Introduction to 21st-century physics

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Lecture notes on introductory mechanics and thermodynamics (ING175)

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0 Introduction

The loss implied in such an acquisition can be estimated only by those who have been compelled to unlearn a science that they might at length begin to learn it.
J. C. Maxwell 1878

Until some decades ago, the 18th-century physical notions typically taught in introductory Bachelor physics courses were enough to prepare an engineer for future specializations and jobs. Students who wanted to venture into modern theories, such as Relativity, were required to **re-learn** some of the most important physical notions – *Energy*, mass, time, entropy above all – which in these theories are quite different from the 18th-century ones. But at that time the modern theories still mostly had only theoretical, not practical, importance. So the re-learning efforts of the curious students could perhaps be justified.

"The achieved performances of atomic clocks and time transfer techniques imply that the definition of time scales and the clock comparison procedures must be considered within the framework of general relativity"
Petit & Wolf 2005

That situation has changed today. Modern theories are an essential part of many everyday technologies, like nuclear reactors and the Global Positioning System[1]; and they are required for developing new technological possibilities, from quantum computers[2] to solar sails[3]. An engineering student (including communication and data engineering) may likely end up in a job that requires an understanding of modern physical notions. The diffusion of large language models[4] will moreover require future engineers who actually **understand** those physical notions, not little monkeys who have been trained to manipulate some equations and to throw some technobabble around. Automated large language models are faster, cheaper, and more precise in doing the latter kind of monkey activities. So why should one hire a human to do the same?

[1] `https://www.gps.gov`; see the entertaining discussion in Taylor & Wheeler 2000 Project A; and also Petit & Wolf 2005; Fliegel & DiEsposti 1996; Ashby 2002; Müller et al. 2008. [2] `https://www.ibm.com/topics/quantum-computing`. [3] see for instance `https://www.planetary.org/sci-tech/lightsail`, `https://www.cubesail.us`. [4] `https://www.ibm.com/topics/large-language-models`.

While moving from the older to the newer notions often requires re-learning efforts and conceptual re-orientations, the move in the opposite direction is less demanding. The modern physical notions are more encompassing than their 18th-century parents. Their understanding leads to an understanding of their older counterparts as approximate and special cases. Students, moreover, have often been hearing quite early from mass media about the new notions; for instance about the equivalence of mass and energy. Owing to this early exposure, students sometimes ask very intelligent questions – "*should the mass of the body be included in its internal energy?*" – when exposed to the old notions.

It is therefore high time that introductory Bachelor physics courses be based on modern physical notions. Students should not be required to waste time and mental effort to learn something that they must unlearn and re-learn, only because of academia's and teachers' inertia.

Some teachers say "it would be too difficult for students to understand modern ideas, because they are too familiar with the old ones. This is why we need to teach the old and slightly incorrect ideas first, and correct them later". I think that this kind of reasoning is scientifically unacceptable and leads to a vicious circle. Students are unfamiliar with new notions only because they were raised by a generation who was taught the old. New notions become familiar after a couple of generations learn them early. This is obvious if you consider that notions such as "energy", "electromagnetic field", "vector" are very familiar today, but were absolutely *un*familiar a couple centuries ago. If we had always taught what's familiar, then we would still be teaching about the elements *air, earth, water, fire*, and that the Sun revolves around the Earth. Arguments in favour of teaching old notions are just pretexts for laziness.

The present lecture notes are an experiment and attempt to introduce classical mechanics and thermodynamics from modern physical notions. The core equations remain the same, but the students should have a broader conception of their meaning and of the symbols that appear in them.

1 Physics?

If you think about it, many things we ordinarily do every day are just some sort of magic. Think of how you can instantaneously see and speak with a person living on another continent, in real time, using just a small widget in the palm of your hand. Think of how you can instantaneously see where you are on the Earth, using the same widget. Think of how fast you can go to another country, by flying in a huge metal thing. Think of how you can command and interact with a purely fictitious animated world when you play on your computer. The list can go on forever. Other things are luckily less ordinary, but still inspire a lot of awe: think of the devastating power unleashed by something roughly as small as a tennis ball, in an atomic bomb.

We can do these astonishing things thanks to our understanding of how the world works. That's Physics.

Many things can be said and have been said about science and physics. Rather than repeating what's been already written in many excellent books, I invite you to take a break here and go read their introductions. Choose as you please; don't limit yourself to popular books; compare what they say.

. . . .

1.1 Several possible formalisms or "languages"

Welcome back! The only thing about physics that I need to say here is that physics can be expressed and written from wildly different points of view, using wildly different principles. Let's call these "different physics languages"; a more technical name is "physics formalisms". One may approach a physics phenomenon or problem in terms of *Lagrangeans*, or *Hamiltonians*, or *fibre bundles*, or *categories*, or *action principles*, or many other formalisms. These formalisms or languages are not completely separated; we know how to translate among them. In "doing" physics, one may jump among formalisms, because some ideas may be easier to express, or some results easier to find, in one formalism than another. No matter which physics

$$\delta\int L\,\mathrm{d}t = 0 \quad L = \frac{1}{2}m\,v^2$$

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}m\,\boldsymbol{v} \quad \boldsymbol{F} = 0$$

Example of two different languages expressing the same physical phenomenon formalism you choose, the results and the concrete applications are still the same. The choice is to a great extent subjective, based on your aesthetic tastes. You see that in "doing" physics you can express your personality and put your own artistic touch; this is why it's such a cool subject (and other subjects are like this too).

In these notes I'm choosing one particular formalism: the one that for me is the most easily *visualizable*; because I believe that visualization can be beneficial in learning new things. Or maybe I'm choosing it just because I like it best. I encourage you to explore how the physics you've learned is expressed in other physics formalisms; maybe you'll like another physics formalism better.

The formalism we'll be using might be called "field theory". Roughly speaking it takes as starting point the ideas of space and time, or better spacetime, in which there are different kinds of "stuff". It expresses the regularity and patterns that we observe in physical phenomena as "budgets" about the different kinds of stuff, and of relations between these kinds. Please don't take the description just given too literally; it's just meant to give you a very vague idea of the field-theoretical viewpoint.

It goes without saying that all these "physics languages" are to a great extent mathematical.

"this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics" Galilei 1623

One reason is that numbers allow us to convey information in a concise and precise way. Imagine you have to tell someone, who doesn't know Bergen, where in Bergen you are right now, to within 10 m. You can do that with a description, "... and there's a building called so-and-so which looks like so-and-so...", which would be lengthy and tricky. Or you can just give two numbers: latitude and longitude:

$$60.369\,40, \ 5.3518 \ .$$

And in these two numbers all digits are important; for instance, the latitude is not 60.369 **47**.

But the most important reason is that mathematics allows us to describe and follow the patters and variety of physical phenomena in a greatly concise and precise way. And to develop

"There is nothing that can be said by mathematical symbols and relations which cannot also be said by words. The converse, however, is false. Much that can be and is said by words cannot successfully be put into equations, because it is nonsense." Truesdell 1966

their relationships in a rigorous way. All our present technology would have been impossible to discover, and would be impossible to realize, without the mathematical language of physics.

I invite you again to read what many good texts say about the relationship between physics and mathematics. No point repeating here what has been said better elsewhere.

1.2 Quantities, primitive and derived

One topic, however, must be briefly discussed because it's important in understanding the notes that follow. It's the distinction between *primitive* and *derived* quantities.

I shall assume that you already know what a *physical quantity* is. Examples are: position, duration, velocity, pressure, energy, temperature.

A *derived quantity* is one that is defined in terms of other quantities. For example, velocity $\boldsymbol{v}$ (more precisely: average velocity) is defined as the ratio between a distance $\boldsymbol{d}$ (a vector) and a time duration $\boldsymbol{t}$:

$$\boldsymbol{v} := \frac{\boldsymbol{d}}{t}$$

where the symbol ":=" means "is defined as" or "is defined by". This means that in principle we could avoid using the word "velocity" and the symbol "$\boldsymbol{v}$" altogether, and instead always speak about distance and duration, using their symbols. It would be extremely inconvenient, but it could be done. The definition of a derived quantity often tells us how that quantity can be measured.

A derived quantity is defined in terms of other quantities, and these may in turn be derived quantities, that is, defined in terms of still other quantities, and so on. But at some point this chain of definitions must come to an end, otherwise we would go around in circles.

A *primitive quantity* is one that we do not define in terms of other quantities. Primitive quantities are the building blocks from which we define all others. That they are not defined in terms of others doesn't mean that we cannot try to explain them. But such explanations must be taken as informal and heuristic. Primitive quantities are often explained through metaphors and by appealing to intuition. You must always be wary of such explanations, because they may fail you spectacularly in some situations.

Often we have a choice about which quantities should be primitive and which should be derived. For instance, *energy* can be defined, in a somewhat complicated way, in terms of quantities like *work* and *heat*, which would then need to be taken as primitive. Or we can take *energy* as primitive, and define *work* and *heat* in terms of it; this second choice turns out to be more convenient to develop the physical theory. It often happens that a quantity is very convenient to build a theory if used as primitive, but difficult to understand intuitively; and vice versa that a quantity is very intuitive, but leads to a complicated theory.

Among quantities which we'll take as primitive are *time*, *space*, *matter*, *energy*, *momentum*, and several others.

1.3 Physical dimensions and units

Measurement is the process by which we determine the value of a physical quantity. Measurements can be extremely complex, and can extremely different even if they are about the same quantity. Consider the ways we can measure the mass of a football, compared to the ways we can measure the mass of the Sun.

To each quantity we associate a **physical dimension**. The term 'dimension' here has nothing to do with physical extension, as in "the dimensions of this box"; be careful not to confuse the two. Usually it's clear which one is meant from the context. Physical dimensions help us avoid making operations that don't make sense with some quantities. For example, it doesn't make sense to sum up the volume of a body of water with its temperature; and indeed the volume has dimension length^3, whereas temperature has dimension *temperature*, and quantities with different dimensions cannot be added up.

With each physical dimension we can associate a measurement **unit**, which expresses a basic standard for comparing the measurement results of similar quantities. For example, we could use the *minute* or the *second* as units to measure *time*.

One can choose a basic set of physical dimensions from which to define all others, and for these a set of standard units. Here we shall follow the International System of Units (SI)[5].

The topics of measurement and physical dimensions, which are studied in *metrology* and in *dimensional analysis*, could occupy an entire

[5] https://www.nist.gov/pml/special-publication-811

course by themselves. I shall assume that you already know their basics notions and that you read about the SI.

The measurement of some physical quantities consists in just one number with associated physical dimension; we shall call such quantity a **scalar**. The measurement of other physical quantities consists instead in a triplet of numbers with associated physical dimension; we shall call such quantity a **vector**.

Note that these two terms have very specific and slightly different meanings in different theories, so don't take their definition used here as universal. For example, in these notes and in Newtonian mechanics we call *energy density* a scalar, but in general relativity it cannot be called a scalar.

2 Time, space, "stuff"

2.1 Time

Time is a primitive quantity. We understand the notion of time intuitively, even if it's difficult to explain (that's why it's taken as primitive). In 1905, with the theory of relativity, part of our everyday intuition about this notion was seriously shaken. For many years afterwards our old intuition could still be used in practice and in applications. But the new, correct intuition is becoming more and more important in everyday life and technologies. For example, GPS navigation, which we use everyday from leisure activities such as hiking or sightseeing to more critical ones such as aeroplane landing, critically depend on the correct notion and intuition of time. Let's see how our traditional intuition goes astray with a concrete experiment.

"In 1976, the International Astronomical Union introduced relativistic concepts of time and the transformations between various time scales and reference systems. [. . .] Now [. . .] it is necessary to base all astrometry, reference systems, ephemerides, and observational reduction procedures on consistent relativistic grounds. This means that relativity must be accepted in its entirety, and that concepts, as well as practical problems, must be approached from a relativistic point of view."
Kovalevsky & Seidelmann 2004

Here's Alice, Bob, and Charlie. They have extremely precise clocks built in exactly the same way. They stay very close to one another and synchronize their clocks. Still keeping close, they go around, maybe on an aeroplane or space ship, and all the time they check their clocks. They notice that their clocks stay perfectly synchronized all the time, no matter where they go and what they do.

At some point they separate, each one going around independently. One of them might stay in place, another might take a helicopter, and another might go for a trip on Mars and back.

Alice and Bob at some point meet again, and compare their clocks. They see that their clocks aren't synchronized anymore; the difference could be as small as microseconds, or as large as years. In fact, if this time discrepancy is large, they would notice that they themselves have aged differently; so time discrepancy doesn't affect the clocks only. Let's say for concreteness that Alice's clock is ahead of Bob's, or equivalently that Bob's is behind Alice's. Note the following aspects:

First, neither Alice or Bob can say "my clock was wrong": neither has noticed anything strange about the "passage of time".

Second, they might wonder what's the time on Charlie's clock. But Charlie is at some distance away. They could decide to contact Charlie via radio, say, and ask "what shows your clock right *now*?". But they would notice that there's a delay, even if extremely small, in the radio transmission; so it's unclear to what time would Charlie's answer apply. If we say "let's account for the radio-signal speed", we see that there's a logical problem: speed is distance divided by time, and here we have a problem in exactly determining what's the "correct" time! So we would be reasoning in circles. Besides, even neglecting these difficulties, Charlie's answer could reveal a time that completely different from Alice's and from Bob's – it could be years ahead or behind both of theirs!

Third, if they now stay together, they will see that their clocks remain exactly synchronized, besides the discrepancy they noticed when they met. This discrepancy doesn't increase or decrease. They may even retrace together Alice's and Bob's previous trips; their clocks still remain synchronized.

The experience just described will occur again any time two or more of them meet. There could be a hundred observers like Alice, Bob, Charlie, initially at the same place and synchronized. Whenever two or more of them meet after having been separated, they will notice discrepancies in their clocks. But their clocks will have exactly the same time lapses as long as they stay together.

Consider for a moment an imaginary world in which these experiments had given a different kind of result. According to Newtonian mechanics, whenever two or more initially synchronized observers like Alice, Bob, Charlie had met, their clocks would have always shown exactly the same time. If one year, 23 days, 8 hours, 9 minutes, and 3.045 399 283 240 992 663 02 seconds have passed for you since you last met Alice, you'd see that exactly the same amount of time has passed for her when you two meet again. If you think about it, in this case it would have beeen somewhat natural to think "right now, the clocks of far-away Alice, Bob, Charlie must show the same time as mine" (even though you have no real experimental way of confirming that).

But that's an imaginary world. In our world is the more complicated situation described initially that holds. Only one conclusion can be drawn from these experimental results: **Time is not some sort of universal quantity. It is, so to speak, "local" to a person or clock, or to a group of persons or clocks that stick together.** This also means that *it doesn't make* *sense* to ask questions like "what can be the time for far-away Charlie, right now?".

The time measured by a specific observer is called the *proper time* of that observer. Luckily we know more about how the proper times of separated observers can differ when they meet again. It turns out – according to our current understanding – that the time differences depend, roughly speaking, on how fast the observers are moving with respect to one another and to matter around the universe, and on how much energy is contained in the regions they travel. The general theory of relativity gives us the equations determining any such proper-time differences.

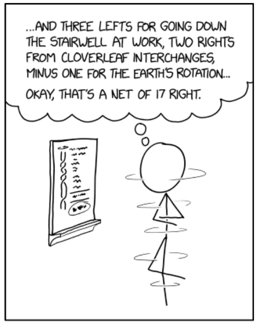
The situation depicted in the experiments above is real. It can be measured, for example, comparing initially synchronized clocks that have been put in aeroplanes flying in different directions. Most importantly, it affects everyday relevant technologies such as the Global Positioning System. Formulae from general relativity appear in your phone's GPS software; see for instance § 20.3.3.3.3 of the Interface Control Document `IS-GPS-200` at `https://www.gps.gov/technical/icwg/`. It must also be taken into account in the establishment and synchronization of time in our everyday equipments:

"The plot for Cesium [. . .] characterizes the best orbiting clocks in the GPS system. What this means is that after initializing a Cesium clock, and leaving it alone for a day, it should be correct to within [. . .] 4 nanoseconds. Relativistic effects are huge compared to this."
Ashby 2003

> International Atomic Time (TAI) is based on more than 250 atomic clocks distributed worldwide that provide its stability, whereas a small number of primary frequency standards provide its accuracy. Universal Coordinated Time, which is the basis of all legal time scales, is derived from TAI. To allow the construction of TAI and the general dissemination of time, clocks separated by thousands of kilometres must be compared and synchronized. [. . .] The achieved performances of atomic clocks and time transfer techniques imply that the definition of time scales and the clock comparison procedures must be considered within the framework of general relativity.
>
> (Petit & Wolf 2005)

In most everyday situations for us, who live on or nearby Earth and move at speeds much smaller than c with respect to one another, the discrepancies between our proper times are so small that cannot be measured with ordinary clocks or with our internal clocks. Consider a person walking 10 m away from you and then immediately walking back to you, at 1 m/s. The time elapsed for you will be 20 s, but for that person will be 19.999 999 999 999 999 889 s, a difference of 10^{-16} s, which is the error of an atomic clock. If human beings still exist in some decades or centuries, with space travel they will probably have to deal more and more with proper-time discrepancies also in everyday life.

For the most part of the rest of these notes, we won't need to deal with differences in proper time. But I recommend that you keep present how time really works, and that these small time discrepancies exist and occur all the time along your *worldline*.



SPACETIME HEALTH TIP: REMEMBER TO CANCEL OUT YOUR ACCUMULATED TURNS AT THE END OF EACH DAY TO AVOID WORLDLINE TORSION.

https://xkcd.com/2882

Time has physical dimension of *time*, symbol T, and we shall for the most part measure it using the unit *second*, symbol s.

2.2 Space

Together with the notion of time, also the notion of space loses some of its traditional intuition. Several observers in motion with respect to one another will generally disagree on the dimensions of an approximately rigid object in their vicinity. For objects that are far away from an observer, the very notion of “distance” becomes tricky has different and non-equivalent definitions; one must be very careful on which definition is being used.

We shall not delve further into these peculiarities of time and space. Keep simply in mind that phenomena happen in *spacetime*, and that there's no way to attribute a universal time, nor a universal position in space, to a physical event. There is one absolute: whoever locally measures the speed of light, will find the value $c \coloneqq 299\,792\,458\,\mathrm{m/s}$. This value is exact by definition, and serves as a way to define a local notion of space and distance.

Space has physical dimension of *length*, symbol L, and we shall for the most part measure it using the unit *metre*, symbol m.

2.3 Coordinate systems

It is necessary to have a way for distinguishing physical events and phenomena and locating them in spacetime. This is achieved through a *coordinate system*. A coordinate system assigns four numerical labels to every point in spacetime. Often these labels have some kind of physical meaning, such as the proper time elapsed for a specific clock, or the distance from some event as measured by a specific observer, but they don't need to.

A coordinate system also solves the problems coming from proper-time and space discrepancies among different observers. We can assign to every physical event a *coordinate time* and a *coordinate spatial position*, which are the same for all observers, because decided by agreement. Coordinate time doesn't have a strict physical meaning, and will generally be different from the proper times registered by different observers. It can nevertheless be used for "doing physics", and it is the time we shall most often use in our equations. A coordinate time commonly used for Earth-physics purposes is Universal Coordinated Time (UTC)[6]. The clock on your phone, and on devices that get synchronized via internet, shows UTC, not your proper time. An observer on Earth at 0 m over sea level, and not moving, measures a proper time exactly equal to UTC (besides small variation coming from the movements of Solar System bodies). Observers at other altitudes or moving with respect to Earth's surface notice that their proper times are slightly different from UTC.

We shall often denotes the four coordinates of a coordinate system by

$$(t, x, y, z)$$

where t is a coordinate time, usually UTC, and (x, y, r) determine a spatial position. The triplet of spatial coordinates is often denoted by the vector $\boldsymbol{r}$:

$$\boldsymbol{r} \coloneqq (x, y, z) \ .$$

It is always important to specify how the coordinate system you're using is defined. The definition of the spatial coordinates (x, y, z) is typically different from problem to problem. We shall typically use coordinates that form $\frac{\pi}{2}\,\mathrm{rad} \equiv 90°$ angles with one another; but their directions and their origin – that is, where they have value $x = y = z = 0\,\mathrm{m}$ – always depend on the problem, so make sure you always specify them.

Whenever we speak of a "region of space" or of a "surface in space", we mean a region at some specific coordinate time t.

[6] https://www.nist.gov/pml/time-and-frequency-division/time-realization/utc nist-time-scale-0/

Some physical phenomena happen along a line, in one dimension. In this case we can omit two of the spatial coordinates, assuming they have some constant, unimportant values. In these cases we can simply write, for instance, (t, x) as our coordinates.

Exercise 2.1

On relation between coord time and proper time for satellite

2.4 Seven primitive quantities

In the physics formalism that we're using there are about seven more quantities, technically called *fields*, that we take as primitive:

matter
electric charge
magnetic flux
energy
momentum
angular momentum
entropy

The dimensions, units, and scalar or vector character of these quantities appear in table 1.

Recall that primitive quantities cannot be defined: we can only try to have an intuitive understanding of them, for example through their properties. Some of the seven primitive quantities are easier to grasp intuitively than others. But they all have two basic properties in common: of each, we can ask – or rather, measure:

- **How much of this quantity is in a particular (3D) region of space at a particular time?**
- **How much of this quantity flows through a particular (2D) surface during a particular time lapse?**

and we can ask these questions of any region of space and any time lapse. The results of the two measurements above are numbers, which in general can be positive or negative, for scalar quantities; or vectors for vector quantities

Owing to the two properties above, each of the seven quantities can be considered as some kind of "stuff" that can be present at each spacetime

Quantity	**Dimension**	**Unit**
Time	time	*second* s
Length	length	*metre* m
Temperature	temperature	*kelvin* K
Matter	amount of substance	*mole* mol
Electric charge	electric charge	*coulomb* C
Magnetic flux	magnetic flux	*weber* Wb
Energy	energy, mass	*joule* J, *kilogram* kg
Momentum	mass·length/time, energy·time/length	$\mathrm{kg \cdot m/s}$, $\mathrm{J \cdot s/m}$
Angular momentum	mass·length^2/time, energy·time	$\mathrm{kg \cdot m^2/s}$, $\mathrm{J \cdot s}$
Entropy	energy/temperature	J/K

Table 1 Dimensions and units of the main physical quantities used in these notes. Quantities in **boldface** are vectors, the others are scalars

point. But don't take the word 'stuff' too literally: I don't necessarily mean concrete objects like a ball, or substances like water.

What's remarkable about these seven quantities is that they are common to all our main physical theories, approximate or not: from Newtonian mechanics to general relativity and quantum theory; from subatomic scales to cosmological scales. The physical meaning and mathematical characterization of these quantities can be slightly different depending on the physical theory and scale. For example, in quantum theory they are mathematically represented by so-called operators rather than functions; and at molecular scales entropy has a meaning connected with probability theory. Yet, these seven quantities are really universal to our present way of doing physics and of describing physical phenomena around and within us.

Other auxiliary quantities, besides these, also appear in some physical theories. Important examples are temperature and metric. Most of these auxiliary quantities don't have the "stuff-like" properties of the main seven quantities; for instance, we cannot ask "how much temperature is in a particular region?". We shall discuss and use some auxiliary quantities later.

2.5 Densities and fluxes

How do we mathematically represent the seven main quantities? We said that, for each, we can ask *how much in this region?* and *how much through this surface during this time?* We must therefore use mathematical objects that allow us to answer these two questions. These objects are the *density* and the *flux vector*. The idea behind them is quite simple.

Density

Take a point in spacetime with coordinates (t_0, x_0, y_0, z_0). At the instant t_0, imagine a very small cuboid centred at (x_0, y_0, z_0), with sides aligned with the coordinate axes x, y, z and of lengths Δx, Δy, Δz. Then we have

$$\text{amount of quantity in cuboid} = n(t_0, x_0, y_0, z_0)\, \Delta x\, \Delta y\, \Delta z\ . \tag{1}$$

where $n(t_0, x_0, y_0, z_0)$ is the **density** at the spacetime point (t_0, x_0, y_0, z_0).

The density tells us how much quantity there's in a unit of volume. As the notation suggests, it is a function of the coordinates, that is, of time and spatial position. This functional dependence reflects the fact that we can have a larger amount of a quantity concentrated in some regions at some times, than in other regions or at other times.

In order to calculate the total amount of the quantity in an arbitrary 3D region, we simply divide it into very small cuboids similar to the one above. For each cuboid we calculate the respective amount; this amount will generally be different from cuboid to cuboid, because the value of the density $n(t, x, y, z)$ will be different at each cuboid's centre. Then we sum up all these amounts.

Curiousity 2.1

You probably recognize this procedure as the description of integration. Indeed we can write:

$$\text{total amount of quantity in 3D region} = \iiint\limits_{\text{3D region}} n(t,x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{2}$$

Clearly we must know the value of the density n at all points within the region, in order to calculate the total amount. Note that if the quantity is absent in some subregion, then $n(\dots) = 0$ there.

For the flux of the quantity through a surface we consider three different cases:

x-Flux

Take again a point in spacetime with coordinates (t_0, x_0, y_0, z_0). Keeping x_0 fixed, imagine a very small rectangular surface centred at (x_0, y_0, z_0), with sides aligned with the coordinate axes y, z and of lengths Δy, Δz. Imagine that this surface exists for a lapse of time Δt around the time t_0. Then we have

$$\left.\begin{array}{l}\text{amount of quantity}\\ \text{through rectangle } \Delta y\,\Delta z\\ \text{towards positive } x\\ \text{during time lapse}\end{array}\right\} = j_x(t_0, x_0, y_0, z_0)\,\Delta t\,\Delta y\,\Delta z\,. \tag{3}$$

where $j_x(t_0, x_0, y_0, z_0)$ is the **x-flux** at the spacetime point (t_0, x_0, y_0, z_0).

The x-flux tells us how much quantity is flowing in a unit of time through a unit of surface parallel to y, z. Also the x-flux is a function of the coordinates, because the flux could be larger through some surfaces at some times, than through other surfaces at other times. If the quantity is flowing in the negative x direction, then j_x will be negative; and clearly if no quantity is flowing through the small rectangle, then $j_x = 0$.

In an analogous way we define a y-flux and a z-flux:

y-Flux

$$\left.\begin{array}{l}\text{amount of quantity}\\ \text{through rectangle } \Delta z\,\Delta y\\ \text{towards positive } y\\ \text{during time lapse}\end{array}\right\} = j_y(t_0, x_0, y_0, z_0)\,\Delta t\,\Delta z\,\Delta y\;. \tag{4}$$

z-Flux

$$\left.\begin{array}{l}\text{amount of quantity}\\ \text{through rectangle } \Delta x\,\Delta y\\ \text{towards positive } z\\ \text{during time lapse}\end{array}\right\} = j_z(t_0, x_0, y_0, z_0)\,\Delta t\,\Delta x\,\Delta y\;. \tag{5}$$

The three fluxes can be grouped into the **flux vector**

$$\boldsymbol{j}(t,x,y,z) \coloneqq \bigl(j_x(t,x,y,z),\ j_y(t,x,y,z),\ j_z(t,x,y,z)\bigr) \tag{6}$$

In order to calculate the total amount of the quantity through an arbitrary sequence of 2D surfaces during a time lapse, we simply divide the total time into very brief time lapses, and for each of these we approximate the surface with very small rectangles aligned along the three axes. For each such lapse and rectangle we calculate the respective flux amount, using j_x, j_y, or j_z depending on the orientation of the rectangle. Then we sum up all these amounts.

Curiousity 2.2

This is also the description of an integration, which can be written

$$\begin{multline}\text{total amount of quantity through sequence of surfaces} =\\ \iiint j_x(t,x,y,z)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}z\;+\\ \iiint j_y(t,x,y,z)\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}y\;+\\ \iiint j_z(t,x,y,z)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \tag{7}\end{multline}$$

We shall not have to calculate density integrals (2) and flux integrals (7) in these notes. In concrete physics problems these integrals are often difficult to calculate, and require advanced mathematical techniques and specialized software. But if you'll ever end up working in fields such as computational fluid dynamics, atmospheric or ocean modelling, or numerical relativity, then you shall probably encounter them again.

The definitions above of density and flux are appropriate if the quantity in question is a scalar. For vector quantities such as momentum and angular momentum, the density is a vector, and each of the three fluxes is a vector as well.

3 Physical laws

3.1 Two kinds of laws

In the previous chapter we introduced the main quantities in term of which we describe physical phenomena. How do we describe the patterns, regularities, and diversities that we observe in these phenomena? This is the role of physical laws. Different physics formalisms or languages express these laws in different way, even if the actual physical phenomenon is exactly the same.

In the formalism we're using we distinguish between two kinds of physical laws:

- **Balance laws**
- **Constitutive relations**

Balance laws, or simply balances, have this name because they express a sort of trade-off or "budget". A special kind of balance laws are called **conservation laws**. We shall see that there's a set of balances which apply to *all* physical phenomena, without exceptions; they represent the basic universal patterns that we observe everywhere.

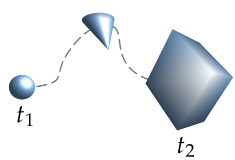
Constitutive relations instead only apply to particular phenomena and only in particular theories. They express the diversity that we observe around us.

3.2 Balance and conservation laws

Balance laws are intuitively and visually easy to grasp. The archetype is as follows:

Consider a closed 2D surface at a (coordinate) time t_1, which encloses a 3D region. A spherical surface delimiting a ball-shaped region is an example. Actually the surface doesn't need to be connected: it could consist of several separate closed surfaces, delimiting distinct 3D regions.

Starting from that particular time, imagine the closed surface moving and deforming, changing shape and size but remaining closed – no holes or cuts – through time, until a later time t_2. At this time we shall have a closed surface that can be different from the initial one, and at a different position. We could start with a spherical surface delimiting a



region the size of a tennis ball, and end with a cubical surface delimiting a region the size of the Sun.

The initial and final surfaces and the sequence in between are completely arbitrary. We choose them depending on the phenomenon that we want to study. They do not need to be the surfaces of real objects. For instance we can consider an imaginary cubical surface in the middle of a room, enclosing the air within; or in the middle of a river, enclosing, for an instant, the water within. But sometimes these arbitrary surfaces do coincide with the boundaries of objects. For instance we can consider the outer surface of a tennis ball, and follow it as the ball moves.

Now consider any one of the seven primitive quantities of § 2.4, except magnetic flux. Remember the two questions or measurements that we could make on such quantity: *how much in this region?* and *how much through this surface during this time?* We therefore make the following measurements:

- N_1: Total amount of quantity contained within the surface at time t_1.
- J: Total amount of quantity flowing *inwards* through the moving surface between times t_1 and t_2.
- N_2: Total amount of quantity contained within the surface at time t_2.

Each of the three amounts above is either a scalar of appropriate physical dimensions that can be positive, negative, or zero; or a vector. For instance, for matter we could have $N_1 = 3.5\,\mathrm{mol}$; for energy we could have $J = -600\,\mathrm{J}$; for momentum we could have $\boldsymbol{N}_2 = (-3.4, 0, +5)\,\mathrm{kg}\cdot\mathrm{m/s}$.

With these definitions we can introduce two kinds of balance laws.

Conservation law

A quantity is said to be **conserved**, or to satisfy a **conservation law**, if the following equality holds:

$$N_2 - N_1 - J = 0 \tag{8}$$

A conservation law allows us to make several kinds of predictions or deductions. For example:

- If we know the amount of quantity within the surface at t_1 and the amount that flowed in through the surface between t_1 and t_2, then we can predict the amount within the surface at t_2:

$$N_2 = N_1 + J \; .$$

- If we know the amount of quantity within the surface at t_1 and the one at t_2, then we can deduce the total amount that flowed in through the surface between the times t_1 and t_2:

$$J = N_2 - N_1 \; .$$

- If we know the amount of quantity within the surface at t_2 and the amount that flowed in through the surface between t_1 and t_2, then we can deduce the amount within the surface at t_1:

$$N_1 = N_2 - J \; .$$

These kinds of predictions and deductions are very powerful: consider that the shape and motion of the surface, as well as the times t_1, t_2, are completely arbitrary. Keep in mind that if the three quantities are scalars, then in general they can be positive, negative, or zero. For example, in the case of matter, if the total amount flowed in through the surface is negative, $J < 0$, then we must also have $N_2 < N_1$.

We unconsciously use some conservation laws all the time in our everyday life. If you seal a glass in a box, and upon dropping the box you hear that the glass shatters, and see that no holes were formed in the box, then you are sure that all the glass pieces now in the box could be glued together to form the original glass in its entirety.

Our intuitive understanding of a conserved quantity is that it cannot be "created" or "destroyed". This leads to the definition of the more general balance law:

Balance law

A quantity is said to be only **balanced**, or to satisfy a **balance law** or simply **balance**, if the following equality holds:

$$N_2 - N_1 - J = R \tag{9}$$

where R is the amount of quantity created within the sequence of surfaces between the times t_1 and t_2.

If the quantity is a scalar, then R can also be negative, in which case we say that some of the quantity has been destroyed.

A balance law allows us to make several kinds of predictions or deductions as well. For example:

- If we know the amount of quantity within the surface at t_1, the amount that flowed in through the surface between t_1 and t_2, and the amount that was created within the surface between t_1 and t_2, then we can predict the amount within the surface at t_2:

$$N_2 = N_1 + J + R\ .$$

And similarly as in the other examples for a conservation law. You see that in order to use a balance we need an extra piece of information.

A balance law is somehow more trivial than a conservation law: if we measure that $N_1 - N_2 + J$ is not zero, we can always say that some amount of quantity must have been created or destroyed between the two times; so a quantity might always be said to satisfy a balance. But a balance is not trivial if we have some specific physical law that tells us in advance how the amount created or destroyed, R, can be calculated.

A conservation law can be seen as a special and powerful case of a balance law, one for which $R = 0$ *always*.

Curiousity 3.1

Conservation and balance laws appear even simpler from the point of view of Relativity Theory. From a four-dimensional spacetime perspective, a 3D volume is a region, called *hypersurface*, having one less dimension than spacetime. But a sequence of 2D surfaces through time is also just a region having one less dimension than spacetime – two spatial dimensions and one temporal one. Thus the distinction between the 3D region at t_1, the sequence of 2D surfaces between t_1 and t_2, and the 3D region at t_2 disappear: they are seen to be just different parts of the same three-dimensional hypersurface. We perceive some parts of this hypersurface as belonging "to the same time", showing their three dimensions all at once; and other parts as extending

through time, showing only two dimensions at any time. In fact, different observers make this division in different ways.
And from a spacetime perspective, the amount of a quantity N_1 or N_2 within a 3D region is seen as a flux through time; so its apparent difference from the flux J also disappear. Their only difference is that one (C) points exclusively in the time direction, while the other (J) also points partially in a spatial direction.

Curiousity 3.2

In the case of magnetic flux, the idea of a conservation law is analogous, but is formulated with one less spatial dimension: we consider a closed 1D curve at t_1, one at t_2, and a sequence in between these times. The magnetic flux turns out to be a quantity for which it's possible to ask how much is "linked" to a closed curved, and how much is crossing a closed curve. One way to understand this is to imagine magnetic flux as a bundle of tubes or lines that are either closed or extend to infinity. It is a very fascinating quantity, and one may wonder if other quantities exist which satisfy similar balances even with one less dimension. But we shall not pursue magnetic flux or similar topics in these notes.

3.3 Balance laws expressed with derivatives

In § 2.5 we saw that each of the main seven quantities is represented by a density and by a flux vector, which allow us to calculate the amount of quantity within a 3D region at a particular time, an flowing through a 2D surface during an time interval. These amounts are exactly what N_1, N_2, J express. Therefore, a conservation or balance law must also give some relationship between density and flux vector.

The way to find such relationship is intuitive and mathematically simple, if only a bit lengthy. The idea is to formulate the balance law for a very simple case, the same simple set-up that we used in § 2.5 to introduce density and flux.

Take a point in spacetime with coordinates (t, x, y, z). We apply the conservation law (8) to the following situation:

N_1: Choose an initial time $t_1 = t - \frac{\Delta t}{2}$, slightly before t, and a cuboid region centred at $(x, y,)$ with sides Δx, Δy, Δz. The total amount of the quantity within this region is given, adapting formula (1), by

$$N_1 = n\big(t - \tfrac{\Delta t}{2}, x, y, z\big)\, \Delta x\, \Delta y\, \Delta z\ . \tag{10}$$

N_2: Choose a final time $t_2 = t + \frac{\Delta t}{2}$, slightly after t, and a cuboid region centred at $(x, y,)$ with sides Δx, Δy, Δz. The total amount of the quantity within this region is given, adapting formula (1), by

$$N_2 = n\big(t + \tfrac{\Delta t}{2}, x, y, z\big)\, \Delta x\, \Delta y\, \Delta z\ . \tag{11}$$

J: We calculate the flux separately through the six rectangular surfaces bounding the small cuboid region:

J_x: First choose a rectangular surface centred slightly to one side of the point (x, y, z): at $(x - \frac{\Delta x}{2}, y, z)$, with sides parallel to y, z and lengths $\Delta y, \Delta z$. We keep this surface constant for the small duration Δt. The x-flux through this rectangle, towards positive x, during Δt, is according to formula (3)

$$J_x^{\text{in}} = j_x\big(t, x - \tfrac{\Delta x}{2}, y, z\big)\, \Delta t\, \Delta y\, \Delta z\ .$$

Now choose the rectangular surface parallel to the previous one, but on the other side of the point (x, y, z), centred at $(x + \frac{\Delta x}{2}, y, z)$. The x-flux through this rectangle, towards positive x, during Δt, is

$$J_x^{\text{out}} = j_x\big(t, x + \tfrac{\Delta x}{2}, y, z\big)\, \Delta t\, \Delta y\, \Delta z\ .$$

Note that the flux J_x^{in} points to the interior of the 3D region, towards its centre (x, y, z); whereas J_x^{out} points to the exterior, away from the centre. The total flux into the region is therefore given by their difference:

$$\begin{aligned} J_x &= J_x^{\text{in}} - J_x^{\text{out}} \\ &= \big[j_x\big(t, x - \tfrac{\Delta x}{2}, y, z\big) - j_x\big(t, x + \tfrac{\Delta x}{2}, y, z\big)\big]\, \Delta t\, \Delta y\, \Delta z \end{aligned} \tag{12}$$

J_y, J_z: An analogous reasoning can be made to find the total flux through the other four surfaces, two parallel to z, y and two to x, y. We find

$$\begin{aligned} J_y &= J_y^{\text{in}} - J_y^{\text{out}} \\ &= \left[j_y\left(t, x, y - \tfrac{\Delta y}{2}, z\right) - j_x\left(t, x, y + \tfrac{\Delta y}{2}, z\right) \right] \Delta t \, \Delta z \, \Delta x \end{aligned} \tag{13}$$

$$\begin{aligned} J_z &= J_z^{\text{in}} - J_z^{\text{out}} \\ &= \left[j_z\left(t, x, y, z - \tfrac{\Delta z}{2}\right) - j_x\left(t, x, y, z + \tfrac{\Delta z}{2}\right) \right] \Delta t \, \Delta x \, \Delta y \end{aligned} \tag{14}$$

The total flux into the region during the time interval Δt is finally the sum of the three above:

$$J = J_x + J_y + J_z \tag{15}$$

Up to now we have only chosen arbitrary surfaces, regions, and a time interval. Now let's assume that the corresponding amounts and fluxes satisfy a balance law like (8). Putting together the puzzle pieces we find

$$N_2 - N_1 - J = 0 \qquad \Longrightarrow$$

$$\left.\begin{aligned} & n\left(t + \tfrac{\Delta t}{2}, x, y, z\right) \Delta x \, \Delta y \, \Delta z - n\left(t - \tfrac{\Delta t}{2}, x, y, z\right) \Delta x \, \Delta y \, \Delta z \\ & - \left[j_x\left(t, x - \tfrac{\Delta x}{2}, y, z\right) - j_x\left(t, x + \tfrac{\Delta x}{2}, y, z\right) \right] \Delta t \, \Delta y \, \Delta z \\ & - \left[j_y\left(t, x, y - \tfrac{\Delta y}{2}, z\right) - j_x\left(t, x, y + \tfrac{\Delta y}{2}, z\right) \right] \Delta t \, \Delta z \, \Delta x \\ & - \left[j_z\left(t, x, y, z - \tfrac{\Delta z}{2}\right) - j_x\left(t, x, y, z + \tfrac{\Delta z}{2}\right) \right] \Delta t \, \Delta x \, \Delta y \end{aligned}\right\} = 0$$

This expression is quite long, but it should be intuitively understandable if you try to identify the individual summed terms.

Now we take the last, long expression – which is an equality – and divide its left and right sides by $\Delta t \, \Delta x \, \Delta y \, \Delta z$. Note that many of the "Δ"

factors simplify out. We arrive at this:

$$\left.\begin{aligned}
&\frac{n\left(t+\frac{\Delta t}{2},x,y,z\right)-n\left(t+\frac{\Delta t}{2},x,y,z\right)}{\Delta t}\\
&+\frac{j_x\left(t,x+\frac{\Delta x}{2},y,z\right)-j_x\left(t,x-\frac{\Delta x}{2},y,z\right)}{\Delta x}\\
&+\frac{j_y\left(t,x,y+\frac{\Delta y}{2},z\right)-j_y\left(t,x,y-\frac{\Delta y}{2},z\right)}{\Delta y}\\
&+\frac{j_z\left(t,x,y,z+\frac{\Delta z}{2}\right)-j_z\left(t,x,y,z-\frac{\Delta z}{2}\right)}{\Delta z}
\end{aligned}\right\} = 0\,.$$

Still a long expression; but examine each fraction: we have a difference between a term calculated at some point, and one calculated at some point plus an increment "Δ"; this difference is then divided by that increment. This is the definition of *partial derivative*, that is, the derivative with respect to one variable, while keeping the other variables fixed. The short symbol for this is "$\frac{\partial\ldots}{\partial\ldots}$", so we can simply write

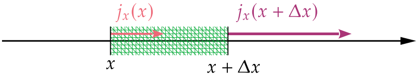
Conservation law in derivative form

$$\frac{\partial n}{\partial t}+\frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}+\frac{\partial j_z}{\partial z}=0 \qquad (16)$$

Let's make a simple check to see if this formula makes sense, in a simple one-dimensional case where we disregard the y, z coordinates (let's say their fluxes are zero), so that the balance law becomes

$$\frac{\partial n}{\partial t}+\frac{\partial j_x}{\partial x}=0\,.$$

Suppose that the derivative of the x-flux is positive: $\frac{\partial j_x}{\partial x}>0$. This means that the x-flux increases as we move from x to $x+\Delta x$, as shown in the side figure. The x-flux at x is bringing some amount of quantity into the region of interest, the x-flux at $x+\Delta x$ is taking a larger amount out of that region. Therefore, during the short time Δt in which this flux takes place, the total amount of quantity within the region will decrease. But the total amount is given by $n(t)$, which is



therefore decreasing with time. This means that its time derivative is negative: $\frac{\partial n}{\partial t} < 0$. This agrees with balance law above: the sum of the two derivatives must be zero, so if one is positive, the other must be negative.

Balance laws like (16) are the basis of many important computational and simulation methods in a huge variety of physical applications: simulation of the ocean around an offshore oil platform, of wind in a wind farm, of air around an aeroplane's wing, of earthquakes, of electromagnetic-wave propagation, of oscillations in a suspension bridge, of percolation of fluids through terrain, of chemical reactions, of energy transfer. . . The list could go on for pages!

The basic idea for using a balance law for a simulation is easy to understand. Take again the simpler one-dimensional formula without y, z coordinates, and rewrite it remembering the basic meaning of the derivative:

$$\frac{n(t+\Delta t, x) - n(t, x)}{\Delta t} + \frac{j_x(t, x + \frac{\Delta x}{2}) - j_x(t, x - \frac{\Delta x}{2})}{\Delta x} = 0$$

Suppose we know all quantities at time t; we can then find the density n at time $t + \Delta t$. Leave it on the left side and bring all other terms on the right side, multiplying them by Δt:

$$n(t+\Delta t, x) = \Delta t \left[n(t, x) - \frac{j_x(t, x + \frac{\Delta x}{2}) - j_x(t, x - \frac{\Delta x}{2})}{\Delta x} \right]$$

which is essentially Euler's method[7].

This formula can then be used iteratively to find the density n at later times. As a simple simulation set-up, we consider a region of space of interest and divide it into cells of width Δx. We also treat time in steps of Δt. Then the iterative procedure goes as follows:

1. *Initial values:* Assign the value of n at each x-cell at the initial time.
2. *For-loop in x:* for each x-cell, calculate the value of n at $t + \Delta t$, using the formula above.
3. Increase time by Δt.
4. Go to 2.

[7] https://mathworld.wolfram.com/EulerForwardMethod.html

For example, suppose we have two x-cells of width $\Delta x = 1$: one centred at x_1 and the other at $x_2 = x_1 + 1$. There are three boundaries: the leftmost at $x_1 - \frac{1}{2}$, one between the cells at $x_1 + \frac{1}{2} \equiv x_2 - \frac{1}{2}$, and the rightmost at $x_2 + \frac{1}{2}$. Then we have the following steps:

1. Assign the values $n(t_0, x_1), n(t_0, x_2)$ of n at x_1 and x_2 at time t_0.
2. Assign the values
$$j_x\big(t_0, x_1 - \tfrac{1}{2}\big)\ ,\quad j_x\big(t_0, x_1 + \tfrac{1}{2}\big)\ ,\quad j_x\big(t_0, x_2 + \tfrac{1}{2}\big)$$
of the fluxes at the three boundaries at time t_0.
3. Calculate the values of n at x_1 and x_2 at time $t_1 = t_0 + \Delta t$, for instance
$$n(t_1, x_1) = \Delta t\left[n(t_0, x_1) - \frac{j_x(t_0, x_1 + \frac{1}{2}) - j_x(t_0, x_1 - \frac{1}{2})}{\Delta x}\right].$$
Note that we have all quantities required on the right side.
4. Assign the values
$$j_x\big(t_1, x_1 - \tfrac{1}{2}\big)\ ,\quad j_x\big(t_1, x_1 + \tfrac{1}{2}\big)\ ,\quad j_x\big(t_1, x_2 + \tfrac{1}{2}\big)$$
of the fluxes at the three boundaries at time t_1.
5. Calculate the values of n at x_1 and x_2 at time $t_2 = t_1 + \Delta t$, for instance
$$n(t_2, x_1) = \Delta t\left[n(t_1, x_1) - \frac{j_x(t_1, x_1 + \frac{1}{2}) - j_x(t_1, x_1 - \frac{1}{2})}{\Delta x}\right].$$
Note that we have again all quantities required on the right side.
6. And so on.

This routine requires us *to know the flux at each x-cell and at each time*. But we shall soon see that this requirement can be bypassed.

The rudimentary routine above is the basis from which more precise and complex simulation routines are developed.

Exercise 3.1

Write a script, in your preferred programming language, that implements the simulation routine above in an x-grid with four cells $x = 1, \ldots, x = 4$. Take a grid size $\Delta x = 1$ and time step $\Delta t = 1$. We can imagine that the quantity in question is electric charge. Use the following initial values for $n(t, x)$ at $t = 0$:

$$n(t{=}0, x{=}1) = 7\ ,\quad n(t{=}0, x{=}2) = 0\ ,\quad n(t{=}0, x{=}3) = 0\ ,\quad n(t{=}0, x{=}4) = 7$$

and assume the flux is, at each time t:

$$j_x(t, x{=}0.5) = 0\,,$$
$$j_x(t, x{=}1.5) = +2\,,\quad j_x(t, x{=}2.5) = 0\,,\quad j_x(t, x{=}3.5) = -2\,,$$
$$j_x(t, x{=}4.5) = 0\,.$$

Questions and tasks:

1. Which time evolution for n do you observe? Does it make sense, given the fluxes above?
2. Feel free to try out or generalize your script with:
   - different initial values
   - different fluxes
   - an explicit formula for the flux, for instance

     $$j_x(t, x) = 8\,x^3 - 60\,x^2 + 118\,x - 45$$

   - time-dependent fluxes
   - more cells
   - two or three dimensions, including y- or z-fluxes

3.4 Constitutive relations

In the previous section we saw that a balance law can be used to predict how the density of a quantity changes with time. But we also saw that in order to make such prediction we need to already know what the *future* fluxes of the quantity will be, throughout the region of interest. This doesn't sound like a great achievement: to predict the future, we need the future. The situation doesn't improve if we have two or more quantities that satisfy balance laws: we shall need to know in advance the fluxes of each.

It turns out, however, that there are physical laws which *relate the densities of some quantities to the fluxes of others*. Such relations can potentially allow us to calculate the time evolution without knowing the fluxes in advance. Roughly, the basic idea is as follows: From the density of one quantity, we calculate the fluxes of the second quantity, and use them to predict its density at a later time. Once we know this density, we use it to calculate the new fluxes of the first quantity, and therefore its density at a later time; and so on. This cross-prediction does not fully eliminate the necessity of specifying some future quantities in advance; but the remaining ones that we need to specify are often in our control. Indeed this is how we can predict how a physical system will respond to influences controlled by us.

Physical laws of this kind turn out not to be universal: they depend on, or are 'constitutive'[8] of, the particular physical phenomenon and the physical theory being used. For this reason they are called **constitutive relations** or **constitutive equations**. In some fields they are called **closure equations**, because they allow us to "close" the system of balance equations in such a way that it can be used for future predictions without knowing in advance the future value of some quantities.

Constitutive relations express the diversity that we observe around us, for example the different behaviours of a drop of water, which obeys some constitutive relations, as compared with a block of wood, which obeys others. They also mark the difference between specialized or approximate physical theories, for example between Newtonian mechanics, which is based on particular approximate constitutive relations, and general relativity, which is based on different and more exact constitutive relations. Depending on the specific scientific field you'll work in, you'll learn some constitutive relations in more detail than others.

Constitutive relations come in a great variety of mathematical forms. Some of them are simple algebraic relations between the density of one quantity and the flux of another. Others involve spatial or time derivatives. Other still involve integrals in space or in time; the latter give rise to physical phenomena that seem to posses "memory".

[8] https://www.merriam-webster.com/dictionary/constitutive

4 The Seven Wonders of the World

We already remarked that the seven primitive quantities introduced in § 2.4 are common to all our main physical theories. Even more remarkable is that each of them obeys a balance law like (9); three actually obey a conservation law like (8):

The seven fundamental balances

Conservation of matter
Conservation of electric charge
Conservation of magnetic flux
Balance of energy
Balance of momentum
Balance of angular momentum
Balance of entropy

These seven balance laws are known, so far, to be satisfied by *all* phenomena and in all our main physical theories, approximate or not: from subatomic scales to cosmological scales, from Newtonian mechanics to general relativity and quantum theory. No exceptions are known.

These balances are truly the "Seven Wonders of the World"[9].

Once you learn these balances, you can apply them no matter what you work with: construction of bridges, control of chemical reactions, operation of GPS navigation and satellites, monitoring of nuclear power plants, sending robots to Mars, or collisions of subatomic particles.

More precisely: matter, electric charge, magnetic flux always satisfy strict conservation laws. For matter we actually have several conservation laws, one for each kind of matter. Energy, momentum, angular momentum, entropy satisfy balance laws, which become conservation laws only in special circumstances, but not always.

[9] Compare with the traditional seven wonders: https://www.britannica.com/topic/Seven-Wonders-of-the-World, https://education.nationalgeographic.org/resource/seven-wonders-ancient-world/.

5 Matter

Matter is probably the easiest quantity to grasp intuitively; it's what we can really call "stuff". It is important to clearly distinguish matter from *mass*: mass can be considered a property of matter, but the two are different: for example, the mass of some amount of matter may change, even if the amount of matter stays the same.

There are different kinds of matter, each of which satisfies a balance relation. The distinction into different kinds depends on the physical theory. In most everyday situations, the distinction corresponds to different chemical elements[10], and each satisfies its own balance. These balances are the basis of stoichiometry[11]. If we observe phenomena such as nuclear fission or fusion, however, we notice that the balances of chemical elements are not really satisfied. With such phenomena we make a different distinction of types of matter, for instance *baryons* and *leptons*, and each satisfies again its own balance. It is unclear whether these balances might be broken in other physical phenomena at smaller scales. In these notes we shall usually consider chemical elements as the different kinds of matter, making some exceptions in discussion of nuclear phenomena.

5.1 Electric charge and magnetic flux

5.2 Magnetic flux

5.3 Energy

5.4 Momentum

5.5 Angular momentum

5.6 Entropy

[10] https://doi.org/10.1351/goldbook.C01022 [11] https://doi.org/10.1351/goldbook.S06026

6 Thermodynamics

6.1 Notes on "quasi-static" processes

In general, none of the statements "reversible $\Rightarrow$ quasi-static" or "quasi-static $\Rightarrow$ reversible" is true.

A counterexample to the second implication are systems with internal state variables, which cannot be made non-dissipative, no matter how slowed-down they are. See the discussion and mathematical analysis in Astarita § 2.5.

A counterexample to the first implication is a system of spins in a crystal lattice. It is possible to *reversibly* bring the system form an equilibrium state to another with opposite temperature by reversing the external magnetic field *as fast as possible* – and therefore *not* through a quasi-static process. In fact it is key here that the process be *not*-quasi-static, but as fast as possible, because a slow change of the external magnetic field would lead to an irreversible process with dissipation. For more details see the discussion in Buchdahl, Lecture 20.

The point is that for some systems a *fast* change can actually prevent the onset of dissipative phenomena, and so the process needs to be fast if we want it to be reversible. Adiabatic processes often also need to be fast (as a curious historical fact, Truesdell & Bharatha, Preface p. xii, remark that "In introducing what we today call an 'adiabatic process', Laplace called it 'a sudden compression', in which he was followed by Carnot").

In fact, clearly non-quasi-static phenomena such as *explosions* can in some circumstances be described by *reversible* processes! This is possible if the explosion involves many shock waves, as explained by Oppenheim, chap. 1 p. 63:

> If there is more than one shock, the losses in available energy are diminished, so that in the limit, with an infinite number of shocks, they become negligible, and the process acquires the character of a thermodynamically optimal, i.e., reversible, change of state. The study of explosion processes reveals that, indeed, they are associated not with one but with a multitude of shocks.

For explosions see also the mathematical analysis by Dunwoody: *Explosion and implosion in a mixture of chemically reacting ideal gases*, where again reversible-process equations are used.

A caveat about reversible and quasi-static associations is given by Ericksen (§ 1.2):

> Some tend to associate nearly reversible processes with those taking place very slowly – the "quasi-static" processes. This probably stems, at least in part, from experience with classical theories of heat conduction, viscosity, and so on. However, a ball made of silly putty behaves almost reversibly when bounced rapidly and various other high polymers have similar predilections. So, it seems prudent to be open-minded in considering what may be reversible processes for particular systems.

He later discusses (§ 3.1) the case of bars subjected to dead loads, for which we can have reversible processes under sudden jumps in elongation. He concludes (p. 46) that "the sudden jump provides an example of a process that is reversible but not reasonably considered to be quasi-static".

But there's an important question that underlies our discussion: what do we actually mean by "quasi-static"? We need to specify a time scale, otherwise the term is undefined. For example, a geological process (say, tectonic motion) can be considered as quasi-static – or even completely static – on time scales of minutes or days; but it is not quasi-static on time scales of millions of years.

Whether a process is reversible or not, within any tolerance needed, is an experimental question. We can measure any relevant quantities, say pressure p and exchanged heat q, under the process, and compare them with those, p^* and q^*, determined by the equations for a reversible process. We may find for example that at all times

$$\left|\frac{p-p^*}{p^*}\right| < 0.001 \,, \quad \left|\frac{q-q^*}{q^*}\right| < 0.001$$

and conclude that the process is reversible, if relative discrepancies of 0.1% or less are negligible in our concrete application.

But suppose that someone tells us “if you want the process to be reversible, you must make sure that it is quasi-static”. Alright, but how much is “quasi-static”? is it OK if the piston moves with a speed of 1 cm/s? or is that too much? How about 1 mm/s? – In fact we may find that for some kind of fluid 1 cm/s is absolutely acceptable for the process to be reversible, whereas for another kind of fluid that speed would lead (at the same temperature) to an irreversible process.

You see how this imprecise situation can lead to circular definitions: "if the process is irreversible, then it means it isn’t quasi-static" – but then we are actually *defining* "quasi-static" in terms of "reversible"! Any statement of the kind "reversible ⇒ quasi-static" or "quasi-static ⇒ reversible" then becomes not a matter of experimental verification, but of pure *semantics*. At this point we can simply get rid of "quasi-static" terminology since it doesn’t bring any new physics to the table. This circularity is admitted for example by Callen in discussing irreversible gas expansion (Problem 4.2-3 p. 99):

> The fact that $dS > 0$ whereas $dQ = 0$ is inconsistent with the presumptive applicability of the relation $dQ = T\,dS$ to all quasi-static processes. We define (by somewhat circular logic!) the continuous free expansion process as being "essentially irreversible" and *non-quasi-static*.

A similar criticism can be read in Astarita, § 2.9, p. 62, where he also provides a mathematical quantification of quasi-static, similar to the one given above for reversibility:

> Often this point is circumvented by bringing in another difficult concept, that of a quasi-static transformation, which proceeds "through a sequence of equilibrium states." Quasi-static is an impressive word, but the only meaning which can be attached to it is the less impressive word "slow" – and how can one speak of slowness without implying the concept of time? How slow is slow enough? If one chooses to develop a thermodynamic theory (rather than a thermostatic one), the answer is easy. For instance, in the case of a system where the state is $V, T, \dot{V}$ [the latter is the rate of change of V], one needs to assume that [the non-equilibrium pressure]

$p(V, T, \dot{V})$ is a Taylor-series expandable at $\dot{V} = 0$ to obtain [that

$$p = p^* + \left.\frac{\partial p}{\partial \dot{V}}\right|_{\dot{V}=0} \dot{V} + \mathrm{O}(\dot{V}^2) ,$$

where $p^* = p(V, T, 0)$ is the pressure at equilibrium]. One then reaches the conclusion that if the condition

$$\dot{V} \ll \frac{p^*}{\partial p / \partial \dot{V}|_{\dot{V}=0}}$$

is satisfied, then indeed the difference between p and p^* is negligibly small as compared to p^*, and thus the process can be regarded as a quasi-static one.

Criticisms against the fuzzy notion of "quasi-static" have appeared in many other works. Truesdell & Bharatha (Preface p. xii), make the historical remark that "the 'quasi-static process' was barely mentioned for the first time in 1853 and was altogether foreign to the early work [in thermodynamics]". See also the mathematical analysis by Serrin: *On the elementary thermodynamics of quasi-static systems and other remarks*.

I also want to point out that "quasi-static" in some works has specific meanings somewhat unrelated to the discussion above. For example that the rate of increase of the total kinetic energy K of the system is negligible, so that the law of energy balance, which in its full generality is

$$\frac{\mathrm{d}(U + K)}{\mathrm{d}t} = Q + W$$

(that is, the rate of increase of internal energy U and kinetic energy is equal to the heat rate Q and work rate W provided to the system) can be approximated by

$$\frac{\mathrm{d}U}{\mathrm{d}t} = Q + W .$$

Or that similar inertial terms in the motion of the system are negligible. See for example the book by Day, chap. 2.

But note that such definitions of "quasi-static" have, again, *no* a-priori relation with reversibility.

Finally, the equation $\mathrm{d}S = Q/T$ is only valid for a process that is:

- reversible (by definition),

- closed (no exchange of mass),
- with a homogeneous surface temperature,
- without bulk heating (such as instead happens in a microwave oven).

Under the last three conditions we have in general that $\mathrm{d}S \geqslant Q/T$; when the equality sign is satisfied, then the process is *defined* as reversible. See Astarita, § 1.5, or Müller & Müller, for the different forms of the second law under different circumstances. This equation may be valid in quasi-static and non-quasi-static processes, as explained above.

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("de *X*" is listed under D, "van *X*" under V, and so on, regardless of national conventions.)

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