Discrete Mathematics

Propositions

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Topics

Proposition

Definition
proposition (or statement):
a declarative sentence that is either true or false

- law of the excluded middle:
  a proposition cannot be partially true or partially false
- law of contradiction:
  a proposition cannot be both true and false

Proposition Examples

propositions

- The Moon revolves around the Earth.
- Elephants can fly.
- $3 + 8 = 11$

not propositions

- What time is it?
- Exterminate!
- $x < 43$

Propositional Variable

- propositional variable: a name that represents the proposition

examples

- p_1: The Moon revolves around the Earth. (T)
- p_2: Elephants can fly. (F)
- p_3: $3 + 8 = 11$ (T)

Compound Propositions

- compound propositions are obtained by applying logical operators
- truth table: a table that lists the truth value of the compound proposition for all possible values of its variables

Negation (NOT)

$\neg p$

p	$\neg p$
T	F
F	T

examples

- $\neg p_1$: The Moon does not revolve around the Earth.
  $\neg T : F$
- $\neg p_2$: Elephants cannot fly.
  $\neg F : T$

Conjunction (AND)

$p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

examples

- $p_1 \wedge p_2$: The Moon revolves around the Earth and elephants can fly.
  $T \wedge F : F$
- $p_1 \wedge p_3$: The Moon revolves around the Earth and $3 + 8 = 11$.
  $T \wedge T : T$

Disjunction (OR)

$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

example

- $p_1 \vee p_2$: The Moon revolves around the Earth or elephants can fly.
  $T \vee F : T$

Exclusive Disjunction (XOR)

$p \veebar q$

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

examples

- $p_1 \veebar p_2$: Either the Moon revolves around the Earth or elephants can fly.
  $T \veebar F : T$
- $p_1 \veebar p_3$: Either the Moon revolves around the Earth or $3 + 8 = 11$.
  $T \veebar T : F$

Implication (IF)

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- also called conditional
- if p then q
- p is sufficient for q
- q is necessary for p
- p: hypothesis
- q: conclusion

Implication Examples

- p_4: $3 < 8$, p_5: $3 < 14$, p_6: $3 < 2$, p_7: $8 < 6$

- $p_4 \to p_5$:
  if $3 < 8$, then $3 < 14$
  $T \to T : T$
- $p_4 \to p_6$:
  if $3 < 8$, then $3 < 2$
  $T \to F : F$
- $p_6 \to p_4$:
  if $3 < 2$, then $3 < 8$
  $F \to T : T$
- $p_6 \to p_7$:
  if $3 < 2$, then $8 < 6$
  $F \to F : T$

Implication Example

- "If I weigh over 70 kg, then I will exercise."

- p: I weigh over 70 kg.
- q: I exercise.
- when is this claim false?

$p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional (IFF)

$p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- p if and only if q
- p is necessary and sufficient for q

Example

- mother tells child:
  "If you do your homework, you can play computer games."
- h: The child does her homework.
- p: The child plays computer games.
- what does the mother mean?

- $h \to p$
- $\neg h \to \neg p$
- $h \leftrightarrow p$

Well-Formed Formula

syntax

- which rules will be used to form compound propositions?
- a formula that obeys these rules: well-formed formula (WFF)

semantics

- *interpretation*: calculating the value of a compound proposition by assigning values to its variables
- truth table: all interpretations of a proposition

Formula Examples

not well-formed

- $\vee p$
- $p \wedge \neg$
- $p \neg \wedge q$

Operator Precedence

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\rightarrow$
5. $\leftrightarrow$

- parentheses are used to change the order of calculation
- implication associates from the right:
  $p \rightarrow q \rightarrow r$ means $p \rightarrow (q \rightarrow r)$

Precedence Examples

- s: Phyllis goes out for a walk.
- t: The Moon is out.
- u: It is snowing.
- what do the following WFFs mean?

- $t \wedge \neg u \rightarrow s$
- $t \rightarrow (\neg u \rightarrow s)$
- $\neg(s \leftrightarrow (u \vee t))$
- $\neg s \leftrightarrow u \vee t$

Metalanguage

- target language: the language being worked on
- metalanguage: the language used when talking *about* the properties of the target language

Metalanguage Examples

- a native Turkish speaker learning English
- target language: English
- metalanguage: Turkish

- a student learning programming
- target language: C, Python, Java, ...
- metalanguage: English, Turkish, ...

Formula Properties

- WFF is true for all interpretations: tautology
- WFF is false for all interpretations: contradiction

- these are concepts of the metalanguage

Tautology Example

$$p \wedge (p \rightarrow q) \rightarrow q$$

p	q	$p \rightarrow q$ (A)	$p \wedge A$ (B)	$B \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Contradiction Example

$p \wedge (\neg p \wedge q)$

p	q	$\neg p$	$\neg p \wedge q$ (A)	$p \wedge A$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Logical Implication and Equivalence

- if $P \rightarrow Q$ is a tautology, then P logically implies Q: $P \Rightarrow Q$
- if $P \leftrightarrow Q$ is a tautology, then P and Q are logically equivalent: $P \Leftrightarrow Q$

Logical Implication Example

$p \wedge (p \rightarrow q) \Rightarrow q$

$p \wedge (p \rightarrow q) \rightarrow q$

p	q	$p \rightarrow q$ (A)	$A \wedge p$ (B)	$B \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Logical Equivalence Example

$\neg p \Leftrightarrow p \rightarrow F$

$\neg p \leftrightarrow p \rightarrow F$

p	$\neg p$	$p \rightarrow F$ (A)	$\neg p \leftrightarrow A$
T	F	F	T
F	T	T	T

Logical Equivalence Example

$p \rightarrow q \Leftrightarrow \neg p \vee q$

$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

p	q	$p \rightarrow q$ (A)	$\neg p$	$\neg p \vee q$ (B)	$A \leftrightarrow B$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Logical Equivalence Example

- implication: $p \rightarrow q$
- *contrapositive*: $\neg q \rightarrow \neg p$
- *converse*: $q \rightarrow p$
- *inverse*: $\neg p \rightarrow \neg q$

$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

p	q	$p \rightarrow q$ (A)	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$ (B)	$A \leftrightarrow B$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Metalogic

- $P_1, P_2, \ldots, P_n \vdash Q$
  There is a proof which infers the conclusion Q from the assumptions $P_1, P_2, \ldots, P_n$.
- $P_1, P_2, \ldots, P_n \vDash Q$
  Q must be true if $P_1, P_2, \ldots, P_n$ are all true.

Formal Systems

- a formal system is consistent if for all WFFs P and Q:
  if $P \vdash Q$ then $P \vDash Q$
- if every provable proposition is actually true

- a formal system is complete if for all WFFs P and Q:
  if $P \vDash Q$ then $P \vdash Q$
- if every true proposition can be proven

Gödel's Theorem

- propositional logic is consistent and complete

Theorem (Gödel's Theorem)

Any logical system that is powerful enough to express arithmetic must be either inconsistent or incomplete.

- liar's paradox: "This statement is false."

Propositional Calculus

1. semantic approach: *truth tables*
   too complicated when the number of primitive statements grow
2. syntactic approach: *rules of inference*
   obtain new propositions from known propositions
   using logical implications
3. axiomatic approach: *Boolean algebra*
   substitute logically equivalent formulas for one another

Laws of Logic

Double Negation (DN)
$\neg(\neg p) \Leftrightarrow p$

Commutativity (Co)
$p \wedge q \Leftrightarrow q \wedge p$ $\quad$ $p \vee q \Leftrightarrow q \vee p$

Associativity (As)
$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $\quad$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

Idempotence (Ip)
$p \wedge p \Leftrightarrow p$ $\quad$ $p \vee p \Leftrightarrow p$

Inverse (In)
$p \wedge \neg p \Leftrightarrow F$ $\quad$ $p \vee \neg p \Leftrightarrow T$

Laws of Logic

Identity (Id)
$p \wedge T \Leftrightarrow p$ $\quad$ $p \vee F \Leftrightarrow p$

Domination (Do)
$p \wedge F \Leftrightarrow F$ $\quad$ $p \vee T \Leftrightarrow T$

Distributivity (Di)
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $\quad$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Absorption (Ab)
$p \wedge (p \vee q) \Leftrightarrow p$ $\quad$ $p \vee (p \wedge q) \Leftrightarrow p$

DeMorgan's Laws (DM)
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\quad$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Equivalence Example

$$
\begin{array}{lll}
 & p \rightarrow q & \\
\Leftrightarrow & \neg p \vee q & \\
\Leftrightarrow & q \vee \neg p & Co \\
\Leftrightarrow & \neg\neg q \vee \neg p & DN \\
\Leftrightarrow & \neg q \rightarrow \neg p &
\end{array}
$$

Equivalence Example

$$
\begin{array}{lll}
 & \neg(\neg((p \vee q) \wedge r) \vee \neg q) & \\
\Leftrightarrow & \neg\neg((p \vee q) \wedge r) \wedge \neg\neg q & DM \\
\Leftrightarrow & ((p \vee q) \wedge r) \wedge q & DN \\
\Leftrightarrow & (p \vee q) \wedge (r \wedge q) & As \\
\Leftrightarrow & (p \vee q) \wedge (q \wedge r) & Co \\
\Leftrightarrow & ((p \vee q) \wedge q) \wedge r & As \\
\Leftrightarrow & q \wedge r & Ab
\end{array}
$$

Duality

- dual of s: s^d
  replace: $\wedge$ with $\vee$, $\vee$ with $\wedge$, T with F, F with T
- principle of duality: if $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$

example

$$
\begin{array}{rl}
s: & (p \wedge \neg q) \vee (r \wedge T) \\
s^d: & (p \vee \neg q) \wedge (r \vee F)
\end{array}
$$

Inference

- establish the validity of an argument
- starting from a set of propositions
- which are assumed or proven to be true

notation

$$
\begin{array}{c}
p_1 \\
p_2 \\
\cdots \\
p_n \\
\hline
\therefore q
\end{array}
\qquad\qquad
p_1 \wedge p_2 \wedge \cdots \wedge p_n \Rightarrow q
$$

Trivial Rules

Identity (ID)

$$\frac{p}{\therefore p}$$

Contradiction (CTR)

$$\frac{F}{\therefore p}$$

Basic Rules

OR Introduction (OrI)

$$\frac{p}{\therefore p \lor q}$$

AND Elimination (AndE)

$$\frac{p \land q}{\therefore p}$$

AND Introduction (AndI)

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \land q}$$

Modus Ponens

Implication Elimination (ImpE)

$$\frac{\begin{array}{c} p \rightarrow q \\ p \end{array}}{\therefore q}$$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia has won the lottery.
- Therefore, Lydia will buy a car.

Modus Tollens

Modus Tollens (MT)

$$\frac{\begin{array}{c} p \rightarrow q \\ \neg q \end{array}}{\therefore \neg p}$$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia did not buy a car.
- Therefore, Lydia did not win the lottery.

Modus Tollens

$$\frac{\begin{array}{c} p \rightarrow q \\ \neg q \end{array}}{\therefore \neg p}$$

1.	$p \rightarrow q$	A
2.	$\neg q \rightarrow \neg p$	$EQ : 1$
3.	$\neg q$	A
4.	$\neg p$	$ImpE : 2, 3$

Fallacies

$$\frac{\begin{array}{c} p \rightarrow q \\ q \end{array}}{\therefore p}$$

$(p \rightarrow q) \land q \not\Rightarrow p$

- $p : F,\ q : T$
  $(F \rightarrow T) \land T \rightarrow F : F$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia has bought a car.
- Therefore, Lydia has won the lottery.

Fallacies

$$\frac{\begin{array}{c} p \rightarrow q \\ \neg p \end{array}}{\therefore \neg q}$$

$(p \rightarrow q) \land \neg p \not\Rightarrow \neg q$

- $p : F,\ q : T$
  $(F \rightarrow T) \land T \rightarrow F : F$

example

- If Lydia wins the lottery, she will buy a car.
- Lydia has not won the lottery.
- Therefore, Lydia will not buy a car.

Implication Introduction

Implication Introduction (ImpI)

$$\frac{p \vdash q}{\therefore\ \vdash p \rightarrow q}$$

- if it can be shown that q is true assuming p is true
- then $p \rightarrow q$ is true *without assuming p is true*
- p is a provisional assumption (PA)
- provisional assumptions have to be discharged

Implication Introduction Example

$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

1.	p	PA
2.	$p \rightarrow q$	A
3.	q	$ImpE : 2, 1$
4.	$\neg q$	A
5.	$q \rightarrow F$	$EQ : 4$
6.	F	$ImpE : 5, 3$
7.	$p \rightarrow F$	$ImpI : 1, 6$
8.	$\neg p$	$EQ : 7$

OR Elimination

OR Elimination (OrE)

$$\begin{array}{c} p \vee q \\ p \vdash r \\ q \vdash r \\ \hline \therefore \quad \vdash r \end{array}$$

- p and q are provisional assumptions

Disjunctive Syllogism

Disjunctive Syllogism (DS)

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

example

- Bart's wallet is either in his pocket or on his desk.
- Bart's wallet is not in his pocket.
- Therefore, Bart's wallet is on his desk.

Disjunctive Syllogism

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

applying OrE:

$$\begin{array}{c} p \vee q \\ p \vdash q \\ q \vdash q \\ \hline \therefore q \end{array}$$

1.	$p \vee q$	A
2.	$\neg p$	A
3.	$p \rightarrow F$	$EQ : 2$
4a1.	p	PA
4a2.	F	$ImpE : 3, 4a1$
4a.	q	$CTR : 4a2$
4b1.	q	PA
4b.	q	$ID : 4b1$
5.	q	$OrE : 1, 4a, 4b$

Hypothetical Syllogism

Hypothetical Syllogism (HS)

$$\frac{\begin{array}{c} p \rightarrow q \\ q \rightarrow r \end{array}}{\therefore p \rightarrow r}$$

1.	p	PA
2.	$p \rightarrow q$	A
3.	q	$ImpE : 2, 1$
4.	$q \rightarrow r$	A
5.	r	$ImpE : 4, 3$
6.	$p \rightarrow r$	$ImpI : 1, 5$

Hypotetical Syllogism Example

Spock to Lieutenant Decker:

It would be a suicide to attack the enemy ship now.
Someone who attempts suicide is not psychologically fit to command the Enterprise.
Therefore, I am obliged to relieve you from duty.

- p: Decker attacks the enemy ship.
- q: Decker attempts suicide.
- r: Decker is not psychologically fit to command the Enterprise.
- s: Spock relieves Decker from duty.

Hypotetical Syllogism Example

$$\frac{\begin{array}{c} p \\ p \rightarrow q \\ q \rightarrow r \\ r \rightarrow s \end{array}}{\therefore s}$$

1.	$p \rightarrow q$	A
2.	$q \rightarrow r$	A
3.	$p \rightarrow r$	$HS : 1, 2$
4.	$r \rightarrow s$	A
5.	$p \rightarrow s$	$HS : 3, 4$
6.	p	A
7.	s	$ImpE : 5, 6$

Inference Examples

$$\frac{\begin{array}{c} p \rightarrow r \\ r \rightarrow s \\ x \vee \neg s \\ u \vee \neg x \\ \neg u \end{array}}{\therefore \neg p}$$

1.	$\neg u$	A	6.	$r \rightarrow s$	A
2.	$u \vee \neg x$	A	7.	$\neg r$	$MT : 6, 5$
3.	$\neg x$	$DS : 2, 1$	8.	$p \rightarrow r$	A
4.	$x \vee \neg s$	A	9.	$\neg p$	$MT : 8, 7$
5.	$\neg s$	$DS : 4, 3$			

Inference Examples

$$\begin{array}{c} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow x \\ \neg x \\ \hline \therefore p \end{array}$$

1.	$\neg x$	A	6.	$(\neg p \vee \neg q) \rightarrow (r \wedge s)$	A
2.	$r \rightarrow x$	A	7.	$\neg(\neg p \vee \neg q)$	$MT : 6, 5$
3.	$\neg r$	$MT : 2, 1$	8.	$p \wedge q$	$DM : 7$
4.	$\neg r \vee \neg s$	$OrI : 3$	9.	p	$AndE : 8$
5.	$\neg(r \wedge s)$	$DM : 4$			

Inference Examples

$$\begin{array}{c} p \rightarrow (q \vee r) \\ s \rightarrow \neg r \\ q \rightarrow \neg p \\ p \\ s \\ \hline \therefore F \end{array}$$

1.	p	A
2.	$q \rightarrow \neg p$	A
3.	$\neg q$	$MT : 2, 1$
4.	s	A
5.	$s \rightarrow \neg r$	A
6.	$\neg r$	$ImpE : 5, 4$
7.	$p \rightarrow (q \vee r)$	A
8.	$q \vee r$	$ImpE : 7, 1$
9.	q	$DS : 8, 6$
10.	$q \wedge \neg q : F$	$AndI : 9, 3$

Inference Examples

If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 20℃, there is no chance for rain. Today the temperature is 22℃ and Lois is wearing her red headband. Therefore, Lois will mow her lawn.

- p: There is a chance of rain.
- q: Lois' red headband is lost.
- r: Lois mows her lawn.
- s: The temperature is over 20℃.

Inference Examples

$$\begin{array}{c} (p \vee q) \rightarrow \neg r \\ s \rightarrow \neg p \\ s \wedge \neg q \\ \hline \therefore r \end{array}$$

1.	$s \wedge \neg q$	A
2.	s	$AndE : 1$
3.	$s \rightarrow \neg p$	A
4.	$\neg p$	$ImpE : 3, 2$
5.	$\neg q$	$AndE : 1$
6.	$\neg p \wedge \neg q$	$AndI : 4, 5$
7.	$\neg(p \vee q)$	$DM : 6$
8.	$(p \vee q) \rightarrow \neg r$	A
9.	?	$7, 8$

References

Required Reading: Grimaldi

- Chapter 2: Fundamentals of Logic
  - 2.1. Basic Connectives and Truth Tables
  - 2.2. Logical Equivalence: The Laws of Logic
  - 2.3. Logical Implication: Rules of Inference

Supplementary Reading: O'Donnell, Hall, Page

- Chapter 6: Propositional Logic