Project-I

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1.1 Exercise1

i) After randomly initializing $x0$ and repeating the experiments. E has the eigenvalues

$$\begin{pmatrix} 1 \\ -0.5000 + 0.2887i \\ -0.5000 - 0.2887i \\ 0 \end{pmatrix}$$

two real and two complex values, the largest eigenvalue is 1. The sequence $E^1 5x0$ normalised using l2 norm, as the number of iterations increases tends to the largest eigenvector of E, the eigenvector corresponding to eigenvalue 1. eigenvector

$$\begin{pmatrix} 0.6447 \\ 0.2478 \\ 0.4154 \\ 0.5920 \end{pmatrix}$$

ii) Eigenvector

$$\begin{pmatrix} 0.2230 \\ 0.3755 \\ 0.4201 \\ 0.7955 \end{pmatrix}$$

Eigenvalues

$$\begin{pmatrix} 0 \\ 0.5614 \\ -0.2807 + 0.2640i \\ -0.2807 - 0.2640i \end{pmatrix}$$

The largest eigenvalue is 0.5614 and the sequence converges again to the largest eigenvector, up to a constant, -1 in my case.

iii) Eigenvalues of E

$$\begin{pmatrix} 1.0000 \\ -0.5000 + 0.2887i \\ -0.5000 - 0.2887i \\ -0.0000 \\ 1.0000 \\ -1.0000 \end{pmatrix}$$

There are two eigenvalues of 1. The sequence after 15 steps does not converge to the largest eigenvector (any of them)

$$\begin{pmatrix} 0.3500 \\ 0.1050 \\ 0.2099 \\ 0.3149 \\ 0.6848 \\ 0.5043 \end{pmatrix}$$

1.2 Exercise2

i) The igenvectors are -0.6446 -0.2887 + 0.5000i -0.2887 - 0.5000i 0.5345 -0.2478 -0.4330 - 0.2500i -0.4330 + 0.2500i -0.8018 -0.4154 0.5774 0.5774 0.0000 -0.5920 0.1443 - 0.2500i 0.1443 + 0.2500i 0.2673 Eigenvalues are

$$\begin{pmatrix} 1.0000 \\ -0.4250 + 0.2454i \\ -0.4250 - 0.2454i \\ -0.0000 \end{pmatrix}$$

The convergence vector is

$$\begin{pmatrix} 0.6447 \\ 0.2478 \\ 0.4154 \\ 0.5920 \end{pmatrix}$$

ii) GREEN 0.2769 -0.3641 - 0.3761i -0.3641 + 0.3761i 0.0044 0.3984 0.5959 0.5959 -0.1091 0.4548 -0.3506 + 0.2950i -0.3506 - 0.2950i -0.0657 0.7468 0.2363 + 0.3242i 0.2363 - 0.3242i 0.9918 Eigenvalues are

$$\begin{pmatrix} 0.6618 \\ -0.2427 + 0.2257i \\ -0.2427 - 0.2257i \\ -0.0264 \end{pmatrix}$$

The convergence vector is

$$\begin{pmatrix} 0.2769 \\ 0.3984 \\ 0.4548 \\ 0.7468 \end{pmatrix}$$

iii) BLUE Eigenvector of M blue

$$\begin{pmatrix} 0.5351 \\ 0.2057 \\ 0.3449 \\ 0.4914 \\ 0.3943 \\ 0.3943 \end{pmatrix}$$

The eigenvalues are

$$\begin{pmatrix} 1.0000 \\ 0.8500 \\ -0.4250 + 0.2454i \\ -0.4250 - 0.2454i \\ 0.0000 \\ -0.8500 \end{pmatrix}$$

The convergence vector is

$$\begin{pmatrix} 0.5200 \\ 0.2012 \\ 0.3358 \\ 0.4778 \\ 0.4257 \\ 0.4085 \end{pmatrix}$$

1.3 Exercise3

i) E is a stochastic column matrix for i) and iii). ii) has a column only with 0 elements. M is a stochastic column matrix for i) and iii) again. ii) has a column only of 0.15 values, so it is not .

ii) Let A be a stochastic column matrix. A and $A^\top$ have the same eigenvalues, so it is enough to show that $A^\top$ has eigenvalue 1, which will imply A also has eigenvalue of 1.

We use the property that for eigenvalue λ of $A^\top$ $det(A^\top - \lambda * I) = 0$. We compute

$$det(A^\top - I) = \det \begin{pmatrix} a_{11} - 1 & a_{21} & ... & a_{n1} \\ a_{12} & a_{22} - 1 & ... & a_{n2} \\ ... & ... & ... & ... \\ a_{1n} & a_{2n} & .... & a_{nn} - 1 \end{pmatrix} \tag{1}$$

$$det(A^\top - I) = \det \begin{pmatrix} \sum_i a_{i1} - 1 & a_{21} & ... & a_{n1} \\ \sum_i a_{i2} - 1 & a_{22} - 1 & ... & a_{n2} \\ ... & ... & ... & ... \\ \sum_i a_{in} - 1 & a_{2n} & .... & a_{nn} - 1 \end{pmatrix} \tag{2}$$

$$det(A^\top - I) = \det \begin{pmatrix} 0 & a_{21} & ... & a_{n1} \\ 0 & a_{22} - 1 & ... & a_{n2} \\ ... & ... & ... & ... \\ 0 & a_{2n} & .... & a_{nn} - 1 \end{pmatrix} \tag{3}$$

For the first equality we added all the columns $2, 3, ...N - 1$ to the first column. For the second equality, the determinant of a matrix with a column of 0 equals 0. This implies $A^\top$ and implicitly A have eigenvalue 1.

Assume that A, $A^\top$ have an eigenvalue greater than 1, λ_0 with eigenvector v. $A^\top v = \lambda_0 v$. Then we have $\sum_i a_{it} * v_i = \lambda_0 v_t$ for every t in 0..N. Let m be the index corresponding to the largest entry in v. then $\sum aik * vi < \sum aijvk = vk < \lambda vk$ which is a contradiction, so 1 is the largest eigenvalue.

1.4 Exercise4

Gradient of

$\nabla f = (M - I)^\top \cdot (M - I) \cdot x + \gamma \cdot (\mathbb{1}^\top x - 1)\mathbb{1}$

$\nabla f = (M - I)^\top \cdot (M - I) \cdot x + \gamma \cdot (\mathbb{1}\mathbb{1}^\top)x - \gamma\mathbb{1}$
Hessian of f $\nabla^2 f = (M - I)^\top \cdot (M - I) + \gamma \cdot \mathbb{1}\mathbb{1}^\top$

Find the Lipshitx constant.. $\nabla f(x_1) - \nabla f(x_2) = [(M - I)^\top \cdot (M - I) + \gamma \cdot (\mathbb{1}\mathbb{1}^\top)](x_1 - x_2)$

there is no line here
$\|\nabla f(x_1) - \nabla f(x_2)\| = \|[(M - I)^\top \cdot (M - I) + \gamma \cdot (\mathbb{1}\mathbb{1}^\top)](x_1 - x_2)\|$
$\|\nabla f(x_1) - \nabla f(x_2)\| \leq \|(M - I)^\top \cdot (M - I) + \gamma \cdot (\mathbb{1}\mathbb{1}^\top)\|\|x_1 - x_2\| = L\|x_1 - x_2\|$, which implies the Lipschitz constant is
$L = \|(M - I)^\top \cdot (M - I) + \gamma \cdot (\mathbb{1}\mathbb{1}^\top)\|$

1.5 Exercise9

$f_\sigma = 0.5 \cdot (Mx - x)^\top \cdot (Mx - x) + \gamma \cdot (\mathbb{1}^\top x - 1)^2 + 0.5 * \sigma * x^\top * x$
$\nabla f_\sigma = (M - I)^\top \cdot (M - I) \cdot x + \gamma \cdot (\mathbb{1}^\top x - 1)\mathbb{1} + \sigma \cdot x$
$\nabla f_\sigma = [(M - I)^\top \cdot (M - I) + \gamma\mathbb{1}^\top \cdot \mathbb{1} + \sigma I] \cdot x - \gamma\mathbb{1}$
$\nabla f_\sigma = \phi_\sigma \cdot x - \gamma\mathbb{1}$

We have $(M - I)^\top \cdot (M - I)$, $\gamma\mathbb{1}^\top \cdot \mathbb{1}$ and σI are symmetric matrices, which implies ϕ_σ is symmetric.

Acknowledgments

References