Sinusoids

$$x(t) = \underbrace{A}_{\text{Amplitude}} \cos(\underbrace{w_0}_{\text{radian frequency}} \overset{\text{time}}{t} + \overset{\text{phase}}{\phi}) = A\cos(2\pi f_0 t + \phi)$$

Ex:

$$x(t) = 10\cos(2\pi(440)t - 0.4\pi)$$

$A = 10 \qquad w_0 = 2\pi(440) \qquad \phi = -0.4\pi$

* It repeats the same pattern of oscillations every $1/440 \approx 0.00227$ sec. (period)

* cosine signals (sinusoids) are important because many physical systems generate signals that can be modeled as sine or cosine functions versus time.

Period

* The frequency of the sinusoid determines its period

$$x(t + T_0) = x(t)$$

$$A\cos(w_0(t + T_0) + \phi) = A\cos(w_0 t + \phi)$$

$$\cos(w_0 t + w_0 T_0 + \phi) = \cos(w_0 t + \phi)$$

Since the cosine function has a period of 2π

$$w_0 T_0 = 2\pi \Rightarrow T_0 = \frac{2\pi}{w_0}$$

$$(2\pi f_0) T_0 = 2\pi \Rightarrow T_0 = \frac{1}{f_0}$$

Euler's Formula

Proof I:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

let $f(\theta) = e^{-j\theta}(\cos\theta + j\sin\theta)$

$$\frac{df}{d\theta} = -je^{-j\theta}(\cos\theta + j\sin\theta) + e^{-j\theta}(-\sin\theta + j\cos\theta)$$

$$= e^{-j\theta}(-j\cos\theta + \sin\theta) + e^{-j\theta}(-\sin\theta + j\cos\theta)$$

$$= e^{-j\theta}(\cancel{-j\cos\theta} + \cancel{\sin\theta} - \cancel{\sin\theta} + \cancel{j\cos\theta})$$

$$= e^{-j\theta}(0)$$

$$\frac{df}{d\theta} = 0$$

If the derivative of f equals to zero, then the original function is a constant

$$f(\theta) = k, \quad \text{for all } \theta$$

for $\theta = 0 \Rightarrow f(0) = e^{-j0}(\cos(0) + j\sin(0))$

$$= 1(1 + 0j) = k$$

$$k = 1$$

$$f(\theta) = e^{-j\theta}(\cos\theta + j\sin\theta) = 1$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Proof II:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \rightarrow \text{Taylor expansion of } e^x$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \rightarrow \text{'' '' '' } \sin(x)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \rightarrow \text{'' '' '' } \cos(x)$$

$$\sin(x) + \cos(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} \cdots$$

$$e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5!} + \cdots$$

$$j^2 = -1$$

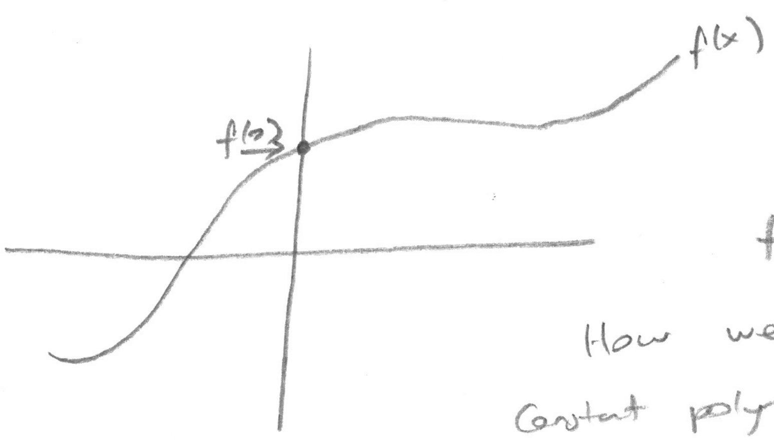
$$e^{jx} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!} - j\frac{x^7}{7!} + \cdots$$

$$\boxed{\cos(x) + j\sin(x) = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!} - j\frac{x^7}{7!} + \cdots = e^{jx}}$$

Proof Of Taylor Expansion: (Intuition of Taylor Expansion)

Khan Academy

Approximate arbitrary function using polynomials

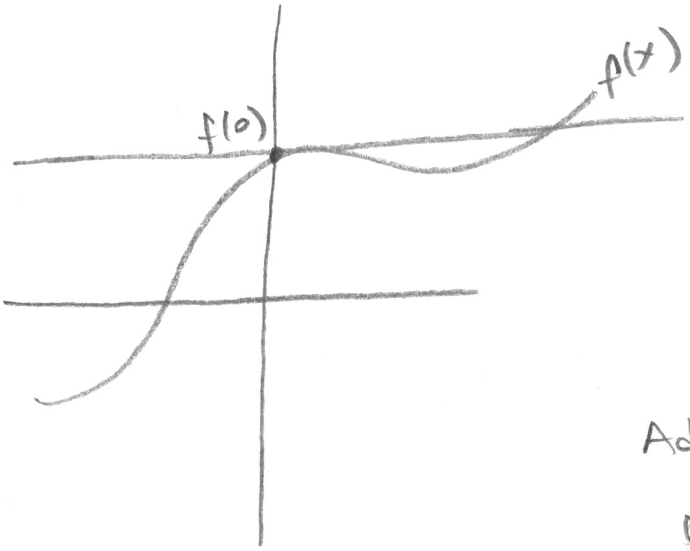


We know

$f(0), f'(0), f''(0), f^{(4)}(0), \ldots$

How we can approximate using polynomial?

Constant polynomial $p(x) = f(0)$



$p(x) = f(0)$ Bad approximation

↳ constant

Add next approximation

We want

$$p'(x) = f'(0)$$

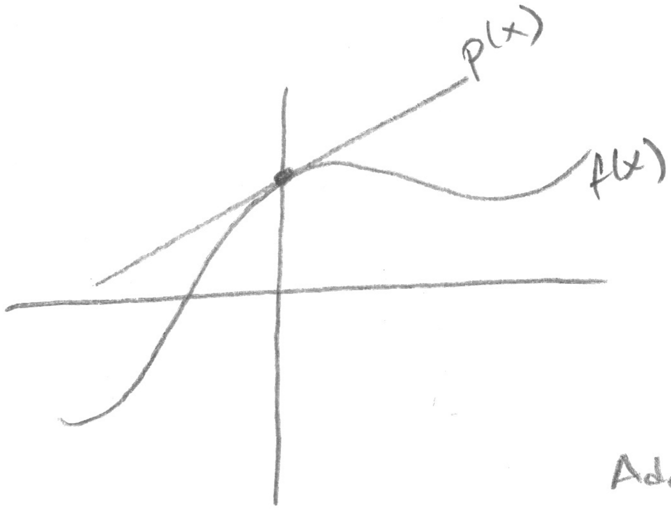
$* \ p(x) = f(0) + f'(0)\,x$

so

$$p(0) = f(0)$$

$$p'(0) = f'(0)$$

same slope



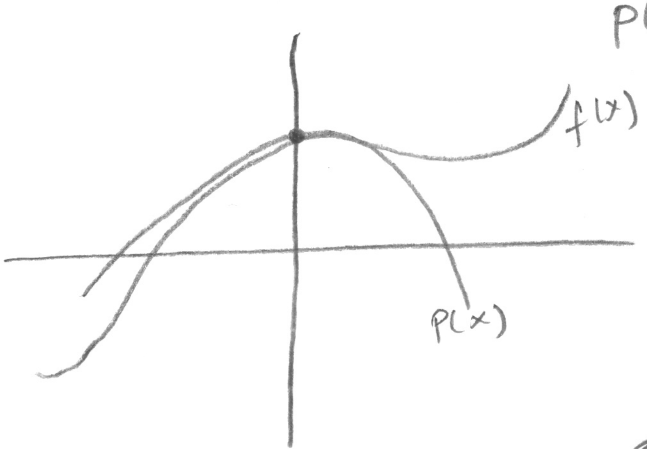
Add next term we want

$$p''(x) = f''(0)$$

$$p(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2}$$

take derivative to see where 2 comes from

same concavity



$p'(x) = f'(0) + f''(0)x \qquad p'(0) = f'(0)$

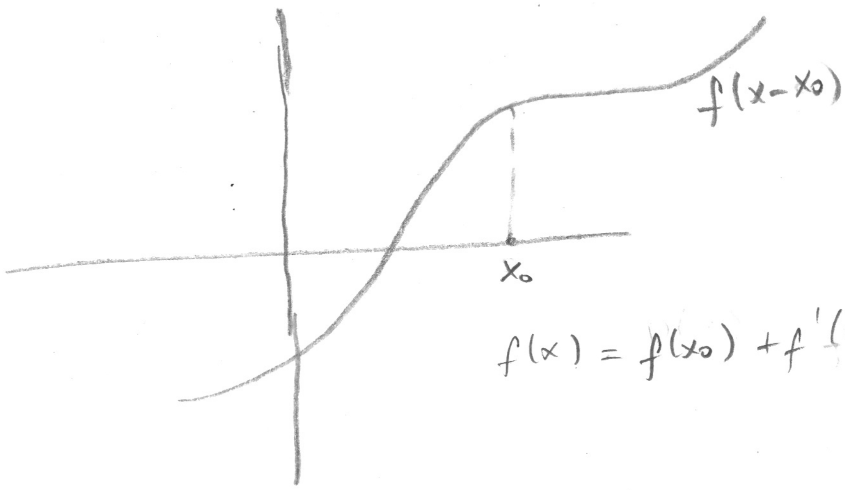
$p''(x) = f''(0)$

Add nth derivative

maclaurin series

$$p(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} +$$

$$f^{(4)}(0)\frac{x^4}{4!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}$$



$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$\frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 +$$

Look Appendix-A1, A2 for real proof of Taylor Expansion

Convert to polar form

a) $z = 0 + j2$

$|z| = 2 \qquad \theta = \arctan\left(\frac{2}{0}\right)$?

$\theta = \frac{\pi}{2}$

$z = 2e^{j\pi/2}$

Convert to Rectangular Form

$z = \sqrt{2}\, e^{j(3\pi/4)}$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{j\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$

$z = -1 + j$

- Evaluate $e^{j(\pi+2\pi m)}$ (m integer)

$$e^{j(\pi+2\pi m)} = \cos(j\pi+2\pi m) + j\sin(j\pi+2\pi m)$$
$$= \cos(\pi) + \cancel{j\sin(\pi)}$$
$$= -1$$

- $z_1 = -4+j3 \quad z_2 = 1-j$

Evaluate: $z_1^* = ?$

$$z_1^* = -4-3j$$

$$\frac{z_1}{z_2} = ? \qquad \frac{-4+3j}{1-j} = \frac{-4+3j-4j-3}{2}$$

$(1+j)$

multiply by conjugate of

$$= \frac{-j-7}{2}$$

- Solve the following for z:

$$z^4 = j \qquad r^4 e^{j4\theta} = e^{j(\frac{\pi}{2}+2\pi l)}$$

Assume z is a some complex number

$z = re^{j\theta}$

$z^4 = r^4 e^{j4\theta}$

$$r = 1$$
$$4\theta = \frac{\pi}{2} + 2\pi l$$
$$\theta = \frac{\pi}{8} + \frac{\pi}{2} l \quad , l = 0, 1, 2, 3$$

- Evaluate j^j

$$(j)^j = (\cos(\tfrac{\pi}{2}) + j\sin(\tfrac{\pi}{2}))^j = (e^{j(\pi/2)})^j$$
$$= e^{-(\pi/2)}$$

P2.9) $x(t) = \overbrace{2\sin(w_0 t + 45^\circ)}^{K} + \overbrace{\cos(w_0 t)}^{L}$

a) Express $x(t)$ as $x(t) = A\cos(w_0 t + \phi)$

<u>Remind</u> $re^{j(w_0 t + \phi)} = r(\cos(w_0 t + \phi) + j\sin(w_0 t + \phi))$

$$\Rightarrow r\cos(w_0 t + \phi) = \mathrm{Re}\{re^{j(w_0 t + \phi)}\}$$

$$\sum_{k=1}^{N} A_k \cos(w_0 t + \phi_k) = \sum_{k=1}^{N} \mathrm{Re}\{A_k e^{j(w_0 t + \phi_k)}\}$$

$$= \mathrm{Re}\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{jw_0 t}\right\}$$

$$= \mathrm{Re}\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{jw_0 t}\right\}$$

very important step

$$= \mathrm{Re}\left\{\left(Ae^{j\phi}\right) e^{jw_0 t}\right\}$$

It is proved using phasor addition

$$= \mathrm{Re}\{Ae^{j(w_0 t + \phi)}\}$$

$$= A\cos(w_0 t + \phi)$$

$$K = 2\sin(w_0 t + 45^\circ) = 2\cos(w_0 t - 45^\circ)$$

$$= 2\cos\left(w_0 t - \frac{\pi}{4}\right)$$

$$= \mathrm{Re}\{2e^{j(w_0 t - \frac{\pi}{4})}\}$$

$$= \mathrm{Re}\{2e^{-j\pi/4} e^{jw_0 t}\}$$

$$L = \cos(w_0 t) = \mathrm{Re}\left\{ e^{jw_0 t} \right\}$$

$$x(t) = K + L = \mathrm{Re}\left\{ e^{jw_0 t}\, 2e^{-\frac{\pi}{4}j} \right\} + \mathrm{Re}\left\{ e^{jw_0 t} \right\}$$

$$= \mathrm{Re}\left\{ e^{jw_0 t}\, 2e^{-\frac{\pi}{4}j} + e^{jw_0 t} \right\}$$

$$= \mathrm{Re}\left\{ e^{jw_0 t} \left(2e^{-\pi/4\, j} + 1 \right) \right\}$$

$$2e^{-\pi/4\, j} = 2\left(\cos(-\pi/4) + j \sin(-\pi/4) \right)$$

$$= 2\left(+\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} \right) = +\sqrt{2} - j\sqrt{2}$$

$$\Rightarrow \quad x(t) = \mathrm{Re}\left\{ e^{jw_0 t} \left(1 + \sqrt{2} - j\sqrt{2} \right) \right\}$$

$$r = \sqrt{(1+\sqrt{2})^2 + (\sqrt{2})^2}$$

$$= \sqrt{1 + 2\sqrt{2} + 2 + 2} \simeq 2.8$$

$$\theta = \arctan\left(-\frac{\sqrt{2}}{1+\sqrt{2}} \right) \simeq -0.53$$

$$x(t) = \mathrm{Re}\left\{ 2.8\, e^{jw_0 t}\, e^{-j0.53} \right\}$$

$$= 2.8 \cos(w_0 t - 0.53)$$

2.12 Give two possible complex-valued sol'ns to the following diff. eqs.

$$\frac{d^2x(t)}{dt^2} = -100\,x(t)$$

$$z(t) = e^{j(10t+\phi)}$$

$$z^*(t) = e^{-j(10t+\phi)}$$

(yotube.com/watch?v=kSSb9cn9vWk)

Formal Definition of Taylor's Theorem: Let $f:[a,b]\to R$, $n\in N$ and assume $f, f', f'', \dots, f^{(n)}$ are cts. on $[a,b]$ and $f^{(n+1)}$ exists on (a,b). Let $x_0\in[a,b]$. Then if $x\neq x_0$ with $x\in[a,b]$ there exists c between x and x_0 such that

Real proof

$$f(x)=\underbrace{f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x)}+\underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^n}_{R_n(x)}$$

$P_n(x)$ → polynomial approximation part

$R_n(x)$ → remainder part

As n increase, $R_n(x)$ will decrease becose $(n+1)!$ gets bigger

Let $x\neq x_0$ in $[a,b]$

set $M := \dfrac{f(x)-P_n(x)}{(x-x_0)^{n+1}}$ $\left(\text{WE WANTS TO SHOW}: M=\dfrac{f^{(n+1)}(c)}{(n+1)!}\right)$

Becose if we can show this, as $n\to\infty$, $M=0$. So, $R_n(x)=0$

Define:

$$\rho:[a,b]\to R$$

$$\rho(t)=f(t)-P_n(t)-M(t-x_0)^{n+1}$$

Take $(n+1)$th derivative of ρ

$$\rho^{(n+1)}(t)=f^{(n+1)}(t)-\underbrace{\cancel{P_n^{(n+1)}(t)}}_{0}-M(n+1)!$$

0 → becose nth order polynomial

$$\rho^{(n+1)}(t)=f^{(n+1)}(t)-M(n+1)!\Rightarrow$$ if we can show $\rho^{(n+1)}(t)=0$, then $M=\dfrac{f^{(n+1)}(t)}{(n+1)!}$

For $k\leq n$,

$$\rho^{(k)}(x_0)=f^{(k)}(x_0)-P_n^{(k)}(x_0)-\underbrace{\cancel{M(x_0-x_0)}}_{0}=0$$

Becase we know that from polynomial approximation up to the nth derivative of f and P $f^{(k)}(t)=P^{(k)}(t)$

Appendix – A1

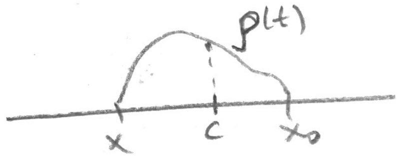
Also we know that

$$\rho(x) = f(x) - p(x) - M(x-x_0)^{n+1}$$

$$M = \frac{f(x)-p(x)}{(x-x_0)^{n+1}}$$

$$\rho(x) = f(x) - p(x) - \frac{f(x)-p(x)}{\cancel{(x-x_0)^{n+1}}} \cancel{(x-x_0)^{n+1}}$$

$$= 0$$



According to Rolle's theorem on $[x, x_0]$ or $([x_0, x])$, there is some point c where $\rho'(c) = 0$

Assume that such a point c_1, there exist another point c_2 that has $\rho''(c_2) = 0$ on $[c_1, x_0]$ (or $[x_0, c_1]$).

If we keep applying this pattern we will get $c_3, c_4, c_5, \dots, c_n$. If we know that $\rho^{(n)}(c_n) = 0$, then there exist a point c in (c_n, x_0) (or (x_0, c_n)) with

$$\rho^{(n+1)}(c) = 0$$

So $0 = \rho^{(n+1)}(c) = f^{(n+1)}(c) - M(n+1)! = 0$

$$\frac{f^{(n+1)}(c)}{(n+1)!} = M$$