# Orthogonal Sets Of Functions

A set of functions $\{\phi_1(x), \phi_2(x), \dots\}$ is an orthogonal set of functions on the interval $[a,b]$ if any two functions in the set are orthogonal to each other.

How do we calculate orthogonality? In vector space we calculate orthogonality using inner product of two vectors.

Example:
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$= \sum_{k=1}^{N} a_k b_k$$

In infinite dimensional space summation becomes integration. If we realize that functions are infinite dimensional vectors in infinite dimensional Hilbert space and if we find they are orthogonal each other we can construct basis set of functions.

So we want to find functions $\phi$

$$(\phi_n, \phi_m) = \int_a^b \phi_n(x)\, \phi_m(x)\, dx = 0 \qquad (n \neq m)$$

We want to show that:
$$\left\{ 1, e^{j\frac{2\pi}{T_0}t}, e^{j\frac{2\pi}{T_0}2t}, \dots \right\}$$ forms orthogonal basis

Because we will use these functions in Fourier series, this set is important

Show that

$$\left(1, e^{j\frac{2\pi}{T_0}t}\right) = 0$$

on set $(0, T_0)$

$$\left(1, e^{j\frac{2\pi}{T_0}2t}\right) = 0$$

$$\left(e^{j\frac{2\pi}{T_0}t}, e^{j\frac{2\pi}{T_0}3t}\right) = 0$$

$\vdots$

$$\int_0^{T_0} 1 . e^{j\frac{2\pi}{T_0}t} dt = 0 \implies$$ This can be shown easily

$$\int_0^{T_0} 1 . e^{j\frac{2\pi}{T_0}t} dt = 0 \implies$$ "

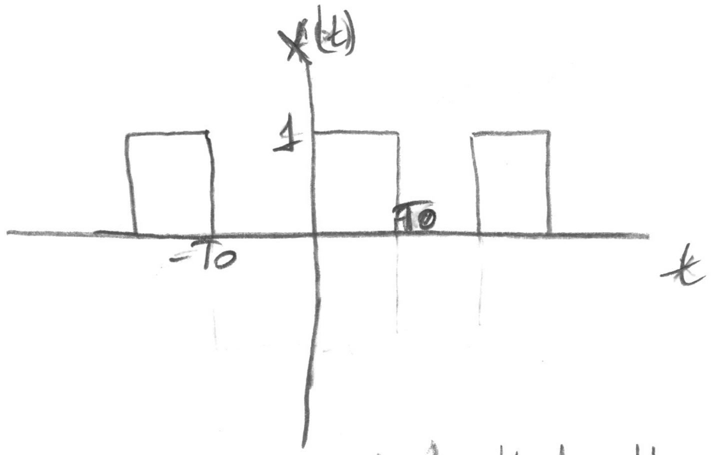
$$\int_0^{T_0} e^{j\frac{2\pi}{T_0}t} . e^{j\frac{2\pi}{T_0}3t} dt = 0 \implies$$ "

$\vdots$

So, these functions are orthogonal to eachother

We ~~use~~ orthogonal functions in finding of coefficients of Fourier Series.

Ex:




Joseph Fourier suggested that the signal could be approximated using sinusoids. But he did not prove it. The theory has been proved after Fourier. It has hard proof.

So, the theory proved and we know that sum of sinusoids gives us the original signal. The question is: which sinusoids do we need to obtain target signal?

We can state Fourier's theory like following mathematically

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left((2\pi/T_0)\, k t + \phi_k\right) \quad , k \text{ is integer}$$

We know that

$$A_k \cos\left(\frac{2\pi}{T_0} k t + \phi_k\right) = Re\left\{A_k e^{j\left(\frac{2\pi}{T_0} k t + \phi_k\right)}\right\}$$

$$= Re\left\{A_k e^{j\frac{2\pi}{T_0} k t} e^{j\phi_k}\right\}$$

If we assume $a_k = \frac{1}{2} A_k e^{j\phi_k} \Rightarrow \quad = Re\left\{a_k e^{j\frac{2\pi}{T_0} k t}\right\}$

$$= a_k e^{j\frac{2\pi}{T_0} k t} + \underset{\substack{\downarrow \\ \text{conjugate of } a_k}}{a_k^*} e^{-j\frac{2\pi}{T_0} k t} = a_k e^{j\frac{2\pi}{T_0} k t} + a_{-k} e^{-j\frac{2\pi}{T_0} k t}$$

If $a_{-k} = a_k^* \Rightarrow$ we can write

So

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

becomes

$$= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

We know that if we multiply two complex exponential signal and integrate multiplication become zero most of the time

$$\int_0^{T_0} e^{j(2\pi/T_0)kt} \, e^{-j(2\pi/T_0)lt} \, dt$$

$$= \int_0^{T_0} e^{j(2\pi/T_0)(k-l)t} \, dt \qquad \text{become } 0$$

If $k = l$ ($k, l$ is integer) $\Rightarrow \int_0^{T_0} e^{j(2\pi/T_0)\overbrace{(k-l)}^{\text{become } 0}t} \, dt$

$$= \int_0^{T_0} e^0 \, dt = \int_0^{T_0} 1 \, dt = t \Big|_0^{T_0} = \underline{T_0}$$

If $k \neq l \Rightarrow \int_0^{T_0} e^{j(2\pi/T_0)\overbrace{(k-l)}^{\text{becomes } m \neq 0}t} \, dt$

$$= \int_0^{T_0} e^{j(2\pi/T_0)mt} \, dt = \frac{e^{j\left(\frac{2\pi}{T_0}\right)mt}}{j\left(\frac{2\pi}{T_0}\right)m} \Bigg|_0^{T_0}$$

$$\left.\frac{e^{j\left(\frac{2\pi}{T_0}\right)mt}}{j\left(\frac{2\pi}{T_0}\right)m}\right|_0^{T_0} = \frac{e^{j\left(\frac{2\pi}{\cancel{T_0}}\right)m\cancel{T_0}} - 1}{j\left(\frac{2\pi}{T_0}\right)m}$$

because k and l is integer, m is integer

so, $e^{j2\pi m}$ always 1.

(If you use Euler formula, you can prove it easily)

$$\text{So}\quad \frac{\overset{1}{\cancel{e^{j2\pi m}}} - 1}{j\left(\frac{2\pi}{T_0}\right)m} = 0 \quad \text{always if } k \neq l$$

So, if we multiply $x(t)$ by a complex sinusoid and integrate the multiplication:

$$\int_0^{T_0} x(t)\cdot e^{-j\left(\frac{2\pi}{T_0}\right)lt}\,dt = \int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j\left(\frac{2\pi}{T_0}\right)kt}\right) e^{-j\left(\frac{2\pi}{T_0}\right)lt}\,dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^{T_0} e^{j\left(\frac{2\pi}{T_0}\right)(k-l)t}\,dt \rightarrow K$$

if $k \neq l$, $K = 0$

if $k = l$, $K = T_0$

$$\sum_{k=-\infty}^{\infty} a_k \left( \int_0^{T_0} e^{j(2\pi/T_0)(k-l)t} dt \right) = a_l T_0$$

So,

$$\int_0^{T_0} x(t) e^{-j\left(\frac{2\pi}{T_0}\right) l t} dt = a_l T_0$$

$$a_l = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\left(\frac{2\pi}{T_0}\right) l t} dt$$

change subscript $l$ to $k$ for convenience

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\left(\frac{2\pi}{T_0}\right) l t} dt$$

If we return at example

$$x(t) = \begin{cases} 1 \text{ for } 0 \leq t \leq \frac{1}{2} T_0 \\ 0 \text{ for } \frac{1}{2} T_0 \leq t \leq T_0 \end{cases}$$

$$a_k = \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} (1)\, e^{-j(2\pi/T_0)kt}\, dt$$

$$a_k = \left(\frac{1}{T_0}\right) \left. \frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k} \right|_0^{1/2\, T_0}$$

$$= \left(\frac{1}{T_0}\right) \frac{e^{-j(2\pi/T_0)k\frac{1}{2}T_0} - e^{-j(2\pi/T_0)k(0)}}{-j(2\pi/T_0)k}$$

$$= \frac{e^{-j\pi k} - 1}{-j2\pi k} \leftarrow e^{-j\pi} \text{ always } -1$$

$$= \frac{(-1)^k - 1}{-j2\pi k} = \frac{1-(-1)^k}{-j2\pi k}$$

$k \neq 0$ because denominator becomes $0$

$$a_0 = \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} (1)\, e^{-j0t}\, dt = \frac{1}{T_0}\int_0^{1/2\, T_0} (1)\, dt = \frac{1}{T_0}\left(\frac{1}{2}T_0\right) = \frac{1}{2}$$

$$a_k = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \pm 5 \dots \\ 0 & k = \pm 2, \pm 4, \pm 6 \dots \\ 1/2 & k = 0 \end{cases}$$

because $\frac{(-1)^k - 1}{-j2\pi k}$

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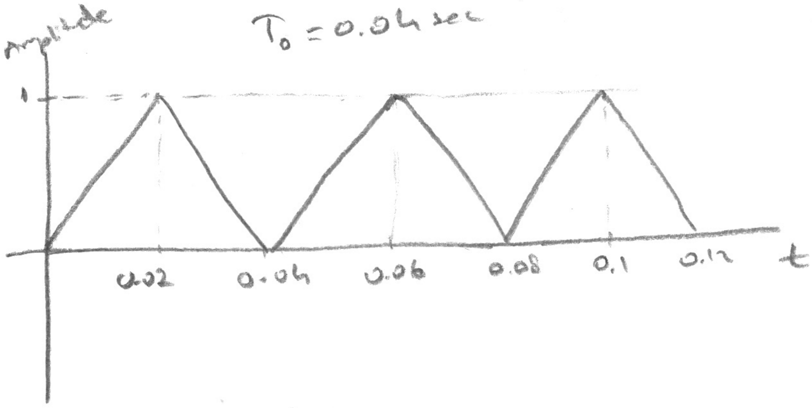
## Ex: Triangle Wave




$$x(t) = \begin{cases} 2t/T_0 & \text{for } 0 \le t \le \frac{1}{2}T_0 \\ 2(T_0 - t)/T_0 & \text{for } \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

We have already found that

$$\underbrace{a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt}_{\text{This means area under curve in a period}}$$

$$a_0 = \frac{1}{T_0} \cdot T_0 \cdot \frac{1}{2} = \frac{1}{2}$$

$$a_k = \frac{1}{T_0}\int_0^{\frac{1}{2}T_0} (2t/T_0)\, e^{-j(2\pi/T_0)kt}\, dt + \frac{1}{T_0}\int_{1/2\, T_0}^{T_0} (2(T_0 - t)/T_0)\, e^{-j(2\pi/T_0)kt}\, dt$$

$$a_k = \frac{e^{-jk\pi} - 1}{\pi^2 k^2}$$

$$a_k = \begin{cases} \frac{-2}{\pi^2 k^2} & k = \pm 1, \pm 3, \pm 5, \dots \\ 0 & k = \pm 2, \pm 4, \pm 6, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

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## Fourier Series In Sines & Cosines

$$x(t) = \underbrace{\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}}_{\text{Exponential}} = \underbrace{a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)}_{\text{Trigonometric}}$$

## Even functions

If $x(t)$ is even (we will call it $x_e(t)$), it can be represented as a sum of cosines of various frequencies via the equation:

$$x_e(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t)$$

$$x_e(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)$$

Multiply both sides and integrate over one period

$$\int_{T_0} x_e(t) \cos(n\omega_0 t)\, dt = \int_{T_0} \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) \cos(m\omega_0 t)\, dt$$

$$= \sum_{n=0}^{\infty} a_n \int_{T_0} \cos(n\omega_0 t) \cos(m\omega_0 t)\, dt$$

$$= \sum_{n=0}^{\infty} a_n \int_{T_0} \frac{1}{2}\Big(\cos((m+n)\,\omega_0 t) + \cos((m-n)\,\omega_0 t)\Big)\, dt$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} a_n \int_{T_0} \big(\cos((m+n)\,\omega_0 t) + \cos((m-n)\,\omega_0 t)\big)\, dt$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} a_n \Big( \overset{m \geq 0 \Rightarrow 0}{\cancel{\int_{T_0} \cos((m+n)\,\omega_0 t)\, dt}} + \int_{T_0} \cos((m-n)\,\omega_0 t)\, dt \Big)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} a_n \underbrace{\int_{T_0} \cos((m-n)\,\omega_0 t)\, dt}$$

If $m \neq n$, equal to $0$

If $m = n$, equal to $\int_T 1\, dt = T$

$$\int_{T_0} x_e(t) \cos(m\omega_0 t)\, dt = \frac{1}{2}\sum_{n=0}^{\infty} a_n \int_{T_0} \cos((m-n)\,\omega_0 t)\, dt$$

$$= \frac{1}{2} a_m T_0$$

$$a_m = \frac{2}{T_0}\int_{T_0} x_e(t) \cos(m\omega_0 t)\, dt$$

Change subscript $\quad a_n = \frac{2}{T_0}\int_{T_0} x_e(t) \cos(n\omega_0 t)\, dt$

In the same way we can find odd functions using sinusoids.

## Odd Functions

Odd functions can be represented by a Fourier Sine Series

$$x_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin(n w_0 t)$$

$$b_n = \frac{2}{T_0} \int_{T_0} x_0(t) \sin(n w_0 t)\, dt$$

## Arbitrary Functions

Any function can be represented of an even and an odd part.

$$x_{odd}(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

$$x_{even}(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x(t) = x_{odd}(t) + x_{even}(t)$$

So, a periodic function can be represented

$$x_T(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n w_0 t) + b_n \sin(n w_0 t)\right)$$

$$a_0 = \frac{1}{T_0} \int_{T_0} x_{T_0}(t)\, dt \qquad a_n = \frac{2}{T_0} \int_{T_0} x_{T_0}(t) \cos(n w_0 t)\, dt, \; n \neq 0$$

$$b_n = \frac{2}{T_0} \int_{T_0} x_{T_0}(t) \sin(n w_0 t)\, dt$$

# Fourier Transform

It is a special case of the Fourier Series when the period $T \to \infty$

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$T_0 a_n = \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

As $T \to \infty$, $\omega_0 = \frac{2\pi}{T_0}$ becomes very small, the quantity $n\omega_0$ becomes cts. quantity.

So, $\omega = n\omega_0$, $T_0 \to \infty$

$T_0 a_n$ becomes function of $\omega$

Forward FT.

$$T_0 a_n = X(\omega)$$

Analysis equation $\longrightarrow$ $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse FT

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} T_0 a_n e^{jn\omega_0 t} \cdot \frac{1}{T_0}$$

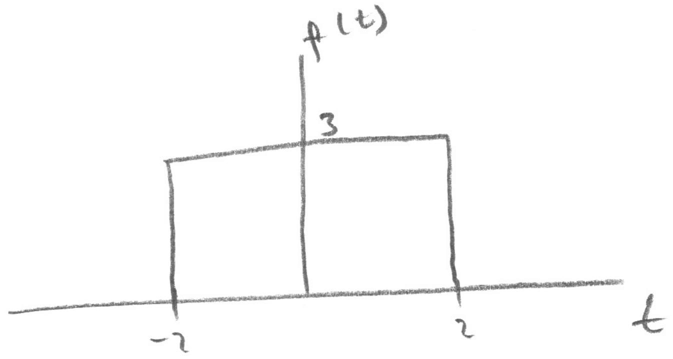
$$x(t) = \sum_{n=-\infty}^{\infty} T a_n e^{jn\omega_0 t} \frac{1}{T} = \sum_{n=-\infty}^{\infty} X(\omega) e^{j\omega t} \frac{d\omega}{dt}$$

Inverse F.T.

Synthesis Equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

Example:




Calculate FT

$$\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-2}^{2} 3 e^{-j\omega t} dt$$

$$= 3 \int_{-2}^{2} e^{-j\omega t} dt = \frac{3}{-j\omega} \left[ e^{-j\omega t} \Big|_{-2}^{2} \right]$$

$$= \frac{3}{-j\omega} \left[ e^{-j\omega 2} - e^{j\omega 2} \right] = \frac{3}{j\omega} \left( e^{2j\omega} - e^{-2j\omega} \right)$$

we know that $\sin\theta = \frac{1}{2j} \left( e^{j\theta} - e^{-j\theta} \right)$

$$\frac{3}{j\omega}\left(e^{2j\omega}-e^{-2j\omega}\right)=\frac{6}{\omega}\left(\frac{1}{2j}\left(e^{2j\omega}-e^{-2j\omega}\right)\right)$$

$$=\frac{6}{\omega}\sin 2\omega \longrightarrow F.T.$$

Graph of this function like this: