# CONVERGENT SEQUENCES OF COMPLETE THEORIES

By

DANA SCOTT

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## ABSTRACT

A general logical principle is proved about the satisfiability of formulas of the first-order predicate calculus containing at most $n$ variables. Applying the principle to geometry it is shown that a first-order formula with at most $n$ variables is true in Euclidean geometry of dimension $n - 1$ if and only if it is true in all higher dimensions. As a consequence, infinite dimensional Euclidean Spaces are indistinguishable by first-order properties. Applying the principle to equivalence relations, a new proof of the decidability of the first-order theory of one equivalence relation is obtained as a corollary. The method has the advantages of being combinatorial and model-theoretic and not requiring the reduction of formulas to any kind of normal form. Similar applications to a few other simple theories are presented.

To the memory of

Jan Kalicki ,

who showed me for the first time abstract mathematics, to whom I owe my interest in logic, who inspired my first original thought, and who died so needlessly at such an early age.

# Introduction

The main result of this work was motivated by a metamathematical question in elementary Euclidean geometry which was proposed to the author by Professor Alfred Tarski: Can infinite dimensional Euclidean Spaces be distinguished by elementary geometrical properties? The answer is no. The Elementary theory of an infinite dimensional space is determined as the limit of the theories of its finite dimensional subspaces, in an appropriate sense, and hence all such spaces have the same properties. The method of proof of this result turned out to have a simple abstract form which is given in all details in Part I. The general theorem is then applied to geometry in Part II. Finally in Part III several questions in some simple combinatorial theories are dealt with by the method yeilding some new proofs of improved forms of known results and several new results about the effective decidability of these theories. The complete orgainization of this work as well as a summary of the theorems can be obtained from the Table of Contents, where all the numbered definitions and theorems are given a brief description.

The author would like to express his warmest thanks to several people who have made this research possible: first to Professor Tarski who suggested the problem, who made several helpful comments on the proofs, and to whom I shall always be grateful for a model of clarity of style, good judgment, and the highest ideals in mathematical research; next to Professor Alonzo Church for his careful examination of the work and many helpful suggestions; and finally to the Department of Mathematics which has been remarkably generous to me during my stay in Princeton, not only in allowing me the oppertunity of studying at Princeton, but also in the support and encouragement given me, which I appreciate very much.

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## Part II Euclidean Geometry

## §0. Notational Conventions

The purpose of this first section is to collect together all the set-theoretical notions and notations which are used without special reference in the sequel. All other definitions are given in their proper places in the relevant sections.

<u>Membership</u> of an element $x$ in a set $X$ is denoted by

$$x \in X ;$$

while <u>non-membership</u> is denoted by

$$x \notin X .$$

The <u>inclusion</u> of a set $X$ in a set $Y$ is denoted by

$$X \subseteq Y .$$

The symbol $0$ denotes the <u>empty set</u> as well as the number <u>zero</u>.

The set consisting of only the elements $x_o, x_1, \ldots, x_{n-1}$ is denoted by

$$\{ x_o, x_1, \ldots, x_{n-1} \}$$

In general the collection of all objects $x$ satisfying a property $P(x)$ is denoted by

$$\{ x \mid P(x) \}$$

Further this notation is extended to the case where if $f(x)$ is a term or function from objects to objects, then

$$\{ f(x) \mid P(x) \}$$

denotes the collection of all objects $f(x)$ where $x$ satisfies the condition $P(x)$. Of course this notation is inexact, since the bound variable $x$ is not explicitly indicated, but the usage is so common and the contexts so simple that no confusion can arise. A similar notation will also be applied to functions of several variables.

The operations of <u>union</u>, <u>intersection</u>, and <u>difference</u> of two sets $X$ and $Y$ are defined as follows:

$$X \cup Y = \{ x \mid x \in X \text{ or } x \in Y \}$$

$$X \cap Y = \{ x \mid x \in X \text{ and } x \in Y \}$$

$$X \sim Y = \{ x \mid x \in X \text{ and } x \notin Y \}$$

If $\mathfrak{X}$ is a collection of sets, then the <u>union</u> of all sets in $\mathfrak{X}$ is defined by the equation

$$\bigcup \mathfrak{X} = \{ x \mid x \in X \text{, for some } X \in \mathfrak{X} \}$$

If $\mathfrak{X}$ is non-empty, then the <u>intersection</u> of all sets in $\mathfrak{X}$ can be defined as

$$\bigcap \mathfrak{X} = \{ x \mid x \in X \text{, for all } X \in \mathfrak{X} \}$$

In case $X_i$ is a sequence of sets with $i$ in an index set $I$, then the union and intersection can be defined as

$$\bigcup_{i \in I} X_i = \bigcup \{ X_i \mid i \in I \}$$

$$\bigcap_{i \in I} X_i = \bigcap \{ X_i \mid i \in I \}$$

For sequences of sets there are two other operations <u>limsup</u> and <u>liminf</u>. Let $I$ be an index set and let $\mathfrak{I}$ be the family of all finite subsets of I. Then if $X_i$ for $i \in I$ are sets, we define

$$\limsup_{i \in I} X_i = \bigcap_{F \in \mathcal{F}} \bigcup_{i \in I \sim F} X_i$$

$$\liminf_{i \in I} X_i = \bigcup_{F \in \mathcal{F}} \bigcap_{i \in I \sim F} X_i$$

Finally if $\limsup_{i \in I} X_i = \liminf_{i \in I} X_i$ we say that the sequence <u>converges</u> and call the result of limsup and liminf the <u>limit</u> and denote it by

$$\lim_{i \in I} X_i$$

In the above we have spoken of function and sequences in an intuitive way, but it is also necessary to have an explicit set-theoretical definition. We begin by defining the <u>ordered pair</u> of two objects x and y by

$$< x,y > = \{\{ x \},\{ x,y \}\}$$

The <u>cartesian product</u> of two sets X and Y can then be defined by the equation

$$X \times Y = \{ \langle x,y \rangle \mid x \in X \text{ and } y \in Y \} .$$

A <u>function</u> F from X to Y is a set satisfying the following two conditions:

(i) $F \subsetneq X \times Y$

(ii) for each $x \in X$ there is one and only one $y \in Y$ such that $\langle x,y \rangle \in F$. The unique y mentioned in (ii) is denoted by $F(x)$. The collection of all functions from a set X to a set Y is denoted by

$$Y^X .$$

Thus, an <u>I-termed sequence</u> of objects from a set $X$ is best interpreted as an element in $X^I$.

The <u>ordinal numbers</u> used in this paper, are only the finite ones and the first infinite one. Each ordinal is to be thought of as the set of all smaller ordinals. Thus:

$$5 = \{ 0,1,2,3,4 \} .$$

where, for example,

$$3 = \{ 0,1,2 \}$$

$$= \{ 0, \{0\},\{0,\{0\}\}\}$$

This point of view makes it possible to identify $X^8$ as the set of all eight-termed sequences of objects in $X$. Notice that the statements $5 < 6$ and $5 \in 6$ have exactly the same meaning under this interpretation of ordinal numbers.

The collection of all finite ordinals is denoted by $\omega$. Thus, $X^\omega$ denotes all ordinary infinite sequences of objects in $X$. It will be useful sometimes to consider not all the infinite sequences but just those with finitely many different terms. This suggests the definition $X^{(\omega)} = \{f \mid f \in X^\omega$ and for some $n \in \omega$ , $f(n) = f(m)$ for all $m \in \omega$ where $n \leq m$

It will be necessary to modify sequences at certain points. Let $f \in X^\omega$ and let $i < \omega$. We define

$$f\ (i/x)$$

to be that sequence $g \in X^\omega$ such that

$$f(j) = g(j) \quad \text{for} \quad j < \omega\ ,\ j \neq i\ ,$$

and

$$g(i) = x.$$

In other words $f$ has been modified only at the $i^{th}$ term.

Subsets of $X^n$ are to be thought of as n-ary relations over the set $X$. We shall use the notation

$$< x_o , \ldots, x_{n-1} >$$

to denote the n-termed sequence of elements $x_1 , \ldots, x_{n-1}$. Thus if $R \subseteq X^n$, then the statement

$$< x_o , \ldots, x_{n-1} > \in R$$

can be interpreted as saying that the elements $x_o , \ldots, x_{n-1}$ stand in the relation $R$. This of course leads to confusion when $n = 2$, because the symbol $< x_o, x_1 >$ can mean either the ordered pair with terms $x_o, x_1$ or the two-termed sequence with terms $x_o, x_1$. These are distinct sets if the above definitions are taken in the strict interpretation. Unfortunately there seems to be no natural way of having ordered pairs and two-termed sequences identified as the same objects, so we shall let this confusion stand, since it can lead to no mistakes in the applications in this paper.

For binary relations there is an operation called relative product which is a simple extension of the notion of functional composition. Let $R, S \subseteq X^2$. We define

$$R \; ; \; S = \{ \langle x,z \rangle \mid \langle x,y \rangle \in R \text{ and}$$
$$\langle y,z \rangle \in S \text{ for some } y \in X \}$$

Finally we shall use the notation

$$\text{card } (X)$$

to denote the cardinal number of the set $X$. We shall not attempt here to identify the cardinal numbers as actual set-theoretical constructs since this leads to certain difficulties in the foundations of set theory. Instead we shall say that if $X$ is finite, then card $(X)$ is the unique finite ordinal having the same number of elements as $X$. In case $X$ is infinite, a statement card $(X)$ = card $(Y)$ simply means that the two sets $X$ and $Y$ can be put into a one-one correspondence.

## Part I. General Logical Theorems

### § 1 Sequences of Theories

There are many general facts about logical theories that depend only on having the formalism include the notation and rules of inference of the sentential calculus and are independent of any additional operations on sentences. A systematic treatment of theories based on this idea was given by Tarski in [8]. We shall essentially follow Tarski's development here recalling the known definitions and theorems needed and proving some additional facts about complete theories.

Throughout this section $S$ shall denote a non-empty set, called the set of <u>sentences</u>, and the symbols $\longrightarrow$ and $\sim$ shall denote a binary and a singulary operation under which $S$ is closed. $\longrightarrow$ will be called <u>implication</u>, and $\sim$, <u>negation</u>. In addition $L$ shall denote a subset of $S$, called the set of <u>logically valid sentences</u>, about which we make the following assumptions:

(i) $L \neq S$ ;

(ii) If $\phi, \psi, \chi \in S$ , then the sentences

$$[\phi \to [\psi \to \phi]] , [[\phi \to [\psi \to \chi]] \to [[\phi \to \psi] \quad [\phi \to \chi]]]$$

and $[[\sim \psi \to \sim \phi] \to [\phi \to \psi]]$ are all in $L$;

(iii) If $\phi, \psi \in S$ and $\phi, [\phi \to \psi] \in L$, then $\psi \in L$.

These assumptions assure us that $L$ is a consistent set of sentences containing all instances of the usual axioms for the sentential calculus and that $L$ is closed under the rule of modus ponens. The operations on sentences of <u>conjunction</u>, <u>disjunction</u>, and <u>equivalence</u> are introduced by the standard definitions.

Definition 1.1. (i) $[\phi \wedge \psi] = \sim [\phi \to \sim \psi]$

(ii) $[\phi \vee \psi] = [\sim \phi \to \psi]$

(iii) $[\phi \leftrightarrow \psi] = [[\phi \to \psi] \wedge [\psi \to \phi]]$

Our main objects of study are theories which can be defined as follows:

Definitions 1.2. A subset $T$ of $S$ is called a (consistent) <u>theory</u> if it satisfies these three conditions:

(i) $T \neq S$ ,

(ii) $L \subseteq T$ ;

(iii) if $\phi, \psi \in S$ and $\phi, [\phi \to \psi] \in T$ , then $\psi \in T$.

As an immediate consequence of the definitions we can state

Theorem 1.3. If $\mathfrak{T}$ is a non-empty class of theories, then $\bigcap\mathfrak{T}$ is a theory. If in addition $\mathfrak{T}$ is a directed class, then $\bigcup\mathfrak{T}$ is a theory.

Thus if $X$ is any set of sentences contained in at least one theory then $X$ is contained is a smallest theory. This suggests a definition

Definition 1.4. If $X$ is any set of sentences, then $Cl\ (X)$, the <u>closure</u> of $X$ under the rules of deduction, is the intersection of all theories containing $X$, if there is at least one, or is the set $S$ otherwise.

Definition 1.5. Let $T_1$ and $T_2$ be two theories.

(i) $T_2$ is an <u>extension</u> of $T_1$ if $T_1 \subseteq T_2$,

(ii) $T_2$ is a <u>finite extension</u> of $T_1$ if there exists a finite set $X$ of sentences such that $T_2 = Cl(T_1 \quad X)$;

(iii) $T_2$ is an <u>infinite extension</u> of $T_1$ if it is an extension but not a finite extension.

It is quite easy to show that all finite extensions of a theory $T$ are of the form $Cl(T \cup \{\phi\})$ where $\phi$ is a sentence in $S$. The last definition needed is that of a complete theory.

Definition 1.6. A theory $T$ is <u>complete</u> if it has no proper extensions.

It is well known that complete theories are the same as those theories $T$ satisfying the condition: for all $\phi \in S$, either $\phi \in T$ or $\sim\phi \in T$.

The first result to be proved here concerns the finite extensions of a certain type of intersection of complete theories.

Theorem 1.7. Let $I$ be an index set and let $T_i$ be a complete theory for each $i \in I$. Let $T = \bigcap_{i \in I} T_i$ and assume further that

$$T \neq \bigcap ( T_i \mid i \in I , i \neq j )$$

for each $j \in I$. Then each of the theories $T_i$ is a finite extension of $T$ and these are the only finite complete extensions of the theory $T$. In fact if, for each $j \in I$, $\Delta_j$ is an arbitrary choice of a sentence in $T_j \sim \bigcup ( T_i \mid i \in I, i \neq j )$, then

$$T_j = Cl(T \cup \{\Delta_j\}) .$$

Proof: Assuming the hypotheses, we see at once that the sets

$$T_j \sim \bigcup \{T_i \mid i \in I , i \neq j\} \text{ are all}$$

non-empty; thus, the choice of $\Delta_j$ can always be made. [This would seem to require the axiom of choice when applied to arbitrary sequences of theories, but it will be seen below that for the proof we would not have to choose more than one $\Delta_j$ at a time.] It is to be shown first that $T_j = Cl(T \cup \{\Delta_j\})$. Clearly $T \cup \{\Delta_j\} \subseteq T_j$ and hence $Cl(T \cup \{\Delta_j\}) \subseteq T_j$. Assume, then, that $\phi \in T_j$. Consider the implication $[\Delta_j \rightarrow \phi]$. Since $\phi \in T_j$, it follows at once that $[\Delta_j \rightarrow \phi] \in T_j$. By our choice of $\Delta_j$, $\Delta_j \notin T_i$ for all $i \in I, i \neq j$, and from the completeness of the theories $T_i$ we conclude that $\sim \Delta_j \in T_i$, and finally that $[\Delta_j \rightarrow \phi] \in T_i$. Since all indices have been taken into account, we have shown that $[\Delta_j \rightarrow \phi] \in T$. An immediate consequence is the statement $\phi \in Cl(T \cup \{\Delta_j\})$ which allows us to conclude that $T_j \subseteq Cl(T \cup \{\Delta_j\})$. Hence, all the $T_j$ are finite extensions of $T$. Suppose, by way of contradiction, that $T_*$

was another finite complete extension distinct from all the $T_j$. We would have $T_* = Cl(T \cup \{\Delta_*\})$ for some sentence $\Delta_*$. Clearly $\Delta_* \notin T_i$ for all $i \in I$, since otherwise $T_*$ would be included in one of the $T_i$ which is impossible due to the completeness of $T_*$. Thus, $\sim \Delta_* \in T_i$ for all $i \in I$ and so $\sim \Delta_* \in T$. Hence, $\Delta_*$ and $\sim \Delta_*$ are in $T_*$ which implies $T_* = S$, a contradiction. This completes the proof.

In general a theory $T$ that is an intersection of complete theories $T_i$ as in the proceding theorem will have many infinite complete extensions. There are some simple conditions, however, under which there is only one infinite complete extension. In order to facilitate the proof of this result and to indicate the existence of at least one infinite complete extension in case the index set $I$ is infinite, a lemma is interpolated at this point.

Lemma 1.8. Let the sequences $T_i$ and $\Delta_i$ satisfy the conditions of Theorem 1.7. Assume further that the index set $I$ is infinite. Then the set $T_\infty = \liminf_{i \in I} T_i$ is a theory that is an extension of $T$ which is contained in no $T_i$. In addition we have

$$T_\infty = Cl(T \cup \{\sim\Delta_i \mid i \in I\})$$

Proof: By definition of the operation liminf we have

$$T_\infty = \bigcup_{F \in \mathfrak{F}} \bigcap_{i \in I \sim F} T_i$$

where $\mathfrak{F}$ is the set of all of the finite subsets of the index set $I$. By theorem 1.3 each $\bigcap_{i \in I \sim F} T_i$ is a theory, and hence $T_\infty$ is seen to be a directed union of theories, which implies that $T_\infty$ is a theory by 1.3.

Thus $T_\infty$ is indeed an extension of $T$. Notice that we have already used the fact that $I$ is infinite. Since $\sim\Delta_j \in \bigcap_{i \in I \sim \{j\}} T_i$ we see that $\{\sim\Delta_i \mid i \in I\} \subseteq T_\infty$. In particular, $T_\infty$ is shown to be not contained in any $T_i$. To complete the proof we need only establish the inclusion $T_\infty \subseteq Cl(T \quad \{\Delta_i \mid i \in I\})$. Let $\phi \in T_\infty$. For some finite set $F \subseteq I$, $\phi \in \bigcap_{i \in I \sim F} T_i$. Consider the implication

$$\psi = [[\sim\Delta_{j_0} \wedge \cdots \wedge \sim\Delta_{j_{n-1}}] \longrightarrow \phi],$$

where $F = \{j_0, \ldots, j_{n-1}\}$. Clearly $\psi \in \bigcap_{i \in I \sim F} T_i$ and also $\psi \in T_j$ for $j \in F$. Hence $\psi \in T$, which yields $\phi \in Cl(T \cup \{\sim\Delta_i \mid i \in I\})$ as was to be shown.

Theorem 1.9. Let the sequence of theories $T_i$ satisfy the hypotheses of 1.7 and let $I$ be an infinite set. Then the following three conditions are equivalent

(i) The theory $T$ has only one infinite complete extension;

(ii) The theory $T_\infty$ is complete;

(iii) The sequence of theories $T_i$ converges.

Proof: To establish the equivalence of (i) and (ii) it is sufficient to notice that if $T'$ is any infinite complete extension of $T$, then $T_\infty \subseteq T'$. This follows at once from Lemma 1.8 because any such $T'$ being distinct from each $T_i$ must contain all sentences $\sim\Delta_i$. Thus if (i) holds, then (ii) must hold, since otherwise $T_\infty$ would have two distinct complete extensions neither of which could be equal to any $T_i$ by Lemma 1.8. While on the other hand, if (ii) holds then (i) must by the inclusion mentioned above. We must prove next the equivalence of

(ii) and (iii). Assume (ii) first. It is to be shown that $\limsup_{i\in I} T_i = T_\infty$. Of course $T_\infty \subseteq \limsup_{i\in I} T_i$ since $T_\infty$ was defined as the liminf. By way of contradiction assume that $\phi \in \limsup_{i\in I} T_i$ and $\phi \notin T_\infty$. Since we have assumed that $T_\infty$ is complete, it follows that $\sim \phi \in T_\infty$. Now we have

$$T_\infty = \bigcup_{F\in \mathfrak{F}} \bigcap_{i\in I\sim F} T_i \qquad ,$$

where $\mathfrak{F}$ is the set of all finite subsets of $I$. Hence, there is an $F_0 \in \mathfrak{F}$ such that $\sim \phi \in T_i$ for all $i \in I \sim F_0$. On the other hand,

$$\limsup_{i\in I} T_i = \bigcap_{F\in \mathfrak{F}} \bigcup_{i\in I\sim F} T_i .$$

Whence, $\phi \in \bigcup_{i\in I\sim F_0} T_i$, and so $\phi \in T_i$ for some $i_0 \in I \sim F_0$. However, $\sim \phi \in T_{i_0}$ by the previous argument and this contradiction establishes the desired conclusion that (ii) implies (iii). Finally assume that (iii) holds. We must show for every formula $\phi \notin T_\infty$ that $\sim \phi \in T_\infty$. Suppose that $\phi \notin T_\infty$. Since $T_\infty = \limsup_{i\in I} T_i$ by assumption, we have

$$\phi \notin \bigcap_{F\in \mathfrak{F}} \bigcup_{i\in I\sim F} T_i$$

Hence, for some $F \in \mathfrak{F}$ and for all $i \in I \sim F$, $\phi \notin T_i$ or equivalently $\sim \phi \in T_i$. This is exactly equivalent to saying that $\sim \phi \in T_\infty$ as was to be shown.

Remark: Notice in the avobe proof of the equivalence of (ii) and (iii), the only assumptions used were that $I$ is infinite and all the $T_i$ were complete. In particular the hypothesis on $T$ that was essential for Theorem 1.7 was not used. Of course, this assumption was needed for the equivalence of (i) and (ii).

This concludes our discussion of theories in the abstract, and we shall now turn to particular theories formulated in the first order predicate logic where we shall find some special sufficient conditions implying that a given theory is the limit of a sequence of related theories.

## § 2. Arithmetical Extensions of Finite Degree

Through out the remainder of this study, the only theories considered will be sets of sentences of the first order logic valid in a relational system or in a class of relational systems. In this section particular attention will be given to the notion of one relational system being an arithmetical extension of another. This notion is due to Tarski and the reader is referred to the recent paper [11] by Tarski and Vaught. Below we shall introduce a generalization of the notion of arithmetical extension and prove two theorems (2.7 and 2.9) which give purely algebraic conditions for being an arithmetical extension of finite degree. In one case (2.7) the result generalized a similar condition given by Vaught and presented in Theorem 3.1 of [11].

To begin we recall the relevant definitions.

Definition 2.1. (i) An <u>n-ary relational system</u> is a pair $\mathfrak{R} = \langle A, R\rangle$ where $A$ is a non-empty set and $R$ is an n-ary relation over $A$, i.e. $R \subseteq A^n$. (ii) An n-ary relational system $\mathfrak{R} = \langle A, R\rangle$ is a <u>subsystem</u> of a relational system $\mathfrak{S} = \langle B. S\rangle$, or $\mathfrak{S}$ is an <u>extension</u>

of $\mathfrak{R}$, if $A \subseteq B$ and $R = S \cap A^n$.

It is assumed that the reader understands the notions of <u>isomorphism</u> of one relational system onto another and the notion of an <u>automorphism</u> of a given relational system.

Associated with the class of n-ary relational systems is the set of all sentences of the first order logic involving identity and an n-place predicate symbol. One defines first the set $F$ of <u>well-formed formulas</u>, then the notion of a <u>free variable</u>, and finally the set $S$ of <u>sentences</u>. As our basic symbols we have an infinite sequence $v_o, v_1, v_2, \ldots$ of variables, an n-placed predicate symbol $\mathbb{P}$, the symbol of logical identity $=$, and the logical connectives $\rightarrow$, $\sim$, $\bigvee$, as well as brackets [ and ]. <u>The atomic formulas</u> are all formulas of the form

$$\mathbb{P} v_{i_o} v_{i_1} \cdots v_{i_{n-1}}$$

and

$$v_i = v_j ,$$

where $i_o, \ldots, i_{n-1}$, $i$, and $j$ are integers. The set $F$ is then the least class of formulas containing the atomic formulas and such that whenever $\phi, \psi \in F$, Then $\sim \phi$, $[\phi \rightarrow \psi]$, and $\bigvee v_i \, \phi \in F$. The symbol $\bigvee v_i$ is supposed to stand for the <u>existential quantification</u> with respect to the variable $v_i$. A occurence of a variable $v_i$ in a formula $\phi$ is termed <u>free</u> if the occurence is not contained in any well-formed portion of $\phi$ beginning with an existential quantification with respect to $v_i$; otherwise the occurence is called <u>bound</u>. A formula in which all occurences of all variables are bound is called a sentence and the set of all sentences is denoted by $S$.

In this formalism of the predicate logic we need add only one new definition of operations on formulas to those already introduced in general in Definition 1.1: the operation of universal quantification. This is considered as an addendum to 1.1:

$$\text{(iv)} \bigwedge v_i \phi = \sim \bigvee v_i \sim \phi \,.$$

It is seen that the same terminology of free and bound variables can be applied to contexts involving universal quantifiers.

For the remainder of this section the integer $n$ will be fixed and we shall speak simply of relational systems and formulas with out making special reference ot n-ary relations or the n-placed predicate symbol.

It should be clear in the above what changes to make if more than one predicate symbol is to be considered. Further, we could proceed along standard lines in defining axiomatically the set $L$ of all logically valid sentences. This will not be necessary in our development, however, because after defining when a sentence is true in a relational system, we can simply take $L$ to be the sentences true in all systems. The particular definition of truth used here is that first given by Tarski as explained in Tarski Vaught [11] Definition 1.1 page 84.

Definition 2.2. Let $\mathfrak{R} = \langle A, R \rangle$ be a relational system, $\phi$ a formula, and $x$ a sequence in $A^{(\omega)}$. Then $x$ satisfies $\phi$ in $\mathfrak{R}$ in case one of the following five conditions holds:

(i) $\phi$ is of the form $v_i = v_j$, $i, j < \omega$, and $x_i = x_j$;

(ii) $\phi$ is of the form $\mathbb{P} v_{i_0} v_{i_1} \ldots v_{i_{n-1}}$, $i_0, \ldots, i_{n-1} < \omega$, and $< x_{i_0}, \ldots, x_{i_{n-1}} > \in R$;

(iii) $\phi$ is of the form $\sim \psi$, where $\psi \in F$ and $x$ does not satisfy $\psi$ in $\mathfrak{R}$;

(iv) $\phi$ is of the form $[\psi \rightarrow \chi]$, where $\psi, \chi \in F$ and either $x$ satisfies $\chi$ in $\mathfrak{R}$ or $x$ does not satisfy $\psi$ in $\mathfrak{R}$;

(v) $\phi$ is of the form $\bigvee v_i \psi$, $i < \omega$, where $\psi \in F$ and there is an element $a \in A$ such that $x(i/a)$ satisfies $\psi$ in $\mathfrak{R}$.

As a direct consequence of this definition we obtain:

Lemma 2.3. (i) Let $\mathfrak{R} = < A, R >$ be a relational system and let $\phi$ be a formula whose only free variables are $v_{i_0}, \ldots, v_{i_{k-1}}$. Let $x, y$ be sequences in $A^{(\omega)}$ such that $x_{i_0} = y_{i_0}, \ldots, x_{i_{k-1}} = y_{k_{k-1}}$. Then $x$ satisfies $\phi$ in $\mathfrak{R}$ if and only if $y$ satisfies $\phi$ in $\mathfrak{R}$; (ii) Let $\mathfrak{S} = < B, S >$ be a system isomorphic to $\mathfrak{R}$ by a function $f$ from $A$ onto $B$. Then $x$ satisfies $\phi$ in $\mathfrak{R}$ if and only if the sequence $fx = < f(x_0), \ldots, f(x_i), \ldots >$ satisfies $\phi$ in $\mathfrak{S}$.

Definition 2.4. (i) A sentence $\phi$ is <u>true</u> in a relational system $\mathfrak{R} = < A, R >$ if and only if every sequence $x \in A^{(\omega)}$ satisfies $\phi$ in $\mathfrak{R}$. (ii) A sentence $\phi$ is <u>true</u> in a class $K$ of relational systems if and only if $\phi$ is true in each $\mathfrak{R} \in K$.

Clearly by Lemma 2.3 we can conclude that $\phi$ is true in $\mathfrak{R}$ if it is satisfied by <u>some</u> sequence $x$; further if $\mathfrak{S}$ is isomorphic to $\mathfrak{R}$, then $\phi$ is true in $\mathfrak{R}$ if and only if $\phi$ is true in $\mathfrak{S}$.

Definition 2.5. Let $\mathfrak{R}$ be a relational system. The set of all sentences true in $\mathfrak{R}$ is denoted by $Th(\mathfrak{R})$ and is called the theory of The set of all sentences true in all relational systems is denoted by $L$.

With the definition of $L$ just given it is quite easy to verify that $S$ and $L$ satisfy all the conditions of Section 1 under the operations $\rightarrow$ and $\sim$ and that each of the sets $Th(\mathfrak{R})$ is a complete theory in the sense of Definition 1.6. It is of course the content of the Completeness Theorem for First-Order Logic that every complete extension of $L$ is of the form $Th(\mathfrak{R})$ for a suitable relational system $\mathfrak{R}$. We shall not go into details about the Completeness Theorem since it is not used in this paper.

It was pointed out above that if $\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic, then $Th(\mathfrak{R}) = Th(\mathfrak{S})$. The condition of isomorphism is much stronger than the condition of having the same theory, which Tarski has called being arithmetically equivalent. In particular, a subsystem of a relational system may have the same theory as the whole system itself. This specific notion is not too useful, but there is a very useful notion of being an arithmetical extension which was introduced by Tarski and is investigated in [11]. For our purposes we need a more general notion which will now be defined.

Definition 2.6. The system $\mathfrak{S} = \langle B, S \rangle$ is called an arithmetical extension of degree $m$ of a system $\mathfrak{R}$ if the following two conditions hold:

(i) $\mathfrak{S}$ is an extension of $\mathfrak{R}$ ;

(ii) for every formula $\phi$ with at most $m$ distinct variables and every sequence $x \in A^{(\omega)}$, $x$ satisfies $\phi$ in $\mathfrak{R}$ if and only if $x$ satisfies $\phi$ in $\mathfrak{S}$ .

Remark: It would be sufficient in the above definition to restrict ourselves to formulas involving only the variables $v_0, \ldots, v_{m-1}$, and then we could use sequences $x \in A^m$. Notice that the definition refers to the total number of variables in the formula, both free variables and bound variables, and that the same variable is permitted to occur in many different quantifiers.

The relation of this notion to that defined by Tarski and Vaught is simple: $\mathfrak{S}$ is an arithmetical extension of $\mathfrak{R}$ if and only if for all $m$, $\mathfrak{S}$ is an arithmetical extension of $\mathfrak{R}$ of degree $m$.

It is now possible ot state and prove one of our main results on arithmetical extensions of finite degree, which is a direct generalization of the Theorem 3.1 of [ ].

Theorem 2.7. The following two conditions are (jointly) sufficient for a system $\mathfrak{S} = \langle B, S \rangle$ to be an arithmetical extension of degree $m$ of a system $\mathfrak{R} = \langle A, R \rangle$ :

(i) $\mathfrak{S}$ is an extension of $\mathfrak{R}$ ;

(ii) for any subset $A'$ of $A$ with less than $m$ elements and for any element $b \in B$, there exists an automorphism $f$ of $\mathfrak{S}$ which leaves $A'$ pointwise fixed and $f(b) \in A$.

Proof: Using the remark following Definition 2.6 we consider only formulas with variables $v_0, \ldots, v_{m-1}$ at most and only sequences in $A^m$. Assuming conditions (i) and (ii) above, the following is the statement to be proved for all formulas $\phi$ of the restricted type:

(*) for all $x \in A^m$, $x$ satisfies $\phi$ in $\mathfrak{A}$ if and only if $x$ satisfies $\phi$ in $\mathfrak{G}$ .

In as much as satisfaction is defined in terms of the complexity of the formula, the proof of (*) must proceed from simple formulas to more complicated ones. It should be obvious that (*) holds for atomic formulas. It is hardly less obvious that if (*) holds for formulas $\phi$ and $\psi$, then it must hold for the formulas $\sim \phi$ and $[\phi \to \psi]$. Suppose now that (*) holds for $\phi$ and consider the formula $\bigvee v_i \phi$ where $i < m$. Assume first that $x \in A^m$, and $x$ satisfies $\bigvee v_i \phi$ in $\mathfrak{A}$. Hence, for some $a \in A$, $x(i/a)$ satisfies $\phi$ in $\mathfrak{A}$. Whence, $x(i/a)$ satisfies $\phi$ in $\mathfrak{G}$, and so $x$ satisfies $\bigvee v_i \phi$ in $\mathfrak{G}$ also. Assume next that $x$ satisfies $\bigvee v_i \phi$ in $\mathfrak{G}$. Thus, there is a $b \in B$ such that $x(i/b)$ satisfies $\phi$ in $\mathfrak{G}$. Let $A' = \{ x_j \mid j < m \; j \neq i \}$ Clearly $A'$ is a subset of $A$ with fewer than $m$ elements. By condition (ii) let $f$ be an automorphism of $\mathfrak{G}$ that leaves $A'$ pointwise fixed and $f(b) \in A$. Obviously $f\, x(i/b) = x(i/f(b))$, and hence, by Lemma 2.3, $x(i/f(b))$ satisfies $\phi$ in $\mathfrak{G}$. But $x(i/f(b)) \in A^m$ and so $x(i/f(b))$ satisfies $\phi$ in $\mathfrak{A}$ and finally $x$ satisfies $\bigvee v_i \phi$ in $\mathfrak{A}$, which completes the proof.

Notice that Theorem 3.1 of [ ] is a direct consequence of the above result. In fact, Theorem 2.7 was suggested to the author after seeing the paper of Tarski and Vaught. Previous to the appearance of [11] the author had worked with the conditions given below in Theorem 2.9 which are somewhat more complicated but which do not require the existence of automorphisms of the whole of the system $\mathfrak{S}$ and can therefore be applied in a somewhat broader class of situations. Before discussing these conditions, we give the simple result that relates the notion of convergence to the notion of arithmetical extensions of finite degree.

Theorem 2.8. Let $I = \{1,2,3,\ldots\}$ and let $\mathfrak{S}$ be a relational system having subsystems $\mathfrak{R}_i$, $i \in I$, such that $\mathfrak{S}$ is an arithmetical extension of degree $i$ of $\mathfrak{R}_i$ for each $i \in I$. Then

$$\mathrm{Th}(\mathfrak{S}) = \lim_{i \in I} \mathrm{Th}(\mathfrak{R}_i)$$

Proof: Let $\phi$ be any formula. Suppose the number of distinct variables in $\phi$ is $m$. By definition 2.6 we see that $\phi \in \mathrm{Th}(\mathfrak{S})$ if and only if $\phi \in \mathrm{Th}(\mathfrak{R}_i)$ for all $i \geq m$. This implies at once the formula

$$\mathrm{Th}(\mathfrak{S}) = \liminf_{i \in I} \mathrm{Th}(\mathfrak{R}_i)$$

By the remark following Theorem 1.9 and the fact that $\mathrm{Th}(\mathfrak{S})$ is complete, it follows that the sequence of theories $\mathrm{Th}(\mathfrak{R}_i)$ converges and the theorem is established.

Theorem 2.9. Let $\mathfrak{S} = \langle B, S \rangle$ be a relational system and let $K$ be a class of subsystems of $\mathfrak{S}$. The following two conditions are (jointly) sufficient for $\mathfrak{S}$ to be an arithmetical extension of degree $m$ of each system in $K$:

(i) for each subset $B'$ of $B$ with at most $m$ elements, there is a system $\mathfrak{R} = \langle A, R \rangle$ in $K$ with $B' \subseteq A$;

(ii) for any two systems $\mathfrak{R}_1 = \langle A_1, R_1 \rangle$ and $\mathfrak{R}_2 = \langle A_2, R_2 \rangle$ in $K$ and for each set $A' \subseteq A_1 \cap A_2$ with fewer than $m$ elements there is an isomorphism $f$ of $\mathfrak{R}_1$ onto $\mathfrak{R}_2$ leaving $A'$ pointwise fixed.

Proof: Consider only formulas involving the variables $v_0, \ldots, v_{m-1}$. For all such formulas $\phi$ it is to be shown that

(**) for all systems $\mathfrak{R} = \langle A, R \rangle$ in $K$ and for all $x \in A^m$, $x$ satisfies $\phi$ in $\mathfrak{R}$ if and only if $x$ satisfies $\phi$ in $\mathfrak{S}$.

The proof is exactly similar to the proof of Theorem 2.7, the only difficulty being in showing that if (**) holds for a formula $\phi$ it also holds for $\bigvee v_i \, \phi$, where $i < m$. Further, the only problem here is in the following implication:

if $\mathfrak{R} = \langle A, R \rangle \in K$ and $x \in A^m$ and $x$ satisfies $\bigvee v_i \, \phi$ in $\mathfrak{S}$, then $x$ satisfies $\bigvee v_i \, \phi$ in $\mathfrak{R}$, where it has been assumed that (**) holds for $\phi$. Assuming the hypotheses of the above implication, we conclude at once that for some $b \in B$, $x(i/b)$ satisfies $\phi$ in $\mathfrak{S}$. Let $B' = \{ x_j \mid j < m, i \neq j\} \cup \{ b\}$. $B'$ has at most $m$ elements and so from (i) we see that there is an $\mathfrak{R}_1 = \langle A_1, R_1 \rangle$ in $K$ with $B' \subseteq A_1$. Obviously $x(i/b) \in A_1{}^m$ and since (**) holds for $\phi$ we find that $x(i/b)$ satisfies $\phi$ in $\mathfrak{R}_1$. Using condition (ii) let $f$ be an isomorphism of $\mathfrak{R}_1$ onto $\mathfrak{R}$ leaving fixed the set $A' = \{ x_j \mid j < m, i \neq j \}$. By lemma 2.3, $x(i/f(b))$ satisfies $\phi$ in $\mathfrak{R}$ and hence $x$ satisfies $\bigvee v_i \, \phi$ in $\mathfrak{R}$ also. This completes the proof.

# Part II. Euclidean Geometry

## § 3. Euclidean Spaces

In section 4 it will be shown that the arithmetical extensions of finite degree appear wholesale in relational systems derived from Euclidean spaces. This state of affairs has considerable import in an axiomatic discussion of Euclidean geometry as is indicated in section 5. In this section we shall be concerned solely with definitions of the basic notions and a few simple geometrical lemmas.

The notion of a finite dimensional Euclidean space is of course clear to any mathematician. However, the results of this paper apply equally well to infinite dimensional spaces, and so it is better to give the explicit definitions. Particularly so, because there are so many kinds of infinite dimensional spaces of all cardinalities, some complete, some not complete, and so on.

Definition 3.1. A <u>Euclidean space</u> is a real vector space together with a positive definite, symmetric, bilinear form called the inner product.

To be even more formal we could say that a Euclidean space is a system $\langle V, +, 0, \cdot, \bullet \rangle$ where $V$ is a non-empty set called the set of <u>points</u> of the space; $\langle V, +, 0 \rangle$ is an abelian group; $\cdot$ is an operation, called <u>scalar product</u>, from the cartesian product of the set of real numbers with the set $V$ which takes values in $V$ and satisfies the following conditions for all reals $\alpha, \beta$ and all $x, y \in V$ :

(i) $1 \cdot x = x$ ;

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$ ;

(iii) $(\alpha+\beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x)$ ;

(iv) $\alpha \cdot (x+y) = (\alpha \cdot x) + (\alpha \cdot y)$ ;

and finally $\bullet$ is an operation from pairs of points in V to real numbers such that for all reals $\alpha$, $\beta$ and all $x, y, z \in V$ :

(v) if $x \neq 0$, then $x \bullet x > 0$

(vi) $x \bullet y = y \bullet x$

(vii) $((\alpha \cdot x) + (\beta \cdot y)) \bullet z = \alpha \cdot (x \bullet z) + \beta \cdot (y \bullet z)$

German capital letters $\mathfrak{V}$ and $\mathfrak{W}$ will be used to denote Euclidean spaces, and the corresponding sets of points will be denoted by V and W.

The inner product naturally carries along with it the notions of <u>perpendicularity</u> and <u>distance</u>. Thinking of points of the space $\mathfrak{V}$ as vectors sticking out of the origin, we say that two vectors x and y are <u>perpendicular</u> if $x \bullet y = 0$. The <u>distance</u> between the points x and y is denoted by $\| x-y \|$ and is defined by the equation

$$\| x-y \| = +\sqrt{(x-y) \bullet (x-y)} \quad ,$$

where $x - y$ on the right-hand side of course stands for the vector combination $(x + (-1) \cdot y)$.

There are two types of subspaces to be considered in a Euclidean space $\mathfrak{V}$ : <u>linear</u> or <u>vector subspaces</u> and <u>affine subspaces</u>. A <u>vector subspace</u> is a subset X of V containing 0 and closed under vector addition and multiplication by arbitrary scalars. A vector

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subspace can at once be considered as a Euclidean space by simply restricting the inner product to the subset. An <u>affine subspace</u> is subset $Y$ parallel to a vector subspace; in more precise terms, $Y$ is an affine subspace if and only if the set $X = \{ y_1 - y_2 \mid y_1, y_2 \in Y \}$ is a vector subspace. Another way to phrase the definition is to require that if $y \in Y$, then the translate of $Y$ by $-y$, $Y - y = \{y_1 - y \mid y_1 \in Y \}$, is a vector subspace. Still a third equivalent condition is to demand that whenever $y_1\, y_2 \in Y$, then $\alpha \cdot y_1 + (1-\alpha) \cdot y_2 \in Y$ for all real numbers $\alpha$. From the point of view of vector spaces, linear subspaces and affine subspaces are different, since a linear subspace must pass through the origin. However, from the point of view of geometry which does not distinguish any particular point in space, the two notions are essentially equivalent. This is made clear by introducing isometries. An <u>isometry</u> between subsets $X$ of a space $\mathcal{V}$ and $Y$ of a space $\mathcal{W}$ is a one-one function $f$ from $X$ that preserves distances ; in other words, if $x_1, x_2 \in X$ then $\| f(x_1) - f(x_2) \| = \| x_1 - x_2 \|$, where the distance on the left refers to the space $\mathcal{W}$ and on the right to the space $\mathcal{V}$ . Isometries preserve all geometric connections between points that are defined without reference to a fixed origin. In particular, if $X$ is any subset of a space $\mathcal{V}$ and $x_0$ is any point in $V$, then obviously $X$ is mapped isometrically onto $X + x_0$ by the function $f(x_1) = x_1 + x_0$ for all $x_1 \in X$. Hence, any affine subspace is isometric with a vector subspace.

The final concept needed from the standard theory of vector spaces is that of <u>dimension</u>. Every vector space has a linear dimension equal to the maximum number of linearly independent vectors contained in the space. The notion of <u>independence</u> can be defined quite simply: a vector $x$ is <u>independent</u> of a set $X$ contained in a space $\mathcal{V}$ if there is a subspace containing $X$ but not containing $x$; a set $Y$ of vectors is an <u>independent set</u> if each $y \in Y$ is independent of $Y \sim \{y\}$. The usual trick of applying some form of the axiom of choice will prove that a vector space always contains a maximal independent set of vectors, and any two such sets have the same cardinality. This cardinal number is the <u>dimension</u> of the space. The fundamental result about Euclidean spaces is that two Euclidean spaces of the same finite linear dimension are in fact isometric. Hence, there is only one type of Euclidean geometry for each finite dimension. As is well-known, this is false for infinite dimensional spaces of dimension greater than $\aleph_0$. The main question that is investigated here is whether one can find any elementary geometric properties that can distinguish between infinite dimensional spaces that are not isometric. The answer is <u>no</u>, as will be shown in the next section.

In order to prove our results about the geometric theories of Euclidean spaces, only two purely geometric results are needed, both of which are very elementary:

Lemma 3.2 If $\mathcal{V}$ is a Euclidean space and $X$ is a set of at most $m$ elements, $m > 0$, then $X$ is contained in some affine subspace of $\mathcal{V}$ of dimension at most $m - 1$.

Lemma 3.3. If $\mathcal{V}$ is a Euclidean space and $X$ and $Y$ are two affine subspaces of the same finite dimension, then there is an isometry of $\mathcal{V}$ onto itself mapping $X$ onto $Y$ and leaving the set $X \cap Y$

pointwise fixed.

Lemma 3.2 is of course quite obvious and 3.3 is really a special case of a theorem of Witt. Let $\mathcal{Z}$ be a finite dimensional subspace of $\mathcal{V}$ containing both $X$ and $Y$. Without loss of generality we can assume $X \cap Y \neq 0$ since otherwise they would be parallel and a translation would take one onto the other. In fact, we can assume $0 \in X \cap Y$, and hence they are both linear subspaces. Let $\mathcal{Z}^{\perp} = \{ w \mid w \in V$ and $w \bullet z = 0$ for all $z \in \mathcal{Z} \}$. One may show that $V = \mathcal{Z} \oplus \mathcal{Z}^{\perp}$ is a direct decomposition of $V$ in the usual vector space sense (See e.g. Halmos [3] p. ) and that any isometry of $\mathcal{Z}$ onto itself can be extended to all of $V$ by leaving $\mathcal{Z}^{\perp}$ entirely fixed. Thus, the problem is strictly finite dimensional, and the rest of the proof can be copied from the more general theorem given in Artin [2] page 121.

## § 4 Geometrical Relations

A geometrically meaningful concept should be invariant under all rigid motions of space. Interpreting space as a Euclidean space, the rigid motions are just the isometries. Finally we can take the invariance property as the only distinguishing feature of geometrical notions.

Definition 4.1. An <u>n-ary geometrical relation</u> over a Euclidean space $\mathcal{V}$ is a subset $R$ of the cartesian power $V^n$ such that for all $x_0, \ldots, x_{n-1} \in V$ and all isometries $f$ of $\mathcal{V}$ onto itself,

$< x_0, \ldots, x_{n-1} > \in R$ if and only if $< f(x_0), \ldots, f(x_{n-1}) > \in R$.

Without further delay, the fundamental result can be given:

Theorem 4.2. If $R$ is an n-ary geometrical relation over a Euclidean space $\mathcal{V}$ and if $X$ is an affine subspace of dimension at least $m - 1$, then the relational system $< V, R >$ is an arithmetical extension of degree $m$ of the system $< X, R \cap X^n >$.

Proof. We need only verify condition (ii) of Theorem 2.7. Let $X'$ be a subset of $X$ with fewer than $m$ elements. Obviously we need only consider the case $m > 1$. Let $Y$ be an affine subspace of dimension exactly $m - 2$ containing $X'$. Let $Y_0$ be a larger subspace of $X$ of dimension exactly $m - 1$ containing $Y$. Consider any point $b \in V$ not already in $X$. Let $Y_1$ be a subspace of $V$ of dimension exactly $m - 1$ containing $Y$ and the point $b$. Since $b \notin X$, we conclude $Y_0 \cap Y_1 = Y$. By Lemma 3.3, let $f$ be an isometry of $\mathcal{V}$ onto itself leaving $Y$ fixed and mapping $Y_1$ onto $Y_0$. The function $f$ is thus an automorphism of $< V, R >$ satisfying the required conditions of leaving $X'$ pointwise fixed and with $f(b) \in X$.

Corollary 4.3. If $R$ is an n-ary geometrical relation over a Euclidean space $\mathcal{V}$ and if $X$ is an infinite dimensional subspace of $\mathcal{V}$, then the relational system $< V, R >$ is an arithmetical extension of $< X, R \cap X^n >$.

Corollary 4.4. If $R$ is an n-ary geometrical relation over a Euclidean space $\mathcal{V}$ and if $X$ and $Y$ are two infinite dimensional subspaces of $\mathcal{V}$ then the relational systems $< X, R \cap X^n >$ and $< Y, R \cap Y^n >$ are arithmetically equivalent.

Corollary 4.5. If $R$ is an n-ary geometrical relation over a Euclidean space $\mathcal{V}$ of infinite dimension and if $X_i$ is an $i$ dimensional subspace of $\mathcal{V}$ for $i = 0,1,2,\ldots$. Then

$$Th(< V, R >) = \lim_i Th(< X_i, R \cap X_i^n >).$$

The preceding theorem and corollaries answer completely the problem of whether there are any properties expressible in first-order logic about a geometrically meaningful relation that can distinguish between infinite dimensional subspaces of a given space: the answer is quite clearly no. It is only a matter of correct formulation to put this result in its proper setting when applied to the class of all Euclidean spaces. One must define first what a geometrically meaningful notion is for the class of all spaces.

Definition 4.6. An <u>n-ary geometrical relation</u> over the class of all Euclidean spaces is a function $R$ that assigns to each space $\mathcal{V}$ a subset $R_{\mathcal{V}}$ of $V^n$ such that if $\mathcal{V}$ and $\mathcal{W}$ are two Euclidean spaces and $f$ is an isometry of $\mathcal{V}$ into $\mathcal{W}$, then for all $x_o, \ldots, x_{n-1} \in V^n$, $< x_o, \ldots, x_{n-1} > \in R_{\mathcal{V}}$ if and only if $< f(x_o), \ldots, f(x_{n-1}) > \in R_{\mathcal{W}}$.

It is to be stressed that the above definition requires that the invariance must hold for all isometries from one space to <u>subspaces</u> of another space. In other words, the assignment of relations must be completely insensitive to dimension. There is obviously no reason to consider assignments of totally unrelated concepts to spaces of different dimensions.

There are many, many examples of geometrical relations as should be at once obvious. Any notion such as betweeness, equidistance, congruence of triangles, midpoints, centers of gravity all have explicit definitions as geometrical relations over the class of all Euclidean spaces. As far as the binary relations go, they can be described very simply: Every binary geometrical relation R corresponds to a set D of nonnegative real numbers such that for any space $\mathcal{V}$ and all $x, y \in V$:

$$\langle x,y\rangle \in R_{\mathcal{V}} \text{ if and only if } \|x-y\| \in D.$$

Conversely, every set D determines a geometrical relation. Hence, there are $2^{\mathfrak{c}}$ distinct binary geometrical relations, where $\mathfrak{c}$ is the power of the continuum.

Using this last definition we can now prove at once from the preceding results

Theorem 4.7. If R is a geometrical relation over the class of all Euclidean spaces, and $\mathcal{V}$ and $\mathcal{W}$ are infinite dimensional Euclidean spaces, then the relational systems $\langle V, R_{\mathcal{V}}\rangle$ and $\langle W, R_{\mathcal{W}}\rangle$ are arithmetically equivalent.

It should be pointed out that even though the above results were formulated with the use of a single geometrical relation, there is no difficulty in extending them to hold for any sequence of geometrical relations, finite or transfinite.

## § 5 Axiomatic Geometry

It will not be our purpose here to give axioms for Euclidean Geometry; this has been done elsewhere (see Tarski [10] and Schwabhäuser [7]).

Rather we shall be concerned with certain facts about the process of axiomatization and in particular about the consequences for a specific system of geometry that we obtain from the results of the preceeding sections.

A natural selection of primitive notions for the theory of geometry is the pair of geometrical relations betweeness and equidistance; these were selected in the paper of Tarski [10], while a much more complicated formalism was used by Schwabhäuser in [7] which was more in line with the Hilbert system . . For dimensions greater than 1, Pieri in [5] has essentially shown that betweeness can be defined interms of equidistance, and hence we can dispense with one relation. However, in order to include one-dimensional geometry it is necessary to either use both relations or to introduce a stronger relation, say the ordering of distances rather than the equality of distances. Inasmuch as Tarski in [10] has given such a simple set of axioms for two dimensions in terms of the two notions, we shall use them for the purposes of discussion.

<u>Betweeness</u> is a ternary geometrical relation, devoted by $B$, such that if $\mathcal{V}$ is a Euclidean space and $x, y, z \in V$, then $\langle x, y, z \rangle \in B_{\mathcal{V}}$ if and only if there is a real number $\alpha \geq 0$ such that $x = \alpha \cdot y + (1-\alpha)z$; in other words, $x$ lies between $y$ and $z$ on the line joining them. Another way to write the condition is by the equation

$$\| y-z \| = \| x-y \| + \| x-z \|$$

<u>Equidistance</u> is a quaternary geometrical relation, denoted by $E$, such that if $\mathcal{V}$ is a Euclidean space and $x, y, z, w \in V$, then $\langle x, y, z, w \rangle \in E_{\mathcal{V}}$ if and only if $\| x-y \| = \| z-w \|$. It is obvious

that these two relations are geometrical over all Euclidean spaces.

Definition 5.1. (i) The <u>elementary geometrical structure</u> of a Euclidean space $\mathcal{V}$ is the relational system $< V, B_{\mathcal{V}}, E_{\mathcal{V}} >$;

(ii) The <u>elementary geometrical theory</u> of a Euclidean space is the theory $Th(< V, B_{\mathcal{V}}, E_{\mathcal{V}} >)$.

Since we know that all Euclidean spaces of the same finite dimension are isometric, it follows that for each dimension $m$ there is only one theory under consideration. This leads us to

Definition 5.2. $G_m$ denotes the unique theory of betweeness and equidistance in Euclidean spaces of dimension $m$ ; in other words, $G_m = Th(< V, B_{\mathcal{V}}, E_{\mathcal{V}} >)$ for all Euclidean spaces $\mathcal{V}$ of dimension $m$.

By Theorem 4.7 all infinite dimensional relational systems made from geometrical relations are arithmetically equivalent. Thus, the next definition is proper.

Definition 5.3. $G_\infty$ denotes the unique theory of betweeness and equidistance in Euclidean spaces of infinite dimension; in other words $G_\infty = Th(< V, B_{\mathcal{V}}, E_{\mathcal{V}} >)$ for all infinite dimensional Euclidean Spaces $\mathcal{V}$.

Finally in analogy with the notation in section 1 we have

Definition 5.4. $G = \bigcap_{m<\omega} G_m$ , in other words, $G$ is the set of sentences about betweeness and equidistance true in all finite dimensions.

From Corollary 4.5 we derive at once

Theorem 5.5. $G_\infty = \lim_{m<\omega} G_m$, in other words, the following four conditions are equivalent about a sentence $\phi$ formulated interms of predicates denoting betweeness and equidistance:

(i) $\phi$ is true in some infinite dimensional space;

(ii) $\phi$ is true in all infinite dimensional spaces;

(iii) $\phi$ is true in all but a finite number of finite dimensions;

(iv) $\phi$ is true in infinitely many finite dimensions.

In consequence we see that not only is there no elementary sentence which can distinguish between infinite dimensional spaces, but also there is not even a sentence true, say, in <u>even</u> finite dimensions but not true in the <u>odd</u> dimensions.

In order to make full use of the general theorems of Section 1, a small lemma is needed.

Lemma 5.6. $G \neq \bigcap \{G_m \mid m<\omega,\ m\neq k\}$, for all $k < \omega$.

Proof. We need to construct a particular sentence $\Delta_k$ such that $\Delta_k \in G_k$ but $\Delta_k \notin G_m$ for all $m < \omega$, $m \neq k$. This requires a simple translation into logical symbols of the sentence: There exists a sequence of $k + 1$ mutually equidistant points which can not be enlarged to a collection of $k + 2$ mutually equidistant points.

With $\Delta_k$ the specific sequence of sentences mentioned in the above proof and with the aid of Theorems 1.7, 1.8, and 1.9 we can prove at once

Theorem 5.7. (i) for each $m < \omega$, $G_m = Cl(G \cup \{\Delta_m\})$;

(ii) the only finite complete extensions of $G$ are the $G_m$;

(iii) $G_\infty = Cl(G \cup \{\sim \Delta_m \mid m<\omega\})$;

(iv) $G_\infty$ is the only infinite complete extension of $G$.

The obvious conclusion of this theorem is that if an axiomatization of Elementary Euclidean Geometry of different dimensions is wanted, then it will be only necessary to axiomatize the incomplete theory $G$ of sentences true in all finite dimensions, since the proper axiomatization of any one particular dimension or of the infinite dimensional theory will result by the simple adjunction of sentences already determined.

Another consequence of these considerations can be drawn from the fact that all complete extensions of $\mathfrak{G}$ have been found. This means that each possible model of the theory $\mathfrak{G}$ is arithmetically equivalent to some standard model based on a Euclidean space. It is a well-known fact that an arbitrary class of relational systems may lead to other relational systems satisfying all sentences true in all systems in the given class, but these new systems need not be equivalent to any of the old. The simplest example is of course the class of all finite relational systems. We have been spared this situation in geometry, however.

Finally the problem of an effective decision method must be considered. In the monograph [9] by Tarski it is shown that the algebra of real numbers has a theory in which it can be effectively decided whether a formula is true or not. As is pointed out there, we then have an effective decision method for geometry; in fact, there will be a uniform method such that for all $m < \omega$ it can be decided whether any sentence $\phi$ is in $G_m$ or not. Having this method we can now state the existence of further decision methods:

Theorem 5.8. The theories $G$ and $\mathfrak{G}_\infty$ are decidable.

Proof. Let $\phi$ be any formula with at most $m + 1$ variables. We have shown in effect that

$$\phi \in G \text{ if and only if } \phi \in G_k \text{ for all } k \leq m ,$$

and that

$$\phi \in \mathfrak{G}_\infty \text{ if and only if } \phi \in G_m .$$

It is then at once obvious that the truth of each of the statements $\phi \in \mathfrak{G}$ and $\phi \in \mathfrak{G}_\infty$ can be checked in a finite number of steps.

In closing this discussion of theories of geometry, one small fact deserves attention. In Lemma 5.6 a sentence $\Delta_m$ was constructed so that $\Delta_m$ was true in m-dimensional space but in no other. Notice that if $\Delta_m$ were written out in full it would require $m + 2$ variables for its formulation. This is a best possible result, for if $\Delta_m$ were equivalent with a sentence in $m + 1$ variables, then since $\Delta_m \in G_m$, $\Delta_m$ would also be in $G_k$ for all $k \geq m$, which is quite impossible.

## Part III Equivalence Relations and Related Theories

### § 6. The Identity Calculus and the Monadic Predicate Calculus.

In Part II of this work the general theorems of Part I were applied to relational systems derived from Euclidean spaces. In this and the following sections certain direct applications of the method will be considered involving much simpler relational systems having to do with equivalence relations and singularly or monadic predicates. The easiest case of all concerns relational systems with no relations at all. That is, one is given a domain of individuals about which the only permissible first order statements are those containing identity as the only predicate. As is well-known (see Ackermann [1] p24) there is an effective decision method for the universal validity of sentences in the pure identity calculus. Indeed it can be shown ( [1] p24 ) that there is a number that can be effectively calculated by examination of the formula such that if the formula is true in a domain with no fewer than that number of elements, then it is true in all larger domains. This situation is very similar to that which we found to hold in Euclidean spaces, where it was not the cardinality but rather the dimension of the domain that was important. This analogy suggested to the author that similar methods could be applied, and the program was successful. Thus, new proofs of old results are obtained by a uniform method and certain new results are also easily proved. An advantage of the method is noted: the procedure is purely model-theoretic and no reduction of formulas to a normal form is required. Thus, for example, Ackermann in [1], chapter III, p. 24 must assume that the formula of the identity calculus is in prenex normal form; while Janiczak in [4] p.137 gives a decision method for the theory of one equivalence relation by the method of elimination of quantifiers for a systematic reduction to a normal form. Both of these devices are completely avoided in the present treatment.

In the following two theorems we assume that we are working in a predicate calculus with identity as the only predicate; in particular the term arithmetical extension should be considered as relativised to this formalism.

Theorem 6.1. Any domain is an arithmetical extension of degree $m$ of any subdomain containing at least $m$ elements.

Proof: We need verify only condition (ii) of Theorem 2.7. Let then $B$ be a domain of elements and $A$ be a subdomain with at least $m$ elements. Let $A' \subseteq A$ where $A'$ has fewer than $m$ elements and let $b \in B$. Any permutation of $B$ is an automorphism of the identity relation over $B$, and since $A' \neq A$, it is clear that there is a permutation $f$ of $B$ leaving $A'$ pointwise fixed and such that $f(b) \in A$. This completes the proof.

Corollary 6.2. A sentence $\phi$ of the identity calculus with at most $m$ variables is true in all domains if and only if it is true in all domains with at most $m$ elements; hence, there is an effective procedure for determining the universal validity of arbitrary sentences of that calculus.

We turn now to the consideration of somewhat more complicated relational systems involving one-placed or singularly predicates. We think of a singularly predicates as simple a subset of the domain and will discuss relational systems of the form $\mathfrak{P} = \langle A, P_0, P_1, \ldots, P_{k-1} \rangle$ where for $i < k$, $P_i \subseteq A$. At first it is more convenient to work with a special class of such systems.

Definition 6.3. A system $\mathfrak{P} = \langle A, P_0, \ldots, P_{k-1} \rangle$ is called a <u>partition system</u> of order $k$ if for each $i < k$, $P_i \subseteq A$, and $A = \bigcup_{i<k} P_i$ and for $i < j < k$, $P_i \cap P_j = 0$.

Obviously every subsystem of a partition system is again a partition system, since there is no requirement that the sets $P_i$ be non-empty.

Definition 6.4. A subsystem $\mathcal{P} = < A, P_o,\ldots,P_{k-1} >$ of a partition system $\mathcal{Q} = < B, Q_o,\ldots,Q_{k-1} >$ is <u>m-dimensional</u> if and only if for each $i < k$ the cardinality of $P_i$ is the minimum of the number $m$ and the cardinality of $Q_i$.

Let the integer $k > 0$ be fixed for the remainder of the discussion. The metamathematical notions will now involve the use of the predicate calculus of the first order with $k$ singularly predicate symbols, say $\mathbb{P}_o,\ldots,\mathbb{P}_{k-1}$ . Clearly if $\mathcal{R} = < A, P_o,\ldots,P_{k-1} >$ is a partition system, then a formula of the formal logic should be interpreted in such a way that the symbol $\mathbb{P}_j$ corresponds to the set $P_i$. Using the notion of dimension introduced above we can now easily prove a theorem on arithmetical extensions which again leads to an effective decision procedure through the use of finite domains.

Theorem 6.5. A partition system is an arithmetical extension of degree $m$ of each of its subsystems of dimension at least $m$.

Proof. As before we shall verify condition (ii) of Theorem 2.7. Let $\mathcal{Q} = < B, Q_o,\ldots,Q_{k-1} >$ be a partition system and let $\mathcal{P} = < A, P_o,\ldots,P_{k-1} >$ be a subsystem of dimension at least $m$. Let $A'$ be a subset of $A$ with less than $m$ elements and let $b \in B$ but $b \notin A$. Since $\mathcal{Q}$ is a partition system, $b \in Q_i$ for some $i < k$. Now $P_i \cap A'$ has fewer than $m$ elements and $b \notin P_i$ so that $Q_i$ has more elements than $P_i$. But $\mathcal{P}$ was by assumption an m-dimensional subsystem of $\mathcal{Q}$ , and therefore $P_i$ must have exactly $m$ elements. Thus, there is an element $b' \in P_i$ with $b' \notin A'$. Let $f$ be the function that is the identity function on all elements of $B$ other than $b$ and $b'$ and set $f(b) = b'$ and $f(b') = b$. Since $f$ restricted to each $Q_j$ is a permutation of these sets, $f$ is an automorphism of $\mathfrak{Q}$. Clearly $f$ leaves $A'$ pointwise fixed and by definition $f(b) \in A$, as was to be shown.

Corollary 6.6. A sentence $\phi$ with at most $m$ variables and involving besides the identity symbol only the singulary predicates $\mathbb{P}_0, \ldots, \mathbb{P}_{k-1}$ is true in all partition systems of order $k$ if and only if it is true in all finite partition systems or order $k$ with at most $k \cdot m$ elements; hence there is an effective procedure for determining the validity of arbitrary sentences in the class of all partition systems.

Proof: We only have to note that an m-dimensional subsystem of a partition system contains at most $k \cdot m$ elements. Hence if $\phi$ fails in any partition, by 6.5 it must fail in an m-dimensional subsystem.

Remark. It should be noted that the number $k \cdot m$ in the above Corollary cannot be improved. For consider the sentence $\phi$ formed as the disjunction over all $i < k$ of the following sentences:

$$\bigvee v_0 \bigvee v_1 \ldots \bigvee v_{m-2} \bigwedge v_{m-1} \Big[\mathbb{P}_i (v_{m-1}) \longrightarrow \big[ v_{m-1} = v_0 \vee v_{m-1} = v_1 \vee \ldots \vee v_{m-1} = v_{m-2} \big]\Big]$$

The sentence $\phi$ contains exactly $m$ variables and expresses the fact that at least one of the sets in the partition contains less than m-elements. Clearly then, $\phi$ is true in all partition systems with fewer than $k \cdot m$ elements, but $\phi$ is not true of all partition systems of order $k$.

Finally let us turn our attention to systems with $k$ singularly predicates which are not restricted by any condition of disjointness or of being a partition. For any such system $\mathfrak{P} = \langle A, P_o, \ldots, P_k \rangle$ there is an associated partition system over $A$ which will be denoted by $\mathfrak{P}_*$. The system $\mathfrak{P}_*$ will have $2^k$ predicates which will be denoted by $P_{\langle i_o, \ldots, i_{k-1} \rangle}$ where $\langle i_o, \ldots, i_{k-1} \rangle$ is any one of the $2^k$ k-tuples of 0's and 1's. To define these predicates we adopt the notation

$$P_j^{\,o} = P_j$$

$$P_j^{\,1} = A \sim P_j \quad \text{for} \quad j < k .$$

Then for each k-tuples $\langle i_o, \ldots, i_{k-1} \rangle$ we set

$$P_{\langle i_o, \ldots, i_{k-1} \rangle} = P_o^{\,i_o} \cap P_1^{\,i_1} \cap \ldots \cap P_{k-1}^{\,i_{k-1}}$$

In order to avoid the very clumsy notation of k-tuples written out as subscripts, the letter $t$ will be used to denote k-tuples. For definiteness sake we set

$$\mathfrak{P}_* = \langle A , \ldots , P_t , \ldots \rangle$$

where the k-tuples $t$ are arranged in lexicographical order with $\langle o, o, \ldots, o \rangle$ on the left and $\langle 1, 1, \ldots, 1 \rangle$ on the far right.

It should be quite clear that $\mathfrak{P}_*$ is a partition system and that $\mathfrak{P}$ and $\mathfrak{P}_*$ essentially contain the same information. To make this last point more precise we have only to note that each $P_j$ can be recaptured from the $P_t$, indeed

$$P_j = P_{\langle o, o, \ldots, o \rangle} \cup \ldots \cup P_t \cup \ldots \cup P_{\langle 1, 1, \ldots, 0, \ldots 1 \rangle} ,$$

Where the union runs over all k-tuples $t$ with a $0$ in the $j^{th}$ position. This simple fact shows us that we can translate any first order sentence about $\mathfrak{P}$ into a first sentence about $\mathfrak{P}_*$. In fact, if $\phi$ is any sentence involving only the singularly predicate symbols $\mathbb{P}_o,\ldots,\mathbb{P}_{k-1}$, then let $\phi_*$ be a sentence involving predicate symbols $\mathbb{P}_t$, where $t$ is a k-tuple of 0's and 1's, that is obtained from $\phi$ by replacing each atomic part $P_j(v_\ell)$ of $\phi$ by the disjunction

$$\mathbb{P}_{<o,o,\ldots,o>}(v_\ell)\vee\ldots\vee\mathbb{P}_t(v_\ell)\vee\ldots\vee\mathbb{P}_{<1,1,\ldots,o,\ldots 1>}(v_\ell)$$

where $t$ runs over all k-tuples with $0$ in the $j^{th}$ position. Notice that $\phi$ is true in $\mathfrak{P}$ if and only if $\phi_*$ is true in $\mathfrak{P}_*$ and that $\phi$ and $\phi_*$ have exactly the same variables. Thus we may apply Corollary 6.6. and obtain at once

Corollary 6.7. A sentence $\phi$ with at most $m$ variables and involving only the singularly predicate symbols $\mathbb{P}_{o},\ldots,\mathbb{P}_{k-1}$ is true in all systems with $k$ arbitrary singularly predicates if and only if it is true in all such systems with at most $2^k\cdot m$ elements; hence, there is an effective procedure for determining the universal validity of arbitrary sentences of the first order logic with $k$ singularly predicate symbols.

Remark. Just as before the number $2^k\cdot m$ cannot be improved.

The above proofs do not greatly differ from the well-known proofs, except that we have systematized the method of proof by the use the general theorem of Part I, section 2. Actually a germ of the idea of this method can be seen in the proofs given by Ackermann in [1] especially pp. 24-26 and pp. 34-37. However, the author was led to the method through the problem of the theory of geometry discussed in Part II and only noticed later the applications to other theories. It should be pointed out that the counting of variables instead of quantifiers has given us stronger results than those given in the Ackermann book.

## § 7. One Equivalence Relation

Closely related to the notion of a partition of a set into disjoint subsets is the idea of an equivalence relation: an equivalence relation partitions the set into disjoint equivalence classes. It might seem that these are the same notions, but this is not the case. Setting aside the point that a partition system, as we have defined it, has only finitely many disjoint sets while an equivalence relation may have infinitely many equivalence classes, it is seen that the automorphisms exhibit the essential difference between the two theories. An automorphism of a partition system can effect only a permutation within each of the sets of the partition, while an automorphism of an equivalence relation can interchange distinct equivalence class of the same cardinality. This extra freedom in the automorphisms of equivalence relations can be taken away simply by adjoining certain individual constants denoting elements of the various equivalence classes; here an automorphism would have to leave the constants fixed and thus could not permute the equivalence classes. This rather informal remark, it is hoped, makes it plausible that any problem about partition systems can be exactly translated into a problem about equivalence relations plus some extra constants. It would follow that any decision procedure given for the theory of equivalence relations would also give a decision procedure for the theory of partition systems. However, a direct proof for partition systems seemed so simple that it was felt to be a better plan to make the results in Theorem 6.5 quite independent of a discussion of equivalence relations, which are somewhat more complicated.

Definition 7.1. An <u>equivalence system</u> is a relational system $\mathcal{R} = \langle A, R \rangle$ where $R$ is a binary relation that is reflexive over $A$ and is symmetric and transitive.

In the study of partition systems it was possible to show that each such system is an arithmetical extension of degree $m$ of certain of its finite subsystems. We shall do the same for equivalence relations, but not so directly, and the reduction will proceed in two stages. It would be possible to accomplish the reduction in one step making use of Theorem 2.9 rather than Theorem 2.7, but the description of the class of subsystems needed is rather involved, and it seems easier to follow when the two steps are separated.

Notice that a subsystem of an equivalence system is again an equivalence system.

If $\mathcal{R} = \langle A, R \rangle$ is an equivalence system, we shall use the notation $R[x]$ for the <u>equivalence class</u> containing an element $x \in A$, in other words

$$R[x] = \{ y \mid y \in A \text{ and } \langle x,y \rangle \in R \} .$$

Notice a permutation $f$ of $A$ is an automorphism of $\mathfrak{R}$ if and only if for all $x \in A$, $f$ is a one-one mapping of $R[x]$ onto $R[f(x)]$.

Definition 7.2. A subsystem $\mathfrak{R} = < A, R >$ of an equivalence system $\mathfrak{S} = < B, S >$ is m-dimensional if for each $x \in A$ the cardinality of $R[x]$ is the minimum of the number $m$ and the cardinality of $S[x]$, and for each $y \in B$ there is an $x \in A$ with $< x, y > \in S$.

This definition is analogous to Definition 6.4 and in 7.3 we give the analogous result about arithmetical extensions; however, an m-dimensional subsystem of an equivalence system need not be finite, and so the results about the effectiveness of the decision process do not follow at once.

Theorem 7.3. An equivalence system is an arithmetical extension of degree $m$ of each of its subsystems of dimension at least $m$.

Proof. As ever, condition (ii) of Theorem 2.7 will be shown to hold. Let $\mathfrak{S} = < B, S >$ be an equivalence system and let $\mathfrak{R} = < A, R >$ be a subsystem of dimension at least $m$. Let $A' \subseteq A$ have less than $m$ elements and let $b \in B$ but $b \notin A$. Let $b' \in A$ with $< b, b' > \in S$. Thus, $R[b'] \neq S[b']$, and so $S[b']$ must have more elements than $R[b']$; hence, $R[b']$ has at least $m$ elements. Consequently there is an element $b'' \in R[b']$ such that $b'' \notin A'$. Let $f$ be that permutation of $B$ that simply interchanges $b$ and $b''$ so that $f(b) = b''$ and $f(b'') = b$. Since $< b, b'' > \in S$, we clearly have an automorphism of $\mathfrak{S}$ leaving $A'$ pointwise fixed, as was to be shown.

Definition 7.4. A subsystem $\mathfrak{R} = < A, R >$ of an equivalence system $\mathfrak{S} = < B, S >$ is of <u>rank m</u> if the following two conditions are satisfied:

(i) $R[x] = S[x]$ for all $x \in A$

(ii) For each cardinal number, the number of equivalence class contained in $A$ of that cardinality is the minimum of the number $m$ and the number of equivalence classes in $B$ of that same cardinality.

Theorem 7.5. An equivalence system is an arithmetical extension of degree $m$ of each of its subsystems of rank at least $m$.

Proof: Condition (ii) of Theorem 2.7 is to be verified as always. Let $\mathfrak{S} = < B, S >$ be an equivalence system and let $\mathfrak{R} = < A, R >$ be a subsystem of rank at least $m$. Let $A' \subsetneq A$ be a subset with less than $m$ elements and let $b \in B$ but $b \notin A$. It follows at once by 7.4 (i) that $S[b] \cap A = 0$. Let $\aleph = \text{card}\ (S[b])$. Thus, the number of equivalence classes included in $B$ of cardinality $\aleph$ is greater than those included in $A$. Hence, the number of such equivalence classes included in $A$ is exactly $m$. Since $A'$ has fewer than $m$ elements, there must exist an element $b' \in A$ such that $A' \cap R[b'] = 0$ and $R[b'] = S[b']$ has exactly $\aleph$ elements. Let $f$ be any permutation of $B$. Leaving all elements outside the set $S[b] \cup S[b']$ fixed and which maps $S[b]$ onto $S[b']$ and $S[b']$ onto $S[b]$. The function $f$ is the required automorphism of $\mathfrak{S}$.

Applying both 7.3 and 7.5 we can finally prove an effective result.

Corollary 7.6. A sentence $\phi$ with at most m variables and involving besides the identity symbol only the binary predicate symbol $\mathbb{P}$ is true in all equivalence systems if and only if it is true in all equivalence systems with at most $m^2(m+1)/2$ elements; hence, there is an effective procedure for determining the validity of arbitrary sentences in the class of all equivalence systems.

Proof. Obviously we need only show that if $\phi$ fails in some equivalence system $\mathfrak{S} = \langle B, S \rangle$, then it fails in some system with at most the required number of elements. First let $\mathfrak{R}_1$ be a subsystem of $\mathfrak{S}$ of dimension m. Obviously from Definition 7.2 such an $\mathfrak{R}_1$ can be constructed with the aid of the Axiom of Choice (see the remark following this proof.) By 7.3, $\phi$ must fail in $\mathfrak{R}_1$. Notice that each equivalence class in $\mathfrak{R}_1$ has at most m elements. Now let $\mathfrak{R}_2$ be a subsystem of $\mathfrak{R}_1$ of rank m. Since we are now working with equivalence classes of bounded finite cardinality, no use of the Axiom of Choice is necessary in the construction of $\mathfrak{R}_2$. By 7.5, $\phi$ must also fail in $\mathfrak{R}_2$. Now look at $\mathfrak{R}_2$. Each equivalence class has at most m elements. For each $i \leq m$, $o < i$, there are at most m equivalence classes of cardinality i. Hence, the total number of elements in $\mathfrak{R}_2$ is at most

$$m \cdot 1 + m \cdot 2 + \dots + m \cdot m = \frac{m^2(m+1)}{2} ,$$

as was to be shown.

Remark: By application of the well-known Skolem-Löwenheim theorem it can be shown that if $\phi$ fails in some equivalence system with at most a denumerable number of elements. Hence, we can always assume that the domain B of $\mathfrak{G}$ is well-ordered and avoid completely the Axiom of Choice in constructing $\mathfrak{R}_1$.

Again we can show that the number $m^2(m+1)/2$ is the best possible. Let m be a fixed integer greater than 0. Suppose that for $j < m$, $j > 0$, $\psi_j$ is a sentence such that it is true in an equivalence system $\mathfrak{R}$ if and only if the number of equivalence class in $\mathfrak{R}$ with exactly j elements is less than m. Suppose, further that $\psi_m$ is a sentence that is true in an equivalence system $\mathfrak{R}$ if and only if the number of equivalence classes in $\mathfrak{R}$ with at least m elements is less than m. Then, if we take as $\phi$ the disjunction

$$\psi_1 \vee \psi_2 \vee \ \cdots \vee \psi_m \quad ,$$

it is readily seen that $\phi$ is true in all equivalence systems with fewer than $m^2(m+1)/2$ elements.

It only remains to be verified that the sentences $\psi_j$ can be written with at most m variables. For each $j \leq m$ consider the formula $\chi_j$ defined as follows:

$$\bigvee v_1 \bigvee v_2 \ldots \bigvee v_{j-1} \ \Big[ \mathbb{P}(v_0, v_1) \wedge \cdots \wedge \mathbb{P}(v_0 \ v_{j-1}) \wedge$$
$$v_0 \neq v_1 \wedge \ v_0 \neq v_2 \wedge \cdots \wedge v_0 \neq v_{j-1} \ \wedge$$
$$v_1 \neq v_2 \wedge \cdots \wedge v_1 \neq v_{j-1} \wedge \ \cdots \wedge v_{j-2} \neq v_{j-1} \Big]$$

where $v_k \neq v_\ell$ has been used as an abbreviation for $\sim v_k = v_\ell$. The formula $\chi_j$ contains at most m variables and has just one free variable $v_0$. From the form of the sentence, it should be clear that a sequence $x$ satisfies $\chi_j$ in an equivalence system $\mathfrak{R}$ if and only if $R[x_0]$ has at least $j$ elements.

Next for each $j < m$ consider the formula $\eta_j$ defined as follows:

$$\bigvee v_1 \bigvee v_2 \dots \bigvee v_{j-1} \bigwedge v_j \Big[ \mathbb{P}(v_0\ v_j) \longrightarrow \Big[ v_1 = v_j \vee v_2 = v_j \vee \dots \vee v_{j-1} = v_j \Big]\Big] .$$

The restriction $j < m$ assures us that $\eta_j$ has at most $m$ variables, and clearly $\eta_j$ has just one free variable $v_0$. Notice that a sequence $x$ satisfies $\eta_j$ in an equivalence system $\mathfrak{R}$ if and only if $R[x_0]$ has at most $j$ elements.

We are now ready to write down the sentences $\psi_j$. For $j = m$ we may take as $\psi_m$ the sentence

$$\bigvee v_1 \bigvee v_2 \dots \bigvee v_{m-1} \bigwedge v_0 \Big[ \chi_m \longrightarrow \Big[ v_0 = v_1 \vee v_0 = v_2 \vee \dots \vee v_0 = v_{m-1} \Big]\Big] ;$$

while for $j < m$ take as $\psi_j$ the sentence

$$\bigvee v_1 \bigvee v_2 \dots \bigvee v_{m-1} \bigwedge v_0 \Big[\Big[ \chi_j \wedge \eta_j \Big] \longrightarrow \Big[ v_0 = v_1 \vee v_0 = v_2 \vee \dots \vee v_0 = v_{m-1} \Big]\Big]$$

It would seem to be quite impossible to write the sentence $\phi$ defined above in prenex normal form and have something containing at most $m$ variables. Even though the $\phi$ is a very long sentence, it is quite naturally written down and contains only $m$ variables. This shows the advantage of our general logical result that does not require the formulas to be in prenex normal form.

The next problem we turn to concerns the combination of the theories of an equivalence relation and a partition system. The reader, if he has read this far, probably has begun to see that there are many variations of the method that can be applied to theories having to do with ways of cutting up a domain into pieces. The main problem is to see if it is possible to find enough permutations of the pieces or the elements in the pieces to prove that for each $m$ such a system is an arithmetical extension of degree $m$ of a finite subsystem, where an upper bound on the number of elements can be given as a simple function of $m$. These variations can be carried on indefinitely, but the computation of the upper bound becomes harder and less interesting. We conclude this section, then, with one final example, and in the last section consider examples with several equivalence relations which are much more intriguing since the general theory is undecidable.

Definition 7.7. An <u>equivalence - partition</u> system of order $k$ is a system $\mathfrak{R} = \langle A, R, P_0, \ldots, P_{k-1} \rangle$ where $\langle A, R \rangle$ is an equivalence system and $\langle A, P_0, \ldots, P_{k-1} \rangle$ is a partition system of order $k$.

Definition 7.8. A subsystem $\mathfrak{R} = \langle A, R, P_0, \ldots, P_{k-1} \rangle$ of an equivalence system $\mathfrak{S} = \langle B, S, Q_0, \ldots, Q_{k-1} \rangle$ is <u>m-dimensional</u> if for each $i < k$ the equivalence system $\langle A \cap P_i , R \cap P_i^2 \rangle$ is an m-dimensional subsystem of the equivalence system $\langle B \cap Q_i , S \cap Q_i^2 \rangle$.

This last definition rests on Definition 7.2, and the content of the notion is that an m-dimensional subsystem should have equivalence classes with at most $m$ elements in common with each set in the partition. In analogy with Theorem 7.3 we can prove:

Theorem 7.9. An equivalence - partition system is an arithmetical extension of degree $m$ of each of its subsystems of dimension at least $m$.

Proof: Condition (ii) of Theorem 2.7 will be shown to hold. Let $\mathfrak{S} = \langle B, S, Q_0, \ldots, Q_{k-1} \rangle$ be an equivalence-partition system, and let $\mathfrak{R} = \langle A, R, P_0, \ldots, P_{k-1} \rangle$ be a subsystem of dimension at least $m$. Let $A' \subsetneq A$ have less than $m$ elements and let $b \in B$ but $b \notin A$. Since $A$ contains representatives of all equivalence classes, let $b' \in A$ with $\langle b, b' \rangle \in S$, and we may assume further that both $b, b' \in Q_i$ for some $i < k$. Thus $R[b'] \cap P_i = R[b'] \cap Q_i \neq S[b'] \cap Q_i$ and hence $R[b'] \cap P_i$ has at least $m$ elements. Consequently there is an element $b'' \in R[b'] \cap P_i$ such that $b' \notin A'$. Let $f$ be that permutation of $B$ that simply interchanges $b$ and $b''$. The function $f$ leaves $S$ and all the partitions $Q_0, \ldots, Q_{k-1}$ in varient and is thus an automorphism of $\mathfrak{S}$ leaving $A'$ pointwise fixed.

Definition 7.10. Let $\mathfrak{R} = \langle A, R, P_0, \ldots, P_{k-1} \rangle$ be an equivalence - partition system, and let $E$ be an equivalence class included in $A$ under $R$. By the <u>trace</u> of $E$ we shall understand the sequence of cardinals $\langle \digamma_0, \digamma_1, \ldots, \digamma_{k-1} \rangle$ such that for $i < k$,

$$\digamma_i = \text{card } (E \cap P_i)$$

Definition 7.11. A subsystem $\mathfrak{R} = \langle A, R, P_0, \ldots, P_{k-1} \rangle$ of an equivalence-partition system $\mathfrak{S} = \langle B, S, Q_0, \ldots, Q_{k-1} \rangle$ is of <u>rank</u> $m$ if the following two conditions are satisfied:

(i) $R[x] = S[x]$ for all $x \in A$ ;

(ii) for each possible trace, the number of equivalence classes contained in $A$ with that trace is the minimum of the number $m$ and the number of equivalence classes contained in $B$ with that trace.

Theorem 7.12. An equivalence - partition system is an arithmetical extension of degree $m$ of each of its subsystems of rank at least $m$.

Theorem 7.5 is a special case of 7.12 and the changes necessary for the more general proof required here are straight·forward and will be omitted.

Corollary 7.13. A sentence $\phi$ with at most $m$ variables and involving besides the identity symbol only the binary predicate symbol $\mathbb{R}$ and the singulary predicate symbols $\mathbb{P}_o, \ldots, \mathbb{P}_{k-1}$ is true in all equivalence-partition systems of order $k$ if and only if it is true in all such systems with at most $k \cdot m^2(m+1)^k/2$ elements; hence, there is an effective procedure for determining the validity of arbitrary sentences is the class of all such systems.

Proof. As in 7.6 it follows from the preceding two theorems that if $\phi$ fails in one such system, then it fails in an equivalence - partition system that is at the same time a subsystem of itself that is of dimension $m$ and rank $m$. Now such systems are finite and we need only find an upper bound on their cardinality. The largest such system would be an equivalence - partition system $\mathfrak{R} = \langle A, R, P_o, \ldots, P_{k-1} \rangle$, where each equivalence class $R[x]$ is such that $R[x] \cap P_i$ has at most $m$ elements and there are exactly $m$ equivalence classes of each possible trace. A particular trace is simply a $k$-termed sequence of integers less than $m + 1$, and hence, denoting by $(m+1)^k$ the set of all such sequences, the number we want to compute is

$$m \cdot \sum_{f\in(m+1)^k} \sum_{i<k} f(i)$$

since this will give the total number of elements required. We have

$$m \cdot \sum_{f\in(m+1)^k} \sum_{i<k} f(i) = m \cdot \sum_{i<k} \sum_{f\in(m+1)^k} f(i)$$

$$= m \cdot \sum_{i<k} (m+1)^{k-1} \sum_{j<m+1} j = m \cdot k \cdot (m+1)^{k-1} \frac{m(m+1)}{2}$$

$$= k \cdot m^2 (m+1)^k/2 .$$

It would be possible to give a sentence showing that the number $k \cdot m^2(m+1)^k/2$ cannot be improved, but the result does not seem worth all the trouble.

## § 8. Several Equivalence Relations

Both Janiczak and Rogers [ 4 and 6 ] have shown that the first-order theory of two equivalence relations is undecidable. This means that there is no effective procedure for determining whether a sentence is true in all systems $\langle A, R, S \rangle$ where $\langle A, R \rangle$ and $\langle A, S \rangle$ are equivalence systems. Stated in terms of an axiomatic theory, the result means that there is no effective procedure for deciding whether a sentence can be deduced on the basis of the axioms and rules of the first-order logic from the sentences:

(i) $\bigwedge x\, \mathbb{R}(x, x)$,

(ii) $\bigwedge x \bigwedge y \bigwedge z \left[\left[\mathbb{R}(x, z) \wedge \mathbb{R}(y, z)\right] \rightarrow \mathbb{R}(x, y)\right]$

(iii) $\bigwedge x\, \mathbb{S}(x, x)$,

(iv) $\bigwedge x \bigwedge y \bigwedge z \left[\left[\mathbb{S}(x, z) \wedge \mathbb{S}(y, z)\right] \rightarrow \mathbb{S}(x, y)\right]$

In addition, Janiczar has shown that these axioms remain undecidable if the theory is strengthened by the addition of the sentence:

(v) $\bigwedge x \bigwedge y \left[\left[\mathbb{R}(x,y) \wedge \mathbb{S}(x, y)\right] \rightarrow x = y\right]$

It is the main result of this section that if the theory is further strengthened by the addition of either of the axioms

(vi) $\bigwedge x \bigwedge y \bigvee z \left[\mathbb{R}(x, z) \wedge \mathbb{S}(z, y)\right]$,

(vii) $\bigwedge x \bigwedge y \left[\bigvee z \left[\mathbb{R}(x, z) \wedge \mathbb{S}(z, y)\right] \rightarrow \bigvee z \left[\mathbb{S}(x, z) \wedge \mathbb{R}(z, y)\right]\right]$,

Then there is an effective decision procedure for the resulting theory. In as much as the sentence (vi) implies (vii) with the aid of (i) - (iv), one need only prove the stronger result; however, the proof for the stronger axiom is so simple that the author prefers to give it first.

Let $\mathfrak{R} = \langle A, R, S \rangle$ be a system of two equivalence relations. To say that $\mathfrak{R}$ satisfies axiom (v) means that the intersection $R \cap S$ is the identity relation over $A$. Two such equivalence relations will be called <u>disjoint</u>. To say that $\mathfrak{R}$ satisfies axiom (vi) means that the relative product $R ; S$ is the universal relation $A^2$. Two such equivalence relations will be called <u>supplementary</u>.

All examples of disjoint and supplementary equivalence relations are very easy to construct: Let $X$ and $Y$ be two non-empty sets. Let $A = X \times Y$ and let $R$ and $S$ be defined as follows:

$$R = \{ \langle\langle x_0, y_0 \rangle , \langle x_1, y_1\rangle\rangle | \ x_0, x_1 \in X \text{ and } y_0, y_1 \in Y \text{ and } x_0 = x_1 \}$$

$$S = \{ \langle\langle x_0, y_0 \rangle , \langle x_1, y_1\rangle\rangle | \ x_0, x_1 \in X \text{ and } y_0, y_1 \in Y \text{ and } y_0 = y_1 \} .$$

Then the system $\mathfrak{R} = \langle A, R, S \rangle$ will satisfy axioms (i) - (vi). In other words, $A$ consists of all points in a rectangle, and $R$ is the equivalence relation of points being in the same row, and $S$ is the relation of points being in the same column. It is quite easy to show that any abstract system satisfying axioms (i) - (vi) is isomorphic to a special system as just described.

From the construction of all examples of disjoint and supplementary equivalence relations just mentioned, it would be possible to show that every question about a sentence being true in all such systems can be reduced in an effective way to a question in the theory of a partition of a set into two sets, where the equivalent sentence would have twice the number of variables. This approach, though quite effective, would not give the most direct decision method. Using the technique developed in this work it is quite easy to compute the exact number of elements of $A$ needed for checking a sentence, once the number of variables is known.

Notice that if $\mathfrak{R} = \langle A, R, S \rangle$ is a system with disjoint and supplementary equivalence relations and if $R[x_0]$ and $R[x_1]$ are two equivalence classes for $R$, then $S$ sets up a one-one correspondence between $R[x_0]$ and $R[x_1]$. First of all, there always exists for each $y \in R[x_0]$ an element $z \in R[x_1]$ such that $< y, z > \in S$ ; moreover, if $< y, z_0 > \in S$ and $< y\ z_1 > \in S$ and $z_0, z_1 \in R[x_0]$, then $< z_0, z_1 > \in S$ and $< z_0\ z_1 > \in R$, and hence $z_0 = z_1$. Thus the correspondence given by S allows us to interchange any two equivalence classes under R by an automorphism of the whole system $\mathfrak{R}$ . The same holds true with R and S in opposite rôles.

Definition 8.1. A subsystem $\mathfrak{R} = < A, R, S >$ of a system $\mathfrak{S} = < B, T, U >$ of two disjoint and supplementary equivalence relations will be called an <u>m-dimensional subsystem of the first kind</u> if

(i) $R[x] = T\ [x]$ for each $x \in A$ ,

(ii) the number of equivalence classes included in A under R is equal to the minimum of the number m and the number of equivalence classes included in B under T.

$\mathfrak{R}$ will be called an <u>m-dimensional subsystem of the second kind</u> if it satisfies similar conditions with R and S, and T and U interchanged.

Remark: It follows from condition (i) that $\mathfrak{R}$ is a system of supplementary equivalence relations.

Theorem 8.2. A system of two disjoint and supplementary equivalence relations is an arithmetic extension of degree m of each of its subsystem of dimension at most m and of either kind.

Proof. Condition (ii) of Theorem 2.7 is to be verified. Let $\mathfrak{S} = < B, T, U >$ be such a system and let $\mathfrak{R} = < A, R, S >$ be a subsystem of dimension at most m of the first kind. The theorem for subsystems of the second kind will have exactly the same proof with $R$ and $S$, $T$ and $U$ interchanged. Let $A' \subsetneq A$ have less than $m$ elements. Let $b \in B$ and $b \notin A$. Thus $T[b] \cap A = 0$ and so there are more $T$ equivalences classes. Hence, $A$ has at least $m$ equivalence classes under $R$, and therefore there must be a $b'' \in A$ with $R[b''] \cap A' = 0$. As remarked above, there will be an automorphism of $\mathfrak{S}$ interchanging the equivalence classes $R[b''] = T[b'']$ and $T[b]$. This automorphism will then map $b$ into $A$ and leave $A'$ pointwise fixed, as was to be shown.

Corollary. 8.3. A sentence $\phi$ with at most $m$ variables and involving besides the identity symbol only the two binary predicate symbols $\mathbb{R}$ and $\mathbb{S}$ is true in all systems of two disjoint and supplementary equivalence relations if and only if it is true in all such systems with at most $m^2$ elements; hence, there is an effective procedure for determining the validity of arbitrary sentences in the class of all such systems.

Proof. If $\phi$ fails in one such system $\mathfrak{S}$ it will fail in a subsystem $\mathfrak{R}_1$ of dimension $m$ of the first kind. Applying 8.2 again, $\phi$ will fail in a subsystem $\mathfrak{R}_2$ of $\mathfrak{R}_1$ which in $\mathfrak{R}_1$ is m-dimensional and of the second kind. Thus, the number of equivalence classes under both relations in $\mathfrak{R}_2$ is at most $m$, which implies at once that the system $\mathfrak{R}_2$ can have at most $m^2$ elements.

Turning our attention now to axiom (vii) we see that the content of this condition for a system of two equivalence relations $< A, R, S >$ is that $R$ and $S$ commute under relative product; that is, $R; S = S; R$.

It is a very simple exercise in the theory of equivalence relations to show that, under this hypothesis, $R ; S$ is the least equivalence relation containing both $R$ and $S$. Thus $R ; S$ breaks $A$ up into blocks compatible with the partitions of both $R$ and $S$. The permutations that we did when $R ; S = A^2$ can now be done on each equivalence class under $R ; S$ separately. The result is that we can show that a sentence $\phi$ with $m$ variables need only be checked in systems $\langle A, R, S \rangle$ where the equivalence classes under $R ; S$ have at most $m^2$ elements. Now not all of these equivalence classes are isomorphic under $R$ and $S$ and indeed there are distinct ones for each pair of integers $i$ and $j$ with $i, j \leq m$, where the number of elements is $i \cdot j$. Our standard argument will then show that, by permuting the equivalence classes of the same type among themselves, ~~that~~ we need at most $m$ of each type. Hence the maximum number of elements needed is

$$m \cdot \sum_{i,j \leq m} i \cdot j = m \; {\frac{m(m+1)}{2}}^{2}$$

This informal argument leads to the result:

Theorem 8.4. A sentence $\phi$ with at most $m$ variables is true in all systems of disjoint and commuting equivalence relations if and only if it is true in all such systems with at most $m^3(m+1)^2/2$ elements.

Remark. It is quite easy to show that the numbers given in 8.3 and 8.4 are the least possible.

As a concluding result, it will be shown that the theory based on axioms (i) - (iv) and (vi), that is, without the condition of disjointness, yields an undecidable theory. In fact we can show that the known undecidable problem about the theory of disjoint equivalence relations reduces to the decision problem for supplementary equivalence relations, and hence, the latter theory must be undecidable. Consider a system $\mathfrak{R} = \langle A, R, S \rangle$ where $R$ and $S$ are disjoint equivalence relations. Let $X = \{ R[x] \mid x \in A \}$ and let $Y = \{ S[x] \mid x \in A \}$ and put $A^* = Y \times Y$. On the set $A^*$ there are two natural equivalence relations $R^*$ and $S^*$, equivalence by rows and columns, as was mentioned above. Now there is an isomorphism mapping $\mathfrak{R}$ onto a subsystem of $\mathfrak{R}^* = \langle A^*, R^*, S^* \rangle$. This isomorphism, call it $f$, is given by the formula

$$f(x) = \langle R[x] \, , \, S[x] \rangle \quad \text{for} \quad x \in A \, .$$

That $f$ is one-one is clear from the disjointness of $R$ and $S$. That $f$ is an isomorphism is obvious. Thus every system of disjoint equivalence relations can be imbedded in a system of disjoint and supplementary equivalence relations. This looks odd because the first theory is undecidable by the results in [4 and 6] while we have shown here that the second theory is decidable. The correct formulation, however, is as follows:

Theorem 8.5. The theory of two disjoint and supplementary equivalence relations together with an arbitrary singulary predicate is an undecidable theory.

This follows from our remarks at once, because the other predicate can be thought of as the image of $A$ in $A^*$ under $f$, showing that the known undecidable theory reduces to the one mentioned above.

It is now quite easy to see what happens when we leave off axiom (v). In a system $\mathfrak{R} = \langle A, R, S \rangle$ satisfying (i) - (iv) and (vi), the intersection $R \cap S$ need not be the identity relation over $A$ but may properly include the identity. But as far as $R$ and $S$ are concerned, elements equivalent by $R \cap S$ are quite undistinguishable. Thus we can think of $R \cap S$ as a "coarse" identity relation, whereas the real identity relation is much "finer". In other words, if we used $R \cap S$ as the identity, then we would be back in the theory of disjoint relations. But since real identity is available, we can distinguish between those elements $x$ such that $(R\cap S)[x]$ has exactly one element, and those $x$ for which this false. This is just like having a one-placed predicate available, so that in formal terms we can say:

Theorem 8.6. The decision problem for the theory of disjoint and supplementary equivalence relations with an additional singulary predicate reduces to the decision problem for the theory of just two supplementary equivalence relations; hence, this latter theory is undecidable.

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