Signals & Systems

2018

Week 4

26.02.18

Last time:

Fourier Series for CT Periodic Signals

- Given a set of harmonically related complex exponentials $e^{jk\omega_0 t}$, $k \in \mathbb{Z}$, we synthesize a signal

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k e^{jk\omega_0 t}}}$$

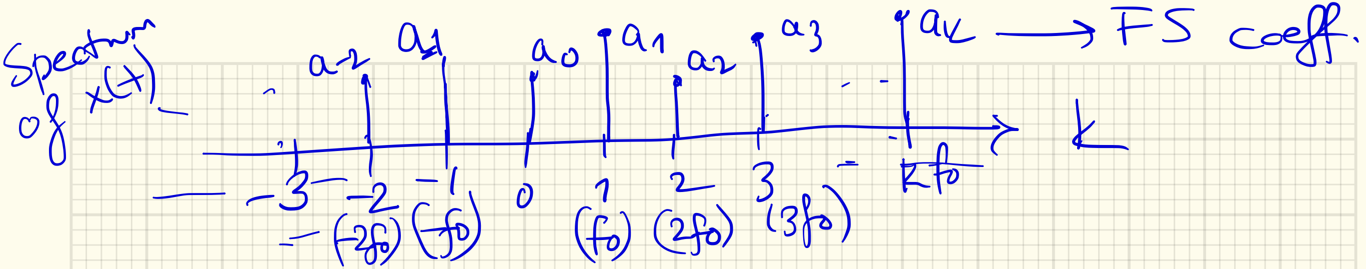
ω_0 = fundamental freq

each term has freq. $k\omega_0$.

then the sum $x(t)$ is also periodic w/ $\underline{T_0} \Longleftrightarrow f_0 = \frac{1}{T_0}$ Hz $\longrightarrow \omega_0 = 2\pi f_0$ rad/s.

- Given $x(t)$: $a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$, $k \in \mathbb{Z}$

- Freq spectrum of $x(t)$: we plot a_k vs kf_0 or k.



<u>Parseval's Relation</u>: The average power of $x(t)$ in one period can be expressed in terms of FS coefficients.

$$P_{av} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- The graph of $|a_k|^2$ as a fn. $kf_0 = f$

$\Rightarrow$ power spectrum of the periodic signal $x(t)$.

Convergence Conditions

For a periodic signal $x(t)$ to have a F.S. representation; it is necessary that

1) Coeff a_k are finite;

If the periodic signal $x(t)$ is absolutely integrable;

$$\int_{T_0} |x(t)|\,dt < \infty \longrightarrow$$ guarantees a_k are finite.

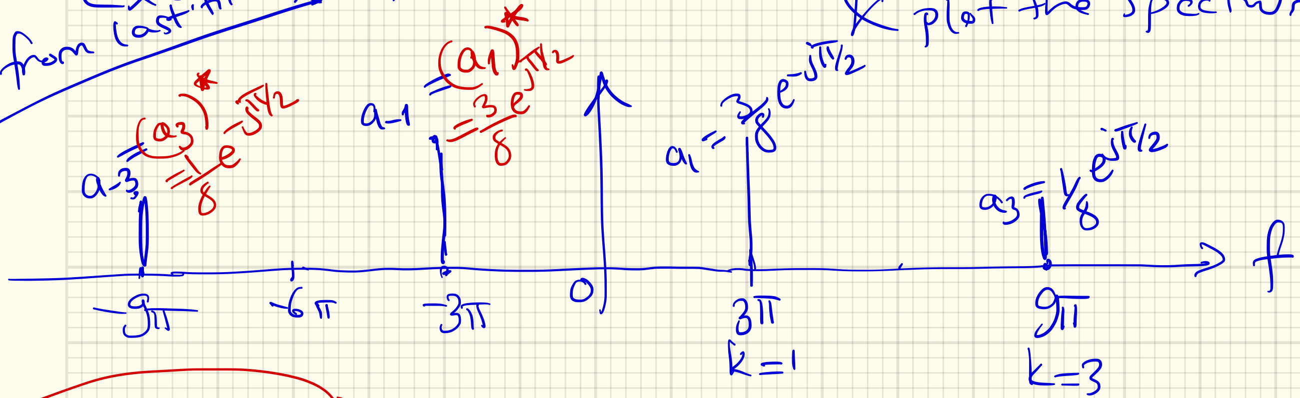
2) When a_k are used to synthesize $x(t)$, the resulting series should converge to $x(t)$

$$x_m(t) = \sum_{k=-m}^{m} a_k e^{jk\omega_0 t}$$;

the periodic $x(t)$ has a finite # maxima & minima & finite discontinuities in one period.

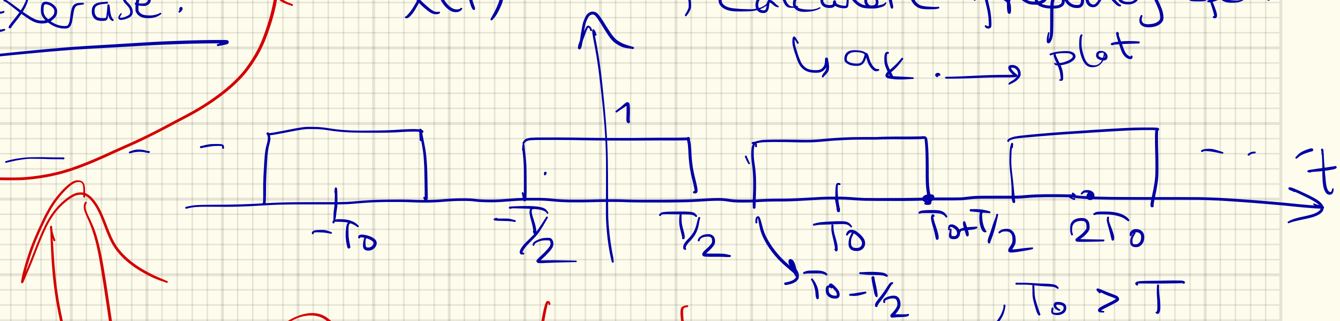
then $x_m(t) \longrightarrow x(t)$ wherever $x(t)$ is continuous.

Exercise from last time: $x(t) = \sin^3(3\pi t)$ Find F.S. coeff. & plot the spectrum



Exercise:

$x(t)$ → Calculate frequency spectrum
↳ a_k → plot



, $T_0 > T$

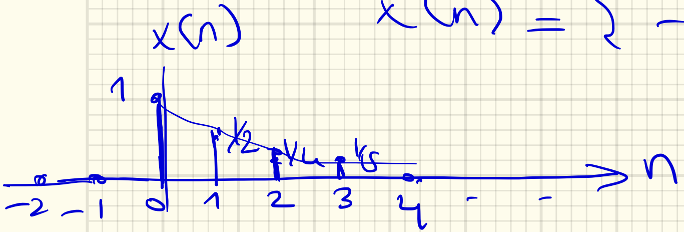
Do it!

Discrete-time (DT) Signals $\longleftrightarrow$ Periodic DT Fourier Series

$x[n]$:

$$x(n) = \begin{cases} \left(\frac{1}{2}\right)^n , & n \geq 0 \\ 0 , & n < 0 \end{cases}$$

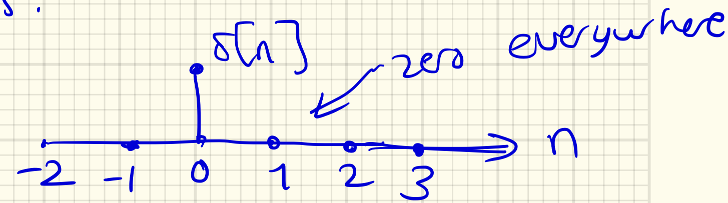
$$x(n) = \{ \cdots \; 0 \;\; 1 \;\; \tfrac{1}{2} \;\; \tfrac{1}{4} \;\; \tfrac{1}{8} \cdots \}$$



Elementary DT Signals:

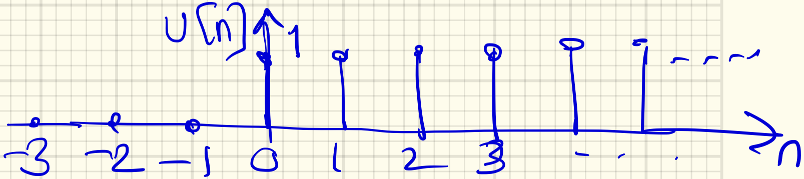
1) Unit Sample Seq.

$$\delta[n] = \begin{cases} 1 , & n = 0 \\ 0 , & n \neq 0 \end{cases}$$



2) Unit Step Seq:

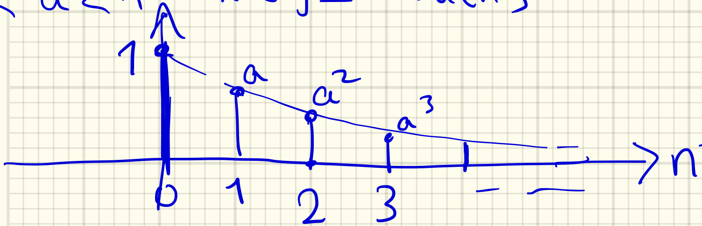
$$u(n) = \begin{cases} 1 , & n \geq 0 \\ 0 , & n < 0 \end{cases}$$



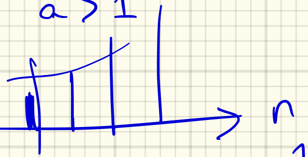
3) Exponential Seq. $x[n] = A a^n$, $-\infty < n < \infty$

A can be real or complex

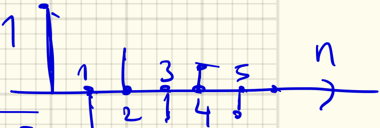
$0 < a < 1$ $\quad x[n] = a^n \cdot u[n]$



$a > 1$



$-1 < a < 0$



$|x[n]|$ decreases in magnitude.

4) DT sinusoidal Sequence:

Recall a CT sinusoid $x(t) = A\cos(\omega t + \phi)$ $\quad \omega$: rad/s

Sample CT sinusoid every T_s sec

$$x(t)\Big|_{t=nT_s} = x(nT_s) = A\cos(\underbrace{\omega T_s}\, n + \phi)$$

T_s: period

T_s: sampling period

$f_s = \frac{1}{T_s}$: sampling frequency

$$\hat{\omega} \triangleq \omega T_s = \frac{\omega}{f_s}$$

$\hat{\omega}$: normalized angular freq.

Def: DT angular frequency

$\hat{\omega}$: $\frac{rad}{s} \cdot s = \underline{radians}$ unit

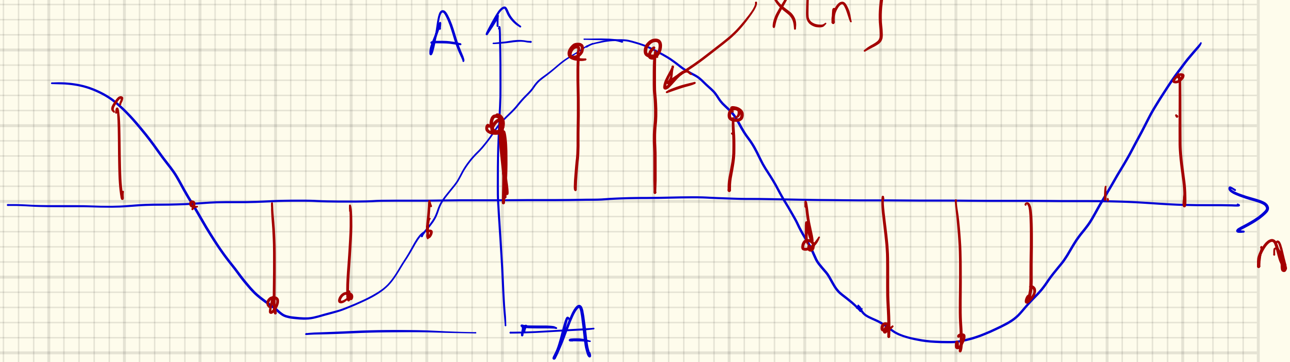
Def: DT sinusoid

$x[n] = A\cos(\hat{\omega}_0 n + \phi)$, $-\infty < n < \infty$.

A, ϕ are real constants

$\hat{\omega}_0$: (angular) frequency of the sinusoid.

↳ unit is radians



★ **DT Sinusoids**: Shift the frequency by $2\pi k$

$\hat{\omega}_1 = \hat{\omega}_0 + 2\pi k$ (Recall $e^{j\hat{\omega} n} = e^{j(\hat{\omega}+2\pi k)n}$)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 1$

$\sin(\hat{\omega}_1 n) = \sin((\hat{\omega}_0 + 2\pi k)n) = \sin(\hat{\omega}_0 n)\cos(kn2\pi) + \cos(\hat{\omega}_0 n)\cancel{\sin(kn2\pi)}$

↳ $= \sin(\hat{\omega}_0 n)$

$\sin(2\pi m) = 0$

$\Rightarrow$ DT Sinusoids are unique over $0 \le \hat{\omega} \le 2\pi$

Rate of oscillation DT sinusoid $\nearrow$ increases for $\hat{\omega}_0 : 0 \to \pi$

decreases for $\hat{\omega}_0 : \pi \to 2\pi$.

Periodic Sequence: A sequence $x(n)$ is periodic if $x[n] = x[n+N]$, $\forall n$. The smallest N for which this eqn holds is the fundamental period

DT Sinusoidal sequence is periodic if

$$\cos(\hat{\omega}_0 n + \phi) = \cos(\hat{\omega}_0 (n+N) + \phi)$$

$$= \cos(\hat{\omega}_0 n + \underbrace{\hat{\omega}_0 N}_{=2\pi k} + \phi)$$

if $\hat{\omega}_0 N = 2\pi k$

$\Rightarrow N = \dfrac{2\pi \hat{k}}{\hat{\omega}_0}$

$$\hat{\omega}_0 = 2\pi \hat{f}_0$$

$\hat{\omega}_0 = \dfrac{2\pi k}{N}$ or $\hat{f}_0 = \dfrac{k}{N}$, $\hat{f}_0$ is a rational number

The normalized frequency $\hat{f}_0 = \frac{f_0}{f_s} = \frac{1/T_0}{1/T_s} = \frac{T_s}{T_0} = \frac{k}{N}$

$\Longrightarrow N . T_s = k . T_0$

In the uploaded

$x_B(t) = \cos\left(\frac{\pi}{5} t\right)$

$x_G(t) = \cos\left(\frac{11\pi}{5} t\right)$

Sampled w/ $T_s = 1$ sec.

Check the corresp. DT sinusoids $x_B[n]$, $x_G[n]$

Calculate $\hat{\omega}_0$, $\hat{f}_0$ - .

Also do the same for $T_s = 0.3$ sec.