L9 TRANSCENDENTAL FUNCTIONS

OVERVIEW

Functions can be classified into two broad groups. Polynomial functions are called ***algebraic***, as are functions obtained from them by addition, multiplication, division, or taking powers and roots. Functions that are not algebraic are called ***transcendental.***

The trigonometric, exponential, logarithmic, and hyperbolic functions are transcendental, as are their inverses.

Transcendental functions occur frequently in many calculus settings and applications, including growths of populations, vibrations and waves, efficiencies of computer algorithms, and the stability of engineered structures.

Inverse Functions

A function that undoes, or inverts, the effect of a function f is called the inverse of f. Many common functions, though not all, are paired with an inverse. Important inverse functions often show up in formulas for antiderivatives and solutions of differential equations. Inverse functions also play a key role in the development and properties of the logarithmic and exponential functions.

Before we define an inverse function we need to know what a ***one to one function*** is.

One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain. Some functions assign the same range value to more than one element in the domain. The function $f(x) = x^2$ assigns the same value, 1, to both of the numbers -1 and +1.The *sines* of ($\frac{\pi}{3}$ and $\frac{2\pi}{3}$) are both $\frac{\sqrt{3}}{2}$.

Other functions assume each value in their range no more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct elements in its domain is called one-to-one. These functions take on any one value in their range exactly once.

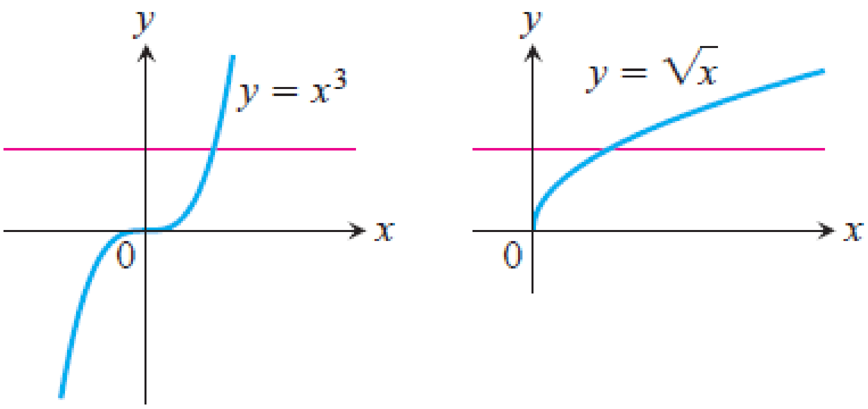
DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain *D* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in *D*.

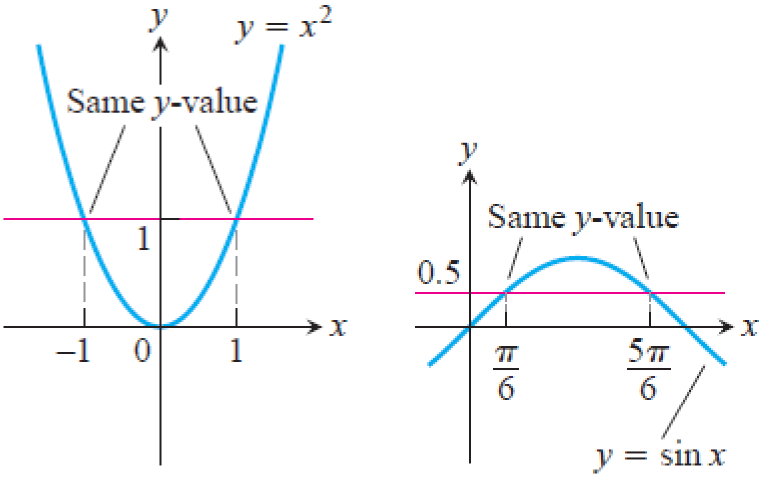
The graph of a one-to-one function *y= f(x)* can intersect a given horizontal line at most once. If it intersects the line more than once, it assumes the same *y*-value more than once, and is therefore not one-to-one.

The Horizontal Line Test for One-to-One Functions

A function *y= f(x)* is one-to-one if and only if its graph intersects each horizontal line at most once.



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or
more horizontal lines more than once.

Inverse Functions

Since each output of a one-to-one function comes from just one input, the effect of the function can be inverted to send an output back to the input from which it came.

DEFINITION

Suppose that f is a one-to-one function on a domain D with range R. The ***inverse function*** f^{-1} is defined by

$$f^{-1}(a) = b \ if \ f(b) = a$$

The domain of f^{-1} is R and the range of f^{-1} is D.

The domains and ranges of f and f^{-1} are interchanged. The symbol f^{-1} for the inverse of f is read "f inverse." The "-1" in f^{-1} is not an exponent:

f^{-1} does not mean 1/f(x).

Finding an Inverse Function

The process of passing from f to f^{-1} can be summarized as a two-step process.

1. Solve the equation *y*= *f(x)* for *x*. This gives a formula where *x* is expressed as a function of *y*.

2. Interchange *x* and *y*, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with *x* as the independent variable and *y* as the dependent variable.

Example

Find the inverse of $y = \frac{1}{2}x + 1$ expressed as a function of x.

Sol.

1. Solve for x in terms of y:

$$y = \frac{1}{2}x + 1$$

$$2y = x + 2$$

$$x = 2y - 2$$

2. Interchange x and y:

$$y = 2x - 2$$

The inverse of the function

$$f(x) = \frac{1}{2}x + 1$$

is the function

$$f^{-1}(x) = 2x - 2$$

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2(f(x)) - 2 = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f\left(f^{-1}(x)\right) = \frac{1}{2}f^{-1}(x) + 1 = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x$$

Example

Find the inverse of the function $y = x^2, where\ x \geq 0$

expressed as a function of x.

Sol.

First solve for *x* in terms of *y*:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x$$

$$|x| = x\ because\ x \geq 0$$

Then interchange *x* and *y*.

$$y = \sqrt{x}$$

So the inverse function is

$$y = \sqrt{x}$$

<u>Derivatives of Inverses of Differentiable Functions</u>

<u>THEOREM</u>

If *f* has an interval *I* as domain and $f'(x)$ exists and is never zero on *I*, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$.

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Or $\frac{df^{-1}}{dx}\Big|_{x=b} = \frac{1}{\frac{df}{dx}\Big|_{x=f^{-1}(b)}}$

Example

The function

$f(x) = x^2$ where $x \geq 0$

And its inverse is $f^{-1}(x) = \sqrt{x}$

Have derivatives

$$f'(x) = 2x$$

$$(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$$

According to the theorem above

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$(f^{-1})'(x) = \frac{1}{2(f^{-1}(x))}$$

$$(f^{-1})'(x) = \frac{1}{2(\sqrt{x})}$$

Logarithms and Exponential Functions

- Natural Logarithm ln x

One solid approach to defining and understanding logarithms begins with a study of the natural logarithm function defined as an integral through the Fundamental Theorem of Calculus. While this approach may seem indirect, it enables us to derive quickly the familiar properties of logarithmic and exponential functions. The functions we have studied so far were analyzed using the techniques of calculus, but here we do something more fundamental. We use calculus for the very definition of the logarithmic and exponential functions.

The natural logarithm of a positive number *x*, written as ln *x*, is the value of an integral.

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

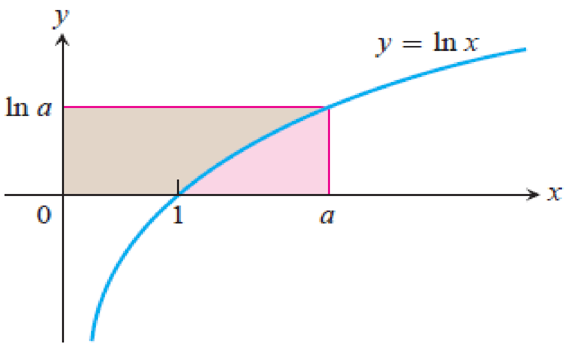


TABLE 7.1 Typical 2-place values of ln x

x	$\ln x$
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

<u>Properties of Logarithms</u>

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$

2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$

3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ (Rule 2 with $a = 1$)

4. *Power Rule*: $\ln x^r = r \ln x$ (r rational)

Example

- Interpreting the Properties of Logarithms

ln 6= ln (2.3)=ln 2+ln 3 (product rule)

ln 4-ln 5=ln (4/5)=ln 0.8 (quotient rule)

ln (1/8)= - ln 8 (reciprocal rule)

ln (1/8) = - ln 8 = - ln 2^3 = - 3 ln 2 (power rule)

- Applying the Properties to Function Formulas

ln 4 + ln sin *x* = ln (4sin *x*)   (product rule)

$\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ (quotient rule)

ln sec *x* = ln (1/cos *x*) = -ln cos *x*   (reciprocal rule)

$\ln \sqrt[3]{x+1} = \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)$ (power rule)

The Derivative of *y* = ln *x*

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

If *u* is a differentiable function of *x* whose values are positive, so that ln *u* is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

To the function *y* = ln *x* gives

$$\boldsymbol{\frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}} \qquad u > 0$$

Example

Find the derivative of *y* with respect to *x*.

1. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$

Sol.

$$\frac{dy}{dx} = \frac{4x^3}{4}\ln x + \frac{x^4}{4}.\frac{1}{x} - \frac{4x^3}{16} = x^3\ln x + \frac{x^3}{4} - \frac{x^3}{4} = x^3\ln x$$

2. $y = x\,(\ln x)^2$

Sol.

$$\frac{dy}{dx} = (\ln x)^2 + x.2\ln x\ .\frac{1}{x} = (\ln x)^2 + 2\ln x$$

3. $y = \ln(\ln x)$

Sol.

$$\frac{dy}{dx} = \frac{1}{\ln x}.\frac{1}{x} = \frac{1}{x\ln x}$$

The number *e*

The number *e* is that number in the domain of the natural logarithm satisfying

ln (*e*) = 1

The number *e* can be calculated as the limit

$$\boldsymbol{e = \lim_{x\to 0}(1 + x)^{1/x}}$$

Its value is calculated with a computer to 15 places accuracy

***e* = 2.718281828459045**.

The Function $y = e^x$

We can raise the number *e* to a rational power *r* in the usual way:

$e^2 = e.e\,,\ \ e^{-2} = \frac{1}{e^2}\,,\ \ e^{1/2} = \sqrt{e}$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take the logarithm, we find that

$$\ln e^r = r \ln e = r.1 = r$$

Since $\ln x$ is one-to-one and

$$\ln(\ln^{-1} r) = r$$

this equation tells us that

$e^r = \ln^{-1} r = \exp r$ for r rational

Generally

For every real number x,

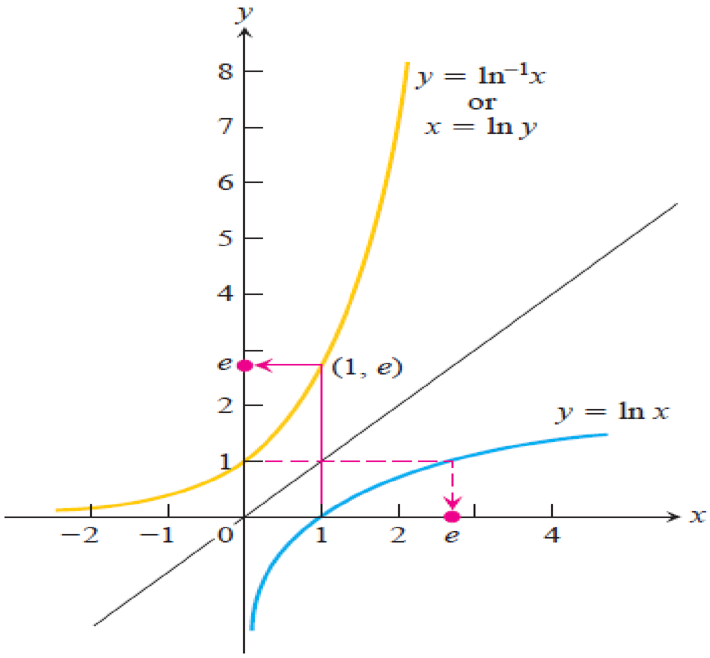
$$\boldsymbol{e^x = \ln^{-1} x = \exp x.}$$

<u>Inverse Equations for e^x and ln x</u>

$$e^{\ln x} = x \quad for\ all\ x > 0$$

$$\ln(e^x) = x \quad for\ all\ \ x$$

The domain of $\ln x$ is $(0,\infty)$, and its range is $(-\infty,\infty)$. So the domain of e^x is $(-\infty,\infty)$, and its range is $(0,\infty)$.



Example Using the Inverse Equations

$\ln e^2 = 2$

$\ln \sqrt{e} = \ln e^{1/2} = \frac{1}{2}$

$\ln e^{\sin x} = \sin x$

$e^{\ln 2} = 2$

$e^{\ln(x^2+1)} = x^2 + 1$

$e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$

Example

Find k if $e^{2k} = 10$

Sol.

Take the natural logarithm of both sides:

$\ln e^{2k} = \ln 10$

$2k = \ln 10$

$$k = \frac{1}{2}\ln 10$$

Laws of Exponents for e^x

For all numbers x, x_1, and x_2 the natural exponential e^x obeys the following laws:

1. $e^{x_1}.e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Example

Applying the Exponent Laws

1. $e^{x+\ln 2} = e^x.e^{\ln 2} = 2e^x$

2. $e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

3. $\frac{e^{2x}}{e} = e^{2x-1}$

4. $(e^3)^x = e^{3x} = (e^x)^3$

The Derivative of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero.

$$\frac{d}{dx}e^x = e^x$$

If u is any differentiable function of x, then

$$\boldsymbol{\frac{d}{dx}e^u = e^u\frac{du}{dx}}$$

Example

Find *dy/dx* for the following

1. $y = 5e^x$

Sol.

$$\frac{dy}{dx} = 5e^x$$

2. $y = e^{\sin x}$

Sol.

$$\frac{dy}{dx} = e^{\sin x}.\frac{d}{dx}(\sin x) = \cos x\,.\,e^{\sin x}$$

a^x and $\log_a x$

For any numbers $a > 0$ and x, the exponential function with base a is:

$$\boldsymbol{a^x = e^{x \ln a}}$$

<u>The Derivative of a^x</u>

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\boldsymbol{\frac{d}{dx} a^u = a^u . \ln a \frac{du}{dx}}$$

Example

Find *dy/dx* for the following

1. $y = 3^x$

Sol.

$$\frac{dy}{dx} = 3^x . \ln 3$$

2. $y = 3^{-x}$

Sol.

$$\frac{dy}{dx} = 3^{-x} . \ln 3 \frac{d}{dx}(-x) = -3^{-x} . \ln 3$$

3. $y = 3^{\sin x}$

Sol.

$$\frac{dy}{dx} = 3^{\sin x}.\ln 3\frac{d}{dx}(\sin x) = \ 3^{\sin x}.\ln 3 \,. \cos x$$

The Inverse of a^x

For any positive number $\neq 1$,

$\log_a x$ is the inverse function of a^x.

$a^{\log_a x} = x \qquad$ for $x > 0$

$\log_a (a^x) = x \qquad$ for all x

And

$$\log_a x = \frac{\ln x}{\ln a}$$

The Derivative of $\log_a x$

To find derivatives involving base a logarithms, we convert them to natural logarithms.

If u is a positive differentiable function of x, then

$$\frac{d}{dx}\ \log_a u = \ \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}.\frac{d}{dx}\ln u = \frac{1}{\ln a}.\frac{1}{u}.\frac{du}{dx}$$

$$\boldsymbol{\frac{d}{dx}\ \log_a u = \ \frac{1}{\ln a}.\frac{1}{u}.\frac{du}{dx}}$$

Example

$$\frac{d}{dx}\ \log_{10}(3x+1) = \ \frac{1}{\ln 10}.\frac{1}{3x+1}.\frac{d}{dx}(3x+1) = \frac{3}{\ln 10}.\frac{1}{3x+1}$$

Example

$$\frac{d}{dx}\left[\log_3\left(\frac{x+1}{x-1}\right)^{\ln 3}\right]=\frac{d}{dx}\left[\ln 3\,.\log_3\left(\frac{x+1}{x-1}\right)\right]=$$

$$=\ln 3\,\frac{d}{dx}\left[\log_3\left(\frac{x+1}{x-1}\right)\right]$$

$$\log_3\left(\frac{x+1}{x-1}\right)=\frac{\ln\left(\frac{x+1}{x-1}\right)}{\ln 3}$$

$$\ln\left(\frac{x+1}{x-1}\right)=\ln(x+1)-\ln(x-1)$$

$$\ln 3\,\frac{d}{dx}\left[\log_3\left(\frac{x+1}{x-1}\right)\right]=\ln 3\,\frac{d}{dx}\left[\frac{\ln(x+1)-\ln(x-1)}{\ln 3}\right]=$$

$$=\frac{\ln 3}{\ln 3}\,\frac{d}{dx}[\ln(x+1)-\ln(x-1)]=\frac{1}{x+1}-\frac{1}{x-1}$$