II / Does Many-Valued Logic Have Any Use?

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INTRODUCTION

I have often asked the question of the title, but seldom have I heard anyone attempt a serious answer. So I shall try again to provoke some discussion. But before looking into details let me ask an historical question: how did Łukasiewicz come to propose his many-valued truth tables? As far as I can make out he said very little in print about the genesis of the idea. And once an idea like this is put forward, it has a life of its own with no one ever asking what right it has to travel so far or where its intellectual visa is. To go far it needs two principal qualities: first, it must be rather simple-minded so as not to tax the patiences of the people it will meet; and in the second place, it must stand in opposition to something 'classical' so we can all have the thrill of the break-through or revolution. None of us wants to be remembered for his bad review of a Beethoven. Speculation in ideas is as risky as speculation in land, but unfortunately the punishment does not come as quickly, and bad shares are still thick on the market.

Among non-classical logics, intuitionism certainly had the second of the two principal virtues, but alas not the first. In cases like this, the plan generally is to fall back on a state of near-religious mysticism in order to make people feel guilty that they are unable to ascend to that higher consciousness. But please do not misunderstand me: even though the intuitionistic logic is not as straightforward as the many-valued brand, I definitely

consider it a good investment and only hope to see soon a more attractive and glossier brochure than those currently available. I think that intuitionism can be brought down to earth and will be found solid, but I am not all that sure about many-valued logic. A quick review of the situation will bring out the weak point that could lead to collapse.

Take the first example of the three-valued system. The table for implication is usually written thus:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

The value $\frac{1}{2}$ is especially piquant. We deny that propositions are to be just *true* or *false*. Pandora's box is open! Where to turn in the face of such uncertainty? And then all of a sudden here is $\frac{1}{2}$ sitting exactly in the middle. What excellent symmetry—surely something classical in that! Now this *is* an idea. But hold; why just *one* intermediate value? why not more? Did Pandora's box contain only one creature? Of course not. This was only a quick snapshot. If you want to see the rest, here they are:

$\rightarrow$	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0
1	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0
$\frac{5}{6}$	1	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
$\frac{2}{3}$	1	1	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{1}{2}$	1	1	1	1	$\frac{5}{6}$	$\frac{2}{8}$	$\frac{1}{2}$
$\frac{1}{3}$	1	1	1	1	1	$\frac{5}{6}$	$\frac{2}{3}$
$\frac{1}{6}$	1	1	1	1	1	1	$\frac{5}{6}$
0	1	1	1	1	1	1	1

You see; there *are* lots. Clearly anything as exact as that must be correct. And if you *still* want more, all of these systems are subsystems of an infinite-valued system. Please don't be afraid of the maths, since it is all quite easy with simple rules of calculation.

No doubt you are annoyed by my sarcasm, but I do ask this:

Before you accept many-valued logic as a long-lost brother, try to think what these fractional truth values could possibly mean. And do they have any use? What is the conceptual justification of 'intermediate' values? This is the weak point of the programme, because we tend to become so fascinated with the patterns and possibilities. Certainly modal logic also suffers from an over-developed formalism. The literature on many-valued systems is ridiculously large,[1] but I am sure it does not match in waste of time that on modalities. I suppose the reason is that the many-valued systems are more rigid, while modal systems permit greater variation. But before we let the formalism run wild, we should ask whether there is any point to it all. A slight nod towards probabilities will definitely not be sufficient for justification either, for it only takes a moment's thought to see that the probability of $[p \rightarrow q]$ *cannot* be a function of the probabilities of p and q. We must seek elsewhere for sensible reasons for considering this new 'logic'. Two directions of investigation will be described.

1 A LOGIC OF ERRORS

I do not mean an erroneous logic, though I am prepared to hear people argue that the idea is wrong. What I am asking is whether we can distinguish some kind of *degrees* of truth. The answer is likely to be 'No' if we expect these degrees to combine with one another in little truth tables. Surely error is connected with both *knowledge* and *belief*; and (Professor Hintikka and the other possible worlders notwithstanding) until we have a coherent logic of these notions, we shall probably not have a convincing treatment of error. But before we are too critical, let us try a little experiment in formalization so that we might get a better grasp of where the difficulties lie.

For simplicity consider a 'model' language with atomic propositional symbols $p, q, r, \ldots$ and one binary connective $\rightarrow$. I have complained before about the presuppositions behind such formal systems,[2] but for this discussion we should put

these worries aside. If A is a well-formed formula of this language, we are going to attempt to explain what could be meant by saying '*the statement* A *is true to within degree of error* i'. An ideographic way of writing this relation between formulae and degrees (whatever they are) is as follows:

$$i \geqslant A.$$

Could the degrees be *ordered* in some way so that some A's are *more* in error than others? Let us try to suppose so. But then what does 'truth' mean? No error? Well, why not? Let us suppose that there is a *minimal* degree of error, called '0'. Thus the degrees will be ordered by a relationship written '$i \geqslant j$', and we suppose that for all degrees i we have $i \geqslant 0$.

The plan is to be a bit more abstract than Łukasiewicz in using ordered sets rather than fractions. In the first place fractions like $\frac{5}{6}$ are too mysterious at the start (we may want to use them later), because they are too much used for other purposes. Ordered sets being more abstract can enjoy a variety of interpretations. Of course, many ordered sets can be isomorphically represented by sets of fractions (or, even better, real numbers), but this may not be significant. In introducing this abstraction and in using the terminology of error note that we have turned things around: the *minimal* degree of error (which we call '0') corresponds to the *maximal* truth value (what Łukasiewicz calls '1'). It will be seen below exactly why this change makes certain properties easier to describe. For the moment we can remark that the change makes the table for $\rightarrow$ less curious:

$\rightarrow$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	1	2	3	4	5
2	0	0	0	1	2	3	4
3	0	0	0	0	1	2	3
4	0	0	0	0	0	1	2
5	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0

Here the *integers* have been used for *counting* the degrees. Fractions are more properly used when a question of *proportion* is to be asked. At the moment we do not know whether there will be more to the idea beyond a simple counting and ordering. In the table we write numerals merely as a convention for conveying this amount of structure. At least we can agree that by placing truth at the 0-end of the scale we make it easy to leave the false-end open. Let us not ask just now why these degrees are *linearly* ordered.

The mystery of the table for $\rightarrow$ is far from being removed, however, by these inconsequential notational changes. We still have to provide a story to explain the pattern and symmetry of the table. We then have to try to sell the story. Note that there must be more than a mere ordering of degrees involved in view of the very regular *shift* from row to row. Before we get to that, the question is: why all those 0's? That is the easy part of the table and does depend only on the ordering. Let us define

$$A \geqslant B \textit{ iff whenever } i \geqslant A, \textit{ then } i \geqslant B.$$

(We can be a little vague about the range of the variable i here since the same definition will apply to any system of degrees.) We read '$A \geqslant B$' both as 'A *implies* B' and as 'A is more in error than B'. This is a metalinguistic relationship and *not* a statement *within* the language. But from the table we see that there is a connection:

$$A \geqslant B \text{ iff } 0 \geqslant A \rightarrow B.$$

That is to say, $A \rightarrow B$ is true iff A implies B. Certainly this is a necessary condition on any *connective* in the language that is going to function as an implication operator. What needs agreement here is the definition of '$A \geqslant B$'; but if degrees were to be reasonable at all, this seems pretty obvious.

To explain the table is to explain '$i \geqslant A \rightarrow B$' which we could read as 'A implies B (not outright but) to within degree of error i'. Can we indeed explain an inexact notion of implication? From the table with the Łukasiewicz pattern, we can observe that the answer depends on the 'distance' between A and B. If A is more false than B, the degree of implication *is* exact, as we have noted. If B is just a 'little' more false than A, then (by the table at least) we want $A \rightarrow B$ to be 'almost' true. If B is very much more in error than A, we want $A \rightarrow B$ to measure this discrepancy on a sliding scale. The assumption here is this: the *distances* between degrees must again be expressible as degrees. Take two degrees j and k. We need to be able to determine this relationship:

$$k \geqslant i + j$$

in order to be able to say (when $k \geqslant j$) that the distance between them is at least up to degree i. (I have written '$+$', but take this as a ternary relationship for now.) Granting this, we can define:

$$i \geqslant A \rightarrow B \textit{ iff whenever } k \geqslant i + j \textit{ and } j \geqslant A, \textit{ then } k \geqslant B$$

If we agree that $k \geqslant j$ iff $k \geqslant 0 + j$, then this definition does indeed reduce to that for "$A \geqslant B$". In other words, when $A \geqslant B$; then $i \geqslant A \rightarrow B$ shows the *shift* needed to move A up to B. We could even make this more ideographic if we wrote:

$$\mathrm{i} \geqslant \mathrm{A} \rightarrow \mathrm{B} \quad \textit{iff} \quad \mathrm{i} + \mathrm{A} \geqslant \mathrm{B},$$

but this is only meant to be suggestive. In any case, if we assume the *numerical* meaning for "$k \geqslant i + j$" as the variables range over the integers (say, in the interval between 0 and 6), then we have a perfect correspondence with the table for $\rightarrow$ handed down to us by Łukasiewicz.[3]

This is hardly as exciting as getting the Ten Commandments, but it seems to make a small amount of sense. Truths (or should I say 'Falsehoods'?) are classified by degrees. The classification is taken to be 'numerical' in the respect of being able to check distances (by subtraction really) against other degrees. Thus we suppose not only a linear ordering but a little *linear algebra* of degrees. Is it completely nutty? I suppose so, but then again why not? Instead of unanalysed atomic propositions (p, q,

$r, \ldots$) could we not have atomic predicate symbols (R, S, $T, \ldots$) and individual constants ($a, b, c, \ldots$) for which an error estimate like:

$$i \geqslant Rab$$

was somehow 'natural'? This means that if we are 'careful' up to i (say, in adjusting our tools), then the test for truth will be positive. If we are more careful or more accurate, then the test may *not* be positive. What is rather too idealized for this story is the assumption that the test will always turn out the same if done with the same accuracy. But if we could live with that assumption, it would seem that there is some chance that this logic is of some use. But it would take very much more development of a kind *not* to be found in the literature at the present time. And why just Łukasiewicz connectives? In this spirit there might be others that were just as interesting.

Since we have no convincing evidence of a use for the Łukasiewicz system to present here as a logic of error, we shall not detail the formal system. However, it should be mentioned that the multiple entailment relation

$$A_0, A_1, \ldots, A_{n-1} \vdash B_0, B, \ldots, B_{m-1}$$

has a simple interpretation by degrees:

whenever $i \geqslant A_t$ *for all* $t < n$, *then* $i \geqslant B_u$ *for some* $u < m$.

The axiomatic treatment of the logic with rules expressed in terms of $\vdash$ seems quite satisfactory, though some problems yet remain.[4]

2 A LOGIC OF RISK?

The second idea for a use for many-valued logic is due to Robin Giles.[5] He would no doubt prefer to talk of a logic of *commitment*, but being less optimistic (as in the case of error) I shall use the word 'risk'. The approach is game-theoretic in

nature.[6] A player *risks* something in asserting a proposition, and we make this risk *cumulative* in the sense that multiple assertion of a proposition carries a corresponding multiplication of the risk. Instead of the material entailment relation we had in the degree interpretation, we shall have a cumulative entailment which we write as:

$$A_0, A_1, \ldots, A_{n-1} \Vdash B_0, B_1, \ldots, B_{m-1}$$

Intuitively speaking this means that a certain position of the game is *safe* for *me*. (*You* are the other player, by the way.) I will let you go first, if you wish. You go ahead and assert *all* the A's, in the meantime I will assert *all* the B's. We will have each risked something, and we will have had to pay each other what we lost by making these assertions. The position was *safe* for me because the loss incurred by me was *less than* your loss, so I have a *net gain* (or at least no loss in case we came out equal).

One might well ask what this has to do with logic and with many-valued logic in particular. The first answer is that there are simple rules about how to *deduce* that a position is safe given that certain others are safe. The table of inference rules illustrates this. The notation is to be read as follows. The Greek letters stand for *sequences* of formulae. The *tilde* (as in '$\tilde{\Gamma}$') means a *permutation* of the sequence. The *comma* (as in "Γ, A") means the *adjunction* of the indicated formulae. The *multiple* (as in '$n.\Gamma$') means the *repetition* of all the formulae the indicated number of times. The symbol '$\varnothing$' is the name of the *empty* sequence.

INFERENCE RULES FOR $\Vdash$

(Z) $\varnothing \Vdash \varnothing$

(A) $$\frac{\Gamma \Vdash \Delta}{\Gamma, A \Vdash \Delta, A}$$

(M) $$\frac{\Gamma \Vdash \Delta}{\Gamma, A \Vdash \Delta}$$

(P) $$\begin{array}{c} \Gamma \Vdash \Delta \\ \Gamma \Vdash \tilde{\Delta} \end{array}$$

$$\text{(T)}\quad \frac{\Gamma \Vdash \Delta \qquad \Delta \Vdash \Theta}{\Gamma \Vdash \Theta}$$

$$\text{(S)}\quad \frac{\Gamma, A \Vdash \Delta, A}{\Gamma \Vdash \Delta}$$

$$\text{(D)}\quad \frac{n.\Gamma \Vdash n.\Delta}{\Gamma \Vdash \Delta}\ \text{(where } n>0\text{)}$$

It can be shown[7] that in the case of a set of unstructured atomic formulae, a relationship:

$$A_0, A_1, \ldots, A_{n-1} \Vdash B_0, B_1, \ldots, B_{m-1}$$

follows from the stated rules if and only if *for all* numerical evaluation functions $\#$ that map (atomic) formulae to *non-negative* real numbers it is the case that:

$$\#A_0 + \#A_1 + \ldots + \#A_{n-1} \geqslant \#B_0 + \#B_1 + \ldots + \#B_{m-1}$$

Thus the rules, in this simple case, generate exactly the positions safe for *all* games. The same would hold if we took certain given entailments as *axioms*. We would then make deductions from them, and we would use only those $\#$ that made them safe.

So far this is not especially interesting. What Giles noticed is that the *connective* of implication can be given a game-theoretic characterization based on this very natural rule:

> *He who asserts* A→B *agrees to assert* B *if his opponent will assert* A.

The table expresses this idea with entailments.

RULES FOR→

$$(\to)\quad \frac{\Gamma \Vdash \Delta \qquad \Gamma, A \Vdash B, \Delta}{\Gamma \Vdash A \to B, \Delta} \qquad \frac{\Gamma, B \Vdash A, \Delta}{\Gamma, A \to B \Vdash \Delta}$$

With hindsight it is not surprising that such rules characterize

Łukasiewicz implication because the operation in any case is just numerical *subtraction*, in the sense that:

$$\#(A \to B) = \#B - \#A$$

provided:

$$\#B \geqslant \#A,$$

otherwise we have:

$$\#(A \to B) = 0.$$

(The two cases here, by the way, technically correspond to the two premisses needed in the first rule for →.) The real point is that *all* the rules are very simple when phrased in the game-theoretic language, and so Giles will argue that on *intuitive grounds* we are led to the Łukasiewicz system. The argument will be even better if Giles can convince us that such entailments are natural for, say, physical theories (his original motivation), *and* if we can find ways of using the formalism to give 'neat' axiomatic theories. This latter will require much more investigation of quantifier calculus, but at least now we have some reason to try. Further, I feel that we can also see definite hope for the usefulness of this kind of logic which was not clear from the original presentation.

NOTES

1. An extensive bibliography can be found in N. Rescher, *Many-valued Logic*, McGraw-Hill, 1969.
2. See 'Background to formalization' in *Truth, Syntax and Modality* (H. Leblanc, ed.), North Holland, 1972.
3. The *form* of this semantical interpretation, which is very much like possible-worlds semantics for modal logic is due independently to the author and A. Urquhart. See for a fuller discussion and for references D. Scott, 'Completeness and axiomatizability in many-valued logic' in *Tarski*

Symposium (L. Henkin, et al., eds.), *Amer. Math. Soc.*, 1974 (in press).
4. More discussion and details are to be found in the papers cited in notes 2 and 3.
5. The papers by Giles are not yet published. The author learned about the approach from two preprints: 'A non-classical logic for physics' (1972) and 'Physics and logic' (1973).
6. The method is very close to that advocated by J. Hintikka, *Logic, Language Games and Information*, Oxford, 1973.
7. The method of proof is that in D. Scott, 'Measurement structures and linear inequalities', *J. Math. Psychology*, vol. 1 (1964), pp. 233–47.

Comment

BY T. J. SMILEY

I agree with Professor Scott in thinking that there may be a use for many-valued logic, even though my own candidate—a use for 3-valued logic in connection with the theory of descriptions—would be different from his. Of his two candidates, he gives more space (if not more support) to the idea of a many-valued logic of error, in which the truth-values of Łukasiewicz's system would stand for different degrees of error. By a degree of error Scott does not mean a probability, and he rightly emphasizes that the logic of probability is not many-valued. I shall argue in reply that Scott's logic of error is not many-valued either, and that he is wrong about errors for the same sort of reason as he is right about probabilities. This leads me to scrutinize his choice of a scale of degrees of error, to see if we could derive a many-valued logic by postulating a different scale.

In §2 I shall say something about logical consequence, starting from the observation that two systems of many-valued logic may have identical truth-values and truth-tables and theorems and still differ over the inferences they count as valid. The systems Scott presents differ in this way from each other and from any system based on the conventional idea of 'designating' some of the truth-values; and I want in particular to compare the conventional idea with Scott's approach to many-valued logic in terms of 'valuations'.

1. A MANY-VALUED LOGIC OF ERROR?

Scott remarks that it only takes a moment's thought to see that the probability of 'if A, B' is not a function of the probabilities of A and B. But the same moment's thought shows that the degree of error of 'if A, B' is not a function of the degrees of error of A and B, for the point is the same in each case: we want to accept 'if A, A' without being forced to accept 'if A, B' for every B that happens to have the same probability or degree of error as A. And if this failure of 'if' to be probability-functional is an objection to a many-valued logic of probability, its failure to be error-functional must surely be an equal objection to a many-valued logic of error.

This objection is not insuperable. After all, isn't it the stock objection to the classical 2-valued logic that 'if' is not truth-functional? And isn't the stock rejoinder that even if the classical system cannot represent 'if', it can represent 'and' and 'not', and with these we can define a truth-functional substitute for 'if', viz. material implication. If this sort of rejoinder is unacceptable then its inability to represent 'if' will indeed be fatal to the claims of a many-valued logic of error (together with, let it be said, every other formal system in existence). So let us accept the rejoinder, and limit the question to the system's ability to represent conjunction and negation.

As a candidate for a many-valued logic of probability we