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Tutorial I

1] Expand $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ into Fourier series.

$\rightarrow$

Compare the interval with $(0, 2l)$

$\therefore \underline{\underline{l = 1}}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$a_0 = \int_0^1 \pi x\, dx + \int_1^2 0\, dx$$

$$= \pi \left[\frac{x^2}{2} \right]_0^1$$

$$a_0 = \frac{\pi}{2}$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos n\pi x\, dx$$

$$= \int_0^1 \pi x \cos n\pi x\, dx + \int_1^2 (0)\, dx$$

$$= \pi \left[(x)\left(\frac{\sin n\pi x}{n\pi} \right) - 1 \left(\frac{-\cos n\pi x}{(n\pi)^2} \right) \right]_0^1$$

$$= \pi \left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2} \right]$$

$$a_n = \frac{1}{n^2\pi} [(-1)^n - 1]$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 (0) \, dx$$

$$= \pi \left[x \left(\frac{-\cos n\pi x}{n\pi} \right) - (1) \left(\frac{-\sin n\pi x}{n^2\pi^2} \right) \right]_0^1$$

$$= \pi \left[\frac{-\cos n\pi}{n\pi} \right]$$

$$b_n = \frac{-(-1)^n}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi} [(-1)^n - 1] \cos n\pi x - \frac{(-1)^n}{n} \sin n\pi x$$

2] Obtain Fourier expansion of $f(x) = x^2$ in $0 < x < a$.
Deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$\rightarrow$

$\therefore 2l = a$

$l = \frac{a}{2}$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$a_0 = \frac{2}{a} \int_0^{a} x^2\, dx$$

$$= \frac{2}{a} \left[\frac{x^3}{3} \right]_0^a$$

$$= \frac{2}{\cancel{a}} \left[\frac{a^{\cancel{3}\,2}}{3} \right]$$

$$a_0 = \frac{2a^2}{3}$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{a} \int_0^{a} x^2 \cos\left(\frac{n\pi x}{a/2}\right) dx$$

$$= \frac{2}{a} \left[x^2 \left(\frac{\sin n\pi x/a}{\frac{2n\pi}{a}} \right) - (2x) \left(\frac{-\cos 2n\pi x/a}{\left(\frac{2n\pi}{a}\right)^2} \right) + (-2) \left(\frac{-\sin 2n\pi x/a}{\left(\frac{2n\pi}{a}\right)^3} \right) \right]_0^a$$

$$= \frac{2}{a} \left[2a \frac{a^2}{4n^2\pi^2} \right]$$

$$a_n = \frac{a^2}{n^2\pi^2}$$

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$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{a}\int_0^{a} x^2 \sin\frac{2n\pi x}{a}\, dx$$

$$= \frac{2}{a}\left[x^2\left(\frac{-\cos\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right) - 2x\left(\frac{-\sin\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}\right) + (-2)\left(\frac{\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^3}\right)\right]_0^a$$

$$= \frac{2}{a}\left[a^2\left(\frac{-a}{2n\pi}\right) + (-2)\left(\frac{a^3}{8n^3\pi^3}\right) + 2\left(\frac{a^3}{8n^3\pi^3}\right)\right]$$

$$= \frac{2}{a}\left[\frac{-a^3}{2n\pi}\right]$$

$$b_n = \frac{-a^2}{n\pi}$$

$$f(x) = \frac{a^2}{3} + \sum_{n=1}^{\infty}\left(\frac{a^2}{n^2\pi^2}\cos\frac{2n\pi x}{a} - \frac{a^2}{\pi n}\frac{\sin 2n\pi x}{a}\right)$$

T. P.: $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

Put $x = 0$

$$0 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{-a^2}{3} = \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{-\pi^2}{3} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right] \quad \text{— } A$$

Put $x = a$,

$$a^2 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$a^2 - \frac{a^2}{3} = \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{2\cancel{a^2}}{3} = \frac{\cancel{a^2}}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{2\pi^2}{3} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right] \quad \text{— } B$$

Add A & B,

$$\frac{-\pi^2}{3} + \frac{2\pi^2}{3} = \frac{2}{1^2} + \frac{2}{2^2} + \frac{2}{3^3} + \dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

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3] Obtain Fourier expansion of $f(x) = \begin{cases} 2, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$

$\rightarrow$

length of interval $= 2l$.
$f(x)$ is neither even nor odd

Length $= 2-(-2) = 4$.

$2l = 4$

$\underline{\underline{l = 2}}$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 2\,dx + \frac{1}{2}\int_{0}^{2} x\,dx$$

$$= \frac{1}{2}\left[2x\right]_{-2}^{0} + \frac{1}{2}\left[\frac{x^2}{2}\right]_{0}^{2}$$

$$= \frac{1}{2}\left[4 + \frac{4}{2}\right]$$

$$\underline{\underline{a_0 = 3}}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\,\cancel{dx}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right]$$

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$$= \frac{1}{2}\left\{\left[2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right]_{-2}^{0} + \left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^2\right\}$$

$$= \frac{1}{2}\left[\frac{\cos n\pi}{\left(\frac{n\pi}{2}\right)^2} - \frac{1}{\left(\frac{n\pi}{2}\right)^2}\right]$$

$$= \frac{1}{2}\left[\frac{4(-1)^n - 4}{n^2\pi^2}\right]$$

$$a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$b_n = \frac{1}{2}\left\{\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx\right\}$$

$$= \frac{1}{2}\left\{\left[2\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right]_{-2}^{0} + \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right) - \left(\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^2\right\}$$

$$= \frac{1}{2}\left\{2\left(\frac{-1}{\frac{n\pi}{2}}\right) - 2\left(\frac{-\cos n\pi}{\frac{n\pi}{2}}\right) + 2\left(\frac{-\cos n\pi}{\left(\frac{n\pi}{2}\right)}\right)\right\}$$

$$b_n = \frac{-2}{n\pi}$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{3}{2} + \sum_{1}^{\infty}\left(\frac{2}{(n\pi)^2}\left((-1)^n - 1\right)\cos\frac{n\pi x}{2} - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)$$

4] Obtain Fourier ~~series~~ expansion of x^2 in $(-l, l)$.
Deduce i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

→

$f(x) = x^2$ is an even function.

$\therefore b_n = 0$.

$l = 2l$.

L of interval $= l - (-l) = 2l$

$\therefore 2l = 2l$

$l = l$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_0 = \frac{2}{l}\int_0^l x^2\,dx$$

$$= \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l$$

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$a_0 = \frac{2l^2}{3}$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_{0}^{l} x^2 \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(x^2)\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2x)\left(\frac{-\cos \frac{n\pi x}{l}}{(n\pi/l)^2}\right) + 2\left(\frac{-\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-(2l)\left(\frac{-l^2 \cos n\pi}{(n\pi)^2}\right)\right]$$

$$a_n = \frac{4l^2(-1)^n}{n^2\pi^2}$$

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{(n\pi)^2} \cos \frac{n\pi x}{l}$$

T.P. : $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

Put $x = 0$,

$$0 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right]$$

$$\frac{-l^2}{3} = \frac{-4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Put $x = l$,

$$l^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$l^2 - \frac{l^2}{3} = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{2l^2}{3} = \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$