered  $\text{IBL}_\infty$ -algebra of bidegree  $(d, \gamma)$  in MV-formalism is a complete filtered MV-algebra  $(V, \Delta, R, \mu, \delta, \varepsilon, \eta)$  satisfying the following conditions:*

- (1)  $R = \mathbb{K}((\hbar))$  is the ring of Laurent series<sup>5</sup> in a formal variable  $\hbar$  of degree  $|\hbar| = 2d$  equipped with the complete, Hausdorff and exhaustive filtration

$$\mathcal{F}^\lambda \mathbb{K}((\hbar)) := \left\{ \sum_{i=-\infty}^{\infty} a_i \hbar^i \in \mathbb{K}((\hbar)) \mid a_i = 0 \text{ for } i < \frac{\lambda}{2\gamma} \right\} \quad \text{for } \lambda \in \mathbb{R}. \quad (\text{D.13})$$

DONE It is  
The multipli-  
cation is not  
continuous:  
The multipli-  
cation is not  
continuous

- (2) There is a complete filtered graded vector space  $W$  such that if we define

$$U := W[1] \quad \text{and} \quad \mathcal{F}^\lambda U := (\mathcal{F}^{\lambda-\gamma} W)[1] \quad \text{for all } \lambda \in \mathbb{R}, \quad (\text{D.14})$$

then  $(V = \hat{S}U, \mu, \delta, \eta, \varepsilon)$  is the completion of the symmetric bialgebra on  $U$  filtered by the induced filtration.

<sup>5</sup>Another notation for  $\mathbb{K}((\hbar))$  is  $\mathbb{K}[[\hbar]][\hbar^{-1}]$  to emphasize that it is the localization of the ring of power series  $\mathbb{K}[[\hbar]]$  at the powers of  $\hbar$ . An element of  $\mathbb{K}((\hbar))$  is a formal power series  $\sum_{i=-\infty}^{\infty} a_i \hbar^i$  with only finitely many non-zero  $a_i \in \mathbb{K}$  for  $i \leq 0$ .



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(3) The BV-operator  $\Delta : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$ , where  $\hat{S}U((\hbar)) := SU \hat{\otimes} \mathbb{K}((\hbar))$ ,<sup>6</sup> decomposes as

$$\Delta = \hbar^{-1} \Delta_0 + \Delta_1 + \hbar \Delta_2 + \hbar^2 \Delta_3 + \cdots, \quad (\text{D.15})$$

where for all  $i \geq 0$  the operator  $\Delta_i : \hat{S}U \rightarrow \hat{S}U$  is a linear differential operator of order  $\leq i$ .<sup>7</sup>

We often denote the data of a complete filtered  $\text{IBL}_\infty$ -algebra in MV-formalism simply by  $(W, \Delta)$ .

A morphism of complete filtered  $\text{IBL}_\infty$ -algebras  $(W^+, \Delta^+)$  and  $(W^-, \Delta^-)$  of bidegree  $(d, \gamma)$  in MV-formalism is a morphism of complete filtered MV-algebras  $\mathfrak{f} : \hat{S}U^+((\hbar)) \rightarrow \hat{S}U^-((\hbar))$  which decomposes as

$$\mathfrak{f} = \hbar^{-1} \mathfrak{f}_0 + \mathfrak{f}_1 + \hbar \mathfrak{f}_2 + \hbar^2 \mathfrak{f}_3 + \cdots, \quad (\text{D.16})$$

where for all  $i \geq 0$  the map  $\mathfrak{f}_i : \hat{S}U^+ \rightarrow \hat{S}U^-$  is  $\mathbb{K}$ -linear and satisfies

$$\hat{S}_j U^+ \subset \ker \mathfrak{f}_i \quad \text{for all } j > i. \quad (\text{D.17})$$

**Notation D.2.4** (Replacing  $*$  with  $\odot$  for  $V = SU$ ). From now on, we will denote the convolution product  $*$  on morphisms of  $SU$  by  $\odot$ . It is namely the same operation on morphisms which is denoted by  $\odot$  in Section 3.2 and in [CFL15].

For a map  $\mathfrak{f} = \sum_{i=-\infty}^{\infty} \mathfrak{f}_{i+1} \hbar^i : \hat{S}U^+((\hbar)) \rightarrow \hat{S}U^-((\hbar))$ , where  $\mathfrak{f}_i : \hat{S}U^+ \rightarrow \hat{S}U^-$  is  $\mathbb{K}$ -linear, we denote

$$\langle \mathfrak{f} \rangle_{klg} := \pi_l \circ \mathfrak{f}_{k+g} \circ \iota_k : \hat{S}_k U^+ \rightarrow \hat{S}_l U^- \quad \text{for all } k, l \geq 0, g \in \mathbb{Z}.$$

This is the notation introduced in [CFL15, Equation (2.14)].

Based on [CFL15], we will now give an equivalent characterization of complete filtered  $\text{IBL}_\infty$ -algebras in MV-formalism and their morphisms in terms of their components  $(\mathfrak{q}_{klg})$  and  $(\mathfrak{f}_{klg})$  within the surface calculus. We will not repeat the interpretation of the algebraic relations in terms of gluing of surface; for this, see [CFL15]. We need the following lemma.

DONE Source and target: One has to define the product on  $\mathfrak{f} : V \rightarrow V'$ , i.e., differenti source and target.

<sup>6</sup>In contrast to  $\mathbb{K}((\hbar))$ , an element of  $\hat{S}U((\hbar))$ , when seen as a power series, can have a non-zero coefficient at every power  $\hbar^i$ .

<sup>7</sup>The definition of a differential operator on p.301 generalizes in a straightforward way to complete filtered algebras.

**Lemma D.2.5** (Differential operators on filtered symmetric bialgebras). *Let  $U$  be a complete filtered graded vector space. Then the following holds for the complete filtered symmetric bialgebra  $\hat{S}U$  and all  $k \in \mathbb{N}_0$ :*

- (a) *A linear homogenous map  $D : \hat{S}U \rightarrow \hat{S}U$  of finite filtration degree is a differential operator of order  $\leq k$  if and only if it can be written as*

$$D = \sum_{i=0}^k D_i \odot \mathbb{1} \quad (\text{D.18})$$

*for linear homogenous maps  $D_i : \hat{S}_i U \subset \hat{S}U \rightarrow \hat{S}U$  of finite filtration degrees (here  $D_i = 0$  on  $\hat{S}_j U$  for  $j \neq i$  is the trivial extension). The maps  $D_i$  are uniquely determined by  $D$ .*

- (b) *A linear homogenous map  $\mathfrak{f} : \hat{S}U \rightarrow \hat{S}U$  of finite filtration degree satisfies  $\hat{S}_{i>k} U \subset \ker \mathfrak{f}$  if and only if it can be written as  $\mathfrak{f} = \mathfrak{f}_0 + \dots + \mathfrak{f}_k$  for linear homogenous maps  $\mathfrak{f}_i : \hat{S}_i U \rightarrow \hat{S}U$  of finite filtration degrees. The maps  $\mathfrak{f}_i$  are uniquely determined by  $\mathfrak{f}$ .*

*Proof.* (a) The claim is implied by the following subclaim and proposition, which generalize in a straightforward way to the filtered case. Recall that an operator  $D : SU \rightarrow SU$  is called a *derivative* of order  $\leq k$  if

$$D(1) = 0 \quad \text{and} \quad \psi_i^D(v_1, \dots, v_i)(1) = 0 \quad \text{for all } i \geq k+1 \text{ and } v_1, \dots, v_i \in V,$$

where  $\psi_i^D$  were defined on page 301.

*Subclaim* (Derivatives and differential operators). A homogenous linear operator  $D : SU \rightarrow SU$  is a differential operator of order  $\leq k$  if and only if the linear operator  $D' := D - D_0 \odot \mathbb{1}$ , where  $D_0 = D \circ \iota_0$ , is a derivative of order  $\leq k$  (recall that  $\iota_k : S_k U \rightarrow SU$  is the inclusion).

*Proof.* Given a derivative  $D'$  of order  $\leq k$  and a linear map  $D_0 : S_0 U \subset SU \rightarrow SU$ , we will check that  $D := D' + D_0 \odot \mathbb{1}$  is a differential operator of order  $\leq k$ . We have

$$\begin{aligned} \underbrace{\psi_{k+2}^{D'}(v_1, \dots, v_{k+2})(1)}_{=0} &= [\psi_{k+1}^{D'}(v_1, \dots, v_{k+1}), L_{v_{k+2}}](1) \\ &= \psi_{k+1}^{D'}(v_1, \dots, v_{k+1})(v_{k+2}) - \underbrace{\psi_{k+1}^{D'}(v_1, \dots, v_{k+1})(1)}_{=0} v_{k+2} \end{aligned}$$

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for all  $v_1, \dots, v_{k+2} \in V$ , and hence  $D'$  is also a differential operator of order  $\leq k$ . For all  $v \in SU$ , it holds

$$(D_0 \odot \mathbb{1})(v) = D_0(1)v,$$

and hence for all  $v, v_1 \in SU$ , we have

$$\begin{aligned} \psi_1^{D_0 \odot \mathbb{1}}(v_1)(v) &= (D_0 \odot \mathbb{1})(v_1 v) - (-1)^{D_0 v_1} v_1 (D_0 \odot \mathbb{1})(v) \\ &= D_0(1)v_1 v - (-1)^{D_0 v_1} v_1 D_0(1)v \\ &= 0. \end{aligned}$$

It follows that  $D_0 \odot \mathbb{1}$  is a differential operator of order 0. Clearly, a sum of differential operators of orders  $\leq k_1$  and  $\leq k_2$  is a differential operator of order  $\leq \max(k_1, k_2)$ . Therefore,  $D = D' + D_0 \odot \mathbb{1}$  is a differential operator of order  $\leq k$ . Moreover, since  $D'(1) = 0$  by definition, we have  $D(1) = D_0(1)$ , i.e.,  $D_0 = D \circ \iota_0$ .

Given a differential operator  $D$  of order  $\leq k$ , we define  $D_0 := D \circ \iota_0$  and  $D' := D - D_0 \odot \mathbb{1}$ . Firstly, it holds

$$D'(1) = D(1) - (D_0 \odot \mathbb{1})(1) = 0.$$

Secondly, we have

$$\psi_i^D(v_1, \dots, v_i) = \psi^{D'}(v_1, \dots, v_i) = 0,$$

and hence  $\psi_i^D(v_1, \dots, v_i)(1) = 0$  for all  $i \geq k+1$  and  $v_1, \dots, v_i \in V$ . Therefore,  $D'$  is a derivative of order  $\leq k$ . (Subclaim)  $\square$

**Proposition D.2.1** ([Mar01, Proposition 3.2]). A linear operator  $D : SU \rightarrow SU$  is a derivative of order  $\leq k$  if and only if it can be written as  $D = \sum_{i=1}^k D_i \odot \mathbb{1}$  for unique  $D_i : S_i U \rightarrow SU$ .

(b) This is clear. We have  $\mathfrak{f}_i = \mathfrak{f} \circ \iota_i$  for all  $i = 0, \dots, k$ .  $\square$

**Proposition D.2.6** (Filtered  $\text{IBL}_\infty$ -algebras in MV-formalism in components). *We have the following equivalences of structures:*

- (a) *The data of a complete filtered  $\text{IBL}_\infty$ -algebra of bidegree  $(d, \gamma)$  in MV-formalism  $(W, \Delta)$  is equivalent to the data of a complete filtered graded vector space  $W$  and a collection of  $\mathbb{K}$ -linear maps  $\mathfrak{q}_{klg} : \hat{S}_k U \rightarrow \hat{S}_l U$  for all  $k, l, g \in \mathbb{N}_0$  which are homogenous, have finite filtration degrees and satisfy the following conditions for all  $k, l, g \geq 0$ :*

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- (1)  $|\mathfrak{q}_{klg}| = -2d(k + g - 1) - 1$ .
- (2)  $\|\mathfrak{q}_{klg}\| \geq -2\gamma(k + g - 1)$ .
- (3) The inequality in (2) is strict whenever  $k = 0$ .
- (4) The sum of maps  $Q_{kg} := \sum_{l=0}^{\infty} \mathfrak{q}_{klg} : \hat{S}_k U \rightarrow \hat{S}U$  converges (in the sense of Lemma D.1.1).
- (5) The following equation holds:<sup>8</sup>

$$\sum_{\substack{h \geq 1 \\ k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k \\ l_1 + l_2 - h = l \\ g_1 + g_2 + h - 1 = g}} \mathfrak{q}_{k_1 l_1 g_1} \circ_h \mathfrak{q}_{k_2 l_2 g_2} = 0. \quad (\text{D.19})$$

- (b) The data of a strict morphism  $\mathfrak{f} : (W^+, \Delta^+) \rightarrow (W^-, \Delta^-)$  of complete filtered  $\text{IBL}_{\infty}$ -algebras of bidegree  $(d, \gamma)$  in MV-formalism, strict meaning that  $\varepsilon^- \circ \mathfrak{f} = \mathfrak{f} \circ \eta^+ = 0$ , is equivalent to a collection of  $\mathbb{K}$ -linear maps  $\mathfrak{f}_{klg} : \hat{S}_k U^+ \rightarrow \hat{S}_l U^-$  for all  $k, l, g \in \mathbb{N}_0$  which are homogenous, have finite filtration degrees and satisfy the following conditions for all  $k, l, g \geq 0$ :

- (1)  $|\mathfrak{f}_{klg}| = -2d(k + g - 1)$ .
- (2)  $\|\mathfrak{f}_{klg}\| \geq -2\gamma(k + g - 1)$ .
- (3) The inequality in (2) is strict whenever  $k = 0$ .
- (4) The sum  $F_{kg} := \sum_{l=0}^{\infty} \mathfrak{f}_{klg} : \hat{S}_k U^+ \rightarrow \hat{S}U^-$  converges.
- (5) It holds  $\mathfrak{f}_{k0g} = \mathfrak{f}_{0lg} = 0$ .

DONE  $r = 0$  in composition: Is there  $r = 0$ ? Because twisting with MC element has this term, but it does not make much sense in the composition! Do the relations hold also for  $k, l = 0$  or we do not require that?

<sup>8</sup>Notice that (D.19) does not contain  $\mathfrak{q}_{00g}$  with  $g \geq 0$  for any  $k, l, g \geq 0$  (otherwise all  $\mathfrak{q}_{klg}$  appear). In fact, it will be clear from the proof that there is no condition on  $\mathfrak{q}_{00g}$  coming from  $\Delta^2 = 0$ ; if we work over a general ring, then  $\mathfrak{q}_{00g}(1)$  for  $g \geq 0$  can be arbitrary odd elements in it. Note that this contradicts [CFL15, p. 47, bottom-most paragraph].

(6) The following equation holds:

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{h_1, \dots, h_r \geq 1 \\ k_1, \dots, k_r, k^-, l_1, \dots, l_r, l^-, g_1, \dots, g_r, g^- \geq 0 \\ k_1 + \dots + k_r = k \\ l_1 + \dots + l_r - k^- + l^- = l \\ g_1 + \dots + g_r + g^- + k^- - r = g}} \mathfrak{q}_{k^- l^- g^-} \circ_{h_1, \dots, h_r} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \\
 &= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{h_1, \dots, h_r \geq 1 \\ k_1, \dots, k_r, k^+, l_1, \dots, l_r, l^+, g_1, \dots, g_r, g^+ \geq 0 \\ k_1 + \dots + k_r + k^+ - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1, \dots, h_r} \mathfrak{q}_{k^+ l^+ g^+}.
 \end{aligned} \tag{D.20}$$

The term  $r = 0$  on the left- or right-hand side is possible only for  $k = 0$  or  $l = 0$  and equals  $\mathfrak{q}_{0l g}^- \circ e : \hat{S}_0 U^+ \rightarrow \hat{S}_l U^-$  or  $e \circ \mathfrak{q}_{k0 g}^+ : \hat{S}_k U^+ \rightarrow \hat{S}_0 U^-$ , respectively, where  $e$  is the  $\hat{S}_0 U^+ \rightarrow \hat{S}_0 U^-$  component of the unit for the convolution product (D.4) (the identity under  $\hat{S}_0 U^\pm \simeq \mathbb{K}$  via  $\eta^\pm$ ).<sup>9</sup>

The equivalences (a) and (b) are given by the formulas

$$\Delta_i = \sum_{\substack{k, g \geq 0 \\ k+g=i}} Q_{kg} \odot \mathbb{1} \quad \text{and} \quad \mathfrak{f}_i = \sum_{\substack{k, g \geq 0 \\ k+g=i}} F_{kg} \quad \text{for all } i \geq 0, \text{ respectively.} \tag{D.21}$$

We remark that the operations  $\circ_{h_1, \dots, h_r}$  are defined in Definition D.4.2 in the next section or in Definition 3.2.2.

Suppose that  $\mathfrak{f}^+$  and  $\mathfrak{f}^-$  are strict composable morphisms of complete filtered  $\text{IBL}_\infty$ -algebras in MV-formalism. Then we have the following:

(c) The composition  $\mathfrak{f} := \mathfrak{f}^- \diamond \mathfrak{f}^+$  is a strict morphism of complete filtered  $\text{IBL}_\infty$ -algebras

<sup>9</sup>If  $\mathfrak{f}$  is not strict, it would be interesting to know whether  $e^\mathfrak{f} \Delta^+ = \Delta^- e^\mathfrak{f}$  is still equivalent to the connected calculus (D.20) or whether there are some other equations possibly involving disconnected gluing. Note that trying to incorporate  $\mathfrak{f}_{0lg}$  to the right-hand side and  $\mathfrak{f}_{k0g}$  to the left-hand side of (D.20) always leads to the disconnected calculus. Also note that  $\mathfrak{f}_{00g}$  for  $g \geq 0$  do not appear in (D.20) for any  $k, l, g \geq 0$ ; otherwise all  $\mathfrak{f}_{klg}$  appear. Finally, note that  $\mathfrak{q}_{klg}^+$  and  $\mathfrak{q}_{klg}^-$  appear for all  $k, l, g \geq 0$  and that it follows from (D.20) that  $e \circ \mathfrak{q}_{00g}^+$  and  $\mathfrak{q}_{00g}^- \circ e$  have to be equal for all  $g \geq 0$ .

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in MV-formalism and its components  $\mathfrak{f}_{klg}$  for  $k, l \geq 1, g \geq 0$  are given by

$$\mathfrak{f}_{klg} = \sum_{\substack{r^-, r^+ \geq 0 \\ k_1^-, l_1^-, \dots, k_{r^-}^-, l_{r^-}^- \geq 1 \\ k_1^+, l_1^+, \dots, k_{r^+}^+, l_{r^+}^+ \geq 1 \\ g_1^+, \dots, g_{r^+}^+, g_1^-, \dots, g_{r^-}^- \geq 0 \\ k_1^- + \dots + k_{r^-}^- = l_1^+ + \dots + l_{r^+}^+ \\ k_1^+ + \dots + k_{r^+}^+ = k \\ l_1^- + \dots + l_{r^-}^- = l \\ g_1^+ + \dots + g_{r^+}^+ + g_1^- + \dots + g_{r^-}^- - r^+ - r^- \\ + k_1^- + \dots + k_{r^-}^- + 1 = g}} \frac{1}{r^+! r^-!} (\mathfrak{f}_{k_1^- l_1^- g_1^-}^-, \dots, \mathfrak{f}_{k_{r^-}^- l_{r^-}^- g_{r^-}^-}^-) \circ_{\text{con}} (\mathfrak{f}_{k_1^+ l_1^+ g_1^+}^+, \dots, \mathfrak{f}_{k_{r^+}^+ l_{r^+}^+ g_{r^+}^+}^+), \quad (\text{D.22})$$

where  $\circ_{\text{con}}$  denotes the connected composition (see Definition D.4.2 in the next section).

*Proof.* (a) Suppose first that  $(W, \Delta)$  is a complete filtered  $\text{IBL}_\infty$ -algebra of bidegree  $(d, \gamma)$  in MV-formalism; i.e., we have  $\Delta = \Delta_0 \hbar^{-1} + \Delta_1 + \Delta_2 \hbar + \dots$  for  $\Delta_i : \hat{\text{S}}U \rightarrow \hat{\text{S}}U$  differential operators of order  $\leq i$ ,  $|\Delta| = -1$ ,  $\|\Delta\| \geq 0$  and  $\|\Delta(1)\| > 0$ . Using Lemma D.2.5, we write  $\Delta_i = \sum_{j=0}^i D_{ij} \odot \mathbb{1}$  for  $\mathbb{K}$ -linear maps  $D_{ij} : \hat{\text{S}}_j U \subset \hat{\text{S}}U \rightarrow \hat{\text{S}}U$  of finite filtration degrees which are uniquely determined by  $\Delta_i$ , and we define  $Q_{kg} := D_{k+g, k} : \hat{\text{S}}_k U \rightarrow \hat{\text{S}}U$  for all  $k, g \in \mathbb{N}_0$ . Next, we define  $\mathfrak{q}_{klg} := \pi_l \circ Q_{kg} : \hat{\text{S}}_k U \rightarrow \hat{\text{S}}_l U$  for all  $k, l, g \in \mathbb{N}_0$ .

In the following computations, we will keep in mind that  $|\hbar| = 2d$  and  $\|\hbar\| = 2\gamma$ . Using the formula  $|f \circ g| = |f| + |g|$ , which is valid for homogenous linear maps  $f$  and  $g$ , we have

$$\begin{aligned} |\Delta| = -1 &\iff \forall i \geq -1 : |\Delta_{i+1}| = -1 - di \\ &\iff \forall k, g \geq 0, k + g = i + 1 : |Q_{kg}| = -1 - di \\ &\iff \forall k, l, g \geq 0, k + g = i + 1 : |\mathfrak{q}_{klg}| = -1 - d(k + g - 1). \end{aligned}$$

Thus (1) follows. Using that  $\|f \circ g\| \geq \|f\| + \|g\|$  and  $\|\sum_i v_i\| \geq \inf_i \|v_i\|$ , we have

$$\begin{aligned} \|\Delta\| \geq 0 &\iff \forall i \geq -1 : \|\Delta_{i+1}\| \geq -2\gamma i \\ &\iff \forall k, g \geq 0, k + g = i + 1 : \|Q_{kg}\| \geq -2\gamma i \\ &\iff \forall k, l, g \geq 0, k + g = i + 1 : \|\mathfrak{q}_{klg}\| \geq -2\gamma(k + g - 1). \end{aligned}$$

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Thus (2) follows. The argument for “ $\implies$ ” on the second line is inductive using that

$$Q_{kg} = \left( \Delta_{k+g} - \sum_{j=0}^{k-1} Q_{j,k+g-j} \odot \mathbb{1} \right) \circ \iota_k$$

for all  $k, g \geq 0$ . Evaluation of  $\Delta(1)$  gives

$$\begin{aligned} \Delta(1) &= \Delta_0(1)\hbar^{-1} + \Delta_1(1) + \Delta_2(1)\hbar + \cdots \\ &= Q_{00}(1)\hbar^{-1} + Q_{01}(1) + Q_{02}(1)\hbar + \cdots \\ &= \left( \sum_{l=0}^{\infty} \mathfrak{q}_{0l0}(1) \right) \hbar^{-1} + \left( \sum_{l=0}^{\infty} \mathfrak{q}_{0l1}(1) \right) + \hbar \left( \sum_{l=0}^{\infty} \mathfrak{q}_{0l2}(1) \right) + \cdots, \end{aligned}$$

and we see that  $\|\Delta(1)\| > 0$  is equivalent to (3). As for (4), it is implied by the Resummation Lemma (Lemma D.1.1). Namely, we have

$$\sum_{l=0}^{\infty} \mathfrak{q}_{klg}(v) = \sum_{l=0}^{\infty} \pi_l(Q_{kg}(v)) \stackrel{D.1.1}{=} Q_{kg}(v) \quad \text{for all } v \in \hat{S}_k U.$$

Thus  $\sum_{l=0}^{\infty} \mathfrak{q}_{klg}$  converges to  $Q_{kg}$ .

On the other hand, given  $\mathfrak{q}_{klg}$  satisfying (1), (2), (3) and (4), we clearly get an operator  $\Delta : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$  which has the decomposition (D.15) and which satisfies  $|\Delta| = 0$ ,  $\|\Delta\| \geq 0$  and  $\|\Delta(1)\| > 0$ .

Assuming (1), (2), (3) and (4), it remains to check that  $\Delta^2 = 0$  is equivalent to (5). This is the same computation as in [CFL15, Section 2], just in the filtered setting and allowing  $k = 0$  or  $l = 0$  (it will turn out that it does not change anything). Recall the notation from Section 3.2 that  $\hat{D} := D \odot \mathbb{1} = \mu(D \hat{\otimes} \mathbb{1})\delta$ . We will be using formulas from Remark 3.2.3 in that section, which are proven in Proposition D.4.4 in the next section. Note that by the Resummation Lemma, elements of the completion can be summed up in any order, even as nested infinite sums, provided one (and hence all) of these sums converges. Therefore, we could, in principle, write just one sum  $\sum$  and think of  $\Delta$  as of the sum  $\sum \hat{\mathfrak{q}}_{klg} \hbar^{k+g-1}$  over all  $k, l, g \geq 0$ .

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We have

$$\begin{aligned}\Delta \circ \Delta &= \sum_{i=-2}^{\infty} \left( \sum_{\substack{k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ k_1 + k_2 + g_1 + g_2 = i+2}} \hat{\mathbf{q}}_{k_1 l_1 g_1} \circ \hat{\mathbf{q}}_{k_2 l_2 g_2} \right) \hbar^i \\ &= \sum_{i=-2}^{\infty} \left( \sum_{\substack{k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ k_1 + k_2 + g_1 + g_2 = i+2}} \sum_{h=0}^{\min(k_1, l_2)} \overline{\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}} \right) \hbar^i,\end{aligned}$$

and we see that

$$\langle \Delta^2 \rangle_{klg} = \sum_{\substack{k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ k_1 + k_2 + g_1 + g_2 = k+g+1}} \sum_{h=0}^{\min(k_1, l_2)} \pi_l \circ \overline{\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}} \circ \iota_k \quad \text{for all } k, l \geq 0, g \geq -1.$$

Because  $\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2} : \hat{S}_{k_1+k_2-h}U \rightarrow \hat{S}_{l_1+l_2-h}U$ , it follows from the definition  $\hat{f} := \mu(f \otimes \mathbb{1})\delta$  that  $\pi_l \circ \overline{\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}} \circ \iota_k = 0$  unless  $0 \leq k_1 + k_2 - h \leq k$ ,  $0 \leq l_1 + l_2 - h \leq l$  and  $l_1 + l_2 + h - (k_1 + k_2 + h) = l - k$ . For fixed  $k, l, h \geq 0, g \geq -1$ , it holds

$$\left\{ \begin{array}{l} k_1, l_1, g_1, k_2, l_2, g_2 \geq 0 \\ 0 \leq k_1 + k_2 - h \leq k \\ 0 \leq l_1 + l_2 - h \leq l \\ l_1 + l_2 - k_1 - k_2 = l - k \\ k_1 + k_2 + g_1 + g_2 = k + g + 1 \end{array} \right\} = \bigsqcup_{s=0, \dots, k} \left\{ \begin{array}{l} k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k - s \\ l_1 + l_2 - h = l - s \\ g_1 + g_2 + h - 1 = g + s \end{array} \right\},$$

and hence

$$\begin{aligned}\langle \Delta^2 \rangle_{klg} &= \sum_{s=0}^k \sum_{h=0}^{g+s+1} \sum_{\substack{k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k - s \\ l_1 + l_2 - h = l - s \\ g_1 + g_2 + h - 1 = g + s}} \pi_l \circ \overline{\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}} \circ \iota_k \\ &= \underbrace{\sum_{h=0}^{g+1} \sum_{\substack{k_1, k_2, l_1, l_2, g_1, g_2 \geq 0 \\ k_1 + k_2 - h = k \\ l_1 + l_2 - h = l \\ g_1 + g_2 + h - 1 = g}} \mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}}_{=:(\square)_{klg}} + \sum_{s=1}^k \pi_l \circ \overline{(\square)_{k-s, l-s, g+s}} \circ \iota_k.\end{aligned}\tag{D.23}$$

DONE This is incorrect!!: The computation is incorrect—it is either  $k' < k$  or  $l' < l$  and so on.

We see immediately that

$$\langle \Delta^2 \rangle_{klg} = 0 \quad \text{for all } k, l \geq 0, g \geq -1,\tag{D.24}$$



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which is equivalent to  $\Delta^2 = 0$ , follows from  $(\square)_{klg} = 0$  for all  $k, l \geq 0, g \geq -1$ . The reverse implication is proven by induction on the linear order  $\prec$  on signatures  $(k, l, g)$  defined in [CFL15, Definition 2.4]. It holds  $(k', l', g') \prec (k, l, g)$ , by definition, if one of the following conditions is satisfied:

- (i)  $k' + l' + 2g' < k + l + 2g$ ,
- (ii)  $k' + l' + 2g' = k + l + 2g$  and  $g' > g$ , or
- (iii)  $k' + l' + 2g' = k + l + 2g$  and  $g' = g$  and  $k' < k$ .

For the last sum in (D.23), we denote  $k_s := k - s$ ,  $l_s := l - s$  and  $g_s := g + s$  and compute

$$(k - k_s) + (l - l_s) + 2(g - g_s) = s + s + 2(-s) = 0.$$

Therefore, case (ii) applies, and so  $(k_s, l_s, g_s) \prec (k, l, g)$  for all  $s = 1, \dots, k$ . This shows the equivalence of  $\Delta^2 = 0$  and  $(\square)_{klg} = 0$  for all  $k, l \geq 0, g \geq -1$ . Finally,  $(\square)_{k,l,-1} = 0$  holds automatically because  $g_1 + g_2 + h = 0$  implies  $h = 0$ , and all terms  $\mathbf{q}_{k_1 l_1 g_1} \circ_h \mathbf{q}_{k_2 l_2 g_2}$  with  $h = 0$  vanish. This is because  $\mathbf{q}_{klg}$  are odd and  $f_1 \circ_0 f_2 = (-1)^{|f_1||f_2|} f_2 \circ_0 f_1$ .

(b) Exactly as in (a), we first prove the equivalence of  $|\mathbf{f}| = 0$ ,  $\|\mathbf{f}\| \geq 0$ ,  $\|\mathbf{f}(1)\| > 0$  and the conditions (D.16) and (D.17) to (1), (2), (3) and (4) under the correspondence (D.21). Clearly,  $\varepsilon^- \circ \mathbf{f} = \mathbf{f} \circ \eta^+ = 0$  is equivalent to (5).

Assuming (1), (2), (3) and (4), it remains to check that the equation  $e^{\mathbf{f}} \Delta^+ = \Delta^- e^{\mathbf{f}}$  is equivalent to (6). This is again the same computation as in [CFL15, Section 2]. We will do it in the weak case and then restrict to the strict case for the induction. We have

$$\begin{aligned} e^{\mathbf{f}} \circ \Delta^+ &= \left( \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i_1, \dots, i_r \geq -1} \sum_{\substack{k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k_1 + g_1 = i_1 + 1, \dots, k_r + g_r = i_r + 1}} \mathbf{f}_{k_1 l_1 g_1} \odot \dots \odot \mathbf{f}_{k_r l_r g_r} \hbar^{i_1 + \dots + i_r} \right) \\ &\quad \circ \left( \sum_{i^+ = -1}^{\infty} \sum_{\substack{k^+, l^+, g^+ \geq 0 \\ k^+ + g^+ = i^+ + 1}} \hat{\mathbf{q}}_{k^+ l^+ g^+}^+ \hbar^{i^+} \right) \\ &= \sum_{i = -\infty}^{\infty} \left( \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{k^+, l^+, g^+, k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^+ + k_1 + \dots + k_r + g^+ + g_1 + \dots + g_r = i + r + 1}} (\mathbf{f}_{k_1 l_1 g_1} \odot \dots \odot \mathbf{f}_{k_r l_r g_r}) \circ \hat{\mathbf{q}}_{k^+ l^+ g^+}^+ \right) \hbar^i \end{aligned}$$

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$$\begin{aligned}
&= \sum_{i=-\infty}^{\infty} \left( \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^+ + k_1 + \dots + k_r + g^+ \\ + g_1 + \dots + g_r = i + r + 1}} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = l^+}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1, \dots, h_r} \mathfrak{q}_{k^+ l^+ g^+}^+ \right) \hbar^i \\
&\quad \underbrace{\hspace{15em}}_{=: (* )_{i,r}^+ : \hat{S}U^+ \rightarrow \hat{S}U^-}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\Delta^- \circ e^{\mathfrak{f}} \\
&= \sum_{i=-\infty}^{\infty} \left( \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{k^-, l^-, g^- \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^- + k_1 + \dots + k_r + g^- \\ + g_1 + \dots + g_r = i + r + 1}} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k^-}} \mathfrak{q}_{k^- l^- g^-}^- \circ_{h_1, \dots, h_r} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \right) \hbar^i. \\
&\quad \underbrace{\hspace{15em}}_{=: (* )_{i,r}^- : \hat{S}U^+ \rightarrow \hat{S}U^-}
\end{aligned}$$

We will consider  $\pi_l \circ (* )_{i,r}^+ \circ \iota_k$  for fixed  $i \in \mathbb{Z}$  and  $k, l \geq 0$ . Because of the definition of  $\circ_{h_1, \dots, h_r}$ , only the terms with  $l_1 + \dots + l_r = l$  and  $k^+ + k_1 + \dots + k_r - h_1 - \dots - h_r = k^+ + k_1 + \dots + k_r - l^+ = k$  survive in  $\pi_l \circ (* )_{i,r}^+ \circ \iota_k$ . Denoting  $g := i - k + 1$ , we have

$$\begin{aligned}
k^+ + k_1 + \dots + k_r + g^+ + g_1 + \dots + g_r &= i + r + 1 & g_1 + \dots + g_r + g^+ + l^+ - r &= g \\
k^+ + k_1 + \dots + k_r - l^+ &= k & \iff & k^+ + k_1 + \dots + k_r - l^+ &= k \\
l_1 + \dots + l_r &= l & & l_1 + \dots + l_r &= l.
\end{aligned}$$

Therefore, it holds

$$\begin{aligned}
\pi_l \circ (* )_{i,r}^+ \circ \iota_k &= \sum_{\substack{k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k^+ + k_1 + \dots + k_r - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g}} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = l^+}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1, \dots, h_r} \mathfrak{q}_{k^+ l^+ g^+}^+.
\end{aligned}$$

Collecting the terms with  $h_i = 0$  and using the graded commutativity of  $\odot$ , we obtain

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$$\begin{aligned}
\pi_l \circ (*)_{i,r}^+ \circ \iota_k &= \underbrace{\sum_{\substack{h_1, \dots, h_r \geq 1 \\ k^+, l^+, g^+, k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ h_1 + \dots + h_r = l^+ \\ k_1 + \dots + k_r + k^+ - l^+ = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r + g^+ + l^+ - r = g}}_{=: (\square^+)^r_{klg}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_r l_r g_r}) \circ_{h_1, \dots, h_r} \mathfrak{q}_{k^+ l^+ g^+}^+ \\
&+ \sum_{r'=0}^{r-1} \binom{r}{r'} \sum_{\substack{h_1, \dots, h_{r'} \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_{r'}, l_{r'}, g_{r'} \geq 0 \\ h_1 + \dots + h_{r'} = l^+ \\ k_1 + \dots + k_{r'} + k^+ - l^+ = k \\ l_1 + \dots + l_{r'} = l \\ g_1 + \dots + g_{r'} + g^+ + l^+ - r = g}} (\mathfrak{f}_{k_1 l_1 g_1}, \dots, \mathfrak{f}_{k_{r'} l_{r'} g_{r'}}) \circ_{h_1, \dots, h_{r'}} \mathfrak{q}_{k^+ l^+ g^+}^+ \\
&\quad \odot \mathfrak{f}_{k_{r'+1} l_{r'+1} g_{r'+1}} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \\
&= (\square^+)^r_{klg} + \sum_{r'=0}^{r-1} \binom{r}{r'} \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ 0 \leq g' \leq g+r-r'}} \left( (\square^+)^{r'}_{k' l' g'} \right. \\
&\quad \left. \odot \sum_{\substack{k_{r'+1} l_{r'+1} g_{r'+1}, \dots, k_r l_r g_r \geq 0 \\ k_{r'+1} + \dots + k_r = k - k' \\ l_{r'+1} + \dots + l_r = l - l' \\ g_{r'+1} + \dots + g_r - (r - r') = g - g'}} \mathfrak{f}_{k_{r'+1} l_{r'+1} g_{r'+1}} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \right). \\
&\quad \quad \quad =: (\triangle)^{r-r'}_{k-k', l-l', g-g'}
\end{aligned}$$

We used here that for fixed  $r \geq 0$ ,  $0 \leq r' \leq r-1$ ,  $k, l \geq 0$  and  $g \in \mathbb{Z}$ , we have

$$\begin{aligned}
\left\{ \begin{array}{l} h_1, \dots, h_{r'} \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_{r'}, l_{r'}, g_{r'} \geq 0 \\ h_1 + \dots + h_{r'} = l^+ \\ k_1 + \dots + k_{r'} + k^+ - l^+ = k \\ l_1 + \dots + l_{r'} = l \\ g_1 + \dots + g_{r'} + g^+ + l^+ - r = g \end{array} \right\} &= \bigsqcup_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ -r' \leq g' \leq g+r-r'}} \left\{ \begin{array}{l} h_1, \dots, h_{r'} \geq 1 \\ k^+, l^+, g^+ \geq 0 \\ k_1, l_1, g_1, \dots, k_{r'}, l_{r'}, g_{r'} \geq 0 \\ h_1 + \dots + h_{r'} = l^+ \\ k_1 + \dots + k_{r'} + k^+ - l^+ = k' \\ l_1 + \dots + l_{r'} = l' \\ g_1 + \dots + g_{r'} + g^+ + l^+ - r' = g' \end{array} \right\} \\
&\quad \times \left\{ \begin{array}{l} k_{r'+1}, l_{r'+1}, g_{r'+1}, \dots, k_r, l_r, g_r \geq 0 \\ k_{r'+1} + \dots + k_r = k - k' \\ l_{r'+1} + \dots + l_r = l - l' \\ g_{r'+1} + \dots + g_r - (r - r') = g - g' \end{array} \right\},
\end{aligned}$$

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where the notation is the vertical version of  $\{\cdot \mid \cdot\}$ . In fact, the summation starts from  $g' = 0$  because if  $g' < 0$ , then  $(\square^+)^{r'}_{k'l'g'} = 0$ . Summing over  $r \in \mathbb{N}_0$ , we get

$$\begin{aligned}
& \sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (* )_{i,r}^+ \circ \iota_k \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} (\square^+)^r_{klg} + \sum_{r=0}^{\infty} \sum_{r'=0}^{r-1} \frac{1}{r'!} \frac{1}{(r-r')!} \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ 0 \leq g' \leq g+r-r'}} (\square^+)^{r'}_{k'l'g'} \odot (\Delta)^{r-r'}_{k-k', l-l', g-g'} \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} (\square^+)^r_{klg} + \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ g' \geq 0}} \left( \sum_{r'=0}^{\infty} \frac{1}{r'!} (\square^+)^{r'}_{k'l'g'} \right) \odot \left( \sum_{t=1}^{\infty} \frac{1}{t!} (\Delta)^t_{k-k', l-l', g-g'} \right),
\end{aligned} \tag{D.25}$$

where we used the substitution  $t = r - r'$  and the fact that if  $g - g' < -t$ , then  $(\Delta)^t_{k-k', l-l', g-g'} = 0$ . Similarly, we obtain

$$\begin{aligned}
& \sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (* )_{i,r}^- \circ \iota_k \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} (\square^-)^r_{klg} + \sum_{\substack{0 \leq k' \leq k \\ 0 \leq l' \leq l \\ g' \geq 0}} \left( \sum_{t=1}^{\infty} \frac{1}{t!} (\Delta)^t_{k-k', l-l', g-g'} \right) \odot \left( \sum_{r'=0}^{\infty} \frac{1}{r'!} (\square^-)^{r'}_{k'l'g'} \right).
\end{aligned} \tag{D.26}$$

Now,  $e^{\mathfrak{f}} \Delta^+ = \Delta^- e^{\mathfrak{f}}$  is equivalent to

$$\sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (* )_{i,r}^+ \circ \iota_k = \sum_{r=0}^{\infty} \frac{1}{r!} \pi_l \circ (* )_{i,r}^- \circ \iota_k \quad \text{for all } k, l \geq 0 \text{ and } i \in \mathbb{Z}. \tag{D.27}$$

By our previous computations, this follows from

$$\sum_{r=0}^{\infty} \frac{1}{r!} (\square^+)^r_{klg} = \sum_{r=0}^{\infty} \frac{1}{r!} (\square^-)^r_{klg} \quad \text{for all } k, l, g \geq 0,$$

which are precisely equations (D.20). In order to prove the reverse implication, one needs to do induction on signatures  $(k, l, g)$  as in (a). It is possible in the strict case by the following lemma.

*Subclaim* (Induction in strict case I). Suppose that  $\mathfrak{f}_{0lg} = \mathfrak{f}_{k0g} = 0$  for all  $k, l, g \geq 0$ . Then for any  $k, l \geq 0$  and  $i \in \mathbb{Z}$ ,  $g = i - k + 1$ , only the terms with  $(k', l', g') \prec (k, l, g)$  contribute to the sums on the right-hand side of (D.25) and (D.26). These sums vanish if  $(k, l, g) = (0, 0, 0)$ .

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*Proof.* Recall the definition of  $\prec$  on p.312. First of all,  $(\Delta)_{k-k', l-l', g-g'}^t \neq 0$  implies

$$D := (k + l + 2g) - (k' + l' + 2g') = (k - k') + (l - l') + 2(g - g') \geq t + t + 2(-t) = 0,$$

where  $k - k', l - l' \geq t$  holds due to strictness. Suppose that  $D = 0$ . Then  $g' \geq g$  must hold because  $k - k' \geq 0$  and  $l - l' \geq 0$ . If  $g' = g$ , then  $l = l'$  and  $k = k'$  must hold; this is a contradiction with  $k - k' \geq t \geq 1$ . Therefore, it holds  $g' > g$ , and case (ii) of the definition of  $\prec$  applies.

If  $(k, l, g) = (0, 0, 0)$ , then  $(\Delta)_{0,0,-g'}^t = 0$  because  $t \geq 1$  and  $k - k' \geq t$ . (*Subclaim*)  $\square$

(c) We have

$$e^{\mathfrak{f}} = \sum_{k,l,g \geq 0} \langle e^{\mathfrak{f}} \rangle_{klg} \hbar^{k+g-1},$$

where

$$\begin{aligned} \langle e^{\mathfrak{f}} \rangle_{klg} &= \sum_{\substack{r \geq 0 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k_1 + \dots + k_r = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r - r + 1 = g}} \frac{1}{r!} \mathfrak{f}_{k_1 l_1 g_1} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \\ &= \mathfrak{f}_{klg} + \sum_{\substack{r \geq 2 \\ k_1, l_1, g_1, \dots, k_r, l_r, g_r \geq 0 \\ k_1 + \dots + k_r = k \\ l_1 + \dots + l_r = l \\ g_1 + \dots + g_r - r + 1 = g}} \frac{1}{r!} \mathfrak{f}_{k_1 l_1 g_1} \odot \dots \odot \mathfrak{f}_{k_r l_r g_r} \end{aligned} \quad (\text{D.28})$$

for all  $(k, l, g) \neq (0, 0, 1)$  with  $k, l \geq 0, g \in \mathbb{Z}$ . For  $(k, l, g) = (0, 0, 1)$ , one has to add the unit  $e$  coming from the summand with  $r = 0$ . We find that

$$\begin{aligned} \langle e^{\mathfrak{f}^-} e^{\mathfrak{f}^+} \rangle_{klg} &= \sum_{r^-, r^+ \geq 0} \frac{1}{r^+! r^-!} (\mathfrak{f}_{k_1^- l_1^- g_1^-}^- \odot \dots \odot \mathfrak{f}_{k_{r^-}^- l_{r^-}^- g_{r^-}^-}^-) \circ (\mathfrak{f}_{k_1^+ l_1^+ g_1^+}^+ \odot \dots \odot \mathfrak{f}_{k_{r^+}^+ l_{r^+}^+ g_{r^+}^+}^+) \\ &\quad \substack{k_1^-, l_1^-, g_1^-, \dots, k_{r^-}^-, l_{r^-}^-, g_{r^-}^- \geq 0 \\ k_1^+, l_1^+, g_1^+, \dots, k_{r^+}^+, l_{r^+}^+, g_{r^+}^+ \geq 0 \\ k_1^- + \dots + k_{r^-}^- = l_1^+ + \dots + l_{r^+}^+ \\ k_1^+ + \dots + k_{r^+}^+ = k \\ l_1^- + \dots + l_{r^-}^- = l \\ g_1^+ + \dots + g_{r^+}^+ + g_1^- + \dots + g_{r^-}^- - r^+ - r^- + k_1^- + \dots + k_{r^-}^- + 1 = g} \end{aligned}$$

for all  $k, l \geq 0, g \in \mathbb{Z}$ . The composition  $\mathfrak{f}^- \diamond \mathfrak{f}^+ = \log(e^{\mathfrak{f}^-} \circ e^{\mathfrak{f}^+})$  is the unique  $\mathfrak{f}$  such that  $e^{\mathfrak{f}} = e^{\mathfrak{f}^-} \circ e^{\mathfrak{f}^+}$ ; this is equivalent to

$$\langle e^{\mathfrak{f}} \rangle_{klg} = \langle e^{\mathfrak{f}^-} e^{\mathfrak{f}^+} \rangle_{klg} \quad \text{for all } k, l \geq 0, g \in \mathbb{Z}. \quad (\text{D.29})$$

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In the strict case, this can be solved for  $(f_{klg})$  by induction on signatures.

*Subclaim* (Induction in strict case II). Suppose that  $f^+$  and  $f^-$  are strict. Then it holds  $(k_i, l_i, g_i) \prec (k, l, g)$  for all  $i = 1, \dots, r$  and  $r \geq 2$  in (D.28) for all  $k, l \geq 0, g \in \mathbb{Z}$ .

*Proof.* We compute

$$k + l + 2g - 2 = (k_1 + l_1 + 2g_1 - 2) + \dots + (k_r + l_r + 2g_r - 2),$$

where  $k_i + l_i + 2g_i - 2 \geq 0$  by strictness. It follows that  $k + l + 2g \geq k_i + l_i + 2g_i$  holds for all  $i = 1, \dots, r$ . If, e.g.,  $k + l + 2g = k_1 + l_1 + 2g_1$ , then  $k_2 = l_2 = \dots = k_r = l_r = 1$ ,  $g_2 = \dots = g_r = 0$ , and hence  $g = 1 - r$ . However, this can not happen as  $r \geq 2$  and  $g \geq 0$ . Therefore, case (i) of  $\prec$  always occurs. (Subclaim)  $\square$

Using this, we can set  $f_{110} = f_{110}^- \circ f_{110}^+$  and  $f_{0lg} = f_{k0g} := 0$  for all  $k, l, g \geq 0$  and solve (D.29) for  $(f_{klg})$  for all  $k, l \geq 1, g \geq 0$  by induction over  $(k, l, g)$ . The solution (D.22) will solve (D.29) for all  $k, l \geq 0, g \in \mathbb{Z}$  (the equations for  $g < 0$  consist of disconnected gluings and can be checked by splitting into connected components). Conditions on the filtration degree are easy to check as in [CFL15, Lemma 8.5].  $\square$

Negative  $g$ :  
How is it with  
negative  $g$ 's?  
I never really  
checked.

We expect that weak complete filtered  $IBL_\infty$ -algebras in MV-formalism with weak morphisms form a subcategory of the category of complete filtered MV-algebras (to see this, it remains to prove a version of (c) of Proposition D.2.6 for weak morphisms). The identity morphism in this category is the continuous  $\mathbb{K}((\hbar))$ -linear extension of the trivial extension of the identity  $\mathbb{1} : \hat{U} \rightarrow \hat{U}$  to  $\hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$  (see (iii) of Remark D.1.10).

In the following, we will compare Definition D.2.3 to the definition of filtered  $IBL_\infty$ -algebras from [CFL15, Section 8].

A *filtered  $IBL_\infty$ -algebra of bidegree  $(d, \gamma)$  over  $\mathbb{K}$  on a complete filtered graded vector space  $W$  according to [CFL15]* is a collection of homogenous  $\mathbb{K}$ -linear maps  $q_{klg} : \hat{S}_k U \rightarrow \hat{S}_l U$  of finite filtration degrees for all  $k, l, g \in \mathbb{N}_0$ , where  $U$  was defined in (D.14), which satisfy, firstly, (1) and (2) of (a) of Proposition D.2.6 with the strict inequality in (2) for all  $(k, l, g)$  from the set

$$\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (2, 0, 0), (0, 2, 0), (0, 0, 1)\} \quad (D.30)$$

and, secondly, the equations (D.24). A *morphism of filtered  $IBL_\infty$ -algebras of bidegree  $(d, \gamma)$  over  $\mathbb{K}$  on complete filtered graded vector spaces  $W^+$  and  $W^-$  according to [CFL15]* is a collection of homogenous  $\mathbb{K}$ -linear maps  $f_{klg} : \hat{S}_k U^+ \rightarrow \hat{S}_l U^-$  of finite filtration degrees for all  $k, l, g \in \mathbb{N}_0$  which satisfy, firstly, (1) and (2) of (b) of Proposition D.2.6

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with the strict inequality in (2) for  $(k, l, g) \in (D.30)$  and, secondly, the equations (D.27).

Recall (D.14) and notice that if we define the filtration  $\mathcal{F}_W^\lambda U := (\mathcal{F}^\lambda W)[1]$  for all  $\lambda \in \mathbb{R}$  and denote the corresponding filtration degree by  $\|\cdot\|_W$ , then it holds  $\|\cdot\| = \|\cdot\|_W + \gamma$  on  $U$ , and for a map  $\mathbf{q}_{klg} : \hat{S}_k U \rightarrow \hat{S}_l U$ , we have

$$\|\mathbf{q}_{klg}\| \geq -2\gamma(k + g - 1) \iff \|\mathbf{q}_{klg}\|_W \geq \gamma(2 - 2g - k - l).$$

Signatures (D.30) together with  $(1, 1, 0)$  correspond to unstable surfaces, i.e., those  $(k, l, g)$  for which  $\chi_{klg} = 2 - 2g - k - l \geq 0$ . Allowing morphisms  $\mathbf{f}$  which have non-zero components of these signatures leads to the appearance of an infinite number of summands in  $\langle \Delta^- e^{\mathbf{f}} \rangle_{klg}$ ,  $\langle e^{\mathbf{f}} \Delta^+ \rangle_{klg}$  and  $\langle e^{\mathbf{f}^-} e^{\mathbf{f}^+} \rangle_{klg}$  (and in the Maurer-Cartan equation (D.38) later); the strict inequalities for filtration degrees of these components seem to be the minimal condition to algebraically handle this situation. Again, the author does not see any technical reason for imposing strict filtration degree conditions for  $\mathbf{q}_{klg}$ . See Figure D.3 for the graphical explanation of the *bubbling*.

We observe the following differences between our approach to weak  $\text{IBL}_\infty$ -algebras and the original approach from [CFL15]:

- (i) *Bubbling*. The definition of [CFL15] is symmetric in inputs and outputs, whereas our definition is not (compare (D.30) and (3) of Proposition D.2.6). We will illustrate this in Example D.2.9 below.
- (ii) *General ring*. The theory of [CFL15] is formulated over a filtered graded commutative ring  $R$ , i.e.,  $W$  is a filtered  $R$ -module and the maps are  $R$ -linear. We can tweak our formalism to handle this situation if  $R$  is a  $\mathbb{K}$ -algebra (e.g., the Novikov ring) by replacing  $\mathbb{K}((\hbar))$  with  $R((\hbar))$  in Definition D.2.3.
- (iii) *Completions*. The BV-formalism of [CFL15] uses the completion  $\hat{S}U$  with respect to the union filtration  $\mathcal{F}_\cup$  of the induced filtration and the filtration by weights (see Section D.1); however, this is not necessarily a filtered bialgebra (see Example D.1.5), Proposition D.1.8 might not apply, and it is not clear whether  $e^{\mathbf{f}}$  (or the multiplication with the exponential of the Maurer-Cartan element later) are well-defined operators on  $\hat{S}U((\hbar))$ .

In the following case, our theory and the theory of [CFL15] agree.

**Proposition D.2.7** (Equivalence of definitions in bounded case over  $\mathbb{K}$ ). *Let  $d \in \mathbb{Z}$  and  $\gamma > 0$ . For  $\zeta \in \mathbb{R}$ , we denote by  $\mathcal{W}_\zeta$  the class of complete filtered graded vector spaces*

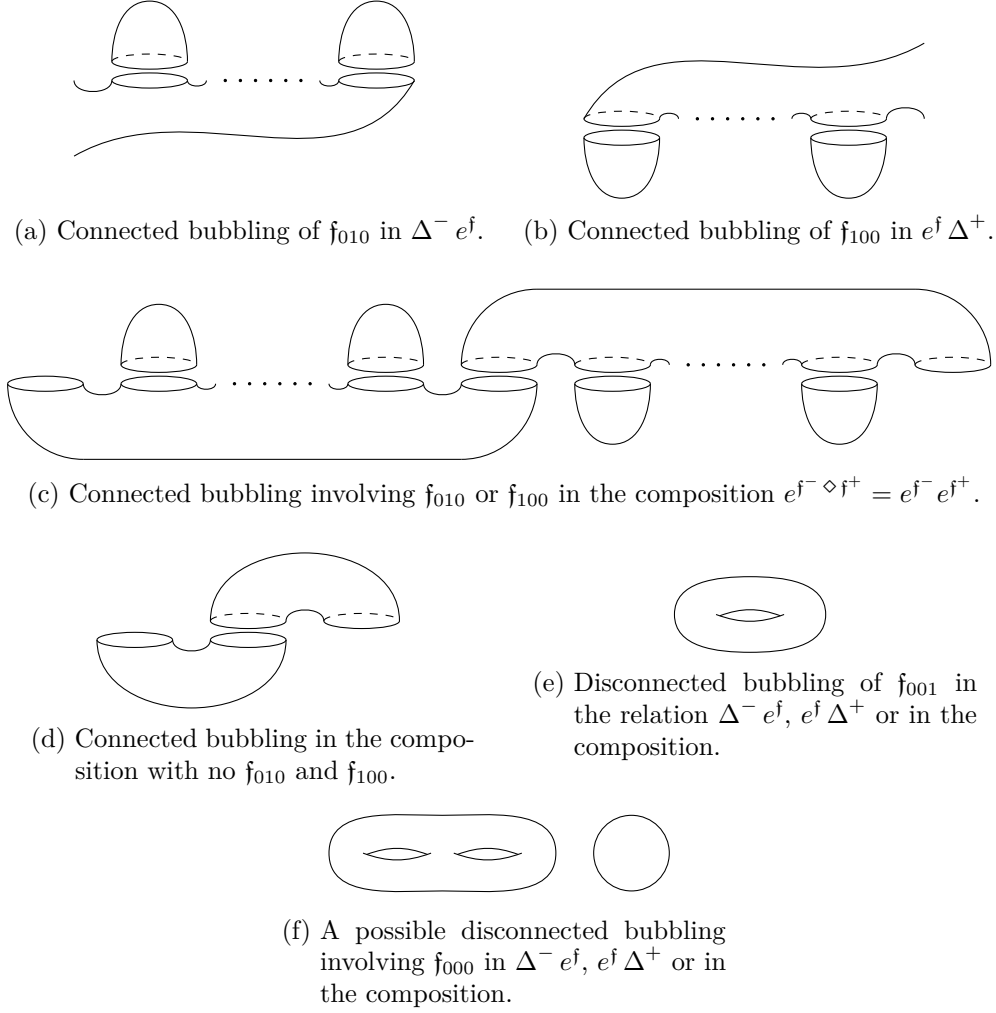


Figure D.3.: Bubbling in  $IBL_\infty$ -relations. If we glue any of the components above to a surface of signature  $(k, l, g)$ , the signature remains the same. Note that since  $g$  is defined via the Euler characteristic, adding a disconnected component without inputs and outputs decreases  $g$  by one.



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$W$  for which there is an  $\alpha > \zeta$  such that

$$\mathcal{F}^\alpha W = W. \quad (\text{D.31})$$

For such  $W$ , we will consider weak  $\text{IBL}_\infty$ -algebras of bidegree  $(d, \gamma)$  over  $\mathbb{K}$  and their weak  $\text{IBL}_\infty$ -morphisms. We have the following:

- (a) Filtered  $\text{IBL}_\infty$ -algebras and morphisms from [CFL15] over  $\mathcal{W}_{-\gamma}$  are also complete filtered MV-algebras and morphisms from Definition D.2.3, respectively.
- (b) The two definitions agree over  $\mathcal{W}_0$ . If in addition  $\gamma \geq 1$ , then also the BV-formalisms are identical.
- (c) The canonical dIBL-structure on the (reduced) dual cyclic bar complex

$$W = \hat{\text{B}}_{\text{cyc}}^* V[2 - n]$$

for a Poincaré duality algebra  $V$  of degree  $n$  from Section 3.4 satisfies (b).

*Proof.* For  $W \in \mathcal{W}_\zeta$ , there is an  $\alpha > \zeta$  such that

$$\|\cdot\| = \|\cdot\|_W + \gamma \geq \alpha + \gamma \quad \text{on } U,$$

and hence

$$\|\mathbf{q}_{klg}(v)\|_U \geq l(\alpha + \gamma) \quad \text{for all } v \in \hat{S}_k U \quad (\text{D.32})$$

and any  $k \geq 0, l \geq 1, g \geq 0$ . It holds also for  $l = 0$  because the filtration on  $\hat{S}_0 U \simeq \mathbb{K}$  is non-negative. We see that if  $\zeta \geq -\gamma$ , which is the case of both (a) and (b), then the sums  $\sum_{l=0}^\infty \mathbf{q}_{klg}$  converge automatically; i.e., (4) of (a) of Proposition D.2.6 holds. It remains to study the relation of the strict filtration degree conditions

$$\forall l, g \in \mathbb{N}_0 : \quad \|\mathbf{q}_{0lg}\| = \|\mathbf{q}_{0lg}(1)\| > -2\gamma(g - 1) \quad (\text{D.33})$$

and

$$\forall (k, l, g) \in (\text{D.30}) : \quad \|\mathbf{q}_{klg}\| > -2\gamma(k + g - 1). \quad (\text{D.34})$$

Here,  $\|\mathbf{q}_{0lg}\| = \|\mathbf{q}_{0lg}(1)\|$  holds because the trivial filtration on  $\mathbb{K}$  is non-negative and it holds  $\|1\| = 0$ . For morphisms, the story is the same, and the rest is implied by Proposition D.2.6.

(a) Assuming (D.34), we have to prove (D.33). It is easy to see that if  $\zeta \geq -\gamma$ , then (D.33) follows from (D.32) except for  $(k, l, g) = (0, 0, 0), (0, 1, 0), (0, 2, 0)$  and

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$(0, 0, 1)$ . These are precisely the signatures from (D.30) with no input; in particular, they are implied by (D.34).

(b) Assuming (D.33), we have to prove (D.34). Clearly, (D.33) implies the strict inequality for all  $(k, l, g) \in (D.30)$  with no input. It remains to check it for  $(1, 0, 0)$  and  $(2, 0, 0)$ . Using  $\zeta \geq -\gamma$ , we obtain

$$\|\mathfrak{f}_{100}(v)\| \geq \|\mathfrak{f}_{100}\| + \|v\| \geq 0 + (\gamma + \alpha) > 0 \quad \text{for all } v \in \hat{S}_1 V,$$

and using  $\zeta \geq 0$ , we obtain

$$\|\mathfrak{f}_{200}(v)\| \geq \|\mathfrak{f}_{200}\| + \|v\| \geq -2\gamma + 2(\gamma + \alpha) > 0 \quad \text{for all } v \in \hat{S}_2 V.$$

Because  $\text{im } \mathfrak{f}_{100}, \text{im } \mathfrak{f}_{200} \subset \mathbb{K}$ , it must hold  $\mathfrak{f}_{100} = \mathfrak{f}_{200} = 0$ , and hence the strict filtration degree condition is automatically satisfied.

The equivalence of BV-formalisms for  $\gamma \geq 1$  follows from Lemma D.1.6 because  $\mathcal{F}^1 U = (\mathcal{F}^{1-\gamma} W)[1] = W[1] = U$ , where  $\alpha > 0 \geq 1 - \gamma$ .

(c) For any  $k \in \mathbb{N}_0$  and  $k \leq \lambda < k + 1$ , we have

$$\begin{aligned} \mathcal{F}_w^\lambda B^{\text{cyc}} V &= B_1^{\text{cyc}} V \oplus \cdots \oplus B_k^{\text{cyc}} V, \\ \mathcal{F}^\lambda B_{\text{cyc}}^* V &= \{\psi \in B_{\text{cyc}}^* V \mid \psi|_{\mathcal{F}^\lambda B^{\text{cyc}} V} = 0\} \\ &\simeq (B_{\text{cyc}}^* V)_{k+1} \oplus (B_{\text{cyc}}^* V)_{k+2} \oplus \cdots, \end{aligned}$$

where  $\mathcal{F}_w$  is the increasing filtration by weights,  $\mathcal{F}$  the dual filtration and  $(B_{\text{cyc}}^* V)_i$  the graded dual to  $B_i^{\text{cyc}} V$ . We also see that  $\mathcal{F}^\lambda B_{\text{cyc}}^* V = B_{\text{cyc}}^* V$  for all  $\lambda < 1$ . Since  $\hat{B}_{\text{cyc}}^* V$  is the completion of  $B_{\text{cyc}}^* V$  in the graded category, we have  $\hat{B}_{\text{cyc}}^* V \in W_0$ . Finally, we know that  $\gamma = 2$  for the canonical dIBL-algebra.  $\square$

*Remark D.2.8* (Generalization over algebra). It is easy to see that (a) of Proposition D.2.7 also holds when we work over a non-negatively filtered augmented unital graded  $\mathbb{K}$ -algebra  $R$ . However, (b) should not generalize as there are no conditions in the filtered MV-formalism implying the strict filtration degree condition for  $(1, 0, 0)$  and  $(2, 0, 0)$ .  $\triangleleft$

*Example D.2.9* (Asymmetry of MV-formalism). How is it possible that there are no strict filtration degree conditions on  $(1, 0, 0)$  and  $(2, 0, 0)$  in the filtered MV-formalism even though these unstable surfaces can obviously bubble off (see Figure D.3)?<sup>10</sup>

<sup>10</sup>Note that this question is relevant only when we work over a general  $\mathbb{K}$ -algebra  $R$ , so that  $\mathfrak{f}_{100}$  and

DONE Possible problems: There might be problem with using (D.32) in the way I am using this! Maybe over  $\mathbb{K}$  there are even more zero operations.

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Heuristically, increasing the number of  $(1,0,0)$ 's glued to the outputs increases the number of times  $\bar{\delta}$  has to be applied to the input to split it and feed it into the new  $(1,0,0)$ 's. The limit conilpotency property (D.6) steps in, increases the filtration degree and provides convergence of the infinite sum. As for the bubbling of  $(2,0,0)$ , we look at (c) and (d) of Figure D.3 and see that there is always an increasing number of  $(0,2,0)$ 's or  $(1,0,0)$ 's in the adjacent components. Now,  $(0,2,0)$  increases the filtration degree due to the strict filtration degree condition, and  $(1,0,0)$  increases the filtration degree due to the limit conilpotency property as explained above. The convergence is again established.

We will now illustrate the bubbling of  $f_{100}$  and  $f_{010}$  in the relations for morphisms with a concrete computation. Let  $f_{100} : \hat{S}_1 U \rightarrow \hat{S}_0 U$ ,  $f_{010} : \hat{S}_0 U \rightarrow \hat{S}_1 U$ ,  $q_{1l0} : \hat{S}_1 U \rightarrow \hat{S}_l U$  and  $q_{kl0} : \hat{S}_k U \rightarrow \hat{S}_l U$  for  $k, l \geq 0$  be  $\mathbb{K}$ -linear maps such that the following sums converge to  $\mathbb{K}((\hbar))$ -linear operators  $\hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$ :

$$\begin{aligned} \Delta &:= \sum_{l=1}^{\infty} \hat{q}_{1l0}, & f &:= f_{100}, \\ \Delta' &:= \sum_{k=1}^{\infty} \hat{q}_{k10} \hbar^{k-1}, & f' &:= f_{010} \hbar^{-1}. \end{aligned}$$

Assume that  $\|f\| = \|f_{100}\| \geq 0$ . Because  $f_{100}(1) = 0$ , convergence of the exponential

$$e^f = \sum_{r=0}^{\infty} \frac{1}{r!} f_{100}^{\odot r} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$$

relies purely on the limit conilpotency property (see the proof of Proposition D.1.8). Assume that  $\|f_{010}\| = \|f_{010}(1)\| > 2\gamma$ , i.e.,  $\|f'\| = \|f'(1)\| = \|f_{010}(1)\| - 2\gamma > 0$ . Clearly, convergence of the exponential

$$e^{f'} = \sum_{r=0}^{\infty} \frac{1}{r!} f_{010}^{\odot r} \hbar^{-r} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar)) \tag{D.35}$$

relies purely on the strict filtration degree condition.

Particular bubbling in composition: How is the bubbling of  $f_{200}^{\odot k}, f_{100}^{\odot k}, f_{030}^{\odot k}$  handled in the composition? Maybe it is because there is a strict filtration degree condition on  $f_{010}$ .

---

$f_{200}$  do not necessarily vanish.

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For all  $v \in S_1 U$ , we have

$$\begin{aligned}
 (e^\dagger \Delta)(v) &= \sum_{l=1}^{\infty} e^\dagger \hat{\mathbf{q}}_{1l0}(v) \\
 &= \sum_{l=1}^{\infty} \frac{1}{l!} \mu^{(l)} \mathbf{f}_{100}^{\otimes l} \bar{\delta}^{(l)} \mathbf{q}_{1l0}(v) \\
 &= \sum_{l=1}^{\infty} \mathbf{f}_{100}^{\otimes l} (\mathbf{q}_{1l0}(v)).
 \end{aligned} \tag{D.36}$$

In this simplest case, the limit conilpotency property takes the form  $\bar{\delta}^{(k)}(S_l U) = 0$  for  $k > l$ , which bounds the number of  $\mathbf{f}_{100}$ 's applied to individual summands. Next, we have

$$\begin{aligned}
 (\Delta' e^{\vee})(1) &= \sum_{r=0}^{\infty} \frac{1}{r!} \Delta' \mathbf{f}_{010}(1)^r \hbar^{-r} \\
 &= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{i=1}^{\infty} \hat{\mathbf{q}}_{i10} \mathbf{f}_{010}(1)^r \\
 &= \sum_{r=1}^{\infty} \sum_{i=1}^r \frac{1}{i!(r-i)!} \mathbf{q}_{i10} (\mathbf{f}_{010}(1)^i) \mathbf{f}_{010}(1)^{r-i} \hbar^{-r+i-1} \\
 &= \left( \sum_{i=1}^{\infty} \frac{1}{i!} \mathbf{q}_{i10} (\mathbf{f}_{010}(1)^i) \right) \underbrace{\left( \sum_{t=0}^{\infty} \frac{1}{t!} \mathbf{f}_{010}(1)^t \hbar^{-t-1} \right)}_{=e^{\vee}(1) \hbar^{-1}}.
 \end{aligned} \tag{D.37}$$

DONE Combinatorial factor: Maybe there is no combinatorial factor! Checked and it is correct.

Clearly, the condition  $\|\mathbf{f}_{010}(1)\| > 0$  is required for the convergence of the sum in the first bracket.  $\triangleleft$

Let us now consider the twisting of a complete filtered  $\text{IBL}_{\infty}$ -algebra in MV-formalism  $(\mathbf{q}_{klg})$  on  $W$  with a Maurer-Cartan element  $\mathbf{n}$ . The *Maurer-Cartan element* is, by definition, a morphism from the trivial  $\text{IBL}_{\infty}$ -algebra 0 to  $W$ . Because  $\hat{S}0 = \hat{S}_0 0 = \mathbb{K}$ , only the  $(0, l, g)$ -components denote by  $\mathbf{n}_{lg} : \mathbb{K} \rightarrow \hat{S}_l U$  for  $l, g \geq 0$  might be non-zero, and thus

$$\mathbf{n} = \sum_{l, g \geq 0} \mathbf{n}_{lg} \hbar^{g-1} : \hat{S}0((\hbar)) \longrightarrow \hat{S}U((\hbar)).$$

Clearly,  $e^{\mathbf{n}}(1) = e^{\mathbf{n}(1)} \in \hat{S}U((\hbar))$ . Consider the left multiplication  $L_{e^{\mathbf{n}}(1)} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$ , and let  $\Delta$  be the BV-operator for  $(\mathbf{q}_{klg})$ . The BV-operator for the twisted  $\text{IBL}_{\infty}$ -algebra  $(\mathbf{q}_{klg}^{\mathbf{n}})$  is defined by

$$\Delta^{\mathbf{n}} := L_{e^{-\mathbf{n}}(1)} \circ \Delta \circ L_{e^{\mathbf{n}}(1)} : \hat{S}U((\hbar)) \longrightarrow \hat{S}U((\hbar)). \tag{D.38}$$

$l = 0$  Maurer Cartan: Kai does not have  $\mathbf{n}_{0g}$ . Notice that having  $\mathbf{n}_{0g}$  causes disconnected twisting!

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The twisting always produces an input-strict  $\text{IBL}_\infty$ -algebra. Indeed, we have

$$\Delta \circ e^n = 0 \iff \Delta(e^{n(1)}) = 0 \iff \Delta^n(1) = 0.$$

This is called the *Maurer-Cartan equation*.

**Proposition D.2.10** (Twisted  $\text{IBL}_\infty$ -algebra). *Let  $(W, \Delta)$  be a complete filtered  $\text{IBL}_\infty$ -algebra in MV-formalism, and let  $\mathbf{n}$  be a Maurer-Cartan element. Then  $\Delta^n$  defined by (D.38) is an input-strict complete filtered  $\text{IBL}_\infty$ -algebra in MV-formalism of the same bidegree as  $(W, \Delta)$ , and for its components  $(\mathbf{q}_{klg}^n)$ , it holds*

$$\mathbf{q}_{klg}^n = \sum_{\substack{r \geq 0 \\ k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ k' - h_1 - \dots - h_r = k \\ l' + l_1 + \dots + l_r - h_1 - \dots - h_r = l \\ g' + h_1 + \dots + h_r - r + g_1 + \dots + g_r = g}} \mathbf{q}_{k'l'g'} \circ_{h_1, \dots, h_r} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_r g_r}) \quad \text{for all } k \geq 1, l, g \geq 0,$$

where  $(\mathbf{q}_{klg})$  and  $(\mathbf{n}_{lg})$  are components of  $\Delta$  and  $\mathbf{n}$ , respectively.

*Proof.* Using  $L_{e^{n(1)}} = e^n \odot \mathbb{1}$ , we compute

$$\begin{aligned} & \Delta \circ L_{e^{n(1)}} \\ &= \sum_{\substack{r \geq 0 \\ k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0}} \frac{1}{r!} \hat{\mathbf{q}}_{k'l'g'} \circ (\mathbf{n}_{l_1 g_1} \odot \dots \odot \mathbf{n}_{l_r g_r} \odot \mathbb{1}) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \\ &= \sum_{\substack{r \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r + k = k'}} \frac{1}{r!} \mathbf{q}_{k'l'g'} \circ_{h_1, \dots, h_r, k} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_r g_r}, \mathbb{1}) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \\ &= \sum_{\substack{r \geq 0 \\ 0 \leq r' \leq r \\ k, k', l', g', l_1, g_1, \dots, l_{r'}, g_{r'} \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ h_1 + \dots + h_{r'} + k = k'}} \frac{1}{(r - r')!} \frac{1}{r'!} \mathbf{n}_{l_{r'+1} g_{r'+1}} \odot \dots \odot \mathbf{n}_{l_r g_r} \\ & \quad \odot (\mathbf{q}_{k'l'g'} \circ_{h_1, \dots, h_{r'}, k} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_{r'} g_{r'}}, \mathbb{1})) \hbar^{g_1 + \dots + g_r - r + k' + g' - 1} \\ &= \left( \sum_{t \geq 0} \frac{1}{t!} \mathbf{n}_{l'_1 g'_1} \odot \dots \odot \mathbf{n}_{l'_t g'_t} \hbar^{g'_1 + \dots + g'_t - t} \right) \\ & \quad \odot \left( \sum_{k, l, g \geq 0} \sum_{\substack{r' \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ k' - h_1 - \dots - h_{r'} = k \\ l' + l_1 + \dots + l_{r'} - h_1 - \dots - h_{r'} = l \\ g' + h_1 + \dots + h_{r'} - r' + g_1 + \dots + g_{r'} = g}} \frac{1}{r'!} \mathbf{q}_{k'l'g'} \circ_{h_1, \dots, h_{r'}, k} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_{r'} g_{r'}}, \mathbb{1}) \hbar^{k + g - 1} \right) \end{aligned}$$

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$$\begin{aligned}
&= \left( \sum_{t \geq 0} \frac{1}{t!} (\mathbf{n}_{l'_1 g'_1} \odot \cdots \odot \mathbf{n}_{l'_t g'_t} \odot \mathbb{1}) \hbar^{g'_1 + \cdots + g'_t - t} \right) \\
&\quad \circ \left( \sum_{k, l, g \geq 0} \sum_{\substack{r' \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ k' - h_1 - \cdots - h_{r'} = k \\ l' + l_1 + \cdots + l_{r'} - h_1 - \cdots - h_{r'} = l \\ g' + h_1 + \cdots + h_{r'} - r' + g_1 + \cdots + g_{r'} = g}} \frac{1}{r'!} \mathbf{q}_{k' l' g'} \circ_{h_1, \dots, h_{r'}} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_{r'} g_{r'}}) \odot \mathbb{1} \hbar^{k+g-1} \right) \\
&= L_{e^{n(1)}} \circ \left( \sum_{k, l, g \geq 0} \sum_{\substack{r' \geq 0 \\ k, k', l', g', l_1, g_1, \dots, l_r, g_r \geq 0 \\ h_1, \dots, h_{r'} \geq 1 \\ k' - h_1 - \cdots - h_{r'} = k \\ l' + l_1 + \cdots + l_{r'} - h_1 - \cdots - h_{r'} = l \\ g' + h_1 + \cdots + h_{r'} - r' + g_1 + \cdots + g_{r'} = g}} \frac{1}{r'!} \mathbf{q}_{k' l' g'} \circ_{h_1, \dots, h_{r'}} (\mathbf{n}_{l_1 g_1}, \dots, \mathbf{n}_{l_{r'} g_{r'}}) \hbar^{k+g-1} \right).
\end{aligned}$$

The claim follows. □

### D.3. BV-complexes for IBL-infinity-algebras

We plan to use the filtered MV-formalism from the previous section to study chain-maps between the following chain complexes.

**Definition D.3.1** (Chain complexes for  $\text{IBL}_\infty$ -algebras). *Given a complete filtered  $\text{IBL}_\infty$ -algebra in MV-formalism  $(U, \Delta)$ , we call the chain complex  $(\hat{S}U((\hbar)), \Delta)$  the BV-complex. If  $(U, \Delta)$  is input-strict, i.e., if  $\Delta(1) = 0$ , then  $(\hat{S}U[[\hbar]], \Delta)$  is a chain complex as well, and we call it the strict BV-complex. If  $(U, \Delta)$  is strict, we also have the chain complex  $(\hat{S}_1 U, \mathbf{q}_{110})$ , and we call it the  $\text{IBL}_\infty$ -chain complex (c.f., Definition 3.2.8).*

Clearly, the definition works also for filtered  $\text{IBL}_\infty$ -algebras from [CFL15] (we are just not sure about Proposition D.3.3 below, and thus we rather stick to our formalism).

*Remark D.3.2* (BV-bicomplex for surfaces). Writing  $\Delta = \Delta_1 + \Delta_2 \hbar + \Delta_3 \hbar^2 + \cdots$  in the input-strict case, we have

$$0 \stackrel{!}{=} \Delta^2 = \Delta_1^2 + \hbar(\Delta_1 \Delta_2 + \Delta_2 \Delta_1) + \hbar^2(\Delta_2^2 + \Delta_1 \Delta_3 + \Delta_3 \Delta_1) + \hbar^3(\cdots) + \cdots.$$

We see that  $\Delta_1^2 = 0$  and  $[\Delta_1, \Delta_2] = 0$ , where  $[\cdot, \cdot]$  is the graded commutator, always hold. Moreover, if  $[\Delta_1, \Delta_3] = 0$ , then also  $\Delta_2^2 = 0$ . Now,  $|\Delta_1| = -1$  always holds, and if  $d = -1$ , then  $|\hbar| = -2$  and  $|\Delta_2| = -1 - 2d = 1$ . We see that if  $[\Delta_1, \Delta_3] = 0$  and  $d = -1$ , then  $(\hat{S}U, \Delta_1, \Delta_2)$  is a (homological) mixed complex. If  $\Delta_k = 0$  for  $k \geq 3$ , then the  $[\hbar^{-1}, \hbar]$ - and  $[[\hbar]]$ -versions of the total complex (see Remark B.2.4) compute

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the BV- and strict BV-homology, respectively. Note that by [CVonb, Proposition 2.2], these versions are quasi-isomorphism invariants of mixed complexes.<sup>11</sup>

In the case of the canonical dIBL-algebra on cyclic cochains of a Poincaré duality algebra of degree  $n$ , we have  $d = n - 3$ , and so  $d = -1$  corresponds to  $n = 2$ .  $\triangleleft$

**Proposition D.3.3** (Morphisms of BV-complexes). *In the category of complete filtered IBL $_{\infty}$ -algebras in MV-formalism, the following holds:*

- (a) *A weak morphism of weak IBL $_{\infty}$ -algebras induces a chain map  $e^{\mathfrak{f}}$  of BV-complexes.*
- (b) *An input-strict morphism of input-strict IBL $_{\infty}$ -algebras induces a chain map of both the BV-complex and its strict version.*
- (c) *The  $(1, 1, 0)$ -part of a strict IBL $_{\infty}$ -morphism of strict IBL $_{\infty}$ -algebras induces a chain map of IBL $_{\infty}$ -chain complexes.*
- (d) *For a Maurer-Cartan element  $\mathbf{n}$ , the left multiplication  $L_{e^{\mathbf{n}(1)}} : \hat{S}U((\hbar)) \rightarrow \hat{S}U((\hbar))$  is an isomorphism of the BV-complexes  $(\hat{S}U((\hbar)), \Delta) \simeq (\hat{S}U((\hbar)), \Delta^{\mathbf{n}})$ .*

*Proof.* Clear.  $\square$

Notice that whereas  $L_{e^{\mathbf{n}(1)}} : \hat{S}U \rightarrow \hat{S}U$  is an isomorphism of chain complexes and it has a well-defined logarithm (i.e., it holds  $\|L_{e^{\mathbf{n}(1)}}\| \geq 0$  and  $\|L_{e^{\mathbf{n}(1)}}(1) - 1\| = \|e^{\mathbf{n}(1)} - 1\| > 0$ ), it is typically not equal to  $e^{\mathfrak{f}}$  for any (weak) IBL $_{\infty}$ -morphism  $\mathfrak{f}$ .

As an example, suppose that  $\mathbf{n}(1) = \hbar^{-1}\mathbf{n}_{10}(1) \in \hat{S}U$  (recall that we interpret  $\mathbf{n}_{10}$  as a map  $\mathbb{K} \rightarrow \hat{S}U$ ) and assume that  $L_{e^{\mathbf{n}(1)}} = e^{\mathfrak{f}}$  for  $\mathfrak{f} = \mathfrak{f}_0\hbar^{-1} + \mathfrak{f}_1 + \mathfrak{f}_2\hbar + \dots$  with  $\hat{S}_i U \subset \ker(\mathfrak{f}_k)$  for all  $i > k$ . In particular, for  $u \in U$ , there is no negative power of  $\hbar$  in  $\mathfrak{f}(u)$ . On the other hand, there is precisely one negative power of  $\hbar$  in  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} (L_{e^{\mathbf{n}(1)}} - e)^{\odot k} u$ , namely  $\mathbf{n}_{10}(1)\hbar^{-1}$  for  $k = 1$ . Therefore,  $L_{e^{\mathbf{n}(1)}}$  does not come from a weak IBL $_{\infty}$ -morphism as long as  $\mathbf{n}_{10}(1) \neq 0$ . This sheds light on the matter of the author's confusion from the preamble about twisting with a Maurer-Cartan element.

*Remark D.3.4* (Some questions). (i) What is the geometric meaning of

$$H(\hat{S}U((\hbar)), \Delta) \simeq H(\hat{S}U((\hbar)), \Delta^{\mathbf{n}}), \quad H(\hat{S}U[[\hbar]], \Delta) \quad \text{and} \quad H(\hat{S}U[[\hbar]], \Delta^{\mathbf{n}}), \quad (\text{D.39})$$

where  $U = B_{\text{cyc}}^* H_{\text{dR}}(M)[3 - n]$  is the (reduced) dual cyclic bar complex of the de Rham cohomology of a closed oriented  $n$ -manifold  $M$  with the filtration dual to the (homological, i.e., increasing) filtration by weights,  $\Delta$  represents the canonical dIBL-structure

<sup>11</sup>A morphism of mixed complexes commutes with both the boundary operator and the differential. A quasi-isomorphism of homological mixed complexes is a morphism of mixed complexes which is a quasi-isomorphism with respect to the boundary operator.

for the intersection pairing on  $H_{\text{dR}}(M)$  and  $\mathfrak{n}$  is the Chern-Simons Maurer-Cartan element? Are some of the homologies (D.39) isomorphic? How about the bicomplex from Remark D.3.2 for surfaces?

(ii) Are there any implications between the following statements (a)–(c)? A strict  $\text{IBL}_\infty$ -morphism of strict  $\text{IBL}_\infty$ -algebras induces

- (a) a quasi-isomorphism of  $\text{IBL}_\infty$ -complexes,
- (b) a quasi-isomorphism of strict BV-complexes and
- (c) a quasi-isomorphism of BV-complexes?

(iii) For a Maurer-Cartan element  $\mathfrak{n}$ , is there a formula for

$$\log L_{e^{\mathfrak{n}}(1)} : \hat{SU}((\hbar)) \longrightarrow \hat{SU}((\hbar))?$$

Notice that  $\|L_{e^{\mathfrak{n}}(1)}\| \geq 0$  and

◁

## D.4. Composition of polynomials in convolution product

We will work on a (non-filtered)  $\mathbb{N}_0$ -graded bialgebra  $V$  over  $\mathbb{K}$  and consider  $\mathbb{K}$ -linear maps  $V \rightarrow V$ . Nevertheless, the results extend in a straightforward way to weight-graded complete filtered bialgebras  $\hat{V}$  and  $\mathbb{K}$ -linear maps of finite filtration degrees, and ultimately to the formal series  $\hat{V}((\hbar))$  and  $\mathbb{K}((\hbar))$ -linear maps of finite filtration degrees.

The theory is based on the following observation.

**Lemma D.4.1** (Iterated compatibility condition). *Let  $(V, \mu, \delta, \eta, \varepsilon)$  be an  $\mathbb{N}_0$ -graded bialgebra. Write  $V = \bigoplus_{i=0}^{\infty} V_i$ ,<sup>12</sup> and for every  $i \in \mathbb{N}_0$ , let  $\iota_i : V_i \rightarrow V$  and  $\pi_i : V \rightarrow V_i$  be the canonical inclusion and projection, respectively. Let  $m, n \in \mathbb{N}$ , and let  $\sigma_{m,n} \in \mathbb{S}_{mn}$  be the permutation of  $mn$  elements depicted in Figure D.4. For  $A \in \mathbb{N}_0^{m \times n}$ , let*

$$F_A := \bigotimes_{i=1}^m \bigotimes_{j=1}^n \pi_{A_{ij}} \circ \iota_{A_{ij}} : V^{\otimes mn} \longrightarrow V^{\otimes mn}.$$

*Then for the iterated product  $\mu^{(m)} : V^{\otimes m} \rightarrow V$  and the iterated coproduct  $\delta^{(n)} : V \rightarrow$*

Add formulas:  
Add here  
formulas for  
 $\mathfrak{n} = \mathfrak{f}_* \mathfrak{m}$ . It  
should be the  
composition  
 $\mathfrak{f} \diamond \mathfrak{m}$  but  
 $\mathfrak{m}$  is weak  
morphism!  
But very  
simple weak  
morphism.  
Maybe the  
induction will  
work. Add  
the formula  
for  $\mathfrak{f}^{\mathfrak{m}} =$   
 $\log(L_{e^{-\mathfrak{m}}(1)} \circ e^{\mathfrak{f}} \circ L_e$   
See Page 99 in  
Diary I.

<sup>12</sup>We use the lower index because it should correspond to the weight in the weight-graded setting of  $V = SU$ .



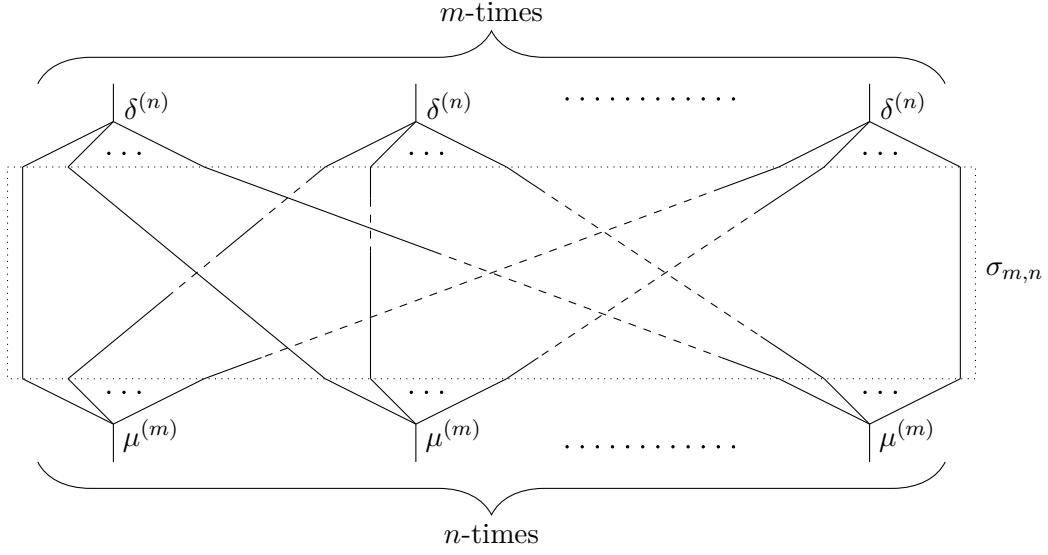


Figure D.4.: The “heart and veins” of the iterated bialgebra compatibility condition  $\delta^{(n)}\mu^{(m)}$ , and the definition of  $\sigma_{m,n}$ .

$V^{\otimes n}$ , the following holds:

$$\delta^{(n)}\mu^{(m)} = \mu^{(m)\otimes n}\sigma_{m,n}\delta^{(n)\otimes m} = \sum_{A \in \mathbb{N}_0^{m \times n}} \mu^{(m)\otimes n}\sigma_{m,n}F_A\delta^{(n)\otimes m}. \quad (\text{D.40})$$

For  $n = m = 2$ , this reduces to the well-known compatibility relation

$$\delta \circ \mu = (\mu \otimes \mu) \circ (\mathbb{1} \otimes \tau \otimes \mathbb{1}) \circ (\delta \otimes \delta). \quad (\text{D.41})$$

*Proof.* Clearly, the second equality of (D.40) holds because  $\sum_{A \in \mathbb{N}_0^{m \times n}} F_A = \mathbb{1}^{mn}$ . As for the first equality, it holds for  $m = n = 2$  from the definition of a bialgebra; the rest is obtained by induction on  $m, n$ , distinguishing the two cases  $\delta^{(n)}\mu^{(m)} = (\delta \otimes \mathbb{1}^{\otimes n-2})\delta^{(n-1)}\mu^{(m)}$  and  $\delta^{(n)}\mu^{(m)} = \delta^{(n)}\mu^{(m-1)}(\mu \otimes \mathbb{1}^{\otimes n-2})$ , respectively. In the first case, the induction step reads

$$\begin{aligned} \delta^{(n)}\mu^{(m)} &= (\delta \otimes \mathbb{1}^{\otimes n-2})\delta^{(n-1)}\mu^{(m)} \\ &= (\delta \otimes \mathbb{1}^{\otimes n-2})\mu^{(m)\otimes n-1}\sigma_{m,n-1}\delta^{(n-1)\otimes m} \\ &= \mu^{(m)\otimes n}(\sigma_{m,2}\delta^{\otimes m} \otimes \mathbb{1}^{\otimes m(n-2)})\sigma_{m,n-1}\delta^{(n-1)\otimes m} \\ &= \mu^{(m)\otimes n}\sigma_{m,n}\delta^{(n)\otimes m}. \end{aligned}$$

On the third line, we used that  $\delta\mu^{(m)} = (\mu^{(m)} \otimes \mu^{(m)})\sigma_{m,2}\delta^{\otimes m}$ , which equals (D.41) for

$m = 2$ , and for  $m \geq 3$ , it is proven by induction with the induction step

$$\begin{aligned}\delta\mu^{(m)} &= \delta\mu^{(m-1)}(\mu \otimes \mathbb{1}^{\otimes m-2}) \\ &= (\mu^{(m-1)} \otimes \mu^{(m-1)})\sigma_{m-1,2}\delta^{\otimes m-1}(\mu \otimes \mathbb{1}^{\otimes m-2}) \\ &= (\mu^{(m-1)} \otimes \mu^{(m-1)})\sigma_{m-1,2}(\mu \otimes \mu \otimes \mathbb{1}^{m-2})(\mathbb{1} \otimes \tau \otimes \mathbb{1} \otimes \mathbb{1}^{m-2})\delta^{\otimes m} \\ &= (\mu^{(m)} \otimes \mu^{(m)})\sigma_{m,2}\delta^{\otimes m}.\end{aligned}$$

The second case is analogous.  $\square$

**Definition D.4.2** (Compositions of monomials in convolution product). *Consider an  $\mathbb{N}_0$ -graded bialgebra  $(V, \mu, \delta, \eta, \varepsilon)$ . Given linear maps  $\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'} : V \rightarrow V$  and a matrix  $A \in \mathbb{N}_0^{r' \times r}$  for  $r, r' \in \mathbb{N}$ , we define the  $A$ -composition  $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}) : V \rightarrow V$  by the formula*

$$\begin{aligned}(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}) \\ := \mu^{(r)} \circ (\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r) \circ \mu^{(r') \otimes r} \circ \sigma_{r',r} \circ F_A \circ \delta^{(r) \otimes r'} \circ (\mathfrak{f}'_1 \otimes \dots \otimes \mathfrak{f}'_{r'}) \circ \delta^{(r')},\end{aligned}$$

where  $\sigma_{r',r}$  and  $F_A$  were defined in Lemma D.4.1.

We call a matrix  $A \in \mathbb{N}_0^{r' \times r}$  connected if for the matrix

$$B := \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

the matrix product  $B^{r+r'}$  has at least one row with all entries non-zero.<sup>13</sup> In this case, we say that the  $A$ -composition  $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'})$  is connected.

Suppose that  $V$  is connected, i.e.,  $V_0 = \langle 1 \rangle$ . Given linear maps  $\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'} : V \rightarrow V$  for  $r, r' \in \mathbb{N}$ , we define the connected composition  $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{\text{con}} (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}) : V \rightarrow V$  by

$$(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{\text{con}} (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}) = \sum_{\substack{A \in \mathbb{N}_0^{(r'+1) \times (r+1)} \\ A \text{ connected} \\ A_{r'+1, r+1} = 0}} (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}) \square_A (\mathfrak{f}'_1, \dots, \mathfrak{f}'_{r'}, \mathbb{1}).$$

Given linear maps  $\mathfrak{f}, \mathfrak{f}_1, \dots, \mathfrak{f}_r : V \rightarrow V$  and parameters  $h_1, \dots, h_r \in \mathbb{N}_0$  for  $r \in \mathbb{N}$ , we define the partial compositions  $\mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r), (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f} : V \rightarrow V$  by

$$\mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r)$$

<sup>13</sup>Such  $A$  represents a connected weighted bipartite graph.

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$$\begin{aligned}
&:= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([(\mu^{(r)} \otimes \mu^{(r)})(F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r})\sigma_r \delta^{\otimes r}] \otimes \mathbb{1}) \\
&\quad (\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1})\delta^{(r+1)}, \\
&(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f} \\
&:= \mu^{(r+1)}(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1})([\mu^{\otimes r} \sigma_r^{-1}(F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r})(\delta^{(r)} \otimes \delta^{(r)})] \otimes \mathbb{1}) \\
&\quad (\mathbb{1} \otimes \tau)(\delta \otimes \mathbb{1})(\mathfrak{f} \otimes \mathbb{1})\delta,
\end{aligned}$$

where we set  $F_{h_1, \dots, h_r} := F_{(h_1, \dots, h_r)} = F_{(h_1, \dots, h_r)}^T = \iota_{h_1} \pi_{h_1} \otimes \dots \otimes \iota_{h_r} \pi_{h_r}$  and  $\sigma_r := \sigma_{r,2}$ , and we omit writing the composition  $\circ$ . For  $r = 1$ , we have  $\mu^{(1)} = \delta^{(1)} := \mathbb{1}$  by definition, and both equations above reduce to

$$\begin{aligned}
\mathfrak{f} \circ_h \mathfrak{f}_1 &= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)(F_h \otimes \mathbb{1}^{\otimes 2})(\delta \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \mathbb{1})\delta \\
&= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(F_h \otimes \mathbb{1}^{\otimes 2})(\mathbb{1} \otimes \tau)(\delta \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \mathbb{1})\delta.
\end{aligned}$$

(c.f., Definition 3.2.2 in Section 3.2 for the case of  $V = S(C[1])$ )

**Proposition D.4.3** (Partial compositions). *Let  $\mathfrak{f}, \mathfrak{f}_1, \dots, \mathfrak{f}_r : V \rightarrow V$  be linear maps on a connected  $\mathbb{N}_0$ -graded bialgebra  $V$ , and let  $h_1, \dots, h_r \in \mathbb{N}_0$  for  $r \in \mathbb{N}$ . Then the following formulas hold:*

$$\begin{aligned}
\mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) &= \sum_{\substack{A \in \mathbb{N}_0^{(r+1) \times 2} \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} (\mathfrak{f}, \mathbb{1}) \square_A (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}), \\
(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f} &= \sum_{\substack{A \in \mathbb{N}_0^{2 \times (r+1)} \\ A = \begin{pmatrix} h_1 & \dots & h_r & \bullet \\ \bullet & \dots & \bullet & 0 \end{pmatrix}}} (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}) \square_A (\mathfrak{f}, \mathbb{1}).
\end{aligned}$$

*Proof.* We compute

$$\sum_{\substack{A \in \mathbb{N}_0^{(r+1) \times 2} \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} (\mathfrak{f} * \mathbb{1}) \square_A (\mathfrak{f}_1 * \dots * \mathfrak{f}_r * \mathbb{1})$$

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$$\begin{aligned}
&= \sum_{\substack{A \in \mathbb{N}_0^{(r+1) \times 2} \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} \mu(\mathfrak{f} \otimes \mathbb{1}) \mu^{(r+1) \otimes 2} \sigma_{r+1,2} F_A \delta^{\otimes r+1} (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1}) \mu^{(r+1) \otimes 2} \sigma_{r+1} (\iota_{h_1} \pi_{h_1} \otimes \mathbb{1} \otimes \cdots \otimes \iota_{h_r} \pi_{h_r} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \iota_0 \pi_0) \delta^{\otimes r+1} \\
&\quad (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1}) \mu^{(r+1) \otimes 2} (F_{h_1, \dots, h_r} \otimes \mathbb{1} \otimes \mathbb{1}^{\otimes r} \otimes \iota_0 \pi_0) \sigma_{r+1} \delta^{\otimes r+1} \\
&\quad (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1}) (\mu \otimes \mathbb{1}) (\mathbb{1} \otimes \tau) ([\mu^{(r) \otimes 2} (F_{h_1, \dots, h_r} \otimes \mathbb{1}^{\otimes r}) \sigma_r \delta^{\otimes r}] \otimes \mathbb{1}) \\
&\quad (\mathfrak{f}_1 \otimes \cdots \otimes \mathfrak{f}_r \otimes \mathbb{1}) \delta^{(r+1)} \\
&= \mathfrak{f} \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r).
\end{aligned}$$

On the line before the last line we used that for a connected bialgebra  $V$ , it holds

$$(\mathbb{1} \otimes \iota_0 \pi_0) \delta(v) = v \otimes \mathbb{1} \quad \text{and} \quad (\iota_0 \pi_0 \otimes \mathbb{1}) \delta(v) = \mathbb{1} \otimes v \quad \text{for all } v \in V.$$

The case of  $(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} \mathfrak{f}$  is treated analogously, using that  $\sigma_{m,n} = \sigma_{n,m}^{-1}$  (this is visible by looking at the highlighted square in Figure D.4, which is symmetric under the rotation by  $180^\circ$ ).  $\square$

We see that the operations  $\circ_{h_1, \dots, h_r}$  and  $\circ_{\text{con}}$ , which are the cornerstone of the surface calculus for  $SU$  in Section D.2, originate naturally from  $\square_A$ . Clearly, we can now replace  $SU$  by any connected weight-graded bialgebra  $V$  and develop a surface calculus for  $BV_\infty$ -algebras over  $V$ .

The following formulas were stated in Remark 3.2.3 in Section 3.2 and come from [CFL15], where they were used in a slightly different form (and without a proof).

**Proposition D.4.4** (Formulas involving partial compositions). *For  $r \in \mathbb{N}$ , let  $\mathfrak{f}, \mathfrak{f}_1, \dots, \mathfrak{f}_r : V \rightarrow V$  be linear maps on a connected  $\mathbb{N}_0$ -graded bialgebra  $V$ . Then the following formulas hold:*

$$\widehat{\mathfrak{f}} \circ \widehat{\mathfrak{f}}_1 = \sum_{h \geq 0} \widehat{\mathfrak{f} \circ_h \mathfrak{f}_1}, \quad (\text{D.42})$$

$$\widehat{\mathfrak{f}} \circ (\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) = \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \cdots + h_r = k}} (\mathfrak{f} \iota_k \pi_k) \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r), \quad (\text{D.43})$$

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$$(\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) \circ \hat{\mathfrak{f}} = \sum_{l \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = l}} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{h_1, \dots, h_r} (\iota_l \pi_l \mathfrak{f}), \quad (\text{D.44})$$

$$\mathfrak{f} \circ_{h_1, \dots, h_{r-1}, 0} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) = \mathfrak{f} \circ_{h_1, \dots, h_{r-1}} (\mathfrak{f}_1, \dots, \mathfrak{f}_{r-1}) * \mathfrak{f}_r, \quad (\text{D.45})$$

$$(\mathfrak{f}_1, \dots, \mathfrak{f}_r) \circ_{0, h_2, \dots, h_r} \mathfrak{f} = \mathfrak{f}_1 * (\mathfrak{f}_2, \dots, \mathfrak{f}_r) \circ_{h_2, \dots, h_r} \mathfrak{f}. \quad (\text{D.46})$$

(Recall that  $\hat{\mathfrak{f}} := \mathfrak{f} * \mathbb{1} : V \rightarrow V$ .)

*Proof.* As for (D.42), we compute

$$\begin{aligned} \hat{\mathfrak{f}} \circ \hat{\mathfrak{f}}_1 &= \mu(\mathfrak{f} \otimes \mathbb{1}) \delta \mu(\mathfrak{f}_1 \otimes \mathbb{1}) \delta \\ &= \mu(\mathfrak{f} \otimes \mathbb{1}) (\mu \otimes \mu) (\mathbb{1} \otimes \tau \otimes \mathbb{1}) (\delta \otimes \delta) (\mathfrak{f}_1 \otimes \mathbb{1}) \delta \\ &= \mu(\mu \otimes \mathbb{1}) (\mathfrak{f} \otimes \mathbb{1}^{\otimes 2}) (\mu \otimes \mathbb{1}^{\otimes 2}) (\mathbb{1} \otimes \tau \otimes \mathbb{1}) (\delta \otimes \mathbb{1}^{\otimes 2}) (\mathfrak{f}_1 \otimes \mathbb{1}^{\otimes 2}) (\delta \otimes \mathbb{1}) \delta \\ &= \mu \left( \underbrace{[\mu(\mathfrak{f} \otimes \mathbb{1}) (\mu \otimes \mathbb{1}) (\mathbb{1} \otimes \tau) (\delta \otimes \mathbb{1}) (\mathfrak{f}_1 \otimes \mathbb{1}) \delta]}_{= \sum_{h \geq 0} \mathfrak{f} \circ_h \mathfrak{f}_1} \right) \delta \\ &= \sum_{h \geq 0} \widehat{\mathfrak{f} \circ_h \mathfrak{f}_1}. \end{aligned}$$

On the second line, we used the compatibility condition; on the third line, we used associativity and coassociativity.

In order to see (D.43), the easiest is to use Proposition D.4.3:

$$\begin{aligned} \hat{\mathfrak{f}}(\mathfrak{f}_1 * \cdots * \mathfrak{f}_r) &= \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k}} \sum_{\substack{A \in \mathbb{N}_0^{r \times 2} \\ A = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \end{pmatrix}}} (\mathfrak{f} \iota_k \pi_k, \mathbb{1}) \square_A (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \\ &= \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k}} \sum_{\substack{A' \in \mathbb{N}_0^{(r+1) \times 2} \\ A' = \begin{pmatrix} h_1 & \bullet \\ \vdots & \vdots \\ h_r & \bullet \\ \bullet & 0 \end{pmatrix}}} (\mathfrak{f} \iota_k \pi_k, \mathbb{1}) \square_{A'} (\mathfrak{f}_1, \dots, \mathfrak{f}_r, \mathbb{1}) \\ &= \sum_{k \geq 0} \sum_{\substack{h_1, \dots, h_r \geq 0 \\ h_1 + \dots + h_r = k}} (\mathfrak{f} \iota_k \pi_k) \circ_{h_1, \dots, h_r} (\mathfrak{f}_1, \dots, \mathfrak{f}_r). \end{aligned}$$

On the second line, the bottom-left  $\bullet$ , i.e., how many “veins” of  $\mathbb{1}$  go into  $\mathfrak{f} \iota_k \pi_k$ , is forced to be 0 because  $\mathfrak{f} \iota_k \pi_k$  has  $k$  inputs and  $h_1 + \dots + h_r = k$ . Equation (D.44) is proven analogously.

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As for (D.45), we prefer to manipulate the expression with bialgebra operations:

$$\begin{aligned}
& \mathfrak{f} \circ_{h_1, \dots, h_{r-1}, 0} (\mathfrak{f}_1, \dots, \mathfrak{f}_r) \\
&= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([\mu^{(r) \otimes 2}(F_{h_1, \dots, h_{r-1}, 0} \otimes \mathbb{1}^{\otimes r})\sigma_r \delta^{\otimes r}] \otimes \mathbb{1}) \\
&\quad (\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1})\delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([(\mu^{(r-1)} \otimes \mu^{(r)})(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r})(\sigma_{r-1} \otimes \mathbb{1}) \\
&\quad (\delta^{\otimes r-1} \otimes \mathbb{1})] \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1})\delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)(\mathbb{1} \otimes \mu \otimes \mathbb{1})([\mu^{(r-1) \otimes 2}(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r-1})\sigma_{r-1} \\
&\quad \delta^{\otimes r-1}] \otimes \mathbb{1}^{\otimes 2})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_r \otimes \mathbb{1})\delta^{(r+1)} \\
&= \mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \mu)(\mathbb{1} \otimes \tau \otimes \mathbb{1})([\mu^{(r-1) \otimes 2}(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r-1})\sigma_{r-1} \\
&\quad \delta^{\otimes r-1}] \otimes \mathbb{1}^{\otimes 2})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_{r-1} \otimes \mathbb{1} \otimes \mathfrak{f}_r)\delta^{(r+1)} \\
&= \mu([\mu(\mathfrak{f} \otimes \mathbb{1})(\mu \otimes \mathbb{1})(\mathbb{1} \otimes \tau)([\mu^{(r-1) \otimes 2}(F_{h_1, \dots, h_{r-1}} \otimes \mathbb{1}^{\otimes r-1})\sigma_{r-1} \\
&\quad \delta^{\otimes r-1}] \otimes \mathbb{1})(\mathfrak{f}_1 \otimes \dots \otimes \mathfrak{f}_{r-1} \otimes \mathbb{1})\delta^{(r)}] \otimes \mathfrak{f}_r)\delta \\
&= \mu(\mathfrak{f} \circ_{h_1, \dots, h_{r-1}} (\mathfrak{f}_1, \dots, \mathfrak{f}_{r-1}) \otimes \mathfrak{f}_r)\delta \\
&= \mathfrak{f} \circ_{h_1, \dots, h_{r-1}} (\mathfrak{f}_1, \dots, \mathfrak{f}_{r-1}) * \mathfrak{f}_r
\end{aligned}$$

Equation (D.46) is proven analogously. □

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