The Φ–Information Conjecture: Information as Structural Tension

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Abstract

We formulate and analyze the Φ–Information Conjecture: the idea that information is not a primitive physical quantity but the non–commutative residue arising from the incompatibility between two rational orders applied to the same system: a spectral rational order (selection of coherent modes) and a geometric resolution order (coarse–graining at finite scale). Formally, for a Hilbert space H and two operators P_N and G_ℓ representing these orders, the Φ–residue is

$$\Delta_\Phi(\ell) = P_N G_\ell - G_\ell P_N,$$

and the Φ–information at scale ℓ is defined as

$$I_\Phi(\ell) = \|\Delta_\Phi(\ell)\|.$$

We present the conjecture in a minimal operator–algebraic setting, prove basic structural properties, work out explicit finite–dimensional examples (including two– and three–dimensional models), and discuss how I_Φ relates (and does not reduce) to classical notions of information such as Shannon entropy. We emphasize that this is a structural conjecture about information as incompatibility of rational orders, not a claim of new fundamental dynamics. A final discussion section addresses natural objections concerning tautology, conservation laws, and the apparent arbitrariness in the choice of operators.

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1 Introduction

The goal of this note is to formulate a concrete and mathematically precise conjecture about the nature of information. The guiding intuition can be stated informally as follows:

> Information is not a substance or a primitive field. Instead, information emerges whenever two rational ways of structuring a system are incompatible.

By a "rational way of structuring" we mean an order that selects, organizes or simplifies the system according to a coherent rule: spectral truncation, coarse–graining in resolution, projection onto a subspace of relevant degrees de freedom, and so on. The central theme of the present work is that such orders are *not* assumed to be unique: different, equally rational, ways of structuring the same system may be mutually incompatible, and this structural tension carries quantitative content.

The Φ–Information Conjecture makes this idea precise using a minimal operator–algebraic framework. We consider:

- a Hilbert space H of possible descriptions,
- a self–adjoint operator D that encodes a "rational structure",
- a spectral projection P_N that selects the first N coherent modes,
- a resolution operator G_ℓ that implements coarse–graining or smoothing at finite scale ℓ.

The information is then identified with the norm of the commutator–like residue

$$\Delta_\Phi(\ell) = P_N G_\ell - G_\ell P_N.$$

The rest of the paper is organized as follows. In Section 2 we define the two rational orders. Section 3 states the Φ–Information Conjecture formally and discusses the choice of norm. Section 4 derives basic properties of $\Delta_\Phi(\ell)$ and $I_\Phi(\ell)$, including remarks on the role of the resolution scale. Section 5 presents explicit finite–dimensional examples (in two and three dimensions), with different scale dependences for the information profile. Section 6 discusses the relation to classical and quantum information measures and indicates concrete physical settings where the framework may apply. Section 7 introduces a minimal dynamical law for Δ_Φ. Section 8 contains a critical discussion of conceptual choices and open problems. We conclude in Section 9.

2 Rational Orders in an Operator Framework

We work on a complex separable Hilbert space H, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

2.1 Spectral rational order

Definition 1 (Spectral rational order)**.** *Let D be a self–adjoint operator on H with discrete spectrum $\{\lambda_k\}_{k\in\mathbb{N}}$ and associated orthonormal eigenvectors $\{e_k\}_{k\in\mathbb{N}}$. For $N \in \mathbb{N}$ we define the* spectral rational order *as the orthogonal projection*

$$P_N : H \to H, \qquad P_N f = \sum_{k=0}^{N} \langle f, e_k\rangle e_k.$$

Intuitively, P_N selects the first $N+1$ coherent modes of the description encoded by D. A central question, revisited in Section 8, is how N should be chosen in concrete applications. At the abstract level, N is a parameter specifying the degree of spectral resolution admitted by the rational order.

2.2 Resolution order

We now introduce a family of bounded operators that implement coarse–graining at finite resolution.

Definition 2 (Resolution order)**.** *Let $\{G_\ell\}_{\ell>0}$ be a family of bounded operators on H such that:*

1. *G_ℓ is self–adjoint and positive for each $\ell > 0$.*
2. *$\|G_\ell\| \le 1$ and $G_\ell \to I$ strongly as $\ell \to 0^+$.*
3. *G_ℓ is contractive: $\|G_\ell f\| \le \|f\|$ for all $f \in H$.*

We call G_ℓ the resolution order *at scale ℓ.*

Typical examples include heat semigroups $G_\ell = e^{-\ell^2 L}$ generated by a positive self–adjoint operator L (for instance, a Laplace–type operator) or Gaussian smoothing kernels in continuous settings. In finite dimension, G_ℓ can be thought of as a scale–dependent mixing or smoothing operation whose dependence on ℓ encodes how much structure is suppressed at that resolution.

Remark 1. *The framework does not require G_ℓ to be a function of the same operator D that defines P_N. In fact, the central phenomenon (non–commutativity) becomes trivial if P_N and G_ℓ are both functions of the same D. We therefore allow D and the generator of G_ℓ to be distinct. Conceptually, D encodes a "rational" mode decomposition, while G_ℓ encodes a geometrically or operationally motivated notion of finite resolution.*

3 The Φ–Residue and the Φ–Information Conjecture

We now introduce the key object.

Definition 3 (Φ–residue)**.** *Given P_N and G_ℓ as above, the* Φ–residue *at scale ℓ is the bounded operator*

$$\Delta_\Phi(\ell) = P_N G_\ell - G_\ell P_N.$$

This is the standard commutator between two bounded operators, but we interpret it as a residue of incompatibility between two rational orders.

3.1 Choice of norm

To turn $\Delta_\Phi(\ell)$ into a scalar "amount" of information, we must choose a norm. The following choice plays a distinguished role.

Definition 4 (Φ–information at scale ℓ)**.** *The* Φ–information *at scale ℓ is defined as the operator norm*

$$I_\Phi(\ell) = \|\Delta_\Phi(\ell)\| = \sup_{\|f\|=1} \|\Delta_\Phi(\ell)f\|.$$

Unless explicitly stated otherwise, $\|\cdot\|$ denotes the operator (spectral) norm induced by the Hilbert space norm on H. In finite dimension, all norms on the space of bounded operators are equivalent, so the qualitative behavior of $I_\Phi(\ell)$ does not depend on the specific choice; the operator norm is adopted for its direct spectral interpretation.

In applications it is also useful to consider a *state–dependent* quantity

$$I_\Phi(\ell; f) = \|\Delta_\Phi(\ell)f\|, \qquad \|f\| = 1,$$

which measures how much information is revealed at scale ℓ for a specific state or vector $f \in H$. When probability distributions or density matrices are encoded into H, this state–dependent version is the natural bridge to Shannon and von Neumann entropies (see Section 6).

3.2 Statement of the conjecture

We can now state the conjecture.

Conjecture 1 (The Φ–Information Conjecture)**.** *Information is not a primitive physical quantity but the non–commutative residue that arises when a system is subjected to two incompatible rational orders. Concretely, for any pair (P_N, G_ℓ) as above, the quantity*

$$I_\Phi(\ell) = \|P_N G_\ell - G_\ell P_N\|$$

measures the amount of information revealed at scale ℓ by the tension between the spectral rational order P_N and the resolution order G_ℓ. Classical information measures (such as Shannon or von Neumann entropy) can be seen, in appropriate limits, as derived from or closely related to $I_\Phi(\ell)$ and its state–dependent versions for suitable choices of (P_N, G_ℓ).

The conjecture is structural: it does not assert that I_Φ is the only reasonable notion of information, nor that it replaces existing measures. It claims that information can be understood as a norm of non–commutativity between rational orders, and that this structural viewpoint unifies a number of familiar phenomena in discrete, continuous, classical and quantum settings.

Normative interpretation The conjecture is not merely the observation that non–commutativity can be quantified through an operator norm. Rather, it asserts a normative structural principle: information is fundamentally relational and arises from the tension between distinct rational orders. Its physical quantification is therefore axiomatically captured by the magnitude of this structural tension, expressed as the norm of the non–commutative residue $\Delta_\Phi(\ell)$. The nontrivial content of the conjecture lies precisely in this identification.

4 General Properties of $\Delta_\Phi(\ell)$ and $I_\Phi(\ell)$

We collect elementary but useful properties.

Proposition 1 (Trivial and non–trivial cases)**.**

1. *If P_N and G_ℓ commute, then $\Delta_\Phi(\ell) = 0$ and $I_\Phi(\ell) = 0$.*
2. *If P_N and G_ℓ do not commute, then $\Delta_\Phi(\ell) \neq 0$ and $I_\Phi(\ell) > 0$.*

Proof. Item (1) is immediate. If $P_N G_\ell = G_\ell P_N$, then $\Delta_\Phi(\ell) = 0$ and hence $I_\Phi(\ell) = 0$. For (2), if the operators do not commute, then $P_N G_\ell - G_\ell P_N$ is non–zero, and the operator norm of a non–zero bounded operator is strictly positive. □

Proposition 2 (Boundedness)**.** *For all $\ell > 0$,*

$$I_\Phi(\ell) \leq 2\|P_N\|\|G_\ell\| \leq 2.$$

Proof. We have

$$\|\Delta_\Phi(\ell)\| \leq \|P_N G_\ell\| + \|G_\ell P_N\| \leq \|P_N\|\|G_\ell\| + \|G_\ell\|\|P_N\| = 2\|P_N\|\|G_\ell\|.$$

Since P_N and G_ℓ are contractions in the examples of interest ($\|P_N\| \leq 1$, $\|G_\ell\| \leq 1$), the bound follows. □

Proposition 3 (Vanishing in the sharp limit)**.** *Assume that $G_\ell \to I$ strongly as $\ell \to 0^+$ and $\|G_\ell\| \leq 1$. Then*

$$\lim_{\ell \to 0^+} I_\Phi(\ell) = 0.$$

Proof. For any $f \in H$,

$$\Delta_\Phi(\ell) f = P_N G_\ell f - G_\ell P_N f = P_N(G_\ell f - f) - (G_\ell P_N f - P_N f).$$

Hence

$$\|\Delta_\Phi(\ell) f\| \leq \|P_N\|\|G_\ell f - f\| + \|G_\ell P_N f - P_N f\|.$$

By strong convergence $G_\ell \to I$, both terms go to zero as $\ell \to 0^+$ for each fixed f. Using uniform boundedness, one shows that $\|\Delta_\Phi(\ell)\| \to 0$, hence $I_\Phi(\ell) \to 0$. □

Remark 2. *Intuitively, as $\ell \to 0^+$ the resolution order becomes indistinguishable from the identity and therefore the incompatibility between the orders vanishes. Information, in the sense of I_Φ, disappears in the continuum (or infinite–resolution) limit. In finite–dimensional toy models the limit $\ell \to 0$ is algebraically trivial, but the nontrivial content lies in how $I_\Phi(\ell)$ varies with ℓ away from this limit. Section 5.1 illustrates different possible scaling behaviors.*

Remark 3 (Physical relevance)**.** *The structural mechanism exhibited in finite dimensions appears naturally in several physical contexts: (i) spin systems with incompatible measurement bases, (ii) signal processing scenarios involving spectral versus spatial filters, and (iii) quantum information settings where projective measurements fail to commute with resolution–dependent channels. These examples illustrate that $\Delta_\Phi(\ell)$ captures a notion of informational tension that is ubiquitous across models, not an artifact of the chosen matrices.*

5 Explicit Finite–Dimensional Examples

To see that the conjecture is non–trivial but mechanically computable, we present explicit finite–dimensional examples with $H = \mathbb{C}^2$ and $H = \mathbb{C}^3$. These models also allow us to study the dependence of $I_\Phi(\ell)$ on the scale parameter ℓ for different functional forms of the misalignment between rational orders.

5.1 Two–dimensional toy model

Let $H = \mathbb{C}^2$ with standard basis (e_1, e_2).

Example 1 (Minimal non–commutative model)**.** *Define*

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

as the projection onto the first coordinate (spectral rational order), and let

$$G_\ell = \begin{pmatrix} \cos\theta(\ell) & -\sin\theta(\ell) \\ \sin\theta(\ell) & \cos\theta(\ell) \end{pmatrix},$$

with a scale–dependent angle $\theta(\ell)$ satisfying $\theta(\ell) \to 0$ as $\ell \to 0^+$. This G_ℓ represents a rotation mixing the two components at finite resolution. The dependence of θ on ℓ encodes how much misalignment is introduced at that scale.

We compute:

$$P_1 G_\ell = \begin{pmatrix} \cos\theta(\ell) & -\sin\theta(\ell) \\ 0 & 0 \end{pmatrix}, \qquad G_\ell P_1 = \begin{pmatrix} \cos\theta(\ell) & 0 \\ \sin\theta(\ell) & 0 \end{pmatrix}.$$

Therefore,

$$\Delta_\Phi(\ell) = P_1 G_\ell - G_\ell P_1 = \begin{pmatrix} 0 & -\sin\theta(\ell) \\ -\sin\theta(\ell) & 0 \end{pmatrix}.$$

The eigenvalues of $\Delta_\Phi(\ell)$ are $\pm\sin\theta(\ell)$, so the operator norm is

$$I_\Phi(\ell) = \|\Delta_\Phi(\ell)\| = |\sin\theta(\ell)|.$$

We see explicitly:

- $I_\Phi(\ell) = 0$ iff $\theta(\ell) = 0$ (the orders commute).
- For small ℓ, if $\theta(\ell) \sim a\ell$ then $I_\Phi(\ell) \sim a\ell$, so information grows linearly with the misalignment between orders.

This simple model already illustrates the core idea: information is a quantitative measure of the incompatibility between a projection and a resolution–dependent rotation.

Scale profiles for the angle $\theta(\ell)$

The dependence of the mixing angle $\theta(\ell)$ on the resolution scale ℓ determines the informational profile

$$I_\Phi(\ell) = |\sin\theta(\ell)|.$$

Several natural classes are:

(1) Linear profile

$$\theta(\ell) = a\ell,$$

so that $I_\Phi(\ell) \sim a\ell$ for small ℓ.

(2) Sublinear profile A slower growth such as

$$\theta(\ell) = a\sqrt{\ell}$$

gives $I_\Phi(\ell) \sim a\sqrt{\ell}$.

(3) Saturating profile A resolution–dependent rotation that approaches a maximal mixing angle $\theta_{\max}$ takes the form

$$\theta(\ell) = \theta_{\max}\left(1 - e^{-\ell/\ell_0}\right).$$

Here $\ell_0 > 0$ is a reference resolution scale marking the onset of saturation: for $\ell \ll \ell_0$ the angle grows approximately linearly, while for $\ell \gg \ell_0$ it approaches the maximal mixing angle $\theta_{\max}$.

5.2 A three–dimensional example

To confirm that nontrivial behavior is not an artifact of dimension two, we now consider a 3–dimensional example.

Let $H = \mathbb{C}^3$ with standard basis (e_1, e_2, e_3).

Example 2 (Projection onto a plane and 3D rotation)**.** *Define the spectral rational order as the projection onto the first two coordinates*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and let G_ℓ be a rotation in the (e_1, e_3)–plane by an angle $\theta(\ell)$, leaving e_2 fixed:

$$G_\ell = \begin{pmatrix} \cos\theta(\ell) & 0 & -\sin\theta(\ell) \\ 0 & 1 & 0 \\ \sin\theta(\ell) & 0 & \cos\theta(\ell) \end{pmatrix}, \qquad \theta(\ell) \to 0 \text{ as } \ell \to 0^+.$$

A direct computation yields

$$PG_\ell = \begin{pmatrix} \cos\theta(\ell) & 0 & -\sin\theta(\ell) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_\ell P = \begin{pmatrix} \cos\theta(\ell) & 0 & 0 \\ 0 & 1 & 0 \\ \sin\theta(\ell) & 0 & 0 \end{pmatrix},$$

and therefore

$$\Delta_\Phi(\ell) = PG_\ell - G_\ell P = \begin{pmatrix} 0 & 0 & -\sin\theta(\ell) \\ 0 & 0 & 0 \\ -\sin\theta(\ell) & 0 & 0 \end{pmatrix}.$$

The nontrivial part of $\Delta_\Phi(\ell)$ acts on the subspace spanned by $\{e_1, e_3\}$ and is identical, in that subspace, to the 2×2 matrix from Example 1. The eigenvalues of $\Delta_\Phi(\ell)$ are therefore $\pm\sin\theta(\ell)$ and 0, so the operator norm is again

$$I_\Phi(\ell) = \|\Delta_\Phi(\ell)\| = |\sin\theta(\ell)|.$$

This shows that the basic mechanism—information as norm of incompatibility between a projection and a scale–dependent rotation—persists in higher dimension and is not specific to qubit–like situations.

5.3 Comments on infinite–dimensional models

In infinite–dimensional settings (for instance, $H = L^2(M)$ for a Riemannian manifold M), one may take:

- $P_N = \chi_{[0,\Lambda_N]}(D_{\text{spec}})$, a spectral projection of some self–adjoint "rational" operator D_{spec},
- $G_\ell = e^{-\ell^2 L}$, a heat kernel generated by a possibly different operator L, for example the Laplace–Beltrami operator.

If $[D_{\text{spec}}, L] \neq 0$, these operators do not commute in general, and $\Delta_\Phi(\ell)$ can be non–zero. An explicit continuous model on the circle S^1 can be constructed along these lines by choosing D_{spec} and L with different eigenbases. The technical details are straightforward but notationally heavier, so we confine ourselves here to the finite–dimensional illustration and the abstract framework; the behavior of $I_\Phi(\ell)$ as $\ell \to 0$ then connects with regularity conditions on the underlying functions.

6 Relation to Classical and Quantum Information

The conjecture is not that $I_\Phi(\ell)$ *is* the Shannon entropy or the von Neumann entropy, but that both can be understood, in suitable settings, as derived from structural incompatibility of rational orders. Here we outline several bridges.

6.1 Discrete coarse–graining and Shannon entropy

Consider a finite probability vector $p = (p_1, \ldots, p_n)$ with $\sum_i p_i = 1$, and a partition of indices into "macroscopic" cells $C_1, \ldots, C_m$. The classical Shannon entropy is

$$S_{\text{Shannon}}(p) = -\sum_i p_i \log p_i.$$

A coarse–grained description at the level of cells uses the probabilities

$$q_j = \sum_{i \in C_j} p_i, \qquad j = 1, \ldots, m.$$

The difference between $S_{\text{Shannon}}(p)$ and $S_{\text{Shannon}}(q)$ measures, in information theory, the loss of microscopic detail under coarse–graining.

In the Φ–framework, one can model this as follows:

- $H = \mathbb{C}^n$ with standard basis $\{e_i\}$.
- P_N projects onto a subspace encoding a "rational" structure (for instance, a specific decomposition into relevant modes).
- G_ℓ averages over the partition cells C_j, playing the role of a resolution–dependent coarse–graining.

The residual operator $\Delta_\Phi(\ell)$ then quantifies how much the rational structure encoded by P_N is incompatible with the coarse–grained description induced by the partition. For reasonable choices of (P_N, G_ℓ), one expects

$$I_\Phi(\ell; p) := \|\Delta_\Phi(\ell) v_p\| \propto \left|S_{\text{Shannon}}(p) - S_{\text{Shannon}}(q)\right|,$$

where $v_p \in H$ encodes the probabilities p_i in a suitable way (for instance, $v_p = (\sqrt{p_1}, \ldots, \sqrt{p_n})$). We do not claim this proportionality is universal, only that it can be exhibited in explicit models; the conjecture is that the loss or gain of Shannon information can be traced back to non–commuting rational orders.

6.2 Von Neumann entropy, measurements and non–commutativity

In quantum theory, information is often measured by the von Neumann entropy of a density matrix ρ,

$$S(\rho) = -\mathrm{Tr}\,(\rho \log \rho)\,.$$

A projective measurement in a given basis corresponds to a family of orthogonal projections $\{Q_j\}$ with $\sum_j Q_j = I$. The incompatibility between the state ρ and the measurement can be captured by commutators $[\rho, Q_j]$. When two alternative measurements correspond to non–commuting operators, one can derive Bell–type constraints and nonclassical correlations.

Within the Φ–framework, one can consider:

- P_N as a rational truncation on the Hilbert space of states, or as the spectral projection associated with a chosen observable;
- G_ℓ as a resolution–dependent channel (for example, a depolarizing or Gaussian channel) or as an effective coarse–graining induced by limited measurement resolution.

Then $\Delta_\Phi(\ell)$ measures how much this channel fails to commute with the rational truncation. For appropriate choices, one may relate $S(\rho)$ and mutual informations to the behavior of

$$I_\Phi(\ell, \rho) = \|\Delta_\Phi(\ell)\rho\|$$

in specific regimes. At a conceptual level, the idea that informational constraints arise from non–commuting observables (as in Bell inequalities) is naturally rephrased here: incompatibility of rational orders—spectral and resolutive—*is* the carrier of information in this structural sense.

6.3 Concrete physical arenas

While the present paper remains deliberately conservative in physical claims, we briefly list settings where the framework appears naturally:

- **Spin systems with incompatible measurements:** P_N can encode a truncation to a subset of spin eigenstates, while G_ℓ represents a noisy or finite–resolution measurement channel. The resulting Δ_Φ captures how much the noisy channel fails to respect the preferred spectral decomposition.
- **Signal processing with spectral vs. spatial filters:** In time–frequency analysis, projections onto band–limited subspaces compete with spatial or temporal windowing. The Φ–residue quantifies the irreducible incompatibility between ideal band–limiting and finite–extent windows.
- **Quantum information with non–commuting bases:** Different choices of P_N and G_ℓ can model preferred code subspaces, error–correcting channels and coarse–grained measurements, with $I_\Phi(\ell)$ quantifying the structural tension between them.

In each case, the same operator–algebraic skeleton appears; the physical content lies in how P_N and G_ℓ are anchored to concrete operations or constraints.

7 Minimal Dynamics for the Φ–Residue

The conjecture so far is static: given P_N and G_ℓ, it defines $I_\Phi(\ell)$ at a fixed scale. To discuss evolution, we introduce a minimal dynamical law.

Let D be a self–adjoint operator that generates a one–parameter unitary group $U(t) = e^{-iDt}$ on H. We define time–dependent operators

$$P_N(t) = U(t)P_N U(t)^*, \qquad G_\ell(t) = U(t)G_\ell U(t)^*,$$

and the corresponding time–dependent residue

$$\Delta_\Phi(\ell, t) = P_N(t)G_\ell(t) - G_\ell(t)P_N(t).$$

Proposition 4 (Heisenberg–type evolution)**.** *$\Delta_\Phi(\ell, t)$ satisfies*

$$\frac{d}{dt}\Delta_\Phi(\ell, t) = i[\Delta_\Phi(\ell, t), D].$$

Proof. This is the usual Heisenberg equation. Since $P_N(t) = U(t)P_N U(t)^*$ and similarly for $G_\ell(t)$, we have

$$\frac{d}{dt}P_N(t) = i[D, P_N(t)], \qquad \frac{d}{dt}G_\ell(t) = i[D, G_\ell(t)].$$

From $\Delta_\Phi(\ell, t) = P_N(t)G_\ell(t) - G_\ell(t)P_N(t)$, a direct calculation yields the stated equation. □

Remark 4. *Proposition 4 does* not *introduce a new fundamental dynamics. It simply states that if P_N and G_ℓ are transported by a unitary evolution generated by D, then their incompatibility evolves exactly as any operator does in the Heisenberg picture. In this minimal setting, the operator norm $I_\Phi(\ell, t) = \|\Delta_\Phi(\ell, t)\|$ is conserved in time because the evolution is unitary. Irreversibility, in the present framework, is therefore associated not with time t but with changes in the resolution scale ℓ and the corresponding coarse–graining of descriptions.*

8 Discussion and Critical Assessment

We summarize and critically assess the conceptual choices of the present formulation, with an eye to natural objections.

8.1 On the alleged "tautology"

A first reaction is that $I_\Phi(\ell) = \|\Delta_\Phi(\ell)\|$ merely relabels the obvious fact that a commutator measures the incompatibility of two operators. In this sense, the framework might appear tautological.

The intended content of the conjecture is not the algebraic fact that $[P_N, G_\ell]$ vanishes if and only if the operators commute, but the *normative identification* of information with this non–commutativity between two specific classes of rational orders: a spectral order and a resolution order. In other words, the conjecture is axiomatic rather than deductive: it posits that the amount of informational tension between two descriptions of a system is fundamentally captured by the norm of their failure to commute. The mathematical structure itself is elementary; the nontrivial content lies in the claim that this particular structure deserves to be called "information" and can illuminate familiar informational quantities in various limits.

8.2 Choice of operators and parameters

A second family of objections concerns the apparent arbitrariness of the choices involved: the scale ℓ, the cutoff N and the specific forms of P_N and G_ℓ.

The scale ℓ in finite dimension In finite–dimensional toy models, the parameter ℓ is introduced via an angle $\theta(\ell)$ controlling the misalignment between orders. It is true that in such models ℓ has no intrinsic meaning beyond this role. The situation is analogous to representing coarse–graining in statistical mechanics by a parameter controlling the width of a convolution kernel: in a small system, the parameter is a modeling choice; in continuum limits it becomes tied to genuine spatial or temporal resolution. The present finite–dimensional examples should therefore be read as pedagogical stand–ins for continuum settings where ℓ is naturally a spatial, temporal or energy–scale of coarse–graining.

The cutoff N The projection P_N depends on N, and different N will in general yield different values of $I_\Phi(\ell)$. In concrete applications, N should be chosen according to physical or informational criteria: an energy cutoff, a complexity bound, a model order in signal reconstruction, an effective number of modes supported by the apparatus, and so on. The present formalism does not dictate a unique value of N; rather, it provides a way to quantify how different choices of spectral truncation (different rational orders) interact with a given resolution order. Clarifying when a cutoff can be considered "rational" in a more objective sense is an interesting open problem.

Why spectral vs. geometric orders One could consider many pairs of non–commuting operators. The reason to focus on (P_N, G_ℓ) is that spectral projections of a self–adjoint D are unitarily invariant and encode intrinsic coherent modes, while diffusive or smoothing operators G_ℓ (heat kernels, Gaussian filters) are canonical from the geometric and operational side. This pair minimizes arbitrariness: P_N is determined by the spectrum of D, independent of coordinate choices, and G_ℓ is tied to the geometry and finite resolution of measurements. Other pairs can be studied, but the spectral / geometric tension appears as a natural first candidate.

8.3 Conservation in time and irreversibility in scale

The minimal dynamics of Section 7 implies conservation of $I_\Phi(\ell, t)$ under unitary time evolution. This may seem at odds with thermodynamic intuition, where informational quantities typically grow or decay in time.

The point of the present framework is different: I_Φ is a structural, scale–dependent quantity associated with the incompatibility of two orders. It is not meant to reproduce thermodynamic entropy as a function of time. Instead, irreversibility is associated with the *coarse–graining of descriptions*, i.e. with changes in ℓ and, possibly, changes in which rational order is privileged. The monotonic behavior of $I_\Phi(\ell)$ with decreasing resolution (increasing ℓ) reflects the fundamental informational irreversibility associated with coarse–graining or renormalization group flow, rather than the standard time evolution of a closed system.

9 Conclusion

We presented the Φ–Information Conjecture: information is the norm of a non–commutative residue $\Delta_\Phi(\ell)$ arising from the interplay between spectral rational order P_N and resolution order G_ℓ. This framework is mathematically minimal yet capable of capturing non–trivial structure. The success of this structural approach hinges on the canonical choice of P_N and G_ℓ, demonstrating that the informational content is determined by the objective tension between spectral coherence and geometric resolution. This provides a new perspective: information is not intrinsic to a system but emerges from the incompatibility between rational descriptions. Its conservation in time is a consequence of unitarity, while its growth is tied to the irreversibility of scale (coarse–graining).

Whether this perspective can be extended to fields, quantum systems, thermodynamics, or cosmological scales remains an open question. The conjecture presented here should be seen as a structural starting point rather than a finished physical theory.

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