MODELS FOR VARIOUS TYPE-FREE CALCULI

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Introduction

The question is this: what is the proper *logical* notion of function? For the discussion it seems to me that we must distinguish logical notions from those axiomatized in mathematical *theories*. Making such a distinction in no way means that, for our own benefit, we should refrain from investigating the coherence of certain of our ideas with the aid of mathematical models. Thus, for example, notions of *proposition* and *propositional connective* can be very greatly illuminated by consideration of the variety of Boolean algebras or more general lattices. Some of us are content to confine attention completely to these models, but this is surely wrong. To make the ascent from mathematics to logic is to pass from the object language to the metalanguage or, as it might be said without jargon, to stop toying around and to start believing in something.

The difference between play and faith is well illustrated by geometry (and, also by category theory). Euclidean geometry can provide a field for a lifetime of *study*—especially with the use of the advanced techniques of algebra and topology—but we can also *believe* that the geometry of physical space *is* Euclidean. Euclid did, Newton did too, but now we know better (some of us, that is). Physics has shown that the flow of time together with the presence of matter dictate a far more complex geometry for 'true' space. Alas, we are not yet sure exactly which this geometry is, and the uncertainty no doubt drives many of us back to pure mathematics. The complexity of the situation makes mathematical study and experimentation essential for progress, but the difficulties cannot keep me from entertaining the belief that space has *some* geometry, a preferred, correct geometry. I, myself, cannot give a very clear or exciting description of this geometry, but I know people who can. As regards logic I have rather more self-confidence.

Turning to category theory we come closer to the topic of this paper. One view of a category is that it represents a 'space' of allowed functions—functions under composition rather than under function value. The generality attained by freeing functions from the tie to the specifics of argument-value pairs and by treating them in relation to one another has excellent mathematical benefits: there are many—what should we call them? *non-Fregean*?—models of the axioms of category theory that cannot be explained through arguments and values. That is fine. I approve. But note that when category theorists speak of *functors* they go straight back to the use of arguments and values. They have this idea of functional correspondence as part of their logic, I claim. I would like to make this logic clearer, and so would they. They often study *functor categories* and thereby apply their own theory to itself. It is an excellent idea, but it is no replacement for logic. If and when the so-called *category of all categories* is fully revealed to us, the story may be different. But I doubt it. Until the gospel is written, I will continue to assert that, in the 'universal' or 'logical' category, function-value can be *defined* in terms of composition, and so the dispute is a 'semantical' one. Euclid was proven wrong, and so it may be with us Fregeans. In the meantime I think I have an interesting contribution to make to the logical study of functions, in particular as it is cast in the framework of the λ-calculus of Church and Curry.

In the introduction to his final publication on the λ-calculus, Church writes:

> Underlying the formal calculus which we shall develop is the concept of a function, as it appears in various branches of mathematics, either under that name or under one of the synonymous names, "operation" or "transformation." The study of the general properties of functions, independently of their appearance in any particular mathematical (or other) domain, belongs to formal logic or lies on the boundary line between logic and mathematics. This study is the original motivation for the calculi—but they are so formulated that it is possible to abstract from the intended meaning and regard them merely as formal systems.
>
> A *function* is a rule of correspondence by which when anything is given (an *argument*) another thing (the *value* of the function for that argument) may be obtained. That is, a function is an operation which may be applied on one thing (the argument) to yield another thing (the value of the function). It is not, however, required that the operation shall necessarily be applicable to everything whatsoever; but for each function there is a class, or range, of possible arguments—the class of things to which the operation is significantly applicable—and this we shall call the *range of arguments*, or *range of the independent variable*, for that function...
>
> It is, of course, not excluded that the range of arguments or range of values of a function should consist wholly or partly of functions. The derivative, as this notion appears in the elementary differential calculus, is a familiar mathematical example of a function for which both ranges consist of functions... Formal logic provides other examples; thus the existential quantifier, according to the present account, is a function for which the range of arguments consists of propositional functions, and the range of values consists of truth values.
>
> In particular it is not excluded that one of the elements of the range of arguments of a function f should be the function f itself. This possibility has frequently been denied, and indeed, if a function is defined as a correspondence between two previously given ranges, the reason for the denial is clear. Here, however, we regard the operation or rule of correspondence, which constitutes the function, as being first given, and the range of arguments then determined as consisting of the things to which the operation is applicable. This is a departure from the point of view usual in mathematics, but it is a departure which is natural in passing from consideration of functions in a special domain to the consideration of functions in general, and it finds support in consistency theorems which will be proved below.[1]

Personally, I see no need "to abstract from the intended meaning" and find mere formal systems merely boring. Was Church worried that mathematicians might not consider logical studies worthwhile? or was he still hurting from the inconsistency that Kleene and Rosser had found[2] in his 'logistic'? The specter of formal inconsistency certainly must lie behind the strange inconsistency of exposition whereby, on the one hand, Church takes such care in explaining that functions need not be everywhere defined, while on the other, he allows so easily for meaningful self-application. The claim for the naturalness in generality is awfully weak, and the formal consistency proof (The Church-Rosser Theorem) is not convincing. I should say that truth is that self-application is allowed because it is so *convenient* for what Church wanted to do. Indeed, he wanted to do much more and could not. The λ-calculus is only a fragment of his inconsistent system, an interesting fragment to be sure, and the consistency proof is comforting. Still, a fragmentary result does not prove that the concept *as intended* is coherent. In fact, I feel that Church went much too far when he further claimed that only the terms with *normal forms* are meaningful. Such an extreme stand seems to me to rather spoil the original insight.

Curry provides a somewhat different view:

> Much of the work in illative combinatory logic has been concerned with proving consistency of very weak systems. There is a reason for this. [Combinatory logic] is concerned with notions of such generality that we have no intuitions concerning them.

[1] A. CHURCH, *The calculi of lambda-conversion*, Annals of Mathematics Studies, vol. 6, 2nd ed., Princeton, 1951, pp. 2–3.

[2] References and historical comments are given H. B. CURRY and R. FEYS, *Combinatory logic*, vol. 1, Amsterdam, 1958, pp. 273–274.

The paradoxes have shown this; indeed these contradictions have arisen because they involve combinatory situations—including a delimitation of the notion of proposition—which had not been made explicit. Furthermore these paradoxes have arisen in the work of such minds as Frege, Church, and Quine. This is an argument for the thesis that logical intuitions are not inborn, but arise from experience. The only way of developing new intuitions is by trial and error. The investigations of weak systems, and possibly the discovery of new inconsistencies, is a means to that end. Thus the weakness of these systems does not show that the methods are unsound. We know from the work of Gödel that we cannot expect to prove by finitary methods the consistency of really useful systems. Indeed it is possible that we shall... continue to prove stronger and stronger systems consistent and weaker and weaker systems inconsistent, but that we shall always be interested in systems for which neither of these alternatives holds. Whether this is so or not, the study of consistency questions advances our knowledge. In the early stages we are interested in constructive consistency proofs for very weak systems. But as our intuitions develop we may expect infinitistic methods to be applied and to bear fruit.[3]

I find it very difficult to agree with this position. It is not that I disparage experience or trial and error, rather I wish to dispute the claim that we have no intuitions. In the next section of this paper an intuitive construction will be attempted which will lead to some quite specific investigations. The means I shall use are 'infinitistic', but it does not proceed by the kind of formalistic experimentation that Curry seems to have in mind. The discovery that I made was the result of considerable trial and error on my part, but it was on a conceptual level and did not unfold by attempting consistency proofs. It will be in terms of rather definite intuitions that I shall try to convince people to be interested in various systems, and, after my intuition about the λ-calculus is explained, I shall attempt to say something new about the nature of the paradoxes in this context.

Before beginning the detailed discussion, there is another general point that needs attention. I refer to the conflict in conception between functions in *extension* versus functions in *intension*. Which is the proper approach for logic? The extensional view of function treats them as abstract objects which are completely determined by their argument-value pairs. As it is often said: a function is uniquely determined by its graph; though there is no reason to *identify* a function with its graph as some kind of class of pairs. In intension, on the other hand, a function is a *rule*; that is, a function is somehow connected up with its *definition*. If this was indeed the 'original intention of the founders of our subject', then I submit that they intended to lead us directly into a confusion of use and mention. Though I cannot

[3] H. B. CURRY, *Combinatory logic*, in Contemporary philosophy. A survey, vol. I (R. Klebansky, ed.), Firenze, 1968, pp. 295–307.

agree completely with Quine that intensional 'logic' is *entirely* a use-mention confusion, it is very easy to fall into such error. I hope some day to see a convincing formulation of intensional logic, but I claim that nothing exists *today* that approaches the level of development of extensional logic. Of course, that is no argument against the use of intensions. My complaint is that so far everything is very fragmentary.[4]

Schönfinkel seemed to have something of the idea that his general functions were rules; because he, like Curry, who made the discovery for himself, wanted to eliminate variables from logic in order to get away from the complications of *substitution*. But substitution is a *syntactical* problem, while logic is a *conceptual* one. I do not see that the self-applicable combinators can be justified as providing an 'analysis' of substitution. Their use may make the act of substitution unnecessary, which is nice, but there is much more to them than that. But I do not find an explanation of just what this is in the work of Curry and his students.[5]

Consider Church's λ-δ-calculus.[6] Values are determined according to whether a term has a normal form or not. (Similar systems have been proposed by Goodman[7] and Kearns.[8]) That strikes me as being an out-and-out use-mention mistake. Now, I do not mean to say that the formal system does not work somehow, I only wonder why I should consider this system interesting. After all, the deductive rules are (in view of Gödel) quite weak. Why make decisions on the basis of such a formalism? What notion has been (partially) axiomatized here? Should the general notion of function be so sensitive to details of choice of description and deduction? Some say *yes*, I say *no*.

There may be another point of view from which intensional comparisons are reasonable. We do have something of an idea of a *proof*, a *computation*, a *process*. A function could be a 'scheme' for a type of process which would become definite when presented with an argument. The value would be extracted as an end result of the process. Two functions that are extensionally

[4] Studies in modal logic have not yet gotten to mathematical problems. The closest approach is investigations in proof theory of intuitionistic logic, but connections with the type-free λ-calculus are not yet very clear.

[5] Up-to-date references can be found in H. B. CURRY, T. R. HINDLEY and T. P. SELDIN, *Combinatory Logic*, vol. 2, Amsterdam, 1972.

[6] A. CHURCH, *op. cit*, pp. 55–60, and CURRY and FEYS, *op. cit.*, p. 93.

[7] N. GOODMAN, *A simplification of combinatory logic*, Journal of Symbolic Logic, vol. 34, pp. 1–12.

[8] T. KEARNS, *A system of reduction*, Journal of Symbolic Logic, vol. 32.

the same might 'compute', however, by quite different processes. (One is *fast*, the other *slow*. We all know examples.) This kind of talk is very loose, though. The stickler is that these 'computations' have to be *abstract* objects; they have to be more than just types as distinguished from tokens, because equivalences of 'meaning' are usually allowed. (Say, the orders of execution of certain steps may be interchangeable.) The mixture of abstract objects needed for a sweeping theory of this kind would obviously have to be very rich, and I worry that it is a quicksand for foundational studies. In this paper the extensional approach is adopted, and if necessary we shall *model* intensional notions in terms of suitably chosen extensional objects. Maybe after sufficient trial and error, we can come to agree that intensions have to be believed in, not just reconstructed; but I have not yet been able to reach that higher state of mind.

1. Analysis

In the beginning there were the two values, the true and the false. Why? One answer certainly is that we need at least two discrete, separated, distinguishable, distinct different objects. That much is clear, but why identify these values with the truth values? The answer is: why not? Though explicit reference to the truth values can be eliminated from many discussions, it need not be. If one insists, one can say that certain abstract values 'represent' the truth values without saying that they *are* the truth values. But since I have no desire to restrict the use of abstract objects, I see no harm in making the identification and feel that it makes life simpler to do so.

1.1. AXIOM. $\mathsf{true} \neq \mathsf{false}$.

Having taken the first step, one must then find ways of using the truth values. An obvious use is in the making (better: recording) of decisions. Decision entails choice, I suppose, and the choice situation will be best expressed by the conditional:

$$z \supset x, y.$$

Once we decide whether the value of z is true or false, we shall then be able to make the corresponding choice of x or y. This is not the material conditional of propositional calculus, because x and y may be any two values, not just truth values.[9] Another way of explaining what we are doing is to say that we allow for *definition by cases*, which is often written as:

[9] The use of the conditional was introduced into algorithmic languages by T. Mc Carthy.

$$(z \supset x, y) = \begin{cases} x, & \text{if } z; \\ y, & \text{if not.} \end{cases}$$

But this is all notation, the real assumptions are given by:

1.2. AXIOM. (i) $(\mathsf{true} \supset x, y) = x$;

(ii) $(\mathsf{false} \supset x, y) = y$.

Such conditional expressions, when iterated, permit the definition of many different truth functions. These functions, along with others, we wish to contemplate as abstract objects. That is effected by *functional abstraction*, as in this example:

1.3. DEFINITION. $(x, y) = \lambda z \cdot (z \supset x, y)$.

Again it could be asked why just this combination is called the *ordered pair*. The answer is pragmatic: by Axiom 1.2 we see that it works; and it is a simple device requiring no novel primitive notions. The pair is, however, treated as a function, which is going beyond mere combinations of truth values.

In general we must take abstraction $(\lambda z \cdot)$ as a variable binding operator which, in the theory, is complemented by the binary operation $f(x)$ of *functional application*. The basic intuition of argument-value functional correspondence is embodied in:

1.4. AXIOM. $(\lambda z \cdot \sim z \sim)(x) = \sim x \sim$.

Strictly speaking 1.4 is an axiom schema, because any 'well-formed' expression can go in for the $\sim$-part we have indicated schematically. Just which the well-formed expressions actually are, we leave vague at the moment. At least, at first glance, we can guess that we mean to allow all those composed out of λ, $\supset$, true, false, and variables with the aid of functional application. But care must be exercised, since such a type-free naive function theory might just be inconsistent. It is not, and it is one of the main aims of this essay to give intuitive reasons why not.

Axioms for equality (=) have not been explicitly stated: equality is assumed to be a common notion with the usual properties. Axiom 1.4 is often taken as the *second* axiom of conversion (β-conversion). The first axiom allows for rewriting of bound variables:

$$\lambda z \cdot (\sim z \sim) = \lambda w \cdot (\sim w \sim),$$

with the usual side conditions on avoiding the clash of free and bound variables (α-conversion). This is a 'grammatical' matter not specific to the ideas under discussion, and we again assume all this as part of our common understanding.[10]

On the borderline between the general and the specific is the *principle of extensionality*:

1.5. AXIOM. *If for all* z, $\sim z \sim\ = \ldots z \ldots$, *then*

$$\lambda z \cdot (\sim z \sim) = \lambda z \cdot (\ldots z \ldots).$$

We give it a number for emphasis. What is without question a special assumption is the *principle of functionality*:

1.6. AXIOM. $f = \lambda x \cdot f(x)$

This axiom is often called the axiom of η-conversion.[11] Both 1.5 and 1.6 could be combined into a single statement:

$$f = g \text{ whenever } f(x) = g(x) \text{ for all } x.$$

This is the usual version of the axiom of extensionality, but it seems better to separate out 1.5, which has a more general logical character, from 1.6, which is quite specific to our concept of function.

Supposing that we are able to carry through with a uniform type-freeness, then the import of Axiom 1.6 is that *everything is a function* (f is a free variable). Such austerity! and purity! No other objects but functions are to be found in our universe. But then how neat things will be. It is very tempting to try to make such an arrangement. The only trouble is that intuition may find itself at a loss to answer: what function is the truth value true? But there is no need to be at a loss:

1.7. AXIOM. (i) $\mathsf{true} = \lambda x \cdot \mathsf{true}$;

(ii) $\mathsf{false} = \lambda x \cdot \mathsf{false}$.

Said otherwise, the truth values are each *identical with their own constant functions*. (Given a, the function with constant value a is $\lambda x \cdot a$.) The justification? Again the answer is why not? Any number of 'atomic' objects could be accommodated in this way. It is a very simple method of preserving the purity of 1.6. I know of other plans for making identifications, but this seems the simplest and most elegant.[12]

[10] See the discussion in CURRY and FEYS, *op. cit.*, especially Chapter 3, and on p. 90.

[11] CURRY and FEYS, *op. cit.*, p. 92.

[12] The identification of atoms with their own constant functions is similar to Quine's idea in *New Foundations* that an individual can be identified with its own unit set.

We must make a digression at this point. The classical versions of the λ-calculus opt for even greater purity: No atomic elements at all. The truth values and pairs are introduced by:

$$\mathsf{true} = \lambda x \cdot \lambda y \cdot x;$$

$$\mathsf{false} = \lambda x \cdot \lambda y \cdot y;$$

$$(x, y) = \lambda z \cdot z(x)(y).$$

These are quite pleasant definitions and I have often enjoyed working with them.[13] I have recently found that they possess one fault, however: they cause these very basic notions to be inextricably tied up with the idea of function (the type of inextricability can be made precise). The truth values and the conditional (or pair) are more primitive, I feel. In any case, we are presently engaged in an *analysis* of the (iterative and self-applicative) notion of function; we have long ago come to terms with the truth values. Thus, on conceptual grounds, I prefer not to define them in this functional manner. The construal of pairs as functions and the identifications of truth values with constant functions are quite harmless evocations of functionality. The danger comes in 1.4 as we shall now see.

Recalling Church's generosity with self-application quoted in the introduction, let us consider an example often used for illustration:

$$\nabla = \lambda x \cdot x(x)(x).$$

In a straightforward and formal manner, we allow '$x(x)(x)$' as a well-formed expression. The problem is to evaluate $\nabla(\nabla)$. By formal application of 1.4 we find:

$$\nabla(\nabla) = \nabla(\nabla)(\nabla) = \nabla(\nabla)(\nabla)(\nabla) = \ldots$$

If 1.4 is the only method of evaluation, then $\nabla(\nabla)$ cannot be reduced to a 'meaningful' normal form. Church seems to have felt that the value should be 'undefined'. I have come to agree, but for a totally different reason. The undefinedness of $\nabla(\nabla)$ will be provable as a theorem. It will not be the case that any two expressions without normal forms are to be put equal; of course, Church did not think so either. If the reader wants, he may say that my method gives a way of making precise discriminations between *degrees of undefinedness*. The value of $\nabla(\nabla)$, as it turns out in my suggested interpretation, is undefined in the worst possible way.

[13] They are used also to good effect in GOODMAN, *op. cit.*

We objectified truth values, because we wanted to 'see' them and to think about them—and because we felt there was something definite about them. The undefined is not often objectified, because there is so little that is definite about it. Indeed it might at first seem that it is almost self-contradictory to objectify it. Let us investigate the matter. A symbol is needed. I have chosen this symbol: $\perp$. When we write:

$$f(x) = \perp,$$

we mean that *the function f is (fully) undefined at the argument x*. Eventually we shall want to prove that $\nabla(\nabla) = \perp$, so that $\perp$ becomes definable in the calculus. For the present we prefer to regard $\perp$ as a new primitive notion.

If $\perp$ is to mean total undefinedness, then degrees of undefinedness can be sorted out with the aid of this definition:

1.8. DEFINITION. $x \sqsubseteq y$ *iff whenever* $f(y) = \perp$, *then* $f(x) = \perp$.

The idea is that y is *better* as regards definedness in case the total *un*definedness for any function at y always implies the same for x. This definition provides a comparison in more than degree: if $x \sqsubseteq y$, then y is a better *version* of x, or, as we might say, x *approximates* y. This can be made more precise:

1.9. PROPOSITION. (i) $x \sqsubseteq x$;

(ii) $x \sqsubseteq y$ and $y \sqsubseteq z$ *imply* $x \sqsubseteq z$;
(iii) $x \sqsubseteq y$ *implies* $f(x) \sqsubseteq f(y)$;
(iv) $f \sqsubseteq g$ *implies* $f(x) \sqsubseteq g(x)$.

PROOF: The first two properties are immediate from the definition. To prove (iii), assume that $x \sqsubseteq y$ and $g(f(y)) = \perp$. By 1.4 we can find a function $h = \lambda z \cdot g(f(z))$, so that $h(y) = \perp$. But then $h(x) = \perp$, and thus $g(f(x)) = \perp$. For (iv), assume $f \sqsubseteq g$. This time define $h = \lambda z \cdot z(x)$. By (iii), $h(f) \sqsubseteq h(g)$; then by 1.4 we find $f(x) \sqsubseteq g(x)$. □

It will be noted that we have made heavy use of Axiom 1.4. The reason is that $\sqsubseteq$ has been defined in 1.8 by quantification over the (unknown) domain of functions. The alternative would have been to assume all of 1.9 axiomatically; however, 1.8 is so simple, it was not really worth the effort. Even so, some additional properties of $\sqsubseteq$ will have to be axiomatic.

1.10. AXIOM. (i) $\perp \sqsubseteq x$;

(ii) $x \sqsubseteq y$ *and* $y \sqsubseteq x$ *imply* $x = y$.

1.11. AXIOM. $f \sqsubseteq g$ *if for all* x, $f(x) \sqsubseteq g(x)$.

The effect of these assumptions is that our universe of objects is *partially ordered* by $\sqsubseteq$ in such a way that the operation of functional application $f(x)$ is *monotonic* in both f and x. Furthermore, the partial ordering makes the ordering among the functions come out *argumentwise*.[14] Clearly 1.11 renders 1.5 and 1.6 superfluous. We stated 1.5 and 1.6 because they were much more intuitive; 1.11 is a very strong assumption. In this type-free theory, functions are also functionals (that is, operators on other functions). As a consequence of 1.11 we would have $h(f) \sqsubseteq h(g)$ provided only that $f(x) \sqsubseteq g(x)$ holds for all x. In a way this means that, in the 'calculation' of $h(f)$, the function f enters only through its function values. This is an intuitive assessment, but I think the remark gives the right impression.

Some examples of how the partial ordering works out should make the situation clearer. The element $\perp$ is an object without 'content'. It might be better to say it carries no *information*—other than its being an element of our universe. Now $\perp \sqsubseteq \lambda x \cdot \perp$ holds by 1.10 (i). Thus $\perp(x) \sqsubseteq \perp$ by 1.9 (iv). It follows at once that $\perp = \lambda x \cdot \perp$; in other words, as a function, $\perp$ is the constantly totally undefined function. A function that is not totally undefined is the function (true, false) which takes on at least two distinct values. Using functions like this we can prove:

1.12. PROPOSITION. (i) $\mathsf{true} \not\sqsubseteq \mathsf{false}$;

(ii) $\mathsf{false} \not\sqsubseteq \mathsf{true}$.

PROOF: Suppose $\mathsf{true} \sqsubseteq \mathsf{false}$, then

$$\mathsf{true} = (\mathsf{true}, \perp)(\mathsf{true}) \sqsubseteq (\mathsf{true}, \perp)(\mathsf{false}) = \perp;$$
$$\mathsf{false} = (\mathsf{false}, \perp)(\mathsf{true}) \sqsubseteq (\mathsf{false}, \perp)(\mathsf{false}) = \perp.$$

Thus $\mathsf{true} = \perp = \mathsf{false}$, which contradicts 1.1. Part (ii) is proved similarly. □

Note that by monotonicity we have the relationships $a \sqsubseteq c$ and $b \sqsubseteq c$, where

$$a = (\mathsf{true}, \perp);$$
$$b = (\perp, \mathsf{false});$$
$$c = (\mathsf{true}, \mathsf{false}).$$

Clearly any arbitrary finite partial ordering could be isomorphically represented using iterated ordered pairs. In an expression like that for a above,

[14] This idea of treating partial functions via monotonicity is taken from R. PLATEK, *New foundations for recursion theory*, Stanford Thesis, 1964.

the $\perp$-part can be regarded as an *incompleteness* which is open to 'improvement'. And indeed a was improved to c. It is *not* the case that b is an improvement of a, however, even though they both contain about the same 'amount' of information. The elements a and b are in fact incomparable, just as true and false are. The point of this example was to show in a simple context why the relationship $a \sqsubseteq c$ is *qualitative* rather than *quantitative*. It has more the nature of a part whole relationship, because in a sense the information content of a is 'included' in that of c.

The possibility of being able to 'improve' information, should suggest the questions of whether improvement can be *iterated* and whether there are *limits* to improvement. Again examples with ordered pairs suffice to make the point. Let $a_0 = \perp$ and proceed inductively:

$$a_{n+1} = (\mathsf{true}, a_n).$$

Inasmuch as $a_0 \sqsubseteq a_1$, we can argue that $a_n \sqsubseteq a_{n+1}$ for all n my monotonicity. Could there be a 'limit' to this sequence. Since $\sqsubseteq$ is a partial ordering, and since the sequence is monotonically increasing, it would be natural to suppose that a limit would be a *least upper bound* in the sense of the partial ordering:

$$a_\infty = \bigsqcup_{n=0}^{\infty} a_n.$$

If this element existed, then in some sense it would satisfy an 'infinite' equation:

$$a_\infty = \big(\mathsf{true}, (\mathsf{true}, (\mathsf{true}, \ldots))\big).$$

Indeed it would be tempting to say that:

$$a_\infty = (\mathsf{true}, a_\infty).$$

But before we can justify such an equation, a further analysis of the notions is required.

First, we must ask about least upper bounds. What is the meaning of this construct intuitively? Think of the 'partial' ordered pairs a and b we had before. Each had a different 'bit' of information the other lacked. Joining them together gives the pair c as a 'whole'. We would like to write:

$$c = a \sqcup b.$$

In this case we already knew the desired join c, whereas in the other example a_∞ was something 'new' that had to be found. Thinking of parts and wholes, there would seem to be no reason why we could not join any number of objects into a whole. Why not try? Trying means assuming a new axiom we could call the *principle of completeness*. (As usual an axiom is justified by explicit reference to a few outstanding particular cases.)

1.13. Axiom. *Every collection of elements has a least upper bound in the sense of the partial ordering.*

The empty collection $\emptyset$ is included and we could just as well define:

$$\perp = \bigsqcup \emptyset,$$

where $\bigsqcup$ is the least-upper-bound symbol. (The small symbol $\sqcup$ is used for a finite number of terms.) This approach makes Axiom 1.10 (i) superfluous. Axiom 1.11 has an interesting consequence with regard to functions:

1.14. Proposition. $f \sqcup g = \lambda x \cdot (f(x) \sqcup g(x))$.

Proof: The function on the right (note that we are assuming that it exists) is clearly an upperbound to f and g in view of 1.11. Suppose h is another. Then $f(x) \sqsubseteq h(x)$ and $g(x) \sqsubseteq h(x)$. Thus $f(x) \sqcup g(x) \sqsubseteq h(x)$ holds for all x. By 1.1 again, we have $\lambda x \cdot f(x) \sqcup g(x) \sqsubseteq h$. $\square$

This simple result shows once more how functions are 'determined' by their argument-value pairs. Here the 'join' of the information contained in two functions is accomplished in a completely argumentwise fashion. The proposition can be generalized to any number of functions even an infinite collection.

A curious question is that of calculating the join of true and false. We might even worry whether these two elements should even possess a join. The a, b, c-example with ordered pairs shows that *certain* joins exist. We have uncritically assumed in Axiom 1.13 that all collections of elements have a join. The trouble is that we are not familiar with the idea of the join of $\{\mathsf{true}, \mathsf{false}\}$. An even worse example would be the join of *all* elements. This 'total' join (assuming we can come to accept it) will be denoted by $\top$. It satisfies the characteristic inequality:

$$x \sqsubseteq \top.$$

Returning now to the truth values, if they are so very *separate* (even: contradictory), their join must be very large. In fact, why not assume that they are as separate as possible? In order to be able to formulate the full force of the *principle of separateness*, we require the notion of the *meet or greatest lower bound* of elements. The symbols to be used are $\bigsqcap$ and $\sqcap$. The definition can be given in terms of $\bigsqcup$:

$$\bigsqcap X = \bigsqcup \{y : y \sqsubseteq x \textit{ for all } x \in X\}.$$

A partial ordering with $\bigsqcup$ is a *complete lattice*, and this reduction of $\bigsqcap$ to $\bigsqcup$, as is well known, is valid in all complete lattices.[15]

1.15. AXIOM. (i) $\mathsf{true} \sqcup \mathsf{false} = \top$;

(ii) $\mathsf{true} \sqcap \mathsf{false} = \bot$.

A good answer to one question usually provokes another query. Now that we are thinking of the relation of $\top$ and $\bot$ to the truth values, we must return to a consideration of the conditional. There are different ways to select the assumptions, but I find a certain simplicity and regularity in the following:

1.16. AXIOM. *Let* $w = (z \supset x, y)$.

(i) *If* $\mathsf{true} \sqsubseteq z$ *and* $\mathsf{false} \not\sqsubseteq z$, *then* $w = x$;

(ii) *If* $\mathsf{true} \not\sqsubseteq z$ *and* $\mathsf{false} \sqsubseteq z$, *then* $w = y$;

(iii) *If* $\mathsf{true} \not\sqsubseteq z$ *and* $\mathsf{false} \not\sqsubseteq z$, *then* $w = \bot$;

(iv) *If* $\mathsf{true} \sqsubseteq z$ *and* $\mathsf{false} \sqsubseteq z$, *then* $w = x \sqcup y$.

The reason I like this choice of convention for $\supset$ is by using the values indicated, the function $\lambda z \cdot (z \supset x, y)$ is the *minimal* extension to the domain of all objects of the natural function on $\{\mathsf{true}, \mathsf{false}\}$ defined by Axiom 1.2. (Note that 1.2 is included now as a special case of 1.16 if we grant 1.12.) In making the extension we have chosen the smallest values possible (in the sense of the partial ordering $\sqsubseteq$) which render the function monotonic.

Other conventions are possible over those of 1.16. For example, clause 1.16 (iv) might seem more reasonable (or at least: more *strict*) if we were to put $w = \top$. Call the strict function $:\supset$. It can be defined in terms of $\supset$, however:

$$(z :\supset x, y) = (z \supset (z \supset x, \top), y).$$

Thus it seems quite sufficient to allow $\supset$ as the more fundamental primitive. And $\supset$ has several useful properties.

1.17. PROPOSITION. *Let* $f = (\mathsf{true}, \mathsf{false})$, *then*

$$f(z) = f(f(z)) \sqsubseteq z$$

holds for all z. $\square$

The proof need not detain us since it is an easy argument by cases supplied by the clauses of 1.16. Note that the function f of 1.17 is *four valued*, because in view of 1.15 and 1.16 the values will fall in $\{\bot, \mathsf{true}, \mathsf{false}, \top\}$. Indeed

[15] The finite inf is of course definable in terms of the general one.

these four values form a lattice among themselves. The property of the function f displayed in 1.17 is usefully called being a *projection*. In words, we are assuming in our axioms that the four-valued lattice generated by the truth values is a projection of the whole universe.

Another example of a projection, which this time projects the universe onto a quite large subuniverse, is definable in terms of variable ordered pairs.

1.18. PROPOSITION. *Let*

$$\mathsf{F} = \lambda f \cdot (f(\mathsf{true}), f(\mathsf{false})),$$

then this function is a projection.

PROOF: It is obvious from the properties of pairs that $\mathsf{F}(f) = \mathsf{F}(\mathsf{F}(f))$. What remains is to prove that:

$$(z \supset f(\mathsf{true}), f(\mathsf{false})) \sqsubseteq f(z).$$

This follows by monotonicity of f by checking out the four cases of 1.16. $\square$

These two results give only the barest indication of the variety of subspaces available and of their relationships. Note that the range of the projection of 1.18 is the whole collection of ordered pairs of elements of the universe.

Axioms 1.15 and 1.16 express in different ways the separation one from the other of the two truth values, true and false. It is possible to insist on an even greater separation of these two elements considering them in relation to all the other elements. Thus in view of 1.18 we could define:

$$t_0 = \bot,$$
$$t_{n+1} = (\mathsf{true}, t_n),$$

and we would have:

$$t_0 \sqsubseteq t_1 \sqsubseteq \ldots \sqsubseteq t_n \sqsubseteq \ldots \sqsubseteq \mathsf{true}.$$

Would it be conceivable that 'in the limit', so to speak, the equation:

$$\mathsf{true} = \bigsqcup_{n=0}^{\infty} t_n$$

could hold? In this case the answer is *no*, because each $t_n(\bot) = \bot$; while this is not the case for the (constant) function true. But could there be other *partial* functions, infinite in number, which when joined together gave true? A more difficult example of a strictly increasing sequence of elements below true is defined by:

$$u_0 = \bot,$$
$$u_{n+1} = \lambda x \cdot (\mathsf{true}, u_n).$$

Here we must go to $u_n(\perp)(\perp) = \perp$ before we see that the limit formula is impossible. But these are special cases. We shall assume for full *isolation* that *no* infinite limit can reach the truth values. This is somehow reasonable, because they are assumed as primitive and are not constructed from other parts. The resulting axiom may be called a *principle of continuity*:

1.19. AXIOM. *Let* $t = \mathsf{true}$ *or* $t = \mathsf{false}$. *Then if* X *is any set of elements and* $t \sqsubseteq \bigsqcup X$, *then* $t \sqsubseteq \bigsqcup X_0$, *for some finite subset* $X_0 \subseteq X$.

It may seem strange to use the word *continuity* in connection with 1.19, but the usage is quite justified.[16] For a given truth value t, the predicate $t \sqsubseteq x$ can be considered as a two-valued predicate of x. But the two values are *separated*, and what we have assumed in 1.19 is that this discrete-valued predicate is *continuous*—in a suitable sense. Let us try to make this sense more precise. The $\bigsqcup$-operation works as a kind of limit operation in this space. This is especially intuitive in forming the $\bigsqcup$ of an increasing sequence of elements. More generally we can consider as limits the $\bigsqcup$ of *directed* sets of elements. (A set X is directed iff for all finite $X_0 \subseteq X$, there exists an $y \in X$ with $\bigsqcup X_0 \sqsubseteq y$. Thus a directed set is nonempty.) To be continuous means to preserve limits; formally:

1.20. DEFINITION. *A function f is continuous iff for all directed sets* X

$$f(\bigsqcup X) = \bigsqcup \{f(x) : x \in X\}.$$

We note that since functions are monotonic, the set on the right in 1.20 is also directed. Now what we have assumed about the truth values is that the two-valued monotonic functions:

$$(\top, \perp) \quad \text{and} \quad (\perp, \top)$$

are both continuous. And so are such functions as (x, y) for any given x and y. Indeed these particular finite-valued functions f have a stronger property: for any set X there is a finite subset $X_0 \sqsubseteq X$ such that

$$f(\bigsqcup X) = f(\bigsqcup X_0).$$

Such functions will be called *discrete*. There are a large number of discrete and finite-valued functions.

1.21. DEFINITION. (i) $\mathsf{I}_0 = (\mathsf{true}, \mathsf{false})$;

(ii) $\mathsf{I}_{n+1} = \lambda f \cdot \lambda x \cdot \mathsf{I}_n\big(f\big(\mathsf{I}_n(x)\big)\big)$;

(iii) $\mathsf{I}_n = \lambda f \cdot f$.

[16] A complete discussion is given in my paper *Continuous lattices*, in *Toposes, Algebraic Geometry and Logic*, Springer Lecture Notes in Mathematics, vol. 274, 1972, pp. 97–136.

1.22. PROPOSITION. *The functions I_n form a strictly increasing sequence of projections below the identity function* I. *They are discrete and finite valued, and the elements of their ranges are hereditarily so.* □

The proof is straightforward from the definitions; the hereditary property means that each element of the range of the function has the same property. In the first instance the facts about I_0 are immediate; the formula 1.21(ii) then makes an inductive argument possible. Thus the range of I_0 is 4-valued and that of I_1 is 36-valued, the latter being in a one-one correspondence with all $\{\perp, \mathsf{true}, \mathsf{false}, \top\}$-valued monotonic functions defined on that same 4-valued lattice. These 36 functions again form a lattice, the diagram of which is shown in the figure. (Note that the bilateral symmetry corresponds to the interchange of the roles of true and false.) The diagram for the range of I_2 would be quite impossible to draw, but all its elements are explicitly definable by very elementary expressions in our language. The same is true of all the I_n for $n > 2$.

The objects in the ranges of the I_n are all finite, discrete—better: *isolated*, in the sense applied to true and false in 1.19. Moreover, they are all expressible. Should there be any other such elements? Maybe. If one wants them, they are conceivable. But—for logic—they are not at all necessary. Certainly for logic we need the true and the false, and, through our analysis, also $\perp$ and $\top$. If logic needs functions (and it does) the other elements follow. Note however in our presentation that 'function' means *partial* function, and, in the finite domain, this also means *monotonic* function. Note too that for the sake of simplicity and elegance, functions of one kind can also be of another kind: the value true is at the same time the *constant* function true. This systematic accumulation (the range of I_{n+1} contains the range of I_n) allows for a move that is not merely one of elegance: namely, the passage to the limit. For it is the case that though there are only a countable number of finite functions in the (combined) ranges of the I_n, there are a *continuum* number of possible limits of these functions. We have already seen a few examples.

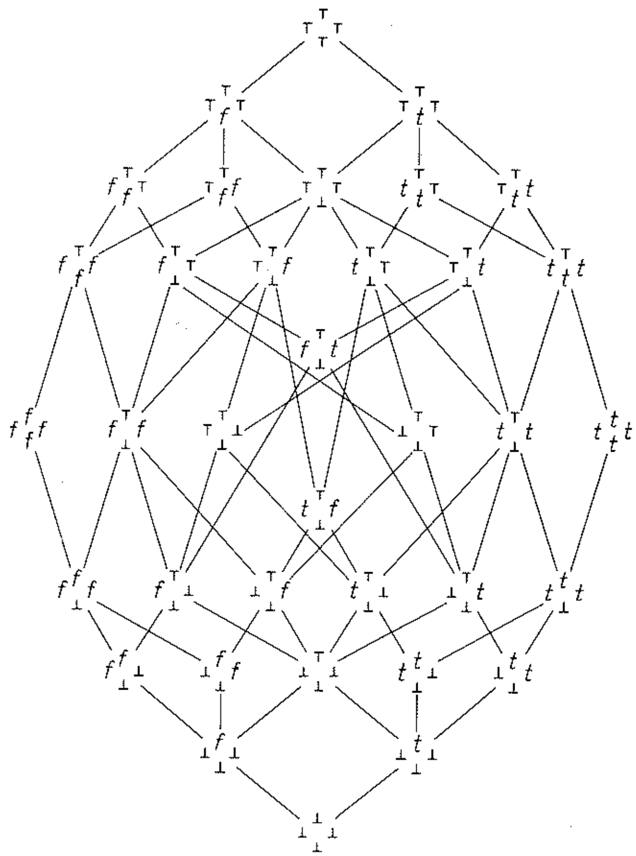
Now the final question is: how much further shall we go? Are there other functions beyond these? *Answer*: Maybe. But do we need them? *Answer*: No. The space of limit points is quite big enough—for a great deal of logic. (This remark will be made more precise later.) So then we must seek a way of limiting ourselves to this quite definite totality. As in classical set theory we need a *principle of foundation* (axiom of foundation):

1.23. Axiom. $\mathsf{I} = \bigsqcup_{n=0}^{\infty} \mathsf{I}_n$.

This is the most obvious and direct form of the axiom; but by utilizing some of its consequences it can be put in a more 'primitive' form. Note first that

$$x = \bigsqcup_{n=0}^{\infty} \mathsf{I}_n(x)$$

follows for all x; hence, as was desired, each element is the limit of finite objects. Furthermore, if x is finite (isolated), then $x = \mathsf{I}_n(x)$ for a suitable choice of n. Thus we see that the finite elements are just those in the ranges of the I_n. Next for an arbitrary x, since each $\mathsf{I}_n(x)$ is a continuous function, the limit is also. Therefore, *all functions of our universe are continuous.* This is the key fact to understanding the whole approach. We could have



The range of I_n.

taken this as an axiom, but it seems better to 'discover' it through contemplation of what would happen by assuming that all elements are limits of finite elements. If we assumed continuity, however, the form 1.23 could take would be more elementary:

$$\mathsf{I}_n \sqsubseteq \mathsf{F} \text{ whenever } \mathsf{I}_0 \sqcup \lambda f \cdot \lambda x \cdot \mathsf{F}\big(\mathsf{F}\,(f(x))\big) \sqsubseteq \mathsf{F}\,.$$

In other words: I is the least fixed point of a certain functional operator. From the assumption that *all* functions are continuous (an assumption which would also imply 1.19), it would follow that the equation of 1.23 holds.

This completes our list of axioms. As presented the axioms are *categorical.* That all models of this (second-order) theory are isomorphic should be clear by now. What may not be so obvious is the *existence* of one model. The question construed in this mathematical way will be discussed in Section 3. It will be seen that there are many similar models of closely related theories. The purpose of the informal discussion of this section was to show that it is reasonable to conceive of something of this nature to believe in. The space we have been investigating has turned out to be far more definite than would have been imagined at first—and I hope it can also be regarded as *natural.* But these are value judgments, it is the 'canonical' character of this space that cannot be denied. In this respect it is much like the real numbers (Euclidean space) or Baire space or Cantor space. Indeed I propose to call it *logical space*, and after a discussion of some necessary model-theoretic niceties, I shall attempt to argue that this is not a misnomer.

2. Summary

As the last section was so long, I propose here to give a quick review of the axioms in a somewhat different order. (The order of Section 1 was necessary, though, for a motivated analysis of the notions.) What turned out was that Logical Space had all along a certain partial ordering $\sqsubseteq$. One basic circle of assumptions can be summarized in:

(I) *Logical space is a nontrivial complete lattice under the partial ordering.*

We have put (I) first here because it is easier to say. What started us off, however, was the concept of *function.* Looked at very abstractly, Logical Space is a function algebra structured by the binary operation of functional application and the reciprocal operator of functional abstraction. We may say that:

(II) *In logical space application and abstraction satisfy the usual laws of conversion and are moreover related to the partial ordering in that the ordering between functions is argumentwise.*

The various monotonic laws are consequences. A certain vagueness resides in the question of what can occur within the *scope* of an abstraction. For the time being we can restrict attention to expressions manufactured from variables, $\perp$, $\top$, $\sqcup$, $\sqcap$, and certain constants by application and abstraction. This restriction will be broadened later.

Next we turn to the truth values:

(III) *Within logical space the* true *and the* false *form, together with* $\perp$ *and* $\top$, *a four-element sublattice of isolated elements which are at the same time identical with their own constant functions.*

It should be remarked that one of the consequences of foundation (Axiom 1.23) is that these four values are the *only* elements satisfying the equation $a = \lambda x \cdot a$.

Having exposed the truth values, the conditional function $\supset$ puts them to work:

(IV) *On logical space the conditional permits a continuous projection of arguments onto the four element sublattice and then an arbitrary assignment of values corresponding to the two truth values.*

Finally the initial projection to the truth values is but the first step of a hierarchy:

(V) *The whole of logical space is generated by limits of the finite elements which are obtained from the truth values by considering truth-valued functions and then functions with these values, etc., arranged as the values of an increasing sequence of projections.*

As a consequence we found that all functions in logical space are continuous. And this answers the query about what is permitted in the scope of the abstraction operator: *any* expression defining a function continuous in the bound variable determines an element of logical space.

The principles summarized in (I)–(V) describe logical space as a space of *partial objects*. The extremes of undefinedness and overdefinedness ($\perp$ and $\top$) have been objectified so that they may enter in an extensional way into the construction of other objects. These combinations are rendered coherent through the partial ordering $\sqsubseteq$. The variety of combination is provided by the truth-values, the conditional, and the functional combinations, which may also be formed into limits. These are operations of a logical nature, and these are all that are permitted. Hence the space can contain only logical elements: it *is* logical space.

3. Construction

The explicit details of a mathematical construction of models for the theory of logical space have been published elsewhere.[17] That presentation, however, is rather severe in style, so it may be well to review here the major points.

The ideas of continuity and limits figure heavily in the investigations; thus, it is not surprising that a discussion of topology emerges. After all, it is the same story in Euclidean space whether one begins with elementary geometry or elementary algebra—and combinatory 'logic' for the most part is the elementary part of the logical algebra of functions. So I look on this work as an inevitable development and only wonder why it took so long to become clear.

A topology can be introduced into any complete lattice. One defines open sets as being those sets U satisfying these two conditions:

(i) *whenever* $x \in \mathrm{U}$ *and* $x \sqsubseteq y$, *then* $y \in \mathrm{U}$;
(ii) *whenever* $\mathrm{X} \subseteq \mathrm{U}$ *is directed,*
and $\bigsqcup \mathrm{X} \in \mathrm{U}$, *then* $\mathrm{X} \cap \mathrm{U} \neq \varnothing$.

The continuous functions in this topology are exactly the continuous functions satisfying the condition of Definition 1.20. The complete lattice becomes a T_0-space, because each point is uniquely determined by its system of neighborhoods. Let us consider for a moment T_0-spaces and their continuous functions in general.

Let D be a fixed space (like logical space) in which functions can take values. Suppose X and Y are any two T_0-spaces, where $\mathrm{X} \subseteq \mathrm{Y}$ as a subspace (in the technical sense of a topological subspace). Consider this diagram:

$$\begin{tikzcd} \mathrm{X} \arrow[r, phantom, "\subseteq"] \arrow[dr, "f"'] & \mathrm{Y} \arrow[d, dashed, "\bar{f}"] \\ & \mathrm{D} \end{tikzcd}$$

Here $f: \mathrm{X} \to \mathrm{D}$ is a given continuous D-valued function. From the point of view of the superspace Y, the function f is only *partially* defined. We could leave it at that, or we could ask whether the value space D has some 'partial' elements sufficient in number to allow for an *extension* of f to

[17] See the paper referenced in note 16.

be made to a continuous $\bar{f}$: $Y \to D$. Indeed, we may ask whether D can be obtained so that this kind of extension is possible no matter what f: $X \to D$ and Y may be. What I discovered as a consequence of my work on logical space is that there exists an enormous variety of such D. Aside from the trivial one-element space, the two-element complete lattice $\{\perp, \top\}$ (with its appropriate topology) is the most immediate example. Any finite lattice is also of this kind, but not every infinite lattice. (A complete and atomless Boolean algebra is unsuitable for this purpose, for example.) I was surprised and pleased to find that any such D with the extension property must be a complete lattice with its lattice topology. The extra condition that the lattice must satisfy is one that makes an intimate connection between the lattice structure and the topology:

(iii) $y = \bigsqcup\{\bigsqcap U\colon y \in U\}$.

This equation must hold for all $y \in D$, where U ranges over all open sets in the sense of (i) and (ii) above. Such lattices I call *continuous lattices*, and their mathematical theory is highly satisfactory.

The study of these lattices is, for me, totally motivated by the desire to found a coherent theory of partial functions (in extension). I feel that I am only doing what is necessary to make things work out properly.

As further evidence that the program is progressing well, I found that I could prove that if $\{X_j\colon j \in \mathscr{J}\}$ is any given system of T_0-spaces (including ordinary sets as discrete spaces, note), then there exists a suitable continuous lattice D such that all the X_j can be embedded in D as subspaces. (Supposing the X_j to be disjoint, then we could make all $X_j \subseteq D$.) Thus by virtue of the extension property of D, if

$$f\colon X_j \to X_j$$

in continuous (which is no restriction in the case of discrete spaces), we may regard f as being a continuous map into D itself, and then a continuous extension

$$\bar{f}\colon D \to D$$

becomes possible. The space D truly is a universe: on that one single space, its homogeneously defined continuous functions can be restricted to produce arbitrary continuous functions on subspaces. We are thus well on our way to a *type-free*, suitably general theory of partial functions. (Certain other steps must be taken though, before the type-freeness allows complete freedom of functional combination.)

Topological spaces are all very well, but we should not become involved in purely topological considerations for their own sake. What is desired

is an application of ideas from a familiar and well-developed mathematical theory to a problem of logical analysis. The set problem is to understand partial functions (especially of higher or even type-free types). In particular, if topology is going to help us, we should stop to think whether *spaces* of functions possess interesting topologies. Such questions have been widely discussed in mathematics and can involve deep results. It is very fortunate that in the present context we need invoke only the easiest notions contained in all the standard text books. This is not to say that future developments will never require more sophisticated techniques, but it is a relief to be able to produce the basic definitions in short order.

As we noted, there is a class of richly endowed spaces D, the continuous lattices, which are at the same time T_0-spaces and complete lattices which enjoy a close connection between the lattice structure and the topology. In suitably chosen such space, part of the endowment consists in the variety of continuous functions $f\colon D \to D$ that are possible. An obvious question, then, is to look at the space $[D \to D]$ of *all* such continuous functions. But which among many is the appropriate topology? There are so many function-space topologies that have been proposed. What we find is that it is the simplest one that works, namely: the *product* topology (often called the topology of *pointwise convergence*). The usefulness of this topology here is due, no doubt, to the assumption of the completeness properties of D which distinguish it as a continuous lattice. That assumption, relatively speaking, requires of D an extensive filling in of gaps. (Naturally, in the finite case, the discrete features of the partial ordering should not be regarded as gaps. In fact, the consecutive intervals are significant jumps serving to isolate the various elements one from another. These isolated elements may similarly exist in infinite lattices.) By so structuring the space of elements, all necessary discriminations between functions can be provided by pointwise properties. (We indicated this in the discussion around Axiom 1.11). Technically speaking, we justify the approach by showing that in general if D and D′ are two continuous lattices, then so is $[D \to D']$ when given the product topology (= the topology associated with the pointwise partial ordering). Furthermore the construct $[D \to D']$ can be shown to have all the right abstract 'Hom'—properties within the category of continuous lattices and continuous functions. Less abstractly, one can conveniently analyze the neighborhood structure of $[D \to D']$ in terms of the neighborhoods in D and D′ and the idea of *approximations* to partial functions.[18]

[18] See *Continuous lattices*, § 3, pp. 111–113.

Having gained some insight into function spaces, we can restate the problem of finding 'type-free' spaces:

Is it possible to find nontrivial spaces D *which can be identified (lattice-theoretically and topologically) with their function spaces* $[\mathsf{D} \to \mathsf{D}]$?

If we can make the D sufficiently large continuous lattices, the extension property together with the possibility of self-application shows that a fully type-free system is possible. (Self-application comes about since under the identification an element may at the same time be considered an argument and a function.)

Among ordinary spaces, as usually studied in topology books, it does not seem possible to make $\mathsf{D} = [\mathsf{D} \to \mathsf{D}]$. However, as I found out in a somewhat indirect way, there are many among T_0-spaces, an underdeveloped part of topology. Indeed, any given system of (ordinary) spaces can be embedded in such a space. The method is one of passage to the limit as suggested in Section 1. There, please note, we simply assumed that the desired limits existed using the argument that there was no need to exclude objects that seemed to fit in well. That level of analysis is shallow and in risk of producing an inconsistency. Here we now outline the construction which assures consistency and possibly even naturalness.

As we already remarked, mere embedding in a continuous lattice, say D_0, is cheap. Let

$$\mathsf{D}_1 = [\mathsf{D}_0 \to \mathsf{D}_0],$$

with the pointwise topology (lattice structure). Is there any chance that D_0 and D_1 can be identified (that is made homeomorphic as spaces and isomorphic as lattices)? It is, in general, very unlikely. But note that D_0 *can* be very naturally embedded in D_1: let to each $x \in \mathsf{D}_0$ correspond the constant function with value x, an element of D_1. This provides a mapping

$$i_0 \colon \mathsf{D}_0 \to \mathsf{D}_1$$

that is a continuous subspace embedding. The map i_0 possesses a partial inverse

$$j_0 \colon \mathsf{D}_1 \to \mathsf{D}_0,$$

which is uniquely determined by the two equations:

$$j_0(i_0(x)) = x,$$

$$i_0(j(x')) \sqsubseteq x',$$

for all $x \in \mathsf{D}_0$ and $x' \in \mathsf{D}_1$. In fact, it turns out that $j_0(x') = x'(\perp)$, the minimum value of the (monotonic) function x'. This can be put in words: each constant can be regarded as a function; but which constant most nearly approximates an arbitrary function? Answer: the minimum value which for monotonic functions can be thought of as the *total variation* of the function. (This only makes sense if one remembers to regard the $\sqsubseteq$-relation in D_0 as an approximation relation.)

And the next step? Easy! Inductively define

$$\mathsf{D}_{n+1} = [\mathsf{D}_n \to \mathsf{D}_n].$$

And embed D_n into D_{n+1}? Yes, but with care. (I consider this the most original step in my construction: once I had this straight all else was forced.)

We were free at first to construe the elements of D_0 as being constant. That gave us an embedding into D_1. But now when we come to D_2 we are not allowed to be so free with D_1, which after all *is* the function space over D_0. We should think of the problem as the embedding of

$$[\mathsf{D}_0 \to \mathsf{D}_0] \text{ into } [\mathsf{D}_1 \to \mathsf{D}_1]$$

while keeping in mind the embedding (already fixed) of D_0 into D_1. A diagram should make matters clearer:

$$\begin{tikzcd} \mathsf{D}_0 \arrow[r, shift left, "i_0"] \arrow[d, "f"'] & \mathsf{D}_1 \arrow[l, shift left, "j_0"] \arrow[d, "f'"] \\ \mathsf{D}_0 \arrow[r, shift left, "i_0"] & \mathsf{D}_1 \arrow[l, shift left, "j_0"] \end{tikzcd}$$

There are obviously two possible functional relations between an f an f':

$$f' = i_0 \circ f \circ j_0,$$

$$f = j_0 \circ f' \circ i_0.$$

The first relationship defines the mapping:

$$i_1 : \mathsf{D}_1 \to \mathsf{D}_2$$

and the second:

$$j_1 : \mathsf{D}_2 \to \mathsf{D}_1.$$

Shortly said: if one knows the approximations between elements (of different spaces), then one knows the corresponding approximations between functions. This procedure can at once be *iterated* to define i_{n+1} and j_{n+1} in terms of i_n and j_n. Each i_n *embeds* D_n into D_{n+1} and j_n *projects* D_{n+1} onto D_n.

We can if we like, identify $x \in D_n$ with $i_n(x) \in D_{n+1}$. For $x' \in D_{n+1}$, we can call $j_n(x') \in D_n$ as the *best* approximation of the higher-type object by a lower-type object. The compositions $j_n \circ i_n$ and $i_n \circ j_n$ behave as in the case $n = 0$ already discussed making the notions of approximation precise. This possibility of *shifting* types was the main clue to the solution of the problem of self-application.

The (higher) types of functions can get very mixed. We have chosen but a selection in the D_n. What we note, however, is that this selection is in some good sense *cofinal* in all the finite types. Indeed D_n is already embedded in D_{n+1} and hence in all D_m for $m > n$. And the later ones project down onto the earlier ones. The types have thus been made *cumulative* and along the way all mixed types can easily be picked up. (The passage from Russell type theory to Zermelo's cumulative types is very analogous—but it is easier and less messy to do with sets.) The accumulation would not have been possible without the aid of the approximation relation $\sqsubseteq$. In this respect partial functions are shown to be more convenient than total functions of the ordinary kind. It remains now to pass to the limit and make precise the concept of *infinite* type.

Suppose a selection $x_n \in D_n$ has been made for $n = 0, 1, 2, \ldots$ and formed into an infinite sequence:

$$x = \langle x_n \rangle_{n=0}.$$

If the x_n have been chosen so as to 'fit together', then we can regard this sequence itself as the limit. But what does it mean, 'fitting together'? For one thing we could ask that the sequence be *increasing*—except the x_n belong to different domains. But this problem is solved by shifting types; to be increasing means just this:

$$i_n(x_n) \sqsubseteq x_{n+1}$$

for all n. This is useful to note, but obviously many *different* increasing sequences can have the *same* limit. The question is whether we can easily distinguish one special sequence out of all the equivalent ones to be *the* limit. The answer is to take the *maximal* one where each x_n is a *best* approximation. That sounds fine if it can be precisely formulated. Recalling that the projections j_n give best approximations, we can try restricting to those sequences x where at each stage

$$(*) \qquad x_n = j_n(x_{n+1}).$$

This is stronger than simply being increasing.

The totality of all sequences satisfying condition $(*)$ forms, by construction the infinite type space D_∞. It can be proved to be a continuous lattice comprehending all the D_n. Its lattice structure for $x, y \in D_\infty$ is given by the relation:

$$x_n \sqsubseteq y_n$$

for all n. Further the mapping $j_{\infty n}: D \to D_n$, where

$$j_{\infty n}(x) = x_n,$$

for $x \in D_\infty$, proves to be the correct projection inverse to the obvious embedding of D_n into D_∞. Indeed there are embeddings and projections i_{mn} and j_{nm} for all pairs n, m with $0 \leqslant m \leqslant n \leqslant \infty$. (Technically: in the category of continuous lattices and continuous maps the space D_∞ is *both* the direct *and* inverse limit of the sequence of spaces $\langle D_n \rangle_{n=0}^{\infty}$ with respect to the two systems $\langle i_n \rangle_{n=0}^{\infty}$ and $\langle j_n \rangle_{n=0}^{\infty}$ of connecting maps.)

That D_∞ somehow exists as a lattice is not so strange; what is pleasant is its *function algebra*. Remember that, by design, D_{n+1} is the function space $[D_n \to D_n]$ over D_n. Thus, for $x, y \in D_\infty$, it always makes sense to write

$$x_{n+1}(y_n)$$

since this belongs to D_n. These elements form a monotonic sequence but may fail to satisfy condition $(*)$. To define $z = x(y)$ in D_∞ we must write:

$$z_n = \bigsqcup_{m=n}^{\infty} j_{mn}\big(x_{m+1}(y_m)\big).$$

This formula makes z automatically best. In this way we extend ordinary functional application on pairs in $D_{n+1} \times D_n$ to an operation or $D_\infty \times D_\infty$. Here is the place where functional application may at last become self-application—if you so desire it.

There are clearly many details to check out, but I hope the idea is clear. The final step is to worry about $[D_\infty \to D_\infty]$. Must we call this $D_{\infty+1}$ and go to higher ordinals? No, for the simple reason that all functions we employ are *continuous*. Thus suppose $f: D_\infty \to D_\infty$ is continuous. Then because all our embeddings and projections are continuous, we can define a sequence $u \in D_\infty$ where:

$$u_0 = j_{\infty 0}\big(f(i_{0\infty}(\perp))\big),$$

$$u_{n+1} = j_{\infty n} \circ f \circ i_{n\infty}$$

We can calculate out that for all $x \in \mathsf{D}_\infty$:

$$f(x) = u(x),$$

where on the left we have ordinary (if you like: set-theoretical) functional application defined on

$$[\mathsf{D}_\infty \to \mathsf{D}_\infty] \times \mathsf{D}_\infty,$$

while on the right we have the newly defined operation on $\mathsf{D}_\infty \times \mathsf{D}_\infty$. That is to say, all the functions (continuous!) in $\mathsf{D}_{\infty+1}$ are already perfectly represented in D_∞: the two spaces are isomorphic! Functional application has been made type-free, continuous, and comprehensive.

In this discussion the initial space D_0 was *any* continuous lattice. In the discussion of Sections 1 and 2 we wanted to be more specific. To obtain the desired specialization we have only to take D_0 as the four-element lattice $\{\bot, \mathsf{true}, \mathsf{false}, \top\}$, and the corresponding D_∞ is a model of the axioms set out above. This gives a very explicit construction of logical space. It seems very special since D_0 can be arbitrary, but it is not.

Suppose that D is an arbitrary continuous lattice with a countable basis for its topology. I conjectured that D could be found as a *retract* of logical space, and this conjecture was quickly proved.[19] Thus logical space is again in another way *universal*. Even better there is a whole *calculus* of retracts that allows one to find them (and the corresponding subspaces) in an effective manner. This requires a bit of explanation.[20]

Let a be a retract of logical space. Its range is a subspace, but we may as well regard it as a (continuous) map of logical space into itself. Therefore we may as well regard it as an *element* of logical space. The characteristic of a retract is that:

$$a \circ a = a.$$

Now *composition* (better: self-composition) is a continuous map, so the retracts of logical space can be thought of as the set of fixed points of a continuous map. Now in general for any complete lattice, the fixed points of a (monotonic) function form a *complete* lattice. So we may say that the retracts of logical space form a complete lattice. This highly abstract discussion is of interest because we can show that the retracts are closed under so many continuous functions.

Here are some examples: Given retracts a and b, define:

[19] This has been established independently by J. Reynolds and Tang.

[20] Retracts are also discussed in *Continuous lattices*.

$$a \times b = \lambda u \cdot \big(a(u, (\mathsf{true})), \; b(u(\mathsf{false}))\big)$$

$$a + b = \lambda u \cdot \big(u(\mathsf{true}) :\supset \big(\mathsf{true}, a(u, (\mathsf{false}))\big), \mathsf{false}, b(u(\mathsf{false}))\big)$$

$$a \to b = \lambda u \cdot \lambda x \cdot b\big(u(a(x))\big).$$

Then each of $a \times b$, $a+b$, $a \to b$ are again retracts and these operations are continuous (and definable). What is the significance? Let the range space of a retraction be denoted by:

$$\mathsf{D}(a) = \{x : a(x) = x\},$$

which in the case of retractions is always a continuous lattice. Then we can prove that

$$\mathsf{D}(a \times b)$$

is always isomorphic with the usual *Cartesian product* of $\mathsf{D}(a)$ with $\mathsf{D}(b)$. Next,

$$\mathsf{D}(a+b)$$

is isomorphic with the least continuous lattice containing the *disjoint sum* of the two lattices $\mathsf{D}(a)$ and $\mathsf{D}(b)$. Finally,

$$\mathsf{D}(a \to b)$$

is isomorphic with the *function space* $[\mathsf{D}(a) \to \mathsf{D}(b)]$. Thus within logical space we have found 'internal' operations corresponding to many important constructions. And the constructs are continuous.

Every continuous function has a least fixed point. Recall the retract I_0 (onto D_0) and find the least fixed point such that:

$$a = I_0 + (a \times a) + (a \to a).$$

On general principles this can be shown again to be a retract. Now $\mathsf{D}(a)$ in this case must be an interesting space. Well, up to isomorphism, we can say that every element of $\mathsf{D}(a)$ is *either* a truth value *or* a pair or a function—and these cases can be distinguished. Note that the components (coordinates or argument-values) are again from the same space $\mathsf{D}(a)$. This means that $\mathsf{D}(a)$ is a model for a whole *new* type-free theory. By choosing combinations of retracts in this way we can construct all sorts of models—explicitly by taking advantage of the universal character of logical space. Once logical space has been constructed, these other constructions become almost automatic.[21]

[21] The use of the calculus of retracts also allows for *proofs* of properties of the subspaces to be given formally from the axioms for logical space.

Conclusion

In Sections 1 and 2 we spoke informally about the axioms for logical space. In Section 3 a construction was supplied proving the consistency of the axioms. This model-theoretic consistency proof applies to the original λ-calculus of Church and Curry in particular, but the axioms we have given are much stronger than any usually considered in combinatory logic. This added strength suggests a change of terminology.

Though the words 'calculus' and 'logic' do have general significance, I would propose calling the systems of Church and Curry *λ-algebra* (or if you like: *combinatory algebra*). This is in analogy to classical algebra. Their theories are *equational* theories of (type-free) functions in combination: the *algebra of functions*—wthether formulated with λ-abstraction or with the so-called combinators. (True, it is a branch of logic like Boolean algebra; but we can usefully apply mathematical techniques.) What I have done is to introduce something new: limits and topology. Therefore, in analogy with classical mathematics, I would like to call the *extended* theory λ-calculus. Any system of rules can be called a calculus, if you like; but analysis (differential and integral calculus) only took wing from the starting point of algebra after the notion of limit was introduced.

This sounds egotistical and is in a way, because I have not discovered anything quite as useful as the integral. But I am not quite mad, as I can show by example. We recall Church's troubles with normal forms and nonnormal forms. *These problems can all be completely analyzed in logical space with the aid of limits.*[22] Not only is $\nabla(\nabla) = \perp$, but one can give a criterion for a λ-expression to evaluate out to $\perp$ that shows exactly how normal forms enter. Even more useful is the analysis of the so-called paradoxical combinator.[23] This is defined as:

$$\mathsf{Y} = \lambda f\cdot\big(\lambda x f\cdot (x(x))\big)\big(\lambda x\cdot f(x(x))\big).$$

This combinator has no normal form but is not meaningless. In fact, for any element f of logical space, $\mathsf{Y}(f)$ is the least fixed point of f, the least solution of the equation:

$$x = f(x).$$

In terms of limits we can write:

[22] This follows from the continuity of the operations (including λ-abstraction) and the construction of logical space.
[23] See CURRY and FEYS, *op. cit.*, pp. 177–178.

$$\mathsf{Y} = \lambda f\cdot \bigsqcup_{n=0}^{\infty} f^n(\top),$$

where f^n denote iterated composition.[24] In other words: a certain infinite series can be written in closed form. Such a result seems to me to be very much in the spirit of classical analysis. There must be many other such results. We must develop the methods of proving them not only for logical space but for the many other analogous spaces that can be similarly constructed.[25]

[24] This especially interesting theorem is due to David Park.
[25] The applications of the calculus that seem most promising are to higher-level computer languages which require complicated recursions on quite involved domains.