A New Category?

Domains, Spaces and Equivalence Relations

Dana S. Scott
School of Computer Science
Carnegie Mellon University

dana.scott@cs.cmu.edu

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1 Introduction.

The familiar categories SET and TOP, consisting of sets and arbitrary mappings and of topological spaces and continuous mappings, have many well known closure properties. For example, they are both *complete* and *cocomplete*, meaning that they have all (small) limits and colimits. They are *well-powered* and *co-well-powered*, meaning that collections of subobjects and quotients of objects can be represented by sets. They are also nicely related, since SET can be regarded as a *full subcategory* of TOP, and the *forgetful functor* that takes a topological space to its underlying set *preserves limits and colimits* (but reflects neither).

The category SET is also a *cartesian closed category*, meaning that the *function-space construct* or the *internal hom-functor* is very well behaved, in the sense that the functor $\cdot \times B$ is adjoint to $B \rightarrow \cdot$ for all objects B. However, it has been known for a long time that in TOP no such assertion is available, because in general it is not possible to assign a topology to the set of continuous functions making this adjointness valid—except under some special conditions on the space B. Many remedies have been proposed, notably, (a) cutting down to *compactly generated spaces*, or (b) expanding the category to the category of *filter spaces* (or a related kind of *limit space*). These are interesting suggestions, but both have some drawbacks. Suggestion (a) applies only to Hausdorff spaces, and suggestion (2)—which the author considers the more interesting—introduces very unfamiliar spaces at the higher types (*i.e.*, after iterating the function-space construct several times). It remains to be seen whether the suggestion of this paper can be regarded as more concrete or more helpful.

The author's solution to the problem of cartesian closedness is motivated by domain theory. The (new?) category is formed from the category TOP_0 of topological T_0-spaces by using spaces together with *arbitrary* equivalence relations, to form the category, to be called called EQU, where the mappings are (suitable equivalence classes of) continuous mappings which preserve the equivalence relations. (A more precise definition will be given below.) Let us call these spaces *equilogical spaces* and the mappings *equivariant*. It seems surprising that this category has not been noticed before—if in fact it has not. It is easy to see that EQU is complete and cocomplete and that it embedds TOP_0 as a full and faithful subcategory (by taking the equivalence relation to be the identity relation).

What is perhaps not so obvious is that EQU is indeed *cartesian closed*. The proof of cartesian closedness outlined here uses old theorems in domain theory originally discovered by the author: in particular, an injective property of algebraic lattices treated as topological spaces and the fact that they form a cartesian closed category (along with continuous functions)[1]. Of course, algebraic lattices are just one of many cartesian closed categories proposed for domain theory—and not the most popular one. They allow, however, for some helpful embeddings of T_0-spaces.

For a long time the author has been distressed that there are *too many* proposed categories of domains and that their study has become too arcane. He hoped that the idea of *synthetic domain theory* would be the natural solution—but that theory has been slowed by many technical problems. The related idea of *axiomatic domain theory* is likewise hampered by the need to overcome technical difficulties. Despite very good work in both these directions a final theory has not emerged—in the author's opinion. Perhaps some of the ideas that have been used in these other ap-

[1] A more abstract, categorical proof can be found in the paper by Birkedal et al. to be found at our WWW site mentioned below

proaches can be transplanted to the study of EQU, which seems to be a rich and fairly natural category with many subcategories. The basic idea is to establish a typed λ-calculus once and for all, and then to single out useful types (or domains) by means of special properties—just as is done in several other branches of mathematics[2].

2 T_0-Spaces and Algebraic Lattices.

Topological spaces will be considered as structures $\mathcal{T} = \langle T, \Omega_\mathcal{T}\rangle$, where T is the set of points of the space, and where $\Omega_\mathcal{T}$ is the set of open sets of $\mathcal{T}$. We shall often write $|\mathcal{T}| = T$, so as not to have to use a special letter for the points of a space.

2.1 Definition. The *neighborhood filter* of a point $x \in |\mathcal{T}|$ of a topological space $\mathcal{T}$ is defined by the equation:

$$\mathcal{T}(x) = \{U \in \Omega_\mathcal{T} \mid x \in U\}.$$

2.2 Definition. The spaces we shall be concerned with are the T_0-spaces, where the topology distinguishes the points. More precisely, a topological space is a T_0-*space* provided that for every pair of distinct points there is an open set that contians one but not the other. Another way to say this condition is to say that for all $x, y \in |\mathcal{T}|$ it is the case that

$$\mathcal{T}(x) = \mathcal{T}(y) \text{ always implies } x = y.$$

The category of all such spaces and continuous mappings between them is denoted by TOP_0.

2.3 Definition. The *specialization ordering* of a topological space $\mathcal{T}$ is defined by:

$$x \leq_\mathcal{T} y \text{ if, and only if, } \mathcal{T}(x) \subseteq \mathcal{T}(y),$$

for all $x, y \in |\mathcal{T}|$.

[2] Along with my students Andrej Bauer, Lars Birkedal, and Jesse Hughes, and our colleague Steven Awodey, we have started on this project, for the time being concentrating on countably based spaces and the connections with interpretations of type theory and higher-order intuitionistic logic by the realizability method.

It is obvious from the definitions that T_0-spaces are *partially ordered* by the specialization ordering. What may not be so obvious to those who have been concerned mostly with Hausdorff spaces is that the specialization ordering can be interesting. For example, a very simple argument shows that *any* given partial ordering can be reobtained from the topology that uses the same set of points and takes as open sets the *upward-closed* sets of elements. More interesting spaces come from complete lattices.

Complete lattices will be considered as structures $\mathcal{L} = \langle L, \leq_{\mathcal{L}} \rangle$, where $\leq_{\mathcal{L}}$ partially orders L and where every subset of L has a least upper bound (equivalently, a greatest lower bound). We write $|\mathcal{L}| = L$, and we use the usual notation for lattice operations (namely, $\vee, \bigvee, \wedge, \bigwedge$). In order to distinguish a special class of complete lattices, the algebraic lattices, we need to pick out special elements of a lattice.

2.4 Definition. A *compact* (sometimes "finite" or "isolated") element of a complete lattice $\mathcal{L} = \langle L, \leq_{\mathcal{L}} \rangle$ is an element e such that whenever $e \leq_{\mathcal{L}} \bigvee S$ for a subset $S \subseteq L$, then $e \leq_{\mathcal{L}} \bigvee S_0$ for a *finite* subset $S_0 \subseteq S$. We denote the set of compact elements of the lattice $\mathcal{L}$ by $\mathcal{K}(\mathcal{L})$.

It is easy to show that the least upper bound of a finite collection of compact elements is again compact. In particular, the least element $\perp_{\mathcal{L}}$ of a complete lattice $\mathcal{L}$ is always compact—but no other elements need be in general. In a finite lattice, all elements are compact. In the lattice $\mathcal{P}A$ of all subsets of a set A, the compact elements are exactly the finite subsets. Algebraic lattices are distinguished as complete lattices with "enough" compact elements.

2.5 Definition. A complete lattice $\mathcal{L}$ is *algebraic* provided that every element is determined by the compact elements it contains. This means that $x, y \in |\mathcal{L}|$ and $\forall e \in \mathcal{K}(\mathcal{L})$**.** $e \leq_{\mathcal{L}} x \Leftrightarrow e \leq_{\mathcal{L}} y$, always imply $x = y$.

It should be noted that in an algebraic lattice, every element is the least upper bound of the compact elements it contains. The typical examples of algebraic lattices come from the study of algebraic systems (hence, the name). More abstractly we define the well known notion of a closure system. We need a preliminary definition.

2.6 Definition. A *directed subset* of a partially ordered set is a subset D such that every finite subset of D has an upper bound in D.

2.7 Definition. An *algebraic closure system* on a set A is a family $\mathcal{C} \subseteq \mathcal{P}A$ closed under **(i)** arbitrary intersections (but note we must set the empty intersection $\bigcap \varnothing = A$), and **(ii)** unions of directed subfamilies.

The most typical examples of algebraic closure systems are the family of subgroups of a group, the family of ideals of a ring, and the family of subspaces of a linear vector space. The downward closed subsets of a partially ordered set form an algebraic closure system, as do the downward closed, directed subsets. The deductively closed sets of formulae of a logical system with finitary inference rules also form an algebraic closure system. However, the closed subsets of a topological space in general *do not* form an algebraic closure system (failing condition (ii)). We also note that the intersection of any number of algebraic closure systems is again such a system (all on a common fixed set).

Associated with an algebraic closure system $\mathcal{C} \subseteq \mathcal{P}A$ is a *closure operator*, where the closure of a subset $X \subseteq A$ is the intersection of all sets in $\mathcal{C}$ containing the set X. The sets in $\mathcal{C}$ are then just the "closed" sets equal to their own closures.

An algebraic closure system is—by definition—a complete lattice (under the set-inclusion relation). It is always an algebraic lattice, since an easy argument shows that the compact elements are the closures of finite sets, and that there are enough of them. Conversely, a well known argument also shows that every abstract algebraic lattice $\mathcal{L}$ is *isomorphically represented* as an algebraic closure system: one takes the underlying set to be the set $\mathcal{K}(\mathcal{L})$ of compact elements and takes the collection of downward-closed, directed sets of compact elements as the algebraic closure system. Hence, thinking about algebraic lattices is the same as thinking about algebraic closure systems—from the point of view of the partial ordering.

What is the connection between lattices and T_0-spaces? We are going to show (in a well known proof) why every complete lattice is such a space, and why the specialization ordering is the original partial ordering.

2.8 Definition. Let $\mathcal{L}$ be a complete lattice. The Σ-*topology* on the lattice is defined as the collection of all upward closed subsets $U \subseteq |\mathcal{L}|$ such that whenever $S \subseteq |\mathcal{L}|$ and $\bigvee S \in U$, then $\bigvee S_0 \in U$ for some finite subset $S_0 \subseteq S$. The collection of all such subsets is denoted by $\Sigma_{\mathcal{L}}$.

2.9 Theorem. *Given a complete lattice $\mathcal{L}$, the structure $\langle |\mathcal{L}|, \Sigma_{\mathcal{L}} \rangle$ is a T_0-space whose specialization ordering is exactly $\leq_{\mathcal{L}}$.*

Proof: That $\Sigma_{\mathcal{L}}$ is closed under arbtrary unions and finite intersections is very easy to check from the definition (but notice that the upward-closedness property has to be invoked). To check the T_0-property, one has only to remark that if $x \in |\mathcal{L}|$, then we have

$$\{y \in |\mathcal{L}| \mid y \not\leq_{\mathcal{L}} x\} \in \Sigma_{\mathcal{L}}.$$

Since the open sets in $\Sigma_{\mathcal{L}}$ are upward closed, then $\leq_{\mathcal{L}}$ is included in the specialization ordering. But the particular open sets displayed above are sufficient to show that the inclusion of the specialization ordering in $\leq_{\mathcal{L}}$ holds as well. **Q.E.D.**

Having established a topology, we can turn complete lattices into a subcategory of the category of T_0-spaces.

2.10 Definition. The category *CLat* of complete lattices consists of such lattices and the functions that are continuous under the Σ-topology.

For complete lattices $\mathcal{L}$ and $\mathcal{M}$, we remark that continuity of a function $f : |\mathcal{L}| \to |\mathcal{M}|$ can be defined in lattice-theoretic terms as a mapping where

$$f(\bigvee_{\mathcal{L}} D) = \bigvee_{\mathcal{M}} f(D),$$

for all directed $D \subseteq |\mathcal{L}|$. Note, too, that continuous functions between complete lattices are *monotone* in the sense of preserving the underlying partial orderings.

A "classical" result about monotone functions and complete lattices concerns fixed points.

2.11 Theorem. [Knaster-Tarski-Davis-Kleene] *A lattice $\mathcal{L}$ is complete if, and only if, every monotone self-function $f : |\mathcal{L}| \to |\mathcal{L}|$ has a fixed point. In a complete lattice the partially ordered set of fixed points of a self-function form a complete lattice, and the least and greatest fixed points are, respectively,*

$$\bigwedge\{x \in |\mathcal{L}| \mid f(x) \leq_{\mathcal{L}} x\} \text{ and } \bigvee\{x \in |\mathcal{L}| \mid x \leq_{\mathcal{L}} f(x)\}.$$

In case the function f is continuous, the least fixed point is given by

$$\bigvee_{n=0}^{\infty} f^n(\perp_{\mathcal{L}}).$$

For the powerset spaces $\mathcal{P}A$ the Σ-topology is very easy to describe: the open sets $\mathcal{U} \subseteq \mathcal{P}A$ are the families of "finite character"; that is, a subset $X \subseteq A$ belongs to $\mathcal{U}$ if, and only if, some finite subset of X belongs to $\mathcal{U}$. This is the same as giving $\mathcal{P}A$ the topology that corresponds to the product topology on 2^A where the two-element set has the topology with one open point and one closed point. The powerset spaces have an important role as being able to embed every T_0-space.

2.12 Theorem. [The Embedding Theorem] *Given a T_0-space $\mathcal{T}$, the mapping $x \mapsto \mathcal{T}(x)$ is a topological embedding of $\mathcal{T}$ into $\mathcal{P}\Omega_{\mathcal{T}}$ considered as a space with the Σ-topology.*

Proof: This is a well known, easy theorem. We have already remarked that the mapping is an injection. To show that it is continuous, we only have to show that the inverse image of the elementary open subsets of $\mathcal{P}\Omega_{\mathcal{T}}$ are open in $\mathcal{T}$. In fact, suppose that $\mathcal{U} = \{U_0, \ldots, U_{n-1}\}$ is a finite subset of $\Omega_{\mathcal{T}}$. This determines as an open subset of $\mathcal{P}\Omega_{\mathcal{T}}$ the following set:

$$\{\mathcal{X} \subseteq \Omega_{\mathcal{T}} \mid \mathcal{U} \subseteq \mathcal{X}\}.$$

All open sets are unions of such sets. The inverse image of the above set is the set

$$\{x \in |\mathcal{T}| \mid \mathcal{U} \subseteq \mathcal{T}(x)\}.$$

But this set is equal to $U_0 \cap \cdots \cap U_{n-1}$, which is obviously open. Thus, the mapping is indeed continuous.

It remains to be proved that inverse mapping from the subspace of $\mathcal{P}\Omega_{\mathcal{T}}$ to the given space $\mathcal{T}$ is also continuous. This amounts to showing that the *image* of an open set in $\Omega_{\mathcal{T}}$ is open in the subspace. Let $U \in \Omega_{\mathcal{T}}$. Its image under the embedding is

$$\{\mathcal{T}(x) \mid x \in U\} = \{\mathcal{T}(x) \mid x \in |\mathcal{T}| \text{ and } U \in \mathcal{T}(x)\}.$$

But this last set is indeed the intersection of the image of the space with the set $\{\mathcal{X} \subseteq \Omega_{\mathcal{T}} \mid U \in \mathcal{X}\}$, which is an open set of $\mathcal{P}\Omega_{\mathcal{T}}$. **Q.E.D.**

Powerset spaces also have another important property concerning continouous functions which is key for the definitions we will give in the next section.

2.13 Theorem. [The Extension Theorem] *If $\mathcal{X}$ is a subspace of a topological space $\mathcal{Y}$, and if $f : |\mathcal{X}| \to \mathcal{P}A$ is continuous, then the function f has a continuous extension to all the points of $\mathcal{Y}$.*

Proof: This theorem was noticed by the author in 1970/71. One proof remarks that the conclusion is obvious in case the set A is a *one-point set*. Since a continuous function $f : \mathcal{X} \to \mathcal{PA}$ is basically just an open subset of $\mathcal{X}$, we just have to recall that every open subset of a subspace is a restriction of (or extends to) an open subspace of a superspace. Then, the theorem must remain true if the range space is replaced by a product of powerset spaces. Since, as a space, $\mathcal{P}A$ is homeomorphic to the product of the spaces $\mathcal{P}\{a\}$ for $a \in A$, the theorem follows for $\mathcal{P}A$.

Another proof gives directly a formula for the extension of the function $f : |\mathcal{X}| \to \mathcal{P}A$ to a function $\bar{f} : |\mathcal{Y}| \to \mathcal{P}A$:

$$\bar{f}(y) = \bigcup\{\bigcap\{f(x) \mid x \in U \cap |\mathcal{X}|\} \mid U \in \mathcal{Y}(y)\},$$

for all $y \in |\mathcal{Y}|$. Of course, it has to be verified that this (well defined) function $\bar{f}$ is a continuous extension of f. **Q.E.D.**

The author also pointed out that the Extension Theorem holds for all continuous retracts of the powerset spaces—these are the continuous lattices. We do not need to have their general theory reviewed here, inasmuch as we can make the necessary later definitions with a special class of retracts—giving the algebraic lattices. Remember that every algebraic closure system $\mathcal{C} \subseteq \mathcal{P}A$ has an associated closure operation $C : \mathcal{P}A \to \mathcal{P}A$. It is not difficult to show that C is a continuous retract. More generally we can consider the class of closure operations as in the following definition.

2.14 Definition. A *continuous closure operation* $C : \mathcal{P}A \to \mathcal{P}A$ is a continuous function such that $\mathsf{I}_{\mathcal{P}A} \subseteq C = C \circ C$.

In the above definition we use the notation of the identity function and the relationship $\mathsf{I}_{\mathcal{P}A} \subseteq C$ as a shorthand for the condition that $X \subseteq C(X)$

holds for all subsets $X \subseteq A$. The main point here is that the above operators give us all algebraic closure systems (and, hence, all algebraic lattices) as A and C vary.

2.15 Theorem. *The set of fixed points of a continuous closure operation is equal to the range of the function and forms an algebraic closure system.*

Proof: The equation $C = C \circ C$ implies that every value of the function is a fixed point (the converse is obvious). Now, the fixed points of a continuous functions in general are closed under directed unions, because the function is continuous.

Suppose now $C : \mathcal{P}A \to \mathcal{P}A$ is a monotone closure operation (continuity need not be invoked again). Suppose that $\{X_i \mid i \in I\}$ is any collection of fixed points. By assumption we know

$$\bigcap_{i \in I} X_i \subseteq C(\bigcap_{i \in I} X_i).$$

But, since, for each $j \in I$, we have

$$\bigcap_{i \in I} X_i \subseteq X_j,$$

we conclude

$$C(\bigcap_{i \in I} X_i) \subseteq C(X_j) = X_j.$$

Therefore,

$$C(\bigcap_{i \in I} X_i) \subseteq \bigcap_{i \in I} X_i.$$

This establishes closure under intersection. **Q.E.D.**

Now, it is simple matter to show that C is a continuous closure operation if, and only if, $C = \mathsf{I} \cup C \circ C$. We can conclude that those operations form a complete lattice by use of facts about the lattice of continuous functions we are about to define. A more refined argument shows that they actually form an algebraic lattice, but we do not need to refer to this interesting fact further here.

2.16 Theorem. *The function space of all continuous functions from one complete lattice $\mathcal{L}$ into another $\mathcal{M}$ again forms a complete lattice under the pointwise ordering of functions.*

Proof: We need some notation. Let $(\mathcal{L} \to \mathcal{M})$ be the structure with the underlying set of all continuous function, where, for $f, g : |\mathcal{L}| \to |\mathcal{M}|$ we define

$$f \leq_{(\mathcal{L}\to\mathcal{M})} g \Leftrightarrow \forall x \in |\mathcal{L}|.\ f(x) \leq_{\mathcal{M}} g(x).$$

To show that the partially ordered set $(\mathcal{L} \to \mathcal{M})$ becomes a complete lattice, we need to define least upper bounds. Let F be a set of continous functions. We introduce the function

$$(\bigvee_{(\mathcal{L}\to\mathcal{M})} F)(x) = \bigvee_{\mathcal{M}} \{f(x) \mid f \in F\},$$

for $x \in |\mathcal{L}|$. It is necessary first to show that this function is continuous (because it preserves least upper bounds of directed sets), and then it easily follows it is the least-upper-bound function in the function-space lattice. **Q.E.D.**

At the time of writing this first draft of this paper, the author is not at all certain whether there is a simple explaination of what the open subsets of $(\mathcal{L} \to \mathcal{M})$ are. All the calculations he has ever done in this space have been carried out using the lattice operations. This remark applies as well to the following theorem.

2.17 Theorem. *The category CLat of complete lattices and continuous functions is a cartesian closed category.*

Proof: (in outline). Basically, we have to show that, given three complete lattices $\mathcal{L}, \mathcal{M},$ and $\mathcal{N}$, there is a one-one correspondence between functions in the two spaces:

$$(\mathcal{L} \times \mathcal{M} \to \mathcal{N}) \text{ and } (\mathcal{L} \to (\mathcal{M} \to \mathcal{N})).$$

In fact this one-one correspondence is an isomorphism of lattices (and a homeomorphism of topological spaces). The nub of the argument is to check that a function of several variables on complete lattices to complete lattices is continuous in *each* of the variables *separately* if, and only if, it is continuous in the variables *jointly* (that is to say, continuous on the product space). This proves to be a direct calculation involving directed sets and least upper bounds. This fact needs to be combined with the fact that if a function f is in the space $(\mathcal{L} \times \mathcal{M} \to \mathcal{N})$, then the function $\Lambda(f)$ defined by

$$\Lambda(f)(x)(y) = f(\langle x, y \rangle),$$

for $x \in |\mathcal{L}|$ and $y \in |\mathcal{N}|$ is in $(\mathcal{L} \to (\mathcal{M} \to \mathcal{N}))$. **Q.E.D.**

Returning to algebraic lattices, there is a direct argument to show that the function space of two algebraic lattices is an algebraic lattice by finding a description of the compact elements of the function space. We give here another argument using algebraic closure operations that relates the function-space construction to powerset spaces.

2.18 Definition. For any set A, let A^* denote the disjoint union of all the finite cartesian powers; that is,

$$A^* = \bigcup_{n=0}^{\infty} A^n.$$

We use the notation $\langle x_0, \ldots x_{n-1}\rangle$ for elements of a power A^n.

2.19 Theorem. *The function space $(\mathcal{P}A \to \mathcal{P}B)$ is a retract of the powerset space $\mathcal{P}(A^* \times B)$ by a continuous closure operation.*

Proof: Given a continuous function $f \in (\mathcal{P}A \to \mathcal{P}B)$ we associate its "graph" $\Phi(f)$ by means of the formula

$$\Phi(f) = \{\langle\langle x_0, \ldots x_{n-1}\rangle, y\rangle \mid y \in f(\{x_0, \ldots x_{n-1}\})\}.$$

It is easy to check that Φ is continuous. Moreover there is a continuous function Ψ defined on $\mathcal{P}(A^* \times B)$ with values in $(\mathcal{P}A \to \mathcal{P}B)$ given by the formula

$$\Psi(F)(X) = \{y \mid \exists s \in X^*.\, \langle s, y\rangle \in F\},$$

for $F \subseteq A^* \times B$ and $X \subseteq A$. Inasmuch as a set is the directed union of its finite subsets, a continuous function $f \in (\mathcal{P}A \to \mathcal{P}B)$ is completely determined by its action on finite sets. This observation shows why we have

$$\Psi(\Phi(f)) = f.$$

In the other direction, it is easy to check that for $F \subseteq A^* \times B$ we have

$$F \subseteq \Phi(\Psi(F)).$$

In this way we find that the composition $\Phi \circ \Psi$ is a continuous closure operation whose range is isomorphic to the complete lattice $(\mathcal{P}A \to \mathcal{P}B)$.
Q.E.D.

The use of the closure operation in the above theorem shows that the function space $(\mathcal{P}A \to \mathcal{P}B)$ is always an algebraic lattice. Now any algebraic lattice $\mathcal{L}$ is isomorphic to the range of a continuous closure L on a powerset $\mathcal{P}A$, and another such lattice $\mathcal{M}$ is isomorphic to the range of a continuous closure M on another powerset $\mathcal{P}B$. We then check that the function space $(\mathcal{L} \to \mathcal{M})$ is isomorphic to the range of the mapping

$$F \mapsto M \circ F \circ L,$$

which proves to be a continuous closure operation on $(\mathcal{P}A \to \mathcal{P}B)$. Hence, the function space of algebraic lattices is again an algebraic lattice. Moreover, we conclude as an application of the previous theorem that the following holds.

2.20 Theorem. *The category ALat of algebraic lattices and continuous functions is a cartesian closed category.*

Though *ALat* is a very nice category, and though the function-space topology can be precisely described in simple and explicit terms, it is still the case that *ALat* is quite a "small" (full) subcategory of TOP_0, and that it is virtually completely disjoint with the important subcategory *SET*. What we are going to try to achieve is a cartesian closed category naturally including all these categories and possessing good properties proved by making use of what we have learned about the category *ALat*.

3 Equilogical Spaces.

We will give two definitions of the category of equilogical spaces and then show that the two categories are equivalent. We will then prove that the category is cartesian closed.

3.1 Definition. The category *EQU* of *equilogical spaces* consists of structures $\mathcal{E} = \langle |\mathcal{E}|, \Omega_{\mathcal{E}}, \equiv_{\mathcal{E}} \rangle$, where $\langle |\mathcal{E}|, \Omega_{\mathcal{E}} \rangle$ is a T_0-space and $\equiv_{\mathcal{E}}$ is an (arbitrary) equivalence relation on the set $|\mathcal{E}|$. The mappings between equilogical spaces are the equivalence classes of continuous mappings between the topological spaces that preserve the equivalence relation (*equivariant* mappings), where the equivalence relation on mappings is defined by

$$f \equiv_{\mathcal{E} \to \mathcal{F}} g \Leftrightarrow \forall x, y \in |\mathcal{E}|.\, x \equiv_{\mathcal{E}} y \Rightarrow f(x) \equiv_{\mathcal{F}} g(y).$$

We remark that it has to be proved that $\equiv_{\mathcal{E}\to\mathcal{F}}$ actually is an equivalence relation, but this is an elementary exercise. It also has to be proved that the equilogical spaces and equivariant maps form a category, but this can also be safely left to the reader.

One odd feature of this definition is that the equivalence relation of an equilogical space may have very little to do with the topology. This means that in some cases the only equivariant mappings between two spaces might be the constant maps, or the only automorphisms of a given space might be the identity—despite a rich underlying topology. Thus, future investigations may suggest limiting the equivalence relations. But, for now, the general properties of the category seem to work out well for arbitrary equivalence relations, so we have not been motivated to make any restrictions.

3.2 Theorem. *The category EQU is complete, co-complete, regular well-powered, and regular co-well-powered* [3]*.*

Proof: The proof proceeds along standard lines making use of the corresponding properties of topological spaces.

Take *products* first. The product (of any number) of topological spaces is a space with a product topology. The product of equivalence relations is an equivalence relation. The projection mappings are clearly equivariant. And, if we have a family of (equivalence classes of) equivariant mappings into the various factor spaces, then (after applying the Axiom of Choice to pick representatives) we can obtain in the usual way *one* equivariant mapping into the product that combines all the separate mappings.

Next, take *equalizers*. Suppose $f, g : |\mathcal{E}| \to |\mathcal{F}|$ are two (representatives of) equivariant mappings. Form the set $\{x \in |\mathcal{E}| \mid f(x) \equiv_{\mathcal{F}} g(x)\}$. Endow this set with the subspace topology and with the restriction of the equivalence relation $\equiv_{\mathcal{E}}$. This structure, along with the obvious inclusion mapping into $\mathcal{E}$, is the desired equailizer. Thus, *EQU* is a complete category.

On to *coproducts*. The coproduct of topological spaces is just a disjoint union of the underlying sets with the topology on the union generated by

[3]The author is indebted to Peter Johnstone for pointing out that, contrary to the assertion made in the first draft of this paper, *EQU* is *not* well powered, for there are fairly simple examples of objects in the category with an unbounded number of non-isomorphic subobjects.

the union of all the topologies. For equivalence relations, we have only to note that the union of equivalence relations on disjoint sets is indeed an equivalence relation. The injection mappings from the separate spaces into the union are obvious, as well as is the lifting property of a family of mappings from the separate spaces into a given target space.

Next, we discuss *coequalizers*. Suppose $f, g : |\mathcal{E}| \to |\mathcal{F}|$ are two (representatives of) equivariant mappings. On $|\mathcal{F}|$ we form the least equivalence relation containing both $\equiv_{\mathcal{F}}$ and the set of pairs $\{\langle f(x), g(x)\rangle \mid x \in |\mathcal{E}|\}$. Using the same topological space as $\mathcal{F}$ does and this equivalence relation, we form the equilogical space $\mathcal{G}$. There is an obvious equivariant mapping $c : \mathcal{F} \to \mathcal{G}$. This structure $\mathcal{G}$, along with the mapping c, is the desired coequailizer. Thus, *EQU* is a cocomplete category.

The properties of being regular well-powered and regular co-well-powered follow from the corresponding properties of TOP_0 and the category of equivalence relations. But, one has to be careful to check that the regular subobjects are obtained by selecting some equivalence classes and taking the union of them to form a subspace; likewise, forming a regular quotient is just making the equivalence relation coarser (putting equivalence classes together). The trouble is that there are subobjects and quotients which are not formed in this simple way. **Q.E.D.**

The proof just given is sketchy in the handling of equivalence classes of maps, and in the construction of the equalizer and coequalizer it has to be checked that the structures suggested have the required universal properties. But, this argument—modulo equivalence classes—is exactly similar to what is done for the catgory TOP_0. We remark that the category of equivalence relations on sets is included here: a set is just a discrete topological space (and these form a full subcategory of TOP_0). Of course, with the aid of the Axiom of Choice, it is quickly shown that the category of equivalence relations is *equivalent* to the category of sets (*via* the obvious use of quotient sets). However, the category *EQU* introduced here is *not* equivalent to the category TOP_0. (For one thing, no topology is being put on the quotient space $|\mathcal{E}|/\equiv_{\mathcal{E}}$. And, if we are right about the properties of *EQU*, this category has a property—cartesian closure—that TOP_0 does not share.)

To investigate *EQU* further, we introduce a closely connected category.

3.3 Definition. The category *PEQU* of *partial equilogical spaces* consists of structures $\mathcal{A} = \langle |\mathcal{A}|, \Omega_{\mathcal{A}}, \equiv_{\mathcal{A}} \rangle$, where $\langle |\mathcal{A}|, \Omega_{\mathcal{A}} \rangle$ is the Σ-topology of an algebraic lattice, and where $\equiv_{\mathcal{A}}$ is a partial equivalence relation, reflexive only on a subset of $|\mathcal{A}|$. The mappings between partial equilogical spaces are the equivalence classes of continuous mappings between the algebraic lattices that preserve the partial equivalence relation, where the equivalence relation on mappings is defined by

$$f \equiv_{\mathcal{A} \to \mathcal{B}} g \Leftrightarrow \forall x, y \in |\mathcal{A}|.\ x \equiv_{\mathcal{A}} y \Rightarrow f(x) \equiv_{\mathcal{B}} g(y).$$

These mappings will also be called *equivariant*.

If we consider the relation $f \equiv_{\mathcal{A} \to \mathcal{B}} g$ as being defined between arbitrary continuous functions, then equivariant maps for the category *PEQU* are the (equivalence classes of) the functions f satisfying $f \equiv_{\mathcal{A} \to \mathcal{B}} f$, since that means that the function preserves the underlying equivalence relation. This remark gives a hint as to how we will define function spaces, but first we want to check the equivalence of categories.

3.4 Theorem. *The categories EQU and PEQU are equivalent.*

Proof: The naturally suggested functor from *PEQU* to *EQU* is the one that takes $\langle |\mathcal{A}|, \Omega_{\mathcal{A}}, \equiv_{\mathcal{A}} \rangle$ and restricts the topology to the subspace on the subset $\{x \in |\mathcal{A}| \mid x \equiv_{\mathcal{A}} x\}$. On this subset the equivalence relation is "total". The mappings are likewise restricted. Call the functor R (for "restriction"). Now, if $f : \mathcal{A} \to \mathcal{B}$ is a map of *PEQU*, then $R(f) : R(\mathcal{A}) \to R(\mathcal{B})$ is clearly valid as a map of *EQU*, and identities and compositions are preserved. (Admittedly, there is some equivalence-class fussing that has to go on here.)

We note first that the functor R is faithful *by definition*. Then, the functor R is full in view of *The Extension Theorem* (because continuous functions between T_0-spaces can be extended to any algebraic lattices embedding them). Finally, the functor R is *surjective* by virtue of The Embedding Theorem (and note that the equivalence relation on the T_0-space does not have to be extended but remains partial). This is enough to show that the categories are equivalent. **Q.E.D.**

The idea of partial equivalence relations is very widely employed. (The author believes he first called general attention to it in the late '60s after extracting it from the studies by G. Kreisel on extensional theories of higher-type functionals in recursion theory.) However, it has been mostly used recently in the context of giving types to (quotients of) subsets of a universal model of some sort. Allowing partial equivalence relations over a large category (such as algebraic lattices) is possibly a new idea. But, certainly, many familiar proofs get reused in the new context. The following theorem is an example of this reuse.

3.5 Theorem. *The category EQU is cartesian closed.*

Proof: In view of the previous theorem, we will show that *PEQU* is cartesian closed. Given structures $\mathcal{A}$ and $\mathcal{B}$ in *PEQU* we define the structure $(\mathcal{A} \to \mathcal{B})$ so that

(i) $|\mathcal{A} \to \mathcal{B}|$ is the set of continuous functions;

(ii) $\Omega_{\mathcal{A}\to\mathcal{B}}$ is the Σ-topology on this algebraic lattice;

(iii) $\equiv_{\mathcal{A}\to\mathcal{B}}$ is the partial equivalence defined previously.

As in the argument for complete lattices, we have to show that, given three structures in *PEQU*, say, $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, there is a one-one correspondence between functions in in the two spaces:

$$(\mathcal{A} \times \mathcal{B} \to \mathcal{C}) \text{ and } (\mathcal{A} \to (\mathcal{B} \to \mathcal{C})).$$

As we know, there is a particular one-one correspondence that is an isomorphism of the underlying algebraic lattices (and a homeomorphism of topological spaces). It only remains to show that the isomorphism preserves the partial equivalence relation on the compound space. This is a "self-proving" theorem, in the sense that once the question is stated it is just a matter of unpacking the definitions to finish it off. **Q.E.D.**

4 Some Subcategories.

We have already remarked that *SET* is embedded as a full and faithful subcategory of *EQU*, and the same goes for TOP_0. Every T_0-space is given the identity relation as its equivalence relation, and this produces the desired embedding into *EQU*. We have to be careful on how these embeddings are interpreted, however. They certainly preserve products and coproducts (and maybe even all limits and colimits), but they do not preserve function spaces. For *SET*, the classical theorem of Cantor tells us that function spaces (starting with a denumerable set such as $\mathbb{N}$, for example, and iterating) become very large in cardinality. This is not so in *EQU*. As regards TOP_0, one can define what it means for given spaces to have an exponential, and the embedding does preserve exponentials that exist.

4.1 Theorem. *For every infinite set A, the space (the algebraic lattice) of all continuous functions $(\mathcal{P}A \to \mathcal{P}A)$ is a retract of $\mathcal{P}A$ by means of a continuous closure operation. The same is true of the product space $(\mathcal{P}A \times \mathcal{P}A)$.*

Proof: We refer back to Theorem 2.19, which tells us that $(\mathcal{P}A \to \mathcal{P}A)$ is such a retract of $\mathcal{P}(A^* \times A)$. But the infinite set A is—by virtue of the Axiom of Choice—in a one-one correspondence with the set $A^* \times A$. Hence, the lattices $\mathcal{P}A$ and $\mathcal{P}(A^* \times A)$ are isomorphic. For the product space, we remark that $(\mathcal{P}A \times \mathcal{P}A)$ is isomorphic to $\mathcal{P}(A + A)$, and so isomorphic to $\mathcal{P}A$ as well. **Q.E.D.**

4.2 Theorem. *Given an infinite set A, the category of partial equivalence relations on $\mathcal{P}A$ is a sub-cartesian-closed full subcategory of PEQU.*

Proof: The idea is that in forming products and function spaces we always embed the necessary algebraic lattices as retracts of $\mathcal{P}A$, and we never have to go outside this space. **Q.E.D.**

Thus, starting with $\mathbb{N}$, the higher-type spaces $\mathbb{N} \to \mathbb{N}$, $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, $((\mathbb{N} \to \mathbb{N}) \to \mathbb{N}) \to \mathbb{N}$, and so on, are all represented *via* equivalence relations on subspaces of one space, $\mathcal{P}\mathbb{N}$, of the cardinality of the continuum.

Even better, if $\mathcal{T}$ is a T_0-space with a countable basis $\mathcal{N} \subseteq \Omega_\mathcal{T}$ for its topology, then the mapping $x \mapsto \mathcal{N} \cap \mathcal{T}(x)$ shows that $\mathcal{T}$ can be regarded as a subspace of $\mathcal{PN}$, which is isomorphic to $\mathcal{P}\mathbb{N}$, if $\mathcal{N}$ is infinite, and to a subspace, otherwise. Therefore, the countably based spaces in *EQU* form a sub-cartesian-closed full subcategory equivalent to a small category. And the same can be said of spaces where the topology has a bound in cardinality on the size of a basis. The whole category *EQU* is thus stratified by the cardinalities of the bases for the topologies. Note, too, that the finite spaces in *EQU* form an interesting sub-cartesian-closed full subcategory, which is probably rather different from just finite T_0-spaces.

The closed subspaces of algebraic lattices (under the Σ-topology) have been much studied. A closed subset of such a lattice is a set that is downward-closed (under the partial ordering) and additionally closed under the formation of least upper bounds of directed subsets. As partially ordered sets, they can also be characterized as algebraic, bounded-complete, directed-closed posets. It is well known that they form a cartesian closed category, where the maps in the category are continuous functions between closed subsets. The Σ-topology also works for them, and they are also a full subcategory of the T_0-spaces. The category is often called *BCPO*.

4.3 Theorem. *The category BCPO can be regarded as a sub-cartesian-closed full subcategory of EQU; i.e., there is a full and faithful functor from BCPO to EQU which preserves the ccc-structure.*

Proof: Working in *PEQU* we can clearly construe the category as being the algebraic lattices with a partial identity relation over a closed subset. If $\mathcal{A}$ and $\mathcal{B}$ are such structures, and if $\mathcal{A}_0$ and $\mathcal{B}_0$ are the corresponding closed subsets, then the function-space construction we are looking at takes all continuous functions between the algebraic lattices (which form an algebraic lattice), and puts a partial equivalence relation on the functions mapping $\mathcal{A}_0$ into $\mathcal{B}_0$ according to whether they restrict down to the same function on $\mathcal{A}_0$. This is a perfectly fine structure in *PEQU*, but it is not directly seen to be in the subcategory *BCPO*. We can call this structure $(\mathcal{A} \to \mathcal{B})$ in *PEQU*, however.

In *BCPO*, the function space is just the set of continuous functions from $\mathcal{A}_0$ into $\mathcal{B}_0$ with the pointwise partial ordering. This space, which we can call temporarily $(\mathcal{A}_0 \to \mathcal{B}_0)$, proves to be in *BCPO*. The Σ-topology has as a *subbase* for the open sets the sets of continuous functions of the form

$$\{f_0 \mid b \leq f_0(a)\},$$

where a is a compact element of $\mathcal{A}_0$, and b is a compact element of $\mathcal{B}_0$. The poset $(\mathcal{A}_0 \to \mathcal{B}_0)$ can be made into an algebraic lattice just by adding a top element, and so we can put ourselves into the image of *BCPO* in *PEQU*.

What we have to show is that the two functions spaces are isomorphic in *PEQU*.

Now, if f is in $(\mathcal{A} \to \mathcal{B})$, we map it to the restriction $f \upharpoonright \mathcal{A}_0$. By definition equivalent functions in $(\mathcal{A} \to \mathcal{B})$ go into equal functions. Conversely, if f_0 is a function in $(\mathcal{A}_0 \to \mathcal{B}_0)$, then we extend it to a continuous function $\bar{f}_0$ on all of the lattice part of $\mathcal{A}$ by setting the values to the top element of $\mathcal{B}$ outside the closed subset $\mathcal{A}_0$. To see that $f_0 \mapsto \bar{f}_0$ is continuous, we take compact elements a of $\mathcal{A}$ and b of $\mathcal{B}$, and we note that

$$b \leq \bar{f}_0(a) \Leftrightarrow a \notin \mathcal{A}_0 \text{ or } \ b \leq f_0(a).$$

Hence, the inverse image of a subbasic open subset of $(\mathcal{A} \to \mathcal{B})$ is either the whole space or a subbasic open subset of $(\mathcal{A}_0 \to \mathcal{B}_0)$. We have continuity, and the two mappings are clearly inverse to one another in *PEQU*. This establishes isomorphism. **Q.E.D.**

5 Acknowledgments and Questions.

The idea that there might be a cartesian closed category like *EQU* occurred to the author at the end of the Fall Semester 1996 while lecturing to a postgraduate class on Domain Theory. He would like to thank that class for being such a lively one and for asking so many questions[4]. Without the task of having to explain the many details and trying to make the motivation for the mathematical concepts convincing—in which he only partly succeeded—he would never have had the stimulation to pursue this direction. The idea in fact came up in the context of polymorphism, which brings us to our first question.

Question 1. Our arguments above show that the (small) category of all spaces in *PEQU* with the *same* underlying space $\mathcal{P}A$ over an infinite set A is cartesian closed. But, since these are just partial equivalence relations on a fixed set, they also form a lattice, where the greatest lower bounds are intersections of partial equivalence relations. By using a retraction of $\mathcal{P}A$ onto $\mathcal{P}A \to \mathcal{P}A$, we make $\mathcal{P}A$ into a model for the untyped λ-calculus

[4]The author would be grateful to receive any further answers, counterexamples, comments or objections by e-mail. Last year various detailed comments from Mike Barr, Peter Johnstone, Peter Freyd, Pino Rosolini, and Martin Escardo were most helpful.

which can be used to give the cartesian structure to the category. How good a model for polymorphism are the partial equivalence relations as types?

Comments. Actually, a very good model is obtained for polymorphism. This can be seen by looking at the construction of the realizability topos over $\mathcal{P}A$ and properties of "modest sets", which have been studied by a large number of authors over that last 15 years. The author's research group[5] is going to study this interpretation in particular detail for the case $\mathcal{P}\mathbb{N}$.

Question 2. Another popular category for domain theory is the category of *SFP* objects or the more general category of *bifinite* domains. These are certain inverse limits of finite posets. Since *EQU* is so complete, will these categories turn out to be sub-cartesian-closed full subcategories of *EQU* as well? What about other categories of domains (say, many of those mentioned in the "Domain Theory" survey of Abramsky and Jung)?

Comments. We have not had time to look into the various categories in detail, but it still seems very likely that these will be well behaved subcategories. So, the advantage we hope to find is being to to put them all into one category.

Question 3. Can the powerset spaces, $\mathcal{P}A$ with the identity equivalence, be defined in *EQU* by a condition expressed entirely in category-theoretical terms? What about the subcategories of algebraic lattices and of continuous lattices? What about *BCPO*? What about *SET*? What about TOP_0?

Comments. Andrej Bauer has checked that the projective objects of *EQU* are just those isomorphic to sets, and the regular projectives are the isomorphs of T_0-spaces. There are indeed going to be characterizations of those other subcategories, but just which are going to be "memorable" needs further study.

Question 4. Which functors have fixed points? (These are solutions to

[5] Progress on our project can be checked on the WWW at the following site: `http://www.cs.cmu.edu/~birkedal/ltc/`. We have also recorded in the thesis proposals of Bauer, Birkedal, and Hughes many more questions for future work than can be explained in the present document.

"domain equations".) Which functors have initial or final algebras?

Comments. This question can be vexing. We know many examples and we know sufficient conditions on functors. We know $\mathcal{D} \cong \mathcal{D} \to \mathcal{D}$ has many solutions (in known subcategories). But it seems quite difficult to find *all* solutions. See the next question.

Question 5. Can Peter Freyd's theory of fixed points of functors of mixed variance be applied to *EQU*? In particular, when $\mathcal{X}$ is a structure in *EQU* with $\mathcal{X} \to \mathcal{X}$ a (special kind of?) retract of $\mathcal{X}$, can this space always be embedded in a space $\mathcal{D} \cong (\mathcal{D} \to \mathcal{D})$?

Question 6. Which structures in *EQU* satisfy the fixed-point property (fpp) for all self-maps? Or is it better to look at the structures all of whose powers satisfy the fixed-point property (the generalized fpp)? The author thought he had a proof once that they would form a cartesian closed full subcategory. Do they? How good is this subcategory for domain theory?

Comments. Yes, we checked this. The spaces in *EQU* satisfying the generalized fpp not only form a cartesian closed subcategory, but they are (by definition) an exponential ideal. The category of Scott domains (closed subspaces of algebraic lattices) and continuous Scott domains (closed subspaces of continuous lattices) also form exponential ideals. Satisfying the generalized fpp is definitely more general, however; but, it seems hard to say exactly how much more general.

Question 7. Many of the proofs indicated in the previous sections required the Axiom of Choice. Can *EQU* be defined in any topos so as to have good properties? Does the construction give an interesting computability theory in the effective topos?

Comments. We have not looked at *EQU* in toposes other than *SET*. But, the realizability interpretion over $\mathcal{P}\mathbb{N}$ leads to finding connections between three toposes which include very satisfactory information on computability. We feel we can thus combine theories of continuity over standard (countably based) spaces with a theory of computability in a common logic giving a basis for type theory and topos logic[6].

[6] See the WWW site mentioned earlier for our detailed plans and progress reports. The papers cited there (availble by ftp) also have many references to other relevant work.

Question 8. The real numbers naturally form a space $\mathbb{R}$ in *EQU*. Is the space $(\mathbb{R} \to \mathbb{R})$ in *EQU* a topological space (say, the continuous function space with the compact-open topology)? And how can we describe higher-type spaces such as $(\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$, and so on? What is the connection to the new theory of "total functions" recently put foward by Ulrich Berger? Is the (definite) Riemann integral

$$\int : \mathbb{R} \times \mathbb{R} \times (\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$$

a map in *EQU*?

Comments. Andrej Bauer has proved that, in view of known theorems on exponentials in TOP_0, $(\mathbb{R} \to \mathbb{R})$ in *EQU* is indeed topological: the space of continuous real functions with the so-called compact-open topology. It is much harder to say what $(\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$ is, because it is not topological. The work of Abbas Edalat can be invoked to settle positively the question about integration. Moreover, integration is a computable mapping. Bauer has also clarified the connections with the work of Berger[7].

[7]See the discussions of the concepts in the thesis proposal of Bauer available on our WWW site.