ALGEBRAS OF SETS BINUMERABLE IN COMPLETE EXTENSIONS OF ARITHMETIC

BY

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A set of integers is usually called *arithmetically definable* if there is a formula $B(x)$ with one free variable (in the notation of elementary first-order arithmetic) such that the set in question consists of just those integers n for which the corresponding instance $B(\Delta_n)$ of the formula is true in the domain of integers. Here the notation Δ_n denotes the n^{th} formal digit in the formalized language of arithmetic. The class of all arithmetically definable sets of integers is of course a denumerable Boolean algebra (or field) of sets. Now aside from the set of all true sentences of arithmetic, there are many other complete and consistent extensions of the axiomatic theory of first-order arithmetic in view of the well-known incompleteness of these axioms. Clearly each such complete extension leads to a denumerable Boolean algebra of sets obtained by using the instances of formulas valid in the complete extension rather than using the true instances. The purpose of this paper is to answer the question: *Which denumerable Boolean algebras of sets are obtainable from complete extensions of arithmetic*? A mathematical characterization of these algebras will be given using some simple notions from recursive function theory.

All formal theories used here are formalized in an applied first-order logic with identity with the individual constants 0 and 1 and the binary operation symbols $+$ and $\cdot$ as the only nonlogical constants. The logical symbols are $\wedge, \vee, \neg, \rightarrow, \longleftrightarrow, \forall, \exists, =$ with $x, y, z, w, x', y', \cdots$ etc. as individual variables. the formal digits Δ_n are defined by recursion so that Δ_0 is 0 and Δ_{n+1} is $(\Delta_n + 1)$. Capital Roman letters will generally denote formulas. Free variables and the substitution of digits for variables will be indicated by an informal parenthesis notation as in $B(x)$ and $B(\Delta_n)$. The standard axiomatic theory of first-order arithmetic is denoted by $\mathbf{P}$ (as in [**6**, p. 52]). A theory in general is simply a set $\mathbf{T}$ of sentences (without free variables) closed under the rules of deduction of first-order logic. We shall be concerned with those theories $\mathbf{T}$ such that $\mathbf{P} \subseteq \mathbf{T}$ and for every sentence A either $A \in \mathbf{T}$ or $\neg A \in \mathbf{T}$ but not both; these are the complete (and consistent) extensions of arithmetic. A set S of integers is *numerable* in such a theory $\mathbf{T}$ if there is a formula $B(x)$ such that $n \in S$ if and only if $B(\Delta_n) \in \mathbf{T}$. Note that the formula $\neg B(x)$ can be used to show that the complement of S is also numerable in $\mathbf{T}$, and hence S is *binumerable* in $\mathbf{T}$ (cf. [**1**, p. 51]). The Boolean algebra of all sets of integers *binumerable* in $\mathbf{T}$ will be denoted by $\mathfrak{B}[\mathbf{T}]$.

Even though our main interest here is in the algebras $\mathfrak{B}[\mathbf{T}]$, it seems to be

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a little more convenient to replace sets of integers by their characteristic functions. The class of characteristic functions of the sets in $\mathfrak{B}[\mathbf{T}]$ will be denoted by $\mathfrak{F}[\mathbf{T}]$. To be more precise, let ω denote the set of (non-negative) integers and let 2^ω denote the class of all $\{0, 1\}$-valued functions defined on ω. If A is a formula of arithmetic, let A^1 be A itself, but let A^0 be $\neg A$. If $\mathbf{T}$ is a complete extension of arithmetic, then a function $f \in \mathfrak{F}[\mathbf{T}]$ if and only if there is a formula $B(x)$ such that $B(\Delta_n)^{f(n)} \in \mathbf{T}$, for all $n \in \omega$. The theorem that will be given below characterizes those subsets of 2^ω that are of the form $\mathfrak{F}[\mathbf{T}]$ for some complete extension $\mathbf{T}$. Obviously we could transform the result to give a characterization of the algebras $\mathfrak{B}[\mathbf{T}]$, but this transformation will be left to the reader.

Let 2^n denote the class of all $\{0, 1\}$-valued sequences (functions) with domain $\{0, \cdots, n-1\}$. If $s \in 2^m$ and $n \leqq m$, then $s \restriction n$ denotes the restriction of s to $\{0, \cdots, n-1\}$. If $f \in 2^\omega$, then $f \restriction n$ is the restriction of f to $\{0, \cdots, n-1\}$. To each $s \in 2^n$ we make correspond a unique integer

$$\|s\| = p_0^{s(0)} \cdot \cdots \cdot p_{n-1}^{s(n-1)} \cdot p_n ,$$

where p_k is the kth prime number starting with $p_0 = 2$. By a *tree* we shall understand a set $\mathcal{T}$ of finite sequences (with arbitrary domains) closed under the formation of restrictions; that is, if $s \in \mathcal{T}$ and $s \in 2^m$, then $s \restriction n \in \mathcal{T}$ for all $n \leqq m$. A tree $\mathcal{T}$ is called *recursive within* a class $\mathcal{F} \subseteqq 2^\omega$, if the set of integers of the form $\|s\|$ for $s \in \mathcal{T}$ is recursive in some finite number of functions $f_0, \cdots, f_{k-1}$ in $\mathcal{F}$ (in the sense of [2, p. 275]). A *path* of a tree $\mathcal{T}$ is a function $f \in 2^\omega$ such that $f \restriction n \in \mathcal{T}$ for all $n \in \omega$. As is well-known, every infinite tree has at least one path. It is now possible to state the main theorem.

THEOREM. *For there to exist a complete extension* $\mathbf{T}$ *of arithmetic such that* $\mathcal{F} = \mathfrak{F}[\mathbf{T}]$ *it is necessary and sufficient that:*

(i) $\mathcal{F}$ *is a denumerable subclass of* 2^ω;

(ii) *every infinite tree recursive within* $\mathcal{F}$ *has a path belonging to* $\mathcal{F}$.

The proof of the necessity of conditions (i) and (ii) will not be given here, but it will be included in [5] along with the proofs of the results mentioned in [4]. For the proof of sufficiency, we shall employ a lemma which seems to be of some independent interest. This lemma slightly generalizes a result of A.Mostowski given in [3]. For the statement of the lemma certain additional terminology is convenient. Let us say that a formula $B(x)$ is *independent* of a formula $A(x)$ if whenever $f \in 2^\omega$ and the set of all sentences consisting of those in $\mathbf{P}$ together with sentences $A(\Delta_n)^{f(n)}$, for $n \in \omega$, is consistent, then the set remains consistent upon the adjunction of sentences $B(\Delta_n)^{g(n)}$, for $n \in \omega$, no matter which $g \in 2^\omega$ is chosen. In other words, on the basis of $\mathbf{P}$ it is impossible to deduce any fact about the relations among the instances of $B(x)$ from any consistent assumption about the instances of $A(x)$. We can define in a similar way the notion of $B(x)$ being independent of a finite number of formulas $A_0(x), \cdots, A_{k-1}(x)$.

LEMMA. *Given a finite number of formulas* $A_0(x), \cdots, A_{k-1}(x)$ *one can effectively find a formula* $B(x)$ *which is independent of them.*

The proof of the lemma will be outlined at the end of the paper. Let us now see how the lemma can be used for the proof of sufficiency. Suppose that $\mathscr{F}$ satisfies conditions (i) and (ii) of the theorem. Enumerate the elements of $\mathscr{F}$ in a sequence $g_0, \cdots, g_k, \cdots$. Let $A_0(x), \cdots, A_k(x), \cdots$ be a (recursive) sequence containing *all* the formulas of arithmetic with one free variable x. Evoke the lemma to introduce (by a recursion) a sequence $B_0(x), \cdots, B_k(x), \cdots$ of formulas such that for each $k \in \omega$ the formula $B_k(x)$ is independent of the finite number of formulas $A_0(x), \cdots, A_k(x), B_0(x), \cdots, B_{k-1}(x)$. By construction it is clear that the set of sentences of the form $B_k(\Delta_n)^{g_k(n)}$, for $k, n \in \omega$, is consistent with $\mathbf{P}$. Hence, there is at least one complete extension $\mathbf{T}$ of $\mathbf{P}$ which contains all these sentences. Obviously for such a $\mathbf{T}$ we have $\mathscr{F} \subseteq \mathfrak{F}[\mathbf{T}]$. What we need to show is that there is at least one $\mathbf{T}$ for which the inclusion is an equality.

To obtain the desired complete extension a new sequence $f_0, \cdots, f_k, \cdots$ of functions in $\mathscr{F}$ will be introduced so that the following set of sentences is consistent:

$$\mathbf{P} \cup \{A_k(\Delta_n)^{f_k(n)}: k, n \in \omega\} \cup \{B_k(\Delta_n)^{g_k(n)}: k, n \in \omega\} .$$

Suppose that $f_0, \cdots, f_{k-1} \in \mathscr{F}$ have already been obtained so that the set

$$\mathbf{U}_m = \mathbf{P} \cup \{A_k(\Delta_n)^{f_k(n)}: k < m; n \in \omega\} \cup \{B_k(\Delta_n)^{g_k(n)}: k < m; n \in \omega\}$$

is consistent. Let $\mathscr{T}_m$ be the set of all functions s such that $s \in 2^r$ for some $r \in \omega$ and such that among the first r proofs (in some standard enumeration of all the proofs of first-order logic) there is no proof establishing the inconsistency of the set of sentences:

$$\mathbf{U}_m \cup \{A_m(\Delta_n)^{s(n)}: n < r\} .$$

It is obvious that $\mathscr{T}_m$ is a tree. Since $\mathbf{U}_m$ is consistent, it is easy to prove that $\mathscr{T}_m$ is an infinite tree. Notice that the predicate of Gödel numbers of proofs which tells which proofs establish the inconsistency of a set of sentences is recursive in the set of Gödel numbers of the sentences. Hence, we see that $\mathscr{T}_m$ is recursive in $f_0, \cdots, f_{m-1}, g_0, \cdots, g_{m-1}$, all of which are in $\mathscr{F}$. Thus from condition (ii) there must be a path of $\mathscr{T}_m$ which is in $\mathscr{F}$; let this path be f_m. By the construction of $B_m(x)$ and the choice of f_m, it follows at once that the set $\mathbf{U}_{m+1}$ is consistent; therefore we can continue obtaining functions. Let $\mathbf{T}$ be the deductive closure of the set $\bigcup_{m \in \omega} \mathbf{U}_m$ of sentences. $\mathbf{T}$ is consistent because each $\mathbf{U}_m$ is consistent and $\mathbf{U}_m \subseteq \mathbf{U}_{m+1}$ for $m \in \omega$. On the other hand $\mathbf{T}$ is complete. For if A is any sentence, then there is a $k \in \omega$ such that $A_k(x)$ is the formula $[A \wedge x = x]$. But $A_k(\Delta_0)^{f_k(0)}$ is equivalent to $A^{f_k(0)}$; hence, this sentence must be in $\mathbf{T}$. Thus $\mathbf{T}$ is a complete extension for which $\mathscr{F} = \mathfrak{F}[\mathbf{T}]$ in view of the construction of the functions f_k.

Finally we must return to the proof of the lemma. It will be enough to show how to obtain a formula $B(x)$ which is independent of a given formula $A(x)$. First of all, let $C_0, \cdots, C_k, \cdots$ be the usual list of all sentences of arithmetic; that is, the sentence with Gödel number k is C_k. The Gödel numbering should be chosen in one of the standard ways so that we can introduce by definition functions $\dot{\rightarrow}$ and $\dot{\neg}$ into the formal theory of arithmetic where $\Delta_k = (\Delta_m \dot{\rightarrow} \Delta_n)$ is provable in $\mathbf{P}$ if and only if C_k is the formula $[C_m \rightarrow C_n]$, and where $\Delta_k = \dot{\neg}\Delta_m$ is provable in $\mathbf{P}$ if and only if C_k is $\neg C_m$. Further, we need to introduce by definition a formula $Pf(x, y)$ with the following meaning: *x is the Gödel number of a proof from the axioms of* $\mathbf{P}$ *of a sentence of the form*

$$[[A(\Delta_0)^{s(0)} \wedge \cdots \wedge A(\Delta_{z-1})^{s(z-1)}] \rightarrow C_y] ,$$

where s is a $\{0, 1\}$*-valued sequence of length z such that whenever* $w < z$, *then* $s(w) = 1$ *if and only if* $A(w)$ *holds.* The predicate $Pf(x, y)$ is of course not a recursive predicate since it involves $A(x)$; however, it can easily be obtained as a modification of the usual recursive proof predicate for the theory $\mathbf{P}$. The particular predicate used here will have the property that if $f \in 2^\omega$; if $\mathbf{P} \cup \{A(\Delta_n)^{f(n)}: n \in \omega\}$ is consistent; and if $\mathbf{T}'$ is the deductive closure of this set of sentences, then $Pf(\Delta_p, \Delta_q) \in \mathbf{T}'$ implies $C_q \in \mathbf{T}'$, and $C_q \in \mathbf{T}'$ implies that $Pf(\Delta_p, \Delta_q) \in \mathbf{T}'$ for some $p \in \omega$. Using this proof predicate we may construct a recursive function d such that for $n \in \omega$ the sentence

$$C_{d(n)} \longleftrightarrow \forall x[Pf(x, \Delta_n \dot{\rightarrow} \Delta_{d(n)}) \rightarrow \exists y \exists z[x = y + (z+1) \wedge Pf(y, \Delta_n \dot{\rightarrow} \dot{\neg}\Delta_{d(n)})]]$$

is provable in theory $\mathbf{P}$ (using, e.g., the method of [**1**, p. 65]). This function d will have the additional property that if $\mathbf{T}'$ is a theory of the type mentioned above, and if either $[C_n \rightarrow C_{d(n)}]$ or $[C_n \rightarrow \neg C_{d(n)}]$ is in $\mathbf{T}'$, then $\neg C_n$ is in $\mathbf{T}'$.

We can now define by recursion a sequence $B_0, \cdots, B_k, \cdots$ of sentences. Suppose $B_0, \cdots, B_{k-1}$ have already been introduced. Corresponding to each $s \in 2^k$ let $e(s)$ be an integer such that $C_{e(s)}$ is the formula

$$[B_0^{s(0)} \wedge \cdots \wedge B_{k-1}^{s(k-1)}].$$

We let B_k be the conjunction of all sentences $[C_{e(s)} \rightarrow C_{d(e(s))}]$ where $s \in 2^k$. Notice that $[C_{e(s)} \rightarrow [B_k \longleftrightarrow C_{d(e(s))}]]$ is a tautology. It will now be very easy to establish by induction on k that if $\mathbf{T}'$ is a theory as mentioned above, then $\mathbf{T}' \cup \{C_{e(s)}\}$ is consistent for all $s \in 2^k$. In other words, if whenever $f \in 2^\omega$ and $\mathbf{U} = \mathbf{P} \cup \{A(\Delta_n)^{f(n)}: n \in \omega\}$ is consistent, then $\mathbf{U} \cup \{B_n^{g(n)}: n \in \omega\}$ will be consistent for any choice of $g \in 2^\omega$. If you examine the choice of sentences B_n, you will see that each of them is a Boolean combination of sentences of the form $C_{d(m)}$ using many different m's. But the sentences $C_{d(m)}$ are essentially substitution instances of the same formula; or at least on the basis of $\mathbf{P}$ we can find a formula $D(x)$ such that $[C_{d(m)} \longleftrightarrow D(\Delta_m)]$ is in $\mathbf{P}$ for all $m \in \omega$, making use of the fact that d is a recursive function. Now the fact that B_n is obtained in a recursive way from the sentences $C_{d(m)}$ by Boolean combinations can be used to construct a formula $B(x)$ such that $[B_n \longleftrightarrow B(\Delta_n)]$ is in $\mathbf{P}$ for all $n \in \omega$. The formula $B(x)$ is the desired formula which is independent of $A(x)$.

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