ERDEMİR

GELECEĞE ŞEKİL VERİYORUZ

2 Vize

olasılık ← 1. vize → 5. hafta

rastgele değişkenler ← 2. vize → (9,10,11) hafta

istatistik

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↳ Tuğba Pamay → Asst.

Probability → Until 1st Midterm

Random Variable(s) → Btw. 1st and 2nd Midterms

Statistics → After 2nd Midterm

Probability

$P: \mathcal{L} \longrightarrow R \cap [0,1]$

mapping (like function)

Sets

↳ $A = \{ a : a \in A \}$ → A consists of element a, a belongs to A.

↳ $\bar{A} = \{ a : a \notin A \}$ → complement of A

Operations on Sets

$A \cup B = \{ x : x \in A \text{ or } x \in B \}$

$A \cap B = \{ x : x \in A \text{ and } x \in B \}$

$\mathcal{L} = A \cup \bar{A} = \{ x : x \in A \text{ or } x \in \bar{A} \}$
↳ Universal set

$\emptyset = A \cap \bar{A} = \{ x : x \in A \text{ and } x \in \bar{A} \}$
↳ empty set

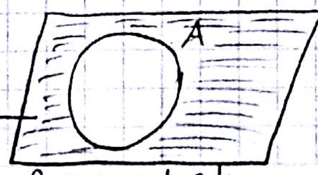
Let there be A_i $i = 1, 2, \dots, n$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \dots \cup A_n = \{ x : x \in A_i \text{ for any } i \}$$

at least one

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_n = \{ x : x \in A_i \text{ for all } i \}$$

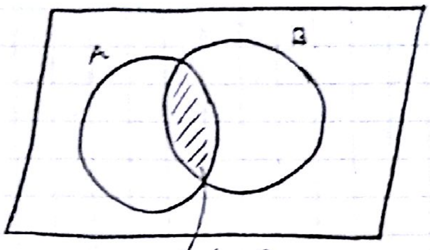
$\bar{A}$



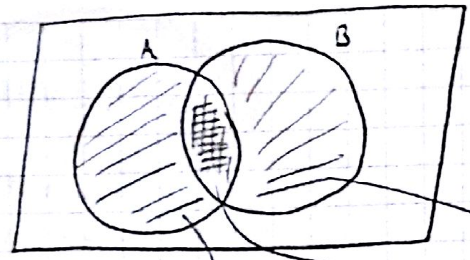
$\mathcal{L}$: Universal set

$A \subseteq \mathcal{L}$

any set including empty set and universal set



$A \cap B$



$A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$

do not intersect

Algebra of Sets

1 → Commutative → $A \cup B = B \cup A$

$A \cap B = B \cap A$

2 → Associative → $A \cup B \cup C = (A \cup B) \cup C$

$= A \cup (B \cup C)$

$= (A \cup C) \cup B$

$A \cap B \cap C = (A \cap B) \cup C$

$= B \cap (A \cap C)$

3 → Distributive

Distribute $\cup$ over $\cap$

~~Distribute over~~

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Distribute $\cap$ over $\cup$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Laws

i) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

ii) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

P "Measures" the likelihood of the (set) event

Universal set ≡ Sample space

Set ≡ Event

Element ≡ Outcome

Outcome of an experiment

More general than human controlled experiment.

Example: - Experiment of mother nature.

$$\mathcal{S} = \{ \text{snowy}, \text{sunny}, \text{rainy}, \text{only cloudy} \}$$

Four single outcome events.

Precipitation (Yağış) = Snowy or Rainy

Event

outcomes

Event of two outcomes

Example: - Stock market experiment

$$\mathcal{S} = \{ \text{Index up}, \text{Index down} \}$$

Two outcomes ($n=2$)

How many possible events?

How many ways of forming sets from two elements?

$2^n = 4$

$$\binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 2^n = \sum_{k=0}^{n} \binom{n}{k}$$

binomial expansion

Axiomatic definition of probability

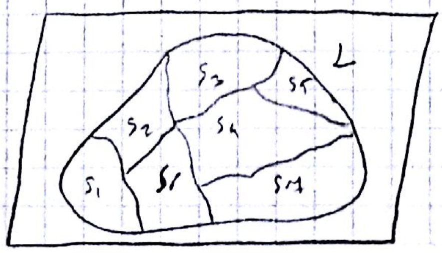
1) $P(A) \geq 0$

any event

2) $P(\mathcal{S}) = 1$ → max. value of probability

3) $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

A and B disjoint (no element in common)



$$\left.\begin{aligned} \mathcal{S} &= \bigcup_{i=1}^{7} S_i \\ S_i \cap S_j &= \emptyset \end{aligned}\right\} \Rightarrow P(\mathcal{S}) = \sum_{i=1}^{7} P(S_i)$$

$$P(A \cup B \cup D) = P(C \cup D) = \underbrace{P(C)}_{P(A) + P(B)} + P(D)$$

Classical Definition:

Very much related to, derivable from the axiomatic definition.

- Assume that the outcomes are equally likely.

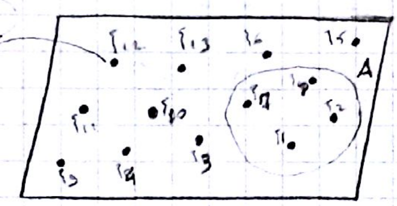
$$P(A) = \frac{n_A}{n}$$

n: # of equally likely outcomes.

n_A: # of equally likely outcomes that belongs to A.

lights

likely equal to being on



$$P(A) = \frac{4}{13}$$

Probability of a light inside A being on is $\frac{4}{13}$.

Axiomatic def.

Let ξ_i be the ith outcome (ith light on)

$P(\xi_i) = ?$

$\Omega = \bigcup_{i=1}^{13} \{\xi_i\}$, $\{\xi_i\} \cap \{\xi_j\} = \emptyset$ → only one ligth on at a particular time.

$$1 = P(\Omega) = \sum_{i=1}^{13} Pr\{\xi_i\} = 13\, Pr\{\xi_i\}$$

$\Rightarrow Pr\{\xi_i\} = \frac{1}{13}$ (only one ligth on at a particular time)

A: The event of any light inside region A.

$$\mathbf{A} = \{\xi_1\} \cup \{\xi_2\} \cup \{\xi_7\} \cup \{\xi_8\}$$

$$P(A) = Pr\{\xi_1\} + Pr\{\xi_2\} + Pr\{\xi_7\} + Pr\{\xi_8\}$$

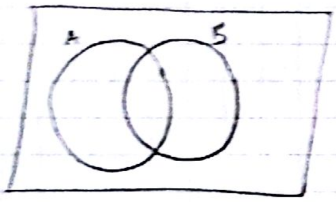
$$= \frac{1}{13} + \frac{1}{13} + \frac{1}{13} + \frac{1}{13} = \frac{4}{13}$$

Frequency Definition:

$P(A) = \lim_{k \to \infty} \frac{k_A}{k}$ $\quad$ k: # trials
$\quad$ k_A: # trials in which event A occurs.

$P(A \cap B)$ Joint probability
(The probability of joint event)

Axiom 3: $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$



$Area(A \cup B) = Area(A) + Area(B) - Area(A \cap B)$

In general when $A \cap B \neq \emptyset$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$\Rightarrow$ Show $P(\emptyset) = \emptyset$

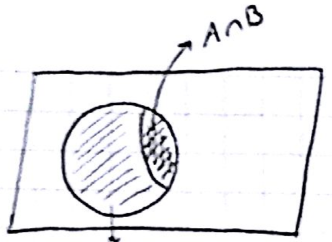
$A \cup \emptyset = A$
$\hookrightarrow$ disjoint $\quad A \cap \emptyset = \emptyset$

$\hookrightarrow P(A \cup \emptyset) = P(A)$
$P(A) + P(\emptyset) = P(A)$
$P(\emptyset) = \emptyset$

$A - B = A \cap \overline{B}$

$A \cup B = \underbrace{(A - B) \cup B}_{\text{Disjoint}} \Rightarrow {}^{*} P(A \cup B) = P(A - B) + P(B)$

$\star \; P(A \cup B) = P(A-B) + P(B)$



$$A = \underbrace{(A-B) \cup (A \cap B)}_{\text{Disjoint}}$$

$\downarrow$ can apply axiom 3

$\star\star \; P(A) = P(A-B) + P(A \cap B)$

Combine $\star$ and $\star\star$ to get

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

HW.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$(A / B) = A - B$

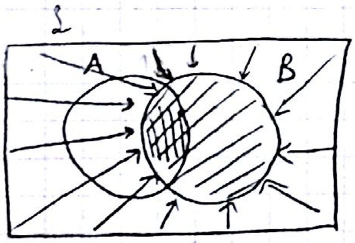
$A/B \neq A|B$

Conditional Probability

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

$\downarrow$ vertical bar

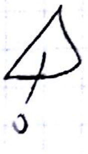
$P(A|B)$: probability of A given B or A conditioned on B having occured



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$\mathcal{E}_B$ shrinks the sample space

After we know that B has occured now sample space is $\Omega' = B$



$$P(A|\Omega) = \frac{P(A \cap \Omega)}{P(\Omega)}$$

Conditional Probability

$P(A|B) \triangleq \dfrac{P(A\cap B)}{P(B)}$

$P(A)$ Mapping $\mathcal{L} \rightarrow$

1) $P(A) \geq \emptyset$
2) $P(\mathcal{E}) = 1$
3) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

Ex: Die throw

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$

$E = \{2, 4, 6\}$

$O = \{1, 3, 5\}$

$H = \{4, 5, 6\}$

$L = \{1, 2, 3\}$

$P(E) = \dfrac{n_E}{n} = \dfrac{3}{6} = \dfrac{1}{2}$

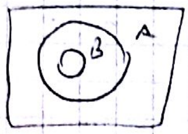
$P(E|H) = \dfrac{P(E\cap H)}{P(H)} \quad \dfrac{n_{E\cap H}/n}{n_H/n} = \dfrac{2/6}{3/6} = \dfrac{2}{3}$

$P(H|E) = \dfrac{P(E\cap H)}{P(E)} = \dfrac{n_{E\cap H}/n}{n_E/n} = \dfrac{2/6}{3/6} = \dfrac{2}{3}$

sıralamayı değiştirmek
başka örneklerde sonucu da
değiştirebilir. Yani $\Rightarrow$ $P(E|H) \neq P(H|E)$

Properties:

i) $P(A|B)$ $\qquad B \subseteq A$



ii) $P(B|A)$

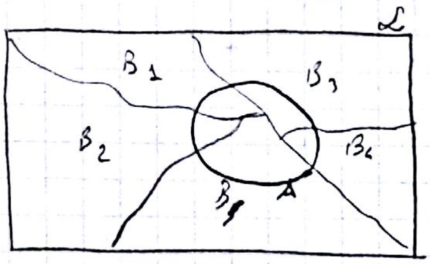
$\longrightarrow$ i) $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

ii) $P(B|A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{P(B)}{P(A)}$

iii) $P(A|B)$ when $A \cap B = \emptyset$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\emptyset}{P(B)} = 0 \quad \text{for } P(B) \neq 0$$

Total Probability



Requirements:

a) $B_i \cap B_j = \emptyset$

b) $\bigcup_i B_i = \mathcal{L}$

$$\underbrace{P(\bigcup_i B_i)}_{\sum_i P(B_i)} = P(\mathcal{L}) = 1$$

A is overlayed on top of the partition of $\{B_i\}$

$$A \cap \bigcup_i B_i = A \cap \mathcal{L} = A$$

$$P(A \cap (\bigcup_i B_i)) = P(A)$$

$$P(\bigcup_i (A \cap B_i)) = P(A)$$

$$\left.\begin{array}{l} C_i = (A \cap B_i) \\ C_j = (A \cap B_j) \end{array}\right\} \text{disjoint} \qquad \sum_i P(A \cap B_i) = P(A)$$

Interpretation: Probability (measure) of the whole is the sum of the probabilities (measures) of the pieces (as long as the pieces do not overlap)

Example: A car of a certain make/model can be manifactured in one of 3 factories I, II, III (plants) if there is a problem/defect in the car a claim (c) is made within the

--- --- .

$P(c \mid I) = 0{,}05$ $\quad P(c \mid II) = 0{,}1 \quad P(c \mid III) = 0{,}08$

claim is made — manifactured in plant I

A customer is only interested in $P(c)$
(He doesnot care in which plant the car is manifactured)

$P(c) = ?$

$P(I) = 0{,}5 \quad P(II) = 0{,}3 \quad P(III) = 0{,}2$

manifactured in I

$P(I) + P(II) + P(III) = 1$ because/therefore more than one plants cannot manifacture the same car.

$C = C \cap \Omega$

$C = C \cap (I \cup II \cup III)$

$= (C \cap I) \cup (C \cap II) \cup (C \cap III)$

restrictions! C to I

These are disjoint

$I \cap II = \emptyset$
$I \cap III = \emptyset$
$II \cap III = \emptyset$

$P(C) = P((C \cap I) \cup (C \cap II) \cup (C \cap III))$

$= \underbrace{P(C \cap I)}_{P(C \mid I) \cdot P(I)} + \underbrace{P(C \cap II)}_{P(C \mid II) P(II)} + \underbrace{P(C \cap III)}_{P(C \mid III) P(III)}$

$P(\bar{C}|I) = 1 - P(C|I) = 1 - 0{,}05 = 0{,}095$

$P(C|I) = 0{,}05$

$$P(C) = P(C\cap I) + P(C\cap II) + P(C\cap III)$$
$$= P(C|I)P(I) + P(C|II)P(II) + P(C|III)P(III)$$
$$= 0{,}05\cdot 0{,}5 + 0{,}1\cdot 0{,}3 + 0{,}08\cdot 0{,}2$$
$$= 0{,}025 + 0{,}03 + 0{,}016$$
$$= 0{,}071$$

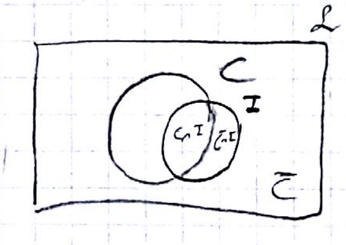
Bayes Theorem

$$P(I|C) = \frac{P(I\cap C)}{P(C)} = \frac{P(C|I)P(I)}{P(C|I)P(I) + P(C|II)P(II) + P(C|III)P(III)}$$
$$= \frac{(0{,}05)(0{,}5)}{0{,}071}$$

$$P(I|\bar{C}) = \frac{P(\bar{C}|I)P(I)}{P(\bar{C}|I)P(I) + P(\bar{C}|II)P(II) + P(\bar{C}|III)P(III)}$$

manufactured in plant I — no claim filed for the car

$$P(C|I) + P(\bar{C}|I) = 1$$
$$P(C\cap I) + P(\bar{C}\cap I) = P(I)$$
$$P(\underbrace{(C\cap I)\cup(\bar{C}\cap I)}_{I}) = P(I)$$



$P(C|I) = 0{,}05 \Rightarrow P(\bar{C}|I) = 0{,}95$

$$P(I|\bar{C}) = \frac{P(\bar{C}|I)P(I)}{P(\bar{C}|I)P(I) + \underbrace{P(\bar{C}|II)}_{1-0{,}1=0{,}9}P(II) + \underbrace{P(\bar{C}|III)}_{1-0{,}08}P(III)}$$
$$= \frac{0{,}95\cdot 0{,}5}{0{,}95\cdot 0{,}5 + 0{,}9\cdot 0{,}3 + 0{,}92\cdot 0{,}2}$$
$$= \frac{0{,}475}{0{,}475 + 0{,}27 + 0{,}184} = \frac{0{,}475}{0{,}929}$$

Example: Multiple choice exam

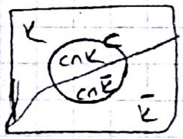
m answers to each question. Student knows the answer with probability p.

K: Knows the answer → if you know answer you mark it correctly

C: Marks the correct answer → you can mark correct answer also without knowing answer (randomly chose)

$$C = (C \cap K) \cup (C \cap \bar{K})$$

$$\hookrightarrow P(C) = P(C \cap K) + P(C \cap \bar{K})$$
$$= P(C|K)P(K) + P(C|\bar{K})P(\bar{K})$$



$P(K|C) = ?$

knows the answer K ↳ marks correctly

$P(C|K) = 1$ → if you know it you mark it correctly with probability 1

$P(C|\bar{K}) = \frac{1}{m}$

$P(K) = p$

$P(\bar{K}) = 1 - p$

$$P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|\bar{K})P(\bar{K})}$$

$$= \frac{1 \cdot p}{1 \cdot p + \frac{1}{m} \cdot (1-p)} = \frac{mp}{mp + 1 - p} = \frac{mp}{(m-1)p + 1}$$

$P(\bar{K}|C)$ small when $p \to 1$

Statistical independence

Events A, B

$$P(A) \overset{?}{=} P(A|B)$$

If yes, A & B are statistically independent.

If no, A & B are statistically dependent.

Assume $P(A) = P(A|B)$

$$\begin{aligned} P(A) \cdot P(B) &= P(A|B)\,P(B) \\ &= P(A \cap B) \\ &= P(B|A)\,P(A) \\ P(B) &= P(B|A) \end{aligned}$$

Statistically independence:

i) $P(A|B) = P(A)$

ii) $P(B|A) = P(B)$

iii) $P(A \cap B) = P(A)\,P(B)$

Example: R: Rainfall exceeding 10 cm/month

S: Summer

$\bar{S}$: Not Summer

$P(R|S) \Rightarrow$ rainfall exceeding 10 cm in summer

$P(R|\bar{S}) \Rightarrow$ " " " not in summer

$P(R) \quad P(R|S) \quad P(R|\bar{S})$

$$P(R|S) < P(R) < P(R|\bar{S})$$

$$P(R) = P(R|S)\,P(S) + P(R|\bar{S})\,P(\bar{S})$$

add upto 1

$P(R) \rightarrow P(R|S)$ ve $P(R|\bar{S})$ arasında bir değer alır.
çünkü $P(S)$ ve $P(\bar{S})$ ile çarpılır!

Event S and R are statistically dependent

U: Stock market goes up in London

D: Stock market goes down in London

$P(R|U) \overset{?}{=} P(R) \overset{?}{=} P(R|D)$

$P(R|U) = P(R)$

Counting

Structures of experiments that are repeated.

Each instance of the experiment is called a trial.

From trial to trial the sample space.

- might stay the same
- might change.

a) Sampling with replacement.

b) Sampling without replacement

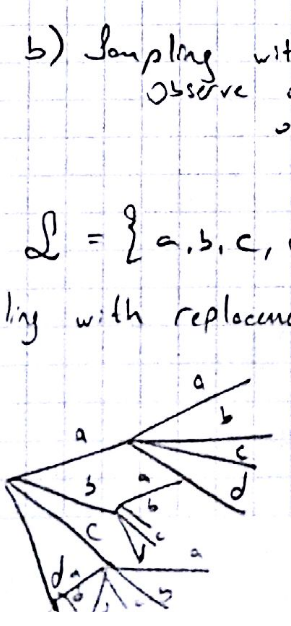
a) Sampling with replacement:
Observe outcome in a trial, can observe the same outcome in the following trials.

b) Sampling without replacement:
Observe outcome in a trial, but cannot observe the same outcome in following trials.

$\Omega = \{a, b, c, d\}$

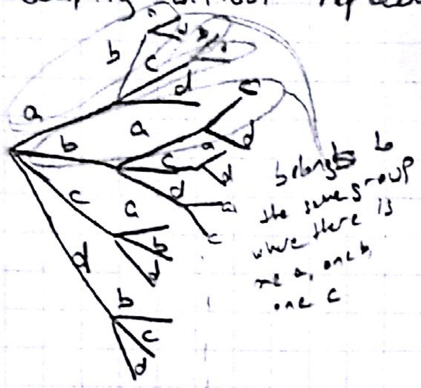
- Sampling with replacement $\quad |\Omega| = n$

In n^k different ways one can select outcomes in k trials.



n^k leaves (paths from the root)

- Sampling without replacement



$? = \frac{n!}{(n-k)!}$ In ? different ways one can select outcomes in k trials.

$$? = n\cdot(n-1)\cdots(n-k+1)$$

$$? = \frac{n!}{(n-k)!}$$

$k!$ permutations map to the same group

Example: abc, acb, bac, bca, cab, cba map to the same group in which there is one a, one b and one c.

How many groups do we have (notice each group is a combination of $k=3$ outcomes.)

How many combinations of $k=3$ outcomes do we have

$$\hookrightarrow \frac{n!}{(n-k)!} \Big/ k! = \binom{n}{k}$$

Example: Suppose there are two a's in our group. In how many different ways can these two a's occur

($k=2$)

$\hookrightarrow$ aa -> 1 way.

Suppose there are one b and one c in our group. In how many different ways can these occur.

($k=2$)

$\hookrightarrow$ bc, cb } 2 ways.

Example: Consider 3 trials. ($k=3$) In how many different ways 3 outcomes can occur.

=) that are all different ? =) 3!

=) two of them are the same ? =) 3

=) all of them are the same ? => 1

Combined experiments:

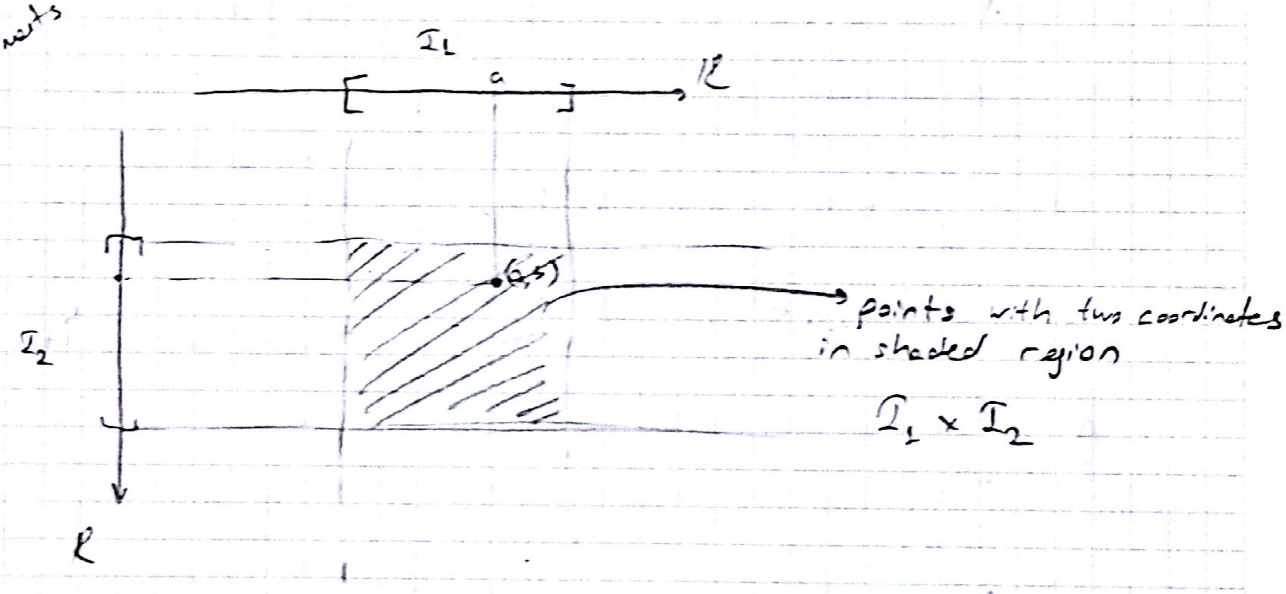
- Repeated in time (trials)
- Performed simultaneously

- Sample space of ith (instance of the) experiment is denoted as Ω_i

sample space of the combined experiments

$\Omega = \Omega_1 \overset{\text{cartesian product}}{\times} \dots \Omega_n$ n (instance of) experiments

$x \in \Omega$ if $x = (x_1, \dots, x_n)$ where $x_i \in \Omega_i$

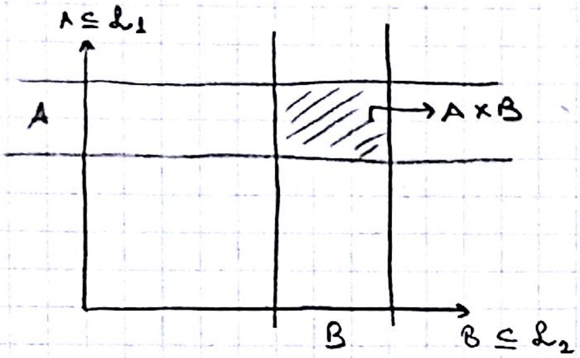


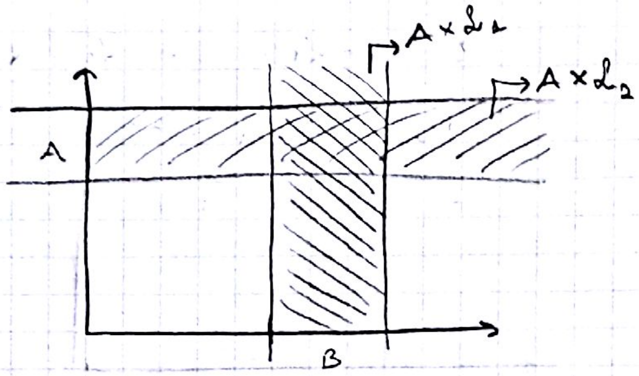
Combined Experiments

$$\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$$

combined space

Sample spaces of the subexperiments





$$A \times B = \underbrace{(A \times \mathcal{S}_2)}_{\tilde{A}} \cap \underbrace{(B \times \mathcal{S}_1)}_{\tilde{B}} \neq A \cap B$$

Ex: $\mathcal{S}_1 = \{T, H\}$ (tail, head) $\quad A = \{H\}$

Coin flip subexperiment

$\mathcal{S}_2 = \{1,2,3,4,5,6\} \quad B = \{1,3,5\}$

Die roll subexperiment

$\tilde{A} = \{H1, H2, H3, H4, H5, H6\}$

$\tilde{B} = \{T1, T3, T5, H1, H3, H5\}$

$\tilde{A} \cap \tilde{B} = \{H1, H3, H5\} = A \times B$

Bernoulli Trials

Sampling with replacement

a) Sample space does not change with trials.

b) The trials are independent of each other.

We perform n trials.

↳ Two possibilities in each trial.

We perform n trials
Two possibilities in each trial

Observe event/outcome A — Observe event/outcome $\bar{A}$

Ex: Flip a coin two times.

$n = 2$

$\Omega_i = \{H, T\} \quad i = 1, 2$

Q: what is the probability that we observe event A k times?

$0 \leq k \leq n$

Part a) In how many different ways k observations of event A can occur?

Ex: $n = 5 \quad k = 2$

$\underline{A}\ \underline{A}\ \underline{\bar{A}}\ \underline{\bar{A}}\ \underline{\bar{A}}$

$\underline{A}\ \underline{\bar{A}}\ \underline{A}\ \underline{\bar{A}}\ \underline{\bar{A}}$

$\underline{A}\ \underline{\bar{A}}\ \underline{\bar{A}}\ \underline{A}\ \underline{\bar{A}}$

$\vdots$

$\binom{n}{k}$

Part b) For each way (ordering) what is the probability?

$P(AA\bar{A}\bar{A}\bar{A}) \overset{?}{=} P(A\bar{A}\bar{A}A\bar{A})$ $\Rightarrow \Omega_i = \{\underbrace{A, \bar{A}}_{\text{two outcomes, two events}}\}$

$P(A \times \bar{A} \times A \times \bar{A} \times \bar{A}) = P(\tilde{A} \cap \tilde{\bar{A}} \cap \tilde{A} \cap \tilde{\bar{A}} \cap \tilde{\bar{A}})$

$\tilde{A} = A \times \Omega_2 \times \Omega_3 \times \Omega_4 \times \Omega_5$

trials are independent

$= P(\tilde{A})\, P(\tilde{\bar{A}})\, P(\tilde{A})\, P(\tilde{\bar{A}})\, P(\tilde{\bar{A}})$

$= [P(\tilde{A})]^k\, [P(\tilde{\bar{A}})]^{n-k}$

$= [P(A)]^k\, [P(\bar{A})]^{n-k}$

Ex: Triple coin flip

$\mathcal{L}_i = \{HT\}$

$\mathcal{L} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

probability of observing two heads in the trials?

$$Pr\{HHT\} = \underbrace{Pr\{Hxx\}}_{\underbrace{Pr\{H\}}_{1/2}} \underbrace{Pr\{xHx\}}_{\underbrace{Pr\{H\}}_{1/2}} \underbrace{Pr\{xxT\}}_{\underbrace{Pr\{T\}}_{1/2}}$$

$$\{HHT\} = \{Hxx\} \cap \{xHx\} \cap \{xxT\}$$

$$\boxed{P(A) = p \quad , \quad P(\bar{A}) = 1-p}$$

$$\begin{aligned} P(A \times \bar{A} \times A \times \bar{A} \times \bar{A}) &= P(\tilde{A} \cap \tilde{\bar{A}} \cap \tilde{A} \cap \tilde{\bar{A}} \cap \tilde{\bar{A}}) \\ &= P(\tilde{A}) P(\tilde{\bar{A}}) P(\tilde{A}) P(\tilde{\bar{A}}) P(\tilde{\bar{A}}) \\ &= [P(\tilde{A})]^k [P(\tilde{\bar{A}})]^{n-k} \\ &= [P(A)]^k [P(\bar{A})]^{n-k} \\ &= p^k \cdot (1-p)^{n-k} \end{aligned}$$

$$A: \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

Binomial formula $\leftarrow$ $\binom{n}{k} p^k (1-p)^{n-k}$ is the probability of observing k event A's in n trials where we do not care about when each A occurs.

Ex: Triple coin flip.

What is the probability of observing two heads?

$n=3$, $k=2$

$\binom{3}{2}$ different orderings of two heads in 3 coin flips.

$p=\frac{1}{2}$ Each ordering carries a probability of $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{3-2}=\frac{1}{8}$

Answer of question is $\binom{3}{2}\cdot\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{3-2}=\frac{3}{8}$

Approximations to the binomial formula.

De Moivre - Laplace (Normal)

$$\binom{n}{k}p^k(1-p)^{n-k} \approx \frac{1}{\sqrt{2\pi np(1-p)}}\exp\left(-\frac{1}{2}\frac{(k-np)^2}{np(1-p)}\right)$$

When can we use it? If we decide to use it what confidence do we have in its accuracy?

i) $|k-\underbrace{np}_{\text{average number of observations of A in n trials.}}| < \sqrt{np(1-p)}$

ii) n is large as well as k.

Q: What is the probability of observing k event A's in n trials where $k_1 \le k \le k_2$

Ex: What is the probability of observing k heads in 10 trials where $2 \le k \le 4$

$Pr\{\{2 \text{ heads in } 10 \text{ trials}\} \cup \{3 \text{ heads in } 10 \text{ trials}\} \cup \{4 \text{ heads in } 10 \text{ trials}\}\}$

disjoint

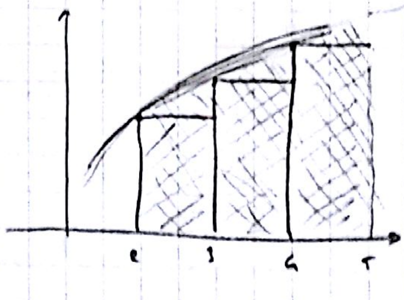
$$\binom{10}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^8 + \binom{10}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^7 + \binom{10}{4}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^6$$

$$= \sum_{k=2}^{4} Pr\left\{ \begin{matrix} k \text{ heads in} \\ 10 \text{ trials} \end{matrix} \right\}$$

Exact expression $\rightarrow$ $$= \sum_{k=2}^{4} \binom{n}{k} p^k (1-p)^{n-k}$$

approximating expression $\rightarrow$ $$= \sum_{k=2}^{4} \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{1}{2}\frac{(k-np)^2}{np(1-p)}\right)$$

$$\approx \int_2^5 \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{1}{2}\frac{(x-np)^2}{np(1-p)}\right) dx$$



$$\text{Answer} = \sum_{k=k_1}^{k_2} \binom{n}{k} p^k (1-p)^{n-k}$$ $\rightarrow$ exact expression

$$= \sum_{k=k_1}^{k_2} \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{1}{2}\frac{(k-np)^2}{np(1-np)}\right)$$ $-$ approximating expression

$$\approx \int_{k_1}^{k_2+1} \frac{1}{\sqrt{2\pi np(1-p)}} \exp\left(-\frac{1}{2}\frac{(x-np)^2}{np(1-np)}\right) dx$$

$$u = \frac{x - np}{\sqrt{np(1-p)}} \qquad du = \frac{dx}{\sqrt{np(1-p)}}$$

$$\int_{\frac{k_1 - np}{\sqrt{np(1-p)}}}^{\frac{k_2+1-np}{\sqrt{np(1-p)}}} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right)}_{\text{standard Gaussian kernel}} du = \Phi\left(\frac{k_2+1-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k_1 - np}{\sqrt{np(1-p)}}\right)$$

$$\Phi(v) = \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du$$

$\Phi(v)$ is tabulated.

v	$\Phi(v)$
-3	0.0013
-2	0.0223
-1	0.1587
0	0.5
1	0.8413
2	0.9772
3	0.9987

$$\Phi(-v) = 1 - \Phi(v)$$

$$\Phi(-3) = 1 - \Phi(3)$$

Ex: Flip a coin 1000 times

What is the probability that we observe between 495 and 505 heads (inclusive)?

$$\sum_{k=495}^{505} \binom{1000}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{1000-k}$$

$$= \sum_{k=495}^{505} \binom{1000}{k} \left(\frac{1}{2}\right)^{1000}$$

$$= \int_{495}^{506} \frac{1}{\sqrt{2\pi \cdot 1000 \cdot \frac{1}{2}(1-\frac{1}{2})}} \exp\left(-\frac{1}{2} \frac{(x-1000\cdot\frac{1}{2})^2}{1000\cdot\frac{1}{2}(1-\frac{1}{2})}\right) dx$$

$$= \int_{495}^{506} \frac{1}{\sqrt{2\pi \cdot 250}} \exp\left(-\frac{1}{2} \cdot \frac{(x-500)^2}{250}\right) dx$$

$$\approx \Phi\left(\frac{506-1000\cdot\frac{1}{2}}{\sqrt{1000\cdot\frac{1}{2}(1-\frac{1}{2})}}\right) - \Phi\left(\frac{495-100\cdot\frac{1}{2}}{\sqrt{1000\cdot\frac{1}{2}(1-\frac{1}{2})}}\right)$$

$\sqrt{250} = 5(3.18)$
$= 15.9$

$$= \Phi\left(\frac{506-500}{\sqrt{250}}\right) - \Phi\left(\frac{495 \cdot 500}{\sqrt{250}}\right)$$

$$= \Phi\left(\frac{6}{15.9}\right) - \Phi\left(\frac{-5}{15.9}\right) \quad 1-\Phi\left(\frac{5}{15.9}\right)$$

$$= \Phi(0.37) + \Phi(0.31) - 1$$

$$= 0.6443 + 0.6217 - 1$$

$$= 0.266$$

Ex: What is the probability of observing a 4 or 5 more than twice in 100 rolls of a fair die?

$n = 100$

$p = P(A) = \frac{n_A}{n} = \frac{2}{6}$ $\quad n_A = \{4,5\}$

$2 < k \leq 100$

$$\sum_{k=3}^{100} \binom{100}{k} p^k \cdot (1-p)^{100-k} = \sum_{k=3}^{100} \binom{100}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{100-k}$$

i) k is not large (k is too small according to n)

ii) $|k - \underbrace{np}_{33.3}| < \sqrt{np(1-p)} \approx \sqrt{\frac{200}{9}} \approx 4.7$

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neither i and iii are satisfied. But if we go ahead with it

$$\int_3^{101} \frac{1}{\sqrt{2\pi \frac{200}{9}}} \exp\left(-\frac{1}{2}\frac{(x-33.3)^2}{(200/9)}\right) dx$$

$$\approx \Phi\left(\frac{101-33.3}{\sqrt{200/9}}\right) - \Phi\left(\frac{3-33.3}{\sqrt{200/9}}\right)$$

$$= \underbrace{\Phi\left(\frac{101-33.3}{4.7}\right)}_{\approx 1} - \Phi\left(\frac{3-33.3}{4.7}\right) = \Phi\left(\frac{67.7}{4.7}\right) - \left(1 - \underbrace{\Phi\left(\frac{30.3}{4.7}\right)}_{\approx 1}\right)$$

$$\approx 1$$

$$\sum_{k=3}^{100} \binom{100}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{100-k} = 1 - \sum_{k=0}^{2}\binom{100}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{100-k}$$

$$= 1 - \left[\binom{100}{0}\cdot\left(\frac{2}{3}\right)^{100} + \binom{100}{1}\cdot\left(\frac{2}{3}\right)^{99}\left(\frac{1}{3}\right) + \binom{100}{2}\left(\frac{2}{3}\right)^{98}\left(\frac{1}{3}\right)^2\right]$$

4. Ders   04.03.2016

Bernoulli Trials, binomial formula

Sampling with replacement

$$Pr\left\{\begin{matrix} k \text{ event A's in} \\ n \text{ trials}\end{matrix}\right\} = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(k-\varphi)^2}{\sigma^2}\right)$$

$\varphi = np$

$\sigma^2 = np(1-p)$

Possion Approximation

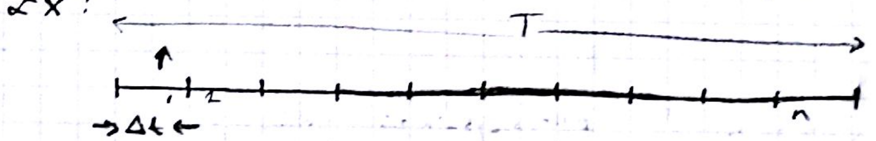
$$b(n,p,k) = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{e^{-a} a^k}{k!} \quad , \; a = np$$

Conditions

i) $n \gg 1$, $p \ll 1$

ii) $np = a$ is neither large nor small.

Ex:



$p = Pr\{\text{One occurence in } \Delta t\}$ is very small when n is very large. (rate of occurences)

$Pr\{\text{Two or more occurences in } \Delta t\} = o(\Delta t)$

$= k_1(\Delta t)^2 + k_2(\Delta t)^2 + \dots$

$$Pr\left\{\begin{matrix} k \text{ occurences} \\ \text{in time } T\end{matrix}\right\} = \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{e^{-a} a^k}{k!} = \frac{e^{-np} \cdot (np)^k}{k!}$$

Ex: At most one car can pass by a certain point on the road in 1 sec. The probability of observing a car crossing the point in 1 sec time segment is 0.001. What is the probability of observing 20 cars pass by the point in 2 munites. time?

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$n = 120$ $\quad np = 0.12$
$p = 0.001$

$$Pr\left\{\begin{matrix}20 \text{ occurrences} \\ \text{in time } T\end{matrix}\right\} = \frac{e^{-0.12}(0.12)^{20}}{20!}$$

Ex: Assume at most 1 customer arrives at a market in 0.01 sec. The probability of 1 arrival in 0.01 sec is 10^{-4}. What is the probability of observing 10 customer arriving in 1 minute?

$$\left.\begin{matrix} n = 60/0.01 = 6000 \\ p = 10^{-4}\end{matrix}\right\} np = 0.6 \qquad Pr = \frac{e^{-0.6}\cdot(0.6)^{10}}{10!}$$

What is the probability of observing the arrival of no customers in 10 sec?

$$\frac{e^{-np}(np)^0}{0!} = \frac{e^{0.1}(0.1)^0}{0!} = e^{0.1}$$

Random Variable

$X : \mathcal{L} \longmapsto \mathbb{R}$

$$Pr\{\zeta : X(\zeta) = +\infty\} = Pr\{\zeta : X(\zeta) = -\infty\} = 0$$

$\{\zeta : X(\zeta) \le x\}$ constitutes an event.

outcome of the sample space

$A_x = \{ \xi : X(\xi) \leq x \}$

$A_y = \{ \xi : X(\xi) \leq y \}$

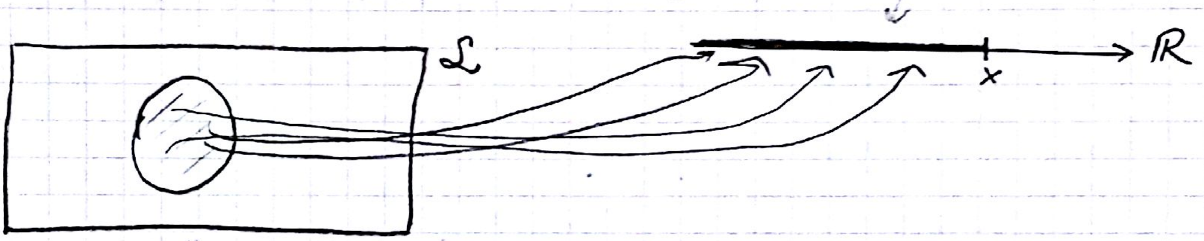
$A_x \cap A_y$ is an event.

$A_x \cap A_y = \{ \xi : X(\xi) \leq \min(x,y) \}$

$A_x \cup A_y = \{ \xi : X(\xi) \leq \max(x,y) \}$

$F_X(x) = Pr\{ \xi : X(\xi) \leq x \} = Pr\{ X \leq x \}$

↳ Cumulative Distribution function



Gerçekleme

⇐ <u>Realizations of random Variable</u>

- Discrete realizations
- Continuous realizations ...

- Discrete Realizations

$Pr\{ \xi : X(\xi) = \underbrace{x_i}_{\text{discrete realization}} \} > \emptyset$ strictly greater than 0.

↳ we can enumerate discrete realization.

Continuous realizations

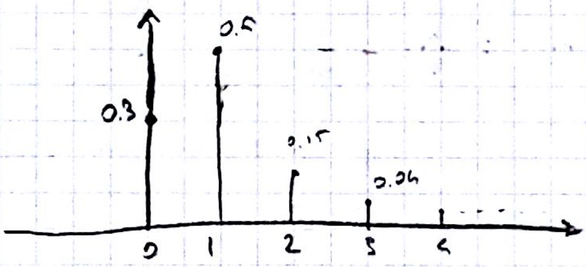
$Pr\{ \xi : X(\xi) = x \} = \emptyset$

Random Variable Types

Realization Type	Discrete		Continuous	
Random Variable Type	Discrete	Hybrid (mixed)		Continuous

Ex: (discrete random variable)

\# of accidents on a road during a day.



$Pr\{\zeta : X(\zeta) = 0\} = 0.3$

$\hookrightarrow Pr\{X = 0\} = 0.3$

Could be accidents

$Pr\{\zeta : X(\zeta) = 0.5) \Rightarrow Pr\{X = 1\} = 0.5$

What is the probability of observing two or more accidents during a day? $\rightarrow Pr\{X \geq 2\}$

$$\overline{\{\zeta : X(\zeta) \geq 2\}} = \{\zeta : X(\zeta) < 2\}$$
$$= \{\zeta : X(\zeta) = 0\} + \{\zeta : X(\zeta) = 1\}$$

$$Pr\{X \geq 2\} = 1 - Pr\{X < 2\}$$
$$= 1 - (Pr\{X = 0\} + Pr\{X = 1\}) = 1 - 0.3 - 0.5$$
$$= 0.2$$

Ex: Double coin flip

$\mathcal{S} = \{HH, HT, TH, TT\}$

Let X represent the number of Heads observed.

$X(HH) = 2$
$X(HT) = 1$
$X(TH) = 1$
$X(TT) = 0$

Since coin is fair $Pr\{H\} = Pr\{T\} = \frac{1}{2}$

Since individual coin flip trials are independent.

$Pr\{HH\} = Pr\{H\} * Pr\{H\} = 1/4$

$Pr\{HT\} = Pr\{H\} \times Pr\{T\} = 1/4$

$Pr\{X=2\} = Pr\{HH\} = 1/4$

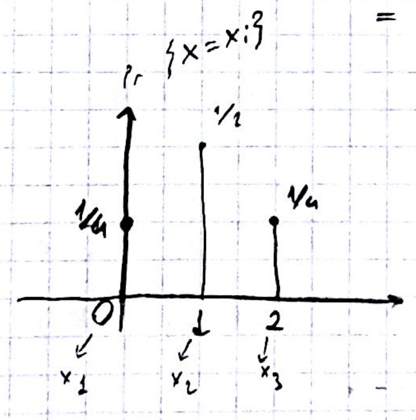
$Pr\{X=0\} = Pr\{TT\} = 1/4$

$Pr\{X=1\} = Pr\{HT, TH\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

What is the probability of observing at least one head?

$Pr\{X \geq 1\} = 1 - Pr\{X < 1\}$

$= 1 - 1/4 = \frac{3}{4} = \underbrace{\frac{1}{2}}_{Pr\{X=1\}} + \underbrace{\frac{1}{4}}_{Pr\{X=2\}}$



$\to P(i) = Pr\{X = x_i\}$: ith probability mass

↳ probability mass function

What is $F_X(x) = Pr\{\zeta : X(\zeta) \leq x\}$ (X: variable; x: real value)

$= Pr\{X \leq x\}$: short notation

$Pr\{X \leq x\}$ (↳ number of heads) $x < \emptyset$ for instance $x = -1$

$Pr\{X \leq x\} = \emptyset \quad x < \emptyset$

What is $F_x(x) = Pr\{\xi : X(\xi) \leq x\}$ real value

$= Pr\{X < x\}$ short notation

$Pr\{X \leq x\}$ $x < \emptyset$
$x = -1$

$F_x(x) = Pr\{X \leq x\} = \emptyset$ $x < \emptyset$

$Pr\{X \leq x\} = Pr\{X = \emptyset\} + Pr\{X < 0\} = \frac{1}{4}$ $0 \leq x < 1$
$\emptyset$

$Pr\{X \leq 0.3\} = Pr\{X = 0\} = \frac{1}{4}$

$\{X \leq 0.6\} = \{X = 0\}$

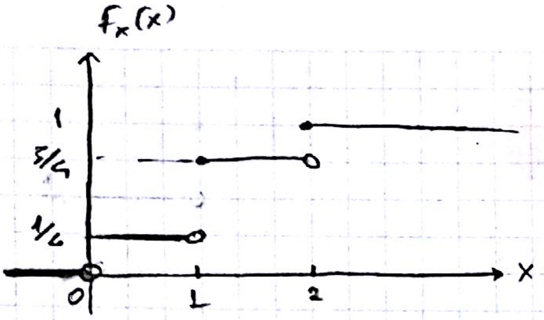
$Pr\{X \leq 1\} = Pr\{\ = 0\} + Pr\{\ = 1\}$ $x = 1$
$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$Pr\{X \leq 1.7\} = Pr\{X = 0\} + Pr\{X = 1\}$
$= 3/4$

$\rightarrow Pr\{X \leq x\} = Pr\{X = 0\} + Pr\{X = 1\}$ $1 \leq x < 2$ small x
$= 3/4$

$Pr\{X \leq 2\} = Pr\{X = 0\} + Pr\{X = 1\} + Pr\{X = 2\}$ small x
$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$ //

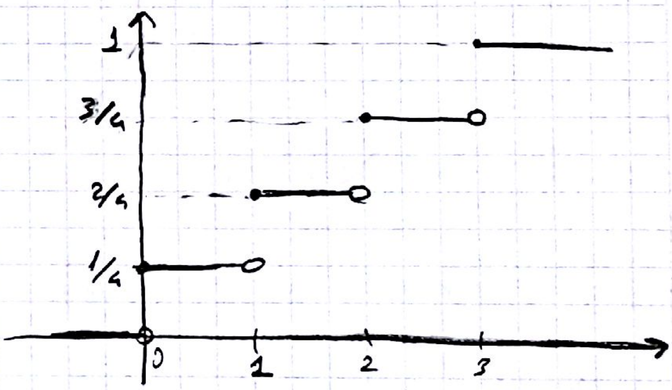
$Pr\{X \leq x\} = Pr\{X = 0\} + Pr\{X = 1\} + Pr\{X = 2\}$ $x = 2$
$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$ //



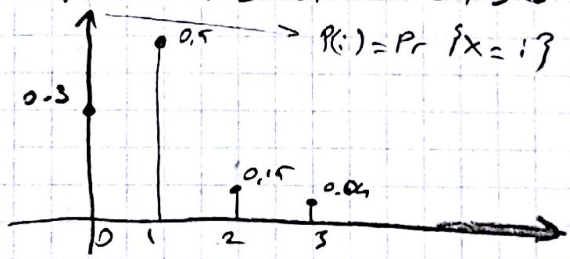
From left to right the function increases in steps or stays the same.

Steps occur at the locations of the discrete realizations.
$F_X(x)$ is nondecreasing.
- as x increases it accumulates probabilities of discrete realizations.

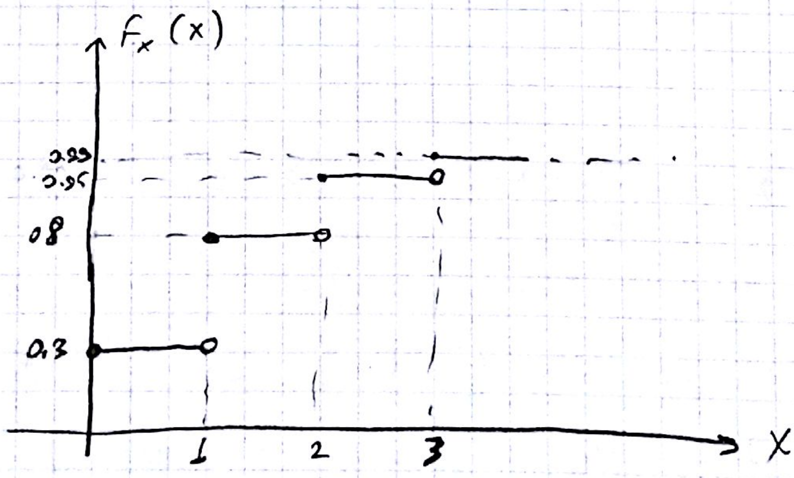
Let $X(HH) = 0$
$X(HT) = 1$
$X(TH) = 2$
$X(TT) = 3$



Ex: Accidents on the road during a day.

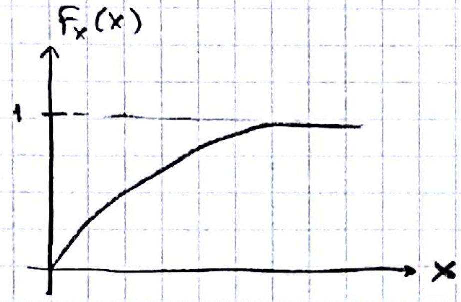


$P(i) = Pr\{X = i\}$ $\quad x_{i+1} = i \quad i = 0, 1, 2$



Ex: Continuous Random Variable.

Lifetime of a battery



lifetime starts with manufacture ≥ 0

$$F_x(x) = P_r\{X \leq x\} = 0 \text{ for } x < 0$$

↳ x'in negatif değerleri için olasılığı 0'a eşit.