Boolean Models and Nonstandard Analysis

by DANA SCOTT[1]

It is certainly debatable whether the crisis over the set-theoretical paradoxes was really a crisis or not. For a long time there was really no impact on the working (or should one say *proving*) mathematician. Only in the last few years have the people in abstract category theory found anything to complain about. It seems to the author that the situation with the continuum hypothesis is much worse. The Russell paradox can be easily dismissed by saying that such all inclusive sets were not intended, but not so with the question of the cardinality of the continuum. Everyone who does modern abstract mathematics uses the continuum as a well-determined set. Only the strict constructivist questions this "obvious" fact. But as Gödel and Cohen have shown us, the cardinality of this continuum is not at all well determined by the current axioms. What to do? Maybe we have to face the fact that there are many distinct theories of the continuum. It is hard to swallow the idea, but, as will be shown below, it is easy to cook up those exotic models out of everyday ingredients.

The Construction of Models for Analysis

We shall employ the Boolean algebraic method presented in my expository paper [4]. As is done there, we shall construct models for analysis (a portion of the higher-order theory of real numbers). For details of checking the logical properties of the models we shall refer to [4] and be content here with descriptive remarks. A thorough exposition of the models for full set theory will be found in the joint paper with Solovay [5].

The idea of constructing Boolean-valued models could have been

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(but was not) discovered as a generalization of the ultraproduct method used now so often to obtain nonstandard models for ordinary analysis. Roughly, we can say that ultraproducts use the *standard* Boolean algebras (the power-set algebras) to obtain models elementarily equivalent to the standard model, whereas the Boolean method allows the nonstandard complete algebras (such as the *Lebesgue algebra* of measurable sets modulo sets of measure zero or the *Baire algebra* of Borel sets modulo sets of the first category). Thus the Boolean method leads to *nonstandard* nonstandard models that are not only not isomorphic to the standard model but are not even equivalent. Nevertheless, they do satisfy all the usual axioms and deserve to be called models of analysis.

To make this comparison clear, let us consider the ultraproduct construction in general. So as not to complicate notation, let us use structures

$$\mathfrak{A} = \langle A, R \rangle,$$

where $R \subseteq A \times A$ is a binary relation. Suppose that for i in an index set I we have corresponding structures

$$\mathfrak{A}_i = \langle A_i, R_i \rangle.$$

We then form the product

$$\underset{i \in I}{X} \mathfrak{A}_i = \langle \underset{i \in I}{X} A_i, S \rangle = \langle B, S \rangle,$$

where S is *not* a binary relation in the ordinary sense. Instead we make S a *Boolean-valued relation.* We define

$$S \colon B \times B \to PI$$

(where P is the power-set operator), so that for $a, b \in B$ we have

$$S(a, b) = \{i \in I \colon a_i R_i b_i\}.$$

Extension of the Definition of Boolean Values

In regard to PI as a complete Boolean algebra, the mapping S defined above allows us to extend the definition of Boolean values from atomic formulas to all logical formulas by means of the following obvious rules (for more detail see [3] and [4]):

$$\begin{aligned}
[\![aSb]\!] &= S(a, b) = \{i \in I \colon a_i R_i b_i\}, \\
[\![a = b]\!] &= \{i \in I \colon a_i = b_i\}, \\
[\![\Phi]\!] &= I - [\![\Phi]\!], \\
[\![\Phi \vee \Psi]\!] &= [\![\Phi]\!] \cup [\![\Psi]\!], \\
[\![\exists x \Phi(x)]\!] &= \bigcup_{a \in B} [\![\Phi(a)]\!].
\end{aligned}$$

Thus every formula $\Phi(a, b, \cdots)$ with constants $a, b, \cdots \in B$ (and without free variables) has a uniquely determined Boolean value

$$[\![\Phi(a, b, \cdots)]\!] \in PI.$$

We call such a formula *valid* (Boolean valid) iff

$$[\![\Phi(a, b, \cdots)]\!] = I.$$

We note that

$$[\![\Phi(a, b, \cdots)]\!] = \{i \in I \colon \mathfrak{A}_i \vdash \Phi(a_i, b_i, \cdots)\};$$

hence it is valid iff it is *true* in *all* the $\mathfrak{A}_i$.

The basic lemma about ultraproducts shows (see [1], Theorem 2.2): Given $\exists x\Phi(x, b, \cdots)$, there exists an element $a \in B$ such that

$$[\![\exists x\Phi(x, b, \cdots)]\!] = [\![\Phi(a, b, \cdots)]\!].$$

That is, even though the value of the existentially quantified formula was defined as a *union* (supremum), it is actually a *maximum*. That is interesting. There are very few homomorphisms of PI (into $\{\mathbf{0}, \mathbf{1}\}$, say) that preserve *all* sups; but it is obvious that any homomorphism of one Boolean algebra into another preserves a max. Therefore, if we regard an ultrafilter D as really being a homomorphism,

$$D \colon PI \to \{\mathbf{0}, \mathbf{1}\},$$

then we can form in the obvious way the *quotient structure*

$$\underset{i \in I}{X} \mathfrak{A}_i / D.$$

This structure in view of the above *maximum principle* is such that any sentence *valid* in $X\mathfrak{A}_i$ is *true* in $X\mathfrak{A}_i/D$. (Of course, this can be sharpened, but the point is to see the conclusion as a consequence of general facts about quotients of Boolean-valued models.) In short, we have divided the ultraproduct construction into two stages: *product* followed by *ultra*. It is the generalization of the product part we wish to emphasize.

Let $\mathfrak{B}$ be an arbitrary complete Boolean algebra. Since $\mathfrak{B}$ is in particular a Boolean σ-algebra, we know that $\mathfrak{B}$ can be represented

$$\mathfrak{B} = \mathfrak{A}/\mathfrak{N}$$

where $\mathfrak{A}$ is a σ-algebra of subsets of a set I and $\mathfrak{N}$ is a σ-ideal of $\mathfrak{A}$. (In [4] the set I was called Ω because it was a measure space. We also change another convention of [4] by using the notation $+, \cdot, \Sigma, \Pi$ for the Boolean operations of $\mathfrak{B}$, saving $\cup, \cap, \bigcup, \bigcap$ for use with their ordinary set-theoretical meanings.)

Now since we are interested in the theory of the real numbers (the set of which is denoted by $\mathbf{R}$) we will form a special product structure. We do not use the full Cartesian power $\mathbf{R}^I$ but only the *subset* $\mathcal{R} \subseteq \mathbf{R}^I$, consisting of the $\mathcal{A}$-*measurable* functions $a\colon I \to \mathbf{R}$. This makes it possible to define the $\mathcal{B}$-values of formulas:

$$\begin{aligned} [\![a = b]\!] &= \{i \in I\colon a_i = b_i\}/\mathfrak{N}, \\ [\![a > b]\!] &= \{i \in I\colon a_i < b_i\}/\mathfrak{N}, \\ [\![a + b = c]\!] &= \{i \in I\colon a_i + b_i = c_i\}/\mathfrak{N}. \end{aligned}$$

Indeed, if $R \subseteq \mathbf{R}^n$ is any n-ary relation with a *Borel* graph, then we can be sure that

$$\{i \in I\colon R(a_i, b_i, \cdots)\} \in \mathcal{A},$$

so that we can define

$$[\![R(a, b, \cdots)]\!] = \{i \in I\colon R(a_i, b_i, \cdots)\}/\mathfrak{N}.$$

Thus any ordinary Borel relation on $\mathbf{R}$ becomes a $\mathcal{B}$-valued relation on $\mathcal{R}$. This makes $\mathcal{R}$ into quite an interesting structure. (Quantified formulas get values just as above. So far $\mathcal{R}$ is only a *first-order* structure having to do with Borel relations. By the same token we extend Borel *functions* $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ to functions $\varphi\colon \mathcal{R}^n \to \mathcal{R}$, as we already indicated with $+\colon \mathbf{R}^2 \to \mathbf{R}$.)

Well, just how interesting a structure is $\mathcal{R}$? First, it is fairly easy to see that $\mathcal{R}$ *is a* ($\mathcal{B}$-*valued*) *real-closed field*. For example, it is clear that

$$a < b \vee a = b \vee a > b$$

is $\mathcal{B}$-valid, and similarly for all the other axioms for ordered fields. Using the extensions of Borel functions to $\mathcal{R}$ we get the roots needed for the axioms of real-closure (see [4] for more details). By quite a different method (see [2]) it can be shown that if $R \subseteq \mathbf{R}^3$ is a Borel relation such that

$$\exists x\, \forall y\, \exists z\, R(x, y, z)$$

is *true* in $\mathbf{R}$, then this formula is *valid* in $\mathcal{R}$. [We can replace x, y, z here each by a string of variables and $R(x, y, z)$ by any quantifier-free combination of Borel relations.] This goes quite a way. Many facts about analytic sets can be expressed in this elementary form. Note, however, that we have only claimed an *implication:* from truth to validity. The converse is *not* correct. There are choices of (nonstandard) algebras $\mathcal{B}$ where the converse fails. (This has to do with Gödel's axiom of constructibility; see [5].)

Of course, analysis wants to have results about arbitrary real functions, and our theory of $\mathcal{R}$ so far has only to do with formulas involving real variables. To bring in the functions we define the set $\mathcal{R}^{\mathcal{R}}$ of allowed functions to be those mappings

$$f\colon \mathfrak{R} \to \mathfrak{R}$$

such that
$$[\![a = b]\!] \leq [\![f(a) = f(b)]\!]$$

for all $a,b \in \mathfrak{R}$. (Here $\leq$ is Boolean inclusion in $\mathfrak{B}$.) All formulas involving function values $f(x)$ and real variables can now be given $\mathfrak{B}$-values for any $f \in \mathfrak{R}$. We can also evaluate equations between functions using the definition

$$[\![f = g]\!] = \prod_{a \in \mathfrak{R}} [\![f(a) = g(a)]\!].$$

The next step is to give $\mathfrak{B}$-values to formulas involving *quantifiers* over functions. And then we can go on to the higher orders. As is shown in [4] (and more fully in [5]) all *axioms* of higher-order real number theory are $\mathfrak{B}$-valid. But for a suitable choice of $\mathfrak{B}$ the continuum hypothesis fails to be valid. The form of this sentence is very simple:

$$\forall h[\exists f\, \forall y[h(y) = 0 \to \exists x[N(x) \wedge y = f(x)]] \vee \exists g\, \forall y\, \exists x[h(x) = 0 \wedge y = g(x)]].$$

Why is it that this fails when the axiom of choice holds?

We remark that this $\mathfrak{B}$-valued model for analysis satisfies the maximum principle. Hence quotients can be taken by ultrafilters forming *really* nonstandard models for analysis in the usual sense of model (see [5] for the proof of the maximum principle and remarks on the Löwenheim–Skolem theorem). We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest?

The answer must be yes, but we cannot yet give a really good argument. Certainly there is *intrinsic* interest in certain of the models. Take the case where $\mathfrak{B}$ is a measure algebra, I is a measure space, $\mathfrak{A}$ is the σ-field of measurable sets, and $\mathfrak{N}$ is the σ-ideal of sets of measure 0. Then $\mathfrak{R}$ is the space of *random variables,* a very well-known space. That it forms a model for real number theory in a precise (although $\mathfrak{B}$-valued) sense must mean something.

Here is one remark that may be useful. In general, one wants to know how various notions familiar from ordinary analysis look in the model. Take the concept of a Borel function; there is a certain formula,

$$\text{Borel } (f),$$

that expresses this in higher-order logic. What can be shown is that if Borel (f) is valid for some $f \in \mathfrak{R}^{\mathfrak{R}}$, then there exists an ordinary Borel function φ: $\mathbf{R}^2 \to \mathbf{R}$ and a particular $a \in \mathcal{R}$ such that

$$f(x) = \varphi(a, x)$$

for all $x \in \mathcal{R}$. In words, we can say that all the nonstandard Borel functions are *quasi-standard* in the sense that they result from standard Borel functions by specializing certain parameters to nonstandard reals. (The details of this theorem were worked out in [2].) There must be other such results.

Of course, the Boolean-valued models have been remarkably successful in solving problems about the existence of certain kinds of unpleasant, complete Boolean algebras (see [5]). But this is a different kind of application from what we usually want nonstandard analysis to do. But maybe there is some hope. Note that in the usual nonstandard models certain standard sets are made subsets of *finite* sets (finite in the model, that is). Then, some fact about finite sets gives us the desired conclusion. The unpleasant Boolean algebras mentioned above can be chosen to make a given standard set *countable* in the Boolean model (see [5]).

In many ways these Boolean models are more like the standard model than the usual nonstandard models (witness the remark about Borel functions). In fact, *countability* behaves very well in the Boolean sense, so that perhaps an argument that uses some formal property of countable sets may reveal something.

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