

The Seven Wonders of the World

Notes on 21st-century physics

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Introduction

The loss implied in such an acquisition can be estimated only by those who have been compelled to unlearn a science that they might at length begin to learn it.

J. C. Maxwell 1878

Until some decades ago, the 18th-century physical notions typically taught in introductory Bachelor physics courses were enough to prepare an engineer for future specializations and jobs. Students who wanted to venture into modern theories, such as Relativity, were required to **re-learn** some of the most important physical notions – *Energy*, mass, time, entropy above all – which in these theories are quite different from the 18th-century ones. But at that time the modern theories still mostly had only theoretical, not practical, importance. So the re-learning efforts of the curious students could perhaps be justified.

That situation has changed today. Modern theories are an essential part of many everyday technologies, like nuclear reactors and the [Global Positioning System¹](#) (see the entertaining discussion in Taylor & Wheeler 2000 Project A, and also Petit & Wolf 2005; Fliegel & DiEsposti 1996; Ashby 2002; Müller et al. 2008); and they are required for developing new technological possibilities, from [quantum computers²](#) to [solar sails³](#) (see for instance the [CubeSail⁴](#) mission). An engineering student (including communication and data engineering) may likely end up in a job that requires an understanding of modern physical notions. The diffusion of [large language models⁵](#) will moreover require future engineers who actually **understand** those physical notions, not little monkeys who have been trained to manipulate some equations and to throw some technobabble around. Automated large language models are faster, cheaper, and more

"The achieved performances of atomic clocks and time transfer techniques imply that the definition of time scales and the clock comparison procedures must be considered within the framework of general relativity"

Petit & Wolf 2005

precise in doing the latter kind of monkey activities. So why should one hire a human to do the same?

While moving from the older to the newer notions often requires re-learning efforts and conceptual re-orientations, the move in the opposite direction is less demanding. The modern physical notions are more encompassing than their 18th-century parents. Their understanding leads to an understanding of their older counterparts as approximate and special cases. Students, moreover, have often been hearing quite early from mass media about the new notions; for instance about the equivalence of mass and energy. Owing to this early exposure, students sometimes ask very intelligent and deep questions – such as “*should the mass of the body be included in its internal energy?*” – when exposed to the old notions.

It is therefore high time that introductory Bachelor physics courses be based on modern physical notions. Students should not be required to waste time and mental effort to learn something that they must unlearn and re-learn, only because of academia’s and teachers’ inertia.

Some teachers say “it would be too difficult for students to understand modern ideas, because they are too familiar with the old ones. This is why we need to teach the old and slightly incorrect ideas first, and correct them later”. I think that this kind of reasoning is scientifically unacceptable and leads to a vicious circle. Students are unfamiliar with new notions only because they were raised by a generation who was taught old ones. New notions become familiar only after a couple of generations learn them at an early stage. This is obvious if you consider that notions such as “energy”, “electromagnetic field”, “vector” are very familiar today, but were absolutely *unfamiliar* a couple centuries ago. If we had always taught what’s familiar, then we would still be teaching about the elements *air, earth, water, fire*, and that the Sun revolves around the Earth. Arguments in favour of teaching old notions are for the most part just pretexts for laziness.

The present lecture notes are an experiment and an attempt to introduce classical mechanics and thermodynamics from modern physical notions. The core equations remain the same, but the students should have a broader conception of their meaning and of the symbols that appear in them.

URLs for chapter *Introduction*

1. <https://www.gps.gov>
2. <https://www.ibm.com/topics/quantum-computing>
3. <https://www.planetary.org/sci-tech/lightsail>
4. <https://www.cubesail.us>
5. <https://www.ibm.com/topics/large-language-models>

Chapter 0

Guide to the student

Maths prerequisites

- Working familiarity with algebra, its operations, and their properties, including square roots; knowledge of exponentials and logarithms is useful.
- Working familiarity with solving equations and inequalities, linear and non-linear.
- Working familiarity with the study of functions of one real variable.
- Working familiarity with derivatives.
- Understanding of what an integral is, even if you won't be required to solve integrals.
- Working familiarity with vector calculus.
- Some familiarity with functions of many variables.
- Understanding of what partial derivatives are.

Physics prerequisites

Just some vague reminiscences of secondary/high-school physics should be enough.

It can be beneficial if you are familiar with basic physics notions such as *velocity*, *mass*, *force*, and similar ones; and if you vaguely remember about Newton's laws.

Structure of this text

👉 Graphical devices

The text includes the following graphical devices:

- Important notions and definitions are also given in **boldface**.
- Important notions:

Something

This is a definition or explanation of Something.

- Warnings and important points that require careful thinking:

❗ Something

Something you must be careful about.

- Exercises:

🧩 Exercise 0.1

This isn't an exercise

- Possible objections you might be thinking about:

📣 This isn't right!

I have an objection here.

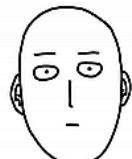
- Discussions and connections with more advanced physics:

❖ Curiosity 0.1 How things really are in quantum physics

Just for your curiosity, you don't need to remember any of this.

👉 Side pictures and quotes

Pictures, graphs, or quotes related to the material are displayed on the right.



This is an image of Saitama, which actually has nothing to do with the text on the left.

👉 Hyperlinks and bibliography

Some words are hyperlinks, like this one about [One Punch Man](#)¹; you also recognize them because they have a little footnote number. The links' URLs are listed at the end of each chapter, just in case you're reading a printed copy and wonder what the link was.

The text gives bibliographic references, like "Einstein [1905](#)", to scientific literature. The references are listed in the final Bibliography on page [66](#).

These references are not part of the course and don't need to be "studied". They are given for two reasons:

- For your own curiosity.
- To back up what's written in the text. In science you should not believe something just because you've read it somewhere. You should, as much as possible, go and check *for yourself how the logic behind the statement is proved and what is the experimental evidence behind the statement*.

👉 Notation and terminology

Mathematical notation, as well as notation for physical dimensions, strictly follows the standards given by the [International System of Units \(SI\)](#)², listed for example in [iso 2009](#) and [iso 2019](#).

"Believe nothing, O monks, merely because you have been told it, or because it is traditional, or because you yourselves have imagined it. Do not believe what your teacher tells you merely out of respect for the teacher."

(attributed to Gautama Buddha)

URLs for chapter 0

1. <https://onepunchman.fandom.com>
2. <https://www.nist.gov/pml/special-publication-811>

Chapter 1

Physics?

If you think about it, many things we ordinarily do every day are some sort of magic. Think of how you can instantaneously see and speak with a person living on another continent, in real time, using just a small widget in the palm of your hand. Think of how you can instantaneously see where you are on the Earth, using the same widget. Think of how fast you can go to another country, by flying in a huge metal thing. Think of how you can command and interact with a purely fictitious animated world when you play on your computer. The list can go on forever. Other things are luckily less ordinary, but still inspire a lot of awe: think of the devastating power unleashed by something roughly as small as a tennis ball, in an atomic bomb.

We can do these astonishing things thanks to our understanding of how the world works. That's Physics.

Many things can be said and have been said about science and physics. Rather than repeating what's been already written in many excellent books, I invite you to take a break here and go read their introductions. Choose as you please; compare what they say; don't limit yourself to popular books.

.....

1.1 Several possible formalisms or “languages”

Physics can be expressed and written from wildly different points of view, using wildly different principles. Let's call these “different physics languages”; a more technical name is “physics formalisms”. One may approach a physics phenomenon or problem in terms of *Lagrangeans*, or

Hamiltonians, or *fibre bundles*, or *categories*, or *action principles*, or many other formalisms. These formalisms or languages are not completely separated; we know how to translate among them. In "doing" physics, one may jump among formalisms, because some ideas may be easier to express, or some results easier to find, in one formalism than another. No matter which physics formalism you choose, the results and the concrete applications are still the same. The choice is to a great extent subjective, based on your aesthetic tastes. You see that in "doing" physics you can express your personality and put your own artistic touch; this is why it's such a cool subject (and other scientific subjects are like this too).

In these notes I'm choosing one particular formalism: the one that for me is the most easily *visualizable*; because I believe that visualization can be beneficial in learning new things. Or maybe I'm choosing it just because I like it best. I encourage you to explore how the physics you've learned is expressed in other physics formalisms; maybe you'll like another physics formalism better.

The formalism we'll be using might be called "field theory". Roughly speaking it takes as starting point the ideas of space and time, or better spacetime, in which there are different kinds of "stuff". It expresses the regularity and patterns that we observe in physical phenomena as "budgets" about the different kinds of stuff, and of relations between these kinds. Please don't take the description just given too literally; it's just meant to give you a very vague idea of the field-theoretical viewpoint.

It goes without saying that all these "physics languages" are to a great extent mathematical.

One reason is that numbers allow us to convey information in a concise and precise way. Imagine you have to tell someone, who doesn't know Bergen, where in Bergen you are right now, to within 10 m. You can do that with a description, "... and there's a building called so-and-so which looks like so-and-so...", which would be lengthy and tricky. Or you can just give two numbers: latitude and longitude:

60.369 40, 5.3518 .

And in these two numbers all digits are important; for instance, the latitude is not 60.369 47.

But the most important reason is that mathematics allows us to describe and follow the patterns and variety of physical phenomena in a greatly concise and precise way. And to develop their relationships in a rigorous

$$\delta \int L dt = 0 \quad L = \frac{1}{2} mv^2$$

$$\mathbf{F} = \frac{d}{dt} m \mathbf{v} \quad \mathbf{F} = 0$$

Example of two different formalisms (red, blue) expressing the same physical phenomenon.

"this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics"

Galilei 1623

way. All our present technology would have been impossible to discover, and would be impossible to realize, without the mathematical language of physics.

I invite you again to read what many good texts say about the relationship between physics and mathematics. No point repeating here what is said better elsewhere.

1.2 Quantities: primitive and derived

One topic must be briefly discussed because it's important for understanding the notes that follow. It's the distinction between *primitive* and *derived* quantities.

I shall assume that you already know what a **physical quantity** is. Examples are: position, duration, velocity, pressure, energy, temperature.

A **derived quantity** is one that is defined in terms of other quantities. For example, velocity v (more precisely: average velocity) is defined as the ratio between a distance d (a vector) and a time duration t :

$$v := \frac{d}{t}$$

where the symbol “:=” means “is defined as” or “is defined by”. This means that in principle we could avoid using the word “velocity” and the symbol “ v ” altogether, and instead always speak about distance and duration, using their symbols. It would lead to very long sentences and formulae and would be extremely inconvenient, but it could be done. The definition of a derived quantity often tells us how that quantity can be measured.

A derived quantity is defined in terms of other quantities, and these may in turn be derived quantities, that is, defined in terms of still other quantities, and so on. But at some point this chain of definitions must come to an end, otherwise we would go around in circles.

A **primitive quantity** is one that we do not define in terms of other quantities. Primitive quantities are the building blocks from which we define all others. That they are not defined in terms of others doesn't mean that we cannot try to explain them. But such explanations must be taken as informal and heuristic. Primitive quantities are often explained through metaphors and by appealing to intuition. You must always be wary of such explanations, because they may fail you spectacularly in some situations.

Often we have a choice about which quantities should be primitive and which should be derived. For instance, *energy* can be defined, in a

“There is nothing that can be said by mathematical symbols and relations which cannot also be said by words. The converse, however, is false. Much that can be and is said by words cannot successfully be put into equations, because it is nonsense.”

Truesdell 1966

somewhat complicated way, in terms of quantities like *work* and *heat*, which would then need to be taken as primitive. Or we can take *energy* as primitive, and define *work* and *heat* in terms of it. This second choice can be more convenient to develop a physical theory. It often happens that a quantity is very convenient for building a theory, if used as primitive; but difficult to understand intuitively. Vice versa, a quantity can be very intuitive but lead to a complicated theory. Among quantities which we'll take as primitive are: *time*, *space* and *length*, *matter*, *energy*, *momentum*, *entropy*, *temperature*, and several others. All will be discussed soon.

1.3 Physical dimensions and units

Measurement is the process by which we determine the value of a physical quantity. Measurements can be extremely complex, and can extremely different even if they are about the same quantity. Consider the ways we can measure the mass of a football, compared to the ways we can measure the mass of the Sun.

To each quantity we associate a **physical dimension**. The term ‘dimension’ here has nothing to do with physical extension, as in “the dimensions of this box”; be careful not to confuse the two. Usually it’s clear which one is meant from the context. Physical dimensions help us avoid making operations that don’t make sense with some quantities. For example, it doesn’t make sense to sum up the volume of a body of water with its temperature; and indeed the volume has dimension *length*³, whereas temperature has dimension *temperature*, and quantities with different dimensions cannot be added up.

With each physical dimension we can associate a measurement **unit**, which expresses a basic standard for comparing the measurement results of similar quantities. For example, we could use the *minute* or the *second* as units to measure *time*.

One can choose a basic set of physical dimensions from which to define all others, and for these a set of standard units. Here we shall follow the **International System of Units (SI)**¹.

The topics of measurement and physical dimensions, which are studied in *metrology* and in *dimensional analysis*, could occupy an entire course by themselves. I shall assume that you already know their basics notions and that you read about the SI.

The measurement of some physical quantities consists in just one number with associated physical dimension; we shall call such quantity a

scalar. The measurement of other physical quantities consists instead in a triplet of numbers with associated physical dimension; we shall call such quantity a **vector**.

! What's scalar or vector depends on the theory

Scalar and *vector* have very specific and slightly different meanings in different theories, so don't take the definitions used here as universal. For example, in these notes and in Newtonian mechanics we call *energy density* a scalar, but in general relativity it cannot be called a scalar.

1.4 What is “fundamental” physics?

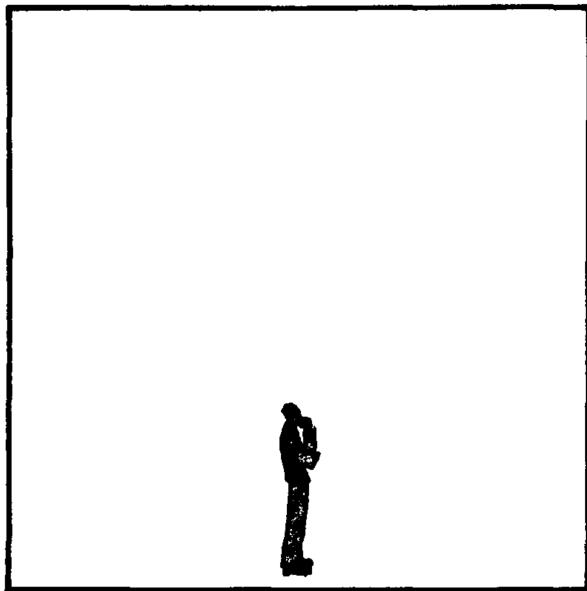
But what's the “ultimate” goal of physics? What's “fundamental” physics? The answer to this question is again subjective – also in this case physics lets you express your proclivities and personality. In the history of physics one can probably identify two main conceptions of “fundamental” physics.

For some physicists it is about finding the ultimate building blocks, so that one day we can say “... and these are the constituents, and they obey these equations”. The history of physics seems to show that this goal is overturned every few generations. And yet every generation says “Now we almost have the complete picture – it's right behind the corner. It's true that previous generations thought they almost had it, and turned out to be wrong. But *this time* is different, this time we have the real deal!”. The theoretical and particle physicist [Geffrey Chew²](#) depicted this situation as in fig. 1.1. For this reason some physicists are a little sceptical about this goal; maybe it's a never-ending structure, with surprises at every deeper look.

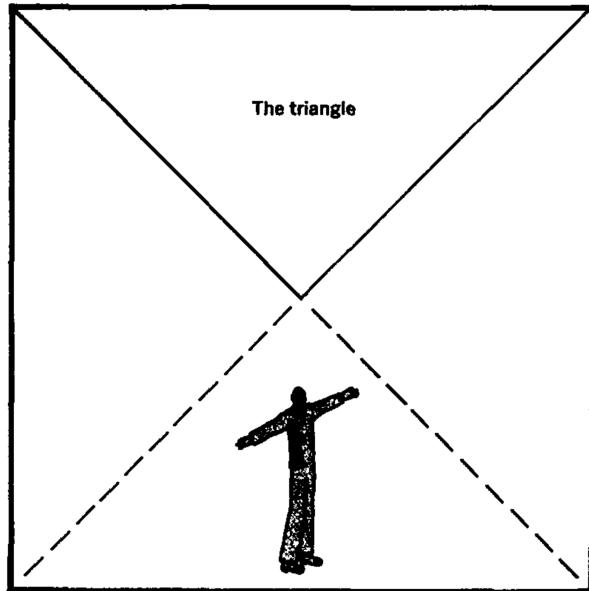
So for other physicists fundamental physics is about finding some point of view or mathematical structure that is rich enough to make useful predictions, and yet flexible enough to accommodate any new patterns or objects that we might discover. In a manner of speaking, it is about finding “patterns of patterns” or “laws about physical laws”.

The two conceptions above are not mutually exclusive, and both are always pursued, even if time-changing fashions may emphasize the one or the other.

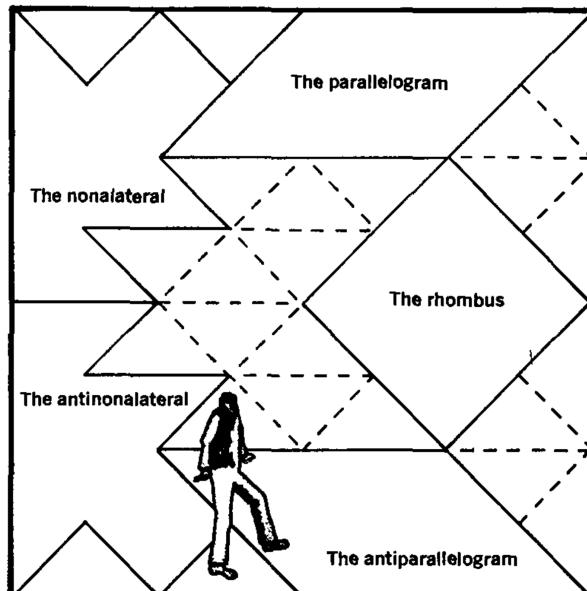
In these notes we take a point of view slightly closer to the second conception. This will also be reflected in the main division between physical laws that we'll draw in ch. 5.



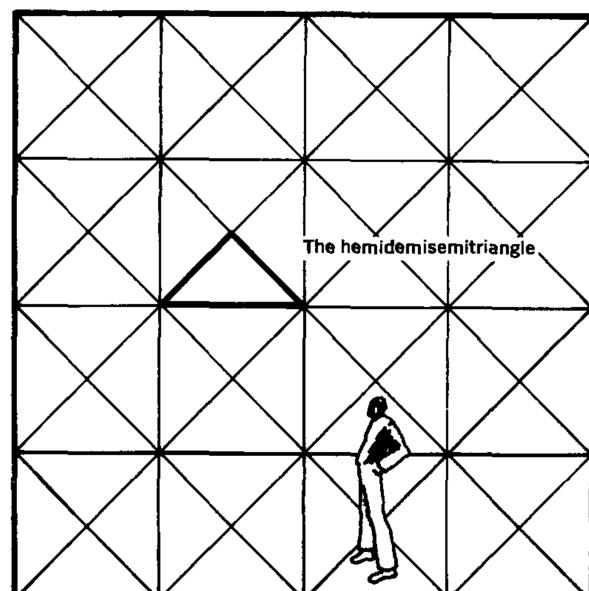
10 000 BC. The inhabitants of the paper square have no conception of the true nature of the universe they inhabit.



1900 AD. Physicists of the square discover a basic subdivision of their universe. They call it the "triangle" and consider it to be the fundamental building block of the universe.



1960 AD. Physicists' conception of their universe is further clouded by new discoveries: the rhombus, the parallelogram, the antiparallelogram, the nonalateral and many others. It is unclear what these discoveries signify.



1970 AD. A new configuration, the "hemidemisemitrangle," is hypothesized, out of which all known configurations of the universe can be constructed. The hemidemisemitrangle is thought to be the fundamental building block of the universe.

Figure 1.1 (Continues on p. 14) *The progress of "fundamental" physics*, from Chew 1970 as reproduced in Truesdell 1987



1930 AD. Physicists discover that the triangle can be split. Its parts are termed the "hemitriangle" and the "demitriangle." These are thought to be the fundamental building blocks of the universe.



1950 AD. Mirror images of the hemitriangle and the demitriangle are discovered. These are termed "antihemimtriangle" and "antidemitriangle."



1975 AD. The hemidemisemimtriangle is discovered. The following year the hemidemisemimtriangle is split.



2000 AD. The inhabitants of this paper square have no conception of the true nature of the universe they inhabit.

URLs for chapter 1

1. <https://www.nist.gov/pml/special-publication-811>
2. <https://www.physics.lbl.gov/rememberinggeoffreychew/>

Chapter 2

Building blocks: Time and space

If we want to describe the motion of a material point, we give the values of its coordinates as a function of time. However, we should keep in mind that for such a mathematical description to have physical meaning, we first have to clarify what is to be understood here by “time”. We have to bear in mind that all our propositions involving time are always propositions about simultaneous events. If, for example, I say that “the train arrives here at 7 o’clock”, that means, more or less, “the pointing of the small hand of my clock to 7 and the arrival of the train are simultaneous events”.

A. Einstein 1905

2.1 Time

Time is a primitive quantity. We understand the notion of time intuitively, even if it’s difficult to explain (that’s why it’s taken as primitive). In 1905, with the theory of relativity, part of our everyday intuition about this notion was seriously shaken. For many years afterwards our old intuition could still be used in practice and in applications. But the new, correct intuition is becoming more and more important in everyday life and technologies. For example, GPS navigation – which we use everyday from leisure activities such as hiking or sightseeing to more critical ones such as aeroplane landing – critically depends on the correct notion and intuition of time.

Let’s see how our traditional intuition goes astray with a concrete experiment. Here’s Alice, Bob, and Charlie. They have extremely precise

clocks built in exactly the same way. They stay very close to one another and synchronize their clocks. Still keeping close, they go around, maybe on an aeroplane or space ship, and all the time they check their clocks. They notice that their clocks stay perfectly synchronized all the time, no matter where they go and what they do.

At some point they separate, each one going around independently. One of them might stay in place, another might take a helicopter, and another might go for a trip on Mars and back.

Alice and Bob at some point meet again, and compare their clocks. They see that their clocks aren't synchronized anymore; the difference could be as small as microseconds, or as large as years. In fact, if this time discrepancy is large, they would notice that they themselves have aged differently; so time discrepancy doesn't affect the clocks only. Let's say for concreteness that Alice's clock is ahead of Bob's, or equivalently that Bob's is behind Alice's. Note the following aspects:

First, neither Alice or Bob can say "my clock was wrong": neither has noticed anything strange about the "passage of time".

Second, they might wonder what's the time on Charlie's clock. But Charlie is at some distance away. They could decide to contact Charlie via radio, say, and ask "what shows your clock right *now*?". But they would notice that there's a delay, even if extremely small, in the radio transmission; so it's unclear to what time would Charlie's answer apply. If we say "let's account for the radio-signal speed", we see that there's a logical problem: speed is distance divided by time, and here we have a problem in exactly determining what's the "correct" time! So we would be reasoning in circles. Besides, even neglecting these difficulties, Charlie's answer could reveal a time that completely different from Alice's and from Bob's – it could be years ahead or behind both of theirs!

Third, if they now stay together, they will see that their clocks remain exactly synchronized, besides the discrepancy they noticed when they met. This discrepancy doesn't increase or decrease. They may even retrace together Alice's and Bob's previous trips; their clocks still remain synchronized.

The experience just described will occur again any time two or more of them meet. There could be a hundred observers like Alice, Bob, Charlie, initially at the same place and synchronized. Whenever two or more of them meet after having been separated, they will notice discrepancies in their clocks. But their clocks will have exactly the same time lapses as long as they stay together.

Consider for a moment an imaginary world in which these experiments had given a different kind of result. According to Newtonian mechanics, whenever two or more initially synchronized observers like Alice, Bob, Charlie had met, their clocks would have always shown exactly the same time. If one year, 23 days, 8 hours, 9 minutes, and 3.045 399 283 240 992 663 02 seconds have passed for you since you last met Alice, you'd see that exactly the same amount of time has passed for her when you two meet again. If you think about it, in this case it would have been somewhat natural to think "right now, the clocks of far-away Alice, Bob, Charlie must show the same time as mine" (even though you have no real experimental way of confirming that).

But that's an imaginary world. In our world is the more complicated situation described initially that holds. Only one conclusion can be drawn from these experimental results: **Time is not some sort of universal quantity. It is, so to speak, "local" to a person or clock, or to a group of persons or clocks that stick together.** This also means that *it doesn't make sense* to ask questions like "what can be the time for far-away Charlie, right now?".

The time measured by a specific observer is called the **proper time** of that observer. Luckily we know more about how the proper times of separated observers can differ when they meet again. It turns out – according to our current understanding – that the time differences depend, roughly speaking, on how fast the observers are moving with respect to one another and to matter around the universe, and on how much energy is contained in the regions they travel. The general theory of relativity gives us the equations determining any such proper-time differences.

The situation depicted in the experiments above is real. It can be measured, for example, comparing initially synchronized clocks that have been put in aeroplanes flying in different directions. Most importantly, it affects everyday relevant technologies such as the Global Positioning System. Formulae from general relativity appear in your phone's GPS software; see for instance § 20.3.3.3 of the Interface Control Document IS-GPS-200 at <https://www.gps.gov/technical/icwg/>. It must also be taken into account in the establishment and synchronization of time in our everyday equipments:

International Atomic Time (TAI) is based on more than 250 atomic clocks distributed worldwide that provide its stability, whereas a small number of primary frequency standards provide its accuracy. Universal Coordinated Time, which is the basis of all legal time scales, is derived from TAI. To

"In 1976, the International Astronomical Union introduced relativistic concepts of time and the transformations between various time scales and reference systems. [...] Now [...] it is necessary to base all astrometry, reference systems, ephemerides, and observational reduction procedures on consistent relativistic grounds. This means that relativity must be accepted in its entirety, and that concepts, as well as practical problems, must be approached from a relativistic point of view."
Kovalevsky & Seidelmann
2004

allow the construction of TAI and the general dissemination of time, clocks separated by thousands of kilometres must be compared and synchronized. [...] The achieved performances of atomic clocks and time transfer techniques imply that the definition of time scales and the clock comparison procedures must be considered within the framework of general relativity.

(Petit & Wolf 2005)

In most everyday situations for us, who live on or nearby Earth and move at speeds much smaller than c with respect to one another, the discrepancies between our proper times are so small that cannot be measured with ordinary clocks or with our internal clocks. Consider a person walking 10 m away from you and then immediately walking back to you, at 1 m/s. The time elapsed for you will be 20 s, but for that person will be 19.999 999 999 999 999 889 s, a difference of 10^{-16} s, which is the error of an atomic clock. If human beings still exist in some decades or centuries, with space travel they will probably have to deal more and more with proper-time discrepancies also in everyday life.

For the most part of the rest of these notes, we won't need to deal with differences in proper time. But I recommend that you keep present how time really works, and that these small time discrepancies exist and occur all the time along your *worldline*.

Time has physical dimension of time and we shall for the most part measure it using the unit *second*, symbol 's'.

2.2 Space

Together with the notion of time, also the notion of space loses some of its traditional intuition. Several observers in motion with respect to one another will generally disagree on the dimensions of an approximately rigid object in their vicinity. For objects that are far away from an observer, the very notion of "distance" becomes tricky has different and non-equivalent definitions; one must be very careful on which definition is being used.

We shall not delve further into these peculiarities of time and space. Keep simply in mind that phenomena happen in **spacetime**, and that there's no way to attribute a universal time, nor a universal position in space, to a physical event. There is one absolute: whoever locally measures the speed of light, will find the value

$$c := 299\,792\,458 \text{ m/s} \quad (2.1)$$

"The plot for Cesium [...] characterizes the best orbiting clocks in the GPS system. What this means is that after initializing a Cesium clock, and leaving it alone for a day, it should be correct to within [...] 4 nanoseconds. Relativistic effects are huge compared to this."

Ashby 2003

This value is exact by definition, and serves as a way to define a local notion of space and distance.

Space has physical dimension of length and we shall for the most part measure it using the unit *metre*, symbol 'm'.

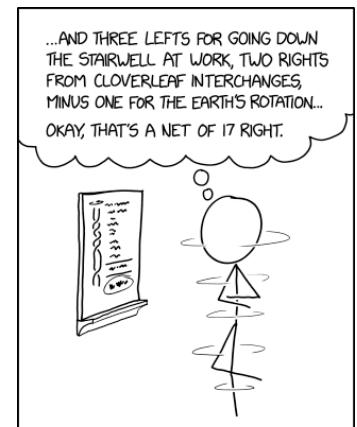
2.3 Coordinate systems and events

It is necessary to have a way for distinguishing physical events and phenomena and locating them in spacetime. This is achieved through a **coordinate system**.

A coordinate system assigns four numerical labels to every point in spacetime. Often these labels have some kind of physical meaning – such as the proper time elapsed for a specific clock, or the distance from some event as measured by a specific observer – but they don't need to.

A coordinate system also solves the problems coming from proper-time and space discrepancies among different observers. We can assign to every physical event a *coordinate time* and a *coordinate spatial position*, which are the same for all observers, because decided by agreement. Coordinate time doesn't have a strict physical meaning, and will generally be different from the proper times registered by different observers. It can nevertheless be used for "doing physics", and it is the time we shall most often use in our equations. A coordinate time commonly used for Earth-physics purposes is [Universal Coordinated Time \(UTC\)](#)¹. The clock on your phone, and on devices that get synchronized via internet, shows UTC, not your proper time. An observer on Earth at 0 m over sea level, and not moving, measures a proper time exactly equal to UTC (besides small variation coming from the movements of Solar System bodies). Observers at other altitudes or moving with respect to Earth's surface notice that their proper times are slightly different from UTC.

Up to now we have often used the word 'event', informally taking its meaning for granted. Let's be more precise: we call **event** or **spacetime point** an extremely small region of space – a point – that only lasts for a very small lapse of time – an instant. The name 'event' is used because typically we approximately identify such a point and instant by means of a physical phenomenon of limited spatial extension and short duration, such as the collision of two subatomic particles. The sudden burst of a very small soap bubble can be considered as an event in some circumstances; but something like "a tennis ball" cannot be considered as an event, mainly because a tennis ball exists for quite a long time, not just for a short instant.



From a four-dimensional spacetime point of view, a tennis ball could be characterized as a line: a **worldline**.

We shall often denote the four coordinates of a coordinate system by the letters

$$(t, x, y, z)$$

where t is a coordinate time, usually UTC, and (x, y, r) determine a spatial position. The triplet of spatial coordinates is often denoted by the vector \mathbf{r} :

$$\mathbf{r} := (x, y, z).$$

It is always important to specify how the coordinate system you're using is defined. The definition of the spatial coordinates (x, y, z) is typically different from problem to problem. We shall typically use coordinates that form $\frac{\pi}{2}$ rad $\equiv 90^\circ$ angles with one another; but their directions and their origin – that is, where they have value $x = y = z = 0$ m – always depend on the problem, so make sure you always specify them.

Whenever we speak of a “region of space” or of a “surface in space”, we mean a 3D or 2D region at some specific coordinate time t .

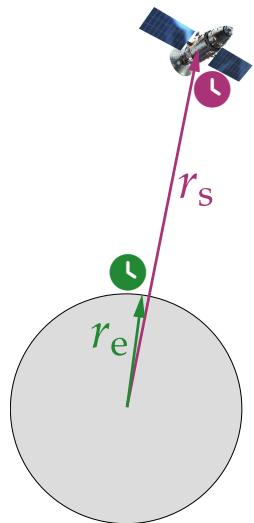
Some physical phenomena happen along a line, in one dimension. In this case we can omit two of the spatial coordinates, assuming they have some constant, unimportant values. In these cases we can simply write, for instance, (t, x) as our coordinates.

Exercise 2.1

Consider a clock at rest on the Earth's surface, at a distance r_e from the Earth's centre; and a clock on a satellite, for instance a GPS satellite or the International Space Station², right above the first clock, at a distance r_s from the Earth's centre. An observer by the clock on Earth measuring a time lapse Δt_e will see the clock on the satellite has having run for a time lapse Δt_s , and vice versa (note that this “vice versa” only holds in this specific situation!). The relation between two time lapses is approximately given by

$$\frac{\Delta t_s}{\Delta t_e} = \frac{\sqrt{1 - \frac{G M}{c^2 r_e}}}{\sqrt{1 - \frac{G M}{c^2 r_s}}}$$

where $G \approx 6.7 \times 10^{-11} \text{ m}^3/(\text{kg s}^2)$, $c = 3.0 \times 10^8 \text{ m/s}$, and the Earth's mass $M = 6.0 \times 10^{24} \text{ kg}$.



1. Take the case of a GPS satellite, with $r_e = 6.4 \times 10^6$ m and $r_s = 2.6 \times 10^7$ m ([NASA data³](#)). If you, on the ground, measure a time lapse of $\Delta t_e = 10$ years, what's the difference, in seconds, with the time lapse Δt_s you see on the satellite?
2. If the time lapses are large compared with the time needed to go from ground to orbit or vice versa, then $\Delta t_s/\Delta t_e$ is also the ratio between the real *ageing* of a person who's been in orbit and one who's been on the ground, when they meet again.

Now consider the case with a black hole instead of Earth. The formula above still apply approximately.

In the movie [Interstellar⁴](#), two astronauts go on Miller's planet, at a distance r_e from the black hole Gargantua, and stay there for 3 hours, leaving one astronaut in orbit at a distance $r_s \approx \infty$ (the distance is large enough that it can be approximated as infinity). When they meet again, the latter astronaut has aged 23 years.

Given that Gargantua's mass is $M = 2.0 \times 10^{38}$ kg, calculate the distance r_e of Miller's planet from the black hole.



2.4 Velocity and acceleration

In some situations the spatial coordinates $\mathbf{r} = (x, y, z)$ may turn out to be functions of the time coordinate t ; the typical example is when we describe how the spatial position of a small body changes with coordinate time. We can write this functional dependence in different ways, for instance

$$\mathbf{r}(t) \quad \text{or} \quad (x(t), y(t), z(t)) .$$

So \mathbf{r} is a vector function of time, which simply means that we have a collection of three functions of time.

If we take the derivative of each coordinate with respect to the time t , we obtain the **coordinate velocity**

$$\mathbf{v}(t) := \frac{d}{dt} \mathbf{r}(t) = \left(\frac{d}{dt} x(t), \frac{d}{dt} y(t), \frac{d}{dt} z(t) \right)$$

which is also a vector.

! Dot-notation for time derivative

The derivative of some quantity with respect to coordinate time is often denoted by a **dot** over the quantity. So we can also write

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$$

The coordinate velocity is usually different from the *physical velocity*, which an observer would measure using proper time and space, for instance using bouncing light rays. In many everyday situations the difference between coordinate and physical velocity is so small that it can be neglected. But in situations involving subatomic particles at high speed, for example, one must take into account that the two velocities are different.

Taking the time derivative once more we obtain the coordinate acceleration, also a vector:

$$\boldsymbol{a}(t) := \frac{d}{dt} \boldsymbol{v}(t) = \frac{d^2}{dt^2} \boldsymbol{r}(t) = \left(\frac{d^2}{dt^2} x(t), \frac{d^2}{dt^2} y(t), \frac{d^2}{dt^2} z(t) \right)$$

which is also a vector.

❖ Curiosity 2.1 Acceleration in relativity theory

In relativity theory, acceleration acquires a special physical significance, because it includes the effect of gravity, and its calculation does not involve just a time derivative. For instance, let's say that you are standing still on the ground, and let's use a coordinate system where x points in front of view, y to your left, and z points upwards. Then your coordinate velocity is $\boldsymbol{v} = (0, 0, 0)$ m/s also according to relativity theory. But the spatial part of your acceleration is approximately $(0, 0, 9.8)$ m/s², not zero!

The definitions and values of acceleration according to relativity theory and according to Newtonian mechanics are therefore quite different even in everyday situations. In these notes we'll mean the Newtonian definition of acceleration, unless stated otherwise.

URLs for chapter 2

1. [https://www.nist.gov/pml/time-and-frequency-division/time-realization/utc
nist-time-scale-0/](https://www.nist.gov/pml/time-and-frequency-division/time-realization/utc-nist-time-scale-0/)
2. <https://www.nasa.gov/international-space-station/>
3. [https://www.nasa.gov/directorates/somd/space-communications-navigation-p
rogram/gps/](https://www.nasa.gov/directorates/somd/space-communications-navigation-program/gps/)
4. <https://www.imdb.com/title/tt0816692/>

Chapter 3

Building blocks: “Stuff”

For Euler, clarity was the hallmark of truth. [. .] To him we owe also the brilliant imagination of the internal pressure in generality, the pressure field as equipollent to the action of the fluid outside any imaginary closed diaphragm upon that within. [. .] I remark upon it in emphasis of the role of imagination and the importance of quantities which can only be thought of and cannot in themselves be measured.

C. A. Truesdell 1956

3.1 Seven primitive quantities with two basic properties

Our physics formalism includes around seven more quantities that we take as primitive:

Seven primitive quantities

matter
electric charge
magnetic flux
energy
momentum
angular momentum
entropy

Technically they are called *fields*, for reasons that we shall see shortly.

Recall that primitive quantities cannot be defined: we can only try to understand them intuitively, for example through their properties. Some

of the seven primitive quantities are easier to grasp intuitively than others. But all seven primitive quantities have two basic properties in common. Of each we can ask, or rather, measure:

📘 The two basic measurements that can be made on the seven quantities

1. How much of this quantity is in a particular three-dimensional region of space at a particular time instant?
2. How much of this quantity flows through a particular two-dimensional surface during a particular time lapse?

We can ask these questions of any region of space and any time lapse, and the surface in the second question can be moving and deforming.

Keep in mind that *the region and surface can be completely imaginary*. This is what makes these questions very powerful for doing physics.

For example, consider a classroom and the people in it. In your imagination you can divide the classroom into two halves, say the front and the rear half. You can then ask or measure simply by counting: 1. how many people are, right now, in the rear half; 2. how many people are crossing the imaginary division between the front and rear half during one minute, starting from now.

The results of the two measurements above are numbers, which in general can be positive or negative, for scalar quantities; or vectors for vector quantities.

Often we consider the second question, about the flow through a surface, in the case of a very short lapse of time, and divide the total flow by that time lapse. This is called the **flux** of the quantity through that surface. So we have an alternative form of the second measurement:

📘 Flux of a substance through a surface

- 2b. How much of this quantity is flowing through a particular two-dimensional surface per unit time, at a particular time instant?

Note that when we measure or speak of a flux, *we must always pay attention on whether the surface we are considering is moving*. If someone asks you what's the flux through a particular surface at a given instant, but they don't tell you what's the velocity of the surface, then the flux is unknown.

As a trivial example, consider a glass surface, and a person on one side of it, moving with a high velocity directed towards the surface. Will

the person crash on the glass? We can't say for sure. The glass surface could be a glass wall in a building, which is not moving; in which case the person will likely crash on it. Or it could be the windscreen of a car, which is moving together with the person, who's the driver; in which case the person won't crash on it.

The basic measurements above can't in general be made, and don't even make sense, for some other quantities. For instance, we cannot ask “what's the total amount of *temperature* in this region?”, or “how much velocity is flowing through this surface?”.

Owing to the two properties above, each of the seven quantities can be considered as some kind of “stuff” that can be present at each spacetime point and fill a region of space. But don't take the word ‘stuff’ too literally: I don't necessarily mean concrete objects like a ball, or substances like water.

What's remarkable about matter, electric charge, magnetic flux, energy, momentum, angular momentum, and entropy, is that *they are common to all our main physical theories*, approximate or not: from Newtonian mechanics to general relativity and quantum theory; from subatomic scales to cosmological scales. The physical meaning and mathematical characterization of these quantities can be slightly different depending on the physical theory and scale. For example, in quantum theory they are mathematically represented by so-called operators rather than functions; and at molecular scales entropy has a meaning connected with probability theory. Yet, these seven quantities are really universal to our present way of doing physics and of describing physical phenomena around and within us.

Let us make a first acquaintance with these seven quantities.

3.2 Matter

Matter is a *scalar* quantity, with SI dimension [amount of substance](#)¹, and measured in units of [moles \(mol\)](#)². Its flux is therefore measured in units of *moles per second* (mol/s). In statistical mechanics and particle physics, matter is often simply counted and so measured in dimensionless units, rather than moles.

Matter is probably the easiest quantity to grasp intuitively; what we ordinarily call “stuff” is matter. It is usually classified into several kinds; the classification depends on the physical phenomena and physical theory

one works with, and is related to whether the kinds of matter can be considered separately conserved, as we'll discuss in § 6.1.

In everyday phenomena not involving [radioactivity](#)³ or [nuclear energy](#)⁴, the different kinds of matter approximately correspond to the [non-radioactive chemical elements](#)⁵: [hydrogen](#)⁶, [helium](#)⁷, [lithium](#)⁸, and so on. Note that some common everyday devices, such as smoke detectors, do involve radioactivity.

In phenomena involving radioactivity or nuclear energy, the different kinds of matter correspond approximately to [baryons](#)⁹, such as protons and neutrons; and [leptons](#)¹⁰, such as electrons. In particle physics, even more subtle classifications of matter are made, into kinds that seem to be conserved, such as electronic-leptons, muonic-leptons, and others. This kind of research is still open, but it seems that the total amount of baryonic and leptonic matter, independently of the kinds into which it can be classified, is always conserved. Note that we're using the term 'matter' in a sense that includes anti-matter, such as positrons.

The total amount of matter in a region *can be negative*. "Negative matter" is what's usually called [anti-matter](#)¹¹. Anti-matter appears in small amounts in everyday life, for example in connection with common radioactivity processes. It is also created and used in medicine, in [positron-emission tomography \(PET\)](#)¹² scans.

Exercise 3.1

According to statements on [symmetrymagazine.org](#)¹³ and [quantum-diaries.org](#)¹⁴,

The average banana (rich in potassium) produces a positron roughly once every 75 minutes.

Unfortunately the original site where this statement was discussed, and the corresponding calculation made, seems not to exist anymore.

1. Do a little research and find out whether this statement is true.
2. From your research, approximately quantify the flux of positrons around an ordinary banana, expressing it in particles/s.



How many positrons do bananas produce?

In these notes we shall usually not consider distinctions between different kinds of matter, making some exceptions for discussions about chemical reactions and nuclear phenomena.

! Matter is different from mass or energy

It is important to clearly distinguish matter from *mass* or *energy*. Mass can be considered a property of matter, but the two are different. In nuclear reactions, for instance, the mass of some amount of matter may change, even if the amount of matter stays the same.

As far as we know, the total amount of energy associated with an amount of matter is always positive, whether the amount of matter is positive or negative (antimatter). This is the reason why antimatter “falls” just like positive matter, a fact that has been experimentally confirmed: see Anderson et al. 2023.

Bookmark icon Matter: notation

The amount of matter in a region is usually denoted with N , and flux of matter with J . In chemistry we usually specify what kind of matter we are speaking about, writing for instance $N_{\text{Ca}} = 5.3 \text{ mol}$, to indicate an amount of 5.3 mol of calcium¹⁵ atoms.

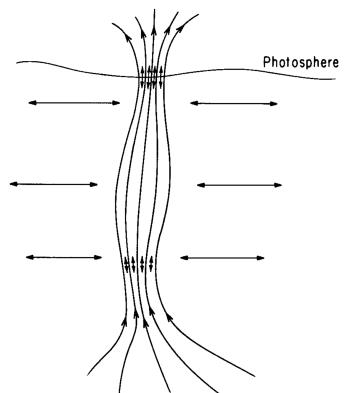
3.3 Electric charge and magnetic flux

Electric charge is a *scalar* quantity, with SI dimension [electric charge](#)¹⁶ (equivalent to electric current×time), and measured in units of [coulombs](#) (C)¹⁷. Flux of electric charge is called *electric current*, and measured in units of [amperes](#) ($\text{A} = \text{C/s}$)¹⁸.

Electric charge is a quantity that is easily grasped in our everyday experience, and doesn't need much comments.

Magnetic flux is a *vector* quantity, with SI dimension [magnetic flux](#)¹⁹, and measured in units of [webers](#) (Wb)²⁰.

The electromagnetic field is most commonly represented by vectors associated to each point in space. But it can also be interpreted and visualized as a collection of moving, oriented tubes or lines, either closed or extending indefinitely; somewhat analogously to how we visualize matter and charge, as moving blobs or points, but with one more dimension. This interpretation goes back to Faraday (1846), Maxwell (1855), and later Dirac (1955) among others, and today is conveniently used in some fields such as [solar physics](#)²¹, for example to study [sunspots](#)²² (see Ryutova 2018). From



"sketch of the magnetic lines of force in a magnetic filament extending up through the photosphere." Parker 1974a

from this point of view, the *voltage* turns out to be the flux of the magnetic flux; it is indeed measured in *volts* (V), equivalent to *webers/second*.

Electromagnetism and this particular visualization of it are very fascinating topics, but we shall not discuss them in these notes.

3.4 Energy

Energy is a *scalar* quantity, with SI dimension energy, and measured in units of *joules* (J); its flux is measured in units of *watts* (W = J/s).

Equivalently we can speak of mass, with SI dimension mass, and measured in units of *kilograms* (kg)²³; its flux is measured in *kilograms per second* (kg/s).

The notion of energy is easily grasped today, and often we speak about it as something that “flows”, is “transported”, “converted”, “stored”, and similar visualizations. This intuition will be enough in these notes. The notion of *mass* is also very intuitive in our everyday life; we associate it with the difficulty in setting objects into motion, or with their weight.

From Relativity Theory, and experimentally, we know that *energy and mass are the same quantity*, and in these notes we shall fully take this point of view.

Some texts say that “mass is a form of energy”, but that’s incorrect: mass *is* energy, and energy in all its possible forms *is* mass.

Here is an example of the fact that any form of energy is mass. Take a common rubber band, and imagine we have an extremely precise weighing scale. We weigh the rubber band, unstretched, and let’s say that we measure a mass of

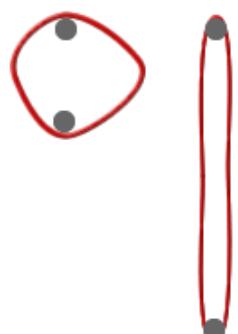
$$0.000\,500\,000\,000\,000\,000\,0\text{ kg}.$$

Now stretch the band a little. According to Newtonian mechanics, it then acquires “elastic energy”, which increases its initial internal energy by, say, 0.3 J. If we now weigh the rubber band again, while stretched, we measure a mass of approximately

$$0.000\,500\,000\,000\,000\,003\text{ kg}.$$

The extremely small difference of 3×10^{-18} kg from the initial mass is exactly the elastic energy that we provided by stretching. Energy has weight; energy is mass.

Now set the unstretched band in motion, so that according to Newtonian mechanics it acquires a kinetic energy of, say, 0.2 J. If we could weigh the



When we stretch a rubber band, its mass increases slightly – even if the amount of rubber remains exactly the same.

band while in motion (but without moving the weighing scale), we would measure a mass of approximately

$$0.000\,500\,000\,000\,000\,002 \text{ kg}.$$

The small difference from the initial mass is the new kinetic energy of the band. Any energy has weight; any energy is mass.

As a final example, consider a 1000 W electric heater, which is radiating 1000 J in one second. The heater is also losing around 0.000 000 000 000 01 kg of mass every second owing to this heat radiation – although it's also acquiring the same amount of mass as electromagnetic energy.

The [atomic bomb](#)²⁴ is a dark example of the fact that mass is energy.

The equivalence between energy and mass is given by the famous formula $E = mc^2$, where c is the speed of light, eq. (2.1). In their respective units this gives

$$1 \text{ kg} = 89\,875\,517\,873\,681\,764 \text{ J} \quad (\text{exactly})$$

$$1 \text{ J} \approx 0.000\,000\,000\,000\,000\,011\,126\,5 \text{ kg}$$

To grasp these numbers, consider that the mass of the rubber band above, 0.5 g, is comparable to the energy released by the [atomic bomb over Hiroshima](#)²⁶.

From the examples above it is clear that the energy changes that we deal with and use in everyday situations are extremely small changes in mass – so small that we can't measure them within the precision of ordinary weighing scales. Vice versa, the masses that we ordinarily deal with are huge amounts of energy. If we spoke only of 'energy' or only of 'mass' in all situations, and used only one unit, either joules or kilograms, we would have to work with very impractical numbers. 'Mass' can be considered a convenient term for 'energy' when huge amounts of it are involved, concentrated in small regions of space, and ordinary small energy changes can be neglected. In these notes we shall often use the expressions 'energy-mass' and 'mass-energy' to remember that these two words denote the same thing.

Energy-mass is present wherever there's matter, charge, or magnetic flux, and vice versa. *Its specific relation with these quantities depends on the coordinate system* (§2.3) that we choose. It is extremely important to keep in mind this dependence. For instance, in the examples above we said that the unstretched rubber band in motion had a mass-energy of 0.000 500 000 000 000 002 kg; but if we choose a coordinate



Hydrogen Bomb Test, 1954
(National Museum of Nuclear
Science & History²⁵)

system in which the band is not moving, its mass-energy would be 0.000 500 000 000 000 000 kg. Owing to this coordinate dependence, and depending on whether matter or magnetic flux is present in a region, we often conveniently partition an amount of energy into different "forms" such as rest energy, internal energy, kinetic energy, gravitational potential energy, electromagnetic energy. This is only a conceptual division.

Also energy flux can be partitioned depending on the coordinate system and on whether there's also a flux of matter. This leads to the important distinction between two forms of energy flux: **heat** and **work**.

We shall see later how to define and calculate these different forms of energy and energy flux.

Energy-mass: notation

The amount of energy in a region is usually denoted with E , or with M if we describe it as mass. Internal energy is denoted U , kinetic energy E_k , potential energy E_p .

The total flux of energy will be denoted Φ , in the form of heat Q , and in the form of work W .

3.5 Momentum

Momentum, also called *linear momentum* to distinguish it from angular momentum, is a *vector* quantity, with SI dimension energy \times time/length, and accordingly measured in units of *joules \times seconds/metres* ($J \cdot s/m$), or equivalently with SI dimension mass \times length/time, and accordingly measured in units of *kilograms \times metres/seconds* ($kg \cdot m/s$). Since it is a vector quantity, it is usually expressed with three numbers, typically the x -, y -, and z -components. In simplified problems where only one or two dimensions are relevant, only the relevant components are reported.

Momentum is a subtle quantity. Textbooks that focus on Newtonian mechanics *define* it as the product of the mass of a body times the body's velocity, usually written " $\mathbf{p} = m\mathbf{v}$ ". This relation, however, is only approximate and valid in some circumstances only; it cannot be used as a general definition. First, it is true that where there's matter in motion there's also momentum; but this momentum is given by the matter's mass \times velocity only as an approximation. Second, there can be momentum also where there's no matter; for instance where there's magnetic flux and charge.

If you prepare a sealed glass box completely empty within (you create a vacuum), then there's momentum within the "empty" box if light or electromagnetic waves are present. We shall discuss later some technological uses of this fact.

It is therefore convenient if you separate your idea of momentum from the idea of "bodies moving", and keep in mind that the latter idea is just a particular case of momentum. Momentum can be associated with motion, though. We shall see that there is an important connection between momentum and our intuitive idea of 'force', which will be useful for getting a better intuition of momentum.

According to Relativity Theory, momentum *is* energy flux (such as heat), and energy flux *is* momentum. If we represent momentum with the symbol \mathbf{P} , and energy flux with the symbol \mathbf{Q} , the equivalence between them is given by

$$\mathbf{Q} = \mathbf{P}c^2 \quad (3.1)$$

Compare this formula with $E = mc^2$. From this point of view, you can think of momentum as "energy in motion". This is consistent with our discussion about mass-energy: since mass is energy, the Newtonian expression " $m\mathbf{v}$ " indicates energy in motion, or a flux of energy. On a sunny day, if you close your eyes and feel the Sun's heat on your face, what you are feeling is actually a flow of momentum. And when you kick a ball, you're setting a huge bundle of energy in motion, and that's why the ball has acquired momentum.

Flux of momentum is what we call **force**.

3.6 Angular momentum

Angular momentum, also called *moment of momentum* or *rotational momentum*, is a *vector* quantity, with SI dimension energy \times time, and accordingly measured in units of *joules* \times *seconds* ($\text{J} \cdot \text{s}$), or equivalently with SI dimension mass \times length²/time, and accordingly measured in units of *kilograms* \times *metres*²/*seconds* ($\text{kg} \cdot \text{m}^2/\text{s}$). It is usually expressed with three numbers, typically the x -, y -, and z -components.

Angular momentum is perhaps an even subtler quantity than momentum. Luckily there's a way of calculating it in terms of momentum, so even if we don't fully grasp it intuitively we can still calculate it. If momentum is denoted by the vector $\mathbf{P} = (P_x, P_y, P_z)$ and the position

vector by $\mathbf{r} = (x, y, z)$, then the angular momentum $\mathbf{L} = (L_x, L_y, L_z)$ with respect to the origin of coordinates is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{P}$$

or equivalently
$$\begin{cases} L_x = y P_z - z P_y \\ L_y = z P_x - x P_z \\ L_z = x P_y - y P_x \end{cases} \quad (3.2a)$$

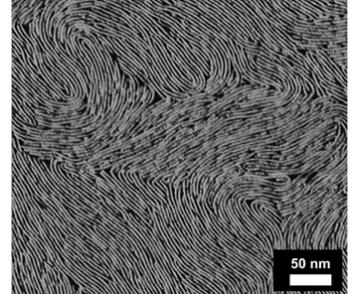
Instead of calling the components " (L_x, L_y, L_z) ", we can also call them " (L_{yz}, L_{zx}, L_{xy}) ", as some books do. The latter names make the formulae above easier to remember:

$$\begin{cases} L_{yz} = y P_z - z P_y \\ L_{zx} = z P_x - x P_z \\ L_{xy} = x P_y - y P_x \end{cases} \quad (3.2b)$$

choose whichever you prefer.

The formula above also suggests that we can somehow think of angular momentum as "energy in rotation". Just like momentum, angular momentum is something that is associated not only with ordinary bodies (matter), but also with charge and magnetic flux, and therefore electromagnetic radiation.

You may wonder: "Do we really need angular momentum? after all it just looks like something constructed from momentum". The answer is yes, we really need it, for two reasons. First, angular momentum obeys an important universal law which is independent from those obeyed by energy and by momentum (Truesdell 1968a tells some of the story of how this was discovered). Second, for some physical phenomena, for example involving liquid polymers²⁷, elementary particles, or electromagnetic radiation, the angular momentum includes an additional part, called *spin* or *intrinsic angular momentum*, that is not related to linear momentum. In the present notes we shall not use this more general kind of angular momentum.



Some liquid polymers (**top**: Liquid Diethoxymethane Polysulfide) need to be described with a special kind of angular momentum, owing to their molecular structure (**bottom**).

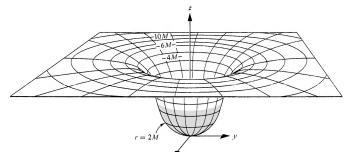
3.7 What are energy, momentum, angular momentum?

From the discussions and formulae above, it seems that energy-mass, momentum, angular momentum are quite closely related to one another.

Relativity Theory indeed shows that they are different aspects of one single geometric object, called *energy-momentum tensor*. They are like its

“shadows”, that we can observe by looking at it from different points of view in time and space (almost literally: they are projections of that object). This is also why their values get intermixed if we change our system of coordinates.

General Relativity gives a new meaning to these quantities: they are *particular curvatures of spacetime*. They express how spacetime is curved in different directions. So whenever we measure, say, the energy or the momentum of some object or of some electromagnetic radiation, we are actually measuring how much that object or radiation is curving spacetime in a particular way. We shall discuss further connections between these three quantities and spacetime in §[X](#).



Energy, momentum, angular momentum are particular curvatures of spacetime.

3.8 Entropy

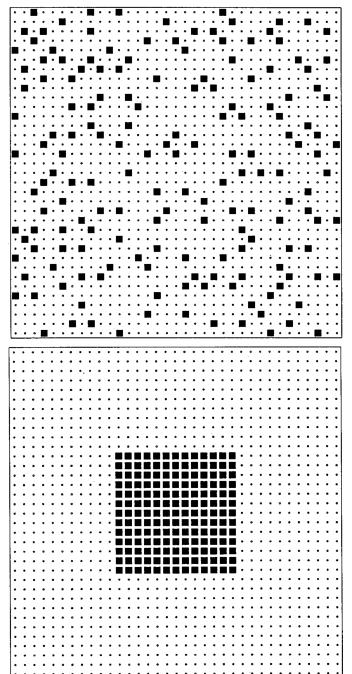
Entropy is a *scalar* quantity, with SI dimension energy/temperature, and measured in units of *joules/kelvins (J/K)*²⁸. From this definition it would seem that entropy is derived from temperature. However, although temperature is taken as primitive by the SI, the [definition of temperature](#)²⁹ actually depends on a fixed value of *Boltzmann's constant*³⁰, which has the dimension of entropy.

Entropy is probably the most difficult quantity to grasp intuitively. Many seemingly intuitive descriptions given in some textbooks are, unfortunately, unhelpful and even misleading. One particularly misleading intuition is that entropy would be a “measure of disorder”. Besides the fact that “disorder” is very vague and subjective, it turns out that some physical phenomena, for example with [liquid crystals](#)³¹, can be considered more “disordered”, and yet have *lower* entropy, than others. See also the example in the side figure. We shall discuss more about such phenomena later on.

In these notes we shall rely on the idea that *entropy expresses a limit on the flux of energy into matter*. Said in simpler but more imprecise words, entropy is a bound on how fast we can heat up a body. We shall develop this idea further later.

One reason why entropy is difficult to grasp intuitively is that it has very different physical and mathematical aspects depending on the spatial scales and physical theory that we use to describe physical phenomena.

In many “continuum” phenomena, that is, phenomena where the molecular constitution of matter is not visible or not taken into account, entropy is treated as a “stuff-like” quantity similar to energy or electric



Microscopic configurations of a lattice gas. **Top:** a configuration coming from a *low*-entropy state. **Bottom:** a configuration coming from a *high*-entropy state (Styer 2000).

charge. But there are difficulties also in this case. For some phenomena, for example involving non-elastic materials such as a simple paper clip, it is possible to introduce several entropies having different values – and not just because of a change in measuring scale – all of which can serve their purpose perfectly fine.

In molecular phenomena involving statistical mechanics, on the other hand, entropy is no longer a physical notion, but a *probabilistic* and *statistical* one, related to guesses and inferences that we make about the physical phenomenon. Yet from many points of view it has roles similar to those of the entropy used in continuum phenomena.

We shall see later that the physical laws for entropy have also a different status with respect to the laws for the other six main quantities: they are, so to speak, “laws about laws”.

3.9 Auxiliary quantities

Besides the seven principal quantities, other auxiliary quantities appear in some physical theories. Important examples are **temperature** and **metric**. Most auxiliary quantities don’t have the “stuff-like” properties of the seven principal quantities. For instance, we cannot ask “what’s the total amount of temperature in this region?”. We shall later discuss and use some auxiliary quantities, especially temperature.

The dimensions, units, and scalar or vector character of these quantities are summarized in table 3.1.

Quantity	SI Dimension	Unit
Time	time	<i>second</i> s
Length	length	<i>metre</i> m
Temperature	temperature	<ikelvin< i=""> K</ikelvin<>
Matter	amount of substance	<i>mole</i> mol
Electric charge	electric charge	<i>coulomb</i> C
Magnetic flux	magnetic flux	<i>weber</i> Wb
Energy	energy, mass	<i>joule</i> J, <i>kilogram</i> kg
Momentum	mass·length/time, energy·time/length	kg · m/s, J · s/m
Angular momentum	mass·length ² /time, energy·time	kg · m ² /s, J · s
Entropy	energy/temperature	J/K

Table 3.1 Dimensions and units of the main physical quantities used in these notes.
Quantities in **boldface** are vectors, the others are scalars

URLs for chapter 3

1. <https://doi.org/10.1351/goldbook.A00297>
2. <https://doi.org/10.1351/goldbook.M03980>
3. <https://www.iaea.org/newscenter/news/what-are-radioactive-sources>
4. <https://www.iaea.org/newscenter/news/what-is-nuclear-energy-the-science-of-nuclear-power>
5. <https://www.ciaaw.org/radioactive-elements.htm>
6. <https://pubchem.ncbi.nlm.nih.gov/element/Hydrogen>
7. <https://pubchem.ncbi.nlm.nih.gov/element/Helium>
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9. <http://hyperphysics.phy-astr.gsu.edu/hbase/Particles/hadron.html#c6>
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13. <https://www.symmetrymagazine.org/2009/07/23/antimatter-from-bananas>
14. <https://www.quantumdiaries.org/2009/07/21/positrons-from-bananas/>
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29. <https://doi.org/10.1351/goldbook.K03374>
30. <https://doi.org/10.1351/goldbook.B00695>
31. <https://www.britannica.com/science/liquid-crystal>

Chapter 4

Densities and fluxes

How do we mathematically represent the seven main quantities? We said that, for each, we can ask *how much in this region?* and *how much through this surface during this time?* We must therefore use mathematical objects that allow us to answer these two questions. These objects are the *density* and the *flux vector*. The idea behind them is quite simple.

Density

Take a point in spacetime with coordinates (t_0, x_0, y_0, z_0) . At the instant t_0 , imagine a very small cuboid centred at (x_0, y_0, z_0) , with sides aligned with the coordinate axes x, y, z and of lengths $\Delta x, \Delta y, \Delta z$. Then we have

$$\text{amount of quantity in cuboid} = n(t_0, x_0, y_0, z_0) \Delta x \Delta y \Delta z . \quad (4.1)$$

where $n(t_0, x_0, y_0, z_0)$ is the **density** at the spacetime point (t_0, x_0, y_0, z_0) .

The density tells us how much quantity there's in a unit of volume. As the notation suggests, it is a function of the coordinates, that is, of time and spatial position. This functional dependence reflects the fact that we can have a larger amount of a quantity concentrated in some regions at some times, than in other regions or at other times.

In order to calculate the total amount of the quantity in an arbitrary 3D region, we simply divide it into very small cuboids similar to the one above. For each cuboid we calculate the respective amount; this amount will generally be different from cuboid to cuboid, because the value of the

density $n(t, x, y, z)$ will be different at each cuboid's centre. Then we sum up all these amounts.

Curiosity 4.1 Y

You probably recognize this procedure as the description of integration. Indeed we can write:

$$\text{total amount of quantity in 3D region} = \iiint_{\text{3D region}} n(t, x, y, z) dx dy dz . \quad (4.2)$$

Clearly we must know the value of the density n at all points within the region, in order to calculate the total amount. Note that if the quantity is absent in some subregion, then $n(\dots) = 0$ there.

Do not confuse flux with movement

Flux

For the flux of the quantity through a surface we consider three different cases:

x-Flux

Take again a point in spacetime with coordinates (t_0, x_0, y_0, z_0) . Keeping x_0 fixed, imagine a very small rectangular surface centred at (x_0, y_0, z_0) , with sides aligned with the coordinate axes y, z and of lengths $\Delta y, \Delta z$. Imagine that this surface exists for a lapse of time Δt around the time t_0 . Then we have

$$\left. \begin{array}{l} \text{amount of quantity} \\ \text{through rectangle } \Delta y \Delta z \\ \text{towards positive } x \\ \text{during time lapse} \end{array} \right\} = j_x(t_0, x_0, y_0, z_0) \Delta t \Delta y \Delta z . \quad (4.3)$$

where $j_x(t_0, x_0, y_0, z_0)$ is the **x-flux** at the spacetime point (t_0, x_0, y_0, z_0) .

The **x-flux** tells us how much quantity is flowing in a unit of time through a unit of surface parallel to y, z . Also the **x-flux** is a function of the coordinates, because the flux could be larger through some surfaces at some times, than through other surfaces at other times. If the quantity is

flowing in the negative x direction, then j_x will be negative; and clearly if no quantity is flowing through the small rectangle, then $j_x = 0$.

In an analogous way we define a y -flux and a z -flux:

y -Flux

$$\left. \begin{array}{l} \text{amount of quantity} \\ \text{through rectangle } \Delta z \Delta y \\ \text{towards positive } y \\ \text{during time lapse} \end{array} \right\} = j_y(t_0, x_0, y_0, z_0) \Delta t \Delta z \Delta y . \quad (4.4)$$

z -Flux

$$\left. \begin{array}{l} \text{amount of quantity} \\ \text{through rectangle } \Delta x \Delta y \\ \text{towards positive } z \\ \text{during time lapse} \end{array} \right\} = j_z(t_0, x_0, y_0, z_0) \Delta t \Delta x \Delta y . \quad (4.5)$$

The three fluxes can be grouped into the **flux vector**

$$\mathbf{j}(t, x, y, z) := (j_x(t, x, y, z), j_y(t, x, y, z), j_z(t, x, y, z)) \quad (4.6)$$

In order to calculate the total amount of the quantity through an arbitrary sequence of 2D surfaces during a time lapse, we simply divide the total time into very brief time lapses, and for each of these we approximate the surface with very small rectangles aligned along the three axes. For each such lapse and rectangle we calculate the respective flux amount, using j_x , j_y , or j_z depending on the orientation of the rectangle. Then we sum up all these amounts.

Curiosity 4.2 T

This is also the description of an integration, which can be written

total amount of quantity through sequence of surfaces =

$$\begin{aligned} & \iiint j_x(t, x, y, z) dt dy dz + \\ & \iiint j_y(t, x, y, z) dt dz dy + \\ & \iiint j_z(t, x, y, z) dt dx dy \quad (4.7) \end{aligned}$$

We shall not have to calculate density integrals (4.2) and flux integrals (4.7) in these notes. In concrete physics problems these

integrals are often difficult to calculate, and require advanced mathematical techniques and specialized software. But if you'll ever end up working in fields such as computational fluid dynamics, atmospheric or ocean modelling, or numerical relativity, then you shall probably encounter them again.

The definitions above of density and flux are appropriate if the quantity in question is a scalar. For vector quantities such as momentum and angular momentum, the density is a vector, and each of the three fluxes is a vector as well.

Chapter 5

Physical laws

5.1 Two kinds of laws

In the previous chapter we introduced the main quantities in term of which we describe physical phenomena. How do we describe the patterns, regularities, and diversities that we observe in these phenomena? This is the role of physical laws. Different physics formalisms or languages express these laws in different way, even if the actual physical phenomenon is exactly the same.

In the formalism we're using we distinguish between two kinds of physical laws:

- **Balance laws**
- **Constitutive relations**

Balance laws, or simply balances, have this name because they express a sort of trade-off or "budget". A special kind of balance laws are called **conservation laws**. We shall see that there's a set of balances which apply to *all* physical phenomena, without exceptions; they represent the basic universal patterns that we observe everywhere.

Constitutive relations instead only apply to particular phenomena and only in particular theories. They express the diversity that we observe around us.

5.2 Balance and conservation laws

Balance laws are intuitively and visually easy to grasp. The archetype is as follows:

Consider a closed 2D surface at a (coordinate) time t_1 , which encloses a 3D region. A spherical surface delimiting a ball-shaped region is an example. Actually the surface doesn't need to be connected: it could consist of several separate closed surfaces, delimiting distinct 3D regions.

Starting from that particular time, imagine the closed surface moving and deforming, changing shape and size but remaining closed – no holes or cuts – through time, until a later time t_2 . At this time we shall have a closed surface that can be different from the initial one, and at a different position. We could start with a spherical surface delimiting a region the size of a tennis ball, and end with a cubical surface delimiting a region the size of the Sun.

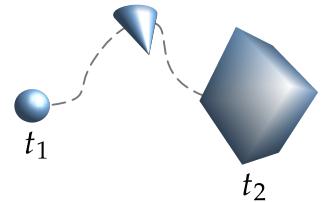
The initial and final surfaces and the sequence in between are completely arbitrary. We choose them depending on the phenomenon that we want to study. They do not need to be the surfaces of real objects. For instance we can consider an imaginary cubical surface in the middle of a room, enclosing the air within; or in the middle of a river, enclosing, for an instant, the water within. But sometimes these arbitrary surfaces do coincide with the boundaries of objects. For instance we can consider the outer surface of a tennis ball, and follow it as the ball moves.

Now consider any one of the seven primitive quantities of § 3.1, except magnetic flux. Remember the two questions or measurements that we could make on such quantity: *how much in this region?* and *how much through this surface during this time?* We therefore make the following measurements:

- N_1 : Total amount of quantity contained within the surface at time t_1 .
- J : Total amount of quantity flowing *inwards* through the moving surface between times t_1 and t_2 .
- N_2 : Total amount of quantity contained within the surface at time t_2 .

Each of the three amounts above is either a scalar of appropriate physical dimensions that can be positive, negative, or zero; or a vector. For instance, for matter we could have $N_1 = 3.5 \text{ mol}$; for energy we could have $J = -600 \text{ J}$; for momentum we could have $\mathbf{N}_2 = (-3.4, 0, +5) \text{ kg} \cdot \text{m/s}$.

With these definitions we can introduce two kinds of balance laws.



Conservation law

A quantity is said to be **conserved**, or to satisfy a **conservation law**, if the following equality holds:

$$N_2 - N_1 - J = 0 \quad (5.1)$$

A conservation law allows us to make several kinds of predictions or deductions. For example:

- If we know the amount of quantity within the surface at t_1 and the amount that flowed in through the surface between t_1 and t_2 , then we can predict the amount within the surface at t_2 :

$$N_2 = N_1 + J .$$

- If we know the amount of quantity within the surface at t_1 and the one at t_2 , then we can deduce the total amount that flowed in through the surface between the times t_1 and t_2 :

$$J = N_2 - N_1 .$$

- If we know the amount of quantity within the surface at t_2 and the amount that flowed in through the surface between t_1 and t_2 , then we can deduce the amount within the surface at t_1 :

$$N_1 = N_2 - J .$$

These kinds of predictions and deductions are very powerful: consider that the shape and motion of the surface, as well as the times t_1, t_2 , are completely arbitrary. Keep in mind that if the three quantities are scalars, then in general they can be positive, negative, or zero. For example, in the case of matter, if the total amount flowed in through the surface is negative, $J < 0$, then we must also have $N_2 < N_1$.

We unconsciously use some conservation laws all the time in our everyday life. If you seal a glass in a box, and upon dropping the box you hear that the glass shatters, and see that no holes were formed in the box, then you are sure that all the glass pieces now in the box could be glued together to form the original glass in its entirety.

Our intuitive understanding of a conserved quantity is that it cannot be “created” or “destroyed”. This leads to the definition of the more general balance law:

Balance law

A quantity is said to be only **balanced**, or to satisfy a **balance law** or simply **balance**, if the following equality holds:

$$N_2 - N_1 - J = R \quad (5.2)$$

where R is the amount of quantity created within the sequence of surfaces between the times t_1 and t_2 .

If the quantity is a scalar, then R can also be negative, in which case we say that some of the quantity has been destroyed.

A balance law allows us to make several kinds of predictions or deductions as well. For example:

- If we know the amount of quantity within the surface at t_1 , the amount that flowed in through the surface between t_1 and t_2 , and the amount that was created within the surface between t_1 and t_2 , then we can predict the amount within the surface at t_2 :

$$N_2 = N_1 + J + R .$$

And similarly as in the other examples for a conservation law. You see that in order to use a balance we need an extra piece of information.

A balance law is somehow more trivial than a conservation law: if we measure that $N_1 - N_2 + J$ is not zero, we can always say that some amount of quantity must have been created or destroyed between the two times; so a quantity might always be said to satisfy a balance. But a balance is not trivial if we have some specific physical law that tells us in advance how the amount created or destroyed, R , can be calculated.

A conservation law can be seen as a special and powerful case of a balance law, one for which $R = 0$ *always*.

Curiosity 5.1 C

onservation and balance laws appear even simpler from the point of view of Relativity Theory. From a four-dimensional spacetime perspective, a 3D volume is a region, called *hypersurface*, having one less dimension than spacetime. But a sequence of 2D surfaces through time is also just a region having one less dimension than spacetime – two spatial dimensions and one temporal one. Thus the distinction between the 3D region at t_1 , the sequence of 2D surfaces

between t_1 and t_2 , and the 3D region at t_2 disappear: they are seen to be just different parts of the same three-dimensional hypersurface. We perceive some parts of this hypersurface as belonging “to the same time”, showing their three dimensions all at once; and other parts as extending through time, showing only two dimensions at any time. In fact, different observers make this division in different ways.

And from a spacetime perspective, the amount of a quantity N_1 or N_2 within a 3D region is seen as a flux through time; so its apparent difference from the flux J also disappear. Their only difference is that one (C) points exclusively in the time direction, while the other (J) also points partially in a spatial direction.

Curiosity 5.2 I

In the case of magnetic flux, the idea of a conservation law is analogous, but is formulated with one less spatial dimension: we consider a closed 1D curve at t_1 , one at t_2 , and a sequence in between these times. The magnetic flux turns out to be a quantity for which it's possible to ask how much is “linked” to a closed curved, and how much is crossing a closed curve. One way to understand this is to imagine magnetic flux as a bundle of tubes or lines that are either closed or extend to infinity. It is a very fascinating quantity, and one may wonder if other quantities exist which satisfy similar balances even with one less dimension. But we shall not pursue magnetic flux or similar topics in these notes.

5.3 Balance laws expressed with derivatives

In § 4 we saw that each of the main seven quantities is represented by a density and by a flux vector, which allow us to calculate the amount of quantity within a 3D region at a particular time, and flowing through a 2D surface during an time interval. These amounts are exactly what N_1 , N_2 , J express. Therefore, a conservation or balance law must also give some relationship between density and flux vector.

The way to find such relationship is intuitive and mathematically simple, if only a bit lengthy. The idea is to formulate the balance law for a very simple case, the same simple set-up that we used in § 4 to introduce density and flux.

Take a point in spacetime with coordinates (t, x, y, z) . We apply the conservation law (5.1) to the following situation:

N_1 : Choose an initial time $t_1 = t - \frac{\Delta t}{2}$, slightly before t , and a cuboid region centred at (x, y, z) with sides $\Delta x, \Delta y, \Delta z$. The total amount of the quantity within this region is given, adapting formula (4.1), by

$$N_1 = n(t - \frac{\Delta t}{2}, x, y, z) \Delta x \Delta y \Delta z . \quad (5.3)$$

N_2 : Choose a final time $t_2 = t + \frac{\Delta t}{2}$, slightly after t , and a cuboid region centred at (x, y, z) with sides $\Delta x, \Delta y, \Delta z$. The total amount of the quantity within this region is given, adapting formula (4.1), by

$$N_2 = n(t + \frac{\Delta t}{2}, x, y, z) \Delta x \Delta y \Delta z . \quad (5.4)$$

J : We calculate the flux separately through the six rectangular surfaces bounding the small cuboid region:

J_x : First choose a rectangular surface centred slightly to one side of the point (x, y, z) : at $(x - \frac{\Delta x}{2}, y, z)$, with sides parallel to y, z and lengths $\Delta y, \Delta z$. We keep this surface constant for the small duration Δt . The x -flux through this rectangle, towards positive x , during Δt , is according to formula (4.3)

$$J_x^{\text{in}} = j_x(t, x - \frac{\Delta x}{2}, y, z) \Delta t \Delta y \Delta z .$$

Now choose the rectangular surface parallel to the previous one, but on the other side of the point (x, y, z) , centred at $(x + \frac{\Delta x}{2}, y, z)$. The x -flux through this rectangle, towards positive x , during Δt , is

$$J_x^{\text{out}} = j_x(t, x + \frac{\Delta x}{2}, y, z) \Delta t \Delta y \Delta z .$$

Note that the flux J_x^{in} points to the interior of the 3D region, towards its centre (x, y, z) ; whereas J_x^{out} points to the exterior, away from the centre. The total flux into the region is therefore given by their difference:

$$\begin{aligned} J_x &= J_x^{\text{in}} - J_x^{\text{out}} \\ &= [j_x(t, x - \frac{\Delta x}{2}, y, z) - j_x(t, x + \frac{\Delta x}{2}, y, z)] \Delta t \Delta y \Delta z \end{aligned} \quad (5.5)$$

J_y, J_z : An analogous reasoning can be made to find the total flux through the other four surfaces, two parallel to z, y and two to x, y . We find

$$\begin{aligned} J_y &= J_y^{\text{in}} - J_y^{\text{out}} \\ &= [j_y(t, x, y - \frac{\Delta y}{2}, z) - j_x(t, x, y + \frac{\Delta y}{2}, z)] \Delta t \Delta z \Delta x \end{aligned} \quad (5.6)$$

$$\begin{aligned} J_z &= J_z^{\text{in}} - J_z^{\text{out}} \\ &= [j_z(t, x, y, z - \frac{\Delta z}{2}) - j_x(t, x, y, z + \frac{\Delta z}{2})] \Delta t \Delta x \Delta y \end{aligned} \quad (5.7)$$

The total flux into the region during the time interval Δt is finally the sum of the three above:

$$J = J_x + J_y + J_z \quad (5.8)$$

Up to now we have only chosen arbitrary surfaces, regions, and a time interval. Now let's assume that the corresponding amounts and fluxes satisfy a balance law like (5.1). Putting together the puzzle pieces we find

$$N_2 - N_1 - J = 0 \quad \Rightarrow$$

$$\left. \begin{aligned} &n(t + \frac{\Delta t}{2}, x, y, z) \Delta x \Delta y \Delta z - n(t - \frac{\Delta t}{2}, x, y, z) \Delta x \Delta y \Delta z \\ &- [j_x(t, x - \frac{\Delta x}{2}, y, z) - j_x(t, x + \frac{\Delta x}{2}, y, z)] \Delta t \Delta y \Delta z \\ &- [j_y(t, x, y - \frac{\Delta y}{2}, z) - j_x(t, x, y + \frac{\Delta y}{2}, z)] \Delta t \Delta z \Delta x \\ &- [j_z(t, x, y, z - \frac{\Delta z}{2}) - j_x(t, x, y, z + \frac{\Delta z}{2})] \Delta t \Delta x \Delta y \end{aligned} \right\} = 0$$

This expression is quite long, but it should be intuitively understandable if you try to identify the individual summed terms.

Now we take the last, long expression – which is an equality – and divide its left and right sides by $\Delta t \Delta x \Delta y \Delta z$. Note that many of the “ Δ ” factors simplify out. We arrive at this:

$$\left. \begin{aligned} &\frac{n(t + \frac{\Delta t}{2}, x, y, z) - n(t - \frac{\Delta t}{2}, x, y, z)}{\Delta t} \\ &+ \frac{j_x(t, x + \frac{\Delta x}{2}, y, z) - j_x(t, x - \frac{\Delta x}{2}, y, z)}{\Delta x} \\ &+ \frac{j_y(t, x, y + \frac{\Delta y}{2}, z) - j_y(t, x, y - \frac{\Delta y}{2}, z)}{\Delta y} \\ &+ \frac{j_z(t, x, y, z + \frac{\Delta z}{2}) - j_z(t, x, y, z - \frac{\Delta z}{2})}{\Delta z} \end{aligned} \right\} = 0 .$$

Still a long expression; but examine each fraction: we have a difference between a term calculated at some point, and one calculated at some point plus an increment “ Δ ”; this difference is then divided by that increment. This is the definition of *partial derivative*, that is, the derivative with respect to one variable, while keeping the other variables fixed. The short symbol for this is “ $\frac{\partial \dots}{\partial \dots}$ ”, so we can simply write

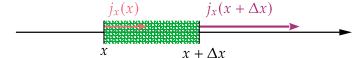
Conservation law in derivative form

$$\frac{\partial n}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0 \quad (5.9)$$

Let's make a simple check to see if this formula makes sense, in a simple one-dimensional case where we disregard the y, z coordinates (let's say their fluxes are zero), so that the balance law becomes

$$\frac{\partial n}{\partial t} + \frac{\partial j_x}{\partial x} = 0.$$

Suppose that the derivative of the x -flux is positive: $\frac{\partial j_x}{\partial x} > 0$. This means that the x -flux increases as we move from x to $x + \Delta x$, as shown in the side figure. The x -flux at x is bringing some amount of quantity into the region of interest, the x -flux at $x + \Delta x$ is taking a larger amount out of that region. Therefore, during the short time Δt in which this flux takes place, the total amount of quantity within the region will decrease. But the total amount is given by $n(t)$, which is therefore decreasing with time. This means that its time derivative is negative: $\frac{\partial n}{\partial t} < 0$. This agrees with balance law above: the sum of the two derivatives must be zero, so if one is positive, the other must be negative.



Balance laws like (5.9) are the basis of many important computational and simulation methods in a huge variety of physical applications: simulation of the ocean around an offshore oil platform, of wind in a wind farm, of air around an aeroplane's wing, of earthquakes, of electromagnetic-wave propagation, of oscillations in a suspension bridge, of percolation of fluids through terrain, of chemical reactions, of energy transfer... The list could go on for pages!

The basic idea for using a balance law for a simulation is easy to understand. Take again the simpler one-dimensional formula without y, z coordinates, and rewrite it remembering the basic meaning of the

derivative:

$$\frac{n(t + \Delta t, x) - n(t, x)}{\Delta t} + \frac{j_x(t, x + \frac{\Delta x}{2}) - j_x(t, x - \frac{\Delta x}{2})}{\Delta x} = 0$$

Suppose we know all quantities at time t ; we can then find the density n at time $t + \Delta t$. Leave it on the left side and bring all other terms on the right side, multiplying them by Δt :

$$n(t + \Delta t, x) = \Delta t \left[n(t, x) - \frac{j_x(t, x + \frac{\Delta x}{2}) - j_x(t, x - \frac{\Delta x}{2})}{\Delta x} \right]$$

which is essentially [Euler's method](#)¹.

This formula can then be used iteratively to find the density n at later times. As a simple simulation set-up, we consider a region of space of interest and divide it into cells of width Δx . We also treat time in steps of Δt . Then the iterative procedure goes as follows:

1. *Initial values:* Assign the value of n at each x -cell at the initial time.
2. *For-loop in x :* for each x -cell, calculate the value of n at $t + \Delta t$, using the formula above.
3. Increase time by Δt .
4. Go to 2.

For example, suppose we have two x -cells of width $\Delta x = 1$: one centred at x_1 and the other at $x_2 = x_1 + 1$. There are three boundaries: the leftmost at $x_1 - \frac{1}{2}$, one between the cells at $x_1 + \frac{1}{2} \equiv x_2 - \frac{1}{2}$, and the rightmost at $x_2 + \frac{1}{2}$. Then we have the following steps:

1. Assign the values $n(t_0, x_1), n(t_0, x_2)$ of n at x_1 and x_2 at time t_0 .
2. Assign the values

$$j_x(t_0, x_1 - \frac{1}{2}), \quad j_x(t_0, x_1 + \frac{1}{2}), \quad j_x(t_0, x_2 + \frac{1}{2})$$

of the fluxes at the three boundaries at time t_0 .

3. Calculate the values of n at x_1 and x_2 at time $t_1 = t_0 + \Delta t$, for instance

$$n(t_1, x_1) = \Delta t \left[n(t_0, x_1) - \frac{j_x(t_0, x_1 + \frac{1}{2}) - j_x(t_0, x_1 - \frac{1}{2})}{\Delta x} \right].$$

Note that we have all quantities required on the right side.

4. Assign the values

$$j_x(t_1, x_1 - \frac{1}{2}), j_x(t_1, x_1 + \frac{1}{2}), j_x(t_1, x_2 + \frac{1}{2})$$

of the fluxes at the three boundaries at time t_1 .

5. Calculate the values of n at x_1 and x_2 at time $t_2 = t_1 + \Delta t$, for instance

$$n(t_2, x_1) = \Delta t \left[n(t_1, x_1) - \frac{j_x(t_1, x_1 + \frac{1}{2}) - j_x(t_1, x_1 - \frac{1}{2})}{\Delta x} \right].$$

Note that we have again all quantities required on the right side.

6. And so on.

This routine requires us *to know the flux at each x -cell and at each time*. But we shall soon see that this requirement can be bypassed.

The rudimentary routine above is the basis from which more precise and complex simulation routines are developed.

Exercise 5.1

Write a script, in your preferred programming language, that implements the simulation routine above in an x -grid with four cells $x = 1, \dots, x = 4$. Take a grid size $\Delta x = 1$ and time step $\Delta t = 1$. We can imagine that the quantity in question is electric charge. Use the following initial values for $n(t, x)$ at $t = 0$:

$$n(t=0, x=1) = 7, \quad n(t=0, x=2) = 0, \quad n(t=0, x=3) = 0, \quad n(t=0, x=4) = 7$$

and assume the flux is, at each time t :

$$\begin{aligned} j_x(t, x=0.5) &= 0, \\ j_x(t, x=1.5) &= +2, \quad j_x(t, x=2.5) = 0, \quad j_x(t, x=3.5) = -2, \\ j_x(t, x=4.5) &= 0. \end{aligned}$$

Questions and tasks:

1. Which time evolution for n do you observe? Does it make sense, given the fluxes above?
2. Feel free to try out or generalize your script with:
 - different initial values
 - different fluxes
 - an explicit formula for the flux, for instance

$$j_x(t, x) = 8x^3 - 60x^2 + 118x - 45$$

- time-dependent fluxes
- more cells
- two or three dimensions, including y - or z -fluxes

5.4 Constitutive relations

In the previous section we saw that a balance law can be used to predict how the density of a quantity changes with time. But we also saw that in order to make such prediction we need to already know what the *future* fluxes of the quantity will be, throughout the region of interest. This doesn't sound like a great achievement: to predict the future, we need the future. The situation doesn't improve if we have two or more quantities that satisfy balance laws: we shall need to know in advance the fluxes of each.

It turns out, however, that there are physical laws which *relate the densities of some quantities to the fluxes of others*. Such relations can potentially allow us to calculate the time evolution without knowing the fluxes in advance. Roughly, the basic idea is as follows: From the density of one quantity, we calculate the fluxes of the second quantity, and use them to predict its density at a later time. Once we know this density, we use it to calculate the new fluxes of the first quantity, and therefore its density at a later time; and so on. This cross-prediction does not fully eliminate the necessity of specifying some future quantities in advance; but the remaining ones that we need to specify are often in our control. Indeed this is how we can predict how a physical system will respond to influences controlled by us.

Physical laws of this kind turn out not to be universal: they depend on, or are '[constitutive](#)'² of, the particular physical phenomenon and the physical theory being used. For this reason they are called **constitutive relations** or **constitutive equations**. In some fields they are called **closure equations**, because they allow us to "close" the system of balance equations in such a way that it can be used for future predictions without knowing in advance the future value of some quantities.

Constitutive relations express the diversity that we observe around us, for example the different behaviours of a drop of water, which obeys some constitutive relations, as compared with a block of wood, which obeys others. They also mark the difference between specialized or approximate

physical theories, for example between Newtonian mechanics, which is based on particular approximate constitutive relations, and general relativity, which is based on different and more exact constitutive relations. Depending on the specific scientific field you'll work in, you'll learn some constitutive relations in more detail than others.

Constitutive relations come in a great variety of mathematical forms. Some of them are simple algebraic relations between the density of one quantity and the flux of another. Others involve spatial or time derivatives. Other still involve integrals in space or in time.

☒ Add examples of laws that reduce to the balances: Bernoulli, Poynting?, mechanical-energy-work theorem

URLs for chapter 5

1. <https://mathworld.wolfram.com/EulerForwardMethod.html>
2. <https://www.merriam-webster.com/dictionary/constitutive>

Chapter 6

The Seven Wonders of the World

We already remarked that the seven primitive quantities introduced in §3.1 are common to all our main physical theories. Even more remarkable is that each of them obeys a balance law like (5.2); three actually obey a strict conservation law like (5.1).

These seven balance laws are known, so far, to be satisfied by *all* phenomena and in all our main physical theories, approximate or not: from subatomic scales to cosmological scales, from Newtonian mechanics to general relativity and quantum theory. No exceptions are known.

The seven fundamental balances

- Conservation of matter
- Conservation of electric charge
- Conservation of magnetic flux
- Balance of energy
- Balance of momentum
- Balance of angular momentum
- Balance of entropy

These balances are truly the “Seven Wonders of the World”¹.

Once you learn these balances, you can apply them to any kind of physical phenomenon or problem you work with: construction of bridges, control of chemical reactions, operation of GPS navigation and satellites,

¹ Compare with the traditional seven wonders: <https://www.britannica.com/topic/Seven-Wonders-of-the-World>, <https://education.nationalgeographic.org/resource/seven-wonders-ancient-world/>.

monitoring of nuclear power plants, sending robots to Mars, or collisions of subatomic particles.

Let us say something more about each.

6.1 Conservation of matter

More precisely: matter, electric charge, magnetic flux always satisfy strict conservation laws. For matter we actually have several conservation laws, one for each kind of matter. Energy, momentum, angular momentum, entropy satisfy balance laws, which become conservation laws only in special circumstances, but not always.

Chapter 7

Matter

Matter is probably the easiest quantity to grasp intuitively; it's what we can really call "stuff". It is important to clearly distinguish matter from *mass*: mass can be considered a property of matter, but the two are different: for example, the mass of some amount of matter may change, even if the amount of matter stays the same.

There are different kinds of matter, each of which satisfies a balance relation. The distinction into different kinds depends on the physical theory. In most everyday situations, the distinction corresponds to different [chemical elements¹](#), and each satisfies its own balance. These balances are the basis of [stoichiometry²](#). If we observe phenomena such as nuclear fission or fusion, however, we notice that the balances of chemical elements are not really satisfied. With such phenomena we make a different distinction of types of matter, for instance *baryons* and *leptons*, and each satisfies again its own balance. It is unclear whether these balances might be broken in other physical phenomena at smaller scales. In these notes we shall usually consider chemical elements as the different kinds of matter, making some exceptions in discussion of nuclear phenomena.

7.1 Electric charge and magnetic flux**7.2 Magnetic flux****7.3 Energy****7.4 Momentum****7.5 Angular momentum****7.6 Entropy**

URLs for chapter 7

1. <https://doi.org/10.1351/goldbook.C01022>
2. <https://doi.org/10.1351/goldbook.S06026>

Chapter 8

Thermodynamics

8.1 Notes on “quasi-static” processes

In general, none of the statements "reversible \Rightarrow quasi-static" or "quasi-static \Rightarrow reversible" is true.

A counterexample to the second implication are systems with internal state variables, which cannot be made non-dissipative, no matter how slowed-down they are. See the discussion and mathematical analysis in Astarita § 2.5.

A counterexample to the first implication is a system of spins in a crystal lattice. It is possible to *reversibly* bring the system from an equilibrium state to another with opposite temperature by reversing the external magnetic field *as fast as possible* – and therefore *not* through a quasi-static process. In fact it is key here that the process be *not-quasi-static*, but as fast as possible, because a slow change of the external magnetic field would lead to an irreversible process with dissipation. For more details see the discussion in Buchdahl, Lecture 20.

The point is that for some systems a *fast* change can actually prevent the onset of dissipative phenomena, and so the process needs to be fast if we want it to be reversible. Adiabatic processes often also need to be fast (as a curious historical fact, Truesdell & Bharatha, Preface p. xii, remark that “In introducing what we today call an ‘adiabatic process’, Laplace called it ‘a sudden compression’, in which he was followed by Carnot”).

In fact, clearly non-quasi-static phenomena such as *explosions* can in some circumstances be described by *reversible* processes! This is possible if the explosion involves many shock waves, as explained by Oppenheim, chap. 1 p. 63:

If there is more than one shock, the losses in available energy are diminished, so that in the limit, with an infinite number of shocks, they become negligible, and the process acquires the character of a thermodynamically optimal, i.e., reversible, change of state. The study of explosion processes reveals that, indeed, they are associated not with one but with a multitude of shocks.

For explosions see also the mathematical analysis by Dunwoody: *Explosion and implosion in a mixture of chemically reacting ideal gases*, where again reversible-process equations are used.

A caveat about reversible and quasi-static associations is given by Erickson (§ 1.2):

Some tend to associate nearly reversible processes with those taking place very slowly – the "quasi-static" processes. This probably stems, at least in part, from experience with classical theories of heat conduction, viscosity, and so on. However, a ball made of silly putty behaves almost reversibly when bounced rapidly and various other high polymers have similar predilections. So, it seems prudent to be open-minded in considering what may be reversible processes for particular systems.

He later discusses (§ 3.1) the case of bars subjected to dead loads, for which we can have reversible processes under sudden jumps in elongation. He concludes (p. 46) that "the sudden jump provides an example of a process that is reversible but not reasonably considered to be quasi-static".

But there's an important question that underlies our discussion: what do we actually mean by "quasi-static"? We need to specify a time scale, otherwise the term is undefined. For example, a geological process (say, tectonic motion) can be considered as quasi-static – or even completely static – on time scales of minutes or days; but it is not quasi-static on time scales of millions of years.

Whether a process is reversible or not, within any tolerance needed, is an experimental question. We can measure any relevant quantities, say pressure p and exchanged heat q , under the process, and compare them with those, p^* and q^* , determined by the equations for a reversible process. We may find for example that at all times

$$\left| \frac{p - p^*}{p^*} \right| < 0.001 , \quad \left| \frac{q - q^*}{q^*} \right| < 0.001$$

and conclude that the process is reversible, if relative discrepancies of 0.1% or less are negligible in our concrete application.

But suppose that someone tells us "if you want the process to be reversible, you must make sure that it is quasi-static". Alright, but how much is "quasi-static"? Is it OK if the piston moves with a speed of 1 cm/s? or is that too much? How about 1 mm/s? – In fact we may find that for some kind of fluid 1 cm/s is absolutely acceptable for the process to be reversible, whereas for another kind of fluid that speed would lead (at the same temperature) to an irreversible process.

You see how this imprecise situation can lead to circular definitions: "if the process is irreversible, then it means it isn't quasi-static" – but then we are actually *defining* "quasi-static" in terms of "reversible"! Any statement of the kind "reversible \Rightarrow quasi-static" or "quasi-static \Rightarrow reversible" then becomes not a matter of experimental verification, but of pure *semantics*. At this point we can simply get rid of "quasi-static" terminology since it doesn't bring any new physics to the table. This circularity is admitted for example by Callen in discussing irreversible gas expansion (Problem 4.2-3 p. 99):

The fact that $dS > 0$ whereas $dQ = 0$ is inconsistent with the presumptive applicability of the relation $dQ = T dS$ to all quasi-static processes. We define (by somewhat circular logic!) the continuous free expansion process as being "essentially irreversible" and *non-quasi-static*.

A similar criticism can be read in Astarita, § 2.9, p. 62, where he also provides a mathematical quantification of quasi-static, similar to the one given above for reversibility:

Often this point is circumvented by bringing in another difficult concept, that of a quasi-static transformation, which proceeds "through a sequence of equilibrium states." Quasi-static is an impressive word, but the only meaning which can be attached to it is the less impressive word "slow" – and how can one speak of slowness without implying the concept of time? How slow is slow enough? If one chooses to develop a thermodynamic theory (rather than a thermostatic one), the answer is easy. For instance, in the case of a system where the state is V, T, \dot{V} [the latter is the rate of change of V], one needs to assume

that [the non-equilibrium pressure] $p(V, T, \dot{V})$ is a Taylor-series expandable at $\dot{V} = 0$ to obtain [that

$$p = p^* + \frac{\partial p}{\partial \dot{V}} \Big|_{\dot{V}=0} \dot{V} + O(\dot{V}^2),$$

where $p^* = p(V, T, 0)$ is the pressure at equilibrium]. One then reaches the conclusion that if the condition

$$\dot{V} \ll \frac{p^*}{\partial p / \partial \dot{V} \Big|_{\dot{V}=0}}$$

is satisfied, then indeed the difference between p and p^* is negligibly small as compared to p^* , and thus the process can be regarded as a quasi-static one.

Criticisms against the fuzzy notion of "quasi-static" have appeared in many other works. Truesdell & Bharatha (Preface p. xii), make the historical remark that "the 'quasi-static process' was barely mentioned for the first time in 1853 and was altogether foreign to the early work [in thermodynamics]". See also the mathematical analysis by Serrin: *On the elementary thermodynamics of quasi-static systems and other remarks*.

I also want to point out that "quasi-static" in some works has specific meanings somewhat unrelated to the discussion above. For example that the rate of increase of the total kinetic energy K of the system is negligible, so that the law of energy balance, which in its full generality is

$$\frac{d(U + K)}{dt} = Q + W$$

(that is, the rate of increase of internal energy U and kinetic energy is equal to the heat rate Q and work rate W provided to the system) can be approximated by

$$\frac{dU}{dt} = Q + W.$$

Or that similar inertial terms in the motion of the system are negligible. See for example the book by Day, chap. 2.

But note that such definitions of "quasi-static" have, again, *no* a-priori relation with reversibility.

Finally, the equation $dS = Q/T$ is only valid for a process that is:

- reversible (by definition),

- closed (no exchange of mass),
- with a homogeneous surface temperature,
- without bulk heating (such as instead happens in a microwave oven).

Under the last three conditions we have in general that $dS \geq Q/T$; when the equality sign is satisfied, then the process is *defined* as reversible. See Astarita, § 1.5, or Müller & Müller, for the different forms of the second law under different circumstances. This equation may be valid in quasi-static and non-quasi-static processes, as explained above.

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(“de X” is listed under D, “van X” under V, and so on, regardless of national conventions.)

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