

Part II: Paper folding

High level:

- usually 2D paper ~ but 1D & 3D⁺ cool too
- no stretching/shrinking/tearing
- no crossing (but touching is O.K.)
- difference from linkages:
creases can go anywhere (fibers, not bars)

Main topics:

- ① definitions/technicalities
 - surprisingly intricate
 - connectivity of configuration space
- ② foldability
 - crease patterns & when they fold
- ③ design
 - given specific geometric goal
(shape/property/etc.)
 - find crease pattern & folding
- ④ geometric construction
 - analog of straight edge & compass
(used all the time by origamists)
 - but more powerful!

Definitions: [Demaine, Devadoss, Mitchell, O'Rourke 2004; Lang 2004]

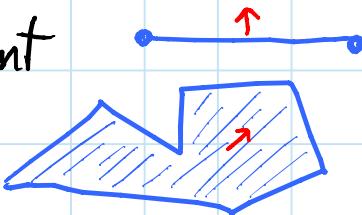
→ see Chap. 11

Piece of paper = "nice" orientable manifold

usually: - 1D line segment

- 2D polygon

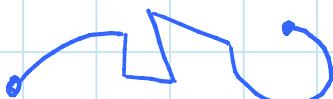
Distinguish "top side"



Folded-state geometry of d -dimensional paper P
= isometric mapping into $(d+1)$ -dim. space $f: P \rightarrow \mathbb{R}^{d+1}$

↳ preserve intrinsic distances (no stretch/shrink)

Free folded state = one-to-one folded-state geometry



↳ no touching ⇒ no crossing

Folded-state order

→ smooth

= for every two distinct noncrossing points p, q of paper
collocated by the geometry — $f(p) = f(q)$ —

specify whether p is on top or bottom side of q :

$$\lambda(p, q) = +1 \text{ or } -1$$

satisfying:

① antisymmetry:



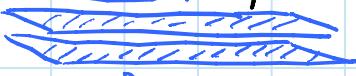
$$\lambda(p, q) \text{ vs. } \lambda(q, p)$$

② transitivity:



p above q above r

③ consistency:



touching regions

④ noncrossing:

no



Folded state = geometry + order

Definitions: (cont'd)

OPEN: simpler definition using infinitesimals?

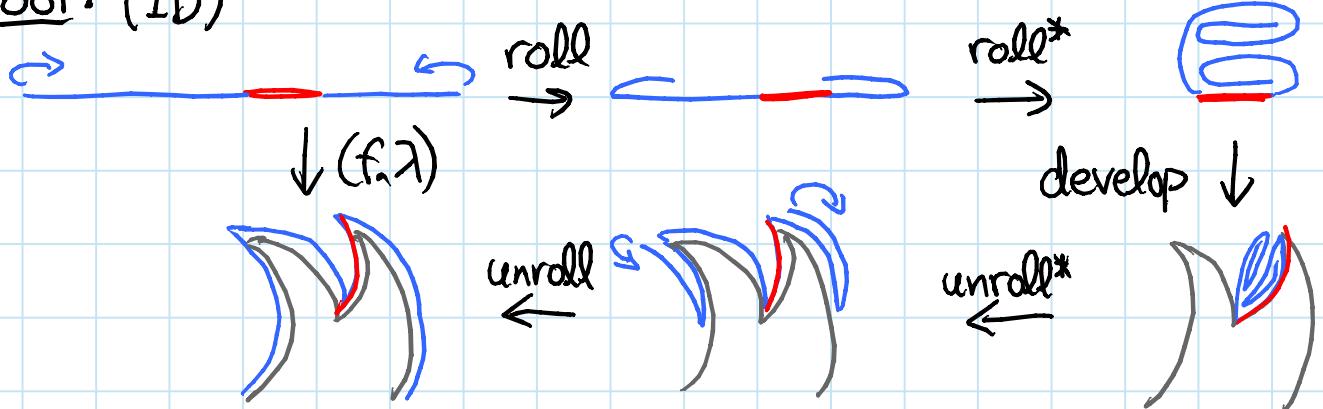
→ see Chap. 11 for continuity def.

Folding motion = continuum of folded states

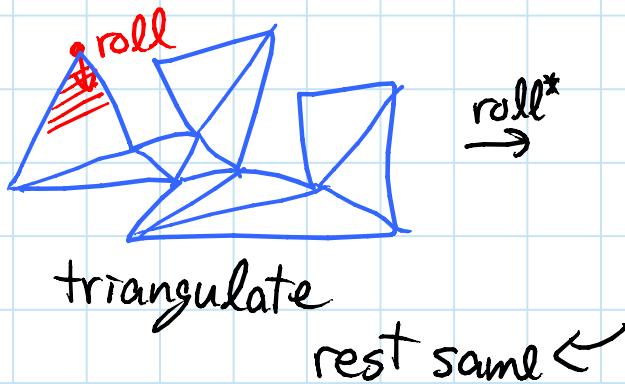
$$m: t \in [0, 1] \mapsto (f_t, \lambda_t)$$

Connectivity of configuration space [Demaine, Devadoss, any polygonal piece of paper P Mitchell, O'Rourke 2004]
has a folding motion into any desired folded state (f, λ)

Proof: (1D)



(2D)



tiny triangle
many layers

□

OPEN: finite # creases, if folded state is free?

OPEN: polygon with holes? (unknotted) polyhedral paper?

Origami terminology: (d -dimensional paper)

Crease pattern = set of crease points on paper
→ $(d-1)$ -dimensional

Flat folding = folded state lying in d dimensions
- call its crease pattern flat foldable
(must use all creases)

Mountain crease = bottom sides touch



Valley crease = top sides touch

Mountain-valley assignment

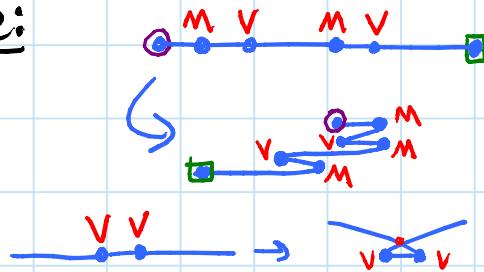
= which creases in crease pattern are mountain/valley
origami notation: $\text{---}\cdot\text{---}/\text{---}\cdot\text{---}$

Mountain-valley pattern = crease pattern
+ mountain-valley assignment

Example: crumple up a piece of paper

1D flat folding: [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2004]

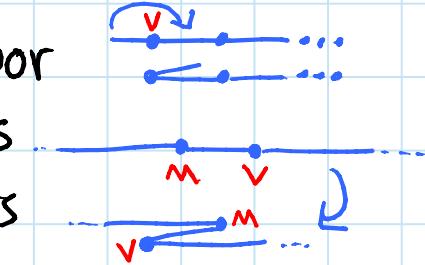
All crease patterns are flat foldable:
zig-zag / accordion fold



Not all mountain-valley patterns:

Two folding operations:

- ① end fold if end length \leq neighbor
- ② crimp two consecutive creases
if length between \leq both neighbors
& one mountain, one valley



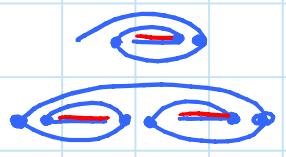
Characterization:

mountain-valley pattern is flat foldable
 \Leftrightarrow there's a sequence of crimps & end folds
 \Leftrightarrow mingling: for any maximal sequence of M's or V's,
adjacent V or M or end on at least one side
is nearer than adjacent M or V:

OPEN: proof without mingling?

1D flat folding (cont'd)

Proof of characterization:

- flat foldable \Rightarrow mingling:
 - sequence of M's or V's form a spiral (or double spiral)
 - at least one end must be short
- mingling \Rightarrow end fold or crimp possible
 - for each maximal sequence of M's or V's write (if "left mingling"
[otherwise
 - & write) if "right mingling"
] otherwise
- \Rightarrow () or [] or ()
 -)(\Rightarrow crimp
 - leading (/ trailing) \Rightarrow end fold
 - if neither: [) [) ... [) (] (] ... (]


- crimp/end fold preserves flat foldability
 - take flat folding
 - move some layers out of crimp
 - \Rightarrow could start with crimp

- induct \Rightarrow sequence of crimps & end folds
 \Rightarrow flat foldable again. \square

2D map folding: [Arkin et al 2004]

↳ rectangular paper with axis-parallel creases

- again every crease pattern is flat foldable:
zig-zag in x then y



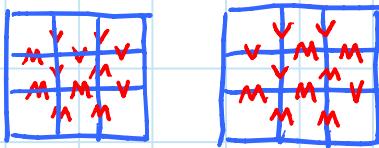
OPEN: characterize flat-foldable mountain-valley patterns — even $2^x n$! [Edmonds 1997]

Simple folds: fold by 180° along one line at once

- in 1D, simple folds are omnipotent:

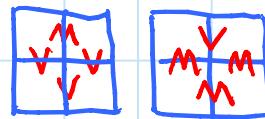
can simulate end folds & crimps

- in 2D, it matters:



Characterization of simple foldability of maps:

- if simply foldable, must be a uniform horizontal/vertical line
↳ all M or all V
- crossing vert./horiz lines must switch M \leftrightarrow V here:
 - local 2×2 patterns:



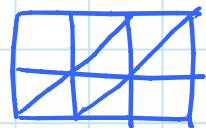
\Rightarrow all uniform horizontal lines must be folded before any vertical lines become uniform, etc.

\Rightarrow sequence of 1D problems on current uniform lines

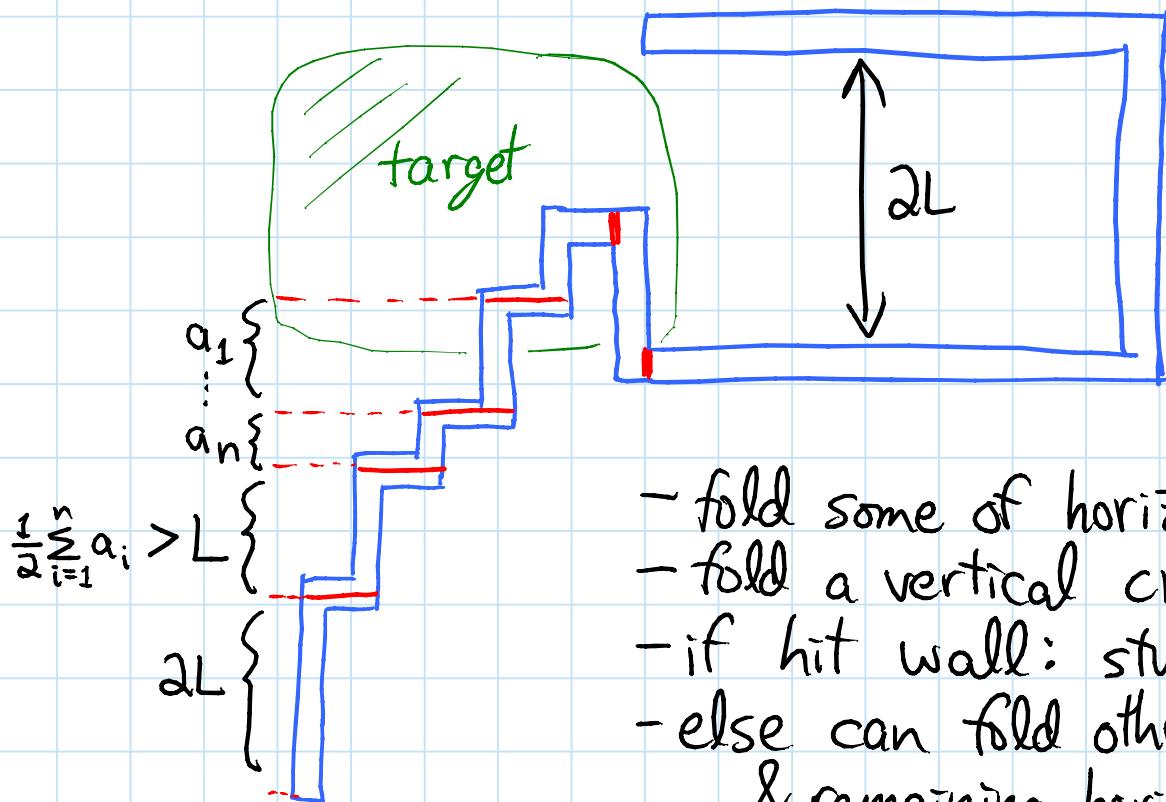
- linear-time algorithm — $O(mn)$ for $m \times n$ map —
by maintaining uniformity for each line,
crimpability & end foldability on lengths

2D map folding (cont'd)

NP-hard if we add 45° diagonal creases
or allow orthogonal paper

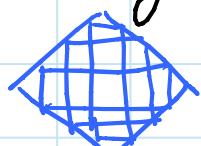
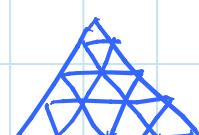


- reduction from Partition: (weakly NP-complete)
given n integers, can you partition them into two groups of equal sum



- fold some of horiz. creases
- fold a vertical crease
- if hit wall: stuck
- else can fold other vertical & remaining horiz. creases

OPEN: orthogonal creases in ortho. convex polygon?
in convex polygon? in nonaligned rectangle?



OPEN: triangular maps?



OPEN: strip with no crossing creases? [Martin Demaine]
(45° , $30/60^\circ$, arbitrary...)