On Knowledge-Soundness of Plonk in ROM from Falsifiable Assumptions

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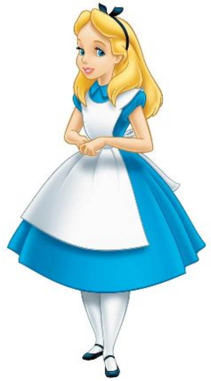
Talk Outlines

1. Plonk preliminaries and limitations in previous security proofs.
2. On the knowledge soundness of the linearization trick.
3. On the knowledge soundness of Plonk.

Ideal Plonk

Indexer $I \rightarrow \{i_k(X)\}$

$P(I, x, w)$



$a_1(X), a_2(X)$

$\xrightarrow{\hspace{3em}}$

$chall_1$

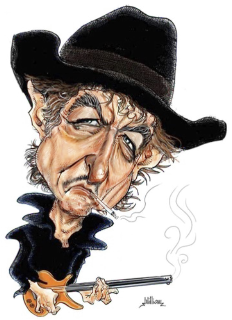
$\xleftarrow{\hspace{3em}}$

$\bullet \quad \bullet \quad \bullet$

$a_{n-1}(X), a_n(X)$

$\xrightarrow{\hspace{3em}}$

$V(I, x)$



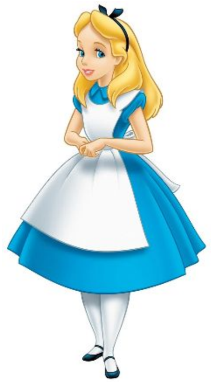
$$\sum_j s_j(\boldsymbol{a}(X), \boldsymbol{i}(X)) \stackrel{?}{=} 0$$

- Completeness: honest prover always convinces the verifier.
- Knowledge Soundness: if the verifier accepts, then the prover knows w.
- Zero-Knowledge: the verifier learns nothing about w.
- Succinctness: constant communication and verification complexity.

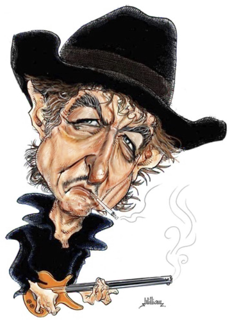
Ideal Plonk with dumb verifier

Indexer $I \rightarrow \{i_k(X)\}$

$P(I, x, w)$



$V(I, x)$



$a_1(X), a_2(X)$

$\longrightarrow$

$chall_1$

$\longleftarrow$

• • •

$a_{m-1}(X), a_m(X)$

$\longrightarrow$

$\xi \leftarrow \$\mathbb{Z}_p$

$$\sum_j s_j(\boldsymbol{a}(\xi), \boldsymbol{i}(\xi)) =^? 0$$

Cryptographic groups

- Bracket notation for additive groups

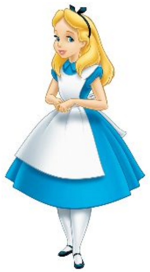
$$\mathcal{G} = \langle g \rangle := [1],$$
$$[x] \in \mathcal{G}: [x] = x[1]\ (= x\ g),$$

- Hardness assumptions

1. $x \leftarrow [x]$ is hard (discrete logarithm assumption)
2. $[x\ y] \leftarrow ([x],[y])$ is hard (CDH assumption)
3. $[1/\sigma] \leftarrow [1,\sigma]$ is hard (SDH assumption)

Polynomial and Rational Functions in Groups

$([1, \sigma, \dots, \sigma^n])$



$[f(\sigma)]$

- $f(X) = \sum_{i=0}^{n} \alpha_i X^i$ poly of degree up to n
  Easy: $[f(\sigma)] = \sum_{i=0}^{n} \alpha_i [\sigma^i]$
- $f(X) = \sum_{i=0}^{m} \alpha_i X^i$ poly of degree $m > n$
  HARD: equivalent to compute $[\sigma^m]$
- $f(X) = \frac{g(X)}{h(X)}, g, h \in Poly, h \nmid g$
  HARD: equivalent to compute $[1/\sigma]$

Variation of CDH

Variation of SDH

Bilinear Pairing Groups

- Three additive cryptographic groups

$$(p, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_T, [1]_1, [1]_2, \cdot)$$

p is the order of each group

1. $[x]_1 \cdot [y]_2 = [x\, y]_T$
2. $[x]_1 \leftrightarrow [x]_2$ is hard (type III pairings: no efficient isomorphism between groups)

KZG Polynomial Commitment Scheme

- $KGen(p, n)$:
  $\sigma \leftarrow \$\mathbb{Z}_p$, $ck = ([1, \sigma, \sigma^2, \ldots, \sigma^n]_1, [1, \sigma]_2)$
- $Com(ck, f)$:
  $C = [f(\sigma)]_1$
- $Open(ck, C, \alpha, f)$
  $\eta = f(\alpha), h(X) = \frac{f(X) - \eta}{X - \alpha}, \pi = [h(\sigma)]_1$
- $Verify(ck, C, \alpha, \eta, \pi) \rightarrow \{0,1\}$
  $$([f(\sigma)]_1 - \eta[1]_1) \cdot [1]_2 = [h(\sigma)]_1 \cdot ([\sigma]_2 - \alpha[1]_2)$$

Why it is secure?

$h(X) \in Poly \Leftrightarrow \eta = f(\alpha)$

Remember the SDH assumption!

KZG has Evaluation-binding

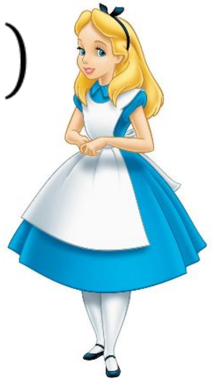
Interactive non-optimized Plonk

SRS $:(I \to \{i_k(X)\},\ [1,\sigma,\dots,\sigma^n]_1,[1,\sigma]_2)$

$P(SRS,x,w)$

$V(SRS,x)$

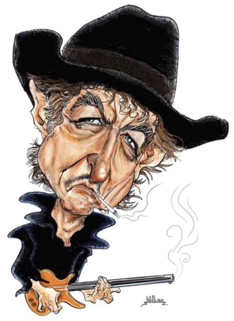
$[a_i]_1 = Com\,(a_i(X))$



$[a_1,a_2]_1$ →

$chall_1$ ←

$\bullet\ \bullet\ \bullet$



$[a_{m-1},a_m]_1$ →

ξ ←

$\eta_i = a_i(\xi)$

$[op_i]_1 = Open\,(a_i(X),\xi)$

$[op_1,\dots,op_m]_1,\eta_1,\dots,\eta_m$ →

$\forall i.\,Verify\ correctness\ of\ \eta_i = a_i(\xi)$

$$\sum_j s_j(\boldsymbol{\eta},\boldsymbol{i}(\xi)) =^? 0$$

Linearization trick

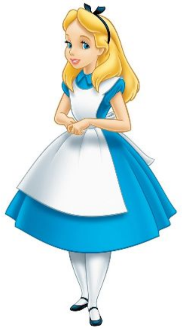
$$\sum_j s_j(\boldsymbol{a}(X), \boldsymbol{i}(X))d_j(X) \stackrel{?}{=} 0$$

$P(SRS, x, w)$

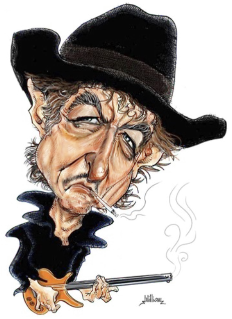
$V(SRS, x)$

$[a_i]_1 = Com\left(a_i(X)\right)$

$[d_i]_1 = Com\left(d_i(X)\right)$



ξ



$[\boldsymbol{a}, \boldsymbol{d}\,]_1$

$\eta_i = a_i(\xi)$

$[op_i]_1 = Open\left(a_i(X), \xi\right)$

$$h(X) = \sum_j s_j(\boldsymbol{a}(\xi), \boldsymbol{i}(\xi))d_j(X)$$

$[op_h]_1 = Open\left(h(X), \xi\right)$

$[op_1, \ldots, op_m, op_h]_1$

$\eta_1, \ldots, \eta_m$

$\forall i.$ *Verify correctness of* $\eta_i = a_i(\xi)$

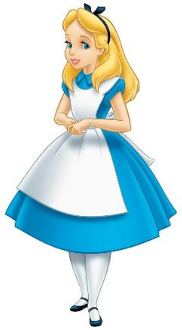
$$[h]_1 = \sum_j s_j(\boldsymbol{\eta}, \boldsymbol{i}(\xi))[d_i]_1$$

Verify correctness of $0 = h(\xi)$

Batch openings (simplified description)

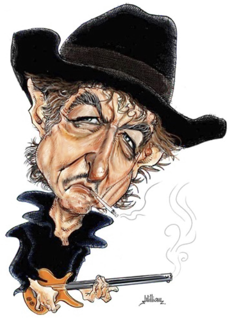
$P(SRS, x, w)$

$V(SRS, x)$



$[a_1]_1 = Com\left(a_1(X)\right)$

$[a_2]_1 = Com\left(a_2(X)\right)$



$[a_1, a_2]_1$

ξ

$\eta_i = a_i(\xi)$

$[op_i]_1 = Open\left(a_i(X), \xi\right)$

η_1, η_2

β

$[op]_1 = [op_1]_1 + \beta[op_2]_1$

$[op]_1$

$$[a_i - \eta_i + \beta(a_2 - \eta_2)]_1 \cdot [1]_2 \stackrel{?}{=} [op]_1 \cdot [\xi - x]_2$$

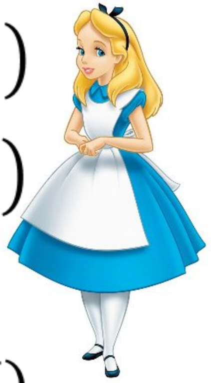
Interactive optimized Plonk $\sum_j s_j(\boldsymbol{a}(X), \boldsymbol{i}(X))d_j(X) \stackrel{?}{=} 0$

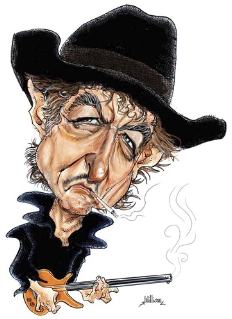
$P(SRS, x, w)$

$V(SRS, x)$

$[a_i]_1 = Com\,(a_i(X))$

$[d_i]_1 = Com\,(d_i(X))$





$[\boldsymbol{a}, \boldsymbol{d}\,]_1$

$\eta_i = a_i(\xi)$

ξ

$h(X) = \sum_j s_j(\boldsymbol{a}(\xi), \boldsymbol{i}(\xi))d_j(X)$

$\eta_1, \dots, \eta_m$

β

- Compute commitment to $h(X)$
- Verify the correctness of all the openings with a single check

$[op]_1$ *batch opening of* $a_i(X)$ *and* $h(X)$

$[op]_1$

Popular Framework (Plonk, Lunar, Marlin)

- An information-theoretic proof model
  - Idealised low-degree protocols
- An extractable polynomial commitment scheme
  - KZG (constant)

Idealized cryptographic groups (AGM, GGM)

Compiler

Succinct interactive ZK argument

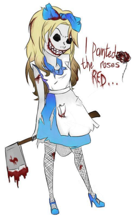
Fiat-Shamir transform

Random Oracle

SNARK

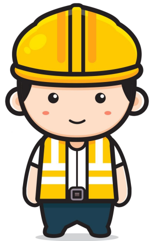
KZG Extractability

$$KGen(p,n) \rightarrow [1, \sigma, \sigma^2, \dots, \sigma^n]_1, [1, \sigma]_2$$



$$[f(\sigma)]_1 \leftarrow A(ck, aux)$$

AGM
extractor



$$\alpha_0, \alpha_1, \dots, \alpha_n : f(X) = \sum \alpha_i X^i$$

- Extraction only from commitment, without an opening
- Plonk security proof based on this assumption

Oblivious Sampling

- Sample group elements without knowing their DL.

$$s \leftarrow D,\ \ D \text{ superpolynomial min-entropy}$$
$$Enc(s) = [x]$$

- DL on $Enc(D)$ is as hard as DL.

$$\Pr[Enc(s) = [x] \mid s \leftarrow D, x \leftarrow A([1], s)] \approx 0$$

Example: encodings on elliptic curves

Extraction from the KZG commitment does not hold in the standard model!!!

[Lipmaa,Parisella,Siim 2023]

New security proof

[Lipmaa,Parisella,Siim 2024]

- KZG is extractable under a falsifiable assumption ARSDH assumption
- New succinct knowledge-sound interactive argument Plonk PIOP (no optimization) in ROM

SNARK	Prover complexity	Verifier complexity	Proof size
Unoptimized Plonk [LPS24]			
Plonk			

No batching
No linearization trick

Fiat-Shamir from knowledge-sound arguments

Fiat-Shamir transform

Succinct interactive ZK argument

SNARK

Random Oracle

Knowledge-soundness [Gabizon,Williamson,Ciobotaru 2019]

[Lipmaa,Parisella,Siim 2024] Special-soundness

Loss Q^{μ}

Ignored in implementation

Loss Q

Assumed in implementation

[Attema,Cramer,Kohl,2021]

Is Plonk special-sound?

Talk Outlines

2. On the knowledge soundness of the linearization trick.

Linearization trick security

$$\sum_j s_j(\boldsymbol{a}(X), \boldsymbol{i}(X))d_j(X) =^? 0$$

$$h(X) = \sum_j s_j(\boldsymbol{a}(\xi), \boldsymbol{i}(\xi))d_j(X)$$

$$[op_h]_1 = Open\ (h(X), \xi)$$

- Secure in AGM
- Insecure in the plain model [Fiore,Faonio,Russo 2024; Lipmaa,Parisella,Siim 2023]
- Knowledge-sound in AGMOS under some conditions on $d_j(X)$-s [Fiore,Faonio,Russo 2024]

Special-soundness of Lin-trick

The linearization trick cannot be special-sound
Even when knowledge-soundness holds in AGMOS

DL-assumption ⇒ Special-soundness is impossible ⇒ Knowledge-soundness is impossible

Important: knowledge-soundness in AGMOS is non-black-box (adversary's random coins are given to the extractor)

Plonk use linearization trick …

Or does it?

Linearization trick

$$\sum_j s_j\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big) d_j(X) =^? 0$$

$$h(X) = \sum_j s_j\big(\boldsymbol{a}(\xi), \boldsymbol{i}(\xi)\big) d_j(X)$$

$$[op_h]_1 = Open\ (h(X), \xi)$$

Plonk

$$\sum_j s_j\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big) d_j(X) + s\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big)\, \tilde{\imath}(X) =^? 0$$

$\tilde{\imath}(X)$ public indexed polynomial

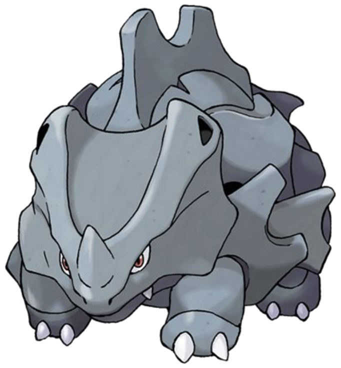
$$h(X) = \sum_j s_j\big(\boldsymbol{a}(\xi), \boldsymbol{i}(\xi)\big) d_j(X) + s\big(\boldsymbol{a}(\xi), \boldsymbol{i}(\xi)\big)\, \tilde{\imath}(X)$$

$$[op_h]_1 = Open\ (h(X), \xi)$$

Talk Outlines

3. On the knowledge soundness of Plonk.

RHINO

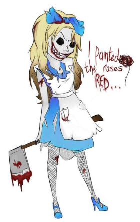


Reduction to a **h**ard assumption **i**f **no**t polynomial

$$s_1\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big) d(X) + s_2\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big)\, \tilde{\imath}(X) =^? 0$$

$\tilde{\imath}(X)$ public indexed polynomial

$$[1, \sigma, \sigma^2, \dots, \sigma^n]_1, [1, \sigma]_2$$



$$\boldsymbol{a}(X), [\tilde{d}]_1$$

$$s_1\big(\boldsymbol{a}(\sigma), \boldsymbol{i}(\sigma)\big)[\tilde{d}] + s_2\big(\boldsymbol{a}(\sigma), \boldsymbol{i}(\sigma)\big)\, \tilde{\imath}(\sigma) =^? 0$$

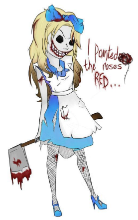
$$d(X) = \frac{s_2\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big)\, \tilde{\imath}(X)}{s_1\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big)}$$

$$d(\sigma) = \tilde{d}$$

RHINO

$$d(X) = \frac{s_2\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big)\, \tilde{\imath}(X)}{s_1\big(\boldsymbol{a}(X), \boldsymbol{i}(X)\big)}$$

$$[1, \sigma, \sigma^2, \ldots, \sigma^n]_1, [1, \sigma]_2$$



$$\boldsymbol{a}(X), [\tilde{d}]_1$$

$$[d(\sigma)]_1 = [\tilde{d}]_1$$

$$s_1\big(\boldsymbol{a}(\sigma), \boldsymbol{i}(\sigma)\big)[\tilde{d}] + s_2\big(\boldsymbol{a}(\sigma), \boldsymbol{i}(\sigma)\big)\, \tilde{\imath}(\sigma) \stackrel{?}{=} 0$$

- $d(X)$ is a polynomial: successfully extract the correct polynomial committed in $[\tilde{d}]_1$
- $d(X)$ is not a polynomial: HARD

Variation of SDH

Interactive Plonk is special-sound

Proof sketch:

1. KZG special-soundness $\Longrightarrow$ Extract all the polynomials $a(X)$
   - Under ARSDH KZG is special-sound [Lipmaa,Parisella,Siim 2024]
   - Batching preserves special-soundness
2. RHINO $\Longrightarrow$ Extract unopened polynomials $d(X)$
   - Under splitRSDH (variation of ARSDH, falsifiable assumption)
3. Plonk idealized protocol is special sound $\Longrightarrow$ Extract a witness
   - First time an idealized proof model is proven special-sound

Plonk is tightly knowledge-sound in the ROM

Thanks for your attention
Questions?

- **[Gabizon,Williamson,Ciobataru 2019]**
  PLONK: Permutations over Lagrange-bases for Oecumenical Noninteractive Arguments of Knowledge
- **[Attema,Cramer,Kohl,2021]**
  A compressed Σ-protocol theory for lattices
- **[Lipmaa,Parisella,Siim 2023]**
  Algebraic Group Model with Oblivious Sampling
- **[Lipmaa,Parisella,Siim 2024]**
  Constant-Size zk-SNARKs in ROM from Falsifiable Assumptions
- **[Fiore,Faonio,Russo 2024]**
  Real-world Universal zkSNARKs are non-malleable

The splitRSDH Assumption

Variant of ARSDH

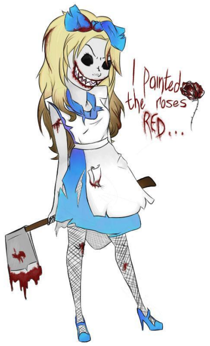
[Lipmaa,Parisella,Siim 2024]

Adversary A

Public polynomials $\psi_i(X)$

$\max \deg \psi_i(X) = n_\psi$

$ck = [1, \sigma, \sigma^2, \ldots, \sigma^n]_1, [1, \sigma]_2$



$\boldsymbol{S}, L(X), [\tilde{\boldsymbol{d}}, \psi]_1$

$$S \subset \mathbb{Z}_p \wedge |S| = n_S > n + n_\psi + 1 \wedge Z_S(X) := \prod_{\alpha \in S}(X - \alpha)$$

$$n + n_\psi < \deg L(X) < n_S$$

$$\sum \left[\tilde{d}_i \psi_i(\sigma)\right]_1 = [\psi Z_S(\sigma)]_1 + [L(\sigma)]_1$$

$\leq n_\psi + n$

$\geq n_S$

Variation of SDH