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AXIOMATIZING SET THEORY

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As long as an idealistic manner of speaking about abstract objects is popular in mathematics, people will speak about collections of objects, and then collections of collections, and collections of collections of . . . of collections. In other words, *set theory is inevitable.* Let us not argue here the question of whether set theory is the ultimate foundation for mathematics. Surely it has already been shown to be an adequate foundation for a reasonably large part of mathematics. Furthermore, even though it may not be the last word, it is a simple theory of perfectly clear intent which would have to be interpretable in whatever theory finally would serve as our foundation. It is the purpose of this essay to demonstrate just how inescapable set theory is by re-examining its axioms.

It must be understood from the start that Russell's paradox is *not* to be regarded as a disaster. It and the related paradoxes show that the naive notion of all-inclusive collections is untenable. That *is* an interesting result, no doubt about it. But note that our original intuition of set is based on the idea of having collections of *already* fixed objects. The suggestion of considering all-inclusive collections only came in later by way of formal simplification of language. The suggestion proved to be unfortunate, and so we must return to the primary intuitions. These intuitions can gain an initial precision through formulating the two basic axioms of *extensionality* and *comprehension*, which we now discuss in detail.

Let the variables $a, b, c, a', b', c', a'', \ldots$ range over *sets* and the variables $x, y, z, x', y', z', x'', \ldots$ range over *arbitrary* objects. The symbol $=$ is used for *identity* and $\in$ for *membership*. Whether it is really interesting or profitable to allow for non-sets in the theory is debatable; but let us not exclude them yet. We agree that the condition $x \in y$ should imply that y is a set, a principle that we can formulate in logical symbols thus:

$$\forall x, y[x \in y \to \exists a[y = a]].$$

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Of course this must be taken as an axiom; but it is so primitive, so much just a convention of grammar, that we will not even give it a name. The first axioms to be named are the following:

EXTENSIONALITY. $\forall a, b[\forall x[x \in a \leftrightarrow x \in b] \rightarrow a = b]$.

COMPREHENSION. $\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \Phi(x)]$.

The extensionality axiom formalizes our idea that a set is nothing more than a collection of objects: It is uniquely determined by its elements. The comprehension axiom formalizes the idea that once a collection a is fixed, we can then extract from a any arbitrary subcollection b. The extraction process is effected by finding a property $\Phi(x)$ which distinguishes b as the subset of a comprehending all those elements having the property. There is no reason to place any restriction on how the property $\Phi(x)$ is formulated: We believe in the existence of arbitrary subsets. It is a great temptation to erase the condition $x \in a$, thus simplifying the axiom schema; but we all know what happens. It is much more profitable to ask: Where does the a come from?

Zermelo answered the question by giving several construction principles for obtaining new a's from old. Fraenkel and Skolem extended the method, and von Neumann, Bernays and Gödel modified it somewhat. Actually this is a rather sad history—because set theory is made to seem so artificial and formalistic. The naive axioms are contradictory. We block the contradiction and thereby emasculate the theory. Therefore to get anywhere we reinstate a few of the principles we eliminated and hope for the best. Now it would be wrong to accuse any of the above men of holding such a simplistic view of the axiomatic process. Nevertheless it is a very widely held view and one that is easy to fall into when considering only the formal axioms without their intuitive justification. Let us try to see whether there is another path to the same theory more obviously based on the underlying intuition.

The truth is that there is only one satisfactory way of avoiding the paradoxes: namely, the use of some form of the *theory of types*. That was at the basis of both Russell's and Zermelo's intuitions. Indeed the best way to regard Zermelo's theory is as a simplification and extension of Russell's. (We mean Russell's *simple* theory of types, of course.) The simplification was to make the types *cumulative*. Thus mixing of types is easier and annoying repetitions are avoided. Once the later types are allowed to accumulate the earlier ones, we can then easily imagine *extending* the types into the transfinite—just how far we want to go must necessarily be left open. Now Russell made his types *explicit* in his notation and Zermelo left them *implicit*. It is a mistake to leave something so important invisible, because so many people will misunderstand you. What we shall try to do here is to axiomatize the types in as simple a way as possible so that everyone can agree that the idea is natural.

Let us proceed on a very primitive basis. In order to obtain the sets to which the comprehension axiom can be applied, we imagine some way of dividing the sets into levels. There will be earlier levels and later levels. Let the sets up to a certain level be thought of as forming a *partial* universe V which is regarded as a legitimate set. We can be generous and assume that all the nonsets, the set-theoretical atoms, belong to all of the levels. In a *later* universe V' we have not only the elements of V, but also V itself to be used to form subcollections of V'; that is, $V \in V'$ as well as $V \subseteq V'$, where we define

$$\forall x, y[x \subseteq y \leftrightarrow \forall z[z \in x \rightarrow z \in y]].$$

Furthermore not only V, but by the same token all the subcollections of V, should also be *elements* of V'.

Once a set is fixed at one level, all its subsets are fixed at a later level—that is certainly the basic idea of the theory of types. We formalize this idea *not* by introducing type indices, but more simply by identifying a level with the collection of all sets (and nonsets) up to that level. We let variables $V, V', V'', \ldots$ range over these levels—that is, we take the idea of a *type level* (as identified with certain sets) as a primitive notion. The "later than" relation is transcribed simply as $V \in V'$. It need hardly be mentioned that we assume that there is at least one level and that each level is a set—axioms that we do not stop to name. What is important is the idea that a given level is *nothing more than* the accumulation of all the members and subsets of all the *earlier* levels (and all the nonsets, if any there be). In formal terms we have this axiom:

ACCUMULATION. $\forall V' \forall x[x \in V' \leftrightarrow \neg \exists a[x = a] \vee \exists V \in V'[x \in V \vee x \subseteq V]]$.

(By the way, just because we use the *variables* $V, V', V'', \ldots$ we should *not* think of the levels *themselves* as being arranged in an ω-type sequence. In general we will want a transfinite sequence. Also note that $V \in V'$ does *not* imply that V' is the *next* level; it may be a much later level.) The purpose of this axiom is to show how the levels *fit together*.

The question of *how far* the levels go out will be postponed for the moment. Of course, however far they do go, our intention is that they eventually capture everything. As an axiom this idea gives the following:

RESTRICTION. $\forall x \exists V[x \subseteq V]$.

In other words, the *whole universe*, if only it were a set, would behave as the ultimate level in the sense of the previous axiom. (Note that this axiom gives the existence of at least one level. It really should have been formulated with the clause $[x \in V \vee x \subseteq V]$, but we shall show below that $[x \in V \rightarrow x \subseteq V]$.) This will turn out to be nothing more or less than the well-known axiom of foundation, which is generally poorly understood. We feel that in the present context it appears as a quite natural expression of the fact that the sets are restricted to the levels.

Well, except for axioms of extent (infinity), what else do we need to explain the behavior of the levels and their subsets? Do we not have to go on to specify that the levels are ordered—even well-ordered? That the levels are comparable? Even that there is a first level? *No, we do not*: All these essential facts will follow from the above apparently primitive assumptions. This at first surprising result shows how little choice there is in setting up the type hierarchy. Furthermore, we will find that the principles of construction (union and power set) also drop out of these axioms; so that really only the principles of extent are needed beyond what we have. It seems safe to say that it would be hard to reduce the axioms of set theory to less than the above and still retain a simple intuitive basis.

As our first deduction, notice that as an immediate consequence of the accumulation axiom we have

(1) $$V \in V' \rightarrow V \subseteq V'.$$

This implies that the "less than" relation among levels is *transitive*. Though it follows from what we will prove below, it is instructive to show that the relation is *irreflexive*:

(2) $$V \notin V.$$

Because suppose the opposite. By the comprehension axiom we can form the set $a = \{x \in V : x \notin x\}$. (We use here the usual notation for set abstraction, which is justified by our axioms.) The set looks familiar, somehow. We have assumed $V \in V$, and by construction $a \subseteq V$. Thus $a \in V$, in view of the accumulation axiom. Now we proceed as with the Russell paradox to derive a contraction. Thus we have put a paradox to work to obtain a useful conclusion which is not at all paradoxical.

To make further progress it is helpful to employ the paradox of the set of all grounded classes just as we employed the Russell paradox. This paradox shows that there is no set consisting of *all* the x such that

$$\forall a[x \in a \rightarrow \exists y \in a[y \cap a = 0]].$$

(Here, 0 denotes the empty set and $y \cap a = \{z \in a : z \in y\}$, the use of both of which is justified by the comprehension axiom.) As in the above argument, we introduce the auxiliary set

$$\|V\| = \{x \in V : \forall a[x \in a \rightarrow \exists y \in a[y \cap a = 0]]\}$$

and ask about the interesting properties of this well-determined subset. The main fact is

(3) $$V \in V' \rightarrow \|V\| \in \|V'\|.$$

Because assume $V \in V'$. Now $\|V\| \subseteq V$, so by the accumulation axiom $\|V\| \in V'$. To establish the desired conclusion we need only show that $\|V\|$ is grounded. Thus, assume $\|V\| \in a$. If $\|V\| \cap a = 0$, we are done. If $\|V\| \cap a \neq 0$, let $x \in \|V\|$ and $x \in a$. By the definition of $\|V\|$, we have in this case again the required $\exists y \in a[y \cap a = 0]$. Thus another paradox has been put to work.

The first reward from the rehabilitation of the grounded classes is this important fact:

(4) $$x \in V \rightarrow x \subseteq V.$$

Because suppose $x \in V$ and let

$$a = \{b : \exists V' \in V[x \in V' \wedge b = \|V'\|]\}.$$

Clearly, by (3), a exists and $a \subseteq \|V\|$. If $a = 0$, then $\forall V' \in V[x \notin V']$. Thus by the accumulation axiom we find

$$\neg \exists c[x = c] \vee \exists V' \in V[x \subseteq V'].$$

The first alternative implies $x \subseteq V$ (by our grammatical convention), while the second implies the same in view of (1). Next, if $a \neq 0$, we can find $b \in a$ with $b \cap a = 0$ because $a \subseteq \|V\|$. But then $b = \|V'\|$ where $V' \in V$ and $x \in V'$. In effect we have already eliminated the case in which $\neg \exists c[x = c]$, so by accumulation we have $V'' \in V'$ with $[x \in V'' \vee x \subseteq V'']$. The first alternative is impossible, because it would imply $\|V''\| \in b$ and $\|V''\| \in a$. From the second alternative we obtain $x \subseteq V'' \subseteq V' \subseteq V$, as desired.

The second reward is the deduction of the full principle of well-foundedness (foundation regularity) for the $\in$-relation:

(5) $$\exists x \Phi(x) \rightarrow \exists x[\Phi(x) \wedge \neg \exists y \in x[\Phi(y)]].$$

Assume $\Phi(z)$; by the restriction axiom, $z \subseteq V$ for some V. As in the last proof, let

$$a = \{b : \exists V' \in V[\exists x \subseteq V'[\Phi(x)] \wedge b = \|V'\|]\}.$$

Again this set exists and $a \subseteq \|V\|$. In case $a = 0$, we easily conclude from (4) by means of the accumulation axiom that $\neg \exists y \in z[\Phi(y)]$. In $a \neq 0$, we choose $b \in a$ with $b \cap a = 0$. Now $b = \|V'\|$ with $x \subseteq V'$ and $\Phi(x)$, for suitable x and V'. From $b \cap a = 0$ we easily conclude as above that $\neg \exists y \in x[\Phi(y)]$ to complete the argument. Note that we can relativize quantifiers and deduce at once from (5) the special case:[1]

(6) $$\exists V \Phi(V) \rightarrow \exists V[\Phi(V) \wedge \neg \exists V' \in V[\Phi(V')]].$$

As an application of (6) we can now show that the levels are indeed linearly ordered:

(7) $$V \in V' \vee V = V' \vee V' \in V.$$

Thus in conjunction with (1) and (6) we see that the levels are in fact *well-ordered*. To prove (7) we argue by contradiction. Assume the negation of the universal generalization of (7), and then pick V so that

$$\neg \forall V'[V \in V' \vee V = V' \vee V' \in V],$$

but so that no $V'' \in V$ has this property. Next let V' be chosen so that

$$V \notin V' \wedge V \neq V' \wedge V' \notin V,$$

[1] As A. Lévy observed to the author, the proof of (4) is, essentially, a proof of (6). Therefore it would seem better to prove (6) first and deduce (4) and (5) as corollaries. Note that this proof of (6) does not use the axiom of restriction.

but so that no $V'' \in V'$ has this property. After these two applications of (6) we shall obtain a contradiction. Suppose $V'' \in V$. Then $V'' \neq V'$ and by transitivity $V' \notin V''$, because $V' \notin V$. Hence by the choice of V, we must conclude $V'' \in V'$. Conversely, suppose $V'' \in V$. Then again $V'' \neq V$ and $V \notin V''$. This time by the choice of V', we must conclude $V'' \in V$. Thus we have shown

$$\forall V''[V'' \in V \leftrightarrow V'' \in V'].$$

Now look at the axiom of accumulation. We easily deduce from this last biconditional that

$$\forall x[x \in V \leftrightarrow x \in V'],$$

which means $V = V'$, a contradiction. Looking back, we certainly have utilized *all* the axioms thoroughly to reach this point: The rest is smooth sailing.

For example, though the existence of the intersection of two sets follows from comprehension, the existence of the union does not. But now from (7) we note that if a and b are given, then for suitable V, V' we have $a \subseteq V \wedge b \subseteq V'$, and $V \subseteq V' \vee V' \subseteq V$. Suppose $V \subseteq V'$, then

$$a \cup b = \{x \in V' : x \in a \vee x \in b\},$$

and similarly in the other case. Note also by (4) that

$$\bigcup a = \{x \in V : \exists c \in a[x \in c]\}.$$

There is, however, a problem about singletons and power sets.

Inasmuch as we have not assumed any axioms of extent, note that the system is trivially consistent with a model consisting of the empty set alone. Hence, we cannot expect to prove the existence of any power sets unconditionally. Nevertheless the situation is not very complicated; what we can prove is

$$\text{(8)} \qquad \exists V[a \in V] \leftrightarrow \exists b \forall c[c \in b \leftrightarrow c \subseteq a].$$

Because, if $a \in V$, then $a \subseteq V' \in V$ for some V'. Therefore, if $c \subseteq a$, then $c \subseteq V' \in V$, and so $c \in V$, too. This means we can take $b = \{c \in V : c \subseteq a\}$ to obtain the desired conclusion. Notice that (8) can be further simplified because in view of the axiom of restriction,

$$\exists V[a \in V] \leftrightarrow \exists y[a \in y].$$

We could call objects x such that $\exists y[x \in y]$ *elements*. Principle (8) tells us that sets that are elements always have power sets. Likewise we could show that elements can always be made into singletons, doubletons, and into *finite* sets in general.

At this point we are free to choose various directions: either moving to the Zermelo-Fraenkel-Skolem theory or to the von Neumann-Bernays-Gödel theory by distinguishing between sets (elements) and classes. The author does not feel that the class theory is particularly useful, but, of course, it is a perfectly good theory. In any case the remarks about the Zermelo theory can always be transposed to the other. The main point is that, whatever is desired, the artificial, "ad hoc" axioms have been completely avoided.

The usual way of axiomatizing the Zermelo-Fraenkel-Skolem theory is to adjoin an axiom of *infinity* and an axiom schema of *replacement*. These two principles can be combined into one seemingly more powerful statement.

REFLECTION. $\exists V \forall x \in V[\Phi(x) \rightarrow \Phi^{(V)}(x)]$.

Here the formula $\Phi(x)$ may contain additional free variables and $\Phi^{(V)}(x)$ represents the result of relativizing all quantifiers in $\Phi(x)$ to elements of V (that is, replace $\exists y$ by $\exists y \in V$, etc.) The principle is called the *reflection axiom* because the *partial* universe V that is asserted to exist *reflects* all the properties $\Phi(x)$ for $x \in V$ that hold relative to the *whole* universe. The author has to admit that the replacement axiom is more elementary and probably more intuitive. But the reflection axiom is so easy to formulate and use, it has all the practical advantages of a good axiom. In any case it is interesting to prove the equivalence with the more primitive axioms, though we will not stop to do so here.

To see why reflection gives infinity we proceed in two steps. First we show

$$\text{(9)} \qquad \forall x \exists V[x \in V].$$

Because let Φ be the formula $\exists y[x = y]$ with x as a free variable which will *not* be quantified upon in the application of reflection. We obtain by reflection

$$\exists V[\Phi \rightarrow \Phi^{(V)}],$$

which means $\exists V[\exists y[x = y] \rightarrow \exists y \in V[x = y]]$, or, more simply, $\exists V[x \in V]$, as desired. In the same way we can show next that

$$\text{(10)} \qquad \exists V[\forall x \in V \exists y \in V[x \in y] \wedge \exists x[x \in V]].$$

Because from (9) the formula $[\forall x \exists y[x \in y] \wedge \exists x[x = x]]$ is provable, and reflection yields (10). Clearly in the well-ordering of the levels the V of (10) is a nonzero limit point and thus represents an infinite set.

We can strengthen the reflection principle to

$$\text{(11)} \qquad \forall a \exists V[a \in V \wedge \forall y, y', y'', \ldots \in V[\Phi(y, y', y'', \ldots) \leftrightarrow \Phi^{(V)}(y, y', y'', \ldots)]].$$

An outline of the proof will have to suffice. In the original reflection axiom we can take the formula to be

$$\forall y, y', y'', \ldots, z[x = \langle y, y', y'', \ldots, z\rangle \rightarrow [\Phi(y, y', y'', \ldots) \leftrightarrow z = 0]]$$
$$\wedge\ \forall y, y', y'', \ldots, z \exists x'[x' = \langle y, y', y'', \ldots, z\rangle] \wedge \exists b[b = a],$$

where $\langle y, y', y'', \ldots, z\rangle$ denotes an unordered tuple. It is best to write out the equation $x = \langle y, y', y'', \ldots, z\rangle$ in the obvious way. From this it is very easy to deduce a statement that clearly implies Fraenkel's replacement schema:

$$\text{(12)} \qquad \forall a \exists V \forall x \in a[\exists y \Phi(x, y) \rightarrow \exists y \in V \Phi(x, y)].$$

By the way, it should be pointed out that the levels can be defined in terms of membership and so are theoretically eliminable. In fact one can prove

$$\begin{aligned} \exists V[a = V] \leftrightarrow \P a \subseteq a \wedge \forall x[\neg \exists c[x = c] \rightarrow x \in 0] \\ \wedge\ \forall a' \in a \exists b \in a \forall c \subseteq b[\P c \in a \wedge [\P c \in b \vee a' \subseteq \P c]] \end{aligned} \tag{13}$$

where we define

$$\P a = \{x : \exists y \in a[x \subseteq y]\}.$$

After a suitable effort we can finally show that the axioms of accumulation and restriction together are equivalent to this single statement:

$$\forall a \exists b \forall c \subseteq b \exists c'[\forall x[x \in c' \leftrightarrow \exists y \in c[x \subseteq y]] \wedge [c' \in b \vee a \subseteq c']].$$

Deductions from this axiom (which were carried out by Montague and the author) are quite lengthy, so there seems to be no technical or conceptual advantage in reducing the number of primitive notions to the minimum. Of course, in some model-theoretic discussions we may want to know that the levels are definable, so the result has some interest.

Looking now to the future we ask: What should the new axioms of set theory be like? We can of course give axioms of infinity that increase the extent of our sets far beyond the inaccessible cardinalities. This chapter of set theory is far from being closed and probably never will be. As yet none of these axioms have settled the continuum hypothesis, though some of them do contradict Gödel's constructibility axiom. It will be very interesting to see what develops along this direction. Also we have not discussed the axiom of choice. No doubt it will always be desirable despite the technical interest of various independence questions involving it and weaker principles. If only it could be deduced from some more primitive principle!

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