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MATHEMATICS

Measurable Cardinals and Constructible Sets

by

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A cardinal number $\mathfrak{m}$ will be called measurable if and only if there is a set X of cardinality $\mathfrak{m}$ and a non-trivial, real-valued, countably additive measure μ defined on all subsets of X. (The term non-trivial can be taken to mean that $\mu(X) = 1$ and $\mu(\{x\}) = 0$, for all $x \in X$). If $2^{\aleph_0} = \aleph_1$, Banach and Kuratowski [1] proved that $\aleph_1$ is not measurable. Ulam [12] proved that if there is a measurable cardinal, then either $2^{\aleph_0}$ is measurable or there exists a 2-valued measurable cardinal (2-valued in the sense that the measure μ can be assumed to take on only the values 0 and 1). Ulam and Tarski showed that no cardinal less than the first strongly inaccessible cardinal beyond $\aleph_0$ can be 2-valued measurable (cf. [12], esp. footnote 1, p. 146). Last year, using some new results of Hanf, Tarski proved [11] that many inaccessibles, in particular the first beyond $\aleph_0$, are not 2-valued measurable (for other proofs cf. [6] and [2]). Even though the least 2-valued measurable cardinal, if it exists at all, now appears to be incredibly large since Tarski's results apply to a seemingly inexhaustible number of inaccessible cardinals, it still seems plausible to many people including the author to assume that such cardinals do exist. However, this assumption has some surprising consequences, for, as shall be outlined below, we can show that the existence of measurable cardinals contradicts Gödel's axiom of constructibility.

We shall work within the system of [4] but shall not follow the notation of [4] too closely. The axiom $V = L$ is assumed in the form of the following statement:

(*) *If M is a class such that*

(i) $M \subseteq PM \subseteq \bigcup_{x \in M} Px$;

(ii) $x - y$, $\bigcup x$, $\breve{x}$, $x|x$, $E|x \in M$, *for all* $x, y \in M$;

then $V = M$.

(Above, the symbol P denotes the power set operation so that PM is the class of all subsets of the class M, and $\bigcup$ the union operation; of course, $\bigcup x = \bigcup_{y \in x} y$). The terms $\breve{x}$ and $x|y$ denote, respectively, the operations of forming the converse

of the relational part of the set x and of forming the relative product of the relational parts of the sets x and y. The class E is the membership relation between sets; hence, $E \restriction x = \{\langle u, v\rangle : u \in v \in x\}$). That the statement (*) is equivalent to $V = L$ follows essentially from the lemma given by Hajnal ([5], p. 133) and the theorem of Shepherdson ([9], p. 186). The possibility of using the specific operations mentioned in condition (ii) of (*) follows from some unpublished results of Tarski.

Let us now assume that measurable cardinals exist. Since the axiom of choice follows from $V = L$, we can identify cardinals with initial ordinals. Let $\omega_\varkappa$, then, be the least measurable cardinal. Since $2^{\aleph_0} = \aleph_1$ follows from $V = L$, we can use the arguments of [12] to conclude that $\omega_\varkappa$ must be the least 2-valued measurable cardinal and that $\omega_\varkappa$ is a strongly inaccessible number; hence, $\omega_\varkappa = \varkappa$. Let $\mu \in \{0, 1\}^{P\varkappa}$ be 2-valued, non-trivial, countably additive measure defined on all subsets of $\varkappa$. (In general if A is a class and b is a set, then A^b denotes the class of all functions with domain b and range included in A). We now employ the measure μ to define certain relations Q_μ and E_μ over the class $V^\varkappa$ as in the theory of the reduced products (ultra products) of relational systems (cf. [3] and [6]).

DEFINITION 1.

(i) $Q_\mu = \{\langle f, g\rangle : f, g \in V^\varkappa \wedge \mu(\{\xi < \varkappa : f(\xi) = g(\xi)\}) = 1\}$;

(ii) $E_\mu = \{\langle f, g\rangle : f, g \in V^\varkappa \wedge \mu(\{\xi < \varkappa : f(\xi) \in g(\xi)\}) = 1\}$.

LEMMA 1. *Q_μ is a congruence relation for E_μ over $V^\varkappa$.*

The proof is very easy and uses only the finite additivity of the measure μ. Our main interest will lie in the structure of the equivalence classes f/Q_μ under the quotient relation E_μ/Q_μ. However, the equivalence classes are not sets and the quotient relation does not really exist. The next lemma gives some facts about relation E_μ which will allow us to replace the equivalence classes by sets thus overcoming this difficulty.

LEMMA 2. (i) *If $\{h \in V^\varkappa : hE_\mu f\} = \{h \in V^\varkappa : hE_\mu g\}$, then $fQ_\mu g$;*

(ii) $\{h \in V^\varkappa : hE_\mu f\} = \{h \in V^\varkappa : \exists k\, [k \in (\bigcup_{\xi<\varkappa} f(\xi) \cup \{0\})^\varkappa \wedge kE_\mu f \wedge hQ_\mu k]\}$;

(iii) $\sim \exists f\, [f \in (V^\varkappa)^\omega \wedge \forall \nu\, [\nu \in \omega \rightarrow f(\nu+1)\, E_\mu f(\nu)]]$.

Statement (i) shows that the equivalence class of f is determined by

$$\{h \in V^\varkappa : hE_\mu f\},$$

This is best proved by contradiction and requires the axiom of choice to find a function h which distinguishes f from g.

Statement (ii) implies that the number of equivalence classes included in the class $\{h \in V^\varkappa : hE_\mu f\}$ is bounded by the cardinality of the set $(\bigcup_{\xi<\varkappa} f(\xi) \cup \{0\})^\varkappa$.

Statement (iii) implies that the relation E_μ is well founded. The proof of (iii) is the first place where the countable additivity of μ is needed in the lemmas. The countable additivity at once reduces (iii) to the corresponding statement for the membership relation E, which follows easily from the axiom of foundation.

Using Lemma 2 we can now prove a statement which shows that $V^\varkappa$ can be mapped onto a class in such a way that the image of Q_μ is the identity relation and the image of E_μ is the membership relation. The method of proof is essentially that of [8] (Theorem 3, p. 147) or of [9] (Theorem 1.5, p. 171); see also [7].

LEMMA 3. *There is a (unique) function σ with domain $V^\varkappa$ such that for $f, g \in V^\varkappa$,*

(i) $\sigma(f) = \{\sigma(h) : h \in V^\varkappa \wedge hE_\mu f\}$;

(ii) $\sigma(f) = \sigma(g)$ *if and only if* $fQ_\mu g$;

(iii) $\sigma(f) \in \sigma(g)$ *if and only if* $fE_\mu g$.

DEFINITION 2. $M = \{\sigma(f) : f \in V^\varkappa\}$.

In other words, the class M is the range of the function σ; it is the class to which we shall apply the hypothesis of $(*)$. We note first:

LEMMA 4. $M \subseteq PM \subseteq \bigcup_{x \in M} Px$.

The first inclusion follows at once from 3 (i) and Def. 2 To prove the second, let $y \in PM$. Using the axiom of choice find $z \in P(V^\varkappa)$ such that $y = \{\sigma(g) : g \in z\}$. Let $f \in V^\varkappa$ be defined so that for $\xi < \varkappa$, $f(\xi) = \{g(\xi) : g \in z\}$. Then $y \in \sigma(f)$. Before we can check the second hypothesis of $(*)$, we need to prove a more general fact about M that can be used in many different ways. In the following $\Phi(v_0, \ldots, v_{k-1})$ will stand for any formula of set theory with free variables $v_0, \ldots, v_{k-1}$ and with *all* quantifiers restricted to V (that is, no bound class variables). Further, $\Phi^{(M)}(v_0, \ldots, v_{k-1})$ is the result of relativising all the quantifiers of $\Phi(v_0, \ldots, v_{k-1})$ to the class M.

LEMMA 5. *If $f_0, \ldots, f_{k-1} \in V^\varkappa$, then $\Phi^{(M)}(\sigma(f_0), \ldots, \sigma(f_{k-1}))$ if and only if $\mu(\{\xi < \varkappa : \Phi(f_0(\xi), \ldots, f_{k-1}(\xi))\}) = 1$.*

The proof proceeds by induction on the number of logical symbols in the formula and is exactly the same proof as that for reduced products (cf. [3], sec. 2). Now by using the proper formulas and Lemma 4 one can easily prove that M satisfies hypothesis (ii) of $(*)$; hence, we have:

COROLLARY 5.1. $V = M$.

To obtain other corollaries, it is useful to have a short notation for the images of the constant functions in $V^\varkappa$ under the mapping σ.

DEFINITION 3. $x^* = \sigma(\{\langle \xi, x\rangle : \xi < \varkappa\})$.

COROLLARY 5.2. *If $x_0, \ldots, x_{k-1} \in V$, then $\Phi^{(M)}(x_0^*, \ldots, x_{k-1}^*)$ if and only if $\Phi(x_0, \ldots, x_{k-1})$.*

Corollary 5.2 is a direct consequence of Lemma 5 obtained by substituting the constant functions for the $f_0, \ldots, f_{k-1}$. Next, if we combine 5.1 with 5.2 using the formula $\Phi(\varkappa)$ that expresses in formal terms that $\varkappa$ is the least 2-valued measurable cardinal, we prove at once:

COROLLARY 5.3. $\varkappa = \varkappa^*$.

To show how a contradiction is reached, we introduce next a special ordinal number that does not correspond to a constant function but is the image of the identity function.

DEFINITION 4. $\delta = \sigma(\{\langle\xi, \xi\rangle : \xi < \varkappa\})$.

LEMMA 6. *If* $\lambda < \varkappa$, *then* $\lambda^* < \delta < \varkappa^*$.

Recalling that less than between ordinals is the same as membership, we see that the inequality $\delta < \varkappa^*$ follows from 3 (iii) and Definitions 3 and 4. The proof of the inequality $\lambda^* < \delta$ reduces simply to the equation $\mu(\xi < \varkappa : \lambda \leqslant \xi\}) = 1$, which follows from the fact that $\varkappa$ is the least 2-valued measurable cardinal.

Notice that from 5.2 it follows at once that the mapping from sets x to sets x^* is one-one; hence, the set $\{\lambda^* : \lambda < \varkappa\}$ must have cardinality $\varkappa$. From 6 it follows that δ must have cardinality at least that of $\varkappa$. On the other hand 5.3 and 6 together imply that $\delta < \varkappa$, which contradicts the choice of $\varkappa$ as an initial ordinal.

In case one does not wish to assume that $V = L$, the above method of proof can be used for the following definite statement: If $\varkappa$ is the least 2-valued measurable cardinal, then $PP\varkappa \in L$.

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