Signals & Systems

Week 6 Spring 2018

12.03.2018

CT F.S.: $x(t)$ periodic w/ $T_0 \to f_0 \to \omega_0$: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$a_k = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jk\omega_0 t} dt$

in 1 period

DTFS: $x[n] = \sum_{k=0}^{N-1} \underbrace{a_k}_{\text{periodic}} e^{+j\frac{2\pi}{N}kn}$

DT periodic signal w/ N

$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$

for periodic signals

What about non-periodic signals?

$x(t)$: non-periodic

CTFT: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

ω:

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{X(j\omega)}_{\text{F.T.}} e^{+j\omega t} d\omega$

Discrete-time Fourier Transform: DTFT:

rad/s $\omega \longrightarrow \boxed{\hat{\omega} = \omega T_s}$ ← sampling period in s.

↑ normalized angular freq : Units = radians

$$\underset{\text{sampled signal}}{x[n]} \longleftrightarrow \underbrace{X(e^{j\hat{\omega}})}_{\text{DTFT}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega} n}$$

<u>Exercise:</u> Show that $X(e^{j\hat{\omega}})$ is periodic w/ 2π.

$$X\left(e^{j(\hat{\omega}+2\pi)}\right) = \sum x[n] \underbrace{e^{-j(\hat{\omega}+2\pi)n}}$$

$$\sum x[n] e^{-j\hat{\omega} n} \cdot \underbrace{e^{-j2\pi n}}_{=1}$$

n are integers

$$\hat{\omega} \in [0, 2\pi)$$

$$(-\pi, \pi]$$

Inverse DTFT: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})e^{j\hat{\omega}n}\,d\hat{\omega}$

any 2π period

$$x[n] \xleftrightarrow{DTFT} X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

Uniqueness of DTFT:

Q: Is DTFT a linear transform? Yes

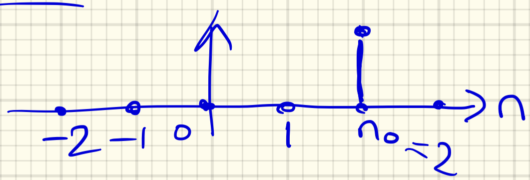
Let $x[n] = a\,x_1[n] + b\,x_2[n]$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty}\left(a\,x_1[n] + b\,x_2[n]\right)e^{-j\hat{\omega}n}$$

show

$$\overset{?}{=} a X_1(e^{j\hat{\omega}}) + b X_2(e^{j\hat{\omega}})$$

Ex: $x[n] = \delta[n-n_0] \xrightarrow{\text{DTFT}} X(e^{j\hat{\omega}}) = ?$



$$= \underbrace{\sum_{n=-\infty}^{\infty} \delta[n-n_0]\, e^{-j\hat{\omega} n}}$$

$$= e^{-j\hat{\omega} n_0}$$

Show Uniqueness of DTFT follows from linearity

Suppose 2 different signals $x_1[n], x_2[n]$ have the same D.T.F.T. Let $x_3[n] = x_1[n] - x_2[n]$

$$X_3(e^{j\hat{\omega}}) = \underbrace{X_1(e^{j\hat{\omega}}) - X_2(e^{j\hat{\omega}})}_{=} = 0 \Rightarrow$$

from defn of IDTFT, $x_3[n]$ has to be zero.

$$\Rightarrow x_1[n] = x_2[n]$$

Existence of DTFT: A sufficient condn for x(n) to have a DTFT $X(e^{j\hat{\omega}})$:

$$|X(e^{j\hat{\omega}})| = \left|\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\hat{\omega}n}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]\, e^{-j\hat{\omega}n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]|\, \underbrace{|e^{-j\hat{\omega}n}|}$$

$$= \sum_{n=-\infty}^{\infty} |x[n]| < \infty \rightarrow \text{DTFT of } x(n) \text{ exists.}$$

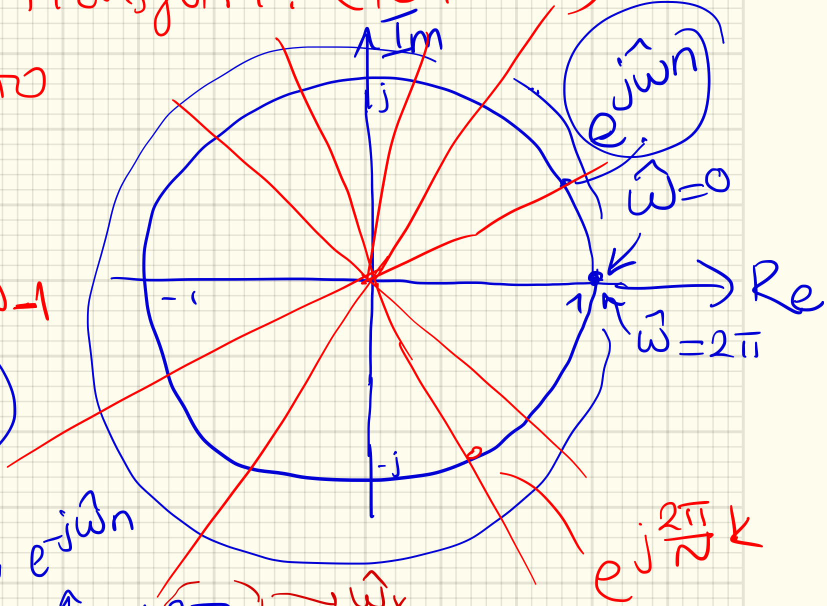
if

Discrete Fourier Transform: (DFT)

Discretize $\hat{\omega}$ into

N values in $(0, 2\pi)$

$$\hat{\omega} = \frac{2\pi}{N} k, \quad k = 0, 1, \dots, N-1$$



~~Rec~~ Recall (DTFT: $X(e^{j\hat{\omega}}) = \sum x(n) e^{-j\hat{\omega}n}$)

discretized freq. → $e^{-j\hat{\omega}n}$

DFT:

$$X[k] = \sum_{n=\cancel{-\infty}\;0}^{\cancel{\infty}\;L-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$\frac{2\pi}{N}k \to \hat{\omega}_k$

$k = 0, 1, \dots, N-1$

: N_pt DFT

$\hat{\omega}_k = \frac{2\pi}{N} k$

range typically finite, x(n) is finite duration

length L

$\Rightarrow X[k]$: both the freq. variable k, (& time variable n) are discrete,

↳ numerically implementable

DFT: basis for Fast Fourier Transform (FFT)

IDFT: $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k}{N} n}, \quad n = 0, 1, \ldots, N-1.$$

* We cannot compute for ∞-length signals, we operate on finite sections of a very long signal

Summary:

Frequency		Time: n (Discrete)	Time: t (cont.)
k	Discrete	DFT, DTFS (Fourier series) $X[k]$, $a_k = a_{k+N}$	Fourier Series CTFS a_k
ω	cont	DTFT $X(e^{j\hat{\omega}})$	CTFT $X(j\omega)$

periodic signals

DFT
DTFT
CTFT

for non-periodic signals

Later, we'll build a library of

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(j\omega)$$

$$x[n] \overset{DTFT}{\Longleftrightarrow} X(e^{j\hat{\omega}})$$

Systems (Chap 5) → SP First
Also check DSP First

Maps an input signal to an output signal:

Systems are used in SP (Signal Processing) for 2 primary purposes:

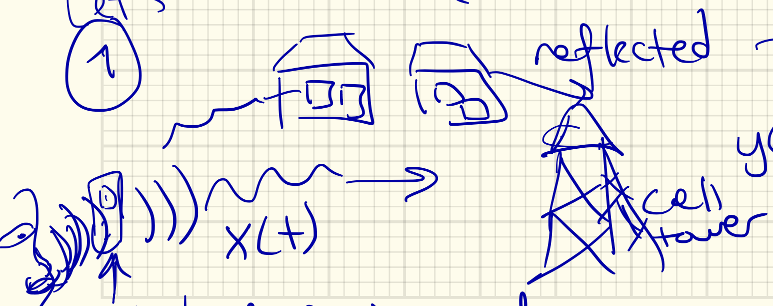
CT : $x(t) \longrightarrow \boxed{} \longrightarrow y(t)$

DT : $x[n] \longrightarrow \boxed{} \longrightarrow y[n]$

1. Model the effects of a physical phenomenon
2. Implement a desired effect on data or signals

Let's take a look at this example: 1

1



cellphone → voice signal

→ echo signal

$$y(t) = x(t) + \alpha\, \overbrace{x(t-\tau)}$$

α: decayed; and delayed by τ

Assume $|\alpha| < 1$

I/O

(1) $x(t) \longrightarrow \boxed{H, h} \longrightarrow$ (★) $y(t) = x(t) + \alpha\, x(t-\tau)$

modeling physical phenomena

received signal at the cell tower

(2) Implement the desired effect : here, it is extracting the xmitted signal

I/O:

$y(t) \longrightarrow \boxed{H_i} \longrightarrow z(t) = y(t) - \alpha\, y(t-\tau) + \alpha^2 y(t-2\tau) - \alpha^3 y(t-3\tau)$

Goal : $z(t) \simeq x(t)$

Insert $y(t)$ (eqn ★)

$z(t) = x(t) - \alpha^4 x(t-4\tau)$

derive this eqn.

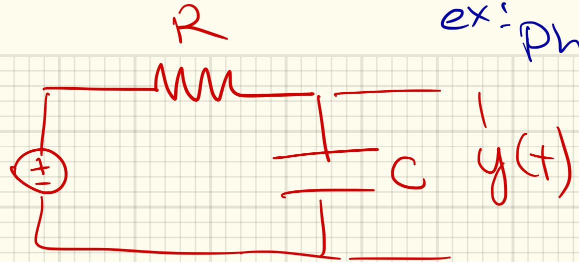
Recall $|\alpha| < 1$

$\alpha^4 \ll 1$

$\to 0$.

$\Rightarrow z(t) \approx x(t)$

ex:



$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

ex: Physical System

$$\frac{dy(t)}{dt} + \frac{1}{RC}\, y(t) = \frac{1}{RC}\, x(t)$$

$\swarrow$ Differential eqn.

DT Difference eqn: Linear const coeff. D.E:

General form of

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

General linear const coeff. Diff Eqn.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

RC circuit: $a_0 = 1/RC$, $b_0 = 1/RC$
$a_1 = 1$

DT Systems

② Implementing a desired effect:

↳ i. Eliminate "noise" → Filters

ii. Extracting desired information from a signal / eliminate the redundant info.

ex: $x[n] \longrightarrow \boxed{\text{DT Sys}} \longrightarrow y[n]$

Define a rule or I/O relation: $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$

3-pt averager.

ex: $y[n] = (x[n])^2$; A squarer system

ex: $y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$; 3pt averager system

but centered ←

ex: 5-pt averager RAF

$$y[n] = \frac{1}{5}\left(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] \right)$$

In general: $(M+1)$ pt. Running Average Filter (RAF)

$$y[n] = \sum_{k=0}^{M} \frac{1}{M+1} x[n-k]$$

b_k : filter coefficients

Generalize to FIR filter: Finite Impulse Response

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$a_0 = 1$

$a_k = 0$
$k \neq 0$

$$x[n] \longrightarrow \boxed{FIR} \longrightarrow y[n]$$

Recall this is a special case: $\displaystyle\sum_{k=-\infty}^{\infty} a_k y[n-k] = \sum_{k=-\infty}^{\infty} b_k x[n-k]$

I/O $\quad x(n) \longrightarrow \boxed{FIR} \longrightarrow y(n)$

ex: $y(n) = 3\,x[n] - x[n-1] + 2\,x[n-2] + x[n-3]$

Filter coeff? $\quad b_k = \{\underset{b_0}{3}, -1, 2, 1\}$ ...

ex: Give

$$\left.\begin{aligned} b_0 &= 2 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \\ b_4 &= -1 \end{aligned}\right\}$$

I/O? FIR.

$$y(n) = 2\,x[n] - x[n-4]$$

Causal System: Causality important for real-time (online) systems.

Def: Output depends on current & past values of the input.

Q: Is this system causal? Yes

eg. $y(10) = 2x[10] - x[6]$

ex: $y(n) = \frac{1}{2}\left(x(n) + x(n+1)\right)$ → Is this causal? No

$y(0) = \frac{1}{2}(x(0) + x(1))$ a future point.

Linear System :

Time-Invariant System :

Causal ✓

Stability

Properties of systems ; We'll continue w/ those next time.