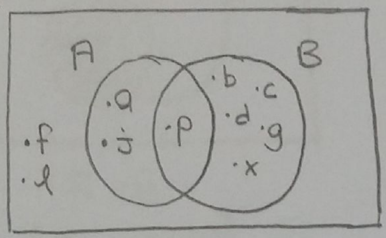
Probability & Statistics / HW#1 Solutions

1) S



a) $\bar{A} = S - A = S - (S \cap A) = \{b, c, d, g, x, f, l\}$

b) $A - B = A - (A \cap B) = \{a, j\}$

c) $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{a, b, c, d, g, j, p, x\}$

d) $B \cap (A \cup \bar{B}) =$

$\bar{B} = S - B = S - (S \cap A) = \{a, j, f, l\}$

$A \cup \bar{B} = \{a, j, p, f, l\}$

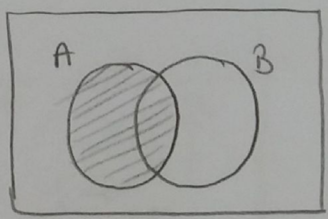
$B \cap (A \cup \bar{B}) = \{p\}$

e) $\bar{A} \cap B = \{x : x \in \bar{A} \text{ and } x \in B\}$

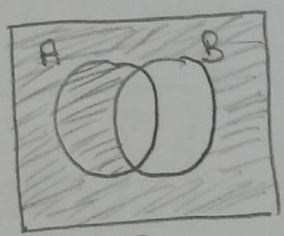
$\bar{A} = \{b, c, d, g, x, f, l\}$

$\bar{A} \cap B = \{b, c, d, g, x\} = B$

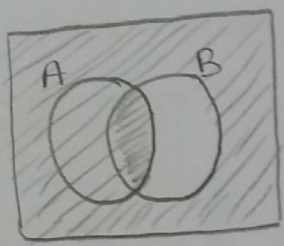
2) i)



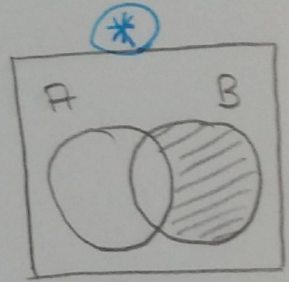
A



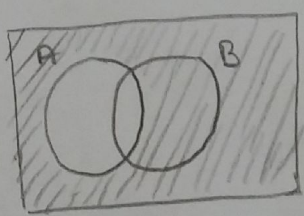
$\bar{B}$



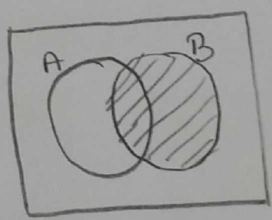
$A \cup \bar{B}$



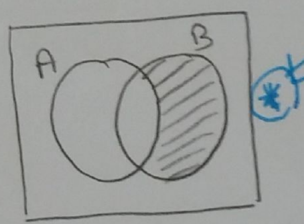
$\overline{(A \cup \bar{B})}$



$\bar{A}$

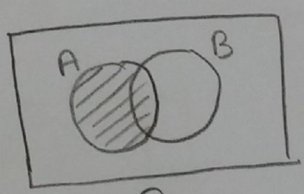


B

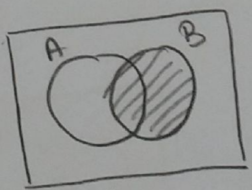


$\bar{A} \cap B$

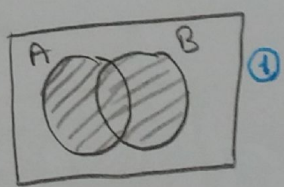
ii)



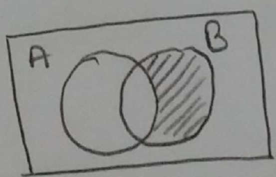
A



B

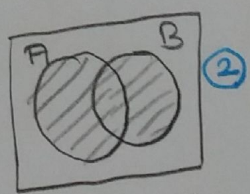


$A \cup B$

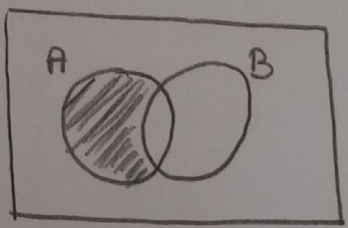


$B - A$

$(B - (A \cap B))$

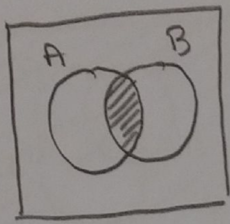


$A \cup (B - A)$

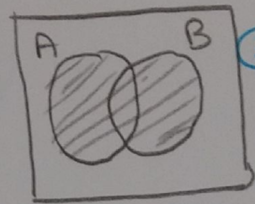


$A - B$

$[A - (A \cap B)]$



$A \cap B$

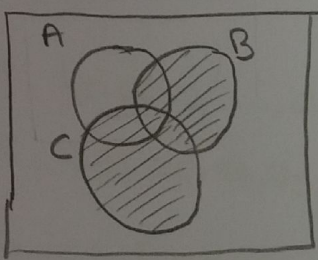


3

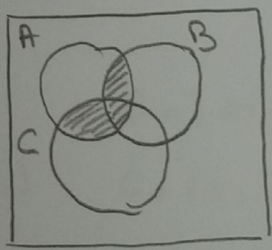
$(A - B) \cup (A \cap B) \cup (B - A)$

So, $\underbrace{A \cup B}_{\text{Diagram 1}} = \underbrace{A \cup (B - A)}_{\text{Diagram 2}} = \underbrace{(A - B) \cup (A \cap B) \cup (B - A)}_{\text{Diagram 3}}$

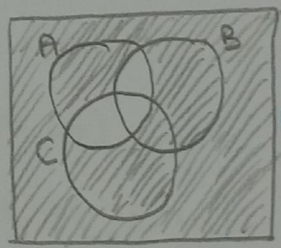
3 a) $\overline{A \cap (B \cup C)} = (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$: This equality is TRUE!



$B \cup C$

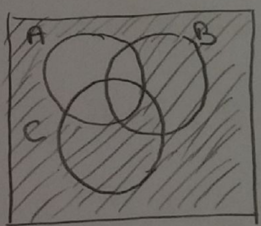


$A \cap (B \cup C)$

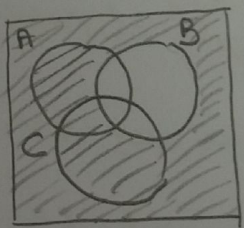


1

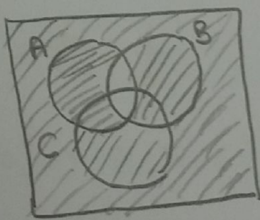
$\overline{A \cap (B \cup C)}$



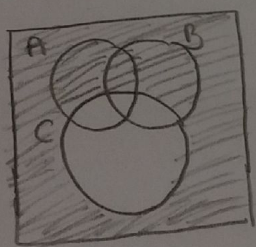
$\bar{A}$



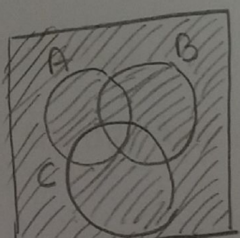
$\bar{B}$



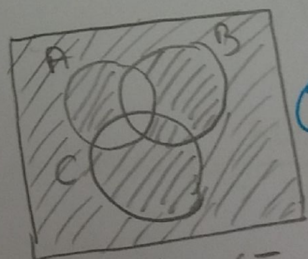
$\bar{A} \cup \bar{B}$



$\bar{C}$



$\bar{A} \cup \bar{C}$



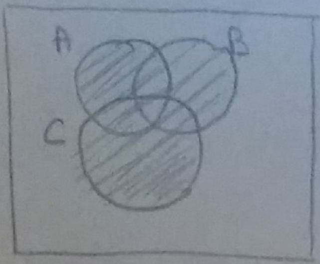
2

$(\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$

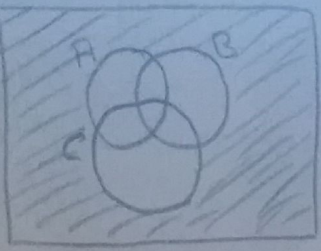
So, $\overline{A \cap (B \cup C)} = (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$

Also; $\overline{A \cap (B \cup C)} = \overline{(A \cap B) \cup (A \cap C)}$ Distributive Law

$= (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \bar{C})$ De morgan Law

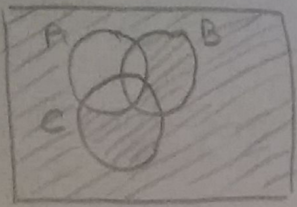


$A \cup B \cup C$

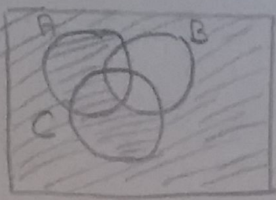


$\overline{(A \cup B \cup C)}$

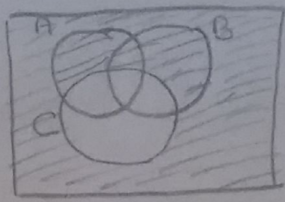
1



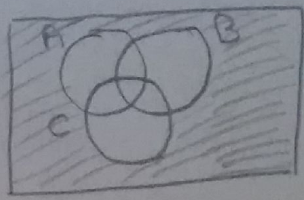
$\bar{A}$



$\bar{B}$



$\bar{C}$



$\bar{A} \cap \bar{B} \cap \bar{C}$

2

So, $\overline{(A \cup B \cup C)} = \bar{A} \cap \bar{B} \cap \bar{C}$ ✓

4) $S = \{a_1, a_2, a_3, \ldots, a_{12}\}$ $\quad B = \{a_1, a_2, a_5, a_7\}$

$A = \{a_1, a_5, a_6, a_9, a_{10}\}$ $\quad C = \{a_2, a_4, a_6, a_7, a_9, a_{11}\}$

a) $A \cup C = \{a_1, a_2, a_4, a_5, a_6, a_7, a_9, a_{10}, a_{11}\} \Rightarrow P(A \cup C) = \dfrac{s(A \cup C)}{s(S)} = \dfrac{9}{12} = \boxed{\dfrac{3}{4}}$

b) $A \cap C = \{a_6, a_9\} \Rightarrow P(A \cap C) = \dfrac{s(A \cap C)}{s(S)} = \dfrac{2}{12} = \boxed{\dfrac{1}{6}}$

c) $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{\frac{2}{12}}{\frac{6}{12}} = \boxed{\dfrac{1}{3}}$

d) $P(C|A) = \dfrac{P(A \cap C)}{P(A)} = \dfrac{\frac{2}{12}}{5/12} = \boxed{\dfrac{2}{5}}$

e) $\bar{C} = \{a_1, a_3, a_5, a_8, a_{10}, a_{12}\}$

$B \cup \bar{C} = \{a_1, a_2, a_3, a_5, a_7, a_8, a_{10}, a_{12}\} \Rightarrow P(B \cup \bar{C}) = \dfrac{s(B \cup \bar{C})}{s(S)} = \dfrac{8}{12} = \boxed{\dfrac{2}{3}}$

f) $B - C = B - (B \cap C) = \{a_1, a_5\} \Rightarrow P(B - C) = \dfrac{s(B-C)}{s(S)} = \dfrac{2}{12} = \boxed{\dfrac{1}{6}}$

g) $A \cap (B \cup \bar{C}) = \{a_1, a_5, a_{10}\}$ $\quad P(A|(B \cup \bar{C})) = \dfrac{P(A \cap (B \cup \bar{C}))}{P(B \cup \bar{C})} = \dfrac{\frac{3}{12}}{8/12} = \boxed{3/8}$

h) $A \cup B = \{a_1, a_2, a_5, a_6, a_7, a_9, a_{10}\}$ $\Rightarrow P(\overline{A \cup B}) = \dfrac{s(\overline{A \cup B})}{s(S)} = \boxed{\dfrac{5}{12}}$

$\overline{A \cup B} = \{a_3, a_4, a_8, a_{11}, a_{12}\}$

5) $B \to 1$ ("B" is uppercase it should be at the beginning of the word) ; So, $\underline{B}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{Total}=6}$

$a \to 2$

$g \to 3$

$e \to 1$

$$\frac{6!}{2!.3!.1!}$$

6) in 18 questions, there exists 13 groups of 6 consecutive questions. In each group, exactly 2 times of each answer type must be contained!

6 questions | 6 questions | 6 questions

There exists 3 groups which consist of 6 consequtive questions. We focus on just the permutation of first group, because the others will be the same!

6 questions

Solution 1 ⇒ Permutation based

$$\frac{6!}{2!.2!.2!} = 90$$

Solution 2 ⇒ Combination based

$$90 = \binom{6}{2}\cdot\binom{4}{2}\cdot\binom{2}{2}$$

← Just one option for the last answer type

Select 2 places from 6 places for an answer type

Select 2 places from the rest of 4 places for one of the other answers type

90 | 90 | 90 ⇒ So, result is $(90)^3$ ✓

7) 5 pairs of shoes ⇒ 10 shoes

4 shoes are drawn randomly

$$\frac{\binom{5}{2}}{\binom{10}{4}} = \frac{\frac{5.4}{2}}{\frac{10.9.8.7}{4.3.2.1}} = \frac{10}{10.3.7} = \frac{1}{21}$$

Each ball can be placed in any one of the r boxes.
So, all possibilities $= r^n$

* Consider first 2 boxes contain k balls totally, so you have to place $(n-k)$ balls to $(r-2)$ boxes. So, you have $(r-2)^{n-k}$ possibilities.

* You can select k balls from n balls in $\binom{n}{k}$ ways.

a) So, the probability of any two boxes contain a total of k balls

"any two boxes" selection: $\binom{r}{2}$

$$\frac{\binom{r}{2}\cdot\binom{n}{k}\cdot(r-2)^{n-k}}{r^n}$$

b) The probability of any two adjacent boxes contain a total of k balls.

number of two adjacent boxes $=(r-1)$

"any two adjacent boxes" selection $= \binom{r-1}{1} = (r-1)$

$$\frac{(r-1)\cdot\binom{n}{k}\cdot(r-2)^{n-k}}{r^n}$$

9) a) 52 cards, 13 of them are hearts!

$$\frac{13}{52}\cdot\frac{39}{51} \checkmark = \frac{39}{204}$$

b) $52-13=39$ cards

probability $= \frac{39}{51}$ ← because first card is selected from 52 cards

c) First one is 6. It can be spade, heart, diamond or club.

$$\frac{1}{4}\cdot\frac{12}{51} + \frac{3}{4}\cdot\frac{13}{51} \cong 0{,}25$$

prob. of first card is spade

If first card is spade, you have 12 choices for the second card

d) The most safe way; $\left(\begin{matrix} A:11 \\ J, Q, K = 10 \end{matrix}\right)$

First Card	Second Card	
2	8, 9, 10, J, Q, K, A	$\rightarrow \frac{4}{52} \cdot \frac{28}{51}$
3	7, 8, 9, 10, J, Q, K, A	$\rightarrow \frac{4}{52} \cdot \frac{32}{51}$
4	6, 7, 8, 9, 10, J, Q, K, A	$\rightarrow \frac{4}{52} \cdot \frac{36}{51}$
5	5, 6, 7, 8, 9, 10, J, Q, K, A	$\rightarrow \frac{4}{52} \cdot \frac{39}{51}$
6	4, 5, 6, 7, 8, 9, 10, J, Q, K	$\rightarrow \frac{4}{52} \cdot \frac{39}{51}$
7	3, 4, 5, 6, 7, 8, 9	$\rightarrow \frac{4}{52} \cdot \frac{27}{51}$
8	2, 3, 4, 5, 6, 7, 8	$\rightarrow \frac{4}{52} \cdot \frac{27}{51}$
9	2, 3, 4, 5, 6, 7	$\rightarrow \frac{4}{52} \cdot \frac{24}{51}$
J, Q, K	2, 3, 4, 5, 6	$\rightarrow \frac{4}{52} \cdot \frac{20}{51} \times 3$
A	2, 3, 4, 5	$\rightarrow \frac{4}{52} \cdot \frac{16}{51}$

+

Result = ... ✓

12) a) Crash event and events of engine failures are dependent of each other.

Engine failure are dependent each other.

$$P(A \cap B) \neq \underbrace{P(A)}_{10^{-5}} \cdot \underbrace{P(B)}_{10^{-5}} \Rightarrow \text{dependent events.}$$

$$10^{-8} \neq 10^{-5} \cdot 10^{-5}$$

b) $P(E_2 \mid E_1) = \dfrac{P(E_1 \cap E_2)}{P(E_1)} = \dfrac{10^{-8}}{10^{-5}} = 10^{-3}$

c) P(crash) =

d) $1 - \{\underbrace{\text{prob. of not crash}}_{1-P(\text{crash})}\}^{10^7}$ (Prob of 1 or more crashes occur in 10^7 flights)

$1 - \{[1 - P(\text{crash})]^{10^7}\}$ ✓

13) a) $P(2016 \mid A) = \dfrac{P(A \mid 2016) \cdot P(2016)}{P(A)} = \dfrac{P(2016 \cap A)}{P(A)}$

$= \dfrac{0{,}6 \times 0{,}1}{0{,}6 \cdot 0{,}2 + 0{,}6 \cdot 0{,}7 + 0{,}6 \cdot 0{,}1}$

b) $P(A \mid 2016) = \dfrac{P(2016 \mid A) \cdot P(A)}{P(2016)} = \dfrac{P(A \cap 2016)}{P(2016)}$

$= \dfrac{0{,}6 \times 0{,}1}{0{,}1 \times 0{,}6 + 0{,}2 \times 0{,}4}$

14) $P(5 \text{ or } dif2) = P(5) + P(dif2) - P(5 \cap dif2)$

$= \dfrac{2}{6} + \dfrac{8}{36} - \dfrac{2}{36} = \dfrac{1}{2}$ ✓

$dif2 = \{(1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4)\}$

$P(dif\ 2) = \dfrac{8}{36}$

15) $1 - P(\text{fail to pass}) = P(\text{pass})$

$N = 400 \times 4 \times 10^{-4} = 0{,}16$

$P(X \geq 2) = 1 - \{P(x=0) + P(x=1)\}$

$= 1 - \left\{ \dfrac{e^{-0{,}16} \cdot 0{,}16^0}{0!} + \dfrac{e^{-0{,}16} \cdot 0{,}16^1}{1!} \right\}$

$= 1 - \{e^{-0{,}16} \cdot 1{,}16\}$ ⇒ Fail to pass

$1 - P(\text{fail to pass}) = 1 - [1 - (e^{-0{,}16} \cdot 1{,}16)]$

$\cong 0{,}98$

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13) a) $P(2016 \cap A) = P(A/2016) \cdot P(A)$

12

$= 0{,}1 \times 0{,}6 = \underline{0{,}06}$

13 b) $P(A|2016) = \dfrac{P(2016|A) \cdot P(A)}{P(2016)} = \dfrac{P(A \cap 2016)}{P(2016)}$

$= \dfrac{0{,}6 \times 0{,}1}{0{,}1 \times 0{,}6 + 0{,}02 \times 0{,}4} = \dfrac{0{,}06}{0{,}06 + 0{,}08} = \dfrac{0{,}06}{0{,}14} \cong \underline{0{,}4286}$

14) $P(5 \text{ or } dif2) = P(5) + P(dif2) - P(5 \cap dif2)$

$= \dfrac{2}{6} + \dfrac{8}{36} - \dfrac{2}{36} = \dfrac{1}{2}$ ✓

$dif2 = \{(1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4)\}$

$P(dif\ 2) = \dfrac{8}{36}$

15) $1 - P(\text{fail to pass}) = P(\text{pass})$

$\lambda = 400 \times 4 \times 10^{-4} = 0{,}16$

$P(X \geq 2) = 1 - \{P(x=0) + P(x=1)\}$

$= 1 - \left\{ \dfrac{e^{-0{,}16} \cdot 0{,}16^0}{0!} + \dfrac{e^{-0{,}16} \cdot 0{,}16^1}{1!} \right\}$

$= 1 - \{e^{-0{,}16} \cdot 1{,}16\}$ ⇒ Fail to pass

$1 - P(\text{fail to pass}) = 1 - [1 - (e^{-0{,}16} \cdot 1{,}16)]$

$\cong 0{,}98$

Binomial Distribution

$\binom{n}{k} p^k (1-p)^{n-k}$

$n = 400$

$k = \{0,1\}$

$p = 4 \times 10^{-4}$

$\underbrace{\binom{400}{0}}_{1} \underbrace{(4 \cdot 10^{-4})^0}_{1} (1 - 4 \cdot 10^{-4})^{400-0} +$

$\underbrace{\binom{400}{1}}_{400} (4 \cdot 10^{-4})^1 (1 - 4 \cdot 10^{-4})^{399} \cong$

$\underline{0{,}988}$

16) a) $P(x=3) = \binom{4}{3} (0{,}001)^3 \cdot (1-0{,}001)^1$

b) $1 - P(x>1) = 1 - [1 - (P(x=0) + P(x=1))]$

$= \underbrace{P(x=0)} + P(x=1)$

$0{,}999 + 0{,}999 \times 0{,}001$

$= 0{,}999999$