Seventy Years of Foundational Studies: A Tribute to Andrzej Mostowski
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The Algebraic Interpretation of Quantifiers: Intuitionistic and Classical

Dana S. SCOTT
Hillman University Professor of Computer Science,
Philosophy, and Mathematical Logic (Emeritus)
Computer Science Department, Carnegie Mellon University
Pittsburgh, Pennsylvania, USA

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"*Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache und dann ist es alsobald ganz etwas Anderes.*"

J.W. von Goethe

1. Introduction

In their 1963 book, *The Mathematics of Metamathematics* [Rasiowa and Sikorski, 1963], the authors present an approach to completeness theorems of various logics using algebraic methods. Perhaps they put the quotation from Goethe over their preface to suggest that mathematicians can be forgiven if they introduce new interpretations of old ideas.

The idea of an algebra of logic can of course be traced back to Boole, but it was revived and generalized by Stone and Tarski in the 1930s; however, the most direct influence on the work of Rasiowa and Sikorski came from their well-known colleague, Andrzej Mostowski, after WW II. Mostowski's interpretation of quantification [Mostowski, 1948] can as well be given for intuitionistic as classical logic.

The present expository paper will briefly review the history and content of these ideas and then raise the question of why there was at the time no generalization made to higher-order logic and set theory. Entirely new light on this kind of algebraic semantics has more recently been thrown by the development of topos theory in category theory. Some reasons for pursuing this generalization will also be discussed in the last section.

In order to dispel any confusion that might arise about the title of this paper, the author would like to point out that there are other meanings to "algebraic" as relating to the semantics of Logic. Perhaps the first interpretation of quantifiers in an algebraic style goes back to consideration of early work in Descriptive Set Theory. Quantifier manipulation was explained in this regard in the well known book on Topology by K. Kuratowski (also quoting Tarski). In 1946 C.J. Everett and S. Ulam published their paper on Projective Algebras. A. Tarski, building on his famous paper on Truth in formalized languages and on his studies of Relation Algebras (along with J.C.C. McKinsey, L. Henkin and many collaborators) developed an elaborate equational theory of Cylindric Algebras (duly quoting Everett and Ulam). In a related but different style, P.R. Halmos and collaborators studied Polyadic Algebras. A generalization of Relation Algebras was proposed by P.C. Bernays and that research has been continued over many years by W. Craig. Most recently, F.W. Lawvere unified many of these ideas-along with the lattice-theoretic semantics explained in the present paper-by showing how quantification can be construed in suitable categories by using the idea of adjoint functors. None of these works are cited in our bibliography here, because the literature is too vast to be explained in such a short paper.

2. Preliminaries, Terminology, Notation

The following standard terminology and notation will be used throughout the paper.

A complete lattice (cLa) is a partially ordered set (poset) $\langle A, \leqslant\rangle$ where every subset has a least upper bound (lub) under $\leqslant$. (Dually we can say, every subset has a greatest lower bound (glb).) A broader notion of a lattice requires only every finite subset to have a lub and a glb. A lub is also called a sup; and a glb an inf.

Theorem 2.1. *If a poset has all lubs, it has all glbs; and conversely.*

The proof is well known: the greatest lower bound of a family is the least upper bound of all the lower bounds of the family.

Theorem 2.2. *Using the following notation, lubs and glbs can be uniquely characterized as follows:*

(LUB) $y \leqslant \bigvee X \iff \forall z[\forall x \in X.\, x \leqslant z \implies y \leqslant z]$
(GLB) $\bigwedge X \leqslant y \iff \forall z[\forall x \in X.\, z \leqslant x \implies z \leqslant y]$
(TOP) $1 = \bigvee A = \bigwedge \emptyset$
(BOT) $0 = \bigwedge A = \bigvee \emptyset$

Theorem 2.3. *Lattices (not necessarily complete) are characterized by these axioms:*

- $\leqslant$ *is a partial order*
- $x \leqslant z \,\&\, y \leqslant z \iff x \vee y \leqslant z$
- $z \leqslant x \,\&\, z \leqslant y \iff z \leqslant x \wedge y$
- $0 \leqslant x \leqslant 1$

Examples of complete lattices abound in set theory. Here is an example of a direct consequence of Theorem 2.1.

Theorem 2.4. *The powerset $\mathcal{P}(A)$ is always a cLa; hence, if a family $\mathcal{C} \subseteq \mathcal{P}(A)$ is closed under all intersections (unions), then $\langle \mathcal{C}, \subseteq\rangle$ is also a cLa.*

We need to remember that in the cLa $\mathcal{P}(A)$ the intersection of the empty family is taken to be A.

3. Closures on a Power Set $\mathcal{P}(A)$

Closure operations $C : \mathcal{P}(A) \to \mathcal{P}(A)$ give interesting examples of complete lattices. There are several kinds of closures axiomatized as follows.

General:

- $X \subseteq C(X) = C(C(X))$
- $X \subseteq Y \Longrightarrow C(X) \subseteq C(Y)$

Topological:

- $X \subseteq C(X) = C(C(X))$
- $C(\emptyset) = \emptyset$
- $C(X \cup Y) = C(X) \cup C(Y)$

Algebraic:

- $X \subseteq C(X) = C(C(X))$
- $C(\bigcup_{X \in \mathcal{W}} X) = \bigcup_{X \in \mathcal{W}} C(X)$ for directed $\mathcal{W} \subseteq \mathcal{P}(A)$

For any closure operation we say that a set $X \subseteq A$ is *closed* provided $C(X) \subseteq X$.

Theorem 3.1. *The family of closed sets of a general closure operation is closed under arbitrary intersections and so forms under inclusion a complete lattice. Moreover, an isomorph of any complete lattice can be found in this way.*

For the easy proof see [Rasiowa and Sikorski, 1963]. The complete lattices formed from topological or algebraic closure operations have additional properties. We will discuss the topological case further below.

4. Implication and Distribution

Definition 4.1. *A lattice operation $\to$ is called an* implication *if and only if it satisfies this axiom:*

- $x \wedge y \leqslant z \Longleftrightarrow x \leqslant y \to z$

Note. The axiom uniquely determines an implication on a lattice if it exists. There is an equational axiomatization where we can define the partial ordering in several ways.

- $$\begin{aligned} x \leqslant y &\Longleftrightarrow x \vee y = y \\ &\Longleftrightarrow x \wedge y = x \\ &\Longleftrightarrow x \to y = 1 \end{aligned}$$

See [Rasiowa and Sikorski, 1963] for details. A lattice with implication is called a *Heyting algebra*. (Ha is used for short.)

Theorem 4.1. *All Ha's are distributive lattices.*

Proof. From basic lattice properties we have $x \wedge y \leqslant (x \wedge y) \vee (x \wedge z)$. Then $y \leqslant x \to ((x \wedge y) \vee (x \wedge z))$. Similarly we have $x \wedge z \leqslant (x \wedge y) \vee (x \wedge z)$. Thus $z \leqslant x \to ((x \wedge y) \vee (x \wedge z))$. It follows that $y \vee z \leqslant x \to ((x \wedge y) \vee (x \wedge z))$. But then $x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$. This is half of the Distributive Law. The other half holds in all lattices. □

Definition 4.2. *A Heyting algebra which has all glbs and lubs (= complete lattice) is called a complete Heyting algebra (or cHa, for short).*

Theorem 4.2. *A complete lattice is a cHa if, and only if, it satisfies:*

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i).$$

The proof of the above equation is a direct generalization of the proof of Theorem 4.1. For the converse we can use the following definition of implication:

$$y \to z = \bigvee \{x |\, x \wedge y \leqslant z\}.$$

The results about distributivity of Heyting algebras were proved independently by Stone [Stone, 1937] and Tarski [Tarski, 1938].

5. Boolean Algebras

The difference between classical and intuitionistic logic lies in the fact that the former satisfies the law of the Excluded Middle. In algebraic terms we have the following.

Definition 5.1. *A Ha (resp. cHa) is a Boolean algebra (or Ba) (resp. complete Boolean algebra or cBa) iff it satisfies* $x \vee (x \to 0) = 1$.

The element $x \to 0$ is called the *negation* of x and is denoted by $\neg x$.

Perhaps we should pause to note this passage from [Rasiowa and Sikorski, 1963] on pp. 8–9:

> The inclusion of two chapters on intuitionism is not an indication of the authors' positive attitude towards intuitionistic ideas. Intuitionism, like other non-classical logics, has no practical application in mathematics. Nevertheless many authors devote their works to intuitionistic logic. On the other hand, the mathematical mechanism of intuitionistic logic is interesting: it is amazing that vaguely defined philosophical ideas concerning the notion of existence in mathematics have lead to the creation formalize logical systems which, from the mathematical point of view proved to be equivalent to the theory of lattices of open subsets of topological spaces. Finally, the formalization of intuitionistic logic achieved by Heting and adopted in this book is not in agreement with the philosophical views of the founder of intuitionism, Brouwer, who opposed formalism in mathematics. Since in treating intuitionistic logic we have limited ourselves to problems which are directly connected with general algebraic, lattice-theoretical and topological methods employed in the book, we have not included the latest results of Beth and Kreisel concerning other notions of satisfiability which we have adopted.

As a matter of fact the topological modeling of intuitionistic formal rules is not really to be regarded as "amazing". For, in view of Theorem 4.2, we conclude at once this theorem and corollary.

Theorem 5.1. *Every sublattice of a cBa (also cHa) closed under* $\wedge$ *and* $\bigvee$ *is a cHa.*

Corollary 5.1. *The lattice of open subsets of a topological space forms a cHa.*

There is also a reverse connection between Heyting algebras and Boelean algebras. An element x of a Heyting algebra is called *stable* (sometimes regular) if it satisfies $x = \neg\neg x$.

Theorem 5.2. *The stable elements of a Ha (cHa) form a Ba (cBa).*

A proof is given in [Rasiowa and Sikorski, 1963], pp. 134–135. Another connection is an embedding theorem.

Theorem 5.3. *Every cHa can be embedded in a cBa so as to preserve* $\wedge$ *and* $\bigvee$.

For a proof see [Johnstone, 1982].

6. Finite Lattices

A finite lattice is complete. Hence, we conclude

Theorem 6.1. *A finite distributive lattice* $\langle A, \leqslant\rangle$ *is a cHa, as is the dual* $\langle A, \geqslant\rangle$.

Perhaps we should note here that 6.1 is not true constructively. (See [Fourman–Scott, 1977] for the explanation.)

The finite Ha's can be analyzed in terms of a special kind of elements. These results are well known. Detailed references can be found in [Rasiowa and Sikorski, 1963].

Definition 6.1. *The set of join irreducible elements of a lattice* $\langle A, \leqslant\rangle$ *is defined as* $Irr(A) = \{x \in A | \; \forall y, z \in A \; [x \leqslant y \vee z \Longrightarrow x \leqslant y \text{ or } x \leqslant z]\}$.

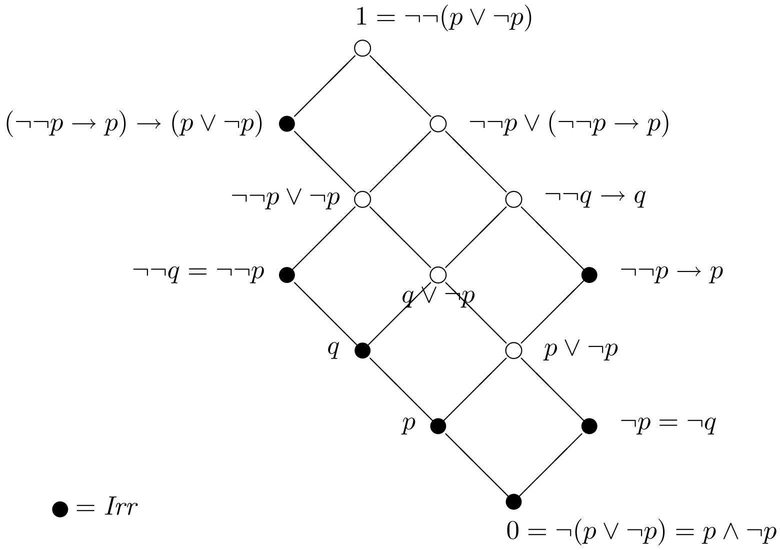
If an element is not irreducible in a distributive lattice, it can be written as the join of two strictly smaller elements. This remark makes it possible to give an inductive proof of the next theorem.

Theorem 6.2. *If* $\langle A, \leqslant\rangle$ *is a finite Ha, then for all elements* $x \in A$,

$$x = \bigvee\{y \in Irr(A) | \; y \leqslant x\}.$$

Here is an example of a finite distributive lattice with the irreducible elements marked.

Example 6.1.



We note that some authors exclude 0 as Irr.

The irreducible elements elements relate directly to what now are known as *Kripke Models*. The following is a classic result.

Theorem 6.3. *Every finite poset $\langle I, \leqslant\rangle$ with 0 is isomorphic to the join irreducibles of a finite Ha.*

Proof. Let $A = \{X \subseteq I |\, 0 \in X \wedge \forall x \in X.\{y \in I|\, y \leqslant x\} \subseteq X\}$. The poset $\langle A, \subseteq\rangle$ is closed under $\cap$ and $\cup$ and hence is a cHa. It is easy to check that $Irr(A) = \{\{y \in I|\, y \leqslant x\}|\, x \in I\}$. $\square$

The connection to Kripke Semantics can be explained as follows. For elements $P, Q \in A$, $x \in I$, we have these formulae:

$$\begin{array}{lcl} P = I & \Longleftrightarrow & \forall x \in I.\, x \in P \\ P \subseteq Q & \Longleftrightarrow & \forall x \in P.\, x \in Q \\ P \cap Q & = & \{x \in I \mid x \in P \wedge x \in Q\} \\ P \cup Q & = & \{x \in I \mid x \in P \vee x \in Q\} \\ P \to Q & = & \{x \in I \mid \forall y \leqslant x[y \in P \Longrightarrow y \in Q]\} \end{array}$$

Working with the finite Ha's – as is well known – leads to a proof of the decidability of intuitionistic propositional logic.

Theorem 6.4. *If a polynomial equation*

$$\sigma(x, y, z, \ldots) = \tau(x, y, z, \ldots)$$

fails in some Ha, then it fails in some finite Ha.

Proof. Let $[\![x]\!]_A$, $[\![y]\!]_A$, $[\![z]\!]_A, \ldots$ be the valuation in a Ha A such that

$$[\![\sigma(x, y, z, \ldots)]\!]_A \neq [\![\tau(x, y, z, \ldots)]\!]_A$$

Let $p_0, p_1, \ldots, p_n$ be all the subpolynomials of σ and of τ. Let B be the finite $\wedge$,$\vee$-sublattice of A generated by the elements $[\![p_i]\!]_A$ for $i \leqslant n$. We need to show

Lemma 6.1. *If $x, y \in B$ and $x \underset{A}{\rightarrow} y \in B$, then $x \underset{B}{\rightarrow} y = x \underset{A}{\rightarrow} y$.*

The easy proof goes back to Definition 4.1. Then, because all implications in σ and τ are in B, we derive

Corollary 6.1. $[\![\sigma]\!]_B = [\![\sigma]\!]_A$ *and* $[\![\tau]\!]_B = [\![\tau]\!]_A$.

Hence, the equation fails in the finite Ha B. This proof was first given in [McKinsey–Tarski, 1948]. □

7. Algebras of Formulae

In this section we recall Mostowski's interpretation of quantifiers as found in algebras of formulae. The *Lindenbaum algebra* of a theory T in classical first-order logic is the algebra formed by equivalence classes of formulae under provable equivalence in the theory. If we denote this provability by $\vdash_T \Phi$, then the equivalence class of Φ is defined as

$$[\![\Psi]\!]_T = \{\Psi \mid \quad \vdash_T \Phi \rightarrow \Psi \quad \text{and} \quad \vdash_T \Psi \rightarrow \Phi\}.$$

These equivalence classes form a partially ordered set by defining

$$[\![\Phi]\!]_T \leqslant [\![\Psi]\!]_T \quad \text{iff} \quad \vdash_T \Phi \rightarrow \Psi.$$

We employ here formulae Φ with possibly *free variables*, but such variable symbols could be taken as additional constants about which the theory T has no special axioms. The theory T might also have other constants and operation symbols. We let $Term$ denote the collection of all individual expressions formed from the variables and constants with the aid of the operation symbols.

In general the Lindenbaum algebra is not a complete Boolean algebra, but it enjoys an amount of completeness shown in crucial result of [Rasiowa–Sikorski, 1950] and of [Henkin, 1949] which we state in 7.2.

Theorem 7.1. *The Lindenbaum algebra of a classical first-order theory is a Ba.*

The proof of this fact in the formulation we have given here is very simple. The axioms for Boolean algebras exactly mimic well known *rules of inference* of logic. Thus, when we define lattice operations in the Lindenbaum algebra, such as the following:

$$[\![\Phi]\!]_T \cap [\![\Psi]\!]_T = [\![\Phi \wedge \Psi]\!]_T,$$

then we have a direct translation between algebraic relationships and deduction rules.

Theorem 7.2. *In the Lindenbaum algebra of a classical first-order theory T (using free variables) we have:*

$$[\![\exists x.\Phi(x)]\!]_T \;=\; \bigvee_{\tau \in \mathit{Term}} [\![\Phi(\tau)]\!]_T \quad \textit{and} \quad [\![\forall x.\Phi(x)]\!]_T = \bigwedge_{\tau \in \mathit{Term}} [\![\Phi(\tau)]\!]_T.$$

Proof. The proof of the quantification results is also quite simple. First, it is clear from rules of logic that

$$[\![\Phi(\tau)]\!]_T \leqslant [\![\exists x.\Phi(x)]\!]_T$$

holds for all terms τ. In other words, the equivalence class of the quantified formula is an *upper bound*. Next, suppose that Ψ provides *any other* upper bound, in the sense that

$$[\![\Phi(\tau)]\!]_T \leqslant [\![\Psi]\!]_T$$

holds for all terms τ. Because we have infinitely many free variables, we can pick one, v, say, not free in Ψ or otherwise in Φ. This means that $\vdash_T \Phi(v) \to \Psi$. But then by logical deduction $\vdash_T \exists x.\Phi(x) \to \Psi$. In terms of equivalence classes we have $[\![\exists x.\Phi(x)]\!]_T \leqslant [\![\Psi]\!]_T$. The LUB property is thus established. The GLB property is proved by a dual argument (or by using negation). □

8. Axiomatizing Free Logic

Before discussing models of intuitionistic predicate logic using cHa's, we need to broaden the idea of *existence* in axiomatic theories. The need to do this is not urgent in classical theory, for we can always say that under normal conditions the value of a function, $f(x)$, say, is given in a natural way; but then if the conditions on x do not hold, then the value of $f(x)$ can be a conventional value, 0, say. This only makes sense, however, using Excluded Middle. In other words, values of terms τ in the formal language are not always going to be required *to exist*.

The method the author has found to be practical is to build existence conditions into the deduction rules for the quantifiers. This has been called *free logic* ([Lambert, 1969], [Mostowski, 1951], [Quine, 1995], [Scott, 1977], [Lambert, 2003]), meaning that logic is freed of implicit existence assumptions. And in connection with this point of view, it seems most convenient to allow free variables also to be used without existence assumptions. Hence, *substitution* for free variables will be allowed for arbitrary terms.

A second move is to weaken equality statements $x = y$ to equivalence statements $x \equiv y$, meaning roughly that if one or the other of x and y exists, then they are identical. This seems to give the most liberal set of rules. Of course, the axioms for propositional logic remain the same as commonly known, so we do not repeat them here.

Definition 8.1. *The axioms and rules for (intuitionistic) equality and quantifiers are as follows:*

(Sub) $$\frac{\Phi(x)}{\Phi(\tau)}$$

(Ref) $$x \equiv x$$

(Rep) $\quad x \equiv y \wedge \Phi(x) \Longrightarrow \Phi(y)$

$(\forall Ins)$ $\quad (\forall x)\Phi(x) \wedge (\exists x)[x \equiv y] \Longrightarrow \Phi(y)$

$(\forall Gen)$ $\quad \dfrac{\Phi \wedge (\exists x)[x \equiv y] \Longrightarrow \Psi(y)}{\Phi \Longrightarrow (\forall y)\Psi(y)}$

$(\exists Ins)$ $\quad \Phi(y) \wedge (\exists x)[x \equiv y] \Longrightarrow (\exists x)\Phi(x)$

$(\exists Gen)$ $\quad \dfrac{\Phi(y) \wedge (\exists x)[x \equiv y] \Longrightarrow \Psi}{(\exists y)\Phi(y) \Longrightarrow \Psi}$

Existence and strict identity can then be defined.

Definition 8.2.

- $Ex \Longleftrightarrow (\exists y)[x \equiv y]$
- $x = y \Longleftrightarrow Ex \wedge Ey \wedge x \equiv y$

Conversely, existence and weak equality can be defined from strong identity.

Theorem 8.1.

- $x \equiv y \Longleftrightarrow [Ex \vee Ey] \Longrightarrow x = y$
- $Ex \Longleftrightarrow x = x$

Read "$E\tau$" as "the value of τ exists", and "$\tau = \sigma$" as "the values of τ and σ are existing and are identical". It actually is a matter of taste which notions to take as primitive as $\equiv$ and $=$ are interdefinable under our conventions, as we have seen.

9. Heyting-Valued Semantics

We take a definition from [Fourman–Scott, 1977] which we use for model theory over a cHa. The work reported in that paper was developed over several years in Scott's graduate seminars at Oxford. Independently, D. Higgs proposed similar definitions—especially for Boolean-valued models. See [Fourman–Scott, 1977] for references.

Definition 9.1. *Let A be a cHa. An A-set is a set M together with a mapping $e : M \times M \to A$ such that for all $x, y, z \in M$, $e(x, y) = e(y, x)$ and*

$$e(x, y) \wedge e(y, z) \leqslant e(x, z).$$

It is called total iff additionally $e(x, x) = 1$ for all $x \in M$. An n-placed predicate on M is a mapping $p : M^n \to A$ where

$$e(x_1, y_1) \wedge \cdots \wedge e(x_n, y_n) \wedge p(x_1, \ldots, x_n) \leqslant p(y_1, \ldots, y_n)$$

holds for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$.

We wish to show how A-sets together with predicates as a structure

$$\langle M, e, p_1, \ldots p_m \rangle$$

give *A-valued models* for intuitionistic first-order free logic.

Structures for logic with operation symbols can be similarly set up but, to interpret operations as functions it is better to consider *complete* A-sets. The details can be found in [Fourman–Scott, 1977].

Definition 9.2. *Given an A-structure*

$$\mathcal{M} = \langle M, e, p_1, \ldots p_m \rangle$$

and a given valuation $\mu : Var \to M$ of free variables, the A-valuation of formulae of free logic is defined recursively as follows:

$$\begin{aligned}
[\![P_i(v_1, \ldots, v_{n_i})]\!]_{\mathcal{M}}(\mu) &= p_i(\mu(v_1), \ldots, \mu(v_{n_i}));\\
[\![v_i = v_j]\!]_{\mathcal{M}}(\mu) &= e(\mu(v_i), \mu(v_j);\\
[\![\Phi \wedge \Psi]\!]_{\mathcal{M}}(\mu) &= [\![\Phi]\!]_{\mathcal{M}}(\mu) \wedge [\![\Psi]\!]_{\mathcal{M}}(\mu);\\
[\![\Phi \vee \Psi]\!]_{\mathcal{M}}(\mu) &= [\![\Phi]\!]_{\mathcal{M}}(\mu) \vee [\![\Psi]\!]_{\mathcal{M}}(\mu);\\
[\![\Phi \Rightarrow \Psi]\!]_{\mathcal{M}}(\mu) &= [\![\Phi]\!]_{\mathcal{M}}(\mu) \to [\![\Psi]\!]_{\mathcal{M}}(\mu);\\
[\![\neg\Phi]\!]_{\mathcal{M}}(\mu) &= \neg[\![\Phi]\!]_{\mathcal{M}}(\mu);\\
[\![\exists v_i \Phi(v_i)]\!]_{\mathcal{M}}(\mu) &= \bigvee_{\nu}(e(\nu(v_i), \nu(v_i)) \wedge [\![\Phi(v_i)]\!]_{\mathcal{M}}(\nu));\\
[\![\forall v_i . \Phi(v_i)]\!]_{\mathcal{M}}(\mu) &= \bigwedge_{\nu}(e(\nu(v_i), \nu(v_i)) \to [\![\Phi(v_i)]\!]_{\mathcal{M}}(\nu)).
\end{aligned}$$

In the above equations, the lattice operations on the right-hand side of the semantic equations are those of the cHa A. And in the last two equations the lubs and glbs are taken over all valuations ν where $\nu(v_j) = \mu(v_j)$ for all variables v_j different from the v_i in the quantifier.

Note that the intention here is that we have a derived semantics for these other two kinds of formulae:

$$\begin{aligned}
[\![Ev_i]\!]_{\mathcal{M}}(\mu) &= [\![v_i = v_i]\!]_{\mathcal{M}}(\mu), \quad \text{and}\\
[\![v_i \equiv v_j]\!]_{\mathcal{M}}(\mu) &= [\![Ev_i \vee Ev_j \to v_i = v_j]\!]_{\mathcal{M}}(\mu).
\end{aligned}$$

And again, because we have defined cHa to exactly mimic the laws and rules of deduction of intuitionistic logic, the proof of the next theorem is obvious.

Theorem 9.1. *The universally valid formulae (with free variables) of an A-structure form a theory in intuitionistic free logic. If A is Boolean, then we have a classical theory.*

10. Completion of Lattices and Completeness of Logic

Lindenbaum algebras give us a certain “formal” interpretation of intuitionistic theories in Ha’s. But a Lindenbaum algebra is not usually complete. What is required to obtain structures over cHa’s is an application of a method called *MacNeille Completion* [MacNeille, 1937].

Theorem 10.1. *Every Ha can be isomorphically embedded into a cHa preserving all existing glbs and lubs.*

When this theorem is proved we will have at once this conclusion.

Corollary 10.1. *Intuitionistic first-order logic is complete with respect to structures over cHa’s; that is, every theory has such a model with the same valid formulae.*

Note that the cHa to be used is not fixed in this statement, as it depends on the choice of theory to make a Lindenbaum algebra.

Proof of Theorem 10.1. Let $\langle A, \leqslant\rangle$ be a Ha. For $X \subseteq A$, define the upper bounds and lower bounds of X by

$$ub(X) = \{y \in A \mid \forall x \in X.\, x \leqslant y\} \quad \text{and} \quad lb(X) = \{x \in A \mid \forall y \in X.\, x \leqslant y\}.$$

Lemma 10.1. *The map $X \mapsto lb(ub(X))$ is a monotone closure on $\mathcal{P}(A)$.*

Proof. Clearly $X \subseteq lb(ub(X))$. Also as ub and lb are antimonotone, so the composition is monotone. We need next to show that

$$ub(X) \subseteq ub(lb(ub(X))).$$

Suppose $y \in ub(X)$ and $x \in lb(ub(X))$. Then, by definition, $x \leqslant y$. Thus $y \in ub(lb(ub(X)))$. Hence, we have a closure operation. □

It is interesting to note in this proof that no special properties of $\leqslant$ were needed. By the general theorems on closures, we can conclude the following.

Lemma 10.2. $\overline{A} = \{L \subseteq A \mid lb(ub(L)) \subseteq L\}$ *is a complete lattice closed under arbitrary intersections.*

Note that $\{\emptyset\}$ is the least element of $\overline{A}$, and A itself $= lb(ub(\{1\})))$ is the largest element. This however does not prove yet that $\overline{A}$ is a cHa. To this end, for $L, M \subseteq A$, define $L \rightarrow M = \{a \in A \mid \forall b \in L.\, a \wedge b \in M\}$.

Lemma 10.3. *For K, L, $M \in \overline{A}$ we have $K \cap L \subseteq M$ iff $K \subseteq L \rightarrow M$.*

Proof. Assume first that $K \cap L \subseteq M$. Suppose $a \in K$ and $b \in L$. Then $a \wedge b \in K \cap L$. Thus $K \subseteq L \rightarrow M$. Conversely, assume $K \subseteq L \rightarrow M$. Suppose $a \in K \cap L$. Then $a \in L \rightarrow M$. But $a = a \wedge a \in M$. Thus $K \cap L \subseteq M$. □

Lemma 10.4. *If L, $M \in \overline{A}$, then $L \rightarrow M \in \overline{A}$.*

Proof. $L \to M = \bigcap_{b \in L} \{a \in A |\ a \wedge b \in M\}$. So, it is sufficient to show for each $b \in L$ that the set

$$K = \{a \in A |\ a \wedge b \in M\} \in \overline{A}.$$

As a shorthand let us write $y \geqslant K$ for $y \in ub(K)$. Suppose $x \in lb(ub(K))$. Then $\forall y \geqslant K.\ x \leqslant y$. Suppose $z \geqslant M$. If $a \in K$, then $a \wedge b \in M$. So $a \wedge b \leqslant z$. Thus $a \leqslant b \to z$. So $b \to z \geqslant K$. Hence, $x \leqslant b \to z$ and then $x \wedge b \leqslant z$. Therefore, $x \wedge b \in lb(ub(M)) \subseteq M$. So $x \in K$. Thus $K \in \overline{A}$. □

Lemma 10.5. *The map $x \mapsto \downarrow x = lb(\{x\})$ is a Ha isomorphism of A into $\overline{A}$ which preserves existing glbs and lubs.*

Proof. The map is one-one and monotone. Also, it is easy to check the following conditions providing a homomorphism.

$$\begin{aligned}
&\downarrow (x \wedge y) &&= \downarrow x \cap \downarrow y;\\
&\downarrow (x \vee y) &&= lb(ub(\downarrow x \cup \downarrow y));\\
&\downarrow x \to \downarrow y &&= \{a |\ \forall b \leqslant x.\ a \wedge b \leqslant y\} = \{a |\ a \wedge x \leqslant y\} = \downarrow (x \to y);\\
&\downarrow \bigwedge_{i \in I} x_i &&= \bigcap_{i \in I} \downarrow x_i;\\
&\downarrow \bigvee_{i \in I} x_i &&= lb(ub(\bigcup_{i \in I} \downarrow x_i)).
\end{aligned}$$

The MacNeille theorem is thus proved. □

The corresponding completeness theorem for classical logic comes down to the next theorem. We use in this proof the well known fact that classical logic results from intuitionistic logic by assuming that every proposition is equivalent to its double negation.

Theorem 10.2. *If A is a Ba, then so is $\overline{A}$.*

Proof. For $L \in \overline{A}$, we calculate

$$\begin{aligned}
\neg\neg L &= \{a |\ \forall c \in L.\ b \wedge c = 0 \Rightarrow a \wedge b = 0\}\\
&= \{a |\ \forall b[\forall c \in L.\ c \leqslant \neg b \Rightarrow a \leqslant \neg b]\}\\
&= \{a |\ \forall b[L \leqslant b \Rightarrow a \leqslant b]\} = lb(ub(L)) = L
\end{aligned}$$

Therefore, in view of Theorem 7.2, every theory T in classical first-order logic has a model in cBa with the same set of valid formulae. This, however, is *not* Gödel's Completeness Theorem. For this we need the famous lemma [Rasiowa–Sikorski, 1950]. But note to this point we did not make use of ultrafilters or the Stone Representation Theorem. □

Lemma 10.1 (Rasiowa–Sikorski). *Given a countable number of infs and sups in a nontrivial Ba, there is an ultrafilter preserving them.*

An ultrafilter is basically a homomorphism from the Ba to the two-element Ba. Without going through the MacNeille completion, however, we can apply the lemma directly to the Lindenbaum algebra of a classical first-order theory to prove

Corollary 10.2 (Gödel). *Every consistent, countable first-order classical theory has a two-valued model.*

Turning our attention now to intuitionitic logic we need the analogue of Theorem 7.2. The proof is essentially same, except that the rules of quantifiers of Section 8 have to be employed.

Theorem 10.3. *In the Lindenbaum algebra of a theory T in intuitionistic first-order free logic (using free variables) we have:*

$$[\![\exists x.\Phi(x)]\!]_T = \bigvee_{\tau \in \mathit{Term}} [\![E\tau \wedge \Phi(\tau)]\!]_T \quad \textit{and}$$

$$[\![\forall x.\Phi(x)]\!]_T = \bigwedge_{\tau \in \mathit{Term}} [\![E\tau \to \Phi(t)]\!]_T.$$

To avoid any misunderstanding, we should stress here (and in 7.2) that the axioms of the theories T should *not* be given with any free variables. The reason is that the proof of 7.2 and 10.3 requires at one point choosing a variable not free in T (as well as in certain formulae). Note that this step also requires that the formulae be finite, while the stock of free variables is infinite.

Theorem 10.4. *Every theory T in intuitionistic first-order free logic has a model in a cHa with the same set of valid formulae.*

Note that if the theory were inconsistent we would have to use the trivial one-element Ha. We also note that in intuitionistic logic there is no Rasiowa–Sikorski Lemma to give us models in "simpler" cHa's other than the one obtained from completing the Lindenbaum algebra.

There are two recent papers on the MacNeille completion method that have been brought to the attention of the author. They are [Harding–Bezhanishvili, 2004] and [Harding–Bezhanishvili, 2007]. As regards completion of Ha's, the first paper has the interesting result that the only non-trivial varieties of Ha's for which the desired result holds are Ha and Ba. Their paper gives many details and also historical references about the method. It should be noted, however, that they are especially interested in topological representations, and in this paper we have not used any representation theory. The second paper gives many results about Boolean modal operators, but undoubtedly their ideas can be adapted to Ha's in some cases. The next section has one such result found by the author.

11. Modal Logic

Starting perhaps with [McKinsey–Tarski, 1944], the algebraic interpretation of logic has been applied to investigations of modal logic. Here we shall not take the time to set out logical axioms but will be content to state a result about cHa's and modal-like operators. The first point to emphasize is that Heyting algebras give us many possibilities (or distinctions) for operators not available in the Boolean world.

We consider four algebraic axioms systems, $(\Box), (\Diamond), (\nabla), (\Delta)$, named by the symbols used for the operator.

($\Box$)
- $\Box 1 = 1$
- $\Box(x \wedge y) = \Box x \wedge \Box y$
- $\Box\Box x = \Box x \leqslant x$

($\Diamond$)
- $\Diamond 0 = 0$
- $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$
- $\Diamond\Diamond x = \Diamond x \geqslant x$

(∇)
- $\nabla(x \wedge y) = \nabla x \wedge \nabla y$
- $\nabla\nabla x = \nabla x \geqslant x$

(Δ)
- $\Delta(x \vee y) = \Delta x \vee \Delta y$
- $\Delta\Delta x = \Delta x \leqslant x$

It is clear that system $(\Diamond)$ is the dual of system $(\Box)$. In Boolean algebra, as is well known, we can pass from one system to the other by defining

$$\Diamond = \neg\Box\neg \quad \text{and} \quad \Box = \neg\Diamond\neg.$$

However, in Heyting algebra the failure of the law of Double Negation blocks this neat conversion (just as we cannot say $\exists = \neg\forall\neg$). As far as the author knows there is no simple correspondence between models of the one system and the other. The same remarks apply to comparing system (∇) to system (Δ).

What we shall prove here is that—in case there is any interest in quantified modal logic—the MacNeille Completion procedure can be applied to Lindenbaum algebras to get algebraic completeness proofs over cHa's for the system ($\Box$). The question for the other systems is left open.

Theorem 11.1. *If A is a Ha and if the operator $\Box : A \to A$ satisfies the $(\Box)$ axioms, then so does the MacNeille completion $\overline{A}$ with the operator $\Box : \overline{A} \to \overline{A}$ defined by*

$$\Box L = \bigvee_{x \in L} \downarrow \Box x$$

for all $L \in \overline{A}$. Moreover, the lattice embedding $x \mapsto \downarrow x$ preserves the $\Box$-operator.

Proof. On A the operator $\Box$ is monotone, and on $\overline{A}$ the corresponding operator $\Box$ is as well. Note that we can also write

$$\Box L = lb(ub(\{\Box x | \, x \in L\}))$$

for all $L \in \overline{A}$. $\Box$

First, $\Box \downarrow 1 = \Box A = A$, as required.

Next, note that since $\Box x \in L$ for all $x \in L$, wwe conclude $\Box L \subseteq L$. By monotonicity we also have $\Box\Box \subseteq \Box L$.

To prove $\Box \subseteq \Box\Box L$, we need to show $\Box x \in \Box\Box L$ for all $x \in L$. By definition we have

$$\Box\Box L = lb(ub(\{\Box y | \, y \in \Box L\})).$$

Suppose $x \in L$. Then if $z \geqslant \{\Box y | \, y \in \Box L\}$, we conclude $z \geqslant \Box\Box x = \Box x$. Therefore $\Box x \in \Box\Box L$ as desired.

Finally, we calculate using the $\wedge$-$\bigvee$-Distributive Law as follows:

$$\begin{aligned}\Box L \cap \Box M &= \bigvee_{\substack{x \in L \\ y \in M}} (\downarrow \Box x \cap \downarrow \Box y) \\ &= \bigvee_{\substack{x \in L \\ y \in M}} \downarrow \Box(x \wedge y) \\ &= \bigvee_{z \in L \cap M} \downarrow \Box z \\ &= \Box(L \cap M)\end{aligned}$$

Thus the defined $\Box : \overline{A} \to \overline{A}$ satisfies the ($\Box$)-equations.

We remark in passing that to obtain a $\Box$-operator on any cHa A, all one needs is a subset $U \subseteq A$ such that $1 \in U$ and $x \wedge y \in U$ for all $y \in U$. We than define

$$\Box x = \bigvee \{y \in U | y \leqslant x\}.$$

The verification of the ($\Box$)-axioms is straightforward. This, unsurprisingly, generalizes how the interior operator is defined in any topological space in terms of basic open sets. Of course, any $\Box$-operator can be so defined in terms of the set

$$U = \{\Box x | \, x \in A\}.$$

The system (∇) at first looks like ($\Box$), but the last inequality is *reversed*. Such operators on cHa's are variously called *j-operators* or *nuclei* (see [Fourman–Scott, 1977] or [Johnstone, 2002–2007], vol. 2, pp. 480ff. The significance of such operators is that they correspond to $\wedge$-$\bigvee$-preserving congruence relations E on a cHa A where we can define ∇_E, given E, by

$$\nabla_E x = \bigvee \{y \in A | \, xEy\}.$$

The other way around, give ∇, we can define E_∇ by

$$xE_\nabla y \quad \text{iff} \quad \nabla x = \nabla y,$$

It follows that $E = E_{\nabla_E}$ and $\nabla = \nabla_{E_\nabla}$ for such ∇ and E.

The ∇-operators on cHa A can also be characterized in terms of fixed-point set

$$U_\nabla = \{x \in A | \, \nabla x = x\} = \{\nabla x | \, x \in A\},$$

Indeed, as for any general closure operation, we can obtain

$$\nabla x = \bigwedge \{y \in U_\nabla | \, x \leqslant y\}.$$

And of course the subset $U_\nabla \subseteq A$ is a subset closed under arbitrary glbs in A, as it is for general closures. However, the first equation in (∇) gives U_∇ an additional property:

- $x \to y \in U_\nabla$ for all $x \in A$ and all $y \in U_\nabla$.

To prove this, it is sufficient to prove in system (∇) the equation:

- $\nabla(x \to \nabla y) = x \to \nabla y$

On the one hand we have $x \to \nabla y \leqslant \nabla(x \to \nabla y)$. On the other hand we see:

$$\nabla(x \to \nabla y) \wedge \nabla x = \nabla(x \wedge (x \to \nabla y)) \leqslant \nabla\nabla y = \nabla y.$$

It follows that

$$\nabla(x \to \nabla y) \leqslant \nabla x \to \nabla y$$

Inasmuch as $x \leqslant \nabla x$, we conclude

$$\nabla x \to \nabla y \leqslant x \to \nabla y,$$

which is what we needed to prove.

Suppose in a cHa A we have $U \subseteq A$ with the $\bigwedge$-closure property and the $\to$-property above. Define

$$\nabla_U x = \bigwedge\{y \in U \,|\, x \leqslant y\}.$$

The key step is to prove the first equation of (∇). Because ∇_U is a closure operation, we have at once:

$$\nabla_U(x \wedge y) \leqslant \nabla_U x \wedge \nabla_U y.$$

Also we see:

$$x \wedge y \leqslant \nabla_U(x \wedge y).$$

But then

$$x \leqslant y \to \nabla_U(x \wedge y) \in U,$$

so

$$\nabla_U x \leqslant y \to \nabla_U(x \wedge y).$$

We conclude

$$y \leqslant \nabla_U x \to \nabla_U(x \wedge y) \in U,$$

so

$$\nabla_U y \leqslant \nabla_U x \to \nabla_U(x \wedge y).$$

This establishes

$$\nabla_U x \wedge \nabla_U y \leqslant \nabla_U(x \wedge y)$$

as required.

If A were a cBa, then ∇-operators are not very interesting. In fact, set $0^* = \nabla 0$, then $\nabla x = 0^* \vee x$, as is easily proved.

System (Δ) is introduced as a "dual" of (∇), but the author has no idea whether it is at all interesting to study.

12. Higher-Order Logic

Here is a principle the author regards – with hindsight – as self-evident.

Principle. *If you understand what first-order models with* arbitrary *predicates are, then you can interpret second-order logic with* predicate quantifiers*. And then you can go on to models of* higher-order logic*, even set theory.*

In what is perhaps the first modern text book in mathematical logic [Hilbert–Ackermann, 1928], second-order logic is presented as a natural step by introducing quantifiers on predicates and relations. Even though the Hilbert–Ackermann book did not define a formal semantics (as was to become standard from the later works of Tarski and Carnap) "meaning" and "logical validity" were intuitively understood. Indeed, Hilbert–Ackermann clearly presented the completeness problems Gödel was soon to solve. Of course, at that stage interpretations of formulae were classical and two valued.

Inasmuch as cHa's generalize cBa's, we will first consider in this section higher-order logic in intuitionistic free logic. For a cHa A, we defined in Section 9 general A-structures containing arbitrary A-valued predicates and relations. For simplicity, in order to fix ideas, let us restrict attention to one-placed predicates. These are functions $p : M \to A$ which have to satisfy an *extensionality condition* requiring $e(x, y) \leqslant (p(x) \leftrightarrow p(y))$, for all $x, y \in M$, and a *strictness condition* requiring

$$p(x) \leqslant e(x, x), \quad \text{for all } x \in M.$$

Let $\mathcal{P}_A(M)$ be the collection of all such p, and call this the *A-valued powerset* of M. To make this an A-set, we need to define (Leibniz) equality on $\mathcal{P}_A(M)$. The definition is just a translation into algebraic terms what we know from logic:

$$e(p, q) = \bigwedge_{x \in M} (p(x) \leftrightarrow q(x))$$

for $p, q \in \mathcal{P}_A(M)$. We note that

$$e(p, p) = 1$$

for all $p \in \mathcal{P}_A(M)$, a "totality" condition that need not hold in M.

Perhaps, to avoid confusion, we should notate the equality on M as e_0, and the equality on $\mathcal{P}_A(M)$ as e_1. We can then use subscripts for the equalities in higher powersets.

Another side remark concerns *totality* in A-sets. Experience (and the study of category theory and topos theory) has shown that for models of intuitionistic logic it is essential to take into account *partial elements* $x \in M$ where $e(x, x) \neq 1$. The reason is that these models come up naturally, and a key example is discussed below.

Given an A-set M, the *completion* $\overline{M}$ of M can be taken to be the collection of all "singletons" $s \in \mathcal{P}_A(M)$; that is to say those one-place predicates such that

$$s(x) \wedge s(y) \leqslant e(x, y)$$

for all $x, y \in M$. There is a natural embedding of M into $\overline{M}$ which we will denote by $\varepsilon : M \to \overline{M}$ and define by

$$\varepsilon(x)(y) = \varepsilon_0(x, y),$$

for all $x, y \in M$. It is clear that $\varepsilon(x) \in \overline{M}$ for all $x \in M$. However, we have to be a little more careful to say how $\overline{M}$ is an A-set, as it is not correct to use e_1 as the equality. Here is the appropriate definition:

$$\overline{e}(s,t) = \bigvee_{x \in M} s(x) \ \wedge \bigvee_{x \in M} t(x) \wedge e_1(s,t).$$

The point here is our definition of being a singleton does not guarantee that an $s \in M$ is *non empty*. The condition to be so is $\overline{e}(s,s) = 1$. But for $x \in M$ we will only have

$$\overline{e}(\varepsilon(x), \varepsilon(y)) = e_0(x,y),$$

for all $x, y \in M$. If $x \in M$ is a properly partial element, then $e_0(x,x) \neq 1$. More details about this notion of completeness are to be found in [Fourman–Scott, 1977], Section 4. We say that an A-set M is *complete* if, and only if, for all $s \in M$ there is a unique $x \in M$ such that $s = \varepsilon(x)$; this means

$$s(y) = e_0(x,y),$$

for all $y \in M$. In general $\overline{M}$ proves to be complete and we can restrict attention to complete A-sets.

The next question is in what sense does the multi-sorted structure

$$\langle M, e_0, p_1, \ldots, \mathcal{P}_A(M), e_1, \alpha_1 \rangle$$

form a model for a second-order logic? Here $\langle M, e_0, p_1, \ldots \rangle$ is a model for first-order logic to which we are adding a new sort, namely, the power set. As a connection between M and $\mathcal{P}_A(M)$ is needed, we let $\alpha_1 : M \times \mathcal{P}_A(M) \to A$ stand for "application" or "membership" defined by

$$\alpha_1(x,p) = p(x)$$

for all $x \in M$ and $p \in P_A(M)$. Then the answer to the question is obvious (it is hoped), because in $\mathcal{P}_A(M)$ there is by design a function p to represent the mapping

$$x \mapsto [\![\Phi(v)]\!]_A(\mu^v_x),$$

where Φ is any formula, of a first- or second-order logic, where μ is a given valuation of the variable, and where μ^v_x is the alteration of μ to make $\mu^v_x(v) = x$. The semantics of multi-sorted formulae of course evaluates quantifiers as glbs or lubs over the appropriate ranges of the variables.

We may recall that early formulation of high-order logic used an inference rule of *substitution of formulae* for predicate variables. The paper [Henkin, 1953] was perhaps the first to show that an axiomatization using a *comprehension axiom* suffices for higher-order logic. Especially as it is difficult to formulate the substitution rule correctly, the Henkin-style has been used ever since.

The author is claiming here that the passage from models of first-order logic to models of second-order logic by adding a new type of objects is an easy one: we just make sure that the new type has the maximal number of objects agreeing with the concept of an A-valued predicate. This is the principle articulated at the head of this section.

By the same token then, it should be just as easy to continue to *third-order* logic. (As a technical convenience it is better to keep to complete A-sets, and so $\mathcal{P}_A(M)$ needs to be replaced by $\overline{P_A(M)}$.) And then fourth-order and higher-order models will folllow by the same principle.

Some examples will perhaps show how the A-valued semantics works out. As a cHa we can take the lattice Ω of open subsets of a convenient topological space T. We then use as first-order elements continuous, real-valued functions f on open sets. That is $f : Ef \to \mathbb{R}$ continuously, where Ef, the domain of definition of f, is in Ω. Equality on such function is defined by

$$e(f, g) = \mathit{Int}\{t \in Ef \cap Eg |\, f(t) = g(t)\}.$$

Using the presentation of [Fourman–Scott, 1977], Section 8, one shows that this set, which we can call $\mathbb{R}_\Omega$, becomes a complete Ω-set. Moreover the usual arithmetic operations of $+$ and $\cdot$ make sense on $\mathbb{R}_\Omega$. The analogue of *ordering* is defined by

$$< (f, g) = \{t \in Ef \cap Eg |\, f(t) < g(t)\}.$$

As a first-order structure $\mathbb{R}_\Omega$ models (most of) the intuitionistic theory of the reals. When using the general method of passing to a second-order structure, one obtains a model also satisfying a very suitable version of *Dedekind completeness* of the ordering. See [Scott, 1968] for an early version.

Now let us modify the construction by letting B be the cBa of measurable sets modulo set of measure zero of a measure space T. The new "reals", $\mathbb{R}_B$, are equivalence classes of real-valued measurable functions (functions made equivalent if they agree up to a set of measure zero). Equality is defined by

$$e([f], [g]) = \{t |\, f(t) = g(t)\}/\mathit{Null},$$

where Null is the ideal of sets of measure zero. In the Boolean B-valued logic, $\mathbb{R}_B$ not only becomes a model of the classical first-order theory of the reals ($=$ real closed fields), but in second-order logic it becomes a model for the Dedekind Completeness Axiom. (One exposition can be found in [Scott, 1967a].) In third-order logic (and higher) it becomes possible to give formally a version of the Continuum Hypothesis. As Solovay discovered, we can give a Boolean-valued version of Paul Cohen's independence proof in this way—provided we take T to be a product space (with a product measure) so that there are a very large number of independent "random variables" ($=$ measurable functions). See [Scott, 1967a] or [Bell, 2005] for further details.

Not only can both classical and intuitionistic higher-order logic be modeled in cHa's (or cBa's) but the iteration of he power set can be pushed into the transfinite obtaining models (and independence proofs) for Zermelo–Fraenkel Set Theory. As was said, in hindsight (after Cohen's breakthrough) this all looks easy and natural. The technical facts about measures, topologies, and Boolean algebras were well set out in [Sikorski, 1960] and were known earlier. So the question the author cannot answer is: Why did not Mostowski or Rasiowa or Sikorski or one of their students in Warsaw extend the algebraic interpretations of the quantifiers to higher-order logic?

13. Conclusion

Our discussion in this paper has concentrated on the lattice-theoretic semantical interpretations of classical and intuitionistic logics championed by the Polish schools of Mostowski (Warsaw) and Tarski (Berkeley). A main point of our exposition is that the well-known algebraic characterizations of cHa's and cBa's exactly mimic the rules of deduction in the respective logics; hence the algebraic completeness proofs are not all that surprising. (The step to classical two-valued models, of course, needs a further argument.) Moreover, once the formal laws are put forward in this way, the connection between intuitionistic logic and topology are not surprising either (*pace* Rasiowa and Sikorski). (But the steps to *special* topological interpretations need further investigation and have not been considered here.)

A second main point here concerns higher-order logic. The author contends that once the situation of first-order is understood, the generalization of the algebraic interpretations is clear. As mentioned in the last section, he does not understand why the Polish schools did not see this already in about 1955. Being a member at that time of the Tarski school, the author also does not understand why *he* did not see this, since the generalization is based on very simple general principles. If this had been done at the time, the question of how the Continuum Hypothesis fares in, say, Boolean-valued interpretations would have come up quickly. Moreover the Polish schools had all the necessary technical information about special cBa's at hand to solve the problem of independence. Ah, well, history, unlike water, does not always find te most direct path.

As a curious sidelight, the idea of the algebraic interpretation in cBa's was already suggested by Alonzo Church in 1953 [Church, 1953]. But neither he nor his later students (including the author) were inspired to look into it further. Instead, it took the breakthrough of *forcing* of Paul Cohen (based on quite different intuitions), which was reinterpreted in terms of Boolean-valued logic by Robert Solovay to close the circle.

And it should not be forgotten that a completely different intuition stemming from *algebraic geometry* was being developed by Grothendieck and his followers. One result was the definition of an *elementary topos* by F. William Lawvere and Myles Tierney around 1970 (for history see [Johnstone, 2002–2007]) which gives the most general notion of the *algebra of higher types* in intuitionistic logic. Topos theory includes not only the interpretations in cHa's, but also the interpretations generalizing Kleene's *realizability*. However, there are many other aspects of topos theory coming from abstract algebra and from algebraic geometry and topology that go far beyond what logicians have ever imagined. The results and literature are too vast to survey here, but the multi-tomed synthesis of Peter Johnstone will provide an excellent and coherent overview. The challenge to logicians now is to use these mathematical techniques to draw (or discover) new conclusions significant for the foundations of mathematics.

14. A Note of Acknowledgement

A first version of this survey was presented at the conference *75 Years of Predicate Logic*, held at the Humboldt University, Berlin, Germany, September 18–21, 2003, on the occasion of the anniversary of the publication of [Hilbert–Ackermann, 1928]. Unfortunately owing to his retirement and subsequent move from Pittsburgh, PA, to Berkeley, CA, the author was unable to complete a manuscript for the proceedings volume, which was eventually published under the title *First-Order Logic Revisited*, Logos-Verlag, Berlin, 2004. The author would, however, like to thank the organizers of that conference for his being able to attend a very interesting workshop and for very warm hospitality in Berlin.

Subsequently the invitation to Poland for the *Trends in Logic* conference in 2005, which was dedicated to the memories of Polish colleagues in Logic whom he knew so well, gave the author the chance to rework the presentation. An invitation to lecture in the Philosophy Department of Carnegie Mellon University in the Spring of 2006 led to further improvements, particularly those concerning the MacNeille Completion and the application to Modal Logic. Thanks go to Steven Awodey for providing that opportunity – and for many years of collaboration in related areas.

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Working on this paper and going over the lists of references gave the author a chance to review how much he owes to the Polish Schools of Logic. He wishes to dedicate this survey to the memories of these teachers and colleagues.

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