Chapter 0

Guide to the student

Maths prerequisites

- Working familiarity with algebra, its operations, and their properties, including square roots; knowledge of exponentials and logarithms is useful.
- Working familiarity with solving equations and inequalities, linear and non-linear.
- Working familiarity with the study of functions of one real variable.
- Working familiarity with derivatives.
- Understanding of what an integral is, even if you won't be required to solve integrals.
- Working familiarity with vector calculus.
- Some familiarity with functions of many variables.
- Understanding of what partial derivatives are.

Physics prerequisites

Just some vague reminiscences of secondary/high-school physics should be enough.

It can be beneficial if you are familiar with basic physics notions such as *velocity*, *mass*, *force*, and similar ones; and if you vaguely remember about Newton's laws.

Structure of this text

Graphical devices

The text includes the following graphical devices:

- Important notions and definitions are also given in **boldface**.
- Important-notion boxes:

Something

This is a definition or explanation of Something.

- Warnings and important points that require careful thinking:

Something

Something you must be careful about.

- Exercises:

Exercise 0.1

This isn't an exercise

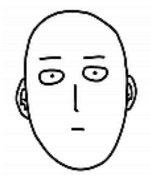
- Discussions and connections with more advanced physics:

How things really are in quantum physics

Just for your curiosity: you don't need to remember any of this.

Side pictures and quotes

Pictures, graphs, or quotes related to the material are displayed on the right.



This is an image of Saitama, which actually has nothing to do with the text on the left.

Hyperlinks and bibliography

Some words are hyperlinks, like this one about One Punch Man[1]; you also recognize them because they have a little footnote number. The links' URLs are listed at the end of each chapter, just in case you're reading a printed copy and wonder what the link was.

The text gives bibliographic references, like "Einstein 1905a", to scientific literature. The references are listed in the final Bibliography on page 103.

These references are not part of the course and don't need to be "studied". They are given for two reasons:

- For your own curiosity.
- To back up what's written in the text. In science you should not believe something just because you've read it somewhere. You should, as much as possible, go and check *for yourself how the logic behind the statement is proved* and *what is the experimental evidence behind the statement*.

"Believe nothing, O monks, merely because you have been told it, or because it is traditional, or because you yourselves have imagined it. Do not believe what your teacher tells you merely out of respect for the teacher."

(attributed to Gautama Buddha)

☛ Notation and terminology

Mathematical notation, as well as notation for physical dimensions, strictly follows the standards given by the International System of Units (SI)[2], listed for example in ISO 2009 and ISO 2019.

URLs for chapter 0

1. https://onepunchman.fandom.com
2. https://www.nist.gov/pml/special-publication-811

Chapter 1

Physics?

If you think about it, many things we ordinarily do every day are some sort of magic. Think of how you can instantaneously see and speak with a person living on another continent, in real time, using just a small widget in the palm of your hand. Think of how you can instantaneously see where you are on the Earth, using the same widget. Think of how fast you can go to another country, by flying in a huge metal thing. Think of how you can command and interact with a purely fictitious animated world when you play on your computer. The list can go on forever. Other things are luckily less ordinary, but still inspire a lot of awe: think of the devastating power unleashed by something roughly as small as a tennis ball, in an atomic bomb.

We can do these astonishing things thanks to our understanding of how the world works. That's Physics.

Many things can be said and have been said about science and physics. Rather than repeating what's been already written in many excellent books, I invite you to take a break here and go read their introductions. Choose as you please; compare what they say; don't limit yourself to popular books.

. . . .

1.1 Several possible formalisms or "languages"

Physics can be expressed and written from wildly different points of view, using wildly different principles. Let's call these "different physics languages"; a more technical name is "physics formalisms". One may approach a physics phenomenon or problem in terms of *Lagrangeans*, or

Hamiltonians, or *fibre bundles*, or *categories*, or *action principles*, or many other formalisms. These formalisms or languages are not completely separated; we know how to translate among them. In “doing” physics, one may jump among formalisms, because some ideas may be easier to express, or some results easier to find, in one formalism than another. No matter which physics formalism you choose, the results and the concrete applications are still the same. The choice is to a great extent subjective, based on your aesthetic tastes. You see that in “doing” physics you can express your personality and put your own artistic touch; this is why it's such a cool subject (and other scientific subjects are like this too).

$$\delta \int L \,\mathrm{d}t = 0 \quad L = \frac{1}{2} m v^2$$

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t} m \boldsymbol{v} \quad \boldsymbol{F} = 0$$

Example of two different formalisms (red, blue) expressing the same physical phenomenon.

In these notes I'm choosing one particular formalism: the one that for me is the most easily *visualizable*; because I believe that visualization can be beneficial in learning new things. Or maybe I'm choosing it just because I like it best. I encourage you to explore how the physics you've learned is expressed in other physics formalisms; maybe you'll like another physics formalism better.

The formalism we'll be using might be called “field theory”. Roughly speaking it takes as starting point the ideas of space and time, or better spacetime, in which there are different kinds of “stuff”. It expresses the regularity and patterns that we observe in physical phenomena as “budgets” about the different kinds of stuff, and of relations between these kinds. Please don't take the description just given too literally; it's just meant to give you a very vague idea of the field-theoretical viewpoint.

It goes without saying that all these “physics languages” are to a great extent mathematical.

“this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics” Galilei 1623

One reason is that numbers allow us to convey information in a concise and precise way. Imagine you have to tell someone, who doesn't know Bergen, where in Bergen you are right now, to within 10 m. You can do that with a description, “... and there's a building called so-and-so which looks like so-and-so...”, which would be lengthy and tricky. Or you can just give two numbers: latitude and longitude:

$$60.369\,40, \;\; 5.3518 \,.$$

And in these two numbers all digits are important; for instance, the latitude is not 60.369**47**.

But the most important reason is that mathematics allows us to describe and follow the patters and variety of physical phenomena in a greatly concise and precise way. And to develop their relationships in a rigorous way. All our present technology would have been impossible to discover, and would be impossible to realize, without the mathematical language of physics.

I invite you again to read what many good texts say about the relationship between physics and mathematics. No point repeating here what is said better elsewhere.

"There is nothing that can be said by mathematical symbols and relations which cannot also be said by words. The converse, however, is false. Much that can be and is said by words cannot successfully be put into equations, because it is nonsense."
Truesdell 1966

1.2 Quantities: primitive and derived

One topic must be briefly discussed because it's important for understanding the notes that follow. It's the distinction between *primitive* and *derived* quantities.

I shall assume that you already know what a **physical quantity** is. Examples are: position, duration, velocity, pressure, energy, temperature.

A **derived quantity** is one that is defined in terms of other quantities. For example, velocity $\boldsymbol{v}$ (more precisely: average velocity) is defined as the ratio between a distance $\boldsymbol{d}$ (a vector) and a time duration $\boldsymbol{t}$:

$$\boldsymbol{v} := \frac{\boldsymbol{d}}{t}$$

where the symbol ":=" means "is defined as" or "is defined by". This means that in principle we could avoid using the word "velocity" and the symbol "$\boldsymbol{v}$" altogether, and instead always speak about distance and duration, using their symbols. It would lead to very long sentences and formulae and would be extremely inconvenient, but it could be done. The definition of a derived quantity often tells us how that quantity can be measured.

A derived quantity is defined in terms of other quantities, and these may in turn be derived quantities, that is, defined in terms of still other quantities, and so on. But at some point this chain of definitions must come to an end, otherwise we would go around in circles.

A **primitive quantity** is one that we do not define in terms of other quantities. Primitive quantities are the building blocks from which we define all others. That they are not defined in terms of others doesn't mean that we cannot try to explain them. But such explanations must be taken as informal and heuristic. Primitive quantities are often explained through metaphors and by appealing to intuition. You must always be wary of such explanations, because they may fail you spectacularly in some situations.

Often we have a choice about which quantities should be primitive and which should be derived. For instance, *energy* can be defined, in a somewhat complicated way, in terms of quantities like *work* and *heat*, which would then need to be taken as primitive. Or we can take *energy* as primitive, and define *work* and *heat* in terms of it. This second choice can be more convenient to develop a physical theory. It often happens that a quantity is very convenient for building a theory, if used as primitive; but difficult to understand intuitively. Vice versa, a quantity can be very intuitive but lead to a complicated theory. Among quantities which we'll take as primitive are: *time*, *space* and *length*, *matter*, *energy*, *momentum*, *entropy*, *temperature*, and several others. All will be discussed soon.

1.3 Physical dimensions and units

Measurement is the process by which we determine the value of a physical quantity. Measurements can be extremely complex, and can extremely different even if they are about the same quantity. Consider the ways we can measure the mass of a football, compared to the ways we can measure the mass of the Sun.

To each quantity we associate a **physical dimension**. The term 'dimension' here has nothing to do with physical extension, as in "the dimensions of this box"; be careful not to confuse the two. Usually it's clear which one is meant from the context. Physical dimensions help us avoid making operations that don't make sense with some quantities. For example, it doesn't make sense to sum up the volume of a body of water with its temperature; and indeed the volume has dimension *length*^{3}, whereas temperature has dimension *temperature*, and quantities with different dimensions cannot be added up.

With each physical dimension we can associate a measurement **unit**, which expresses a basic standard for comparing the measurement results of similar quantities. For example, we could use the *minute* or the *second* as units to measure *time*.

One can choose a basic set of physical dimensions from which to define all others, and for these a set of standard units. Here we shall follow the International System of Units (SI)[1] (see also NIST Special Publication 811[2]).

The topics of measurement and physical dimensions, which are studied in *metrology* and in *dimensional analysis*, could occupy an entire course by themselves. I shall assume that you already know their basics notions and that you read about the SI.

The measurement of some physical quantities consists in just one number with associated physical dimension; we shall call such quantity a

scalar. The measurement of other physical quantities consists instead in a triplet of numbers with associated physical dimension; we shall call such quantity a **vector**.

What's scalar or vector depends on the theory

Scalar and *vector* have very specific and slightly different meanings in different theories, so don't take the definitions used here as universal. For example, in these notes and in Newtonian mechanics we call *energy density* a scalar, but in general relativity it cannot be called a scalar.

1.4 Informal tips about units and maths

Importance of units

Units are very important and must always be written for several reasons.

First, a number without units doesn't tell us anything. If I tell you "the place is at a distance 100 from here", you have no idea how far the place is. "100" what? 100 metres? 100 kilometres? These are completely different distances.

Second, units give us useful information about mathematical objects and their physical relationships and measurement. If you see the expression "3 m/s", then there's a strong possibility that that's a velocity. If you see the expression "5 J/m^2", then you have a hint that it could be measured by measuring an energy and then dividing by an area.

Third, because of the previous reason, keeping track of units often allows us to quickly catch errors in solving a physical problem.

Variables and units

When a physical quantity is denoted by a symbol or variable, keep in mind that a unit is "contained" in the symbol, so to speak. For example if the variable t denotes a time, then it includes some time unit, say seconds. This becomes apparent when we write the value of the symbol, for instance "$t = 120\ \mathrm{s}$". The unit is not predetermined, but it must correspond to the dimension of that quantity. We could for instance write "$t = 2\ \mathrm{min}$" instead; the two expressions are completely equivalent.

This fact must be kept in mind when combining symbols. For example, if d is a distance and t is a time, then writing $v = d/t$ tells us that v is a velocity, and it has appropriate units that come from d and t, for instance m/s.

Units otherwise behave just like *literal constants* for all mathematical purposes, just like the letter 'a' in the expression '$a\,x$'. This is why they can, for instance, be simplified:

$$3\,\mathrm{mol/s}\cdot 5\,\mathrm{s} = 3\,\frac{\mathrm{mol}}{\cancel{\mathrm{s}}}\cdot 5\,\cancel{\mathrm{s}} = 15\,\mathrm{mol}\,.$$

Mathematical functions and units

Particular care must be taken with trigonometric and exponential functions, such as sin(), cos(), tan(), exp(), log(); **these functions only admit a dimensionless argument** (which for the trigonometric ones corresponds to *radians*). So there cannot be units like 's' or 'm' within these functions: we must make sure that any units present within cancel out.

This makes sense, because we wouldn't know how to interpret the argument otherwise. Suppose you read "cos(60 s)" somewhere: how much is that? If we say "just discard the unit", we would have

$$\cos(60\,\mathrm{s}) \stackrel{?}{=} \cos(60) \approx -0.95$$

but wait: 60 s ≡ 1 min, so we could equivalently write "cos(1 min)". Then, according to the hypothetical rule "just discard the unit", we would have

$$\cos(60\,\mathrm{s}) \equiv \cos(1\,\mathrm{min}) \stackrel{?}{=} \cos(1) \approx +0.54$$

a completely different result!

For this reason an expression like '$cos(t)$', with t denoting time, doesn't make sense: there's a time unit in the argument of cos(). If we want to express an oscillation with time, we must write instead something like

$$\cos\left(\frac{t}{T}\right)$$

where T is the period of the oscillation, a symbol which also includes a time unit, which simplifies with the one in t. If the period of the oscillation is $T = 1$ s then we can also simply write

$$\cos(t/\mathrm{s})$$

This expression is now unambiguous: suppose that $t = 60\,\mathrm{s} \equiv 1\,\mathrm{min}$, then

$$\begin{aligned}\cos(t/\mathrm{s}) &= \cos(60\,\mathrm{s/s}) = \cos(60) \approx -0.95\\ &= \cos(1\,\mathrm{min/s}) = \cos(1\cdot 60\,\cancel{\mathrm{s}}/\cancel{\mathrm{s}}) = \cos(60) \approx -0.95\end{aligned}$$

Also remember that **the results of trigonometric and exponential functions are dimensionless numbers**, so an expression like '$3\cos(\dots)$' denotes a pure number, with no units. If you want to express that the result is a length, the appropriate units must appear. We can for instance write

$$L\cos(\dots)$$

where L is a length, and therefore includes some kind of length unit such as 'm'. If this length is, say, $L = 2\,\mathrm{m}$ we can also simply write

$$2\cos(\dots)\,\mathrm{m}$$

Units and derivatives

When we follow the rules above, all other mathematical operations automatically take care of everything. The derivative, for instance, is calculated in the usual way, treating any visible units as literal constants. Let's see a concrete example. This expression

$$x(t) = 2\cos(t/\mathrm{s})\,\mathrm{m}$$

says that the position of some object oscillates with time, between the values $-2\,\mathrm{m}$ and $+2\,\mathrm{m}$. When $t = 0\,\mathrm{s}$, the position is $x = +2\,\mathrm{m}$. The position $x = -2\,\mathrm{m}$ is reached when the argument of cos() is π, that is

$$t/\mathrm{s} = \pi \quad\Rightarrow\quad t \approx 3.14\,\mathrm{s}\,.$$

The velocity of the object (see § 2.4) is given by the derivative of this expression with respect to t. Let's calculate it treating all unit symbols as literal constants:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(2\cos(t/\mathrm{s})\,\mathrm{m}\Big) = 2\underbrace{\left[-\sin(t/\mathrm{s})\cdot\frac{1}{\mathrm{s}}\right]}_{\text{chain rule}}\mathrm{m} = -2\sin(t/\mathrm{s})\,\mathrm{m/s}$$

and you see that the correct units for velocity have automatically appeared.

1.5 What is “fundamental” physics?

But what’s the “ultimate” goal of physics? What’s “fundamental” physics? The answer to this question is again subjective – also in this case physics lets you express your proclivities and personality. In the history of physics one can probably identify two main conceptions of “fundamental” physics.

For some physicists it is about finding the ultimate building blocks, so that one day we can say “. . . and these are the constituents, and they obey these equations”. The history of physics seems to show that this goal is overturned every few generations. And yet every generation says “*Now* we almost have the complete picture – it’s right behind the corner. It’s true that previous generations thought they almost had it, and turned out to be wrong. But *this time* is different, this time we have the real deal!”. The theoretical and particle physicist Geffrey Chew[3] depicted this situation as in fig. 1.1. For this reason some physicists are a little sceptical about this goal; maybe it’s a never-ending structure, with surprises at every deeper look.

So for other physicists fundamental physics is about finding some point of view or mathematical structure that is rich enough to make useful predictions, and yet flexible enough to accommodate any new patterns or objects that we might discover. In a manner of speaking, it is about finding “patterns of patterns” or “laws about physical laws”.

The two conceptions above are not mutually exclusive, and both are always pursued, even if time-changing fashions may emphasize the one or the other.

In these notes we take a point of view slightly closer to the second conception. This will also be reflected in the main division between physical laws that we’ll draw in ch. 5.

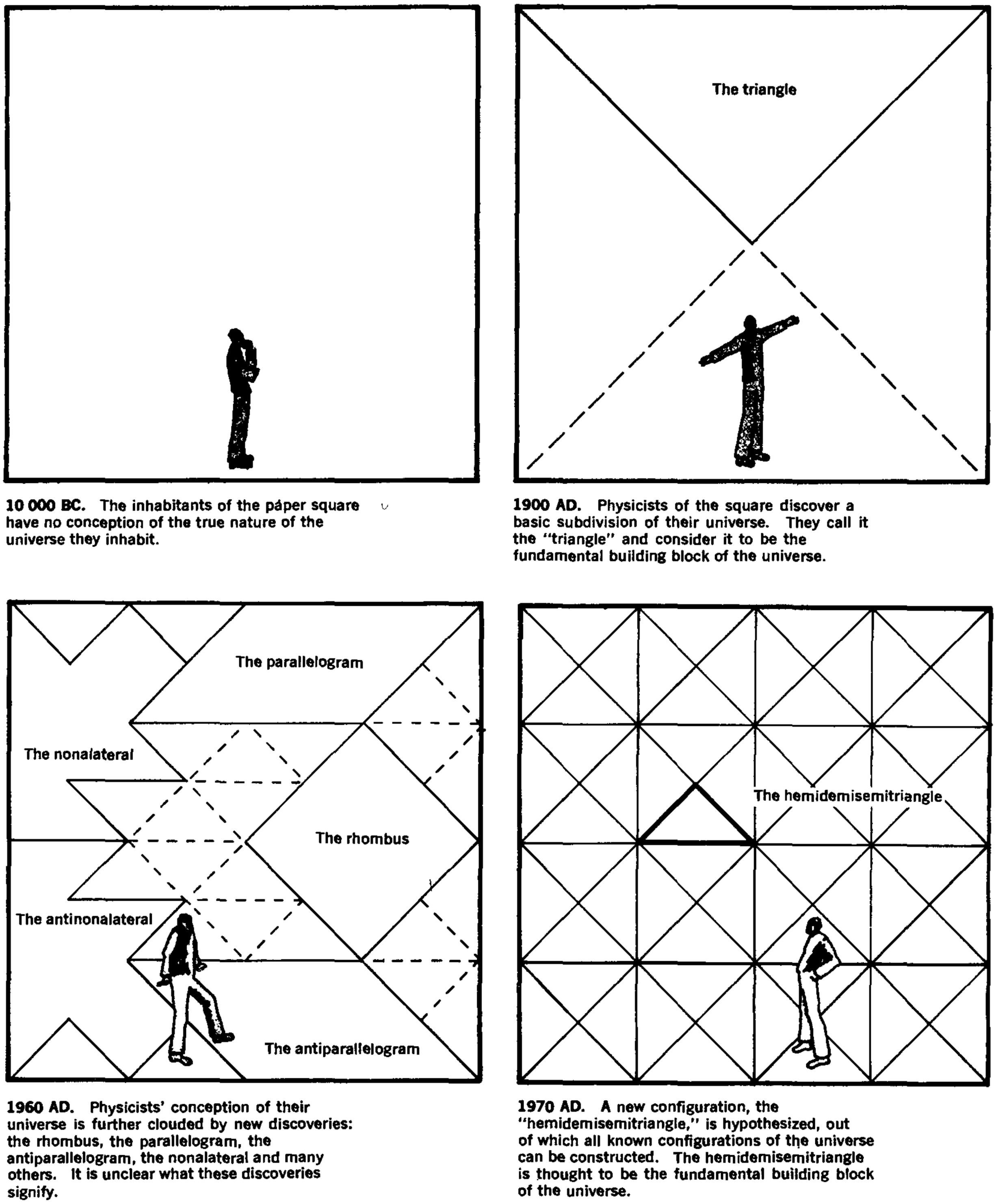
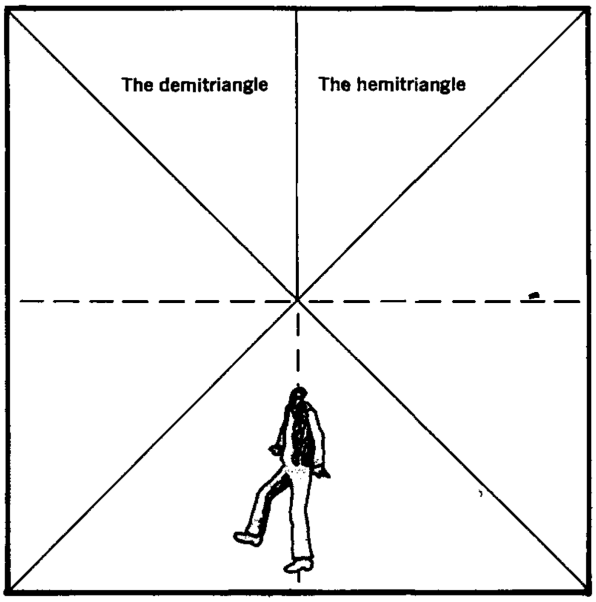
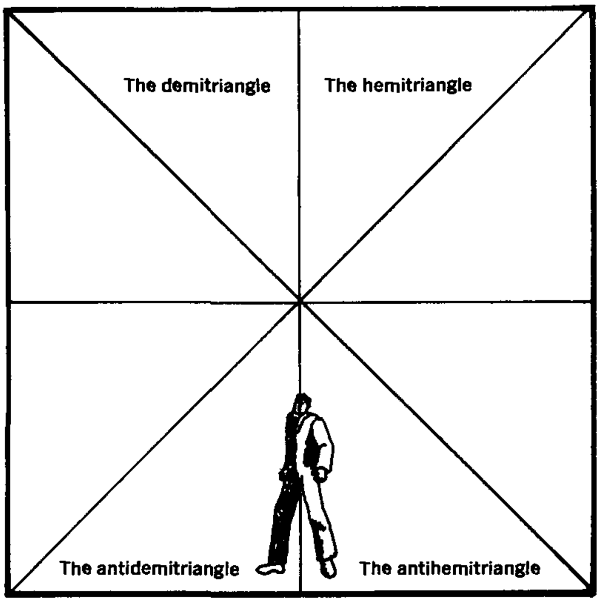


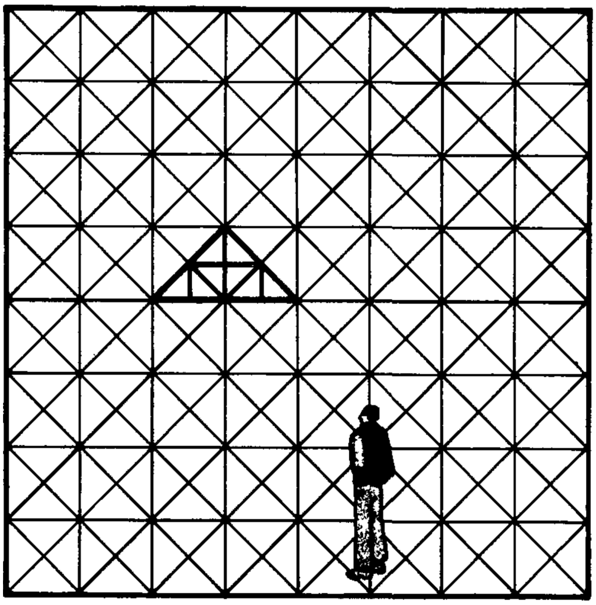
Figure 1.1 (Continues on p. 17) *The progress of "fundamental" physics*, from Chew 1970 as reproduced in Truesdell 1987



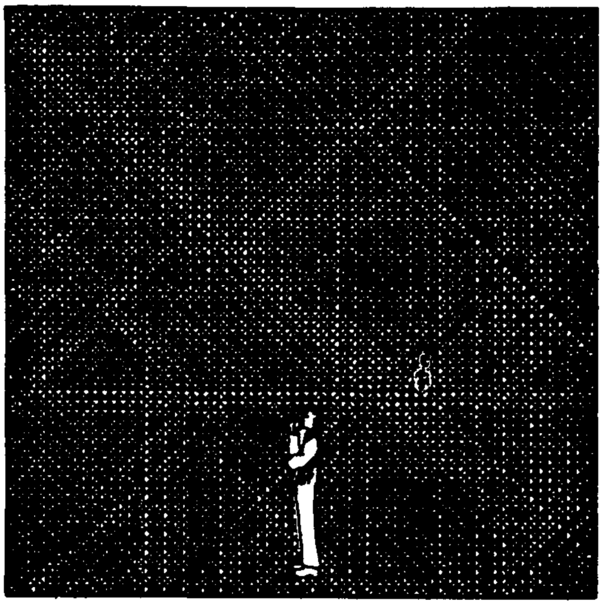
1930 AD. Physicists discover that the triangle can be split. Its parts are termed the "hemitriangle" and the "demitriangle." These are thought to be the fundamental building blocks of the universe.



1950 AD. Mirror images of the hemitriangle and the demitriangle are discovered. These are termed "antihemitriangle" and "antidemitriangle."



1975 AD. The hemidemisemitriangle is discovered. The following year the hemidemisemitriangle is split.



2000 AD. The inhabitants of this paper square have no conception of the true nature of the universe they inhabit.

URLs for chapter 1

1. https://www.nist.gov/pml/owm/metric-si/si-units
2. https://www.nist.gov/pml/special-publication-811
3. https://www.physics.lbl.gov/rememberinggeoffreychew/

Chapter 2

Building blocks: Time and space

If we want to describe the motion *of a material point, we give the values of its coordinates as a function of time. However, we should keep in mind that for such a mathematical description to have physical meaning, we first have to clarify what is to be understood here by "time". We have to bear in mind that all our propositions involving time are always propositions about* simultaneous events. *If, for example, I say that "the train arrives here at 7 o'clock", that means, more or less, "the pointing of the small hand of my clock to 7 and the arrival of the train are simultaneous events".*

A. Einstein 1905a

2.1 Time

Time is a primitive quantity. We understand the notion of time intuitively, even if it's difficult to explain (that's why it's taken as primitive). In 1905, with the theory of relativity, part of our everyday intuition about this notion was seriously shaken. For many years afterwards our old intuition could still be used in practice and in applications. But the new, correct intuition is becoming more and more important in everyday life and technologies. For example, GPS navigation – which we use everyday from leisure activities such as hiking or sightseeing to more critical ones such as aeroplane landing – critically depends on the correct notion and intuition of time.

Let's see how our traditional intuition goes astray with a concrete experiment. Here's Alice, Bob, and Charlie. They have extremely precise clocks built in exactly the same way. They stay very close to one another and synchronize their clocks. Still keeping close, they go around, maybe on an aeroplane or space ship, and all the time they check their clocks. They notice that their clocks stay perfectly synchronized all the time, no matter where they go and what they do.

At some point they separate, each one going around independently. One of them might stay in place, another might take a helicopter, and another might go for a trip on Mars and back.

Alice and Bob at some point meet again, and compare their clocks. They see that their clocks aren't synchronized anymore; the difference could be as small as microseconds, or as large as years. In fact, if this time discrepancy is large, they would notice that they themselves have aged differently; so time discrepancy doesn't affect the clocks only. Let's say for concreteness that Alice's clock is ahead of Bob's, or equivalently that Bob's is behind Alice's. Note the following aspects:

First, neither Alice or Bob can say "my clock was wrong": neither has noticed anything strange about the "passage of time".

Second, they might wonder what's the time on Charlie's clock. But Charlie is at some distance away. They could decide to contact Charlie via radio, say, and ask "what shows your clock right *now*?". But they would notice that there's a delay, even if extremely small, in the radio transmission; so it's unclear to what time would Charlie's answer apply. If we say "let's account for the radio-signal speed", we see that there's a logical problem: speed is distance divided by time, and here we have a problem in exactly determining what's the "correct" time! So we would be reasoning in circles. Besides, even neglecting these difficulties, Charlie's answer could reveal a time that completely different from Alice's and from Bob's – it could be years ahead or behind both of theirs!

Third, if they now stay together, they will see that their clocks remain exactly synchronized, besides the discrepancy they noticed when they met. This discrepancy doesn't increase or decrease. They may even retrace together Alice's and Bob's previous trips; their clocks still remain synchronized.

The experience just described will occur again any time two or more of them meet. There could be a hundred observers like Alice, Bob, Charlie, initially at the same place and synchronized. Whenever two or more of them meet after having been separated, they will notice discrepancies in their clocks. But their clocks will have exactly the same time lapses as long as they stay together. See the illustration in fig. 2.1

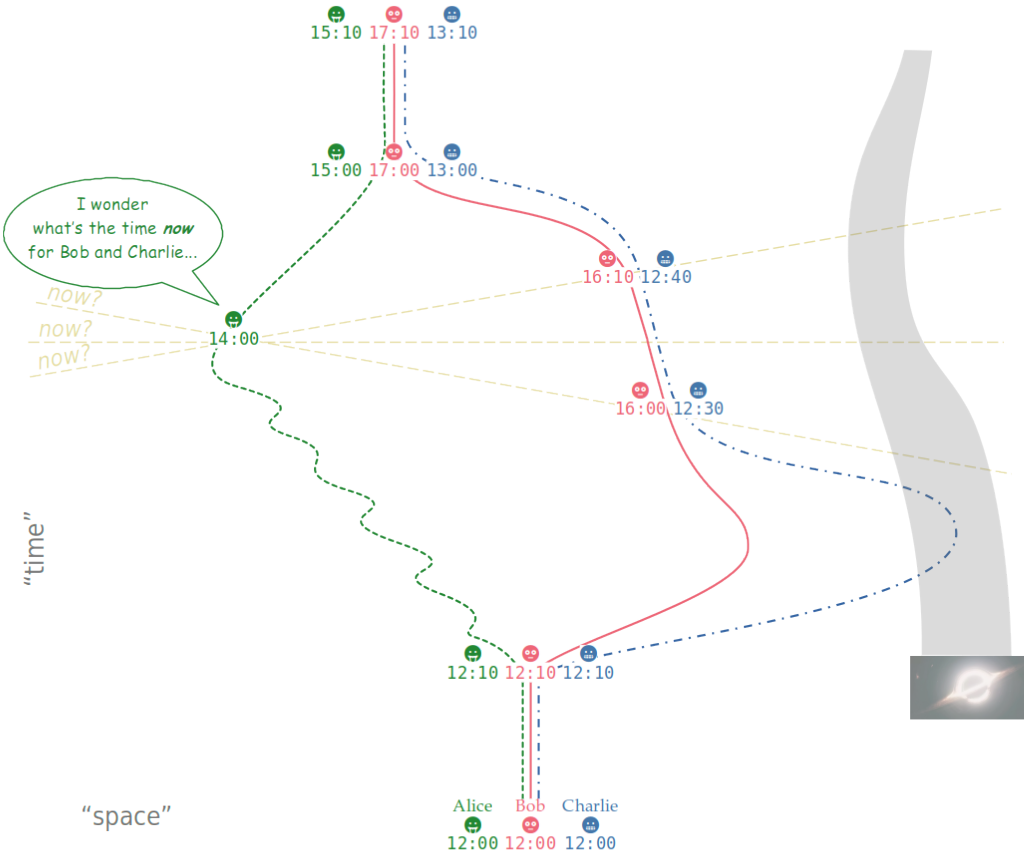


Figure 2.1 Illustration of the experiences of Alice (dashed), Bob (solid), Charlie (dot-dashed) with time. The page represents a two-dimensional spacetime, and is followed from bottom to top.
Bottom: Alice, Bob, Charlie stay close and observe their clocks are perfectly synchronized from 12:00 to 12:10, then they separate.
Right: Charlie visits a region near a strong mass-energy source. Upon meeting again with Bob, the two notice their clocks differ: 16:00 for Bob, 12:30 for Charlie. Yet this clock difference stays the same while they travel together for 10 min.
Left: Alice wanders around travelling at high speed with respect to the fixed stars. At some point she wonders what's the time "now" for Bob and Charlie. Obviously this question doesn't make sense, because (1) when Bob and Charlie are together their clocks differ – impossible to say what's "the" time at their position; (2) not clear which instant in Bob & Charlie's trajectory should be considered as "now" (yellow dashed lines).
Top: When all three meet again, their clocks have completely different readings; and they themselves have aged differently. But their clocks run again at the same rate as long as they stay close.

Consider for a moment an imaginary world in which these experiments had given a different kind of result. According to Newtonian mechanics, whenever two or more initially synchronized observers like Alice, Bob, Charlie had met, their clocks would have always shown exactly the same time. If one year, 23 days, 8 hours, 9 minutes, and 3.045 399 283 240 992 663 02 seconds have passed for you since you last met Alice, you'd see that exactly the same amount of time has passed for her when you two meet again. If you think about it, in this case it would have beeen somewhat natural to think "right now, the clocks of far-away Alice, Bob, Charlie must show the same time as mine" (even though you have no real experimental way of confirming that).

But that's an imaginary world. In our world is the more complicated situation described initially that holds. Only one conclusion can be drawn from these experimental results: **Time is not some sort of universal quantity. It is, so to speak, "local" to a person or clock, or to a group of persons or clocks that stick together.** This also means that *it doesn't make sense* to ask questions like "what can be the time for far-away Charlie, right now?".

The time measured by a specific observer is called the **proper time** of that observer. Luckily we know more about how the proper times of separated observers can differ when they meet again. It turns out – according to our current understanding – that the time differences depend, roughly speaking, on how fast the observers are moving with respect to one another and to matter around the universe, and on how much energy is contained in the regions they travel. The general theory of relativity gives us the equations determining any such proper-time differences.

"In 1976, the International Astronomical Union introduced relativistic concepts of time and the transformations between various time scales and reference systems. [...] Now [...] it is necessary to base all astrometry, reference systems, ephemerides, and observational reduction procedures on consistent relativistic grounds. This means that relativity must be accepted in its entirety, and that concepts, as well as practical problems, must be approached from a relativistic point of view."
Kovalevsky & Seidelmann 2004

The situation depicted in the experiments above is real. It can be measured, for example, comparing initially synchronized clocks that have been put in aeroplanes flying in different directions. Most importantly, it affects everyday relevant technologies such as the Global Positioning System. Formulae from general relativity appear in your phone's GPS software; see for instance § 20.3.3.3.3 of the Interface Control Document `IS-GPS-200` at https://www.gps.gov/technical/icwg/. It must also be taken into account in the establishment and synchronization of time in our everyday equipments:

> International Atomic Time (TAI) is based on more than 250 atomic clocks distributed worldwide that provide its stability, whereas a small number of primary frequency standards provide its accuracy. Universal Coordinated Time, which is the basis of all legal time scales, is derived from TAI. To

> allow the construction of TAI and the general dissemination of time, clocks separated by thousands of kilometres must be compared and synchronized. [. . .] The achieved performances of atomic clocks and time transfer techniques imply that the definition of time scales and the clock comparison procedures must be considered within the framework of general relativity.
> (Petit & Wolf 2005)

In most everyday situations for us, who live on or nearby Earth and move at speeds much smaller than c with respect to one another, the discrepancies between our proper times are so small that cannot be measured with ordinary clocks or with our internal clocks. Consider a person walking 10 m away from you and then immediately walking back to you, at 1 m/s. The time elapsed for you will be 20 s, but for that person will be 19.999 999 999 999 999 889 s, a difference of 10^{-16} s, which is the error of an atomic clock. If human beings still exist in some decades or centuries, with space travel they will probably have to deal more and more with proper-time discrepancies also in everyday life.

"The plot for Cesium [. . .] characterizes the best orbiting clocks in the GPS system. What this means is that after initializing a Cesium clock, and leaving it alone for a day, it should be correct to within [. . .] 4 nanoseconds. Relativistic effects are huge compared to this."
Ashby 2003

For the most part of the rest of these notes, we won't need to deal with differences in proper time. But I recommend that you keep present how time really works, and that these small time discrepancies exist and occur all the time along your *worldline*.

Time has physical dimension of time and we shall for the most part measure it using the unit *second*, symbol 's'.

2.2 Space

Together with the notion of time, also the notion of space loses some of its traditional intuition. Several observers in motion with respect to one another will generally disagree on the dimensions of an approximately rigid object in their vicinity. For objects that are far away from an observer, the very notion of "distance" becomes tricky has different and non-equivalent definitions; one must be very careful on which definition is being used.

We shall not delve further into these peculiarities of time and space. Keep simply in mind that phenomena happen in **spacetime**, and that there's no way to attribute a universal time, nor a universal position in space, to a physical event. There is one absolute: whoever locally measures the speed of light, will find the value

$$c := 299\,792\,458 \text{ m/s} \tag{2.1}$$

This value is exact by definition, and serves as a way to define a local notion of space and distance.

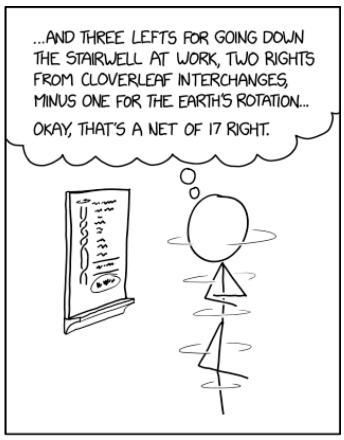
Space has physical dimension of length and we shall for the most part measure it using the unit *metre*, symbol 'm'.

2.3 Coordinate systems and events

It is necessary to have a way for distinguishing physical events and phenomena and locating them in spacetime. This is achieved through a **coordinate system**.

A coordinate system assigns four numerical labels to every point in spacetime. Often these labels have some kind of physical meaning – such as the proper time elapsed for a specific clock, or the distance from some event as measured by a specific observer – but they don't need to.

A coordinate system also solves the problems coming from proper-time and space discrepancies among different observers. We can assign to every physical event a *coordinate time* and a *coordinate spatial position*, which are the same for all observers, because decided by agreement. Coordinate time doesn't have a strict physical meaning, and will generally be different from the proper times registered by different observers. It can nevertheless be used for "doing physics", and it is the time we shall most often use in our equations. A coordinate time commonly used for Earth-physics purposes is Universal Coordinated Time (UTC)[1]. The clock on your phone, and on devices that get synchronized via internet, shows UTC, not your proper time. An observer on Earth at 0 m over sea level, and not moving, measures a proper time exactly equal to UTC (besides small variation coming from the movements of Solar System bodies). Observers at other altitudes or moving with respect to Earth's surface notice that their proper times are slightly different from UTC.



SPACETIME HEALTH TIP: REMEMBER TO CANCEL OUT YOUR ACCUMULATED TURNS AT THE END OF EACH DAY TO AVOID WORLDLINE TORSION.
https://xkcd.com/2882

Up to now we have often used the word 'event', informally taking its meaning for granted. Let's be more precise: we call **event** or **spacetime point** an extremely small region of space – a point – that only lasts for a very small lapse of time – an instant. The name 'event' is used because typically we approximately identify such a point and instant by means of a physical phenomenon of limited spatial extension and short duration, such as the collision of two subatomic particles. The sudden burst of a very small soap bubble can be considered as an event in some circumstances; but something like "a tennis ball" cannot be considered as an event, mainly because a tennis ball exists for quite a long time, not just for a short instant.

From a four-dimensional spacetime point of view, a tennis ball could be characterized as a line: a **worldline**.

We shall often denote the four coordinates of a coordinate system by the letters

$$(t, x, y, z)$$

where t is a coordinate time, usually UTC, and (x, y, r) determine a spatial position. The triplet of spatial coordinates is often denoted by the vector $\boldsymbol{r}$:

$$\boldsymbol{r} \coloneqq (x, y, z) \ .$$

It is always important to specify how the coordinate system you're using is defined. The definition of the spatial coordinates (x, y, z) is typically different from problem to problem. We shall typically use coordinates that form $\frac{\pi}{2}\,\mathrm{rad} \equiv 90°$ angles with one another; but their directions and their origin – that is, where they have value $x = y = z = 0\,\mathrm{m}$ – always depend on the problem, so make sure you always specify them.

Whenever we speak of a "region of space" or of a "surface in space", we mean a 3D or 2D region at some specific coordinate time t.

Some physical phenomena happen along a line, in one dimension. In this case we can omit two of the spatial coordinates, assuming they have some constant, unimportant values. In these cases we can simply write, for instance, (t, x) as our coordinates.

2.4 Velocity and acceleration

In some situations the spatial coordinates $\boldsymbol{r} = (x, y, z)$ may turn out to be functions of the time coordinate t; the typical example is when we describe how the spatial position of a small body changes with coordinate time. We can write this functional dependence in different ways, for instance

$$\boldsymbol{r}(t) \quad \text{or} \quad \big(x(t),\ y(t),\ z(t)\big) \ .$$

So $\boldsymbol{r}$ is a vector function of time, which simply means that we have a collection of three functions of time.

If we take the derivative of each coordinate with respect to the time t, we obtain the **coordinate velocity**

$$\boldsymbol{v}(t) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}x(t),\ \frac{\mathrm{d}}{\mathrm{d}t}y(t),\ \frac{\mathrm{d}}{\mathrm{d}t}z(t)\right)$$

which is also a vector.

Dot-notation for time derivative

The derivative of some quantity with respect to coordinate time is often denoted by a **dot** over the quantity. So we can also write

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \big(\dot{x}(t),\ \dot{y}(t),\ \dot{z}(t)\big)$$

The coordinate velocity is usually different from the *physical velocity*, which an observer would measure using proper time and space, for instance using bouncing light rays. In many everyday situations the difference between coordinate and physical velocity is so small that it can be neglected. But in situations involving subatomic particles at high speed, for example, one must take into account that the two velocities are different.

Taking the time derivative once more we obtain the coordinate acceleration, also a vector:

$$\boldsymbol{a}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}(t) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t),\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t),\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t)\right)$$

which is also a vector.

Acceleration in relativity theory

In relativity theory, acceleration acquires a special physical significance, because it includes the effect of gravity, and its calculation does not involve just a time derivative. For instance, let's say that you are standing still on the ground, and let's use a coordinate system where x points in front of view, y to your left, and z points upwards. Then your coordinate velocity is $\boldsymbol{v} = (0, 0, 0)\ \mathrm{m/s}$ also according to relativity theory. But the spatial part of your acceleration is approximately $(0, 0, 9.8)\ \mathrm{m/s^2}$, not zero!

The definitions and values of acceleration according to relativity theory and according to Newtonian mechanics are therefore quite different even in everyday situations. In these notes we'll mean the Newtonian definition of acceleration, unless stated otherwise.

URLs for chapter 2

1. https://www.nist.gov/pml/time-and-frequency-division/time-realization/utc-nist-time-scale-0/

Chapter 3

Building blocks: "Stuff"

if the skill of the mathematician has enabled the experimentalist to see that the quantities which he has measured are connected by necessary relations, the discoveries of physics have revealed to the mathematician new forms of quantities which he could never have imagined for himself.

J. Clerk Maxwell 1870

3.1 Seven primitive quantities with three basic properties

Besides time and space, our physics formalism includes around seven more quantities that we take as primitive:

Seven primitive quantities

matter
electric charge
magnetic flux
energy
momentum
angular momentum
entropy

Technically they are called *fields*, for reasons we shall shortly see.

Recall that primitive quantities cannot be defined: we can only try to understand them intuitively, for example through their properties. Some of the seven primitive quantities are easier to grasp intuitively than others.

But all seven primitive quantities have three basic properties in common. First of all, of each we can ask, or rather, measure:

The two basic measurements that can be made on the seven quantities

1. **How much of this quantity is in a particular three-dimensional region of space at a particular time instant?**
2. **How much of this quantity flows through a particular two-dimensional surface during a particular time lapse?**

We can ask these questions of any region of space and any time lapse, and the surface in the second question can be moving and deforming. The results of the two measurements above are numbers, which in general can be positive or negative, for scalar quantities; or vectors for vector quantities.

Often we consider the second question, about the flow through a surface, in the case of a very short lapse of time, and divide the total flow by that time lapse. So we have an alternative form of the second measurement:

Flux of a substance through a surface

2b. How much of this quantity is flowing through a particular two-dimensional surface per unit time, at a particular time instant?

This is called the **flux** of the quantity through that surface.

The third property common to the seven quantities regards the two measurements above.

Extensivity of the seven quantities

If we consider two or more separate volumes, the amount of quantity in the total volume is equal to the sum of the amounts in the separate volumes.
Analogously for the flux through separate surfaces.

We say that each of the seven quantities is an **extensive** quantity

The basic measurements above can't in general be made, and don't even make sense, for some other quantities. For instance, we cannot ask "what's the total amount of *temperature* in this region?", or "how much velocity is flowing through this surface?".

Thanks to the three properties above, each of the seven quantities can be intuitively visualized as some kind of "stuff" that can be present at each spacetime point, fills regions of space, and flows through surfaces. This visualization is useful, but also comes with some warnings which we'll discuss later.

What's remarkable about matter, electric charge, magnetic flux, energy, momentum, angular momentum, and entropy, is that *they are common to all our main physical theories*, approximate or not: from Newtonian mechanics to general relativity and quantum theory; from subatomic scales to cosmological scales. The physical meaning and mathematical characterization of these quantities can be slightly different depending on the physical theory and scale. For example, in quantum theory they are mathematically represented by so-called operators rather than functions; and at molecular scales entropy has a meaning connected with probability theory. Yet, these seven quantities are really universal to our present way of doing physics and of describing physical phenomena around and within us.

Let us make a first acquaintance with these seven quantities. The discussion that follows is meant as an introduction; we shall repeat and say more about each quantity in later chapters.

3.2 Matter

Matter is a *scalar* quantity, with SI dimension amount of substance[1], and measured in units of *moles* (mol)[2]. Its flux is therefore measured in units of *moles per second* (mol/s). In statistical mechanics and particle physics, matter is often simply counted and so measured in dimensionless units, rather than moles.

Matter is probably the easiest quantity to grasp intuitively; what we ordinarily call "stuff" is matter. It is usually classified into several kinds; the classification depends on the physical phenomena and physical theory one works with, and is related to whether the kinds of matter can be considered separately conserved, as we'll discuss in § 6.1.

In everyday phenomena not involving radioactivity[3] or nuclear energy[4], the different kinds of matter approximately correspond to the non-radioactive chemical elements[5]: hydrogen[6], helium[7], lithium[8], and so on. Note that some common everyday devices, such as smoke detectors, do involve radioactivity.

In phenomena involving radioactivity or nuclear energy, the different kinds of matter correspond approximately to *baryons*[9], such as protons and neutrons; and *leptons*[10], such as electrons. In particle physics, even more subtle classifications of matter are made, into kinds that seem to be conserved, such as electronic-leptons, muonic-leptons, and others. This kind of research is still open, but it seems that the total amount of baryonic and leptonic matter, independently of the kinds into which it can be classified, is always conserved. Note that we're using the term 'matter' in a sense that includes anti-matter, such as positrons.

The total amount of matter in a region *can be negative*. "Negative matter" is what's usually called anti-matter[11]. Anti-matter appears in small amounts in everyday life, for example in connection with common radioactivity processes. It is also created and used in medicine, in positron-emission tomography (PET)[12] scans.

In these notes we shall usually not consider distinctions between different kinds of matter, making some exceptions for discussions about chemical reactions and nuclear phenomena.

Matter: notation

The amount of matter in a region is usually denoted with N, and flux of matter with J. In chemistry we usually specify what kind of matter we are speaking about, writing for instance $N_{\mathrm{Ca}} = 5.3\,\mathrm{mol}$, to indicate an amount of 5.3 mol of calcium[13] atoms.

Matter is different from mass or energy

It is important to clearly distinguish matter from *mass* or *energy*. Mass can be considered a property of matter, but the two are different. In nuclear reactions, for instance, the mass of some amount of matter may change, even if the amount of matter stays the same.

As far as we know, the total amount of energy associated with an amount of matter is always positive, whether the amount of matter is positive or negative (antimatter). This is the reason why antimatter "falls" just like positive matter, a fact that has been experimentally confirmed: see Anderson et al. 2023.

Exercise 3.1

According to statements on symmetrymagazine.org[14] and quantum-diaries.org[15],

> The average banana (rich in potassium) produces a positron roughly once every 75 minutes.

Unfortunately the original site where the this statement was discussed, and the corresponding calculation made, seems not to exist anymore.

1. **Do a little research and find out whether this statement is true.**
2. **From your research, approximately quantify the flux of positrons around an ordinary banana, expressing it in** particles/s**.**

How many positrons do bananas produce?

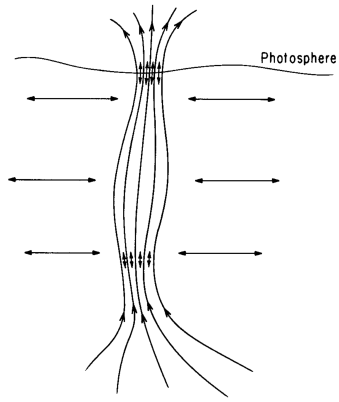
3.3 Electric charge and magnetic flux

Electric charge is a *scalar* quantity, with SI dimension electric charge[16] (equivalent to electric current×time), and measured in units of *coulombs* (C)[17]. Flux of electric charge is called *electric current*, and measured in units of *amperes* (A = C/s)[18].

Electric charge is a quantity that is easily grasped in our everyday experience, and doesn't need much comments.

Magnetic flux is a *vector* quantity, with SI dimension magnetic flux[19], and measured in units of *webers* (Wb)[20].

The electromagnetic field is most commonly represented by vectors associated to each point in space. But it can also be interpreted and visualized as a collection of moving, oriented tubes or lines, either closed or extending indefinitely; somewhat analogously to how we visualize matter and charge, as moving blobs or points, but with one more dimension. This interpretation goes back to Faraday (1846), Maxwell (1855), and later Dirac (1955) among others, and today is conveniently used in some fields such as solar physics[21], for example to study sunspots[22] (see Ryutova 2018). From this point of view, the *voltage* turns out to be the flux of the magnetic flux; it is indeed measured in *volts* (V), equivalent to *webers/second*.



"sketch of the magnetic lines of force in a magnetic filament extending up through the photosphere." Parker 1974a

Electromagnetism and this particular visualization of it are very fascinating topics, but we shall not discuss them in these notes.

3.4 Energy

Energy is a *scalar* quantity, with SI dimension energy, and measured in units of *joules* (J); its flux is measured in units of *watts* (W = J/s).

Equivalently we can speak of mass, with SI dimension mass, and measured in units of *kilograms* (kg)[23]; its flux is measured in *kilograms per second* (kg/s).

The notion of energy is extremely important today, and central in many world-wide important discussions and worries – think of the "energy crisis", "renewable energy", and so on. It is somewhat funny that despite this importance it's actually difficult to answer 'what *is* energy, really?'. Often we speak about energy as something that "flows", is "transported", "converted", "stored", and similar visualizations. This intuition will be enough in these notes. The notion of *mass* is also very intuitive in our everyday life; we associate it with the difficulty in setting objects into motion, or with the weight of objects.

From Relativity Theory and experimentally we know that *energy and mass are the same quantity*, and in these notes we shall emphasize this experimental fact.

Energy and mass are the same

Let's see some examples of why it is impossible to make a real distinction between energy and mass. The following examples have been simplified in some of their aspects, but their main point is valid.

Heated gas Imagine we have a box with a given amount of gas, say 1 mol of oxygen molecules. Using an extremely precise weighing scale, we observe that the mass of the gas is, say, exactly

$$0.031\,999\,540\,000\,000\,000\ \mathrm{kg}\ .$$

Now we heat the gas, providing 60 J, while making sure that not a single molecule of oxygen gets in or out of the box. The temperature of the gas increases by around 3 K. We observe that the weight measured by the scale increases while we heat the gas, reaching the new value

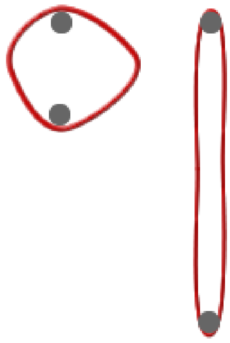
$$0.031\,999\,540\,000\,000\,668\ \mathrm{kg}\ .$$

Clearly the mass has increased, but no molecules were added! The additional mass is the 60 J that we provided to the gas. Energy has weight, energy is mass.

Stretched or moving rubber band Take a common rubber band, and imagine again that we have an extremely precise weighing scale. The rubber band, unstretched, has a mass of exactly

$$0.000\,500\,000\,000\,000\,000\,000\ \mathrm{kg}\ .$$

Now we stretch the band a little. According to Newtonian mechanics it acquires 'elastic energy', which increases its initial internal energy by, say, 0.3 J. Now we weigh the rubber band again, while stretched, and observe a mass of approximately



When we stretch a rubber band, its mass increases slightly – even if the amount of rubber remains exactly the same.

$$0.000\,500\,000\,000\,000\,003\,338\ \mathrm{kg}\ .$$

The extremely small difference of around 3×10^{-18} kg from the initial mass is exactly the elastic energy that we provided by stretching. Energy has weight; energy is mass.

Now set the unstretched band in motion, so that according to Newtonian mechanics it acquires a kinetic energy of, say, 0.2 J. If we could weigh the band while in motion (but without moving the weighing scale), we would measure a mass of approximately

$$0.000\,500\,000\,000\,000\,002\,225\ \mathrm{kg}\ .$$

The small difference from the initial mass is the new kinetic energy of the band.

This case is actually connected with the example of the gas above. If we observed the gas at a molecular level, we would interpret the energy of 60 J provided to it as additional kinetic energy of its molecules. The increase in weight was exactly this additional kinetic energy.

Fission and atomic bombs The atomic bomb[25] is a dark example of the fact that mass is energy. In the case of nuclear fission, if we weigh the amount of nuclear material, say within a box, before and after fission, we observe that its mass has decreased. But we also observe that a great amount of energy has been released out of the box.

Hydrogen Bomb Test, 1954 (National Museum of Nuclear Science & History[24])

Electric heater As a final example consider a 1000 W electric heater, which is radiating 1000 J in one second. The heater is also losing around 0.000 000 000 000 01 kg of mass every second owing to this heat radiation – although it's also acquiring the same amount of mass as electromagnetic energy.

"we are led to the more general conclusion: The mass of a body is a measure of its energy content; if the energy changes by L, the mass changes in the same sense by $L/9\cdot10^{20}$, if the energy is measured in ergs and the mass in grams." Einstein 1905b

The equivalence between energy and mass is given by the famous formula $E = mc^2$, where c is the speed of light, eq. (2.1). In their respective units this gives

$$1\ \mathrm{kg} = 89\,875\,517\,873\,681\,764\ \mathrm{J} \quad \text{(exactly)}$$
$$1\ \mathrm{J} \approx 0.000\,000\,000\,000\,000\,011\,126\,5\ \mathrm{kg}$$

To grasp these numbers, consider that the mass of the rubber band in the example above, 0.5 g, is comparable to the energy released by the atomic bomb over Hiroshima[26].

The practical used of the words 'mass' and 'energy'

From the examples above it becomes clear that energy and mass are two names for the same thing. But it also becomes clear that in our daily experience we deal with energy-mass in two different ways:

On the one hand, we deal with huge (atom-bomb-like) amounts of energy packed in very small volumes: the huge amount of energy that goes together with objects such as pens, keys, bicycles, cars, houses, and so on. We move, push, pull these huge energy amounts from one place to another, and even put them in our pockets. These energy amounts change a little all the time; see the examples above. But the change is so small as to be often undetectable with ordinary scales, and negligible for practical purposes. We call this energy 'mass' and measure it with a unit – kg – that doesn't lead to ridiculously large numbers. And we also agree to neglect the imprecision and fluctuation in its measurement, say any imprecision under 0.000 001 %.

On the other hand, we also deal with the small energy changes and exchanges in all these objects. These exchanges that are very important for our daily life: they keep us warm, make our cells work, make our laptops work. In dealing with these energy exchanges, we don't care about the huge energy reservoirs they come from. So we agree to measure them with a unit – J – that doesn't lead to ridiculously small numbers. And we also agree not to be precise about the total amount in the reservoir from which these energy bits come from.

As an analogy, consider when we speak about the amount of people in different countries. We can say that in Norway there are 5 millions, and in India 1500 millions, so in India there are 300 times more people. By this we don't mean that in Norway there are *exactly* 5 000 000 people and that India has *exactly* 300.000 000 times more people. These numbers are changing slightly all the time; but we don't care about differences of 10 or even 10 000 people. At the same time, think of when you have three dear friends or relatives visiting you from abroad: the amount of 3 people is now for you very important; and you don't care about how much this amount is compared to the total population of your country.

The distinction above is of course not clear-cut. In dealing with some physical phenomena, for example with few molecules or in particle physics, the distinction become too blurry and not useful anymore. And indeed in these situations one often uses the terms 'mass' and 'energy' interchangeably, as well as a common unit for both (for instance *electronvolts*[27]).

In these notes we shall often use the expressions 'energy-mass' and 'mass-energy' to remember that these two words denote the same thing.

Different 'forms' of energy

We often speak of different *forms* of energy-mass. The most important for us will be **internal energy**, **kinetic energy**, **gravitational potential energy**, to be discussed later; another important one is *electromagnetic energy*.

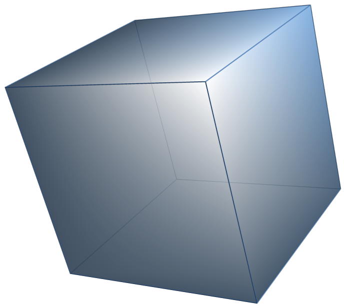
In ch. 5 we shall see that the differences among these energy forms come from the way they are calculated from other quantities, such as matter or magnetic flux and electric charge. For example, if we know that in a volume there's an amount of a particular kind of matter, then we know that there must also be an amount of energy given by a particular formula. And if that matter is moving, then we have to add to the total an extra amount of energy given by another formula. And if in that volume there's a gravitational field, then another extra amount must be added, given by yet another formula. Similarly if we know that an electromagnetic field is in that volume.

We also speak of different forms of flux of energy. The most important for us will be **heat** and **work**, and possibly **convection**. The difference is again in how these fluxes are calculated depending on whether there are also fluxes of matter.

Amounts and forms of energy are coordinate- and observer-dependent

An aspect of energy that must always be kept in mind is that **the amount of energy depends on the coordinate system we're using**. If someone points at a specific region of space at a particular instant, and asks "how much energy is there?", we *cannot* give an answer until a coordinate system is specified. Once the coordinate system has been chosen, then a precise and unambiguous answer can be given. The same is true for the flow of energy through a surface. This also means that observers using different coordinates will usually assign different amounts of energy to the same regions of spacetime.

This is an important difference between energy on one side, and matter and electric charge on the other side. *For matter and electric charge, the questions above can be answered unambiguously independently of any coordinate system*. But not so for energy. The reason of this quirky difference is ultimately connected with the fact that time and space are also observer-dependent.



"How much energy is there in this volume at this instant?" This question cannot be answered until we have specified which coordinate system we're using.

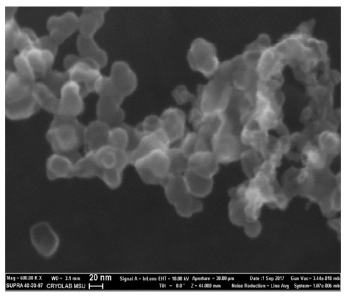
This coordinate-dependence is not a problem: we must always specify our coordinate system anyway, in order to agree on the time and position of physical events. But it can cause problems when we calculate *changes* in energy. If we first calculate or measure some amount of energy in a coordinate system, then we calculate or measure another amount at some later time in a *different* coordinate system, the difference between the two amounts has no meaning whatsoever. In fact it could happen that there's no energy change at all in one coordinate system or in the other: any change we found was just an artefact of mixing up coordinates.

Amounts of energy-mass are coordinate-dependent

Never change coordinate system in the middle of energy calculations!

Also the distinction among different forms of energy is coordinate-dependent. For instance, in one coordinate system we can say that a given volume contains only internal energy; but in another coordinate system that same volume can be said to contain internal and kinetic energy.

The distinction among different forms of energy also depends on the observation scale and on the theory used. For instance, we can observe and model the gas in a container on a scale of metres, seeing it as a uniform flux; in this case we say that there's internal energy in the container. But if we observe and model the gas as a collection of molecules, on a microscopic scale, then we say that there's internal energy (of the single molecules) and kinetic energy in the container (the total energy being the same, unless we have changed coordinate system).



Hydrocarbon fuel particles[28]. The small blobs have size of around 2×10^{-8} m.

The same is true of energy flux: what we call 'heat' on one observation scale is 'work' on a finer scale.

Energy-mass: notation

The amount of energy in a region is usually denoted with E, or with M if we describe it as mass. Internal energy is denoted U, kinetic energy E_k, potential energy E_p.

The total flux of energy will be denoted by $\varPhi$; the flux in the form of heat, by Q; and in the form of work, by W.

Exercise 3.2

In an hour, 14 people exit through a door. Taking the average human weight to be 62 kg (Walpole et al. 2012), what's the average *energy* flux, in J/s, through that door?

3.5 Momentum

Momentum, also called *linear momentum* or *translational momentum* to distinguish it from angular momentum, is a *vector* quantity. Its SI dimension and units can be written in several equivalent ways; we shall keep in mind especially these three:

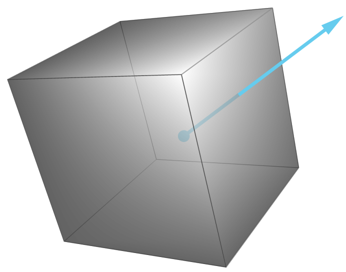
$$\begin{matrix}\text{force} \times \text{time} \\ \mathrm{N \cdot s}\end{matrix} \equiv \begin{matrix}\text{mass} \times \text{length/time} \\ \mathrm{kg \cdot m/s}\end{matrix} \equiv \begin{matrix}\text{energy} \times \text{time/length} \\ \mathrm{J \cdot s/m}\end{matrix}$$

Since it is a vector quantity, it is usually expressed with three numbers, typically the x-, y-, and z-components. In simplified problems where only one or two dimensions are relevant, only the relevant components are reported.

Momentum is a subtle quantity, even subtler than energy. Textbooks that focus on Newtonian mechanics *define* it as the product of the mass and the velocity of a body, usually written "$\boldsymbol{p} = m\boldsymbol{v}$". This relation, however, is only valid in special circumstances, and cannot be used in many everyday technological applications, especially when electromagnetism is involved. And that relation is actually only an approximation even in the circumstances where it's used.

It is therefore convenient to separate our idea of momentum from the idea of "objects" moving, keeping in mind that the latter idea is just a particular case of momentum. Yet, *momentum is indeed associated with translational motion* of matter and of electromagnetic fields. *Translational* motion is the kind of motion that leads to a *new position* in space. For instance, when you walk from one place to a different one, you have performed translational motion (note that translational motion doesn't need to be in a straight line).

Just as energy is often mentally visualized as a sort of fluid (although this visualization comes with many warnings), also momentum can actually be visualized as a sort fluid; but you must imagine it as a "fluid of vectors". Given a particular volume at a particular instant in time, and given a coordinate system, we can speak of the total amount of momentum within that volume. This amount is represented by a *vector*. You can imagine a continuous collection of vectors filling the volume, possibly with different directions and small magnitudes; the total momentum is the sum of all these vectors. This visualization obviously comes with many warnings, but it can be very useful if we are careful.



The amount of momentum within a volume is represented by a (3D) vector

And just as energy, the total amount of momentum is *coordinate-dependent*. So we have the same warning here:

Amounts of momentum are coordinate-dependent

Don't change coordinate system in the middle of calculations of momentum!

Flux of momentum is what we call **force**. A static force, like the one you exert when you hold a bag, is often mentally visualized as a static vector. In ch. 4 we shall discuss a different visualization, in which force is represented as a sort of flow of vectors.

Momentum: notation

The amount of momentum in a region is usually denoted with $\boldsymbol{P}$. The flux of momentum is also called force and denoted with $\boldsymbol{F}$.

Momentum and energy flux are the same

According to Relativity Theory, momentum *is* energy flux (such as heat), and energy flux *is* momentum. If we represent momentum with the symbol $\boldsymbol{P}$, and energy flux with the symbol $\boldsymbol{Q}$, the equivalence between them is given by

$$\boldsymbol{Q} = \boldsymbol{P}c^2 \tag{3.1}$$

Compare this formula with $E = mc^2$. From this point of view, you can think of momentum as "energy in motion". This is consistent with our discussion about mass-energy: since mass is energy, the Newtonian expression "$m\boldsymbol{v}$" indicates energy in motion, or a flux of energy. On a sunny day, if you close your eyes and feel the Sun's heat on your face, what you are feeling is actually a flow of momentum. And when you kick a ball, you're setting a huge bundle of energy in motion, and that's why the ball has acquired momentum.

3.6 Angular momentum

Angular momentum, also called *moment of momentum* or *rotational momentum*, is a *vector* quantity. Its SI dimension and units can be written in several equivalent ways; we shall keep in mind especially these three:

$$\begin{matrix} \text{force} \times \text{length} \times \text{time} \\ \mathrm{N \cdot m \cdot s} \end{matrix} \equiv \begin{matrix} \text{mass} \times \text{length}^2/\text{time} \\ \mathrm{kg \cdot m^2/s} \end{matrix} \equiv \begin{matrix} \text{energy} \times \text{time} \\ \mathrm{J \cdot s} \end{matrix}$$

It is usually expressed with three numbers, typically the x-, y-, and z-components.

Angular momentum is probably an even subtler quantity than momentum. Just as momentum is associated with translational motion, angular momentum is associated with *rotational* motion. Rotational motion is the kind of motion that leads to a *new orientation* in space, rather than to a new position. For instance, if you turn to your left or to your right while standing in place, you have performed rotational motion.

There isn't a clear-cut distinction between translational and rotational motion: usually they involve each other to some degree. A translational motion can be interpreted as a rotation around a point that is very far away; and a rotation of an extended object can be interpreted as small translational motions of its parts.

This is the reason why in many situations we can calculate angular momentum in terms of momentum. If the momentum in a *small* volume is denoted by the vector $\boldsymbol{P} = (P_x, P_y, P_z)$, and the position vector by

$\boldsymbol{r} = (x, y, z)$, then the angular momentum $\boldsymbol{L} = (L_x, L_y, L_z)$ *with respect to the origin of coordinates*, in that same volume, is given by the vector product

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P}$$

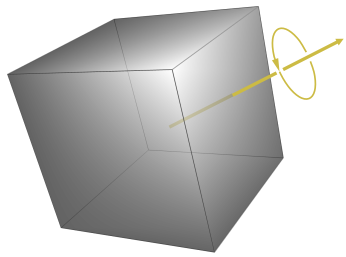
$$\text{or equivalently} \qquad \begin{cases} L_x = y\,P_z - z\,P_y \\ L_y = z\,P_x - x\,P_z \\ L_z = x\,P_y - y\,P_x \end{cases} \tag{3.2a}$$

Instead of calling the components "(L_x, L_y, L_z)", we can also call them "(L_{yz}, L_{zx}, L_{xy})", as some books do. The latter names make the formulae above easier to remember:

$$\begin{cases} L_{yz} = y\,P_z - z\,P_y \\ L_{zx} = z\,P_x - x\,P_z \\ L_{xy} = x\,P_y - y\,P_x \end{cases} \tag{3.2b}$$

Choose whichever you prefer.

Angular momentum is something that is associated not only with ordinary bodies (matter), but also with electromagnetic fields. Just like momentum, also angular momentum can be visualized as a "fluid of vectors". Given a particular volume at a particular instant in time, and given a coordinate system, we can speak of the total amount of angular momentum within that volume. This amount is represented by a *vector*, and is *coordinate-dependent*:



The amount of angular momentum within a volume is represented by a (3D) vector. (Curious about the little circulation symbol around the vector? Then read on.)

! Amounts of angular momentum are coordinate-dependent

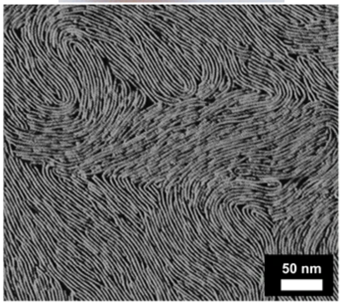
Don't change coordinate system in the middle of calculations of angular momentum!

The flux of angular momentum is also called the *torque*, and bears a relation to the flux of momentum similar to the formula above. If the force – flux of momentum – is denoted by the vector $\boldsymbol{F} = (F_x, F_y, F_z)$ and the position vector by $\boldsymbol{r} = (x, y, z)$, then the flux of angular momentum, or torque, $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$ *with respect to the origin of coordinates* is given by

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

$$\text{or equivalently} \qquad \begin{cases} \tau_x = y\,F_z - z\,F_y \\ \tau_y = z\,F_x - x\,F_z \\ \tau_z = x\,F_y - y\,F_x \end{cases} \qquad \text{or} \qquad \begin{cases} \tau_{yz} = y\,F_z - z\,F_y \\ \tau_{zx} = z\,F_x - x\,F_z \\ \tau_{xy} = x\,F_y - y\,F_x \end{cases} \tag{3.3}$$

You may wonder: “Do we really need angular momentum? after all it just looks like something constructed from momentum”. The answer is yes, we really need it, for two reasons. First, angular momentum obeys an important universal law which is independent from those obeyed by energy and by momentum (Truesdell 1968a tells some of the story of how this was discovered). Second, for some physical phenomena, for example involving liquid polymers[29], elementary particles, or electromagnetic radiation, the angular momentum includes an additional part, called *spin* or *intrinsic angular momentum*, that is *not* related to linear momentum. In the present notes we shall not use this more general kind of angular momentum.



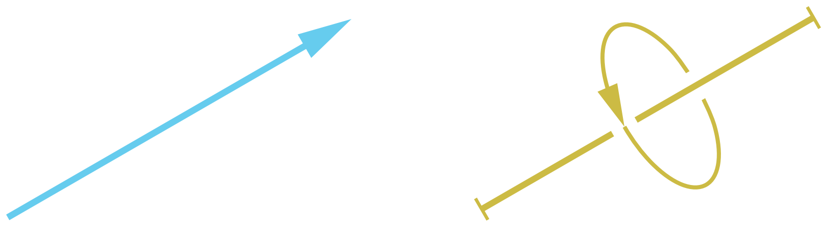
Some liquid polymers (**top:** Liquid Diethoxymethane Polysulfide) need to be described with a special kind of angular momentum, owing to their molecular structure (**bottom**).

Angular momentum: notation

The amount of angular momentum in a region is usually denoted with $\boldsymbol{L}$. The torque or flux of angular momentum is denoted with $\boldsymbol{\tau}$.

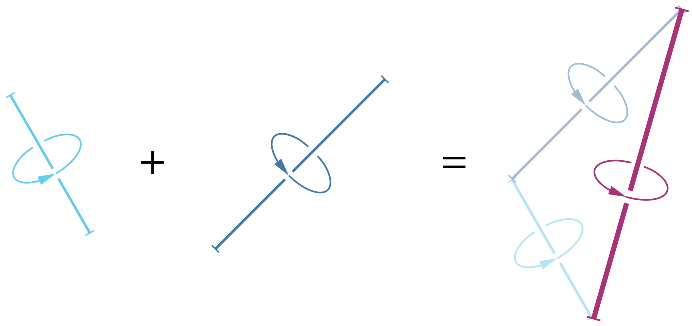
Angular momentum as a twisted vector

In order to represent angular momentum we can use a kind of vectors different from the arrow-like ones (called *polar* vectors) with which you are probably familiar. They are called **twisted vectors**, or also *pseudo*-vectors or *axial* vectors or *outer-oriented* vectors. Twisted vectors represent rotations, and therefore have an orientation, not along them, but *around* them:

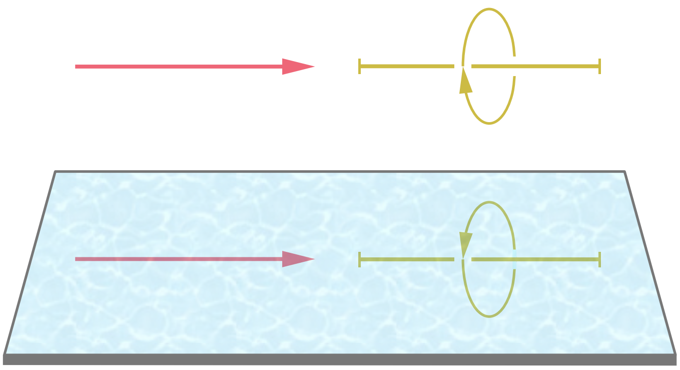


Their length still represents the magnitude of the vector. They make it immediately clear what is the axis of rotation, and what is the sense of rotation.

The sum of twisted vectors is analogous to the sum of ordinary vectors, with the parallelogram rule:

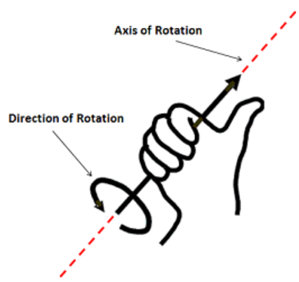


Ordinary vectors and twisted vectors behave very differently if we look at their images through a mirror parallel to their axis: the orientation of ordinary vectors appears unchanged, whereas the orientation of twisted vectors appears *reversed*:



this phenomenon reflects the behaviour of rotations under reflections.

For some mysterious reason many books are afraid of using twisted vectors, and rely on ordinary vectors instead, introducing the “right-hand rule” to determine the sense of rotation from the arrow of the ordinary vector. If you’ve ever asked yourself “why the right hand, and not the left hand?”, the answer is that it’s purely a convention; one could have introduced a left-hand rule instead. Using twisted vectors we don’t need these arbitrary conventions and mnemonics: the sense of rotation is unequivocally indicated by the twisted vector.



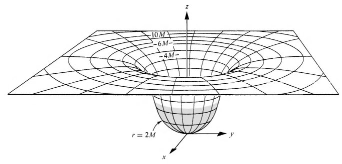
Use whichever vector representation you prefer. In these notes we shall use a hybrid notation, as in the side picture of p. 41, so as to make everybody (or no one) happy.

What are energy, momentum, angular momentum?

From the discussions and formulae above, it seems that energy-mass, momentum, angular momentum are quite closely related to one another. For all three, the amount in a volume or through a surface is undefined unless we specify a coordinate system. And we shall see later that all three satisfy balance laws but not necessarily conservation laws.

Relativity Theory indeed shows that energy, momentum, angular momentum are different aspects of one single geometric object, called *energy-momentum tensor*. They are like its "shadows", that we can observe by looking at it from different points of view in time and space. This is also why their values get intermixed if we change our system of coordinates.

General Relativity gives a new meaning to these quantities: they are *particular curvatures of spacetime*. They express how spacetime is curved in different directions. So whenever we measure, say, the energy or the momentum of some object or of some electromagnetic radiation, we are actually measuring how much that object or radiation is curving spacetime in a particular way.

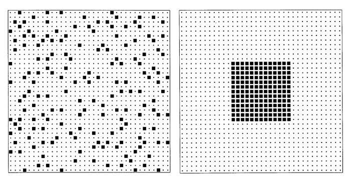


Energy, momentum, angular momentum are particular curvatures of spacetime.

3.7 Entropy

Entropy is a *scalar* quantity, with SI dimension energy/temperature, and measured in units of *joules/kelvins* (J/K)[30]. From this definition it would seem that entropy is derived from temperature. However, although temperature is taken as primitive by the SI, the definition of temperature[31] actually depends on a fixed value of *Boltzmann's constant*[32], which has the dimension of entropy.

Entropy is probably the most difficult quantity to grasp intuitively. Many seemingly intuitive descriptions given in some textbooks are, unfortunately, unhelpful and even misleading. One particularly misleading intuition is that entropy would be a "measure of disorder". Besides the fact that "disorder" is very vague and subjective, it turns out that some physical phenomena, for example with liquid crystals[33], can be considered more "disordered", and yet have *lower* entropy, than others. See also the example in the side figure. We shall discuss more about such phenomena later on.



Microscopic configurations of a lattice gas. **Left**: a configuration coming from a *low*-entropy state. **Right**: a configuration coming from a *high*-entropy state (Styer 2000).

In these notes we shall rely on the idea that *entropy expresses a limit on the flux of energy into matter*. Said in simpler but more imprecise words, entropy is a bound on how fast we can heat up a body. We shall develop this idea further later.

One reason why entropy is difficult to grasp intuitively is that it has very different physical and mathematical aspects depending on the spatial scales and physical theory that we use to describe physical phenomena.

In many "continuum" phenomena, that is, phenomena where the molecular constitution of matter is not visible or not taken into account, entropy is treated as a "stuff-like" quantity similar to energy or electric charge. But there are difficulties also in this case. For some phenomena, for example involving non-elastic materials such as a simple paper clip, it is possible to introduce several entropies having different values – and not just because of a change in measuring scale – all of which can serve their purpose perfectly fine.

In molecular phenomena involving statistical mechanics, on the other hand, entropy is no longer a physical notion, but a *probabilitistic* and *statistical* one, related to guesses and inferences that we make about the physical phenomenon. Yet from many points of view it has roles similar to those of the entropy used in continuum phenomena.

We shall see later that the physical laws for entropy have also a different status with respect to the laws for the other six main quantities: they are, so to speak, "laws about laws".

Entropy: notation

The amount of entropy in a region is usually denoted with S. We shall see that the flux of entropy is tightly related to heat, and we won't need a special symbol for it.

3.8 Auxiliary quantities

Besides the seven principal quantities, other auxiliary quantities appear in some physical theories. Important examples are **temperature** and **metric**. Most auxiliary quantities don't have the "stuff-like" properties of the seven principal quantities. For instance, we cannot ask "what's the total amount of temperature in this region?". We shall later discuss and use some auxiliary quantities, especially temperature.

The dimensions, units, and scalar or vector character of all quantities mentioned so far are summarized in table 3.1.

Quantity	**SI Dimension**	**Unit**
Time	time	*second* s
Length	length	*metre* m
Temperature	temperature	*kelvin* K
Matter	amount of substance	*mole* mol
Electric charge	electric charge	*coulomb* C
Magnetic flux	magnetic flux	*weber* Wb
Energy	energy, mass	*joule* J, *kilogram* kg
Momentum	force · time, mass · length/time, energy · time/length	N · s, kg · m/s, J · s/m
Angular momentum	force · length · time, mass · length^2/time, energy · time	N · m · s, kg · m^2/s, J · s
Entropy	energy/temperature	J/K

Table 3.1 Dimensions and units of the main physical quantities used in these notes. Quantities in **boldface** are vectors, the others are scalars

URLs for chapter 3

1. https://doi.org/10.1351/goldbook.A00297
2. https://doi.org/10.1351/goldbook.M03980
3. https://www.iaea.org/newscenter/news/what-are-radioactive-sources
4. https://www.iaea.org/newscenter/news/what-is-nuclear-energy-the-science-of-nuclear-power
5. https://www.ciaaw.org/radioactive-elements.htm
6. https://pubchem.ncbi.nlm.nih.gov/element/Hydrogen
7. https://pubchem.ncbi.nlm.nih.gov/element/Helium
8. https://pubchem.ncbi.nlm.nih.gov/element/Lithium
9. http://hyperphysics.phy-astr.gsu.edu/hbase/Particles/hadron.html#c6
10. http://hyperphysics.phy-astr.gsu.edu/hbase/Particles/lepton.html#c1
11. https://www.britannica.com/science/antimatter
12. https://www.britannica.com/topic/positron-emission-tomography
13. https://pubchem.ncbi.nlm.nih.gov/element/Calcium
14. https://www.symmetrymagazine.org/2009/07/23/antimatter-from-bananas
15. https://www.quantumdiaries.org/2009/07/21/positrons-from-bananas/
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26. https://www.britannica.com/story/atomic-bombing-of-hiroshima
27. https://home.cern/tags/13-tev
28. https://doi.org/10.4209/aaqr.2019.04.0177
29. https://www.britannica.com/science/polymer
30. https://doi.org/10.1351/goldbook.C01365
31. https://doi.org/10.1351/goldbook.K03374
32. https://doi.org/10.1351/goldbook.B00695
33. https://www.britannica.com/science/liquid-crystal

Chapter 4

Fluxes and volume integrals

For the sake of persons of these different types, scientific truth should be presented in different forms, and should be regarded as equally scientific, whether it appears in the robust form and the vivid colouring of a physical illustration, or in the tenuity and paleness of a symbolical expression.

J. Clerk Maxwell 1870

In ch. 3 we said that the main seven quantities – matter, electric charge, magnetic flux, energy, momentum, angular momentum, and entropy – have three common properties related to their measurement:

(1) We can measure the amount of quantity within a three-dimensional region, at a specific time instant

(2a) We can measure the amount of quantity flowing through a two-dimensional surface during a time lapse...

(2b) ...or alternatively we can measure the amount of quantity flowing through a two-dimensional surface *per unit time*, at a particular time instant

(3) The amount in a volume consisting of separate volumes is equal to the total of the separate amounts. Similarly for the flux through separate surfaces.

Let's give definite names to the measurements (1) and (2b):

volume integral and flux

We can call measurement (1) the **volume integral** of the quantity, although we won't use this term very often.

Most important, we call measurement (2b) the **flux** of the quantity.

These two notions and measurements are very intuitive; that's also why it's convenient to base our physics upon them. In this chapter we straighten some details about their definition and also about our intuition.

4.1 Control volumes and control surfaces

A first question that comes to mind is: how are the volumes and surfaces chosen?

The choice of volume (and therefore of the surface bounding it) for a volume integral, and the choice of surface for a flux are *completely arbitrary*, and they can be *completely imaginary*. Since they are under our control, they are called **control volumes** and **control surfaces**.

For example, consider a classroom and the people in it. In your imagination you can divide the classroom into two halves, say the front and the rear half. You can then ask or measure simply by counting: (1) how many people are, right now, in the rear half; (2) how many people are crossing the imaginary division between the front and rear half during one minute, starting from now.

A control volume and a control surface don't need to be static: they can move and deform.

In the case of a control volume, movement doesn't matter: the volume integral in a control volume *does not depend on the instantaneous motion of the volume*. In fact we can even imagine a control volume that exists for just one time instant.

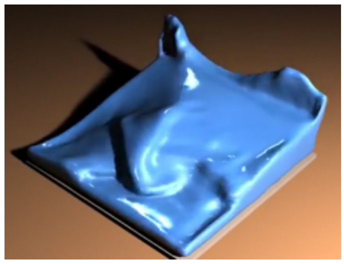
In the case of a control surface the situation is different. The flux through it *depends also on the motion of the surface*. As a trivial example, consider a glass surface, and a person on one side of it, moving with a high velocity directed towards the surface. Will the person crash on the glass? We can't say for sure. The glass surface could be a glass wall in a building, which is not moving; in this case the person will likely crash on it. Or it could be the windscreen of a car, which is moving together with the person, who's the driver; in this case the person won't crash on it.

The golf ball is moving leftwards. Will it hit the metal surface? We don't know unless we know how the surface is moving.

So we can't just imagine a surface that exists for one time instant: we need to imagine it for a very short time lapse, and be able to say how it's moving. If someone asks you what's the flux through a control surface at a given instant, but they don't tell you what's the motion of the surface, then the flux is unknown.

The fact that we can choose control volumes and control surfaces arbitrarily gives us a lot of power in solving physics problems and in making predictions. Typically they are chosen so as to simplify the equations that describe the physical situation, simulate physical phenomena in a more precise way, and focus on details of interest.

When we study some solid object, like a football, a rocket, or a planet, we typically choose a control volume that tightly encloses the object. When we study something flowing or moving, such as a fluid material or an electromagnetic field, we typically divide the space of interest into small control volumes and surfaces, constructing a *mesh*; this mesh can even be refined in regions that are of special interest.

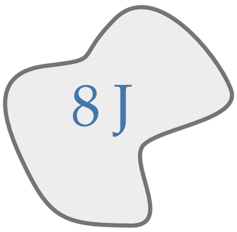


Clever choice of control volume and surfaces allow us to model and predict complex motions of fluids; see also animation[1] (Wojtan et al. 2009)

4.2 Volume integrals: intuition and visualization

Scalar quantities

A volume integral for a scalar quantity, for example energy, can be represented like this:



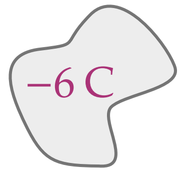
we have eliminated one spatial dimension for simplicity, considering the analogous two-dimensional idea. The volume is in light grey, delimited by a closed darker grey boundary, and we're indicating that the volume integral, that is, the amount of energy within, is 8 J.

As a visualization device, this representation can be useful. But let's straighten out some of its aspects:

- Recall that this is a snapshot at a given time instant. So there are 8 J of energy in the volume at that instant, but we don't know the situation earlier or later: there could be a different amount of energy, the region might be at a different position and have a different shape, or it might not even exist.

- Recall that some scalar quantities, such as electric charge and in some situations matter (antimatter), can have negative amounts.

- We must not surmise that the amount of quantity is uniformly distributed within the volume. In fact there could be negative amounts of it in some subvolumes and positive in others. In particular, even if there is a zero amount of quantity in a volume, some subvolumes could have non-zero amounts: some positive and some negative, so that the total is zero.



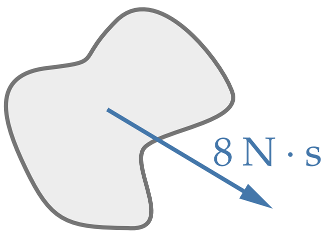
A region with a negative amount of charge

Exercise 4.1

The volume integral of matter in a particular volume is equal to 36 mol. Can we conclude that the volume doesn't contain antimatter?

Vector quantities

A volume integral for a vector quantity, for example momentum, can be represented as follows (we still simplify our visualization to two dimensions):



Momentum is a vector quantity, so the total amount in the volume above is a vector. The picture shows the direction and orientation of this vector, and the magnitude of $8\,\mathrm{N} \cdot \mathrm{s}$ is explicitly reported.

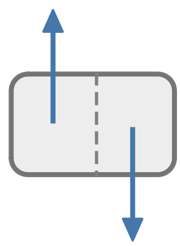
Vector magnitudes and opposite vectors

Remember that the *magnitude* of a vector is always positive, and that $\nwarrow = -1 \cdot \searrow$

The visual representation above is useful, if we keep in mind remarks analogous to the scalar case:

- This is a time snapshot.
- The application point of the vector representing the volume integral is unimportant: for instance, it doesn't need to be placed at the centre of the volume. The vector refers to the volume as a whole, not to some specific point within.
- Different subvolumes could have amounts represented by different vectors; only the total vector is represented above.

This last remark will be especially important when we discuss some physical phenomena involving rotation. As an example, look at the side picture: the volume integral for the whole region is *zero*, but its left and right subregions have *non-zero and opposite* volume integrals.

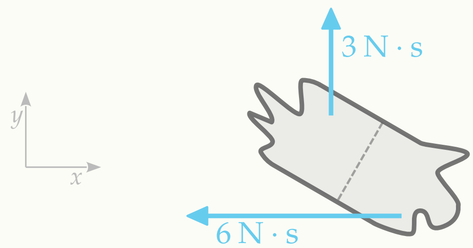


The whole region has zero volume integral. The left and right subregions have non-zero and opposite volume integrals.

Exercise 4.2

Remember the third property (p. 48) of our seven main quantities: The amount in a volume consisting of separate volumes is equal to the total of the separate amounts.

We have a region consisting of two subregions; the amounts of momentum in each subregion are shown below.



1. Write the total momentum in each subregion in component form, (P_x, P_y), according to the coordinate system shown.
2. Calculate the momentum in the whole region; represent it graphically as vector and write it in component form.

Adding vectors in General Relativity

We are used to the idea of adding vectors placed at different points in space: we only have to first move each – keeping it parallel to itself – to a common point, and then add them all at that point with the usual rule.

This operation *cannot* be done in General Relativity: the notion of parallelism doesn't apply anymore in a simple way, owing to the curvature of spacetime. The addition would lead to different results depending on how we transported the vectors. So in General Relativity we can only sum vectors that are placed at the same spacetime point.

How can this operation be possible in Newtonian mechanics and in practical applications, then? After all, General Relativity surely applies here! The answer is that the discrepancies of vector transportation are small enough in the neighbourhood of the Earth, as the curvature of spacetime is very small here.

4.3 Fluxes of scalar quantities: intuition and visualization

The direction, reckoning, and representation of scalar fluxes

Earlier we took the intuitive example of a flow of people through an open door; we might ask, for instance, how many people crossed the door in a minute. But one more detail about this flow is important: in which direction did the persons crossed the door? For example, if the door leads to a classroom, we may need to know whether the people who crossed the door got in or out, so as to know if there are seats left in the classroom.

In order to do this we can: 1. Assign a crossing direction to the door, calling for instance 'positive' the direction from outside to inside the classroom. 2. Count as 'positive' each person who crosses the door in the positive direction, and count as 'negative' each person who crosses the door in the opposite direction. The total tells us the *net* number of people who *entered* the classroom. If the total is positive, then more people got in than out; if the total is negative, then more people got out than in.

One important aspect of this example and terminology is the following symmetry:

- What crossing direction is called 'positive' is fully arbitrary, just a matter of agreement.
- If we decide to call 'positive' the other crossing direction, then the total will change sign. But the physical situation is of course still the same.

Therefore the sentences "+5 persons entered the room" and "−5 persons exited the room" are saying exactly the same thing. This somewhat trivial fact about fluxes will, later on, turn out to be connected with a famous law. So let's remember:

Every flux is equivalent to a flux of opposite amount in the opposite direction

A flux in a particular surface-crossing direction is equivalent to a flux of *opposite* sign in the *opposite* crossing direction.

"LEX III. Actioni contrariam semper & æqualem esse reactionem : sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi."
Newton 1726

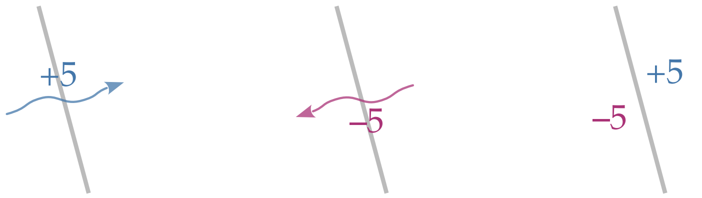
Another important aspect of the example above is that if we're told "−5 persons exited the room", we don't know how exactly this happened: it could be that 5 persons got into the room during that minute; or that 10 persons got in and 5 got out; and so on. This is why we call '−5' the *net* amount.

Finally, consider a similar example but with a quantity that can ordinarily also be negative, such as electric charge. Call 'positive' the crossing direction from outside to inside the room. If we're told that a net charge amount of −5 crossed the door in the positive direction in one minute, then this could have happened in several ways:

- a charge of −5 was brought into the room
- a charge of +5 was brought out of the room
- a charge of −2 was brought into the room during the first 30 s, and a charge of +3 was brought out in the remaining 30 s
- a charge of −2 was brought into the room during the whole minute, and a charge of +3 was brought out at the same time
- . . . and many other possible combinations.

The purpose of the examples and scenarios above is just to make you aware of some aspects of what we shall call "flux", which are trivial but important when considering fluxes of physical quantities.

How can we graphically represent the flux of a quantity, in such a way as to take care of these aspects? Consider these three representations:



The grey straight line represents a surface (simplified to two dimensions) through which we're measuring a flux. In the first and second picture, the wavy arrows represent arbitrary crossing directions called 'positive'. The blue arrow represents that we're calling 'positive' the left-to-right direction; the red arrow represents that we're calling 'positive' the right-to-left direction. The signed number represents the net amount according to the positive direction – so the first and second picture represent the same thing. One possible drawback of these two pictures is that they may suggest that a given amount is actually moving from left to right or vice versa; but we have seen that in general we don't know this. The third picture tries to avoid this misleading suggestion by not showing any arrows; it is meant to represent that on the left side the amount of quantity has changed by -5, and on the right side by $+5$.

The third representation above has one more advantage. Remember from § 4.1 that a flux through a surface may occur because the surface itself is moving. The wavy arrows in the first two representations above may misleadingly suggest that some amount of quantity is "moving" in their direction, or that the surface itself is moving in that direction. The third, arrow-less representation is less misleading.

In these notes we shall settle on the third representation above, but feel free to use the one you prefer – as long as you are aware of all the important aspects of a flux.

What a flux does and doesn't tell

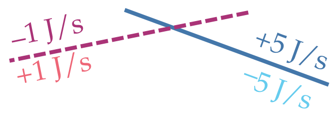
A flux tells us the net change in the amounts of a quantity on the two sides of a surface. These two amounts are equal in magnitude but have opposite signs.

A flux does *not* tell us:

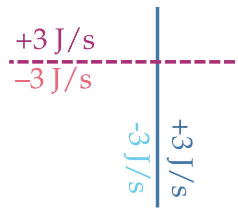
- whether the quantity "was in motion"
- . . . and if the quantity was in motion, what was its sign
- whether the surface was in motion

How does a scalar flux change, if we change the surface?

We know that the flux through a surface consisting of two smaller surfaces is the sum of the fluxes through the smaller surfaces. But what if we consider a *different* surface, maybe intersecting the original one? It's important to keep in mind that **a flux refers to a particular surface, and can be very different if we consider a different surface, even if it's close to the original one**.



Consider for instance the picture on the side. We have two intersecting surfaces (as usual simplified by removing one dimension), both almost horizontal. The energy flux through the solid blue surface, in a roughly upward direction, is $+5\,\mathrm{J/s}$. The energy flux through the dashed red surface, again in a roughly upward direction, is instead $-1\,\mathrm{J/s}$.



As another example, the new side picture shows two similar intersecting surfaces, one fully vertical in solid blue, and one fully horizontal in dashed red. The energy flux through the first in a rightward direction is $+3\,\mathrm{J/s}$, and so is the energy flux in an upward direction through the second surface.

Flux units

Remember that the flux of a quantity is defined as an amount of that quantity *per time lapse*. Therefore the physical dimension of the flux is

$$\frac{\text{quantity}}{\text{time}}$$

and its units will be the units of the quantity divided by seconds:

units of fluxes of the four scalar quantities

quantity	matter	electric charge	energy	entropy
flux units	mol/s	C/s	J/s	J/(K · s)
equivalent units		*ampere* A	*watt* W	

After our discussion about the peculiarity of fluxes it's quite easy to work with the fluxes of the four main scalar quantities: matter, electric charge, energy, entropy. Let us add some reminders and remarks about the fluxes of matter and energy.

Matter flux

Remember that *anti*matter "counts as −1" for calculating amounts of matter. If 1 mol of positrons (anti-electrons) crosses a surface from left to right in 1 s, the left-to-right flux equals −1 mol/s – note the minus sign. The fact that antimatter is given special names can lead to ambiguities. For instance, if someone asks "what's the left-to-right flux of *positrons*?", maybe we should answer "1 mol/s", since the question concerned specifically positrons. It's somewhat like asking "what's the flux of *negative* electric charge?". In these ambiguous situations is best to add some explanatory words.

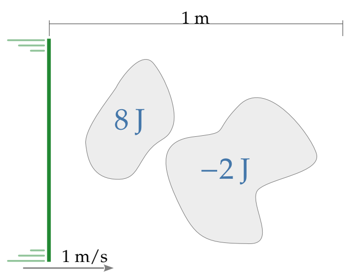
Energy flux

In § 3.4 we mentioned that energy flux can be categorized into different kinds, depending on whether there are fluxes of other quantities through the same surface. We study the exact definitions and formulae later on. The total flux is given by the sum of all these kinds. For instance, through a horizontal surface we can have a downward energy flux of 3 J/s as *heat*, and a downward flux of −1 J/s as *work*. The total downward flux is then 2 J/s. The energy flux that you will calculate in the fourth exercise below is called *energy convection*.

Exercise 4.3

For each question, answer in an *unambiguous* way and sketch a picture representing the flux.

1. The two sides of a particular surface are called 'up' and 'down'. During 0.2 s, an energy of +3 J flows from the up- to the down-side, and an energy of −4 J flows from the down- to the up-side. How much is the flux of energy through the surface?
2. Through the same surface, at a later time, 2 mol of neutrons flow from the up- to the down- side in 0.01 s, and 2 mol of neutrons flow from the down- to the up-side during the same time. How much is the flux of matter through the surface?
3. The two sides of a surface are called 'in' and 'out'. During 0.01 s there is a flow of 1000 electrons from the in-side to the out-side, and also a flow of 1000 *positrons* (anti-electron) in the same direction. How much is the flux of matter through the surface?
4. The side picture shows a surface moving from left to right at a (constant) velocity of 1 m/s. The space to its right has two static regions with some amount of energy as shown. How much is the flux of energy through the surface in 1 s?



4.4 Fluxes of vector quantities: intuition and visualization

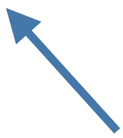
The direction, reckoning, and representation of vector fluxes

The flux of a vector quantity is also a vector, because it is given by an amount of that quantity, which is a vector, divided by time, which is a scalar.

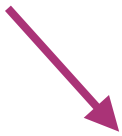
The discussion of § 4.3 about the arbitrary choice of a crossing direction and the minus signs that appear when we reverse it also applies, in an analogous way, to vector fluxes. Also the discussion about the minus sign remains the same; we must only remember that a minus sign changes the sense or orientation of a vector:

$$\nwarrow \quad = \quad -1 \cdot \searrow$$

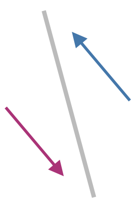
Take for instance a horizontal surface. Suppose that we call 'positive' the *upward* crossing direction, and that with this convection the flux is represented by the vector



Then if we decide to call 'positive' the *downward* crossing direction instead, the same flux is represented by the opposite vector

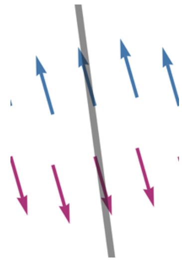


The situation in the case of vector fluxes can require a little more thinking and attention, because it's easy to get confused between the crossing direction for the surface, and the direction suggested by the vector flux – which are completely separate things. For this reason a good graphical representation of a vector flux is this, analogous to the third one for scalar fluxes on p. 54:



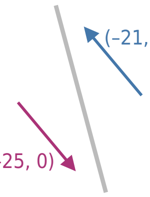
The grey straight line represents a surface through which we're measuring the flux of a vector quantity; on the left side of the surface, the amount of quantity has changed by  and on the right side by . Obviously these two changes have equal magnitude but are opposite.

As a mental image of a vector flux, you can imagine a flow of vectors crossing the surface in one direction, together with a flow of opposite arrows crossing the surface in the opposite direction. See this link[2] as an example.



Imagine the blue upper arrows moving from left to right, and the red lower arrows moving from right to left. Animated version here[3].

Since a vector is represented by three numbers (which can be positive or negative), a vector flux can also be interpreted as the collection of three distinct numerical fluxes:

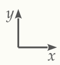


the picture says that the x-component of the quantity has changed on the right side of the surface by -21, and on the left side by $+21$; the y-component has changed on the right side by $+25$, and on the left side by -25; and the z-component hasn't changed on either side.

Another aspect of vector fluxes that we must try not to get confused about is the application point of the vector representing the flux (the base point of the arrow). This is a time snapshot. Just like in the case of vector volume integrals (p. 52), **the application point of the vector representing the flux is unimportant. The vector refers to one side of the surface as a whole.**

Exercise 4.4

A horizontal surface is given, and there is a flux of a vector quantity through it (for the moment we neglect units):

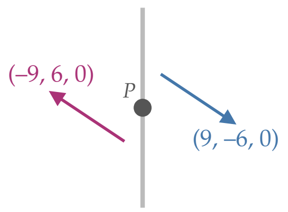
1. If we take the *downward* crossing direction as 'positive', the flux xyz-components are $(5, 5, 0)$. Represent this flux graphically, in the way discussed in the present section. Use the coordinate system  were y points upward.
2. Taking the same crossing direction, represent graphically the flux $(0, -2, 0)$ instead.

3. Taking the same downward crossing direction, we are now told that there is a flux with components $(1, -2, 3)$. What are the components of this flux if we take the *upward* crossing direction as positive?

How does a scalar flux change, if we change the surface?

We saw that the flux of a scalar quantity can be very different if we take a slightly different surface. The same is true of the flux of a vector quantity: in particular, **the vectors representing the fluxes through two slightly different surfaces can point in completely different directions**.

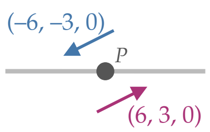
Here is an example. Take a fixed point P. Now take a small vertical surface passing through P. The flux of a vector quantity (momentum for example) through this quantity can be as in this picture:



(animated version[4])

it has components $(9, -6, 0)$, with magnitude around 10.8, if we take the rightward crossing direction as positive.

Now forget about that surface, and take instead a small horizontal surface passing through the same point P. The flux – of the same quantity – through this surface can be as in this picture:



(animated version[5])

it has components $(-6, -3, 0)$, with magnitude around 6.7, if we take the upward crossing direction as positive.

Clearly the vectors representing the fluxes through these two surfaces are different: they point in different directions, and have even different magnitudes.

If we have a vector flux through a particular surface, and we're asked about the flux through a different surface, we might be tempted to "move" through the new surface the vectors representing the flux through the old one. *We must fully resist this temptation*. It only leads to mistakes.

Each surface has a unique flux

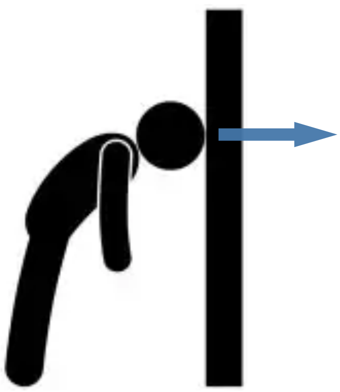
A flux, scalar or vector, through a particular surface, in general doesn't tell you anything about the flux though another surface, even if the other surface is only slightly different from the first one.

4.5 Flux of momentum: force

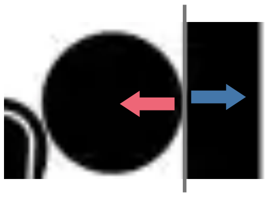
We already mentioned in § 3.5 that **flux of momentum is what we call 'force'**. Owing to the importance of the notion of *force* in the many branches of physics which rely on Newtonian mechanics, we must discuss this connection in depth. This connection, as well as the connection to Newton's laws, will become even clearer when we discuss the balance of momentum in § .

The notion of force is very intuitive. We associate it to the sensations that we feel in our skin, flesh, and even bones when, for instance, we push against a wall, twist a door knob, push backwards on the ground with our feet to run, or other similar actions. This force is typically represented by a vector, having the direction and orientation of the "push" or "pull", and magnitude expressing its intensity. *Such a force vector is exactly the vector expressing the flux of momentum*. The two are the same. A force can therefore also be visualized as a flow of momentum. This mental representation can be illuminating in some physical problems.

As a concrete example, imagine a person pushing against a wall. In terms of force, we say that *the person is exerting a force on the wall, and the force vector on the wall has a person→wall orientation*. This is usually depicted, for example, like this:

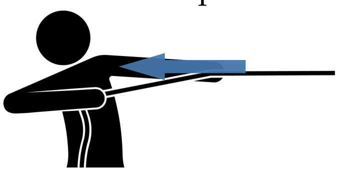


In terms of momentum flux, we imagine a surface separating the person and the wall. If we take a person→wall crossing direction, the flux vector also has a person→wall orientation. Because of the symmetry of flux, if we take a wall→person crossing direction instead, then the flux vector also changes direction. This momentum flux can be depicted (zooming in) like this:

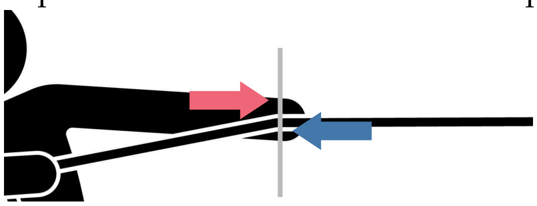


The picture says: on the side of the wall, momentum is changing by an amount having person→wall orientation; and, by symmetry of the flux, on the side of the person's head, momentum is changing by an amount having opposite, wall→person orientation.

Let's take an example involving pulling instead of pushing. Imagine a person pulling a rope fastened somewhere. In terms of force we say that *the person is exerting a force on the rope, and the force vector on the rope has a rope→person orientation*. This can be depicted like this:



In terms of momentum flux, we imagine a vertical surface between the person's hand and the rope. If we take a hand→rope crossing direction, the flux vector has a rope→hand orientation – note the difference from the previous example. Because of the symmetry of flux, if we take a rope→hand crossing direction instead, then the flux vector has opposite direction: hand→rope. This momentum flux can be depicted like this:



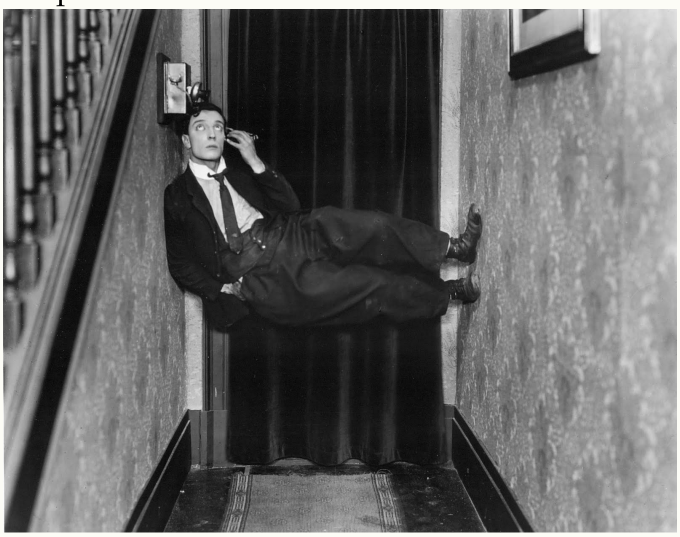
The picture says: on the side of the rope, momentum is changing by an amount having rope→person orientation; and, by symmetry of the flux, on the side of the hand, momentum is changing by an amount having opposite, hand→rope orientation.

A final example illustrates a situation in between the previous two. Consider the foot of a running person, as it pushes on the ground. In terms of momentum flux, we imagine a horizontal surface between the runner's foot and the ground. If we take a downward crossing direction, the flux vector is oriented towards the back of the foot. Because of the symmetry of flux, if we take an upward crossing direction, then the flux vector has opposite direction, towards the front of the foot, the same as the running direction. This momentum flux can be depicted like this:

The picture says: on the side of the ground, momentum is changing by an amount having leftward orientation; and, by symmetry of the flux, on the side of foot, momentum is changing by an amount having rightward orientation.

Exercise 4.5

Using your intuition, try to guess the various momentum flows between this person and the walls:



(Buster Keaton in 'The Electric House'[6])

Netwon's Third Law!

From the examples above, we see that thinking of force as momentum flux automatically leads to *Newton's third law*: if one side is gaining/losing momentum with a given orientation, by symmetry the other side is gaining/losing momentum with the opposite orientation. So if one side is experiencing a force with a given orientation, the other side must be experiencing a force with the opposite orientation.

We see that Newton's third law is the expression of the symmetry of fluxes *for the specific case of the flux of momentum (force)*. But we realize that this property is more general: it applies not only to force, but also to the flux of all other quantities, even scalar ones.

"LEX III. Actioni contrariam semper & æqualem esse reactionem : sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi."

"LAW III. To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

Newton 1726

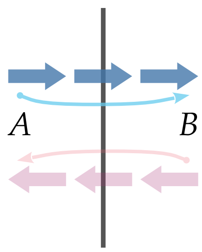
4.6 Pressure, tension, shear force

The examples of the previous section demonstrated a variety of possible orientations of the momentum-flux vector with respect to the surface through which it occurs. Obviously all orientations are possible. Special names are given, however, to three specific kinds of momentum flux: *pressure*, *tension*, and *shear force*.

Consider a surface and calls its sides A and B. Now consider a flux of momentum as seen from A to B.

Pressure

If the momentum-flux from A to B is a vector also oriented from A to B, then we call the momentum flux a **pressure**, or a *compressive momentum flux*.



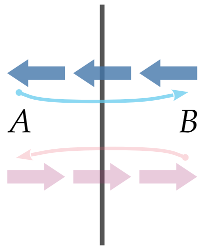
(click the picture for an animated version)[7]

By symmetry, there is an analogous relationship for the momentum flux as seen from B to A.

Pressure is the kind of momentum flux that we exert when we *push* on an object, and that air exerts on all objects it surrounds.

Tension

If the momentum-flux from A to B is a vector oriented in the opposite way, from B to A, then we call the momentum flux a **tension**, or a *tensile momentum flux*.



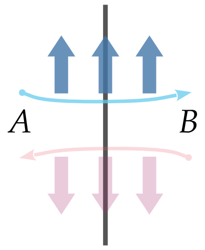
(click the picture for an animated version)[8]

By symmetry, there is an analogous relationship for the momentum flux as seen from B to A.

Tension is the kind of momentum flux that we experience in our bones when we *pull* an object, and that occurs in any section of a stretched rubber band.

Shear force

If the momentum-flux from A to B is a vector oriented along the surface, then we call the momentum flux a **shear force**, or a *shearing momentum flux*.



(click the picture for an animated version)[9]

By symmetry, there is an analogous relationship for the momentum flux as seen from B to A.

Shear force is the kind of momentum flux that we experience under our feet when we walk, and that occurs between a car's wheels and the ground.

In general, a momentum flux won't have any of the three special directions above, but rather a combination of them.

Exercise 4.6

Using your intuition, try to identify the various momentum fluxes that occur in the different parts of a tower crane. Which of the fluxes are (approximately) compressive, tensile, and shearing?

4.7 Intuition and visualization for vector quantities

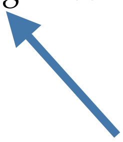
Let's now consider the vector quantities momentum and angular momentum.

Volume integrals

Fluxes

For the visualization and mental representation of the flux of a vector quantity we must pay attention to analogous warnings as for a scalar quantity. Consider again our example surface

and suppose we are told that there's a flux of momentum through it, from left to right, having the following direction and orientation:

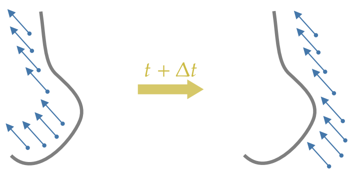


with magnitude 6 N.

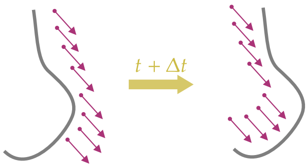
Vector flux and crossing direction are two separate things

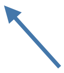
The vector that represents the flux, like the one above, and the surface-crossing direction, say left-to-right, are completely separate and independent things. In particular, they can have completely different directions.

We could visualize the flux above as follows:

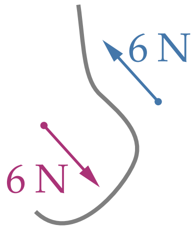


with the familiar warning that the surface could be moving. But the following visualization also corresponds to exactly the same flux:



What happens in either illustration is that there's a change equal to a vectorial amount  of magnitude 6 N on the *right* of the surface, and a change equal to the *opposite* vectorial amount  of the same magnitude on the *left* of the surface.

A less misleading representation of the flux in our example could therefore be like this:

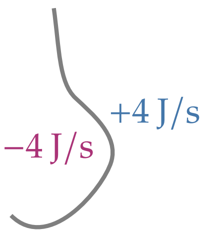


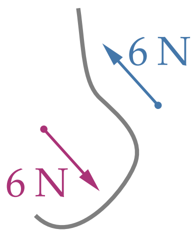
This picture says that there's a momentum change of a particular direction, orientation, magnitude on the left side of the surface, and an opposite momentum change on the right side of the surface. Note that **the vectors indicated on the two sides must be opposite and have equal magnitude**, otherwise the picture doesn't make sense.

An animated representation or mental visualization can also be quite illuminating. You can see one at this link[10].

4.8 The symmetry of fluxes

Our discussion about fluxes of scalar and vector quantities, and their graphical representation, for instance





shows an important mathematical and physical symmetry of the flux of a quantity.

The *sign* of the flux is determined by the direction in which we imagine to cross the surface. In the first picture above, if we are imagining to cross the surface from left to right, then the energy flux is $+4\,\mathrm{J/s}$; and if we are imagining to cross the surface from right to left, then the energy flux is $-4\,\mathrm{J/s}$. Analogously for the vector flux in the second picture above, where the minus sign corresponds to flipping the vector.

The crossing direction is arbitrary, completely left to us (just like the surface itself is arbitrary). It is therefore always important, when we report a flux through a surface, to state which crossing direction we have agreed upon.

We can state this symmetry as follows:

Every flux is the same as the opposite flux in the opposite direction

A flux in a particular surface-crossing direction is equivalent to a flux of opposite sign in the opposite crossing direction.

This statement may sound familiar. Indeed *it includes Newton's famous third law, the "principle of action and reaction", as a special case*. Newton's third law is the expression of the symmetry of vector fluxes, but for the specific case of the flux of momentum, that is, force. Now we see that this property is more general: it applies not only to force, but also to the flux of angular momentum (torque), and to the flux of scalar quantities as well.

"LEX III. Actioni contrariam semper & æqualem esse reactionem : sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi."

"LAW III. To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

Newton 1726

4.9 Total flows

Recall that a flux is defined as the amount of a quantity crossing a surface in a short time lapse Δt, divided by that time lapse. This is why the dimension of a flux is $\frac{\text{dimension of quantity}}{\text{time}}$. Denoting the flux by, say, J, this definition also means that the amount of quantity crossing the surface in Δt is equal to $J\,\Delta t$.

Take now a surface existing between two time instants t_1 and t_2; during this time lapse it could also be moving and changing shape. Choose a crossing direction through the surface. At each intermediate time instant t we can then measure the flux of a quantity crossing the surface in that direction, at that instant; denote it by $J(t)$.

The total amount of quantity that cross the surface between times t_1 and t_2 can be found by dividing the time interval into very short time lapses of length Δt. For each short time lapse we know that the amount is $J(t)\,\Delta t$. The total is obtained by adding these small amounts. As Δt is considered shorter and shorter, this sum is by definition an integral:

Total flow of quantity through a surface between two times

The total amount of quantity crossing a surface (relative to a specified crossing direction) between times t_1 and t_2 is given by

$$\int_{t_1}^{t_2} J(t)\,\mathrm{d}t \;. \tag{4.1}$$

The meaning of the integral above should be clear for any scalar quantity, for which the flux is also a scalar. In the case of a vector quantity, for instance momentum, the flux is also a vector. How is the integral of a vector calculated? The procedure is simple: a vector is represented by three components, and we calculate the integral for each component obtaining three results, which are the components of a new vector.

Take the case of momentum, whose flux (force) we denote $\boldsymbol{F} = (F_x, F_y, F_z)$. The integral of this flux is then

$$\int_{t_1}^{t_2} \boldsymbol{F}(t)\,\mathrm{d}t \coloneqq \left(\int_{t_1}^{t_2} F_x(t)\,\mathrm{d}t\;,\; \int_{t_1}^{t_2} F_y(t)\,\mathrm{d}t\;,\; \int_{t_1}^{t_2} F_z(t)\,\mathrm{d}t\right) . \tag{4.2}$$

There can be zero total flow with non-zero flux

The result of the integral above could be zero – which means that no *net* amount of quantity crossed the surface between t_1 and t_2. Yet the flux $J(t)$ can be non-zero, even at all times. Of course it needs to be positive at some times, and negative at others.

As a simple example, consider a room's door. During one minute, three people enter through the door; during the next minute, three people (not necessarily the same!) exit. The total flow of people is zero, but the flux was non-zero during the first minute, and non-zero during the second minute.

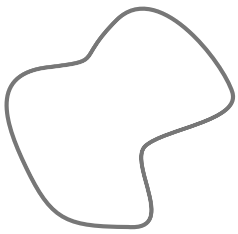
Exercise 4.7

- What is the physical dimension of the total flow of a quantity?
- Suppose that we calculate the integral above in the case of matter,

finding a total of $\int_{t_1}^{t_2} J(t)\,\mathrm{d}t = 7\,\mathrm{mol}$. Now we change our mind and choose the opposite crossing direction for that surface. How does the result above change?

4.10 Closed surfaces, influxes, effluxes

We shall often consider **closed** surfaces, that is, surfaces that don't have a boundary or border, like the surface of a sphere or of a cube. A closed surface delimits a specific three-dimensional volume, and we can naturally speak of its **interior** and its **exterior**. An example (simplified by removing one dimension as usual) is the surface delimiting the volume of our previous examples:



We can give two crossing directions to a closed surface: out-in, from exterior to interior; or in-out, from interior to exterior. A flux through the surface is usually called **influx** if we are considering the out-in crossing direction, and **efflux** or *outflux* if we are considering the in-out crossing direction. Obviously

$$\text{influx} \equiv -\text{efflux}$$

It makes sense to speak of influx or efflux only for a *closed* surface.

The influx and efflux, unless specified otherwise, are fluxes through the *whole* surface. In the picture above, the influx of 2 mol/s can be happening everywhere across the surface, or just across some portions of it.

Exercise 4.8

Take an imaginary cylindrical surface enclosing one control rod[11] in a nuclear-fission reactor[12]. Let's say that in a reactor there are 20

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("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

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