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LOGIC WITH DENUMERABLY LONG FORMULAS AND FINITE STRINGS OF QUANTIFIERS *

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Consider the most obvious and immediate extension of ordinary predicate logic to a logic involving infinitely long expressions: namely, extend the propositional part of the logic by allowing denumerable conjunctions and disjunctions. In making an extension to an infinitary logic, Tarski originally suggested [58b] that infinite strings of quantifiers also be allowed. In some ways, in the denumerable case, this plan is not reasonable. Following the terminology of Karp [a], let us call Tarski's logic $\mathbf{L}_{\omega_1,\omega_1}$, where the two subscripts indicate that both the conjunctions *and* the strings of quantifiers can be of any (denumerable) length $<\omega_1$. The logic we are discussing in this paper is called $\mathbf{L}_{\omega_1,\omega_0}$, where the second subscript indicates that the strings of quantifiers prefixing any subformula of a formula must be *finite* in length. $\mathbf{L}_{\omega_0,\omega_0}$ is the ordinary finitary predicate logic.

As was pointed out by Tarski [58b], in the logic $\mathbf{L}_{\omega_1,\omega_1}$ it is possible to characterize the class of well-ordering relations by a *single* sentence of $\mathbf{L}_{\omega_1,\omega_1}$. For this reason, and also in view of the behavior of $\mathbf{L}_{\omega_1,\omega_1}$ as regards questions of completeness and axiomatization, it seems to the author that $\mathbf{L}_{\omega_1,\omega_1}$ behaves more like *second-order* logic. This is why he feels that Tarski's extension to the countably infinitary language cannot be regarded as a suitable generalization of first-order logic. The purpose of this paper is to summarize the main results presently known about $\mathbf{L}_{\omega_1,\omega_0}$ in order to show that this logic *does* deserve to be called a reasonable infinitary *first-order* logic.

As a first simple example, let us consider formulas of $\mathbf{L}_{\omega_1,\omega_0}$ involving identity and one additional binary predicate $<$. We use $\rightarrow$, $\neg$, $\wedge$, $\vee$, $\leftrightarrow$, $\forall$, $\exists$ for the usual logical connectives and quantifiers, while an in-

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finite conjunction and an infinite disjunction of formulas A_ξ, $\xi<\alpha$, are denoted respectively by

$$\bigwedge_{\xi<\alpha} A_\xi \text{ and } \bigvee_{\xi<\alpha} A_\xi.$$

If we wanted to be more formal, we could consider infinite conjunctions as infinite well-ordered concatenations of symbols of the form

$$[\bigwedge A_0 A_1 \dots A_\xi \dots],$$

where we string out the formulas A_ξ in order as ξ ranges through the ordinals less than α. Of course, in $\mathbf{L}_{\omega_1,\omega_0}$, the length α of the conjunction must be an ordinal less than ω_1. The more abbreviated notation is more convenient, however.

Now, using x and y for individual variables, let us define by transfinite recursion on ordinals $\alpha<\omega_1$ the following sequence of formulas:

$$E_\alpha = \forall y[y<x \leftrightarrow \exists x[x \simeq y \wedge \bigvee_{\xi<\alpha} E_\xi]].$$

Note how as α increases the quantifiers in E_α will become more and more nested even though only two distinct variables occur in the formulas. Each E_α has x as the only free variable. The use of equality in the right half of the formula E_α was only a device to avoid substitution. Allowing the usual inexact notation indicating the free variables and substitutions, what is meant is that these formulas should be logically valid:

$$\forall x[E_\alpha(x) \leftrightarrow \forall y[y<x \leftrightarrow \bigvee_{\xi<\alpha} E_\xi(y)]].$$

Next let $\mathfrak{A}=\langle A, <\rangle$ be a relational system where $<$ simply orders A. Which elements of A will satisfy E_α in $\mathfrak{A}$? The answer, which is easily verified by transfinite induction, is that an element satisfies E_α in $\mathfrak{A}$ if and only if the initial segment of $\mathfrak{A}$ determined by the element is well-ordered by $<$ in type α. (One must realize that the definition of satisfaction of infinitary formulas is as obvious as for finitary formulas —except a transfinite recursion is involved because the formulas are obtained by a transfinite process, as exemplified by the formulas E_α.)

If the set A is countable, then there is a (unique) maximal well-ordered initial segment of $\mathfrak{A}$; say it is of type α. Then the following sentence will be true in $\mathfrak{A}$:

$$[\neg \exists x E_\alpha \wedge \bigwedge_{\xi<\alpha} \exists x E_\alpha].$$

If $\mathfrak{A}$ is actually well-ordered of type α, then the sentence

$$\forall x \bigvee_{\xi<\alpha} E_\xi$$

is true in $\mathfrak{A}$. Indeed, any well-ordered system will satisfy all of the sentences

(*) $$[\neg \exists x E_\alpha \to \forall x \bigvee_{\xi<\alpha} E_\xi]$$

for arbitrary $\alpha<\omega_1$. Conversely, if $\mathfrak{A}$ is a *countable* ordered system satisfying all of the sentences (*), then $\mathfrak{A}$ is a well-ordered system. There are, however, uncountable ordered systems that are not well-ordered but still satisfy all of the sentences (*): namely, it is shown in Karp [*], and was independently established by Lopez-Escobar, that a well-ordered system of type ω_1 is *equivalent* in $\mathbf{L}_{\omega_1,\omega_0}$ to an ordered system of type $\omega_1+\omega_1\cdot\omega_0{}^*$, which obviously is not well-ordered. (*Equivalence* of course means that the two systems satisfy the same sentences of $\mathbf{L}_{\omega_1,\omega_0}$.) It follows at once that well-orderings cannot be characterized by any set of sentences of $\mathbf{L}_{\omega_1,\omega_0}$. We also see the analogy between the logics $\mathbf{L}_{\omega_i,\omega_0}$, $i=0,1$: for systems of cardinality less than $\aleph_i$, $i=0, 1$, characterizations seem to be possible, but for higher cardinalities this is not so. This analogy will be discussed later, when it is shown it holds in general and not only for well-ordered systems.

Before leaving orderings, let us think for a moment about systems $\mathfrak{A}=\langle A, <\rangle$ satisfying these sentences:

(**) $$\forall x \forall y[x<y \vee x \simeq y \vee y<x]$$
$$[\forall x[\forall y[y<x \to \exists x[x \simeq y \wedge A]] \to A] \to \forall x A]$$

where A is an arbitrary formula with x as the only free variable. A more readable version of the second type of formula would be

$$[\forall x[\forall y[y<x \to A(y)] \to A(x)] \to \forall x A(x)],$$

which is nothing more than the schema of transfinite induction. It is easy to argue, using a transfinite induction in the metalanguage, that if $\mathfrak{A}$ satisfies all the sentences (**), then $\mathfrak{A}$ satisfies all the sentences (*). Thus, among *countable* systems at least, the axioms (**) do characterize the well-orderings. Non-standard models of (**) must be uncountable.

We have just seen from the *semantical* view point that the sentences

(*) follow from the set of sentences (**). The deduction of (*) from (**) can also be done *syntactically*, using either logical axioms or Gentzen-style rules. Both versions will be described below.

To make the statement of the systems simpler, we employ a restricted number of logical primitives. In the first instance we shall use only $\rightarrow$, $\neg$, $\bigwedge\!\!\!\bigwedge$, $\forall$, and $\simeq$. Any number of finitary predicate or operation symbols can be used as non-logical constants. The logical axioms are given next.

LOGICAL AXIOMS FOR $\mathbf{L}_{\omega_1,\omega_0}$

$[A \rightarrow [B \rightarrow A]]$,

$[[A \rightarrow [B \rightarrow C]] \rightarrow [[A \rightarrow B] \rightarrow [A \rightarrow C]]]$,

$[[\neg B \rightarrow \neg A] \rightarrow [A \rightarrow B]]$,

$[\bigwedge\!\!\!\bigwedge_{\xi<\alpha} A_\xi \rightarrow A_\eta]$, where $\eta<\alpha<\omega_1$,

$[\forall v A(v) \rightarrow A(t)]$,

$t \simeq t$,

$[t = u \rightarrow [A(u) \rightarrow A(t)]]$,

where A, B, C are arbitrary formulas, v is a variable, t and u are terms, and where A(t) and A(u) denote proper substitutions of terms t and u for free occurrences of the variable v in A.

In addition to these axioms, the following rules are required:

INFERENCE RULES FOR $\mathbf{L}_{\omega_1,\omega_0}$

$$\frac{A, [A \rightarrow B]}{B},$$

$$\frac{[A \rightarrow B_0], [A \rightarrow B_1], \ldots, [A \rightarrow B_\xi], \ldots, \xi<\alpha<\omega_1}{[A \rightarrow \bigwedge\!\!\!\bigwedge_{\xi<\alpha} B_\xi]},$$

$$\frac{[A \rightarrow B(v')]}{[A \rightarrow \forall v B(v)]},$$

where in the last rule the variable v′ is not free in A and every occurrence of v′ in B is free for v in B.

We say that a sentence A *follows syntactically* or is *provable* from a set Φ of sentences if there is a proof of A from Φ; that is, a transfinite sequence of sentences $B_0, B_1, \ldots, B_\xi, \ldots, B_\beta$ where $\beta<\omega_1$ such that

$B_\beta = A$ and for each $\xi \leqslant \beta$, the formula B_ξ is either a logical axiom, a member of Φ, or follows from previous formulas in the sequence by one of three rules of inference. There is no need to consider proofs that are non-denumerably long, since the only infinitary rule of inference makes its conclusion from only denumerably many premises. One other restriction we shall make on the notion of a proof is that every formula B_ξ occurring in a proof has only *finitely* many free variables. This is reasonable because given a sentence A, *every subformula of* A *has only finitely many free variables.* This statement is proved directly by transfinite induction on the length of the formula A and is closely related to the fact that any quantifier prefix of a subformula can involve only finitely many variables. This fact will be important when we discuss the extension of the Löwenheim-Skolem theorem.

Further, we say that a sentence A *follows semantically* from a set Φ of sentences if in every system in which all sentences in Φ are true, the sentence A is also true; that is, every model of Φ is a model of A. We can now state the first main result.

THE COMPLETENESS THEOREM FOR $\mathbf{L}_{\omega_1,\omega_0}$. *A sentence* A *follows syntactically from a countable set* Φ *of sentences if and only if* A *follows semantically from* Φ.

Taking Φ to be the empty set, we see that a sentence A is logically provable using the above axioms and rules if and only if it is universally valid. This is all in strict analogy with the case of $\mathbf{L}_{\omega_0,\omega_0}$, except for the restriction on the countability of Φ. That this restriction cannot be removed was shown by Ryll-Nardzewski, who constructed a remarkable *complete* and *consistent* set Ψ of sentences which has no model (that is, for every sentence A, either A or $\neg$ A but not both follows syntactically from Ψ, but there is no system in which all sentences in Ψ are true.) Unfortunately, Ryll-Nardzewski's ingenious construction is too complicated to explain here. Instead, we will have to be satisfied by the less interesting example presented in the next paragraph.

We employ non-logical constants 0, +, $\prec$, <. The set Θ of sentences we want consists in the first place of the usual axioms for *ordered abelian groups* written in terms of 0, +, and $\prec$. To these we adjoin the *Archimedean axiom*, which is the sentence

$$\forall x[0 \prec x \rightarrow \forall y \bigvee\!\!\!\bigvee_{n<\omega} [y \prec n \cdot x]],$$

where $n \cdot x = x + \ldots + x$ (n times). Next we bring in the relation $<$ with the adjunction of the following non-denumerable number of sentences:

$$\forall x[E_\alpha(x) \wedge y < x \rightarrow y \prec x], \text{ and}$$

$$\exists x E_\alpha(x), \text{ for each } \alpha < \omega_1,$$

and the usual axioms on $<$ for a simple ordering. Obviously any model of these sentences would have to be an Archimedean-ordered group with a subset ordered in type ω_1 by the group ordering. As is well known, this is impossible. On the other hand, every countable subset of Θ clearly has a model where the group is just the ordered group of rationals. Further, we see that Θ is not as satisfactory as Ryll-Nardzewski's set Ψ, because Θ has *no* complete and consistent extensions. This is established by noticing that from Θ we can prove for each $\alpha < \omega_1$ the sentence

$$\exists x[E_\alpha(x) \wedge \forall y[E_\alpha(y) \rightarrow x \doteq y]].$$

In a complete extension of Θ, these *unique* elements determined by the formulas E_α would have to generate an Archimedean-ordered group whose group table is completely determined syntactically by the sentences provable from the given extension. In other words, every complete extension would automatically produce in terms of the syntactically definable elements an Archimedean-ordered group of the kind we already know cannot exist.

The proof of the completeness theorem for $\mathbf{L}_{\omega_1, \omega_0}$ was first given by Karp [a]. Lopez-Escobar and the author have verified that there is a very straightforward adaptation of the standard Henkin-style completeness proof to $\mathbf{L}_{\omega_1, \omega_0}$. The success of the method depends heavily on the fact that the sentences of a given countable set have only countably many different subformulas each of which have only finitely many free variables. This same circumstance makes it at once evident that the next result is correct.

THE DOWNWARD LÖWENHEIM-SKOLEM THEOREM FOR $\mathbf{L}_{\omega_1, \omega_0}$. *Every countable set of sentences which has a model of infinite cardinality has models of all smaller infinite cardinalities.*

The proof is immediate when one realizes that the subformulas require Skolem functions for their initial quantifiers, which are each functions of only a finite number of arguments. The corresponding upward theorem is not so obvious.

THE UPWARD LÖWENHEIM-SKOLEM THEOREM FOR $\mathbf{L}_{\omega_1, \omega_0}$. *Every countable set of sentences which has a model of cardinality $\beth_{\omega_1}$ has models of all higher cardinalities.*

Here $\beth_{\omega_1}$ is the cardinality of the set V_{ω_1} of all sets of rank less than ω_1. The sequence of sets V_α is defined by transfinite recursion; indeed for any ordinal α,

$$V_\alpha = \bigcup_{\xi < \alpha} \mathrm{P} V_\xi,$$

where $\mathrm{P}A = \{X : X \subseteq A\}$. In the upward theorem the cardinal number $\beth_{\omega_1}$ cannot be improved. This is easily demonstrated using the relational systems $\mathfrak{V}_\alpha = \langle V_\alpha, \in_\alpha \rangle$, where $\in_\alpha$ is the usual membership relation restricted to V_α. To this end consider formulas with the non-logical constant $\in$. Define by recursion the sequence V_α of formulas where

$$\mathrm{V}_\alpha = \bigvee_{\xi < \alpha} \forall y[y \in x \rightarrow \exists x[x \doteq y \wedge \mathrm{V}_\xi]].$$

Clearly in the system $\mathfrak{V}_\alpha$ the formula $\mathrm{V}_\beta(x)$, $\beta \leqslant \alpha$, defines the subset V_β of V_α. Now take the formula

$$[\forall x \forall y[\forall z[z \in x \leftrightarrow z \in y] \rightarrow x \doteq y] \wedge \forall x \bigvee_{\beta < \alpha} \mathrm{V}_\beta(x)].$$

One can verify by induction that every model of this sentence is isomorphically embeddable in the system $\mathfrak{V}_\alpha$, if $\alpha < \omega_1$. That is, no model of this sentence has cardinality greater than $\beth_\alpha = \overline{\overline{V}}_\alpha$. Since $\beth_{\omega_1} = \lim_{\alpha < \omega_1} \beth_\alpha$, the desired conclusion follows.

For the proof of the upward theorem one uses the result of Morley [*] which shows that the same result with the same cardinal $\beth_{\omega_1}$ holds for ω-logic. By ω-logic we understand the two-sorted predicate logic where variables of one of the sorts in all interpretations run over the set of integers. One can have any type of predicate symbol with mixed arguments and one must have in addition individual constants for each integer. Then given a countable set Φ of sentences of $\mathbf{L}_{\omega_1, \omega_0}$, one constructs a corresponding set $\Phi^{(\omega)}$ of sentences of ω-logic, which uses the same predicate symbols on the non-integer sort of variables that Φ uses, but which in addition has additional mixed predicates. The sentences in $\Phi^{(\omega)}$ can be so arranged that an $\mathbf{L}_{\omega_1, \omega_0}$-model of Φ gives an ω-model of $\Phi^{(\omega)}$ using the same elements and relations but has suitable mixed relations adjoined; and, conversely, every ω-model of $\Phi^{(\omega)}$ gives an

$\mathbf{L}_{\omega_1,\omega_0}$-model of Φ by removing the mixed relations. The details of the construction need not trouble us here.

Returning to syntactical questions, we can also give a Gentzen-style formulation of logical derivability for $\mathbf{L}_{\omega_1,\omega_0}$. In the following I shall use sets rather than sequents and shall write $\Phi \vdash \Psi$ to mean, roughly, that the conjunction of the formulas in Φ implies the disjunction of the formulas in Ψ. In fact let us write $\Phi \models \Psi$ for the semantical notion meaning every model of Φ is a model of at least one sentence of Ψ. The relation $\models$ will be used only between countable sets of sentences, while $\vdash$ can stand between countable sets of *formulas*. The relation $\vdash$ is precisely described as the least relation satisfying the particular rules given below in which we write $\Phi, \mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_\xi, \ldots, \xi<\alpha \vdash \Psi$ short for $\Phi \cup \{\mathrm{A}_\xi : \xi<\alpha\} \vdash \Psi$. To simplify things we take $\neg$, $\bigwedge$, $\forall$, and $\bumpeq$ as the only logical constants.

Cut-free rules for $\mathbf{L}_{\omega_1,\omega_0}$.

$$\Phi, \mathrm{A} \vdash \Psi, \mathrm{A}.$$

$$\frac{\Phi, \mathrm{A} \vdash \Psi}{\Phi \vdash \Psi, \neg \mathrm{A}}. \qquad \frac{\Phi \vdash \Psi, \mathrm{A}}{\Phi, \neg \mathrm{A} \vdash \Psi}.$$

$$\frac{\Phi, \mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_\xi, \ldots, \xi<\alpha \vdash \Psi}{\Phi, \bigwedge_{\xi<\alpha} \mathrm{A}_\xi \vdash \Psi}.$$

$$\frac{\Phi \vdash \Psi, \mathrm{A}_0 \quad \Phi \vdash \Psi, \mathrm{A}_1 \quad \ldots \quad \Phi \vdash \Psi, \mathrm{A}_\xi \quad \ldots \quad \xi<\alpha}{\Phi \vdash \Psi, \bigwedge_{\xi<\alpha} \mathrm{A}_\xi}.$$

$$\frac{\Phi, \mathrm{A(t)} \vdash \Psi}{\Phi, \forall \mathrm{v} \mathrm{A(v)} \vdash \Psi}. \qquad \frac{\Phi \vdash \Psi, \mathrm{A(v')}}{\Phi \vdash \Psi, \forall \mathrm{v} \mathrm{A(v)}}.$$

$$\Phi \vdash \Psi, \mathrm{t} \bumpeq \mathrm{t}.$$

$$\frac{\Phi, \mathrm{A(t)} \vdash \Psi, \mathrm{B(t)}}{\Phi, \mathrm{t} \bumpeq \mathrm{u}, \mathrm{A(u)} \vdash \Psi, \mathrm{B(u)}} \qquad \frac{\Phi, \mathrm{A(t)} \vdash \Psi, \mathrm{B(t)}}{\Phi, \mathrm{u} \bumpeq \mathrm{t}, \mathrm{A(u)} \vdash \Psi, \mathrm{B(u)}}$$

where in the rule for the introduction of $\forall$ on the right we must obey the usual restriction on free variables. In addition, we can assume in any step in a deduction of a $\vdash$-relationship that the total number of free variables occurring in the formulas in the sets on both sides of the $\vdash$ is finite.

The cut elimination theorem for $\mathbf{L}_{\omega_1,\omega_0}$. *If Φ and Ψ are countable sets of sentences of $\mathbf{L}_{\omega_1,\omega_0}$, then $\Phi \vdash \Psi$ can be established by the cut-free rules if and only if $\Phi \models \Psi$ holds.*

As a consequence of the elimination theorem we have:

The interpolation theorem for $\mathbf{L}_{\omega_1,\omega_0}$. *If Φ and Ψ are countable sets of sentences of $\mathbf{L}_{\omega_1,\omega_0}$ and if $\Phi \vdash \Psi$, then there is a sentence* A *involving as non-logical constants only those occurring in at least one sentence of Φ and at least one sentence of Ψ such that $\Phi \vdash \mathrm{A}$ and $\mathrm{A} \vdash \Psi$.*

These last two results were established by Lopez-Escobar and will appear in his dissertation. Earlier in [61] and [a] Maehara and Takeuti gave cut-elimination theorems for logics $\mathbf{L}_{\theta,\theta}$, where θ is strongly inaccessible, and these proofs encouraged Lopez-Escobar to see if the standard type of argument could be carried over to $\mathbf{L}_{\omega_1,\omega_0}$. Also it must be mentioned that Engeler in [63] gave a Schütte-style formulation of $\mathbf{L}_{\omega_1,\omega_0}$, but in that system formulas with *infinitely many* free variables seem to enter in an essential way. In particular, the author does not see whether Engeler's result can be applied to the proof of the interpolation theorem. Lopez-Escobar feels that the use of only finitely many free variables at each stage of a deduction is quite important in the proof of the interpolation theorem using the method of the cut-elimination theorem. This question, of course, never comes up in $\mathbf{L}_{\omega_0,\omega_0}$.

Once the interpolation theorem is established, then Beth's definability theorem is deduced as usual. Lopez-Escobar has also verified that the more refined results for $\mathbf{L}_{\omega_0,\omega_0}$ about positive and negative sentences also carry over to $\mathbf{L}_{\omega_1,\omega_0}$.

The next step in seeing how results for $\mathbf{L}_{\omega_0,\omega_0}$ generalize to $\mathbf{L}_{\omega_1,\omega_0}$ would be to turn to the initial results of model theory. Some difficulties are at once encountered, however. In the first place, what should one mean by a *universal class* for $\mathbf{L}_{\omega_1,\omega_0}$? The most obvious answer seems to be this: Consider the class of formulae built up from quantifier-free formula by applying the operators $\forall$, $\bigwedge$, $\bigvee$ in any order. As an interesting example (suggested by Lopez-Escobar) take as non-logical constants the individual constant 0 and the singulary function symbol P. The models we have in mind are trees with finitary branching and with paths of length $\leqslant \omega$. These systems can be characterized by the following three axioms, where 0 is interpreted as the *root* of the tree and P(x) as the immediately preceding node to x. (Draw a picture, and all will be clear.) We take

$$\forall \mathsf{x}[\mathsf{x} \simeq 0 \leftrightarrow \mathsf{P}(\mathsf{x}) \simeq \mathsf{x}],$$

$$\forall \mathsf{x} \bigvee_{n<\omega} \mathsf{P}^n(\mathsf{x}) \simeq \mathsf{P}^{n+1}(\mathsf{x}),$$

where $\mathsf{P}^n(\mathsf{x}) = \mathsf{P}(\mathsf{P}(\ldots \mathsf{P}(\mathsf{x}) \ldots))$ (n-times), and finally

$$\bigwedge_{n<\omega} \bigvee_{m<\omega} \forall \mathsf{x}_0 \ldots \forall \mathsf{x}_m [\bigwedge_{k \leqslant m} \mathsf{P}^n(\mathsf{x}_k) \simeq \mathsf{P}^{n+1}(\mathsf{x}_k) \rightarrow \bigvee_{i<j \leqslant m} \mathsf{x}_i \simeq \mathsf{x}_j].$$

Note that this last sentence is our first example of a sentence with an unbounded number of variables. It cannot be simplified directly since we have *no prenex normal form* in $\mathbf{L}_{\omega_1, \omega_0}$. Indeed in view of the results in Tarski [58a] there is no set of universal sentences with a bounded number of variables which are equivalent to the above three sentences. This is so because it is trivial to give a structure $\mathfrak{A} = \langle A, 0, P \rangle$ which is not a model of our three sentences, but every finitely generated subsystem is.

Now the difficulty encountered with universal sentences is this: Suppose a class of systems $\mathscr{K}$ can be characterized by a single sentence of $\mathbf{L}_{\omega_1, \omega_0}$, and suppose $\mathscr{K}$ is closed under the formation of subsystems; then can $\mathscr{K}$ be characterized by a single universal sentence (or even a set of such sentences)? This question has not yet been resolved. The initial stumbling block goes back to the failure of the compactness theorem for $\mathbf{L}_{\omega_1, \omega_0}$, since as we have seen there are uncountable consistent sets of sentences without models. At the present time other topics of model theory seem to share the same fate.

Let us now turn to a general discussion of countable systems in relation to $\mathbf{L}_{\omega_1, \omega_0}$. The author has established the next two results which generalize the corresponding well-known facts about finite systems and $\mathbf{L}_{\omega_0, \omega_0}$. The relational systems considered are to have only a countable number of finitary relations or operations.

THE COUNTABLE ISOMORPHISM THEOREM FOR $\mathbf{L}_{\omega_1, \omega_0}$. *Two countable systems satisfying the same sentences of* $\mathbf{L}_{\omega_1, \omega_0}$ *are isomorphic; indeed, for a given countable system* $\mathfrak{A}$ *one can obtain a single sentence* A *true of* $\mathfrak{A}$ *such that all countable systems satisfying* A *are isomorphic to* $\mathfrak{A}$.

THE COUNTABLE DEFINABILITY THEOREM FOR $\mathbf{L}_{\omega_1, \omega_0}$. *For a finitary relation* S *on the domain of* $\mathfrak{A}$ *to be* $\mathbf{L}_{\omega_1, \omega_0}$ *definable in the countable system* $\mathfrak{A}$ *it is necessary and sufficient that every automorphism of* $\mathfrak{A}$ *be an automorphism of* S.

The proofs of these results will be given elsewhere. The isomorphism theorem has an interesting application to the theory of Borel sets. Let N be a denumerable set and consider the product space $2^{N \times N}$. With the ordinary product topology this space is homeomorphic to Cantor's discontinuum. From our point of view, the space can be considered as the space of all binary relations on the set N. Now every permutation of N induces a permutation of $2^{N \times N}$. The subsets of $2^{N \times N}$ invariant under these permutations are called *invariant sets*. The invariant sets form a Boolean algebra whose *atoms*, the *minimal invariant sets*, are just the *isomorphism types* of binary relations on N. The isomorphism theorem at once implies that *the minimal invariant sets are Borel sets*. This is clear because the set of all relations in $2^{N \times N}$ satisfying a given sentence A of $\mathbf{L}_{\omega_1, \omega_0}$ is obviously a Borel subset of $2^{N \times N}$.

Furthermore, in view of the interpolation theorem, as was pointed out by Ryll-Nardzewski to the author, the invariant Borel subsets of $2^{N \times N}$ are *exactly* the subsets of $2^{N \times N}$ that can be characterized by single sentences of $\mathbf{L}_{\omega_1, \omega_0}$. Indeed, by virtue of the downward Löwenheim-Skolem theorem, two sentences of $\mathbf{L}_{\omega_1, \omega_0}$ determine the same subset of $2^{N \times N}$ if and only if they are logically equivalent. This remark has a pleasant application: The family of invariant Borel subsets of $2^{N \times N}$ is a sub-σ-field of the σ-field of all Borel sets. Assertion: there is a countably additive $\{0,1\}$-valued measure on this sub-σ-field that *cannot* be extended to a Borel measure. Proof: consider those sentences of $\mathbf{L}_{\omega_1, \omega_0}$ that are true of the well-ordering of type ω_1. This determines a $\{0,1\}$-valued measure on the invariant Borel sets in the obvious way. By making use of our ω_1-sequence of formulas E_α it is easily checked that this measure cannot be extended to a countably additive, real-valued measure on all the Borel sets.

As another application of the isomorphism theorem the author obtained the solution to a problem of Kuratowski [48, p. 377]. The problem amounts to this: Let Q be the set of rationals and consider the product space 2^Q. Again this is homeomorphic to Cantor's discontinuum, but now we interpret its points as being sets of rational numbers. Two sets of rationals are called *isomorphic* if there is a one-one order-preserving mapping of one set onto the other (this mapping is *not* defined on the complements of the sets!). Kuratowski's question: Is the subset of 2^Q

of all sets of rationals isomorphic to a given set always a Borel subset of 2^Q? For well-ordered sets of rationals the affirmative answer is given by Kuratowski [48, § 26, XII, 1]. But from our general isomorphism theorem the complete affirmative answer is readily derived. Some of the details of these deductions can be found in Scott [64].

There is also another aspect of the isomorphism theorem that may prove to be of interest. This involves the characterization of elementary equivalence of Fraïssé [54c] and Ehrenfeucht [61]. I will give a somewhat different description which was suggested to me by some recent work of Hanf. If $\mathfrak{A}$ is a relational system with universe A (and, say, a finite number of finitary relations), and if $a \in A^m$ where $m < \omega$, then we let $\mathfrak{A}[a]$ denote the relational system induced from $\mathfrak{A}$ on the set $m = \{0,1, \ldots, m-1\}$ by the function a; that is, the relations of $\mathfrak{A}[a]$ are the inverse images of the relations of $\mathfrak{A}$ under the function a. With this notation we now define $\tau_\alpha(\mathfrak{A}, a)$ by transfinite recursion on α for all systems $\mathfrak{A}$, all $m < \omega$, all $a \in A^m$:

$$\tau_\alpha(\mathfrak{A}, a) = \begin{cases} \mathfrak{A}[a], & \text{if } \alpha = 0; \\ \{\tau_\beta(\mathfrak{A}, a^\frown\langle x\rangle) : x \in A\}, & \text{if } \alpha = \beta + 1; \\ \{\tau_\xi(\mathfrak{A}, a) : \xi < \alpha\}, & \text{if } \alpha \text{ is a limit } > 0. \end{cases}$$

In addition we let $\tau_\alpha(\mathfrak{A}) = \tau_\alpha(\mathfrak{A}, 0)$ where $0 \in A^0$ is the empty sequence. By the way, if $a \in A^m$ and $x \in A$, then $a^\frown\langle x\rangle$ is that sequence $b \in A^{m+1}$, where $b_m = x$ and $b_i = a_i$ for $i < m$.

From the results of Fraïssé and Ehrenfeucht it can be shown that $\tau_{\omega_0}(\mathfrak{A}) = \tau_{\omega_0}(\mathfrak{B})$ if and only if the two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent in $\mathbf{L}_{\omega_0, \omega_0}$. From the results of Karp [*] we see that $\tau_{\omega_1}(\mathfrak{A}) = \tau_{\omega_1}(\mathfrak{B})$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences with *arbitrary* conjunctions and disjunctions, finitary strings of quantifiers, but with quantifier depth strictly less than ω_1. (See Karp [*] for a fuller explanation.) From the isomorphism theorem we see for countable systems that $\tau_{\omega_1}(\mathfrak{A}) = \tau_{\omega_1}(\mathfrak{B})$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. That is, for countable A the system of sets $\tau_\xi(\mathfrak{A})$, $\xi < \omega_1$, is a complete system of isomorphism invariants. Whether this remark has any useful applications remains to be seen. At least it is interesting that the result can be formulated without reference to the formal languages in a purely set-theoretical way.

In conclusion, then, it seems fair to say that $\mathbf{L}_{\omega_1, \omega_0}$ is a useful infinitary logic, and probably many other applications are waiting to be found. Further, the author feels he has fully justified the contention that $\mathbf{L}_{\omega_1, \omega_0}$ is the proper generalization of $\mathbf{L}_{\omega_0, \omega_0}$ to denumerable formulas. In fact we have seen several reasons for claiming that $\mathbf{L}_{\omega_1, \omega_0}$ plays the same role for $\mathbf{L}_{\omega_0, \omega_0}$ that the theory of Borel sets and σ-fields plays for the ordinary fields of sets.

Note added in proof:

Recently, Jerome Malitz has obtained the following results which show that the analogy between the languages $\mathbf{L}_{\alpha, \beta}$ for $\alpha > \omega_0$, $\beta \geqslant \omega_0$, and $\mathbf{L}_{\omega_0, \omega_0}$ is not as strong as might be hoped. For example, Beth's theorem (and hence Craig's interpolation theorem) fails for all languages $\mathbf{L}_{\alpha, \beta}$ whenever $\alpha \geqslant \beta \geqslant \omega_1$. For $\mathbf{L}_{\varkappa, \varkappa}$ where $\varkappa$ is strongly inaccessible this contradicts the interpolation theorem in Maehara–Takeuti [61]. In fact, there is a sentence of $\mathbf{L}_{\omega_1, \omega_1}$ which implicitly defines a relation that is not explicitly definable in any language $\mathbf{L}_{\alpha, \beta}$. In addition, one can show that Craig's theorem fails for $\mathbf{L}_{\alpha, \omega_0}$ whenever $\alpha > \omega_1$; a valid sentence $\sigma \to \varrho$ of $\mathbf{L}_{\omega_2, \omega_0}$ exists such that no sentence γ of any language $\mathbf{L}_{\alpha, \omega_0}$ has the property that both $\sigma \to \gamma$ and $\gamma \to \varrho$ are valid. This answers a question posed in Lopez-Escobar's dissertation. Another problem which Lopez–Escobar raised is that of the preservation of $\mathbf{L}_{\omega_1, \omega_0}$ equivalence under cardinal sums; i.e., if $\mathfrak{A}$ is equivalent to $\mathfrak{A}'$ in $\mathbf{L}_{\omega_1, \omega_0}$ does it follow that the cardinal sum of $\mathfrak{A}$ and $\mathfrak{L}$ is $\mathbf{L}_{\omega_1, \omega_0}$ equivalent to the cardinal sum of $\mathfrak{A}'$ and $\mathfrak{L}$ for all $\mathfrak{L}$. Again the answer is no, and in fact $\mathbf{L}_{\alpha, \beta}$ equivalence is preserved under cardinal sums if and only if α is strongly inaccessible.