notes on efficient coding of color

John Pearson[1,2,3,4*], others[1]

1 Department of Neurobiology
2 Department of Biostatistics & Bioinformatics,
3 Department of Psychology & Neuroscience,
4 Department of Electrical & Computer Engineering,
Duke University, Durham NC 27710, USA

*** Corresponding author: john.pearson@duke.edu**

Abstract

stuff

Linear treatment

In (Atick, Li, *et al.* 1992), the authors discover red-green opponency from a treatment of the L and M channels of color vision in the linear, infinite retina approximation. Here, we rederive this result using the approach of (Jun *et al.* 2022).

We begin with the optimization objective used in (Jun *et al.* 2021). We would like to *maximize*

$$\mathbb{E}_x \log \frac{\det\left(\mathbf{G}\mathbf{W}^\top(\mathbf{C}_x + \mathbf{C}_{n_{\text{in}}})\mathbf{W}\mathbf{G} + \mathbf{C}_{n_{\text{out}}}\right)}{\det\left(\mathbf{G}\mathbf{W}^\top \mathbf{C}_{n_{\text{in}}}\mathbf{W}\mathbf{G} + \mathbf{C}_{n_{\text{out}}}\right)} + \sum_j \lambda_j \mathbb{E}_x r_j, \tag{1}$$

where the Lagrange multiplier λ_j is mean to enforce the constraint $\mathbb{E}_x r_j \leq 1$. From (Jun *et al.* 2022), the determinant in the numerator is the determinant of a matrix with entries

$$\begin{aligned} F_{ij} &= \gamma_i\gamma_j \mathbf{w}_i^\top(\mathbf{C}_x + \sigma_{\text{in}}^2 \mathbb{1})\mathbf{w}_j &(2)\\ &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{z}_i - \mathbf{z}_j)} |v(k)|^2 (C_x(k) + \sigma_{\text{in}}^2) &(3)\end{aligned}$$

in the continuum limit for RFs i and j located at $\mathbf{z}_i$ and $\mathbf{z}_j$, where we have assumed that $\mathbf{C}_x(\mathbf{z}, \mathbf{z}') = \mathbf{C}_x(\mathbf{z} - \mathbf{z}')$ (that is, the spectrum is translationally invariant) and we write $\mathbf{v}_i \equiv \gamma_i \mathbf{w}_i$ for the unnormalized kernel and $\mathbf{v}(k)$ for its Fourier transform (assuming rotational symmetry). In the case of color, the covariance matrix $\mathbf{C}_x$ has additional indices for color channels ($a, b = 1 \ldots 3$), and we write

$$F_{ij} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{z}_i - \mathbf{z}_j)} \sum_{ab} v_a^*(k)(C_{ab}(k) + \sigma_a^2\delta_{ab})v_b(k). \tag{4}$$

In writing this, we have dropped subscripts on C and σ^2 and assumed that noise in the inputs is uncorrelated across color channels.

Now, just as in the single channel case, we can compute the log determinant in the objective by diagonalizing $\mathbf{F}$ in its eigenbasis and performing an integral over its eigenvalues, making the first term in (1) equal to

$$\int_{G_0} \frac{d^2\mathbf{k}}{(2\pi)^2} \log \frac{\frac{\text{vol}(G_0)}{(2\pi)^2}\sum_{\mathbf{g}\in G}\sum_{ab} v_a^*(\mathbf{k}+\mathbf{g})v_b(\mathbf{k}+\mathbf{g})(C_{ab}(\mathbf{k}+\mathbf{g}) + \sigma_a^2\delta_{ab}) + \varepsilon^2}{\frac{\text{vol}(G_0)}{(2\pi)^2}\sum_{\mathbf{g}\in G}\sum_a |v_a(\mathbf{k}+\mathbf{g})|^2\sigma_a^2 + \varepsilon^2}, \tag{5}$$

with σ^2 the variance of input noise, ε^2 the variance of output noise, and G the dual lattice and G_0 its unit cell as in (Jun *et al.* 2022).

Single mosaic case

In the case of a single mosaic, we have a prototypical RF $v_a(k)$, which we assume to be radially symmetric. Writing $v_a(k) = v(k)u_a(k)$ with $\sum_a u_a^*(k)u_a(k) = 1$, we then have

$$\sum_{ab} v_a^*(\mathbf{k}+\mathbf{g})v_b(\mathbf{k}+\mathbf{g})C_{ab}(\mathbf{k}+\mathbf{g}) = |v(\mathbf{k}+\mathbf{g})|^2 C'(\mathbf{k}+\mathbf{g}), \tag{6}$$

where we denote by $C'(k)$ the effective one-dimensional spectrum as filtered by the normalized RF at each channel and frequency. If, in addition, we assume that the input noise is the same in every channel, we have $\sum_{ab} v_a^*(\mathbf{k}+\mathbf{g})v_b(\mathbf{k}+\mathbf{g})\sigma_a^2\delta_{ab} = |v(\mathbf{k}+\mathbf{g})|^2\sigma^2$, and we can follow the derivation in Appendix A.2 of (Jun *et al.* 2022) to reproduce the result of (Atick & Redlich 1992).

Multiple mosaics

Of course, this solution above only allows us to solve for $v(k)$ once we have fixed $u_a(k)$, the color profile of the RF at each spatial frequency. Here, we consider the case of multiple mosaics[1].

As argued in Appendix A.4 of (Jun *et al.* 2022), the optimal choice of RFs for each mosaic occurs when these RFs are as nearly as possible orthogonal to each other in the inner product defined by $C(k) + \sigma^2$. In (Jun *et al.* 2022), this was accomplished by requiring that mosaics not overlap in their spacetime frequency responses: $v^*(k,\omega)v'(k,\omega) = 0$ for v and v' in different mosaics. Here, we have another set of degrees of freedom — separate color channels — at our disposal, and the relevant conditions become

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{z}_i-\mathbf{z}_j)} \sum_{ab} v_a^*(k)(C_{ab}(k) + \sigma_a^2\delta_{ab})v_b'(k) = 0 \tag{7}$$

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{z}_i-\mathbf{z}_j)} \sum_{a} \sigma_a^2 v_a^*(k)v_a'(k) = 0 \tag{8}$$

for all relevant off-diagonal blocks in the determinants to vanish when v and v' are in different mosaics.

To address this problem, we consider the generalized eigenvalue problem for the two matrices $\mathbf{C}(k)$ and $\mathbf{C}_{\text{in}}(k) = \text{diag}(\boldsymbol{\sigma}^{\mathbf{2}})$:

$$\mathbf{A}\boldsymbol{\phi}_i = \lambda_i\mathbf{B}\boldsymbol{\phi}_i \tag{9}$$

for generalized eigenvectors $\boldsymbol{\phi}_i$ (Ghojogh *et al.* 2019). Letting $\mathbf{A} = \mathbf{C}$ and $\mathbf{B} = \mathbf{C}_{\text{in}}$, we can imagine a set of eigenvectors $\mathbf{u}_i(k)$ parameterized by k satisfying

$$\mathbf{u}_i^*(k)\mathbf{C}(k)\mathbf{u}_j(k) = \lambda_j(k)\mathbf{u}_i^*(k)\mathbf{C}_{\text{in}}(k)\mathbf{u}_j(k) = \lambda_j\delta_{ij}. \tag{10}$$

That is, in matrix form (suppressing for the moment the k parameterization)

$$\mathbf{CU} = \mathbf{C}_{\text{in}}\mathbf{U}\boldsymbol{\Lambda} \tag{11}$$

$$\mathbf{U}^\top\mathbf{C}_{\text{in}}\mathbf{U} = \mathbb{1} \tag{12}$$

$$\mathbf{U}^\top\mathbf{CU} = \boldsymbol{\Lambda}, \tag{13}$$

so that if we let the RFs for the mosaics be a linear combination of these basis vectors we have each mosaic's RF represented in the color channel basis as a column of $\mathbf{UV}$, and the matrix products in the numerator and denominator of (5) become

$$\mathbf{V}^\top\mathbf{U}^\top(\mathbf{C} + \mathbf{C}_{\text{in}})\mathbf{UV} = \mathbf{V}^\top(\Lambda + \mathbb{1})\mathbf{V} \tag{14}$$

$$\mathbf{V}^\top\mathbf{U}^\top\mathbf{C}_{\text{in}}\mathbf{UV} = \mathbf{V}^\top\mathbb{1}\mathbf{V}, \tag{15}$$

[1] In this linear response case, of course, each mosaic comprises both ON and OFF response types, making it equivalent to *two* mosaics in the nonlinear case.

respectively. This then gives rise to a new objective in the multi-mosaic case:

$$\int_{G_0} \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_m \left[\log \frac{\sum_{\mathbf{g}\in G} \lambda'_m(\mathbf{k}+\mathbf{g}) + \varepsilon^2}{\sum_{\mathbf{g}\in G} \upsilon_m(\mathbf{k}+\mathbf{g}) + \varepsilon^2} \right], \tag{16}$$

where the λ'_m are the eigenvalues of $\mathbf{V}^\top(\mathbf{\Lambda}+\mathbb{1})\mathbf{V}$, υ_m are the eigenvalues of $\mathbf{V}^\top\mathbf{V}$ (both parameterized by k), and, as in (Jun *et al.* 2022) we have chosen units in which $\mathrm{vol}(G_0) = (2\pi)^2$. Moreover, the constraints on total power can be phrased in terms of λ', so that the combined effective objective becomes:

$$\int_{G_0} \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_m \left[\log \frac{\sum_{\mathbf{g}\in G} \lambda'_m(\mathbf{k}+\mathbf{g}) + \varepsilon^2}{\sum_{\mathbf{g}\in G} \upsilon_m(\mathbf{k}+\mathbf{g}) + \varepsilon^2} \right] - \sum_m \sum_{\mathbf{g}\in G} \nu_m(\mathbf{k}+\mathbf{g})\lambda'_m(\mathbf{k}+\mathbf{g}), \tag{17}$$

Acknowledgments

We thank just about everybody.

References

1. Atick, J. J., Li, Z. & Redlich, A. N. Understanding retinal color coding from first principles. *Neural computation* **4,** 559–572 (1992).
2. Atick, J. J. & Redlich, A. N. What does the retina know about natural scenes? *Neural computation* **4,** 196–210 (1992).
3. Ghojogh, B., Karray, F. & Crowley, M. Eigenvalue and generalized eigenvalue problems: Tutorial. *arXiv preprint arXiv:1903.11240* (2019).
4. Jun, N. Y., Field, G. & Pearson, J. Efficient coding, channel capacity, and the emergence of retinal mosaics. *Advances in neural information processing systems* **35,** 32311–32324 (2022).
5. Jun, N. Y., Field, G. D. & Pearson, J. Scene statistics and noise determine the relative arrangement of receptive field mosaics. *Proceedings of the National Academy of Sciences* **118,** e2105115118 (2021).

Supplemental Material

stuff goes here