SHEAVES AND LOGIC

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PREAMBLE

It seems fair to say that the connection between sheaves and logic originates with Lawvere [29] . In generalizing the notion of sheaf to define sheaves over a site (rather than a topological space), Grothendieck and his coworkers were led to the study of categories of sheaves or *topoi* [1] . Lawvere realized that these categories have the structure necessary to interpret higher-order logic and, together with Tierney [31] , gave an elementary axiomatization of a class of categories which includes the Grothendieck topoi and which provide interpretations for higher-order intuitionistic logic. Various formalizations of these interpretations have been given ([38] , [9]). Although it was soon shown that an elementary topos is "the same thing" as a theory in higher-order intuitionistic logic [9] , the importance of Lawvere's insight lies in the many new *mathematically interesting* models of intuitionistic logic provided by various categories of sheaves. (To any "geometric" first-order theory we can also associate a topos, its "classifying topos" [34] . It is to be hoped that this "geometrization of logic" will prove fruitful.)

Scott's paper in this volume [46] describes a system of higher-order logic which may be interpreted in any topos [9] . Here we describe the models of this logic given by sheaves over a complete Heyting algebra (cHa). These sheaf models subsume the more familiar Beth, Kripke and topological interpretations of intuitionistic logic [6] , [42] , which correspond to interpretations in sheaves over the appropriate cHa of "truth values". They also provide a uniform way of extending these interpretations to higher-order logic and thus help to explain the models of analysis of Scott [44] , Moschovakis [35] and van Dalen [4] . Once we go beyond first-order logic, these sheaf models are more general than Beth, Kripke or topological models [12] . Models over a site (= a small category with a Grothendieck topology [1]) provide yet more generality. Peter Freyd has pointed out that the axiom of choice fails in the topos of $\mathbb{Z}$-sets with finite orbits. This model is similar to the well-known Fraenkel-Mostowski permutation models in that it includes urelements. (Using sites one can make the axiom of choice fail in the well-founded part of a Grothendieck topos (Freyd) and give a topos-theoretic account of Cohen's permutation models.) Sites also arise naturally in first-order model theory once we take into account the comparisons between various models made possible by geometric morphisms. For a presentation of this theory see Makkai and Reyes [34] . However, we find a full-blown categorical presentation is often too abstract and results in very heavy machinery's being brought to bear on very simple problems. By restricting our attention to the special case of models over a cHa , we hope to make what is simple look simple. Models over cHa show clearly the link with traditional models for intuitionistic logic and are sufficient for many applications.

We develop a general theory, of sheaves over a cHa, which is itself intuitionistically valid. In principle, our treatment is formalizable in a system like that we are modelling. Our insistence on working constructively has metamathematical significance: our treatment can be interpreted, wholesale, in the models for higher-order logic built on a given Heyting algebra, or more generally, in any elementary topos. This bonus (extra theorems as a reward for working constructively) is exploited in §9 to show how our treatment includes the theory of taking sheaves for a topology on an elementary topos. Other applications of this idea have been given by Mulvey [36] and Rousseau [43] (using in each case an interpretation in a spatial topos).

Our paper is organized in three chapters. In addition, we refer to Scott [46] as Chapter 0 . Scott's paper provides not only a description of the formal systems we are modelling, but also a discussion of the logical principles we use in our constructive but informal mathematics. Aside from these logical principles, we use the natural numbers (0.7.14) to deal with various questions of finiteness. Although the Axiom of Choice must be abandoned in our constructive mathematics (0.7.15) , a *finite* version is *provable*. By a finite set, we mean one indexed by a natural number (this, by the way, is exactly what Kuratowski-finite means in the presence of a natural-number object [28]). If $S_i \subseteq A$ for $i < n$ and for each $i < n$ we know $\exists x . x \in S_i$ (that is, S_i is *inhabited*) then it is easy to show that there is a function $f : \{ i \mid i < n \} \to A$ with $f(i) \in S_i$ for $i < n$. The proof (as in classical logic) is by induction on the natural number n .

Our first chapter is devoted to (complete) Heyting algebras (cHa) which play a rôle in intuitionistic logic analogous to that of Boolean algebras in classical logic. Heyting was the first to codify the formal rules which characterize them as algebras, which is why we use his name. Heyting algebras have been studied classically (under various names: locale, frame, pseudo-Boolean algebra, etc.), notably by Benabou [2] , Dowker and Papert [5] , Isbell [21] , MacNab [33] , and Simmons [47] ,largely because they arise mathematically as the lattice of open sets of a topological space. This chapter is basically a collection of known results, though we have occasionally had to do some work to ensure that they are intuitionistically valid. The basic examples we introduce in §1 come from topology (lattices of open sets) and algebra (lattices of ideals). Other examples can be obtained by taking quotients of these. Most of the theory deals with these quotients. The main result (due to Dowker and Papert [5]) is that the quotients of a given cHa themselves form a cHa. The proof we give in §2 is essentially that of Isbell [21] expressed constructively. Not every cHa arises as the open-set lattice of a topological space. Classically, those that do (the cHa with *enough points*) correspond to sober spaces (a notion defined in SGA4 [1] in terms of irreducible closed sets). In §3 we give a positive definition of sober and show that this duality also holds constructively. We also give examples of non-spatial cHa .

Of course Heyting algebras also arise logically as the lattice of truth values of an intuitionistic theory (for the connection with first-order logic see Rasiowa and Sikorski [42]). This is why a study of cHa is fundamental to what follows. See 6.14 for an example of the logical information we can glean from our theory.

The first-order structures we introduce in Chapter II are based on Ω-sets or sheaves which are defined in §4 . We start from Ω-sets which are Heyting-valued sets analogous to the classical Boolean-valued sets and were studied independently by Higgs [18] . We single out complete Ω-sets as those in which we can interpret definite descriptions and find that they correspond exactly to sheaves. Specializing to the case where Ω is the lattice $\mathcal{O}(X)$ of opens of some space, we apply the theory of §3 to give a new presentation of the well-known equivalence between sheaves and étale spaces [14] . In fact the essential idea here is already implicit in [18] , however, we stumbled on this proof as a corollary of the representation of internal spaces given in §8 . We introduce the operations and relations needed to structure our Ω-sets in §5 . These lead naturally to the proper notion of morphism of Ω-sets. With this notion, we can rephrase two of our earlier results, saying that the categories of Ω-sets, sheaves over Ω and (when $\Omega = \mathcal{O}(X)$) etale space over X are equivalent. While in categorical spirit we continue to characterize the category of sheaves over a quotient of $P(1)$ as a full subcategory of *Ens* . We shall apply this characterization in §9 . The interpretation of first-order logic is straightforward. We give simple examples from category theory, algebra and analysis to show how it may be used to express interesting properties of sheaves. We then in §6 deal with the logical effects of changing cHa along a morphism and the related direct and inverse image functors. This theory goes back to Joyal, and has been used by Tierney [49,50] to give an elegant construction of classifying topoi. We give an example showing how it may be used to *force* geometric axioms on a given structure.

We have not mentioned applications of sheaves to classical model theory (e.g., Ellerman [7] and Kaiser [26]). However, we believe that the general theory we present provides the proper abstract setting for such work. In particular, even if one is interested only in classical logic, intuitionistic logic is useful. Logical considerations have also provided a new stimulus in representation theory (Johnstone [22] , Kennison [27] , Mulvey [37]) which one might have thought immune to intuitionism.

In Chapter III we consider higher-order structures. Section 7 deals concretely with the construction of higher types in the topos of sheaves on a cHa, and uses them to provide an interpretation of higher-order logic. As examples, in §8 , we introduce the Baire space and the Dedekind reals, and, in the case where Ω is spatial, give their representations as sheaves of germs of continuous functions. Generalizing these examples gives the representation of internal sober spaces announced in [10] . As a corollary we obtain a completely general representation of internal continuous real functions, extending results of Scott [44] and Rousseau [43] . In our final section §9 we recall the definitions of topos and geometric morphism. The category of sheaves on Ω is a topos with a geometric morphism to *Ens* . Internalizing this result gives us the basic facts about sheaves for a topology on an elementary topos [23] . We also give a representation of sheaves on an internal cHa as sheaves on the cHa of global sections.

These notes date back to seminars on sheaves and logic organized by Scott in Oxford starting in the autumn of 1972. Most of the basic ideas (Ω-sets and

singletons, for example) first arose in '72-'73. However, much of this paper is more recent (the representation of sober spaces for example was inspired by a visit (by Fourman) to Montreal in February 1976). Although we have tried to give credit where it is due, it is impossible to catalogue the various contributions made to our thinking by the many people who have helped us to understand this subject. Our thanks are due to all those who participated in the Oxford seminars. The contributions of Robin Grayson, Martin Hyland and Chris Mulvey have in particular left their marks here. We are also grateful to Andre Joyal and Bill Lawvere, whose influence has been profound though infrequent.

CHAPTER I. COMPLETE HEYTING ALGEBRAS

We begin Section 1 with an abstract, lattice-theoretic definition, and then step by step bring in examples from and applications to logic. Completeness is assumed from the start as our main interest is interpretations of *quantified* (higher-order) logic. Constructions of complete Heyting algebras (cHa's) from topology and algebra are discussed in detail. In Section 2 morphisms and quotients are given quite a full theory for two reasons: the concept of a cHa is very useful in understanding intuitionistic topology independently of possible logical applications, and quotients provide new, non-topological cHa's (they are related to forcing and to Grothendieck topologies on sites). In Section 3 the theory is applied to give the definition and basic properties of sober spaces - a topic which has somewhat greater importance in intuitionistic mathematics than in the classical case.

1. DEFINITIONS AND FIRST EXAMPLES

A complete Heyting algebra (cHa) is a special kind of complete lattice. Since we will often regard them as *models* of systems of propositions, we use for the lattice-theoretic operations the logical notation: $\wedge$, $\vee$, $\bigwedge$, $\bigvee$. When we come to interpret a formal language, this means that the same symbols are being used in two different ways; but the context will always make clear whether we are speaking logic or algebra. The zero (bottom) element and unit (top) element are, respectively, denoted by $\bot$ and $\top$; these correspond to the logical *false* and *true* . The partial ordering is denoted by $\leq$ and elements by p, q, r, etc. There are many standard references on lattice theory, and we do not rehearse the well-known definitions.

1.1. DEFINITION. A cHa is a complete lattice Ω satisfying the $\wedge, \bigvee$ -distributive law:

$$p \wedge \bigvee_{i \in I} q_i = \bigvee_{i \in I} (p \wedge q_i)$$

for all $p \in \Omega$ and all systems $\{q_i \mid i \in I\} \subseteq \Omega$.

In any lattice, the partial order relation $p \leq q$ is equivalent to either of the equations $p \wedge q = p$ or $p \vee q = q$. We note that the $=$ in the distributive law can be replaced by $\leq$, since the other inclusion $\geq$ holds in any complete lattice. The order $p \leq q$ can be read logically as the *relation* of implication.

An *operation*, $p \to q$, of implication will be introduced in the next section.

We consider empty and binary meets ($\top$ and $\wedge$) and arbitrary joins ($\bigvee$) as *primitive* in our definitions of cHa. This choice determines the meanings of such concepts as *subalgebra*, *homomorphism*, and the like. Other operations can, and will be defined in terms of the primitive ones. For example, if $P \subseteq \Omega$, then we define the meet by:

$$\bigwedge P = \bigvee \{ q \mid q \leq p \text{ for all } p \in P \} \quad .$$

Finite meets can be defined more explicitly in terms of $\wedge$ and $\top$:

$$\bigwedge_{i<n} x_i = \top \wedge x_0 \wedge \dots \wedge x_{n-1} \quad \text{(by induction on } n \text{).}$$

This makes it clear that taking finite meets and arbitrary joins as primitive would not alter our notions of homomorphism, etc. This may seem obvious, but we have to be a little careful intuitionistically; since, for example, a subset of a finite (that is, finitely indexed) set is *not* in general finitely indexed. When we wish to stress that we are taking finite meets and arbitrary joins as primitive, we talk of $\wedge, \bigvee$-morphisms, $\wedge, \bigvee$-quotients, etc. (The empty meet $\top$ is included *tacitly*.)

A great temptation, which must be resisted in intuitionistic reasoning, is to use the Axiom of Choice. For example, consider a lattice Λ and a subset $L \subseteq \Lambda$. The *closure* of L under $\bigvee$ consists, intuitionistically just as classically, of those elements of Λ expressible as $p = \bigvee_{j \in J} q_j$ with $q_j \in L$. We are asked to show that a sup $\bigvee_{i \in I} p_i$ is again of this form. The classical universal-algebraic proof requires us to show that $\bigvee_{i \in I} p_i = \bigvee_{i \in I} \bigvee_{j \in J} q_{ij}$, where the indicated representation has to be *chosen* over the (infinite) index set I. In this case we can avoid the axiom of choice since each p_i has a *canonical* representation, $p_i = \bigvee \{ q \in L \mid q \leq p_i \}$, and we can easily show that

$$\bigvee_{i \in I} p_i = \bigvee \{ q \in L \mid \exists\, i \in I .\, q \leq p_i \} \quad .$$

In an arbitrary complete lattice Λ we can consider the set D of those $p \in \Lambda$ satisfying the distributive law of 1.1 . This set of *distributive* elements of Λ is evidently closed under finite meets. It turns out that the $\bigvee$-closure of D in Λ is a cHa. With no more work we can say slightly more:

1.2. PROPOSITION. Let Λ and D be as above. If $L \subseteq D$ is a subset closed under finite meets then Ω, the $\bigvee$-closure of L in Λ, is a cHa.

Proof. By definition, Ω is a complete lattice where the $\bigvee$ in Ω is the same as that in Λ. Also meets of finite subsets *of L* are the same as those in Λ. (In particular, $\bot$ and $\top$ belong to Ω.) In general, the meet in Ω will *not* be the same as that in Λ, so we write $\bigwedge\!\!\!\bigwedge$ for meet in Ω. Since for $q \in \Omega$ we have:

$$q = \bigvee \{ p \in L \mid p \le q \} \quad ,$$

we can calculate the meet of $Q \subseteq \Omega$ as follows:

$$\bigwedge\!\!\!\bigwedge Q = \bigvee \{ p \in L \mid p \le q \text{ for all } q \in Q \} \quad .$$

Next, to show Ω is distributive we need only show, for $r, s_i \in \Omega$, that the conditions $p \le r$ and $p \le \bigvee_{i \in I} s_i$ for $p \in L$, always imply $p \le \bigvee_{i \in I} (r \wedge\!\!\!\wedge s_i)$. Because $p \in L$, we know that $p \le \bigvee_{i \in I} (p \wedge s_i)$. But since the $s_i \in \Omega$, we can write:

$$p \wedge s_i = \bigvee \{ p \wedge q \mid q \in L \ , \ q \le s_i \} \quad .$$

And as L is closed under $\wedge$, we see that $p \wedge s_i \le r \wedge\!\!\!\wedge s_i$; and so we have:

$$p \le \bigvee_{i \in I} (p \wedge s_i) \le \bigvee_{i \in I} (r \wedge\!\!\!\wedge s_i) \quad . \qquad \square$$

We shall use 1.2 later. For the moment we turn to some easy examples.

1.3. THE POWER SET. If X is any set, then the power set, $P(X) = \{ Y \mid Y \subseteq X \}$, is obviously a cHa. We say "obviously" because we *assume* intuitionistic logic in defining and proving properties of $\cap$ and $\bigcup$. Thus, we accept the law:

$$\phi \wedge \exists x . \psi(x) \to \exists x \, [\phi \wedge \psi(x)] \quad ,$$

but not the dual law:

$$\forall x \, [\phi \vee \psi(x)] \to \phi \vee \forall x . \psi(x) \quad .$$

This *is* valid classically, so then the dual lattice to $P(X)$ would be a cHa. This is not so intuitionistically. We should remark that the dual distributive law does not imply classical logic (as an interpretation in a three-element chain shows). But more of this later.

1.4. TOPOLOGIES. Given a cHa, a $\top, \wedge, \bigvee$-closed subalgebra is again a cHa - the definition is equational. So what are these subalgebras of $P(X)$? Answer: *topologies* on X . If X is a topological space, then the lattice $\mathcal{O}(X)$ of open subsets of X is just such a subalgebra of $P(X)$. This provides many excellent models for logic because (even intuitionistically) we know so many non-trivial topological spaces.

Suppose X is the space of real numbers. Then the dual of $\mathcal{O}(X)$ is *not* a cHa. This is easy to calculate out, because in any topological space we find for $q_i \in \mathcal{O}(X)$:

$$\bigwedge_{i \in I} q_i = int(\bigcap_{i \in I} q_i) \ .$$

Now if we take the open intervals $p = (0,1)$ and $q_n = (1 - 1/n, 2)$, then we see:

$$p \cup int \bigcap_{n=0}^{\infty} q_n = (0,1) \cup (1,2) \quad ;$$

while on the other hand,

$$int \bigcap_{n=0}^{\infty} (p \cup q_n) = (0,2) \quad .$$

Many topological properties of spaces may be expressed entirely in terms of the lattice of open sets. These notions can then be carried over to all cHa's. For example, take *(quasi) compactness* : If $C \subseteq \Omega$ and $\bigvee C = \top$, then there exist $x_0, \ldots, x_{n-1} \in C$ with $x_0 \vee \ldots \vee x_{n-1} = \top$. Another example is *zero-dimensionality:* every element is the sup of clopen elements (where p is clopen iff for some q we have $p \wedge q = \bot$ and $p \vee q = \top$); that is, the clopen elements form a *basis*. Intuitionistically, however, one has to be careful. Formulating *connectedness* as: whenever $p \wedge q = \bot$ and $p \vee q = \top$, then $p = \top$ or $q = \top$, is fairly weak.

The equational definition of cHa also makes other properties obvious: closure under direct product, for example. Thus, $P(X)$ is just isomorphic to $P(\mathbb{1})^X$ with the pointwise operations. In a different direction, for topological spaces X and Y , we know $\mathcal{O}(X) \times \mathcal{O}(Y)$ is a cHa . In fact, it is isomorphic to $\mathcal{O}(X+Y)$, with the obvious topology on the disjoint union. All of this works intuitionistically, even with infinitely many factors.

1.5. KRIPKE MODELS. As a special case of 1.4 we could take a $\bigcap, \bigcup$-closed family of subsets of X (with the convention that $\bigcap \emptyset = X$ in $P(X)$). Call such a family K . This is "K" for Kripke, since we shall show that these are the well-known Kripke models. K is a cHa (and, in classical logic, so is its dual). It is quite special, however. Because, define the "forcing" relation $\Vdash$ on X by:

$$j \Vdash i \quad \text{iff} \quad \forall p \in K\, [i \in p \ \text{ implies } \ j \in p].$$

This relation is reflexive and transitive (and, by passing to equivalence classes, we could even assume it was a partial ordering; but we will not bother). Note that if we have such a relation, we can define a $\bigcap, \bigcup$-closed family by the equation:

$$K(X) = \{ p \subseteq X \mid \forall i \in p\ \forall j \Vdash i \,.\, j \in p \} \quad .$$

But if we *start* with K ; pass to $\Vdash$; then pass back we get the *same* K . The reason is that:

$$p = \bigcup \{ \{ j \mid j \Vdash i \} \mid i \in p \} \quad ,$$

if p satisfies the above definition; but obviously:

$$\{ j \mid j \Vdash i \} = \bigcap \{ p \in K \mid i \in p \} \in K \quad ,$$

by the definition of $\Vdash$ in terms of K . Thus, any such p belongs to K .

There is in fact a one-one correspondence between reflexive and transitive relations on X and $\bigcap, \bigcup$-closed families $K(X) \subseteq P(X)$.

It is easy to show that Kripke models are really more special than topological models: in Kripke models points have minimal neighbourhoods. More precisely, the sets $p = \{ j \mid j \Vdash i \}$, which we can call *point-like*, can be characterized abstractly in $K(X)$ as those which satisfy:

$$p \leq \bigvee_{i\in I} q_i \quad \text{implies} \quad \exists i \in I \,.\, p \leq q_i \quad ,$$

for all systems $\{q_i\}_{i\in I}$ in the cHa. What it means to be a Kripke model (up to isomorphism) is that *every element is the sup of the point-like elements contained within it.*

Somewhat intermediate between topological and Kripke models are the *distributive algebraic lattices*; we need some definitions.

1.6. DEFINITION. An element p of a (complete) lattice Λ is called <u>finite</u> iff whenever a set $Q \subseteq \Lambda$ is such that $p \leq \bigvee Q$, then $p \leq \bigvee Q_0$ for some finitely indexible subset Q_0 of Q .

We note that, obviously, point-like elements are finite. The finite joins of finite elements are again finite elements. (This includes $\bot$ as finite, which is *not* point-like.) We shall see later that there are lattices with many finite elements but *without* point-like elements.

1.7. DEFINITION. An <u>algebraic lattice</u> is a complete lattice in which every element is the join of its finite subelements (that is, the finite elements form a *basis*).

A Kripke model is thus an algebraic lattice. There are, however, non-distributive algebraic lattices: the distributive ones are the ones we want for our present purposes.

1.8. PROPOSITION. An algebraic lattice satisfying the finite distributive law is a cHa.

Proof. First of all we will verify a weaker infinite distributive law that holds in *all* algebraic lattices. We say that Q is *directed* if every finite subset of Q has an upper bound in Q . Then for directed Q we can prove:

$$p \wedge \bigvee Q \;=\; \bigvee\{ p \wedge q \mid q \in Q \} \quad .$$

Because if e is finite and $e \leq p$ and $e \leq \bigvee Q$, then $e \leq q$ for some $q \in Q$. (Why?) Thus $e \leq \bigvee\{ p \wedge q \mid q \in Q \}$. And that is enough to make the above equation work in an algebraic lattice.

Now if Q is an arbitrary set, then

$$\bigvee Q = \bigvee\{ \bigvee Q_0 \mid Q_0 \subseteq Q , \; Q_0 \text{ finite} \} \quad .$$

That is, every sup is also a directed sup. Now if we have assumed that

$$p \wedge \bigvee Q_0 = \bigvee\{ p \wedge q \mid q \in Q_0 \}$$

for all finite(ly indexible) sets Q_0 , *and* that the lattice is algebraic, then it is a cHa. □

1.9. THE IDEAL LATTICE. With an algebraic structure with finitary relations, the *subalgebras* form an algebraic lattice with the "finite" subalgebras being really the *finitely generated* subalgebras. Usually these lattices are not distributive. (This example, by the way, is the reason for the name "algebraic lattice".) Another example of the same sort is the lattice of *ideals* of a lattice. More generally, let $E, \bot, \vee$ be a semilattice (that is, $\bot \vee x = x = x \vee x$, $x \vee y = y \vee x$, $x \vee (y \vee z) = (x \vee y) \vee z$ all hold identically). We define $\leq$ on E in the usual way. By an ideal of E we understand a subset $p \subseteq E$ where:

(i) $\bot \in p$;

(ii) $x , y \in p$ always implies $x \vee y \in p$;

(iii) $x \leq y \in p$ always implies $x \in p$.

In words: p is a directed and downward closed subset of E . The totality of such forms an algebraic lattice $\hat{E}$, where the finite elements are just the principal ideals:

$$(x) = \{ y \in E \mid y \leq x \} \quad .$$

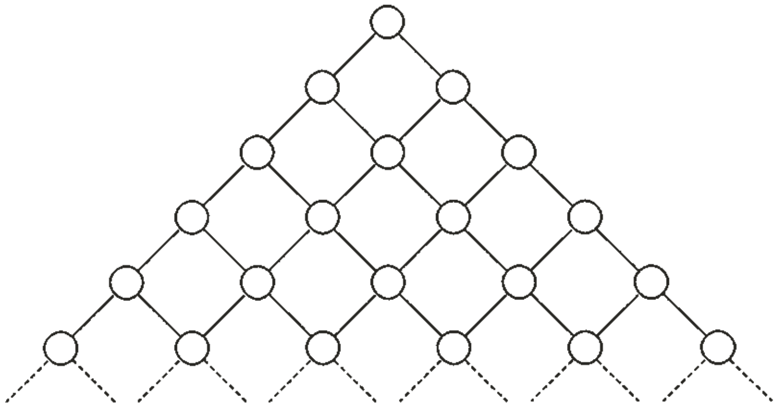
Indeed, the map $x \mapsto (x)$ is an isomorphic embedding of the semilattice E into the lattice of ideals, as is easily proved. Thus *any* semilattice can turn up as the semilattice of finite elements of an algebraic lattice $\hat{E}$.

When is the lattice of ideals distributive? Answer: just when E is. But, since E need not be a lattice, we have to formulate the distributive law in E as a *refinement property*:

$$x \leq y \vee z \text{ implies } \exists u \leq y \; \exists v \leq z . \; x = u \vee v \quad .$$

This is a necessary and sufficient condition for the lattice of ideals to be a cHa. Thus if E is a distributive lattice, this is satisfied.

Consider in this connection the semilattice in the picture. This is in fact a distributive lattice. Note that *every* element is the join of two smaller elements (except for $\bot$, of course). Thus, there are *no* point-like elements in the lattice of ideals.



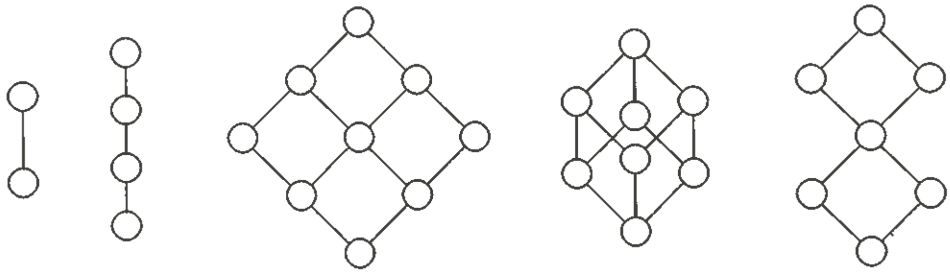
A Distributive Lattice

What we have just defined in 1.9 is a kind of completion, but it only preserves *finite* joins. We shall see later how to take quotients in order to be able to preserve other joins.

Another remark: Suppose E is *finite* with a *decidable* equality relation (that is, just in a one-one correspondence with an initial segment of the integers). Then, if it is distributive, it is a distributive lattice; because E will be closed under the joins of decidable subsets. In particular, we can write:

$$x \wedge y \;=\; \bigvee \{ z \mid z \leq x \;,\; z \leq y \} \quad ,$$

because for that subset we can (by assumption) say whether or not a given z belongs. The usual finite examples of which we can draw complete pictures are of this kind:



Various Finite Distributive Lattices

What may come as a bit of a shock, however, is the fact that, in general, *none* of these are cHa's. The reason is that they are *not complete* in the lattice-theoretic sense. If we want cHa's, we have to pass to the lattices of ideals, which are in general infinite structures.

1.10. PROPOSITION. The two element lattice $\{\bot, \top\}$ is complete iff the following law of logic holds generally: $\neg\phi \vee \neg\neg\phi$.

Proof. Suppose $\{\bot, \top\}$ is complete, where $\bot \neq \top$. Consider any (mathematical) proposition ϕ . Define a subset:

$$P = \{x \mid x = \top \text{ and } \phi\} .$$

(Note: This is not in general a decidable subset.) If we had $\bigvee P \in \{\bot, \top\}$ well defined, there would be two cases: the first is $\bigvee P = \bot$ and the second is $\bigvee P = \top$. In the first case, *not* $\top \in P$ follows, so $\neg\phi$ (that is, *not* ϕ). In the second case, if $\neg\phi$, then $P = \emptyset$ and $\bigvee P = \bot$, which is impossible. Therefore $\neg\neg\phi$.

In the converse direction, suppose this weaker version of the Law of the Excluded Middle does indeed hold. Let $P \subseteq \{\bot, \top\}$ be any subset. As far as sups are concerned, the element $\bot$ is unnecessary. Thus, if it exists,

$$\bigvee P = \{x \mid x = \top \text{ and } \top \in P\} .$$

Now either $\neg \top \in P$ or $\neg\neg \top \in P$. In the first case $P = \emptyset$ and $\bigvee P = \bot$. In the second case we want to show $\bigvee P = \top$. By definition this means:

$$\forall p\, [\, \forall q \in P\;\; q \leq p \text{ implies } p = \top\,] ,$$

where p, q range over $\{\bot, \top\}$. The value $p = \top$ is trivial, so this is equivalent to $\forall q \in P . q \leq \bot$ implies $\bot = \top$, that is, $\neg \forall q \in P . q \leq \bot$. In this, the value $q = \bot$ is trivial, so this is equivalent to $\neg[\, \top \in P$ implies $\top \leq \bot\,]$, that is, $\neg\neg \top \in P$. But this is our assumption, so $\bigvee P$ exists. □

The weak excluded middle of 1.10 is equivalent in intuitionistic logic to De Morgan's law for distributing negation across conjunction. In Johnstone [24] the reader will find 15 equivalent conditions including 1.10 . Scott had noted 1.10 independently, as well as the fact that De Morgan's law implies that *every* finite, decidable semilattice is a complete lattice. The reason for this rests on the fact that if $\{\bot, \top\}$ is complete, then so is the direct power $\{\bot, \top\}^E$. But this is just the lattice of decidable subsets of E , our given finite, decidable semilattice. As we argued before, E is a lattice because it has joins of all its decidable subsets. But the map $\mu(x) = \{y \in E \mid x \nleq y\}$ is a $\vee$-preserving representation of E into the *complete* lattice of decidable subsets of E . Which sets are of this form? Well, they all have the property that $\top \notin s$ and $x \wedge y \in s$ iff $x \in s$ or $y \in s$, for all $x, y \in E$. The totality $\mathcal{E}$ of all such sets is clearly closed under arbitrary unions, and so $\mathcal{E}$ is also a complete lattice. But for $s \in \mathcal{E}$, the inf $x = \bigwedge\{y \in E \mid y \notin s\}$ is such that $\mu(x) = s$. So E and $\mathcal{E}$ are *isomorphic*; therefore E itself is complete.

Since we have spent so much time on lattices in this section, we may as well prove a few more general results. As we noted, the ideals of a lattice do not in general form a cHa. With congruence relations, it is another story.

1.11. PROPOSITION. The congruence relations of any lattice form a cHa.

Proof. The congruence relations of any (finitary) algebra form an algebraic lattice; this is well known and easy to check, since congruence relations are rather like subalgebras of the cartesian square – except as relations they are equivalence relations. As a lattice, the congruence relations are closed under $\bigcap$. The join operation is not $\bigcup$, however. In the case of *two* congruence relations R and S , the join $R \vee S$ is the least *transitive* relation containing both R and S .

Now to prove that we have a cHa , we have only to show the *finite* distributive law holds, in view of 1.8 . This comes down to:

$$T \cap (R \vee S) \subseteq (T \cap R) \vee (T \cap S) \quad ,$$

where, say, R, S and T are congruence relations for a lattice $E, \wedge, \vee$ (we do not even need $\perp$ or $\top$). Suppose, then, xTy and $x(R \vee S)y$. The second means that $xRz_0Sz_1Rz_2 \ldots z_{2n}Rz_{2n+1}Sy$ for a suitable finite sequence of elements z_i . Now note that if zRw , then

$$y \vee (x \wedge z)\ (T \cap R)\ y \vee (x \wedge w) \quad .$$

And similarly for S . Thus we can prove:

$$y \vee x = y \vee (x \wedge x)\ ((T \cap R) \vee (T \cap S))\ y \vee (x \wedge y) = y \quad .$$

By the same argument we can prove:

$$x \vee y\ ((T \cap R) \vee (T \cap S))\ x \quad .$$

Thus $x\ ((T \cap R) \vee (T \cap S))\ y$, as we wanted. □

As a special case of 1.11 , consider a Boolean ring $B, 0, +, \cdot$. (This is a ring where $x^2 = x$ for all $x \in B$; we do not assume there is a unit 1 . B becomes a distributive lattice with $x \wedge y = x\,y$ and $x \vee y = x + y + x \cdot y$.) As is well known, the lattice congruences of B are the same as the ring congruences (because the ring equation $x + y = z$ is equivalent to the conjunction of the two lattice equations $x \wedge y \wedge z = 0$ and $(x \wedge y) \vee z = x \vee y$); and, moreover, ring congruences correspond to *ideals*. let $I(B)$ be the lattice of ideals of B ; by 1.11 and the remarks just made, $I(B)$ is a cHa – this can, of course, be easily proved directly. (Note, however, that in general the ideals of a commutative ring do not form a distributive lattice; we return to this question in 2.15) . In $I(B)$ meets are just intersections; while joins are the closures of the union of ideals under + . The principal ideals form a clopen basis for the lattice, since the complement of (x) in $I(B)$ is the ideal $\{ y \in B \mid x \cdot y = 0 \}$; thus the lattice is *zerodimensional* . The principal ideals may be identified lattice-theoretically as the finite elements of the algebraic lattice $I(B)$: evidently everyone is finite, and the converse follows from the fact that the principal ideals form a basis.

1.12. PROPOSITION. The zero-dimensional algebraic cHa's are, up to isomorphism, exactly the ideal lattices of Boolean rings.

Proof. We have just seen how this works in one direction. So, let Ω be a cHa of the kind mentioned. The clopens of any cHa of course form a Boolean *algebra* (that is, Boolean ring with unit). Let B be the set of *finite* clopens of Ω ; we want to show that B is a Boolean ring, and Ω is isomorphic to $I(B)$.

Let $p \in \Omega$ be finite and let $k \leq p$ be clopen. We show k is finite. Because k is complemented, we can write $p = k \vee k'$ where $k \wedge k' = \bot$. Suppose now that $k \leq \bigvee Q$ where $Q \subseteq \Omega$. Without loss of generality we may suppose $q \leq k$ for $q \in Q$. Now we have $p = \bigvee(Q \cup \{k'\})$. Since p is finite we can write $p = q_0 \vee \ldots \vee q_{n-1}$ where $q_i \in Q$ or $q_i = k'$ for each $i < n$. As these possibilities are not mutually exclusive we cannot immediately define a finite subset of Q covering k . Intuitionistically, we must appeal to the finite axiom of choice. The set $A_i = \{ 1 \mid q_i \in Q \} \cup \{ 0 \mid q_i = k' \}$ is inhabited for each $i < n$. Let f be a choice function so $f(i) \in A_i$ for $i < n$. A *decidable* subset of a finite set *is* finite so $\{ q_i \mid f(i) = 1 \}$ is finite. We show that this set of elements covers k . Write:

$$p = \bigvee\{ q_i \mid i < n \} = \bigvee\{ q_i \mid f(i) = 1 \} \cup \bigvee\{ q_i \mid f(i) = 0 \} .$$

Of course, if $f(i) = 1$ then $q_i \in Q$ and if $f(i) = 0$ then $q_i = k'$ so, multiplying by k on both sides and distributing, we have:

$$\begin{aligned} k &= \bigvee\{ q_i \wedge k \mid f(i) = 1 \} \cup \bigvee\{ q_i \wedge k \mid f(i) = 0 \} \\ &= \bigvee\{ q_i \mid f(i) = 1 \} \vee \bigvee\{ \bot \mid f(i) = 0 \} \\ &= \bigvee\{ q_i \mid f(i) = 1 \} \cup \bot = \bigvee\{ q_i \mid f(i) = 1 \} . \end{aligned}$$

So we did find a finite subset of Q covering k , and this proves k is finite. (We thank Robin Grayson for showing us this trick and apologize to the classical reader for the pain we have had to inflict to make the proof intuitionistic.) Looking back, we have shown in particular that the finite clopens form an ideal in the algebra of clopens; hence, B is a Boolean ring.

The argument of the last paragraph also shows that in view of the assumptions on Ω , the finite clopens form a basis; thus,

$$p = \bigvee\{ k \in B \mid k \leq p \}$$

holds for all $p \in \Omega$. But $\{ k \in B \mid k \leq p \}$ is an ideal of B . If I were any ideal of B , we would only have to let $p = \bigvee I$ to get $I = \{ k \in B \mid k \leq p \}$

by virtue of the finiteness of all elements of B. This establishes the isomorphism between Ω and $I(B)$. □

If B is a Boolean algebra, then in $I(B)$ we have $\top = (1)$ finite. If in 1.12, Ω has $\top$ finite, then *all* clopens are finite. Having $\top$ finite is the abstract notion of compactness. We may thus state:

1.13. COROLLARY. The compact, zero-dimensional cHa's are all algebraic and are, up to isomorphism, the ideal lattices of Boolean algebras.

Isbell [21] provides an interesting characterization of zero-dimensional cHa's. We reformulate his result in order to give an intuitionistic presentation. If Λ is any complete lattice, call an element $x \in \Lambda$ *linear* (say $x \in L$) iff it satisfies the *dual* distributive law

$$x \vee \bigwedge_{i\in I} y_i = \bigwedge_{i\in I} (x \vee y_i)$$

for all $y_i \in \Lambda$. (The reader should be warned that Isbell [21] has half the lattices upside down from our conventions.) The set L of linear elements is obviously closed under finite joins. We remark that *if* Λ is a cHa, then every clopen element is linear. Let K be the set of clopens, let $k \in K$, and let $k' \in K$ be its complement. Argue as follows:

$$k \wedge (k \vee \bigwedge_{i\in I} y_i) = k = k \wedge \bigwedge_{i\in I} (k \vee y_i) ; \qquad \text{and}$$

$$\begin{aligned} k' \wedge (k \vee \bigwedge_{i\in I} y_i) &= k' \wedge \bigwedge_{i\in I} y_i \\ &= k' \wedge \bigwedge_{i\in I} (k' \wedge y_i) \\ &= k' \wedge \bigwedge_{i\in I} (k' \wedge (k \vee y_i)) \\ &= k' \wedge \bigwedge_{i\in I} (k \vee y_i) . \end{aligned}$$

Thus,

$$k \vee \bigwedge_{i\in I} y_i = \bigwedge_{i\in I} (k \vee y_i) .$$

1.14. THEOREM (Isbell). A complete lattice Λ is a zero-dimensional cHa iff the linear elements form a "cobasis" (in the sense that for all $p, q \in \Lambda$, we have $p \leq q$ iff whenever $t \in L$ and $t \vee p = \top$, then $t \vee q = \top$). In such lattices the linear elements are just the clopens.

Proof. If Λ is any zero-dimensional cHa, then, since K forms a basis, we can write:

(1) $\quad p \leq q \leftrightarrow \forall k \in K\, [k \leq p \rightarrow k \leq q]$.

But, as we already indicated, $K \subseteq L$ and of course $k \leq p$ iff $k' \vee p = \top$. Thus, the elements of K form a cobasis, and so Λ satisfies the condition.

Conversely, assume Λ is complete with L as a cobasis. We need to prove:

$$p \wedge \bigvee_{i\in I} q_i \le \bigvee_{i\in I} (p \wedge q_i) \quad .$$

By assumption it suffices to show that for $t \in L$, the equation $t \vee (p \wedge \bigvee_{i\in I} q_i) = \top$ implies $t \vee \bigvee_{i\in I} (p \wedge q_i) = \top$. Assume the first. As t is linear, we see that $t \vee p = \top$ and $t \vee \bigvee_{i\in I} q_i = \top$. Now calculate:

$$\begin{aligned} t \vee \bigvee_{i\in I} (p \wedge q_i) &= t \vee \bigvee_{i\in I} (t \vee (p \wedge q_i)) \\ &= t \vee \bigvee_{i\in I} ((t \vee p) \wedge (t \vee q_i)) \\ &= t \vee \bigvee_{i\in I} (t \vee q_i) \\ &= t \vee \bigvee_{i\in I} q_i = \top \quad . \end{aligned}$$

Thus, Λ is a cHa. We wish to argue next that $L \subseteq K$.

Suppose $t \in L$. Define $t' = \bigwedge \{ y \mid t \vee y = \top \}$. By linearity, $t \vee t' = \top$. To show that $t \wedge t' = \bot$, assume that $s \vee (t \wedge t') = \top$ for an element $s \in L$. Then $(s \vee t) \wedge (s \vee t') = \top$, and so $s \vee t = \top$; hence, $t' \le s$. But $s \vee t' = \top$, so $s = \top = s \vee \bot$. This proves that $t \wedge t' \le \bot$ by the cobasis property. We have shown, therefore, that $t \in K$. Furthermore $t' \in K$ and $t \vee p = \top$ iff $t' \le p$. Thus the cobasis property implies (1) above, which in turn implies that every element is the sup of clopens; or, in other words, Λ is zero-dimensional with $K = L$. □

Note the easy corollary: in a complete Boolean algebra (cBa) both distributive laws hold (both the algebra and its dual are cHa's). Isbell's Theorem will be applied in the proof of 2.20 in a very nice way.

2. MORPHISMS AND QUOTIENTS

The reader must have felt an urge in the last section to ask several times whether there is some relationship between various of the examples we presented. Many of these questions are best expressed by making the class of all cHa's into a category.

2.1. DEFINITION. A <u>morphism</u> between cHa's is a map that preserves finite meets and arbitrary joins.

If Ω and Ω' are two cHa's, then by 2.1 for a map $f^* : \Omega \to \Omega'$ to be a cHa-morphism we must have:

(i) $f^*(\top) = \top'$;

(ii) $f^*(p \wedge q) = f^*(p) \wedge' f^*(q)$;

(iii) $f^*(\bigvee_{i\in I} p_i) = \bigvee'_{i\in I} f^*(p_i)$.

(Generally we drop the dash (') on the operations of Ω' , since there is little chance of confusion. However, for elements we will usually write $p', q', r', \ldots \in \Omega'$.)

2.2. EXAMPLE. Let X and Y be topological spaces and let $f : Y \to X$ be continuous. Then defining

$$f^*(p) = \{ t \in Y \mid f(t) \in p \} ,$$

which is just inverse image, we verify in the known way that $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ is a cHa-morphism. And this motivates 2.1 . (That f^* is well-defined is of course just the *definition* of continuity.)

This remark includes $P(X)$ and the Kripke models $K(X)$ just by taking the appropriate topology on X . In the Kripke model sense, where $\langle X, \Vdash \rangle$ and $\langle Y, \Vdash' \rangle$ are two partially ordered sets, a "continuous" map $f : Y \to X$ is just a function such that:

$$j \Vdash' i \text{ always implies } f(j) \Vdash f(i) .$$

Not all morphisms come from continuous maps, however.

2.3. EXAMPLE. Let E and E' be two semilattices "distributive" in the sense explained in 1.9 . We wish to consider the ideal lattices $\hat{E}$ and $\hat{E}'$, which are cHa's. Let $f : E \to E'$ be a function preserving $\perp$ and $\vee$. On ideals $p \in \hat{E}$ we define:

$$f^*(p) = \{ y' \in E' \mid \exists x \in p \;\; y' \leq f(x) \} .$$

Clearly, $f^*(p) \in \hat{E}'$. Is it a cHa-morphism? Since f is monotone, f^* maps the principal ideals to principal ideals; in this way, just as we can think of E embedded isomorphically into $\hat{E}$, we can regard f^* as an extension (indeed, *the* extension) of f to $\hat{E}$.

It is obvious from the definition that f^* preserves *directed* unions of ideals (that is, directed joins in $\hat{E}$). To check other preservation properties, we note first that for $p, q \in \hat{E}$:

$$p \vee q = \{ x \vee y \mid x \in p , y \in q \} ,$$

owing to the distributivity of E . Therefore,

$$f^*(p \vee q) = f^*(p) \vee f^*(q) .$$

It is also clear that $f^*((\perp)) = (\perp)$, where we are thinking of principal ideals, so f^* preserves all *finite* sups. It thus follows easily that f^* preserves *all* sups.

Meet is just intersection in $\hat{E}$. Since f^* is monotone, all we have to show is:

$$f^*(p) \cap f^*(q) \subseteq f^*(p \cap q) .$$

Suppose $z' \leq f(x)$ with $x \in p$ and that $z' \leq f(y)$ with $y \in q$. Is $z' \in f^*(p \cap q)$? Now there just does *not* seem to be any reason why this should be so in general - especially as we did not assume that E and E' are lattices and that f is a lattice morphism. The property we need is:

(*) $z' \leq f(x)$ and $z' \leq f(y)$ imply $z' \leq f(u)$ for some $u \leq x$ and $u \leq y$.

Obviously this holds in the lattice case.

But there is another problem: what is $f^*(\top)$ (where actually $\top = E$ as an ideal)? It will not be $\top' \in \hat{E}'$ unless we also assume:

(**) $\forall y' \in E' \; \exists x \in E \, . \, y' \leq f(x)$.

This says, more or less, that the image of E is *cofinal* in E'.

Perhaps this is not an especially good example: not only do we have to make rather strong assumptions, but not every morphism from $\hat{E}$ into $\hat{E}'$ need be obtained in this way. However, if $f^* : \hat{E} \to \hat{E}'$ *does* map finite elements to finite elements, its restriction to E (which determines it) will satisfy (*) and (**); so we have not completely wasted our time with this discussion. □

Here is a curious little problem about cHa's: if $f^* : \Omega \to \Omega'$ is a cHa-morphism, is the image $f^*(\Omega)$ also a cHa ? The usual proof from universal algebra is to show that it is a $\top, \wedge, \bigvee$-subalgebra of Ω' ; but the usual proof is *not* intuitionistically valid, because it depends on the Axiom of Choice in showing that the range of f^* is closed under $\bigvee$. The problem is avoided, however, by using special lattice properties that give us a canonical choice of p when we want $p' = f(p)$, namely, the maximal solution of that equation.

2.4. PROPOSITION. Given two complete lattices Λ and Λ', there is a one-one correspondence between $\bigvee$-morphisms $f^* : \Lambda \to \Lambda'$ and $\bigwedge$-morphisms $f_* : \Lambda' \to \Lambda$ which is determined by the adjointness relationship:

$$f^*p \leq p' \text{ iff } p \leq f_*p'$$

for all $p \in \Lambda$ and $p' \in \Lambda'$.

Proof. Suppose we are given f^*. *Define* f_* by the equation

$$f_*p' = \bigvee\{ p \mid f^*p \leq p' \} .$$

This makes it trivial that $f^*p \leq p'$ implies $p \leq f_*p'$. Suppose then that $p \leq f_*p'$ holds. We calculate:

$$\begin{aligned} f^*p &\leq f^*f_*p' && (f^* \text{ is monotone}) \\ &= f^* \bigvee\{ q \mid f^*q \leq p' \} && (\text{by definition}) \\ &= \bigvee\{ f^*q \mid f^*q \leq p' \} && (f^* \text{ preserves } \bigvee) \end{aligned}$$

$\leq p'$ (by lattice properties) .

This proves adjointness. It is easy to see that adjointness uniquely determines f_* .

For $\wedge$-preservation, we argue as follows:

$$\begin{aligned} p \leq f_* \textstyle\bigwedge_i p_i' \quad &\text{iff} \quad f^*p \leq \textstyle\bigwedge_i p_i' \\ &\text{iff} \quad \forall\, i\,.\, f^*p \leq p_i' \\ &\text{iff} \quad \forall\, i\,.\, p \leq f_*p_i' \\ &\text{iff} \quad p \leq \textstyle\bigwedge_i f_*p_i' \quad . \end{aligned}$$

Hence, the two $\wedge$ -expressions are equal.

As this whole proof can be dualized word for word, we see that f_* determines f^* . Note too that adjointness *requires* the preservation properties. □

It must be admitted that 2.4 is a special case of a well-known result about adjoint functors in category theory.

2.5. COROLLARY. The image of a cHa under a cHa-morphism is again a cHa.

Proof. If $f^* : \Omega \to \Omega'$ is a cHa-morphism, then, as we said, it is clear that $f^*(\Omega)$ contains $\top'$ and is closed under $\wedge'$. Suppose $p_i' \in f^*(\Omega)$ for all $i \in I$. Now if $p' \in f^*(\Omega)$, $p' = f^*(p)$ for *some* $p \in \Omega$. It is a consequence from adjointness that, generally, $f^*f_*p' \leq p'$. But we have $f^*p \leq p'$, so $p \leq f_*p'$ and then $p' = f^*p \leq f^*f_*p'$. Thus, $p' = f^*f_*p'$, and this means that f_*p' is a well-determined *preimage* of p' . Thus, even with infinitely many p_i' , we can write:

$$\textstyle\bigvee_{i\in I} p_i' \;=\; \bigvee_{i\in I} f^*f_*p_i' \;=\; f^*(\bigvee_{i\in I} f_*p_i') \in f^*(\Omega) \quad . \qquad \square$$

Example 2.2 continued. We saw for continuous $f : Y \to X$ that $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ is inverse image. What then is f_* ? No, it is not *direct* image since f need not even be an open map. Intuitionistically there does not seem much more to say than what we said in 2.4 and its proof, but classically we can simplify the definition:

$$\begin{aligned} f_*p' &= \bigcup\{\, p \mid f^*p \subseteq p' \,\} \\ &= \bigcup\{\, p \mid \forall\, t \in Y \;\; f(t) \in p \text{ implies } t \in p' \,\} \\ &= \bigcup\{\, p \mid \forall\, t \in Y \;\; t \notin p' \text{ implies } f(t) \notin p \,\} \\ &= \bigcup\{\, p \mid f(Y \sim p') \subseteq X \sim p \,\} \\ &= \bigcup\{\, p \mid p \subseteq X \sim f(Y \sim p') \,\} \\ &= \mathit{int}(X \sim f(Y \sim p')) \quad . \end{aligned}$$

In case f were a *closed* map we could leave off the *int* - and then we get the Boolean dual of the direct image. □

2.6. IMPLICATION. A lattice Ω is a cHa iff $r \mapsto p \wedge r$ is always a $\bigvee$-preserving function. $\bigvee$-preserving functions have adjoints, so what is it in this case? Whatever it is it is uniquely determined by:

$$p \wedge r \leq q \quad \text{iff} \quad r \leq p \to q \quad .$$

This is a corollary of what we have already proved. Furthermore, we can have such a relationship iff Ω is a cHa. (We speak only of complete lattices here.) From this point of view, this is the algebraic significance of the *implication operation* in a cHa.

We can go back to topological spaces and ask what is going on. Classically, in any case, we can show that:

$$p \to q \;=\; int((X \sim p) \cup q) \quad .$$

In Kripke models we can even show intuitionistically that:

$$p \to q \;=\; \{\, i \mid \forall\, j \Vdash i\,.\, j \in p \text{ implies } j \in q \,\} \quad .$$

By the way, it is well known that all the formal properties of $\to$ follow from that adjointness relation. If we specialize q to $\bot$ we get *negation*: $\neg p \;=\; p \to \bot$. The lattice-theoretic meaning of negation is that $\neg p$ is the largest element of Ω *disjoint* from p . Classically, in the topological case we find from the above:

$$\neg p \;=\; int(X \sim p) \quad .$$

But this holds intuitionistically as well. (If we wanted an intuitionistic version of $p \to q$, we could take the interior of the "implication" between the two sets, namely, $\{\, x \in X \mid x \in p \text{ implies } x \in q \,\}$.) □

2.7. OPEN MAPS. In the topological case if $f : Y \to X$ is an *open* map, then the *direct* image is a well-defined mapping

$$f_! : \mathcal{O}(Y) \to \mathcal{O}(X) \quad .$$

It is a $\bigcup$-preserving map, and we can easily prove that:

$$f_! p' \leq p \quad \text{iff} \quad p' \leq f^* p \quad .$$

It follows then that f^* is a $\bigwedge$-preserving map, and we can use this (or the existence of an opposite adjoint $f_!$) as a *definition* of an *open* cHa-morphism. (Such morphisms are also called *essential*.) In the Kripke model case, all "continuous" maps ($\Vdash$-preserving maps) are "open". But this does not mean that *all* cHa-morphisms $f^* : K(X) \to \Omega'$ from a Kripke model are open. This will become clear in the next section when we speak about sober spaces and representations. □

As soon as one has algebras and morphisms, it is natural to ask about *free algebras*. The following goes back to Benabou [2] .

2.8. DEFINITION. Let I be any set. Let Φ_I be the cHa (actually, a Kripke model) of all sets of finite subsets of I closed under taking supersets.

2.9. PROPOSITION. In the category cHa , Φ_I is the free algebra on I-generators.

Proof. Let E be the family of finite subsets of I (finite in the sense of finitely indexible). If $e \in E$ and $f : I \to \Omega$ is any function, then the finite meet $\bigwedge_{i\in e} f(i)$ is well-defined. This means we can extend f from I to a map $\bar{f} : \Phi_I \to \Omega$ by the formula:

$$\bar{f}(p) = \bigvee_{e\in p} \bigwedge_{i\in e} f(i) \quad .$$

Now we note that the sets $[i] = \{ e \in E \mid i \in e \}$ generate Φ_I in view of the formula:

$$\begin{aligned} p &= \bigcup_{e\in p} \bigcap_{i\in e} [i] \\ &= \bigcup_{e\in p} \{ e' \mid e \subseteq e' \} \quad . \end{aligned}$$

Furthermore, we see that $\bar{f}([i]) = f(i)$; so, if we can show that $\bar{f}$ is a cHa-morphism, it not only is an extension of f but it is uniquely determined.

Since the $\top$-element of Φ_I contains the empty set, we see $\bar{f}(\top) = \top$. For binary products, we note first that:

$$\begin{aligned} p \wedge q &= p \cap q \\ &= \{ e \cup d \mid e \in p , d \in q \} \qquad , \quad \text{and so} \\ \bar{f}(p \wedge q) &= \bigvee_{e\in p, d\in q} (\bigwedge_{i\in e} f(i) \wedge \bigwedge_{j\in d} f(j)) \\ &= \bigvee_{e\in p} \bigwedge_{i\in e} f(i) \wedge \bigvee_{d\in q} \bigwedge_{i\in e} f(j) \\ &= \bar{f}(p) \wedge \bar{f}(q) \quad . \end{aligned}$$

Finally, $\bar{f}$ preserves $\bigvee$, because $\bigvee$ *is* $\bigcup$ in Φ_I , and the required formula is obvious from the definition of $\bar{f}$. □

As an instant corollary we see that every cHa Ω is the *quotient* of a Kripke model Φ_I , where we have only to take $I = \Omega$. Then the map $\bar{f} : \Phi_I \to \Omega$, where $f : I \to \Omega$ is the identity function, is a surjection. Perhaps a more interesting representation is given by considering the poset Ω as a Kripke model with $p \Vdash q$ iff $p \leq q$. We then have an $\wedge,\bigvee$-surjection $K(\Omega) \to \Omega$ given by $U \mapsto \bigvee U$.

Quotients are sometimes easier to think of in terms of congruence relations, especially in making a construction when we may not know in advance what the quotient structure really is. Thus, if $f^* : \Omega \to \Omega'$ is a morphism, then in the usual way $\{(p,q) \mid f^*p = f^*q\}$ is the congruence relation on Ω determined by f^*. As f^* has certain properties, so will the congruence relation.

2.10. DEFINITION. The class of all <u>cHa-congruence relations</u>, $C(\Omega)$, on the cHa Ω, consists of all equivalence relations preserving $\wedge$ and $\bigvee$ in the sense that:

(i) pRp' and qRq' imply $(p \wedge q)R(p' \wedge q')$; and

(ii) $\forall\, i \in I\ \ p_i R q_i$ implies $\bigvee_{i \in I} p_i \; R \; \bigvee_{i \in I} q_i$,

for all elements of Ω.

We note that $C(\Omega)$ is a complete lattice, because it is closed under $\bigcap$. What other properties it has will emerge: we have to be careful intuitionistically when infinitary operations are involved. We did not have to be very careful in 1.11 . One of the first questions is : will Ω/R be a cHa ? The answer is *yes*, but we have to proceed as we did in 2.4 and 2.5 .

If we look at 2.4 , we see we can transfer the congruence relation determined by the values $f^*p \in \Omega'$ back to values f_*f^*p which remain in Ω. Indeed, we see

$$f^*p = f^*f_*f^*p ,$$

so we can show:

$$f^*p = f^*q \text{ iff } f_*f^*p = f_*f^*q .$$

The map $p \mapsto f_*f^*p$ has nice properties. It preserves $\top$ and $\wedge$, since both f^* and f_* do. From the above equation, it is seen to be idempotent. But also by adjointness we find that $p \le f_*f^*p$. (This can be called *increasing*, or more colourfully: *inflationary*.) And these are all the properties we need.

2.11. DEFINITION. The class of all <u>J-operators</u>, $J(\Omega)$, on the cHa Ω, consists of all self-maps $J : \Omega \to \Omega$ where:

(i) $p \le Jp = JJp$; and

(ii) $J(p \wedge q) = Jp \wedge Jq$,

hold for all elements of Ω.

We remark that J-operators are *not* cHa-morphisms as, in general, they are not $\bigvee$-preserving in the whole of Ω. But they are closely related to morphisms: the range of J will turn out to be a cHa and, with the codomain so restricted, we then obtain a morphism. Some lemmas are helpful in seeing this.

2.12. LEMMA (Tarski). The fixed points of any monotone operator on a complete lattice form a complete lattice.

Proof. Suppose $f : \Lambda \to \Lambda$ is monotone, and Λ is complete. Consider the poset:

$$\Pi = \{ p \in \Lambda \mid p = f(p) \} .$$

We wish to show that, in itself, Π is a complete lattice. So let $p_i \in \Pi$ for $i \in I$. We will define a join $\bigvee_{i \in I} p_i \in \Pi$. (The dual proof giving the meet works just as well.) Let:

$$\bigvee_{i \in I} p_i = \bigwedge \{ q \mid f(q) \leq q \text{ and } \forall i \in I \; p_i \leq q \} .$$

Now by completeness this exists in Λ. Call the element r for short. Of course $\forall i \in I . p_i \leq r$; and thus $\forall i \in I . p_i \leq f(r)$, because f is monotone and all the $p_i \in \Pi$. In the same way we show that if q satisfies the condition in the definition of r, then so does $f(q)$. But $r \leq q$ for all such q, so $f(r) \leq f(q) \leq q$ for *all* these q. It follows that $f(r) \leq r$ by definition. This shows that r itself satisfies the condition; hence, we conclude $r \leq f(r)$ because $f(r)$ satisfies the condition. Therefore, $r \in \Pi$. Any other element $s \in \Pi$ which is an upper bound to the p_i clearly satisfies the condition; whence $r \leq s$. This makes r a *least* upper bound. □

As a corollary of 2.12 we note that the posets $\{ p \mid p \leq f(p) \}$ and $\{ p \mid f(p) \leq p \}$ are also complete lattices; because if $f : \Lambda \to \Lambda$ is monotone, then so are the two maps:

$$p \mapsto p \wedge f(p) \quad \text{and} \quad p \mapsto p \vee f(p) .$$

Their fixed-point sets are, respectively, the two sets we want.

2.13. LEMMA. The fixed points of a multiplicative operator on a cHa form a cHa.

Proof. Suppose $f : \Omega \to \Omega$ is multiplicative (i.e., 2.11 (ii)), and Ω is a cHa. Then f is monotone. So the fixed points form a complete lattice by 2.12. Since f is multiplicative, the fixed-point set Π is closed under $\wedge$. Thus, to prove that Π is a cHa, we need only verify for elements of Π:

$$p \wedge \bigvee_{i \in I} q_i \leq \bigvee_{i \in I} (p \wedge q_i) .$$

Now for each $i \in I$:

$$p \wedge q_i \leq \bigvee_{i \in I} (p \wedge q_i) .$$

By 2.6 we can write (use $\to$ as an operation):

$$q_i \leq p \to \bigvee_{i \in I} (p \wedge q_i) .$$

Consider for a moment any $r, s \in \Pi$. Again by 2.6, we can prove that $r \wedge (r \to s) \leq s$. Since f is monotone and multiplicative, we find $f(r) \wedge f(r \to s) \leq f(s)$ and thus:

$$f(r \to s) \leq f(r) \to f(s) = r \to s .$$

We then see that

$$f(p \to \bigvee\!\!\!\!\bigvee_{i\in I} (p \wedge q_i)) \le p \to \bigvee\!\!\!\!\bigvee_{i\in I} (p \wedge q_i) \quad .$$

In view of the proof of 2.12 , this is enough to show that

$$\bigvee\!\!\!\!\bigvee_{i\in I} q_i \le p \to \bigvee\!\!\!\!\bigvee_{i\in I} (p \wedge q_i) \quad .$$

By 2.6 , again, we get our conclusion. □

2.14. PROPOSITION. The range of a J-operator on a cHa is a cHa, and the mapping so construed is a cHa-morphism.

Proof. Given $J : \Omega \to \Omega$ as in 2.11 , and noting that the range of J is the same as the fixed-point set of J , we see by 2.13 that the range is a cHa. The map

$$J : \Omega \to J(\Omega)$$

maps $\top$ to $J(\top)$ and it is multiplicative. Thus we need only show it preserves $\bigvee$. Indeed, in $J(\Omega)$ we can easily show that

$$\bigvee\!\!\!\!\bigvee_{i\in I} q_i = J(\bigvee_{i\in I} q_i) \quad ,$$

whenever the $q_i \in J(\Omega)$. For arbitrary $p_i \in \Omega$, we can also easily show:

$$J(\bigvee_{i\in I} p_i) = J(\bigvee_{i\in I} Jp_i) \quad ,$$

and this gives the answer. □

We note that for $p \in \Omega$ and $p' \in J(\Omega)$, we have:

$$p \le p' \quad \text{iff} \quad Jp \le p' \quad .$$

This means that $J_* p' = p'$; so we have shown that *every* J-operator comes from a morphism, as was illustrated in the discussion leading up to 2.11 . To complete the story we also have to connect the J-operators with the cHa-congruence relations. Before doing so, however, a few more remarks about multiplicative operators may be helpful.

The content of 2.13 does not seem to be contained in 2.14 , because there can be multiplicative operators with fixed-point sets that are not the fixed-point sets of any J-operator. (Assuming that the four-element Boolean algebra is complete, consider its one non-trivial automorphism.) If $M : \Omega \to \Omega$ is an *inflationary* multiplicative operator, on the other hand, then we can define a J with the same fixed points. Indeed, let

$$Jp = \bigwedge \{ q \mid p \le q = Mq \} \quad ;$$

by 2.12 , Jp is just the least fixed point of M covering p . Thus, $p \le Jp = MJp = JJp$; and if $p = Mp$, then $p = Jp$. It only remains to prove that J is multiplicative. Now $p \wedge q \leq J(p \wedge q)$ holds, so

$$q \leq p \to J(p \wedge q) \quad .$$

Note that since $p \wedge (p \to r) \leq r$, we can assert:

$$Mp \wedge M(p \to J(p \wedge q)) \leq MJ(p \wedge q) \;=\; J(p \wedge q) \quad .$$

Because $p \leq Mp$, we can derive from this:

$$M(p \to J(p \wedge q)) \leq p \to J(p \wedge q) \quad ,$$

which must be an equality in fact. Thus:

$$Jq \leq p \to J(p \wedge q) \quad , \quad \text{and so}$$

$$p \leq Jq \to J(p \wedge q) \quad .$$

Exactly as before, we prove:

$$M(Jq \to J(p \wedge q)) \leq Jq \to J(p \wedge q) \quad , \quad \text{and so}$$

$$Jp \leq Jq \to J(p \wedge q) \quad .$$

This shows that J is multiplicative and therefore a J-operator.

2.15. RADICAL IDEALS. Here is a different kind of example taken from (intuitionistic!) algebra. Let A be a *commutative ring with unit*. In classical mathematics we can show that A has lots of prime ideals, and this is very helpful for many arguments. The existence proofs are non-constructive, however, and the lack of prime ideals is a great bother. We can still consider the *lattice* of ideals, nevertheless, but it is hardly ever distributive. Closely related to the (non-existant) prime ideals are the so-called *radical ideals*. These rather conservative ideals are well-behaved because they have preserved their roots. For ideals $p \in A$, all we need is this square condition:

$$\forall x \in A \quad x^2 \in p \quad \text{implies} \quad x \in p \quad ,$$

and then the same will hold for n^{th} powers. Let $\surd(A)$ be the set of all such radical ideals. Obviously it is a complete lattice (even: algebraic), and one can verify the distributive law directly by a straight-forward calculation. Thus, $\surd(A)$ is a cHa.

We can look at this example in another way. Let Ω be the family of all subsets $p \in A$ such that

$$\forall x \in p \; \forall a \in A \,.\, ax \in p \quad .$$

Ω is $\cap, \cup$-closed, and thus Ω is a cHa. Define Jp as the least radical ideal generated by p . This mapping $J : \Omega \to \Omega$ clearly satisfies 2.11 (i) . An inductive proof shows that $z \in Jp$ iff there is a finite sequence $\{x_0, \ldots, x_{n-1}\} \subseteq p$ and an integer m such that $z^m = x_0 + \ldots + x_{n-1}$. Thus if $z \in Jp$ and $z \in Jq$ we have $z^{\ell} = y_0 + \ldots + y_{k-1}$ for $y_i \in q$. Multiplying these together we find $z^{m+\ell} \in J(p \wedge q)$, so $z \in J(p \wedge q)$. This shows that J is multiplicative on the lattice Ω. As $\sqrt{(A)} = J(\Omega)$, we have another proof that this is a cHa. □

2.16. PROPOSITION. The complete lattices $\mathcal{C}(\Omega)$ and $\mathcal{J}(\Omega)$ are isomorphic, and the one-one correspondence is provided by this relationship for $R \in \mathcal{C}(\Omega)$ and $J \in \mathcal{J}(\Omega)$:

$$p \wedge q \, R \, q \quad \text{iff} \quad q \leq Jp \ ,$$

for all $p, q \in \Omega$.

Proof. Perhaps the reader did not note that $\mathcal{J}(\Omega)$ is a complete lattice. This can be verified directly by defining:

$$J \leq K \quad \text{iff} \quad \forall\, p, Jp \leq Kp \ , \quad \text{and}$$

$$(\textstyle\bigwedge_{i \in I} J_i)p = \bigwedge_{i \in I} (J_i p) \ .$$

(The only trick is to show that the pointwise meet is an idempotent operator.)

Given a cHa-congruence R, we define

$$J_R p = \bigvee \{ q \mid pRq \} \ .$$

Since R preserves $\bigvee$, we see that $J_R p$ is the *maximal element* of the congruence class of p. It follows that J_R satisfies 2.11 (i) and:

$$pRq \quad \text{iff} \quad J_R p = J_R q \ .$$

Since $pRJ_R p$ and $qRJ_R q$, we see also:

$$(p \vee q) \, R \, (J_R p \vee J_R q) \ .$$

In consequence:

$$J_R p \vee J_R q \leq J_R(p \vee q) \ .$$

Thus J_R is monotone. By the same style of argument we can show:

$$J_R p \wedge J_R q \leq J_R(p \wedge q) \ ,$$

but the converse $\leq$ holds by monotonicity. This proves that J_R is multiplicative. We can now easily check that J_R satisfies the relationship stated above. Clearly this relationship between R and J_R uniquely determines J_R.

In the converse direction, suppose that $J \in \mathcal{J}(\Omega)$ and define:

$$pR_J q \quad \text{iff} \quad Jp = Jq \ .$$

By 2.14 this is indeed a cHa-congruence. Note that if $(p \wedge q)R_J q$, then $Jp \wedge Jq = Jq$; so $q \leq Jq \leq Jp$. In the other way, if $q \leq Jp$, then $Jq \leq JJp = Jp$. So $Jp \wedge Jq = Jq$, and thus $(p \wedge q)R_J q$. That is, R_J and J satisfy the relationship. Suppose S is some other congruence so related to J; then if pSq,

$p \wedge q \, S \, q$ also, so $q \le Jp$. Similarly $p \le Jq$ and $Jp = Jq$ follows. If $Jp = Jq$, then both $p \wedge q \, S \, q$ and $p \wedge q \, S \, p$; so $p \, S \, q$. This means that $S = R_J$.

Perhaps this has been worked out in too much detail; in any case we find:

$$R_{J_R} = R \quad \text{and} \quad J_{R_J} = J \quad ,$$

which gives us the one-one correspondence. It is easy to see that $R \subseteq S$ implies $J_R \le J_S$. Suppose that $J \le K$. If pR_Jq , then $Jp = Jq$. Since $p \le Jq$, we have $Kp \le KJq \le KKq = Kq$; also $Kq \le Kp$. Thus $R_J \subseteq R_K$ holds. The one-one correspondence is order preserving and hence is a lattice isomorphism. □

Contained in what has just been proved is the fact that Ω/R is a cHa because it is obviously isomorphic to $J_R(\Omega)$. Before going further, it might be well to look at some easy examples of congruences, their J-maps, and the corresponding quotient cHa's. In obtaining these examples and further ones, we often wish to form joins of operators in the operator lattice. These joins *exist* by 2.16 , but the proof there does not make them very explicit. When needed, the following may be helpful:

2.17. LEMMA. If Ω is a cHa and $\{M_i\}_{i \in I}$ is a family of inflationary, multiplicative operators, then the least J-operator pointwise greater than or equal to the M_i is defined by the formula:

$$Jp \;=\; \bigwedge \{ q \mid p \le q = M_i q \text{ , all } i \in I \} \quad .$$

The proof is just like the argument given in the discussion following 2.13 , because the trick is to show that J is multiplicative. The reader can check the details.

2.18. ELEMENTARY J-OPERATORS. In these examples Ω is a given cHa.

(i) *The closed quotient.* The operator is defined by:

$$J_a p = a \vee p \quad .$$

This is obviously a J-operator, and the congruence relation is:

$$a \vee p = a \vee q \quad .$$

The set of fixed points (quotient lattice) is:

$$\{ p \in \Omega \mid a \le p \} \quad .$$

Classically speaking in the spatial case where $\Omega = \mathcal{O}(X)$, the quotient corresponds to the topology on the *closed* subspace complementary to the open set a . This quotient makes the element a "false" and is the least such.

(ii) *The open quotient.* The operator is defined by:

$$J^a p = a \to p \quad .$$

The congruence relation is:

$$a \wedge p = a \wedge q \quad (\text{equivalently, } a \leq p \leftrightarrow q) \quad .$$

The set of fixed points is thus isomorphic to

$$\Omega_a = \{ p \in \Omega \mid p \leq a \} \quad .$$

Intuitionistically speaking in the spatial case, this quotient corresponds to the topology on the open subspace a . This quotient makes a "true" and is the least such.

(iii) *The Boolean quotient.* The operator is defined by:

$$B_a p = (p \to a) \to a \quad .$$

The congruence relation is:

$$p \to a = q \to a \quad .$$

The set of fixed points is:

$$\{ p \in \Omega \mid (p \to a) \to a \leq p \} \quad .$$

In case $a = \bot$, this is the well-known $\neg\neg$-quotient giving the (complete) Boolean algebra of "stable" elements. For arbitrary a , we could first form Ω/J_a and follow this by the $\neg\neg$ -quotient to obtain Ω/B_a . (In general, if $J \leq K$, then Ω/K is a quotient of Ω/J .)

We remark that in general Ω/J is a cBa iff $J = B_{J\bot}$. Further, if Ω is already Boolean, then *every* J-operator on Ω is of the form $B_a = J_a$.

(iv) *The forcing quotient.* The operator is a combination of previous ones:

$$(J_a \wedge J^b)p = (a \vee p) \wedge (b \to p) \quad .$$

The congruence relation is a conjunction:

$$a \vee p = a \vee q \quad \text{and} \quad b \wedge p = b \wedge q \; .$$

The set of fixed points is:

$$\{ p \in \Omega \mid b \to p \leq a \to p \} \quad .$$

The point of the quotient is that it provides the *least* J-operator such that $Ja \leq Jb$; that is, we take the least quotient that "forces" $a \to b$ to be true. If we want to force a sequence of statements $a_i \to b_i$, for $i < n$, the operator needed is $\bigvee_{i<n} (J_{a_i} \wedge J^{b_i})$. It is important to note that in general sup's of J-operators cannot be calculated pointwise. We shall see below, however, that it is possible to find a finite expression for this particular sup. (We owe this remark to John Cartmell.)

(v) *A mixed quotient.* The interest of this example lies in the fact that it has a neat finite definition:

$$(B_a \wedge J^a)p = (p \to a) \to p \quad .$$

The congruence relation is:

$$(p \to a) \to p = (q \to a) \to q \quad ,$$

which is equivalent to the conjunction:

$$a \wedge p = a \wedge q \text{ and } p \to a = q \to a \quad .$$

The set of fixed points is:

$$\{ p \in \Omega \mid (p \to a) \to p \le p \} \quad .$$

It is difficult to make this set vivid except to say that it is the set of elements p satisfying Pierce's Law (for a fixed a). □

If we take a polynomial in $\to$, $\wedge$, $\vee$, $\perp$, say $f(p,a,b,\ldots)$, it is a decidable question whether for all $a, b, \ldots$ it defines a J-operator. This does not, however, help us very much in cataloguing such operators (nor in seeing what good they are!) Some techniques can be developed from the following formulae, which were pointed out to us by Cartmell, see also [33] and [47].

2.19. PROPOSITION. In the following, K is an arbitrary J-operator, $\top$ is the constant function (the greatest J-operator with the most trivial quotient), and $\perp$ is the least J-operator (namely, the identity function on Ω):

(i) $J_a \vee J_b = J_{a \vee b}$	(ii) $J^a \vee J^b = J^{a \wedge b}$
(iii) $J_a \wedge J_b = J_{a \wedge b}$	(iv) $J^a \wedge J^b = J^{a \vee b}$
(v) $J_a \wedge J^a = \perp$	(vi) $J_a \vee J^a = \top$
(vii) $J_a \vee K = K \circ J_a$	(viii) $J^a \vee K = J^a \circ K$
(ix) $J_a \vee B_a = B_a$	(x) $J^a \vee B_b = B_{a \to b}$

Proof. Equations (i) - (iv) are easy calculations; while (v) comes down to showing $p = (a \vee p) \wedge (a \to p)$. Formula (vi) is a direct consequence of (vii) (equally, of (viii)).

To prove (vii) we use 2.17 since it gives us the definition:

$$(J_a \vee K)p = \bigwedge \{ q \mid p \le q = a \vee q = Kq \} \quad .$$

Now $p \le K(a \vee p) = a \vee K(a \vee p) = KK(a \vee p)$; also if q satisfies the condition in the curly brackets, then $a \vee p \le q$, so $K(a \vee p) \le q$. Thus the right hand side works out to $K(a \vee p)$.

To prove (viii) we use 2.17 again to write:

$$(J^a \vee K)p \;=\; \bigwedge\{\, q \mid p \le q = a \to q = Kq \,\} \quad .$$

Now $p \le a \to Kp = a \to (a \to Kp)$. Of course $a \to Kp \le K(a \to Kp)$, and in general $K(a \to b) \le a \to Kb$; so $K(a \to Kp) \le a \to KKp = a \to Kp$. Suppose q satisfies the condition in the curly brackets. Then $Kp \le Kq = q$, so $a \to Kp \le a \to q = q$. Thus the right hand side works out to $a \to Kp$.

Formula (ix) simply says that $J_a \le B_a$, which is clear. For (x) we calculate by (viii) :

$$\begin{aligned}(J^a \vee B_b)p &= a \to ((p \to b) \to b)\\ &= (a \to (p \to b)) \to (a \to b)\\ &= (p \to (a \to b)) \to (a \to b)\\ &= B_{a \to b}p \quad . \qquad \square\end{aligned}$$

The next result is due to Dowker and Papert [5] and was also proved in Isbell [21] . Isbell's proof (based on 1.14) is particularly simple involving no lengthy calculation. One can of course employ 2.17 and work out the proof of the distributive law directly.

2.20. THEOREM (Dowker-Papert-Isbell). The lattice $\mathcal{J}(\Omega)$ is a zero-dimensional cHa.

Proof. By 2.19 (vii) we can prove that J_a is linear in $\mathcal{J}(\Omega)$; because if the K_i are J-operators, then:

$$\begin{aligned}(J_b \vee \textstyle\bigwedge_{i\in I} K_i)p &= (\textstyle\bigwedge_{i\in I} K_i)J_bp\\ &= \textstyle\bigwedge_{i\in I} (K_iJ_bp)\\ &= (\textstyle\bigwedge_{i\in I} K_i \circ J_b)p\\ &= (\textstyle\bigwedge_{i\in I} (J_b \vee K_i))p \quad .\end{aligned}$$

Similarly, by 2.19 (viii) we find:

$$\begin{aligned}(J^a \vee \textstyle\bigwedge_{i\in I} K_i)p &= a \to \textstyle\bigwedge_{i\in I} K_ip\\ &= \textstyle\bigwedge_{i\in I} (a \to K_ip)\\ &= \textstyle\bigwedge_{i\in I} (J^a \circ K_i)p\\ &= (\textstyle\bigwedge_{i\in I} (J^a \vee K_i))p \quad ,\end{aligned}$$

so J^a is linear also. Therefore $J^a \vee J_b$ is linear. But remark that for $a, b \in \Omega$,

$$J^a \vee J_b \vee K \;=\; \top \quad \text{iff} \quad a \le Kb \quad .$$

Thus the linear elements of $\mathcal{J}(\Omega)$ do indeed form a cobasis. Now 1.14 applies. $\square$

By the techniques we have illustrated in the foregoing it is easy to see that the map $a \mapsto J_a$ is a cHa-morphism embedding Ω into $J(\Omega)$. As each J_a is clopen in $J(\Omega)$, this is an isomorphism iff Ω is Boolean. This situation is discussed further by Isbell [21] and Simmons [47] . In general, the construction of $J(\Omega)$ can be iterated transfinitely. Isbell also shows that this construction is functorial. Unfortunately, we have been unable to find any application of these facts in our present context so we stop here.

2.21. THE ALGEBRA OF PROPOSITIONS. We close this section with some remarks on $P(\mathbb{1})$, a cHa that intuitionistically plays the role assumed by the two-element Boolean algebra classically. To begin with, we have already taken note of the correspondence between propositions ϕ and elements $\{0 \mid \phi\} \in P(\mathbb{1})$; this is an isomorphism and makes $P(\mathbb{1})$ *the* algebra of propositions in intuitionistic logic. We can also prove easily that $P(\mathbb{1})$ is *initial* in the category of cHa's, because there is a *unique* morphism from $P(\mathbb{1})$ into a given Ω defined by $p \mapsto \bigvee\{\top_\Omega \mid 0 \in p\}$. (Keep in mind that the $p \in P(\mathbb{1})$ are just the subsets of $\{0\}$.) The reason this morphism is unique is that in $P(\mathbb{1})$ we have $p = \bigvee\{\top \mid 0 \in p\}$; that is, $\{\top\}$ is a basis of $P(\mathbb{1})$. (Note that $\{\bot\}$ is a cobasis.)

One way to say that a cHa Ω is non-trivial is to say that $\neg(\bot = \top)$, but this is rather weak intuitionistically. A better notion is to say that the morphism from $P(\mathbb{1})$ into Ω is an *embedding*. Such cHa's we term *proper*. The condition comes down to the schema:

$$\bigvee\{\top_\Omega \mid \phi\} = \top_\Omega \text{ in } \Omega \text{ implies } \phi ,$$

or in quantified form:

$$\forall p \in P(\mathbb{1}) \, [\bigvee\{\top_\Omega \mid 0 \in p\} = \top_\Omega \text{ implies } 0 \in p] .$$

(We could have also had "iff" in place of "implies".) Obviously if we have an embedding, the above condition holds. Conversely, if $\bigvee\{\top_\Omega \mid 0 \in p\} = \bigvee\{\top_\Omega \mid 0 \in q\}$, this implies by the condition: $0 \in p$ iff $0 \in q$. But as $\mathbb{1}$ has only one element, this is the same as $p = q$.

A quotient of $P(\mathbb{1})$ is proper iff the congruence is trivial (the identity), so $P(\mathbb{1})/\neg\neg$ is proper iff $P(\mathbb{1})$ is Boolean (that is, our logic is classical).

In the case of power set algebras like $P(X)$, such an algebra is proper iff X is inhabited. This is obviously stronger than saying $\neg X = \emptyset$.

Recall that Lawvere and Tierney [31] defined a *topology* on $P(\mathbb{1})$ as an operator $j : P(\mathbb{1}) \to P(\mathbb{1})$ satisfying:

(1) $j \circ j = j$;

(2) $j \circ \wedge = \wedge \circ \langle j, j \rangle$;

(3) $j \circ true = true$.

Evidently, every J-operator in the sense of 2.11 is a topology. But in view of the special character of $P(\mathbb{1})$, we can quickly argue that every topology is a J-operator. The only part missing is to show j inflationary. To prove $p \leq j(p)$, assume $0 \in p$. Thus $p = \top$, so $j(p) = \top$ and $0 \in j(p)$. Therefore $0 \in p$ implies $0 \in j(p)$, which is what we wanted. Of course this abbreviated definition only works on $P(\mathbb{1})$, that is why in 2.11 we say inflationary in place of (3) above.

3. POINTS AND SOBER SPACES

Much of our intuition about cHa comes from the example of the opens of a topological space. To see just how far this example is typical, we review the duality between the categories cHa and Top (with $\wedge,\bigvee$- maps and continuous maps respectively). To make the classical theory constructive is just a matter of using the right definitions. No new ideas are needed. We think of a topology as defined by opens as in 1.4 . A map between spaces is continuous iff the corresponding inverse image map takes opens to opens. We have a functor

$$\mathcal{O} : \mathrm{Top} \to \mathrm{cHa}^{op}$$

since $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an $\wedge,\bigvee$-map for each continuous $f : X \to Y$. Given a point t of a topological space X we have an $\wedge,\bigvee$-map $t : \mathcal{O}(X) \to P(\mathbb{1})$ given by $0 \in t(U)$ iff $t \in U$ or, equivalently, $t(U) = \{ 0 \mid t \in U \}$. In fact, this is the $\wedge,\bigvee$-map associated to the continuous map from the one point space to X which singles out t .

3.1. DEFINITION. A <u>point</u> t of a cHa Ω is an $\wedge,\bigvee$-map

$$t : \Omega \to P(\mathbb{1}) \quad .$$

We topologize the set $pt(\Omega)$ of points of Ω , taking as opens the sets $U^* = \{ t \mid 0 \in t(U) \}$ where $U \in \Omega$. Since $U^* \cap V^* = (U \cap V)^*$ and $\bigcup_{i \in I} (U_i^*) = (\bigvee_{i \in I} U_i)^*$, this indeed gives a topology on $pt(\Omega)$. Any $\wedge,\bigvee$-map $F : \Omega \to \Omega'$ induces a map $F^* : pt(\Omega') \to pt(\Omega)$ by composition. Since $(F^*)^{-1}(U^*) = (F(U))^*$, for $U \in \Omega$, we see that F^* is continuous and we have another functor

$$pt : \mathrm{cHa}^{op} \to \mathrm{Top} \quad .$$

Now taking $U \in \Omega$ to $U^* \in \mathcal{O}(pt(\Omega))$ gives an $\wedge,\bigvee$- map

$$\varepsilon_\Omega : \Omega \to \mathcal{O}(pt(\Omega))$$

and taking $t \in X$ to the corresponding point of $\mathcal{O}(X)$ gives a continuous map

$$\eta_X : X \to pt(\mathcal{O}(X)) \quad .$$

We are interested in the cases where these maps are isomorphisms.

3.2. SOBER SPACES AND cHa WITH ENOUGH POINTS. We say a space X is <u>sober</u> iff η_X is an isomorphism, that is, if for every $\wedge\bigvee$-map $f : \mathcal{O}(X) \to P(\mathbb{1})$ there is a unique $t \in X$ such that for all $U \in \mathcal{O}(X)$ we have $t \in U$ iff $0 \in f(U)$. We say a cHa Ω has <u>enough points</u> iff ε_Ω is an isomorphism, that is if to see for $U, V \in \Omega$ that $U = V$ it suffices to check that for every $\wedge\bigvee$-map $f : \Omega \to P(\mathbb{1})$ we have $f(U) = f(V)$.

Evidently, if X is a topological space then $\mathcal{O}(X)$ has enough points and if Ω is a cHa then $\mathrm{pt}(\Omega)$ is sober. What about the spaces we know, are they sober? Classically, $T_2 \Rightarrow$ sober $\Rightarrow T_0$, while T_1 and sober are incomparable. Intuitionistically, sobriety is far more rare (we shall later present a model where Q is not sober!), though of course sober $\Rightarrow T_0$ trivially. Sober spaces are hard to find. We give some examples.

3.3. CONTINUOUS LATTICES. Martin Hyland [20] has pointed out that every continuous lattice is sober in the induced topology (see Scott [45]). We recall that in a lattice L we say x is <u>way below</u> y (written $x \ll y$) iff whenever $y \le \bigvee_{i\in I} z_i$ then $x \le \bigvee_{i\in F} z_i$ for some finite $F \subseteq I$. A <u>continuous lattice</u> is a complete lattice in which every element if the join of the elements way below it: $y = \bigvee\{x \mid x \ll y\}$. The <u>induced topology</u> has as basic opens the sets $0_d = \{x \mid d \ll x\}$ for $d \in L$. For $U \in \mathcal{O}(L)$ we have $U = \bigvee\{0_d \mid d \in U\}$ and if S is a directed set such that $\bigvee S \gg y$ then for some $x \in S$ we have $x \gg y$. (For this and other facts about continuous lattices we refer the reader to Scott [45].)

Now let L be a continuous lattice, we write $\cap \bigcup$ for the operations on the cHa $\mathcal{O}(L)$ of opens and $\wedge\bigvee$ etc., for the lattice operations. To any point $t : \mathcal{O}(L) \to P(\mathbb{1})$ there corresponds a <u>superfilter</u> $F = \{U \in \mathcal{O}(L) \mid 0 \in t(U)\}$ such that if U and V belong to F then $U \cap V \in F$ and if $\bigcup_{i\in I} U_i \in F$ then $U_i \in F$ for some $i \in I$. Now let $y = \bigvee\{\bigwedge U \mid U \in F\}$. If $U \in F$ then $0_d \in F$ for some $d \in U$ thus we can write $y = \bigvee\{\bigwedge 0_d \mid 0_d \in F\}$ note that this is a *directed* join since F is closed under $\cap$. We now show that $y \in 0_d$ iff $0_d \in F$. If $0_d \in F$ then because $0_d = \bigcup\{0_f \mid f \in 0_d\}$, we have $0_f \in F$ for some $f \gg d$. Thus $y \ge \bigwedge 0_f \ge f \gg d$ and $y \in 0_d$. Conversely, if $y \in 0_d$ then $y \gg d$. Because we have y as a directed join, we can write $\bigwedge 0_e \gg d$ for some e with $0_e \in F$. Then $0_e \subseteq 0_d$ and $0_d \in F$. To see that the y we have found is uniquely determined by the property $y \in U$ iff $U \in F$ for $U \in \mathcal{O}(L)$ (which it certainly enjoys), recall that $y = \bigvee\{d \mid y \in 0_d\}$ in any continuous lattice.

3.4. R IS SOBER. We sketch a proof that the Dedekind reals are sober. These are constructed as the set of pairs $\langle U,L\rangle \in P(Q) \times P(Q)$ of subsets of rationals such that U is an inhabited, open, upper cut, L an inhabited, open, lower cut, U and L are disjoint and for rationals $p < q$ either $p \in L$ or $q \in U$. R is topologized by taking as a basis the rational opens (corresponding to open intervals):

$$(p,q) \;=\; \{ \langle U,L\rangle \in R \mid q \in U \text{ and } p \in L \} \quad .$$

Now suppose we have an $\wedge,\bigvee$-map $f : \mathcal{O}(R) \to P(\mathbb{1})$. We define the real number corresponding to this point of $\mathcal{O}(R)$. Let

$$U \;=\; \{ q \in \mathbb{Q} \mid \exists\, p \in \mathbb{Q} \,.\, 0 \in f(p,q) \}$$

and $L \;=\; \{ p \in \mathbb{Q} \mid \exists\, q \in \mathbb{Q} \,.\, 0 \in f(p,q) \}$.

We show U is an open upper cut. If $q \in U$ and $q' > q$ then for some p we have $0 \in f(p,q)$ and, since $(p,q) \subseteq (p,q')$, also $0 \in f(p,q')$ so $q' \in U$. Thus U is an upper cut. Again let $q \in U$ so for some p again $0 \in f(p,q)$. Now observe that $(p,q) \;=\; \bigvee\{ (p,q-1/n) \mid n \in \mathbb{N} \}$ because any real in (p,q) is itself defined by *open* cuts. So, f being an $\wedge,\bigvee$-map, we have $\exists\, n \in \mathbb{N} \,.\, 0 \in f(p,q-1/n)$ whence $q - 1/n \in U$ for some $n \in \mathbb{N}$ and U is open. By a similar argument, L is an open lower cut. If $p \in L$ and $p \in U$ then we would have $0 \in f(p,q)$ and $0 \in f(r,p)$ where, since $f(\emptyset) = \emptyset$, we have $r < p < q$. But this is absurd since then $0 \in f(p,q) \wedge f(r,p) \;=\; f((r,p) \wedge (p,q)) = f(\emptyset) = \emptyset$. Finally, suppose $p < q$ then $R \;=\; \bigcup\{ (p,p+n) \cup (q-n,q) \mid n \in \mathbb{N} \}$. Since $f(R) = \{0\}$ and f preserves $\bigcup$ we must have for some $n \in \mathbb{N}$ either $0 \in f(p,p+n)$, in which case $p \in L$, or $0 \in f(q-n,q)$, when $q \in U$. So we have defined a real. Looking back, the reader will see that the definition we gave was precisely that forced upon us by the requirement that for U a basic rational open

$$\langle U,L\rangle \in U \quad \text{iff} \quad 0 \in f(U) \quad .$$

Since the basic rational opens *are* a basis, this will now hold for all $U \in \mathcal{O}(R)$. □

We now present some examples of cHa *without points*. It turns out that this is simpler intuitionistically than classically. Of course, the trivial cHa ($\top = \bot$) can never have any point. To avoid such degeneracy, we insist that our examples be proper. If we accept intuitionism then no complete Boolean algebra (cBa) can have a point - this would imply the law of the excluded middle as follows: Suppose Ω is a Boolean algebra and $t : \Omega \to P(\mathbb{1})$ a point. Composing with the unique map $\hat{\ } : P(\mathbb{1}) \to \Omega$ we must have for $p \in P(\mathbb{1})$ that $t(\hat{p}) = p$ since $P(\mathbb{1})$ is initial in cHa. Now let U be a complement for $\hat{p}$ in Ω. Since $\wedge\bigvee$-maps preserve Boolean complements, $t(U)$ is a complement for p in $P(\mathbb{1})$. We shall easily find models later where $\neg\, \forall\, p \in P(\mathbb{1})\ (\, p \vee \neg p = \top \,)$.

3.5. POINTLESS cHa. (i) We have seen that any cHa is embedded in its algebra of J-maps. Thus the $\wedge\bigvee$-map $P(\mathbb{1}) \to J(P(\mathbb{1}))$ is one-one and each element J_p of the image of this map is complemented. Now consider the J-map $\neg\neg$ *on* $J(P(\mathbb{1}))$.

Each J_p is a fixed point of $\neg\neg$ and so the composite map

$$P(\mathbb{1}) \to J(P(\mathbb{1})) \to J(P(\mathbb{1}))/\neg\neg$$

is one-one and $J(P(\mathbb{1}))/\neg\neg$ is a proper cBa. If this algebra has a point then mathematics is classical.

(ii) It seems worthwhile to give a construction of a pointless cHa which works classically as well as intuitionistically. Let X be a sober topological space. We shall kill its points leaving a pointless cHa $S(X)$. For $U \in \mathcal{O}(X)$ define

$$F(U) = \bigcup \{ W \in \mathcal{O}(X) \mid \exists\, x_1, \dots, x_n \ (W \setminus \{x_1, \dots, x_n\})^{\circ} \subseteq U \}$$

(where $Y^{\circ} \in \mathcal{O}(X)$ is the interior of $Y \in (X)$). This operation F is multiplicative since

$$(W \setminus \{x_1, \dots, x_n\})^{\circ} \cap (V \setminus \{y_1, \dots, y_m\})^{\circ} = (W \cap V \setminus \{x_1, \dots, x_n, y_1, \dots, y_m\})^{\circ} .$$

As F is obviously inflationary, we may identify the fixed points of F as a quotient cHa $J : \mathcal{O}(X) \to S(X)$. We call the elements of $S(X)$ <u>coperfect</u> <u>opens</u> of X. (Classically, if X is T_1, they are the complements of perfect closed subsets of X.) Now any point of $S(X)$ would give by composition a point f of $\mathcal{O}(X)$ and, as X is sober, a point $t \in X$ such that, for $U \in \mathcal{O}(X)$, we have $t \in U \leftrightarrow 0 \in f(U)$. Consider $U = (X \setminus \{t\})^{\circ}$; we have $\neg\, 0 \in f(U)$, but $J(U) = X$ (we killed this point), which contradicts $f(X) = \{0\}$ since f factors through J. Thus, $S(X)$ has no points. In general, $S(X)$ is proper. For example when $X = \mathcal{R}$, each open $\{ x \in \mathcal{R} \mid 0 \in p \} = \bigvee \{ \mathcal{R} \mid 0 \in p \} = \hat{p}$ is a fixed point of F for $p \in P(\mathbb{1})$. □

The map $\varepsilon_{\Omega} : \Omega \to \mathcal{O}(\mathrm{pt}(\Omega))$ identifies two elements of Ω iff they agree on all points of Ω. If $f : \Omega \to \mathcal{O}(X)$ is an $\wedge\bigvee$- map, then for any $U, V \in \Omega$ which agree on all points of Ω we must have $f(U) = f(V)$, since $f(U) = \{ t \mid 0 \in t(f(U)) \}$. Thus, for any such map we have a factorization

$$\begin{tikzcd} \Omega \arrow[r] \arrow[dr] & \mathcal{O}(\mathrm{pt}(\Omega)) \arrow[d, dashed] \\ & \mathcal{O}(X) \end{tikzcd}$$

which is unique since $\mathcal{O}(\mathrm{pt}(\Omega))$ is a quotient of Ω. This universal property tells us that ε is the counit of an adjunction. A similar argument shows that η is the unit. (See MacLane [32] p. 81 .)

3.6. THEOREM. The functor $\mathcal{O}$ is left adjoint to pt, that is, there is a natural isomorphism

$$\mathrm{Top}^{\mathrm{op}}[X, \mathrm{pt}(\Omega)] \cong \mathrm{cHa}[\Omega, \mathcal{O}(X)] \quad ,$$

and η and ε are the unit and counit of this adjunction.

Proof. We have seen that ε has the universal property of a counit. So we have shown that $\mathcal{O}$ is left adjoint to pt . To check that η is the unit of this adjunction is routine. □

If X is sober, then $pt(\mathcal{O}(X)) \cong X$; and if Ω has enough points, then $\mathcal{O}(pt(\Omega)) \cong \Omega$ by definition. Thus we can state a corollary (see MacLane [32] , IV.3 , Theorem 1).

3.7. COROLLARY. The category Sob of sober spaces is dual to the category Pts , of cHa with enough points. □

We shall make use of this duality to give a description of sheaves as étale spaces in the next section and again in §8 to give a representation for internal sober spaces. For this, apart from the abstract duality, we need to know what happens to open inclusions.

3.8. LEMMA. Let X be a topological space, $\Omega = \mathcal{O}(X)$ and $U \hookrightarrow X$ an open inclusion. This morphism is mapped by $\mathcal{O}$ to the quotient $\Omega \to \Omega_U = \{ V \in \Omega \mid V \leq U \}$ by the open quotient topology J^U (up to a canonical isomorphism).

Proof. Evidently $\mathcal{O}(U) \cong \Omega_U$, and the map dual to the inclusion takes $W \in \Omega$ to $W \cap U \in \Omega_U$; but we saw in 2.18 that this is just the quotient of Ω by J^U . □

CHAPTER II. FIRST-ORDER STRUCTURES

Here we introduce models for intuitionistic logic with partial elements. Heyting algebras give models for intuitionistic propositional logic. The models we introduce are the obvious extensions of these to predicate logic. To make this extension it is natural to introduce Heyting-valued sets or Ω-sets. Section 4 is devoted to a study of these and their relation to sheaves. Every presheaf determines an Ω-set. Sheaves correspond to *complete* Ω-sets (which arise inevitably when one wishes to interpret descriptions), and every presheaf or Ω-set has a completion. Sheaves or Ω-sets may be thought of as collections of elements "varying continuously" over some domain. In Section 5, we describe operations and relations on Ω-sets; the requirement of extensionality turns out to be just the requirement that these be defined locally. We interpret the logic of Chapter 0 as the logic of local properties of variable elements; by using complete Ω-sets or sheaves, we are able to interpret descriptions. In section 6, we consider briefly the effect of changing cHa. The most important cases arise when we have an $\wedge\bigvee$ map $f^* : \Omega \to \Omega'$. We then have constructions of the so-called inverse and direct images of structures, which have interesting logical properties.

4. Ω-SETS AND SHEAVES

A fundamental example is the *sheaf of germs of continuous functions* on a space X. To fix ideas, we consider first *real-valued* functions and denote this sheaf by R_X. We will not define straightaway *germs* and *stalks* but will work directly with (a representation of) the *sections* of this sheaf. Because we wish to emphasize the interpretation of logical formulae in these sheaves, we generally find it is easier to introduce them via sections; also when we discuss abstract cHa's not of the form $\mathcal{O}(X)$, we cannot intuitionistically expect points over which to have these germs, so this is a second reason for preferring sections.

The <u>set of sections</u>, denoted by $|R_X|$, of R_X is the collection of continuous maps $a : U \to \mathbb{R}$ with open domain $U \subseteq X$. We write $Ea = U = \mathrm{dom}\ a$, and call this the <u>extent</u> of a. The set $Ea \in \mathcal{O}(X)$ measures the "time" for which a "exists"; we regard a, then, as a *variable quantity* defined over X, but we have to agree that for one reason or the other a is only partially defined. *Sometimes* such partial elements can be extended to *global* elements where $a \subseteq b$ and $Eb = X$, but this is not always possible.

Given $a, b \in |R_X|$, we can measure how much they *coincide* by defining:

$$[\![a = b]\!] = \mathit{int}\{ t \in Ea \cap Eb \mid a(t) = b(t) \} \quad .$$

The interior operator is applied here because the properties of elements we are to be concerned with are *local* properties; thus, when $t \in [\![a = b]\!]$, the functions have to coincide in a neighbourhood of the point t. By interpreting $\mathcal{O}(X)$ as a truth-value algebra, then $[\![a = b]\!]$ is the truth value of the statement "a = b". Note that $Ea = [\![a = a]\!]$. We call $[\![\cdot = \cdot]\!]$ the <u>equality</u> map.

Given $a \in |R_X|$ and an arbitrary $U \in \mathcal{O}(X)$, we can easily define the <u>restriction</u> $a \restriction U$ of a as the obvious function $a \restriction U : U \cap Ea \to \mathbb{R}$. In the sequel we will abstract the "algebraic" properties of restrictions, extents, and the equality map to define the general notions of sheaf and Ω-set.

As a second, slightly more abstract example, let A be any commutative ring with 1. Let B be the Boolean algebra of idempotents of A, so that $B = \{ e \in A \mid e^2 = e \}$ and $e \wedge f = e \cdot f$ and $e \vee f = e + f - e \cdot f$, for $e, f \in B$. Consider the cHa $I(B)$ of ideals of B. We can use ideals to measure the equality of elements of A. Say $a, b \in A$, and define:

$$[\![a = b]\!] = \{ e \in B \mid ea = eb \} \quad .$$

Obviously $[\![a = b]\!] \in I(B)$, and $[\![a = b]\!] = \top$ iff $a = b$. But in general $[\![a = b]\!]$ may be a nontrivial ideal different from both $\bot = (0)$ and $\top = (1)$. Here $Ea = \top$ for all $a \in A$, so *all* elements are global. This happens sometimes with Ω-sets but not with sheaves. The exact relationship between these concepts will be fully explained below. In the example at hand, note the *absence* of points: intuitionistically the Boolean algebra B may have *no* prime filters. It was easy to define the Ω-set for $\Omega = I(B)$, but there may be no "underlying" space of points.

We turn now to the abstract definitions, remark that our examples satisfy them, and show they begin to give models for the formal logic of Chapter 0.

4.1. DEFINITION. Let Ω be a cHa. An Ω-<u>set</u> A is a set $|A|$ equipped with an Ω-valued equality $[\![\cdot = \cdot]\!] : |A| \times |A| \to \Omega$ satisfying:

(i) $[\![a = b]\!] = [\![b = a]\!]$ (symmetry);

(ii) $[\![a = b]\!] \wedge [\![b = c]\!] \leq [\![a = c]\!]$ (transitivity).

On any Ω-set we define <u>extents</u> and <u>equivalence</u> by:

(iii) $Ea = [\![a = a]\!]$, and

(iv) $[\![a \equiv b]\!] = (Ea \vee Eb) \to [\![a = b]\!]$.

Discussion. In our two examples ($|A| = |R_X|$ and $|A|$ = the ring A), the elements are completely distinguished by $[\![\cdot = \cdot]\!]$ in the sense that $[\![a = b]\!] = \top$ iff $a = b$. This need not always be the case, and we return to the question later. If A is an Ω-set and $M : \Omega \to \Omega'$ is a multiplicative operator between cHa's, then A becomes an Ω'-set with the equality $M[\![a = b]\!]$. The multiplicative map $int : P(X) \to \mathcal{O}(X)$ is a case in point, as is seen by reference to the definition of equality on R_X. Often throwing on such an M will collapse elements, and our definition does not exclude this. Nevertheless, any Ω-set satisfies the Axioms for Equality of Chapter 0, §2, and so the concept has model-theoretic interest.

The connection between restrictions and equality and equivalence in R_X is particularly simple:

$$[\![a \equiv b]\!] = \bigcup \{ U \in \mathcal{O}(X) \mid a \upharpoonright U = b \upharpoonright U \} \quad , \text{ and}$$

$$[\![a = b]\!] = \bigcup \{ U \subseteq Ea \cap Eb \mid a \upharpoonright U = b \upharpoonright U \} \quad .$$

Our definition of *presheaf* allows us to generalize this:

4.2. DEFINITION. A presheaf over a cHa Ω is a set $|A|$ equipped with maps $E : |A| \to \Omega$ and $\upharpoonright : |A| \times \Omega \to |A|$ of *extent* and *restriction* satisfying:

(i) $a \upharpoonright Ea = a$

(ii) $a \upharpoonright p \upharpoonright q = a \upharpoonright (p \wedge q)$

(iii) $E(a \upharpoonright p) = Ea \wedge p$

for all $a \in |A|$ and all $p, q \in \Omega$. On any presheaf we define *equality* by:

(iv) $[\![a = b]\!] = \bigvee \{ p \leq Ea \wedge Eb \mid a \upharpoonright p = b \upharpoonright p \}$.

4.3. PROPOSITION. Every presheaf A over Ω is an Ω-set, and we have for $a, b \in |A|$ and $p \in \Omega$:

$$[\![a \upharpoonright q = b]\!] = [\![a = b]\!] \wedge q \quad .$$

Proof. Symmetry of $[\![\cdot = \cdot]\!]$ as defined in 4.2 (iv) is obvious. For transitivity, apply the distributive law to $[\![a = b]\!] \wedge [\![b = c]\!]$. We get the sup of elements $p \wedge q$ where $p \leq Ea \wedge Eb$, $q \leq Eb \wedge Ec$, $a \upharpoonright p = b \upharpoonright p$, and $b \upharpoonright q = c \upharpoonright q$. From 4.2 we see $a \upharpoonright (p \wedge q) = c \upharpoonright (p \wedge q)$. Also immediate is $p \wedge q \leq Ea \wedge Ec$. Thus $p \wedge q \leq [\![a = c]\!]$, and transitivity now follows. To prove the formula for restrictions, note that

$$[\![a \upharpoonright q = b]\!] = \bigvee \{ p \leq Ea \wedge Eb \wedge q \mid a \upharpoonright p = b \upharpoonright p \} \leq [\![a = b]\!] \wedge q \quad .$$

On the other hand, if p is in the set defining $[\![a = b]\!]$ in 4.2 (iv), then $p \wedge q$ is in the set of the above sup. By the distributive law, then, $[\![a = b]\!] \wedge q \leq [\![a \upharpoonright q = b]\!]$. □

The notions of Ω-set as given in 4.1 and presheaf over Ω in 4.2 are in some ways incomparable without further conditions. For example, we cannot put a presheaf structure on every Ω-set because restrictions need not exist. These restrictions could be added "formally", and we discuss this below. In the case of presheaves, the axioms of 4.2 allow for the passage from elements to restrictions, a cutting down operation, but they give no information about *building up*. Thus the formula for $[\![a \equiv b]\!]$ that holds for R_X does not hold in general: two elements might be "locally" equal and have the same extents but not be actually equal. We thus introduce in 4.6 a uniqueness condition that avoids this difficulty. Some auxiliary definitions and a lemma are helpful in stating the exact requirement.

4.4. DEFINITIONS. In a presheaf A for $a, b \in |A|$, we say a is a restriction of b, or b is an extension of a, and write $a \leq b$, iff $a = b \restriction Ea$. If $B \subseteq |A|$ is any set of elements, we say B is compatible iff the elements are pairwise compatible in the sense that if $b, b' \in B$, then $b \restriction Eb' = b' \restriction Eb$. A join for B is a minimal upper bound for B under $\leq$.

The proof of the following is straight forward.

4.5. LEMMA. Let A be a presheaf; then:

(i) $\leq$ partially orders $|A|$;

(ii) $b \leq a$ implies $Eb \leq Ea$, for all $a, b \in |A|$;

(iii) If $B \subseteq |A|$, then $a \in |A|$ is a join for B iff $Ea = \bigvee\{Eb \mid b \in B\}$ and $b \leq a$ for all $b \in B$.

(iv) $a \restriction \bigvee_{i \in I} p_i$ is a join of $\{a \restriction p_i \mid i \in I\}$ for $a \in |A|$ and $\{p_i \mid i \in I\} \subseteq \Omega$;

(v) Every bounded subset of $|A|$ is compatible and has a join and a (unique) meet under the partial ordering. □

4.6. DEFINITIONS. A presheaf is separated iff joins of subsets (when they exist) are unique. An Ω-set A is separated iff $[\![a \equiv b]\!] = \top$ always implies $a = b$.

4.7. PROPOSITION. Let A be a presheaf. Then the following are equivalent:

(i) A is separated;

(ii) $a \restriction p_i = b \restriction p_i$, all $i \in I$, always implies $a \restriction \bigvee_{i \in I} p_i = b \restriction \bigvee_{i \in I} p_i$;

(iii) $a \restriction [\![a = b]\!] = b \restriction [\![a = b]\!]$, all $a, b \in |A|$;

(iv) $a \restriction [\![a \equiv b]\!] = b \restriction [\![a \equiv b]\!]$, all $a, b \in |A|$;

(v) $p \leq [\![a \equiv b]\!]$ iff $a \restriction p = b \restriction p$, all $a, b \in |A|$, $p \in \Omega$;

(vi) A is separated as an Ω-set ;

(vii) $a \le b$ iff $Ea \le \llbracket a = b \rrbracket$, all $a, b \in |A|$.

Furthermore, if A is separated, we can write for $a, b \in |A|$:

(viii) $\llbracket a \equiv b \rrbracket = \bigvee \{ p \in \Omega \mid a{\upharpoonright}p = b{\upharpoonright}p \}$,

(ix) a and b are compatible iff $\llbracket a = b \rrbracket = Ea \wedge Eb$.

Proof. That (i) implies (ii) follows from 4.5 (iv) . That (ii) implies (iii) follows from Definition 4.2 (iv) . To prove (iii) implies (iv) , note that $a{\upharpoonright}\llbracket a \equiv b \rrbracket = a{\upharpoonright}(Ea \wedge \llbracket a \equiv b \rrbracket) = a{\upharpoonright}(Ea \wedge \llbracket a = b \rrbracket) = a{\upharpoonright}\llbracket a = b \rrbracket$, because $\llbracket a \equiv b \rrbracket$ is defined by 4.1 (iv) . To prove (iv) implies (v) assume first that $p \le \llbracket a \equiv b \rrbracket$. Then $a{\upharpoonright}p = a{\upharpoonright}\llbracket a \equiv b \rrbracket \wedge p = b{\upharpoonright}\llbracket a \equiv b \rrbracket \wedge p = b{\upharpoonright}p$. Conversely, suppose $a{\upharpoonright}p = b{\upharpoonright}p$, then $Ea \wedge p = Eb \wedge p \le Ea \wedge Eb$. Therefore, since $a{\upharpoonright}(Ea \wedge p) = b{\upharpoonright}(Ea \wedge p)$, we see $Ea \wedge p \le \llbracket a = b \rrbracket$ and $Eb \wedge p \le \llbracket a = b \rrbracket$. It follows that $(Ea \vee Eb) \wedge p \le \llbracket a = b \rrbracket$, and so $p \le \llbracket a \equiv b \rrbracket$. Next (v) at once implies (vi) . Assume (vi) . To show (vii) assume $a \le b$. Then $a = b{\upharpoonright}Ea$, and so $Ea \le Eb$. But $a = a{\upharpoonright}Ea$; thus $Ea \le \llbracket a = b \rrbracket$ by definition. In the converse direction, assume $Ea \le \llbracket a = b \rrbracket$. If $p \le Ea$, then $a{\upharpoonright}p = b{\upharpoonright}p$ iff $a{\upharpoonright}p = b{\upharpoonright}Ea{\upharpoonright}p$. Also $Ea \wedge Eb = Ea \wedge E(b{\upharpoonright}Ea)$, so $\llbracket a = b \rrbracket = \llbracket a = b{\upharpoonright}Ea \rrbracket$ by definition 4.2 (iv) . Therefore $Ea \le \llbracket a = b{\upharpoonright}Ea \rrbracket$. But $E(b{\upharpoonright}Ea) \le \llbracket a = b{\upharpoonright}Ea \rrbracket$ too; thus $\llbracket a \equiv b{\upharpoonright}Ea \rrbracket = \top$. By (vi) , $a = b{\upharpoonright}Ea$ follows; that is, $a \le b$. Now assume (vii) . To prove (i) let a_0 and a_1 both be joins of $B \subseteq |A|$. We wish to show $a_0 \le a_1$. By (vii) it is enough to show $Ea_0 \le \llbracket a_0 = a_1 \rrbracket$. Now by 4.5 (iii) , $Ea_0 = \{ Eb \mid b \in B \}$. But for $b \in B$, $b \le a_0$ and $b = a_0{\upharpoonright}Eb$. Similarly, $b = a_1{\upharpoonright}Eb$ and $Eb \le Ea_0 \wedge Ea_1$. Thus $Eb \le \llbracket a_0 = a_1 \rrbracket$. Therefore $Ea_0 \le \llbracket a_0 = a_1 \rrbracket$ as was to be shown, and the circle is complete. Clearly (v) implies (viii) , because by (v) we see $\llbracket a \equiv b \rrbracket$ is a maximum not just a sup. Finally for (ix) , if $a{\upharpoonright}Eb = b{\upharpoonright}Ea$, then $\llbracket a = b \rrbracket = \llbracket a = b \rrbracket \wedge Ea =$ $\llbracket a = b{\upharpoonright}Ea \rrbracket = \llbracket a = a{\upharpoonright}Eb \rrbracket = \llbracket a = a \rrbracket \wedge Eb = Ea \wedge Eb$. On the other hand, if $\llbracket a = b \rrbracket = Ea \wedge Eb$, then we just substitute into 4.7 (iii) to show that $a{\upharpoonright}Eb = b{\upharpoonright}Ea$. □

4.8. EXAMPLES. We began the section with "interesting" examples, but it is perhaps helpful to insert here some particularly easy constructs.

(i) *The space of propositions.* We have already regarded Ω as a *model* for propositional algebra, but we can also think of it as an Ω-set. Define $\llbracket p = q \rrbracket = (p \leftrightarrow q)$. All elements are global and it is not a presheaf. With this definition Ω is separated.

(ii) *The one-point set.* Again use Ω as the underlying set, but this time define $[\![p = q]\!] = p \wedge q$. Equally well we can define $Ep = p$ and $p \upharpoonleft q = p \wedge q$. Now we have a presheaf where everything is a restriction of one global element $\top$. Clearly $\top$ is the principal element and there is only one of these. Note that in this example $[\![p \equiv q]\!] = (p \leftrightarrow q)$. Indeed, any Ω-set becomes a *new* Ω-set if $[\![\cdot = \cdot]\!]$ is replaced by $[\![\cdot \equiv \cdot]\!]$.

(iii) *The constant set.* Let A be any set, and define $|A_\Omega| = A$ where by definition:

$$[\![a = b]\!] = \bigvee \{ \top \mid a = b \} .$$

Classically, we could say $[\![a = b]\!] = \top$, if $a = b$, and $\bot$ otherwise. This is not, however, intuitionistic. A_Ω is an Ω-set but not a presheaf. If Ω is proper (see 2.21), then A_Ω is separated. If we wanted a presheaf, we would have to define $|A_\Omega| = A \times \Omega$ where

$$E\langle a,p\rangle = p \quad \text{and} \quad \langle a,p\rangle \upharpoonleft q = \langle a, p \wedge q\rangle .$$

(In case $A = \{0\}$, we just obtain the previous example.) This, however, is *not* separated; for consider $\langle a,\bot\rangle$ and $\langle b,\bot\rangle$, where $\neg\, a = b$.

(iv) *The Cartesian product.* Let A and B be two presheaves over Ω . We define $A \times B$ over Ω , but perhaps not in the way expected. Set

$$|A \times B| = \{ \langle a,b\rangle \mid a \in |A| , b \in |B| , Ea = Eb \} .$$

Then define:

$$E\langle a,b\rangle = Ea \ (= Eb) , \quad \text{and}$$

$$\langle a,b\rangle \upharpoonleft p = \langle a \upharpoonleft p, b \upharpoonleft p\rangle .$$

This is separated if A and B are. We return to this construction in 5.15 when we justify calling it a product. Of course, we *could* use *all* the ordered pairs, and define

$$E\langle a,b\rangle = \langle Ea,Eb\rangle , \quad \text{and}$$

$$\langle a,b\rangle \upharpoonleft \langle p,q\rangle = \langle a \upharpoonleft p, b \upharpoonleft q\rangle .$$

But then this is a presheaf over $\Omega \times \Omega$ not over Ω . The reader can easily work out the meaning of $[\![\cdot = \cdot]\!]$ here for himself.

(v) *Presheaves as functors.* For any presheaf A over Ω and for $p \in \Omega$, define the sections over p by:

$$A(p) = \{ a \in |A| \mid Ea = p \} .$$

Restriction as a map $a \mapsto a \upharpoonleft q$, where $q \le p$, gives us maps $\rho^p_q : A(p) \to A(q)$. If $q = p$, this is the identity. If $r \le q \le p$, then the triangle commutes:

$$\begin{tikzcd} A(p) \arrow[r] \arrow[dr] & A(q) \arrow[d] \\ & A(r) \end{tikzcd} .$$

As every partially ordered set, in this case Ω , is a category, we can say in short that the presheaf gives us a *functor*, which we also call A and write $A : \Omega^{op} \to \mathcal{S}ets$.

Conversely, given any such functor, we can recover the presheaf in our sense. Regard the sets $A(p)$ for $p \in \Omega$ as pairwise disjoint. Define $|A| = \bigcup_{p\in\Omega} A(p)$. Define $Ea = p$ to mean $a \in A(p)$ for $a \in |A|$. And finally if $a \in |A|$ and $q \in \Omega$, define

$$a \upharpoonright q = \rho^p_{p\wedge q}(a) \in A(p \wedge q) ,$$

where $Ea = p$. It is easy to verify that this makes a presheaf which redetermines the same functor. It is certainly fair to say that the view of presheaves as functors led Grothendieck to far-reaching generalizations of the notion of sheaf and to the notion of topos introduced in SGA4 [1] .

When is the presheaf *separated* ? Let us say that $\{ p_i \mid i \in I \}$ *covers* p provided that $p = \bigvee_{i\in I} p_i$. Then the functor $A : \Omega^{op} \to \mathcal{S}ets$ gives rise to a separated presheaf iff whenever $\{ p_i \mid i \in I \}$ covers p , then the obvious map $A(p) \to \mathsf{X}_{i\in I} A(p_i)$ into the cartesian product obtained from the restriction maps $\rho^p_{p_i}$ is a *monomorphism* (one-one embedding). When we come later to the notion of sheaf, this remark will have to be phrased in a more subtle way, as the simple cartesian product is too crude.

(vi) *Kripke models as presheaves.* Let X be a set equipped with a reflexive and transitive relation $\Vdash$. In 1.5 we defined a propositional model $\Omega = K(X)$ as the cHa consisting of all $p \subseteq X$ such that $j \Vdash i \in p$ always implies $j \in p$. Suppose now we have a family of sets $\{ D_i \mid i \in I \}$, each equipped with an equivalence relation. We write $i \Vdash a = b$ to mean that a and b belong to D_i and are equivalent therein. By definition we have a <u>Kripke</u> <u>model</u> iff whenever $j \Vdash i \Vdash a = b$ then $j \Vdash a = b$. This implies in particular that if $j \Vdash i$, then $D_j \supseteq D_i$. Each Kripke model gives us a $K(X)$-set D where $|D| = \bigcup_{i\in I} D_i$ and we define

$$[\![a = b]\!] = \{ i \in I \mid i \Vdash a = b \} .$$

In general Kripke models in this form are not separated. □

The presheaf ($\mathcal{O}(X)$-set) R_X is separated by definition, but it has an additional, quite important property: *to define a continuous function on an open set* U *, it is sufficient to define it locally.* More precisely, suppose we have a cover where $U = \bigcup\{ U_i \mid i \in I \}$. Suppose also the $f_i : U_i \to \mathbb{R}$ are given. We could not hope to glue these separate functions together into a whole, unless there were some *compatibility conditions*. The obvious one is that $f_i = f_j$ *on* $U_i \cap U_j$ for all $i, j \in I$. Then we can (by a union, so to speak) define a (unique) function $f : U \to \mathbb{R}$ such that $f_i = f \restriction U_i$, all $i \in I$. The existence of such compatible joins is exactly the definition we can state in an abstract form.

4.9. DEFINITION OF SHEAF. A presheaf is a <u>sheaf</u> iff every compatible family of elements has a *unique* join.

Discussion. The definition implies at once that every sheaf is a separated presheaf. The converse does not hold: R_X is a sheaf; the collection of *bounded* functions in $|R_X|$ forms a separated presheaf which, in general, is not a sheaf. We can now, if we wish, write unambiguously $\bigvee_{i \in I} a_i$ for the join of elements in a sheaf - provided we know that $\{ a_i \mid i \in I \}$ is compatible. □

It is more complicated to say when a separated Ω-set is a sheaf, because here restrictions need not exist. Of course we could say what restrictions are to be in the Ω-set, because

$$a = b \restriction p \quad \text{iff} \quad Ea = [\![a = b]\!] \wedge p \quad ,$$

and the right hand side is meaningful in Ω-set terms. But, since it is necessary to secure not only the existence of restrictions but also the existence of joins, it is better to put both existence conditions into one uniform statement. To be able to make a succinct statement, it is helpful here and in other places to use the terminology of singletons.

4.10. DEFINITION. Let A be an Ω-set. A <u>singleton</u> of A is a map $s : |A| \to \Omega$ such that

(i) $s(a) \wedge [\![a = b]\!] \leq s(b)$, and

(ii) $s(a) \wedge s(b) \leq [\![a = b]\!]$,

for all $a, b \in |A|$.

Discussion. Suppose we already have an element $c \in |A|$ fixed. It determines a map $a \mapsto [\![a = c]\!]$ which is obviously a singleton (transitivity!); call this map $\tilde{c}$. Note that $\tilde{c} = \tilde{d}$ iff $[\![c \equiv d]\!] = \top$. Thus an Ω-set is separated iff $\tilde{c}$ always uniquely determines c , or better: a singleton determines *at most* one element. But must a singleton determine an element?

4.11. DEFINITION. An Ω-set is <u>complete</u> iff every singleton determines a *unique* element.

Discussion. The definition means that for every singleton $s : |A| \to \Omega$ there is one and only one element $c \in |A|$ with $s = \tilde{c}$; that is, $s(a) = [\![a = c]\!]$ for all $a \in |A|$. Any complete Ω-set is separated, but not conversely. □

4.12. PROPOSITION. Every sheaf as an Ω-set is complete.

Proof. Let $s : |A| \to \Omega$ be a singleton, where A is assumed to be a sheaf. Define $B = \{ a \restriction s(a) \mid a \in A \}$. Note that by 4.10 (ii) , we have $s(a) \le Ea$, so $E(a \restriction s(a)) = s(a)$. Using 4.7 (iii) we then calculate:

$$\begin{aligned} a \restriction s(a) \restriction E(b \restriction s(b)) &= a \restriction s(a) \wedge s(b) \\ &= a \restriction [\![a = b]\!] \wedge s(a) \wedge s(b) \\ &= b \restriction [\![a = b]\!] \wedge s(a) \wedge s(b) \\ &= b \restriction s(b) \restriction E(a \restriction s(a)) \ . \end{aligned}$$

This proves that B is compatible. Let $b = \bigvee B$. Certainly $a \restriction s(a) \le b$ and so $s(a) \le [\![a = b]\!]$. Thus $s(a) = s(a) \wedge [\![a = b]\!] \le s(b)$. As this holds for all $a \in |A|$, $Eb = \bigvee \{ s(a) \mid a \in |A| \} \le s(b)$. Whence, $[\![a = b]\!] = [\![a = b]\!] \wedge s(b) \le s(a)$. This proves that $s = \tilde{b}$. □

4.13. THEOREM. Sheaves and complete Ω-sets come to the same thing.

Proof. Start with a sheaf A . We have just seen that as an Ω-set it is complete. But if we have a complete Ω-set, how do we regard it as a sheaf? Well, we know what $Ea = [\![a = a]\!]$ means already. To define restrictions for $a \in |A|$ and $p \in \Omega$, note that the map $b \mapsto [\![a = b]\!] \wedge p$ is a singleton. Hence, there is a unique element, called $a \restriction p$, such that $[\![a \restriction p = b]\!] = [\![a = b]\!] \wedge p$. We need to verify the equations of 4.2 .

Since $[\![a = b]\!] = [\![a = b]\!] \wedge Ea$, we see $a \restriction Ea = a$. Since $[\![a \restriction p \restriction q = b]\!] = [\![a = b]\!] \wedge p \wedge q = [\![a \restriction (p \wedge q) = b]\!]$, we see $a \restriction p \restriction q = a \restriction (p \wedge q)$. Because $E(a \restriction p) = [\![a \restriction p = a \restriction p]\!] = [\![a = a]\!] \wedge p = Ea \wedge p$, we find all of (i) - (iii) of 4.2 are true. Thus, the Ω-set A can be regarded as a presheaf. Is it separated - as a presheaf? This will follow if we can verify equation 4.2 (iv) .

If $p \le Ea \wedge Eb$ and $a \restriction p = b \restriction p$, then $p = p \wedge [\![a = a]\!] = [\![a = a \restriction p]\!] = [\![a = b \restriction p]\!] = [\![a = b]\!] \wedge p$. Thus $p \le [\![a = b]\!]$. In the other direction, if $p \le [\![a = b]\!]$, then $p \le Ea \wedge Eb$ at once. But note that $[\![a \restriction p = c]\!] = [\![a = c]\!] \wedge p = [\![a = c]\!] \wedge [\![a = b]\!] \wedge p = [\![b = c]\!] \wedge [\![a = b]\!] \wedge p = [\![b = c]\!] \wedge p = [\![b \restriction p = c]\!]$. Since $a \restriction p$ and $b \restriction p$ determine the same singleton, they must be equal. So the desired formula holds; and by 4.7 it follows that A as a presheaf *redetermines* the same equality as the original Ω-set, and separation must be true.

It remains to prove that A has joins. Let $B \subseteq |A|$ be a compatible subset. Define a map $s : |A| \to \Omega$ by the formula:

$$s(a) = \bigvee\{ [\![a = b]\!] \mid b \in B \} \quad .$$

To show $s(a) \wedge [\![a = a']\!] \leq s(a')$ is very easy by the distributive and transitive laws. We also apply distributivity to calculate:

$$s(a) \wedge s(a') = \bigvee\{ [\![a = b]\!] \wedge [\![a' = b']\!] \mid b, b' \in B \} \quad .$$

Because B is compatible, we see by 4.7 (ix) :

$$\begin{aligned} [\![a = b]\!] \wedge [\![a' = b']\!] &= [\![a = b]\!] \wedge [\![a' = b']\!] \wedge Eb \wedge Eb' \\ &= [\![a = b]\!] \wedge [\![a' = b']\!] \wedge [\![b = b']\!] \\ &\leq [\![a = a']\!] \quad . \end{aligned}$$

Thus $s(a) \wedge s(a') \leq [\![a = a']\!]$, so s is a singleton.

Now let $\tilde{c} = s$. If $b \in B$, then

$$[\![c = b]\!] = \bigvee\{ [\![b = b']\!] \mid b' \in B \} \quad .$$

Clearly $Eb = [\![b = b]\!] \leq [\![c = b]\!]$, so $b \leq c$ by 4.7 (vii) . That proves c is an upper bound to B . But $Ec = [\![c = c]\!] = \bigvee\{ [\![c = b]\!] \mid b \in B \} = \bigvee\{ Eb \mid b \in B \}$, because for $b \in B$, we know $Eb \leq [\![b = c]\!] \wedge [\![b = b]\!] = Eb$. Thus by 4.5 (iii) c is a join of B , and separation means it is unique. Therefore A is a sheaf.

We have shown that, starting with a complete Ω-set A , we can define restrictions so that A is a sheaf that redetermines the same Ω-set by the standard recipe. But just to make sure nothing goes wrong, suppose you are given first A as a sheaf. By 4.12 it becomes a complete Ω-set. But by 4.3 we know all along that restrictions are indeed determined as singletons, so we get the same sheaf back again using the new recipe employed in the last paragraphs. Hence, the theorem. □

In 4.13 the phrase "come to the same thing" is a trifle inexact. In the next section when we define morphisms of sheaves and Ω-sets, we can be more precise by showing that there is an equivalence of categories.

4.14. MORE EXAMPLES. This time we want sheaves. We saw that $\mathcal{R}_X$ was a sheaf, but the commutative ring A as an $I(B)$-set was not a sheaf since all elements were global (that is, there were no restrictions). Our next project will be to give a general method for adding restrictions and joins. In some cases, however, a direct construction is very simple.

(i) *The sheaf of propositions.* We define $\underset{\sim}{\Omega}$ as an Ω-set so that $|\underset{\sim}{\Omega}| = \{ \langle p,q \rangle \in \Omega \times \Omega \mid p \leq q \}$. For $a \in |\underset{\sim}{\Omega}|$ we also write $a = \langle |a|, Ea \rangle$; that is Ea is just the second coordinate of a . Equality is defined by:

$$[\![a = b]\!] = (|a| \leftrightarrow |b|) \wedge Ea \wedge Eb \quad .$$

We leave as an exercise the verification that this is a separated Ω-set in which

$[\![a \equiv b]\!] = (|a| \leftrightarrow |b|) \wedge (Ea \leftrightarrow Eb)$, and

$a \restriction p = \langle |a| \wedge p, Ea \wedge p \rangle$.

It should also be verified that a and b are compatible iff $Ea \wedge Eb \leq |a| \leftrightarrow |b|$. If $B \subseteq |\Omega|$ is a compatible set, then the join is given by:

$\bigvee B = \langle \bigvee\{ |b| \mid b \in B \}, \bigvee\{ Eb \mid b \in B \} \rangle$.

In this way, by direct verification, $\underset{\sim}{\Omega}$ is seen to be a sheaf.

(ii) *Constant sheaves.* Alas, over an arbitrary Ω , an explicit description of all sections of a constant sheaf is messy since Ω is not necessarily proper and we cannot decide whether $p = \bot$ or not. For the moment, then, we restrict ourselves to constant sheaves over $P(\mathbb{1})$, where the description is simple. In the following paragraphs, in any case, we will show how any Ω-set can be "completed" to a sheaf (this construct also incorporates a suitable quotient by an equivalence relation).

Let A be any set and construe it as a presheaf over $P(\mathbb{1})$, where (compare 0.7.12):

$|A| = \tilde{A} = \{ a \subseteq A \mid \forall x, y \in a . x = y \}$.

Further, define:

$Ea = \{ 0 \mid \exists x . x \in a \}$,

$a \restriction p = \{ x \in a \mid 0 \in p \}$, and

$[\![a = b]\!] = \{ 0 \mid \exists x [x \in a \wedge x \in b] \}$.

It is easy to see that:

$[\![a \equiv b]\!] = \{ 0 \mid a = b \}$,

so the presheaf is separated. Now $a, b \in |A|$ are compatible iff $\forall x \in a \, \forall y \in b . x = y$. The join of a compatible family $B \subseteq |A|$ is going to be just its set-theoretical union; thus, A so construed is a sheaf.

(iii) *Sheaves as functors.* We now return to the view of presheaves as functors introduced in 4.8 . To characterize, in a purely category-theoretic way, those functors which give rise to sheaves, we need some terminology. A <u>crible</u> of $p \in \Omega$ is a set K of elements $q \leq p$ below p which is closed downwards: $q' \leq q \in K$ implies $q' \in K$. We say a crible K of p <u>covers</u> p iff $p = \bigvee K$. Now if A is a presheaf and K a crible of p the sets $A(q)$ for $q \in K$ form, together with the restriction maps, a system over which we can take the inverse limit $\varprojlim_{q \in K} A(q)$. The presheaf A is a sheaf iff whenever K covers p the canonical map $A(p) \to \varprojlim_{q \in K} A(q)$ is an isomorphism. To see this note that if $p = \bigvee_{i \in I} p_i$, then

$[p_i \mid i \in I] = \{ q \mid q \leq p_i$ for some $i \in I \}$, the crible generated by the family $\{ p_i \mid i \in I \}$, covers p . Moreover, since we have pairwise intersections $p_{ij} = p_i \wedge p_j \in [p_i \mid i \in I]$, the limit over $[p_i \mid i \in I]$ is just the limit over $\{ p_{ij} \mid i,j \in I \}$, which is the collection of compatible families $\{ a_i \in A(p_i) \mid i \in I \}$. In Grothendieck's hands, this way of looking at things was generalized to give the notion of *site*, an abstract category with a notion of covering crible or Grothendieck topology [1] . The word "crible" is commonly translated by the English "sieve". It describes an agricultural sieve (more properly, a *riddle*) presumably used to separate germs from stalks (cf., Courbet [3]).

(iv) *The sheaf of germs of holomorphic functions.* Suppose X is a complex analytic manifold (in a simple case X might be $\mathbb{C}^n$ where $\mathbb{C}$ is the complex field). C_X is the sheaf of germs of complex-valued continuous functions on (open subsets of) X , which is constructed just like R_X ; it has interesting subsheaves, since X is more than just a topological space. Thus, let H_X be the sheaf obtained from holomorphic functions: We let H_X be the collection of holomorphic maps $f : U \to \mathbb{C}$ where $U \subseteq X$ is open. As usual $Ef = \operatorname{dom} f = U$, and for $f, g \in |H_X|$,

$$[\![f = g]\!] = \mathit{int}\{ t \in Ef \cap Eg \mid f(t) = g(t) \} \quad .$$

Of course $f \restriction U$ is just f restricted to $U \cap \operatorname{dom} f$. As being holomorphic is a local property, it is easy to verify that H_X so defined is a sheaf over $O(X)$.

(v) *The sections of a continuous map.* Suppose we are given a continuous map $\pi : Y \to X$ from one topological space into another. We define an $O(X)$-set π_X as the maps $a : U \to Y$, where $\pi \circ a$ is the identity on $U = Ea$. Again:

$$[\![a = b]\!] = \mathit{int}\{ t \in Ea \cap Eb \mid a(t) = b(t) \} \quad .$$

Not only is it easy to prove that π_X is a sheaf, but it is clear that it generalizes the construction of R_X , since this can be regarded as the sections of the projection function $\pi_2 : \mathbb{R} \times X \to X$.

(vi) *The power-sheaf.* There is no difficulty in proving that the product $A \times B$ of sheaves, as defined in 4.8 (iv) , is again a sheaf. A more interesting construction is that of the *power-sheaf* $P(A)$ of an Ω-set A . The elements of $|P(A)|$ are pairs $\langle P,p \rangle$, where $p \in \Omega$ (the underlying cHa), and $P : |A| \to \Omega$ satisfies the *extensionality* and *strictness* conditions:

(1) $P(a) \wedge [\![a = b]\!] \leq P(b)$, and

(2) $P(a) \leq Ea \wedge p$,

for all $a, b \in |A|$. Of course $E\langle P,p \rangle = p$. We define:

$$[\![\langle P,p \rangle = \langle Q,q \rangle]\!] = p \wedge q \wedge \bigwedge \{ (P(a) \leftrightarrow Q(a)) \mid a \in |A| \} \quad .$$

It follows that:

It follows that:

$$[\![\langle P,p\rangle \equiv \langle Q,q\rangle]\!] \;=\; (p \leftrightarrow q) \wedge \bigwedge\{ (P(a) \leftrightarrow Q(a)) \mid a \in |A| \} \quad , \quad \text{and}$$

$$\langle P,p\rangle \restriction q \;=\; \langle Q, p \wedge q\rangle \quad , \quad \text{where}$$

$$Q(a) \;=\; P(a) \wedge q \quad , \quad \text{all} \;\; a \in |A| \quad .$$

Also, $\langle P,p\rangle$ and $\langle Q,q\rangle$ are compatible iff $p \wedge q \leq (P(a) \leftrightarrow Q(a))$, for all $a \in |A|$. The join of a compatible family $\{ \langle P_i, p_i\rangle \mid i \in I \}$ is just done pointwise (also $(\bigvee_{i \in I} P_i)(a) \;=\; \bigvee_{i \in I} P_i(a)$). Admittedly, it takes a few applications of the distributive law to prove $P(A)$ is a sheaf. In Section 7 we shall show, via the interpretation of logic, that $P(A)$ plays the role of the power set in the category of sheaves over Ω . $\square$

We now turn to the question of how any Ω-set can be completed (and, hence, how any presheaf can be "sheafified"). The original Ω-set (if separated) will be found as a *generating* sub-Ω-set of its completion.

4.15. DEFINITIONS. Let A and B be Ω-sets. We say A is a <u>sub-Ω-set</u> of B , and write $A \subseteq B$, to mean that $|A| \subseteq |B|$ and the equality on A is just the restriction of that on B . We say $A \subseteq B$ <u>generates</u> B iff for each $b \in |B|$ we have:

$$Eb = \bigvee\{ [\![a = b]\!] \mid a \in |A| \} \quad .$$

In the case of presheaves $A \subseteq B$ (<u>subpresheaf</u>) means that on $|A|$ restrictions and extents are inherited from B . (Note: any subset of an Ω-set gives a sub-Ω-set, but subpresheaves must be closed under restrictions.)

4.16. PROPOSITION. Let $A \subseteq B$ be Ω-sets, then:

(i) A generates B iff every singleton of A extends *uniquely* to a singleton of B ;

(ii) If B is a presheaf, then A generates B iff every element of B is a join of restrictions of elements of A ;

(iii) If B is a presheaf and Θ is a basis for Ω , then if we let $|A| = \{ a \in |B| \mid Ea \in \Theta \}$, then A generates B .

Proof. (i). Suppose A generates B and $s : |A| \to \Omega$ is a singleton. Define $t : |B| \to \Omega$ by the formula:

$$t(b) \;=\; \bigvee\{ s(a) \wedge [\![a = b]\!] \mid a \in |A| \} \quad .$$

Clearly t extends s and t is a singleton. Suppose t' were any other singleton extending s . Then

$$t'(b) \;=\; t'(b) \wedge Eb \;=\; \bigvee\{ t'(b) \wedge [\![a = b]\!] \mid a \in |A| \}$$

$$= \bigvee \{ t'(a) \wedge [\![a = b]\!] \mid a \in |A| \} = t(b) .$$

So t is unique.

Assume now A-singletons have unique extensions. Let $b \in |B|$. The function where $s(a) = [\![a = b]\!]$ is a singleton on A . The formula for t above gives an extension. But the map $b' \mapsto [\![b' = b]\!]$ is also an extension. Thus by uniqueness:

$$[\![b' = b]\!] = \bigvee \{ [\![a = b]\!] \wedge [\![a = b']\!] \mid a \in |A| \} .$$

Substituting b for b' gives the desired formula of 4.15 .

For (iv) , assume that B is a presheaf. If A generates B , we can write:

$$b = b \restriction Eb = \bigvee \{ a \restriction [\![a = b]\!] \mid a \in |A| \} .$$

In the other direction, suppose that:

$$b = \bigvee \{ a_i \restriction p_i \mid i \in I \} ,$$

where the $a_i \in |A|$ and the $p_i \in \Omega$. Now since $a_i \restriction p_i \leq b$, we have $E(a_i \restriction p_i) = [\![a \restriction p_i = b]\!]$. Thus $Eb = \bigvee \{ E(a_i \restriction p_i) \mid i \in I \} = \bigvee \{ [\![a \restriction p_i = b]\!] \mid i \in I \}$, so A generates B .

For (iii) , assume B is a presheaf and Θ is a basis for Ω . Define A as shown. Since

$$b = \bigvee \{ b \restriction p \mid p \in \Omega , p \leq Eb \} ,$$

we can use (ii) to conclude that A generates B . □

4.17. DEFINITION OF SHEAFIFICATION. For any Ω-set A the <u>sheafification</u> $\hat{A}$ of A is the Ω-set defined by:

$$|\hat{A}| = \{ s : |A| \to \Omega \mid s \text{ is a singleton} \} , \text{ and}$$

$$[\![s = t]\!] = \bigvee \{ s(a) \wedge t(a) \mid a \in A \} ,$$

for all $s , t \in |\hat{A}|$.

The proof that A is an Ω-set is left as an exercise; what is more interesting is:

4.18. THEOREM. For any Ω-set A , we have $\hat{A}$ complete and generated by $\{ \tilde{a} \mid a \in |A| \}$. Furthermore,

$$[\![\tilde{a} = \tilde{b}]\!] = [\![a = b]\!] ;$$

so if A is separated, it can be regarded as a sub-Ω- set of $\hat{A}$. If B is any other sheaf generated by A , then B is isomorphic to $\hat{A}$.

Proof. First observe that for $b \in |A|$ and $s \in |\hat{A}|$ we have (recalling the

$\tilde{a}$-notation from 4.10) :

$$[\![s = \tilde{b}]\!] = \bigvee \{ s(a) \wedge [\![a = b]\!] \mid a \in |A| \} = s(b) ;$$

whence, $[\![\tilde{a} = \tilde{b}]\!] = [\![a = b]\!]$ follows. But we also have:

$$Es = [\![s = s]\!] = \bigvee \{ s(a) \mid a \in |A| \}$$

$$= \bigvee \{ [\![s = \tilde{a}]\!] \mid a \in A \} ;$$

therefore, $\{ \tilde{a} \mid a \in |A| \}$ generates $\hat{A}$.

Suppose now that σ is a singleton on $\hat{A}$. Define $s(a) = \sigma(\tilde{a})$. This is obviously a singleton on A . Also $\sigma(\tilde{a}) = [\![s = \tilde{a}]\!]$, so we see two singletons on $\hat{A}$ which agree on $\{ \tilde{a} \mid a \in A \}$. Thus σ is determined by an element of $\hat{A}$, which is unique in view of the generating subset.

It is quite clear that if $\{ \hat{a} \mid a \in A \}$ generates B , where B is a sheaf, there is a one-one, $[\![\cdot = \cdot]\!]$-preserving correspondence between the elements of B and the singletons in $\tilde{A}$. □

4.19. EXAMPLES. (i) *The sheaf of germs of real-valued continuous functions.* We know from previous remarks that R_X is a sheaf. The subpresheaf B_X of *bounded* functions is not a sheaf (if X has any interest as a topological space); however, it is evident that every real-valued function is locally bounded, so B_X generates R_X .

(ii) *Presheaves as sheaves.* In general, a presheaf is not a sheaf. However... Let A be a presheaf over Ω and let $K(\Omega)$ be the cHa of downwards closed subsets of Ω , or cribles of τ . (We use the notation of 4.14 (iii) .) Define a presheaf A^* over $K(\Omega)$ by $A^*(K) = \varprojlim_{q \in K} A(q)$ with the obvious restrictions. Now it is easy to check that A^* is a sheaf and that $A^*([q]) = A(q)$. Furthermore, if B is any $K(\Omega)$-sheaf, it arises in this way from the Ω-presheaf whose sections over $p \in \Omega$ are given by $B([p])$. Thus we may say that presheaves over Ω come to the same thing as sheaves over $K(\Omega)$. This type of equivalence is best formulated as an equivalence of categories. We shall return to this point later.

(iii) *The sheaf of propositions.* Ω as a sheaf (cf. 4.14 (i)) is generated by its global sections, an Ω-set isomorphic to Ω as an Ω-set with = taken as $\leftrightarrow$. For very similar reasons we see that the powersheaf $P(X)$ is generated by its global sections: indeed, every element *is a restriction* of a global section! Such strong generation properties may very well fail in a sheaf like R_X , where even if global elements generate it may not be true that every continuous function on an open set can be extended continuously to the whole space.

(iv) *Rings and Boolean ideals.* We began this section with an example of a ring A as an $I(B)$-set, using the cHa of ideals of the Boolean algebra of idempotents. All elements are global; A is not a sheaf. So, the question is: what is $\hat{A}$? We will have some interesting answers to this question in the next section where we can regard $\hat{A}$ as a ring in its own right. For the moment, we just ask about the nature of the elements of $\hat{A}$. From the general construction we know $\hat{A}$ is generated by the global sections corresponding to elements $a \in A$. Are there any others?

A global element is a singleton $s \in \hat{A}$ where $Es = \bigvee\{ s(a) \mid a \in A\} = \top = (1)$, as an ideal. Now the sup of ideals is just the ideal generated by the union, so there must be idempotents $e_0 \in s(a_0)$, ... , $e_{n-1} \in s(a_{n-1})$, where $\Sigma_{i<n} e_i = 1$. As we are in a Boolean algebra B , we can as well suppose that $e_i \cdot e_j = 0$ for $i \neq j$. Let $b = \Sigma_{i<n} e_i \cdot a_i$. We wish to prove that $s = \tilde{b}$, and this will show that A provides *all* the global elements of $\hat{A}$.

To this end, note that $e_i b = e_i a_i$. Thus if $e \in [\![a = b]\!]$, then $ea = eb$, so $e_i ea = e_i eb = e_i ea_i$. Thus $e_i e \in [\![a = a_i]\!]$. But $e_i e \in s(a_i)$; therefore, $e_i e \in s(a)$. As this holds for all i , we find by sums that $e \in s(a)$; thus, $[\![a = b]\!] \in s(a)$. In the other direction, if $e \in s(a)$, then $e_i e \in s(a) \wedge s(a_i)$, so $e_i e \in [\![a = a_i]\!]$. This means, for all $i < n$, $e_i ea = e_i ea_i$. Again summing up, $ea = eb$; that is $e \in [\![a = b]\!]$; and $s = \tilde{b}$ is proved. Following Mulvey [37] we call this sheaf $k(A) = \hat{A}$.

(v) *Rings and radical ideals.* In 2.15 we remarked that the collection $\Omega = \sqrt{}(A)$ of radical ideals of a commutative ring formed a cHa. As first pointed out to us by Tierney, this cHa can be used in a natural way to make A into a Ω-set. This has simularities to the previous example, but it is less trivial. We return to it again in Section 6 when we discuss a larger Ω-set, spec(A) , with very interesting algebraic properties. Returning to A , define this time:

$$[\![a = b]\!] = \{ t \in A \mid t^n a = t^n b \text{ for some } n > 0 \} .$$

What we have on the right hand side is the least radical ideal generated by the ideal $\{ t \in A \mid ta = tb \}$; this is a measure of how nearly a and b are equal, and $[\![a = b]\!] = (1)$ iff $a = b$. Thus A is separated over Ω . Of course all elements are global, so A is no sheaf. We show in this case, too, that A gives exactly the global elements of $\hat{A}$: sheafification adds no new global elements. (This is quite different from $R_X = \hat{\mathcal{B}}_X$, for example.)

So suppose $s \in \hat{A}$ is global with $Es = \bigvee\{ s(a) \mid a \in A\} = (1)$. We obtain, as before, $1 = \Sigma_{i<n} u_i$ for suitably chosen $u_i \in s(a_i)$, $i < n$. Now $u_i \cdot u_j$ $u_i \cdot u_j \in s(a_i) \wedge s(a_j) \leq [\![a_i = a_j]\!]$; whence, for a suitably large m we can say $(u_i u_j)^m (a_i - a_j) = 0$. (The same m for all $i, j < n$ can be found by taking the maximum.) By taking a very large power k of $\Sigma_{i<n} u_i$ and multiplying out, we can write:

$$1 = \Sigma_{i<n} z_i u_i^m$$

for suitable coefficients. Call $u_i^m = v_i$ for short. We have $v_i \in s(a_i)$ and $v_i v_j (a_i - a_j) = 0$. Define

$$b = \Sigma_{i<n} z_i v_i a_i \quad .$$

It is easy to prove $v_i b = v_i a_i$.

As before , if $t \in [\![a = b]\!]$, then $t^p a = t^p b$. Then $t^p v_i z_i a = t^p v_i z_i a_i$, so $t^p v_i z_i \in [\![a = a_i]\!]$. But $t^p v_i z_i \in s(a_i)$ also, so since s is a singleton, $t^p v_i z_i \in s(a)$. Summing up, we get $t^p \in s(a)$, so $t \in s(a)$. In a similar way (just like the other proof) $t \in s(a)$ implies $t \in [\![a = b]\!]$. It follows that $s = \tilde{b}$ as we wish. □

Our examples have for the most part been Ω-sets rather than presheaves. There is a reason for this: not only do we think Ω-sets arise in practice more often, but they have the technical advantage that to obtain $\hat{A}$ one only has to define A - a generating subset. Even to evaluate logical formulae (Section 5) it is only necessary to do the work on generating subsets.

In passing from Ω-sets to presheaves it is only necessary to add restrictions. This could be done by forming pairs $\langle a,p \rangle$ with $p \le Ea$ - but they have to be divided into equivalence classes using the relation between $\langle a,p \rangle$ and $\langle b,q \rangle$ that $p = q$ and $p \le [\![a = b]\!]$. This is tiresome, and it seems marginally better to employ singletons. Similarly in passing from presheaves to sheaves it would be possible to "complete" by adjoining the joins of compatible subsets - somewhat in the style of the ideal completion of a lattice. Again some effort has to be expended to obtain separation, so we feel that singletons are better here also.

Our discussion of the theory of sheaves so far has been rather algebraic (model-theoretic), so to tie up the presentation with more well-known treatments we show how to obtain the usual geometric definition for sheaves over $\mathcal{O}(X)$. The ideas will be used later in Chapter III when we give a representation of internal topological spaces in the sheaf category.

4.20. DEFINITION. An étale space over a space X is a continuous map $\pi : E \to X$ which is a local homeomorphism; that is, every point of E has a neighbourhood mapped homeomorphically onto its image by the restriction of π . We call E the total space of π . (Note that we do *not* assume, as is done in some books, that π is surjective.)

In 4.14 (iv) we have already remarked that the sections of *any* continuous $\pi : E \to X$ (not just local homeomorphisms) form a sheaf. We shall now prove the classical theorem that every sheaf over a space comes (up to isomorphism) not just from a continuous map but from an étale space. The problem, given a sheaf A abstractly, is to construct the total space. The proof given below, though just the classical method in disguise, is perhaps useful to know since it avoids taking any limits and uses instead the duality between cHa's and sober spaces. For the classical proof, see Godement [14].

4.21. LEMMA. Let A be a presheaf over Ω. The collection of global elements of the power sheaf $P(A)$ forms a cHa Π under pointwise $\wedge$ and $\bigvee$. If for $p \in \Omega$ we let $\pi^{-1}(p)$ be the map taking $a \in |A|$ to $Ea \wedge p \in \Omega$ then $\pi^{-1} : \Omega \to \Pi$ is an $\wedge\bigvee$-map. The singletons $\{\tilde{a} \mid a \in |A|\}$ form a basis for Π and

(i) $\tilde{a} \wedge \pi^{-1}(p) = (a \restriction p)^{\sim}$;

(ii) $\tilde{a} \leq \tilde{b}$ iff $a \leq b$;

(iii) $\tilde{a} \wedge \tilde{b} \leq \pi^{-1}[\![a = b]\!]$.

Proof. The global elements of $P(A)$ are just the maps $P : |A| \to \Omega$ where for all $a, b \in |A|$ we have $P(a) \wedge [\![a = b]\!] \leq P(b)$ and $P(a) \leq Ea$. Since these are closed under pointwise $\wedge$ and $\bigvee$ they form a cHa with $a \mapsto \perp$ as zero and $a \mapsto Ea$ as unit. As π^{-1} commutes with these pointwise operations, it is an $\wedge\bigvee$-map. Now $\tilde{a} \leq P$ iff $P(a) = Ea$ so the singletons form a basis for Π because $p \leq P(a)$ iff $(a \restriction p)^{\sim} \leq P$ (remember, P is strict). Now (i) is $[\![a = b]\!] \wedge p = [\![a \restriction p = b]\!]$ rewritten, (ii) is $Ea \leq [\![a = b]\!]$ iff $a \leq b$, in new guise, and (iii) follows from the transitivity of $=$. $\square$

4.22. THEOREM. Let X be a space and A a sheaf over $\mathcal{O}(X)$. Then A is isomorphic to the sheaf of sections of an étale space $\pi : E \to X$ where $\mathcal{O}(E)$ is isomorphic to the cHa Π of global elements of the power sheaf $P(A)$.

Proof. Suppose first that X is sober (we shall return later to the general case) and that $\Omega = \mathcal{O}(X)$. Then $X \cong pt(\Omega)$ and the duality between cHa and spaces gives a continuous map $\pi : pt(\Omega) \to X$. This will turn out to be our étale space. We now show that an element of $|A|$ gives a section of π. For each $a \in |A|$ let $a^{-1}(P) = P(a)$ for $P \in \Pi$, further, let $\Omega_a = \{p \in \Omega \mid p \leq Ea\}$. Now $a^{-1} : \Pi \to \Omega_a$ is an $\wedge\bigvee$-map and $a^{-1}(\pi^{-1}(p)) = p \wedge Ea$ for $p \in \Omega$. Dualizing, we get a section $a : Ea \to pt(\Pi)$ of π.

This is the general picture, it remains to demonstrate, firstly that π is a local homeomorphism, and, secondly that A is precisely the sheaf of sections of π. Now if $P \leq \tilde{a}$ then $P(b) \leq [\![a = b]\!]$ so $P(b) = [\![a = b]\!] \wedge P(a)$ for $b \in |A|$. Thus we have a factorization $a^{-1} : \Pi \to \Pi_a \to \Omega_a$ where the first factor is an open quotient (see 2.18), and the second an isomorphism. Dualizing, we see that a maps Ea homeomorphically to the open set $\tilde{a}^* \cong pt(\Pi_a)$. Thus π must map $\tilde{a}^*$

homeomorphically to Ea and as the $\tilde{a}$ generate Π , we have shown that π is a local homeomorphism. Furthermore, Π must have enough points because each $\Pi_a \cong \Omega_a \cong \mathcal{O}(Ea)$ certainly does.

If we are given a section $\sigma : U \to pt(\Pi)$ of π over $U \in \mathcal{O}(X)$ we dualize to get an $\wedge\bigvee$- map $\sigma^{-1} : \Pi \to \mathcal{O}(U)$ such that $\sigma^{-1}(\pi^{-1}(p)) = p \wedge U$ for $p \in \mathcal{O}(X)$. Now for $s \in |A|$ define $s(a) = \sigma^{-1}(\tilde{a}) \in \mathcal{O}(X)$. We claim that s is a singleton of A . Firstly, $s(a) \wedge s(b) = \sigma^{-1}(\tilde{a} \wedge \tilde{b}) \leq \sigma^{-1}(\pi^{-1}[\![a = b]\!]) = [\![a = b]\!]$. Secondly, $s(a) \wedge [\![a = b]\!] = \sigma^{-1}(\tilde{a} \wedge \pi^{-1}[\![a = b]\!]) = \sigma^{-1}((a \,|[\![a = b]\!])^{\sim}) \leq \sigma^{-1}(\tilde{b}) = s(b)$. Further, $\bigvee \{ s(a) \mid a \in |A| \} = \sigma^{-1}(\top) = U$. So if A is a sheaf, there is a unique $a \in |A|$ with $[\![a = b]\!] = \sigma^{-1}(\tilde{b})$ for $b \in |A|$ or (equivalently, since the $\tilde{b}$ form a basis for Π) with $P(a) = \sigma^{-1}(P)$ for each $P \in \Pi$. Thus, the sections of π correspond exactly to the elements of $|A|$. For $a, b \in |A|$ we have

$$[\![a = b]\!] = a^{-1}(\tilde{b}) = \{ t \in Ea \wedge Eb \mid a(t) = b(t) \} ,$$

since $(\tilde{b})^* = \{ b(t) \mid t \in Eb \}$ and a and b are both sections of π . Also, since π is a local homeomorphism, we have a local section through every point of Π .

When X is not sober, we cannot write $X \cong pt(\mathcal{O}(X))$. We obtain the desired space by pulling back

$$\begin{array}{ccc} E & \longrightarrow & pt(\Pi) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow[\eta_X]{} & pt(\mathcal{O}(X)) \end{array} \quad .$$

As is well-known, the pull back of a local homeomorphism is a local homeomorphism. Explicitly we can write

$$E \cong \{ \langle a(\eta_X(t)),t\rangle \mid a \in |A| \text{ and } t \in Ea \}$$

with $\mathcal{O}(E) \cong \Pi$ consisting of the opens

$$U^* = \{ \langle e,t\rangle \in E \mid e \in U \} \qquad \text{for } U \in \mathcal{O}(pt(\Pi)) \quad .$$

Sobrification preserves open inclusions. By the universal property of $\eta_p : p \to pt(\mathcal{O}(p))$, any section of π over $p \in \mathcal{O}(X)$ gives a section of the original map $pt(\Pi) \to pt(\mathcal{O}(X))$ over $pt(\mathcal{O}(p))$. By the universal property of pullback every section over $pt(\mathcal{O}(p))$ arises in this way from a unique section of π over p . As $\mathcal{O}(X) \cong \mathcal{O}(pt(\mathcal{O}(X)))$, the sections of π are exactly the sections of the original map. □

If a sheaf A is represented in this way as the sheaf of sections of an étale space $\pi : E \to X$, the *stalk* of A at t is the set $A_t = \pi^{-1}\{t\}$. For $a \in |A|$ and $t \in Ea$ we call $a(t) \in A_t$ the <u>germ</u> of a at t . Note that

$$A_t = \{ a(t) \mid t \in Ea \} \quad \text{and}$$
$$a(t) = b(t) \quad \text{iff} \quad t \in [\![a = b]\!] \ .$$

We conclude this section with a discussion which would classically be trivial. Since $\{\top\}$ is a basis for $P(\mathbb{1})$, global sections generate any $P(\mathbb{1})$-sheaf. Classically, $P(\mathbb{1})$ is the only non-trivial such cHa.

4.23. PROPOSITION. Let Ω be a cHa. Every Ω-sheaf is generated by its global sections iff $\{\top\}$ is a basis for Ω.

Proof. Consider the Ω-set A with underlying set $\{p\}$ for some $p \in \Omega$ and $[\![p = p]\!] = p$. Any map $s : \{p\} \to \Omega$ with $s(p) \le p$ is a singleton of A. Such a singleton is inhabited iff $s(p) = \top$ (and hence $p = \top$). So, the set of global sections of $\hat{A}$, the completion of A, is

$$\{ s : \{p\} \to \Omega \mid s(p) = \top \ \text{and} \ p = \top \} \ .$$

If global sections generate $\hat{A}$ then $[\![p = p]\!] = \bigvee\{ s(p) \mid s$ is a global section$\}$ and we have $a = \bigvee\{ \top \mid a = \top \}$. □

The cHa for which $\{\top\}$ is a basis can be described in other ways. For example, an $\wedge$-semilattice L can be embedded as a semilattice of $P(\mathbb{1})$ iff $a \le b$ iff $a = \top$ implies $b = \top$ for $a, b \in L$. For cHa this latter conditions is equivalent to $\{\top\}$ being a basis. We shall be interested primarily in the particular case where Ω is a quotient of $P(\mathbb{1})$.

5. FIRST-ORDER LOGIC

In this section we use sheaves to provide interpretations for first-order logic. To do this we must say what a structure is, which amounts to introducing operations and predicates. This may be done in various ways which turn out to be equivalent. Here we start from a definition which is precisely that necessary to ensure the validity of the axioms of first-order logic. This is given in terms of Ω-sets. When we see what it means for sheaves, it turns out that it is equivalent to demanding that our operations and predicates be defined locally. Our Ω-valued models or sheaves are analogous to Boolean-valued models for classical logic, and similar to Mostowski's topological models [42]. The introduction of partial elements allows us a greater variety of models and means that, using complete Ω-sets or sheaves, we may interpret descriptions. The recognition of these models as sheaves provides us with many models from mathematics.

5.1. OPERATIONS AND RELATIONS. We define an n-ary operation F on an Ω-set A to be a map $F : |A|^n \to |A|$ which is $\equiv$-extensional in the sense that

$$\bigwedge_{i<n} [\![a_i \equiv b_i]\!] \leq [\![F(a_0,\ldots,a_{n-1}) \equiv F(b_0,\ldots,b_{n-1})]\!] \quad .$$

An n-ary relation R on A is a map $|A|^n \to \Omega$ $\equiv$-extensional is the sense that

$$\bigwedge_{i<n} [\![a_i \equiv b_i]\!] \wedge R(a_0,\ldots,a_{n-1}) \leq R(b_0,\ldots,b_{n-1}) \quad .$$

(We could define many-sorted operations and relations similarly, each sort being interpreted by an Ω-set.) We call a collection of operations and relations on A a structure on A .

These definitions will ensure that the axiom of equivalence

(eq) $\qquad \phi(x) \wedge x \equiv y \;\to\; \phi(y)$

becomes valid (by the usual induction on the structure of ϕ). The distinctions discussed in 0.3 have their counterparts in the models. Thus we say an operation F is *strict* iff $EF(a_0,\ldots,a_{n-1}) \leq \bigwedge_{i<n} Ea_i$, and *total* iff equality holds. A predicate R is *strict* iff $[\![R(a_0,\ldots,a_{n-1})]\!] \leq \bigwedge_{i<n} Ea_i$.

5.2. EXAMPLES. (i) Let j be a J-map on Ω . Then $(p \leftrightarrow q) \leq (jp \leftrightarrow jq)$ for $p, q \in \Omega$, so we can consider j as an operation on $\underset{\sim}{\Omega}$. We have $[\![Ep]\!] = [\![Ejp]\!] = \top$ and $[\![p = q]\!] \leq [\![jp = jq]\!]$ from which $\equiv$-extensionality follows formally. In fact j is a strict total operation on $\underset{\sim}{\Omega}$. A similar argument shows that the operations $\wedge$, $\vee$, $\to$, $\neg$ on Ω also become strict total operations on $\underset{\sim}{\Omega}$.

(ii) In the last example, because the equality on $\underset{\sim}{\Omega}$ is non-trivial we had to some work to verify extensionality. If A is a set, any operation on A may be viewed as an operation on the simple Ω-set A_Ω . Any relation on A gives a relation on A_Ω as follows:

$$[\![R(a_0,\ldots,a_{n-1})]\!] = \{ \top \mid R(a_0,\ldots,a_{n-1}) \} \quad .$$

This may be trivial but it will prove useful nonetheless.

(iii) Any (strict) continuous function $f : \mathbb{R}^n \to \mathbb{R}$ gives a strict operation on R_X by composition. That is

$$[f(a_0,\ldots,a_{n-1})](t) = f(a_0(t),\ldots,a_{n-1}(t)) \quad .$$

Extensionality is immediate since $[\![a \equiv b]\!] = \mathit{int}\{ t \mid a(t) \equiv b(t) \}$. The arithmetic operations $+$ and $\times$ in particular arise in this way. They are strict *and* total. The same definition can be applied to a partial continuous function $g : \mathbb{R}^n \to \mathbb{R}$ to give a partial operation on R_X : Recall that x^{-1} is defined for $x \neq 0$ so applying this construction to the inverse function we get a partial operation on R_X with $Ea^{-1} = \{ t \in X \mid a(t) \neq 0 \}$.

Any predicate on $\mathbb{R}^n$ gives a predicate on R_X as follows:

$$[\![R(a_0,\dots,a_{n-1})]\!] = \mathit{int}\{ t \in X \mid R(a_0(t),\dots,a_{n-1}(t)) \} .$$

Particular examples are the predicates of apartness $\neq$, strong ordering $<$, weak ordering $\leq$ and of course equality $=$.

(iv) The Ω-set $k(A)$ associated to a ring A , discussed in 4.1.9 , itself carries a ring structure. We check that the ring operations on $A = |k(A)|$ are extensional. Consider addition. We have trivially $E(a+b) = Ea \wedge Eb = \top$. Less trivially, recall $[\![a = b]\!] = \{ e \in B \mid ea = eb \}$ so that, since if $ea = ea'$ then $e(a+b) = e(a'+b)$, we have $[\![a = a']\!] \leq [\![a+b = a'+b]\!]$. From this, and the corresponding fact for b , it follows that $+$ is a strict total operation on $k(A)$. The verification that $\times$ is also a strict total operation is similar.

Since we like to work with sheaves, but find it easier often to present a generating Ω-set, we must show how to extend any structure on a generating set.

5.3. LEMMA. Let A be a complete Ω-set and $B \subseteq A$ a sub-Ω-set. Any structure on B can be extended to one on A . Furthermore, if $|B|$ generates A then any *strict* operation or relation on B has a *unique* strict extension to A .

Proof. Extending relations is easy: for $a \in |A|$ define

$$R(a) = \bigvee \{ R(b) \wedge [\![a \equiv b]\!] \mid b \in |B| \} .$$

We check two things: Firstly, since R was extensional on B , the new R is an extension of the old. Secondly the new R is extensional,

$$\begin{aligned}[\![a \equiv a']\!] \wedge R(a) &= \bigvee \{ [\![a \equiv a']\!] \wedge R(b) \wedge [\![a \equiv b]\!] \mid b \in |B| \} \\ &\leq \bigvee \{ R(b) \wedge [\![a' \equiv b]\!] \mid b \in |B| \} = R(a') .\end{aligned}$$

For an operation F we define $F(a)$ for $a \in |A|$ to be the unique element of $|A|$ such that for all $a' \in |A|$ we have

$$[\![a' = F(a)]\!] = \bigvee \{ [\![a' = F(b)]\!] \wedge [\![a \equiv b]\!] \mid b \in B \} = s(a')$$

which exists since the term on the right is a singleton. Again it is straightforward to check that the new F is a $\equiv$-extensional extension of the old.

If $|B|$ generates A then any section $a \in |A|$ is locally (where it exists) equal to a section of B . Since strict operations and relations are determined by the properties $R(a) \wedge Ea = R(a)$ and $F(a) \restriction Ea = F(a)$, the extensions we have defined are unique. □

This will enable us later to extend any structure on an Ω-set A to its completion $\hat{A}$. Since $P(A)$, the power sheaf, is defined in terms of strict, extensional predicates, it also means that $P(A) \cong P(\hat{A})$. Now we take a look at presheaves. Any presheaf is an Ω-set. We characterize operations and relations on a separated presheaf in terms of the presheaf structure.

5.4. THEOREM. Let A be a separated presheaf. A map $F : |A|^n \to |A|$ is an operation iff $F(a_0,\dots,a_{n-1}) \restriction p = F(a_0 \restriction p,\dots,a_{n-1} \restriction p) \restriction p$ identically for $p \in \Omega$ and $a_i \in |A|$. A map $R : |A|^n \to \Omega$ is a relation iff $R(a_0,\dots,a_{n-1}) \wedge p = R(a_0 \restriction p,\dots,a_{n-1} \restriction p) \wedge p$ identically.

Proof. Since $p \le [\![a \equiv a \restriction p]\!]$, if R is a predicate, the last equation holds. Conversely, if $a_i \restriction p = b_i \restriction p$ then this equation implies that $R(a_0,\dots,a_{n-1}) \wedge p = R(b_0,\dots,b_{n-1}) \wedge p$. Now by 4.7 (viii) we see that R is extensional. The proof for operations is similar. One way is easy again. In the other direction, suppose $a_i \restriction p = b_i \restriction p$ then, by the condition, $F(a_0,\dots,a_{n-1}) \restriction p = F(b_0,\dots,b_{n-1}) \restriction p$ so that $p \le [\![F(a_0,\dots,a_{n-1}) \equiv F(b_0,\dots,b_{n-1})]\!]$. Again, we appeal to 4.7 to complete the proof. □

We shall often want to define operations and relations on a presheaf which is not separated. Strict operations and relations are easy to present in this way.

5.5. LEMMA. To give strict operation or relation on a presheaf A, it suffices to give a map of the appropriate kind which is *strict*

$$EF(a_0,\dots,a_{n-1}) \le \bigwedge_{i<n} Ea_i \quad \text{or} \quad R(a_0,\dots,a_{n-1}) \le \bigwedge_{i<n} Ea_i$$

and *local*

$$F(a_0,\dots,a_{n-1}) \restriction p = F(a_0 \restriction p,\dots,a_{n-1} \restriction p) \quad \text{or, for relations,}$$

$$R(a_0,\dots,a_{n-1}) \wedge p = R(a_0 \restriction p,\dots,a_{n-1} \restriction p) \quad .$$

Proof. To simplify notation, take the case $n = 1$. We must show that $[\![a \equiv b]\!] \le [\![F(a) \equiv F(b)]\!]$. This is equivalent to $[\![a \equiv b]\!] \wedge (EF(a) \vee EF(b)) \le [\![F(a) = F(b)]\!] \le [\![F(a) = F(b)]\!]$. Since F is strict, this reduces to $[\![a = b]\!] \wedge (EF(a) \vee EF(b)) \le [\![F(a) = F(b)]\!]$. Now take $p \in \Omega$ such that $a \restriction p = b \restriction p$ and $p \le EF(a)$ then $F(a) \restriction p = F(a \restriction p) = F(b \restriction p) = F(b) \restriction p$. From this we deduce firstly that $p \le EF(b)$ and secondly that $p \le [\![F(a) = F(b)]\!]$. However, such p cover the left-hand side of our putative inequality so we are done. Strictness of course is immediate.

The proof for relations is analogous. □

Strict *total* operations have a special interest. It turns out that they are also particularly easy to present. Let A be a presheaf viewed as a functor as in 4.8 (v). Any strict total operation on A given by $F : |A|^n \to |A|$ gives, for each $p \in \Omega$, a map $F_p : (A(p))^n \to A(p)$. These maps commute with the restriction maps, so we have a natural transformation between functors $F : A^n \to A$: The square

$$\begin{array}{ccc} A(p)^n & \xrightarrow{F_p} & A(p) \\ \downarrow & & \downarrow \\ A(q)^n & \xrightarrow[F_q]{} & A(q) \end{array}$$

commutes for $q \leq p$.

Any such natural transformation gives us a strict total operation: define this by $F(a_0,\ldots,a_{n-1}) = F_p(a_0 \restriction p,\ldots,a_{n-1} \restriction p)$ where $p = \bigwedge_{i<n} Ea_i$. For example, if we have a category C of algebras and homomorphisms any functor $F : \Omega^{op} \to C$ gives a presheaf (take underlying sets) equipped with the appropriate operations: if the restrictions are homomorphisms, then the operations are natural.

Before going on with the logic, which will enable us to say something more interesting about our examples, we pause to introduce morphisms.

5.6. MORPHISMS. Let A and B be presheaves over Ω . A <u>morphism</u> $f : A \to B$ of presheaves is a map $f : |A| \to |B|$ such that $Ef(a) = Ea$ and $f(a \restriction p) = f(a) \restriction p$ for $a \in |A|$ and $p \in \Omega$.

By our previous remarks, a morphism $f : A \to B$ is just a strict total operation from A to B . Every sheaf is in particular a presheaf. Thus we have a category $Sh(\Omega)$ of sheaves on Ω . We shall consider its properties from time to time. We note in passing that for $p \in \Omega$ we have a functor $Sh(\Omega) \to \mathcal{E}ns$ taking A to $A(p)$. We write Γ for the *global section functor* which arises when $p = \top$. To give a morphism $f : A \to B$ of presheaves is to give maps $f_p : A(p) \to B(p)$ for $p \in \Omega$, which commute with restrictions. Thus a morphism is just a natural transformation between A and B viewed as functors. If B is a sheaf, it suffices to give such maps for p in some basis for Ω (again commuting with restrictions): every element of $|A|$ is a join of elements a_i with Ea_i a member of the basis and joins in B are uniquely defined in terms of extents and restrictions.

5.7. PRESHEAVES AS SHEAVES (bis). We also have a category $\mathcal{P}sh(\Omega)$ of presheaves over Ω , of which $Sh(\Omega)$ is a full subcategory. We do not dwell on the properties of $\mathcal{P}sh(\Omega)$ as such, since it is equivalent to the sheaf category $Sh(K(\Omega))$ where $K(\Omega)$ is the cHa of downward closed subsets of Ω described in 4.19 (ii) . There it was shown that each Ω-presheaf A corresponds to an $K(\Omega)$-sheaf A^* with $A^*([p]) \cong A(p)$ for $p \in \Omega$. To see that we have the claimed equivalence of categories, it suffices to remark that the elements $[p]$ for $p \in \Omega$ form a basis for $K(\Omega)$.

If we tried to define a morphism of Ω-sets to be a strict total operations we should have an unwieldy notion (or at least an unwieldy category), since B may fail to have elements which morally should be there. Instead, we define a morphism of Ω-sets to be the *graph* of a strict total operation.

5.8. DEFINITION. A morphism $f : A \to B$ of Ω-sets is a relation $f : |A| \times |B| \to \Omega$ which is singlevalued $f(a,b) \wedge f'(a,b') \leq [\![b = b']\!]$, strict and total $[\![a = a]\!] = \bigvee \{ f(a,b) \mid b \in |B| \}$.

It is easily seen that any strict total operation f gives a morphism $f(a,b) = [\![f(a) = b]\!]$. This of course is how we think of any morphism even when there are not enough elements to give a value $f(a) \in |B|$ for each $a \in |A|$. With this in mind, we define the composite of two morphisms $f : A \to B$ and $g : B \to C$ by $g \circ f(a,c) = \bigvee \{ f(a,b) \wedge g(b,c) \mid b \in |B| \}$. Now the whole point of sheaves is that they do have the elements they ought to have so if A and B are sheaves, sheaf maps from A to B are exactly Ω-set morphisms from A to B.

5.9. THEOREM (Higgs [18]). The category of Ω-sets and the Ω-set morphisms is equivalent to $Sh(\Omega)$.

Proof. If B is a sheaf and $f : A \to B$ an Ω-set morphism then defining $f(a)$ to be the unique element of $|B|$ such that $f(a,b) = [\![f(a) = b]\!]$ for all $b \in |B|$ gives a strict total operation corresponding to f. Thus for sheaves our two notions of morphism are equivalent but by Lemma 5.3, morphisms $f : A \to B$ of Ω-sets are just the same as morphisms between their completions. $\square$

5.10. THEOREM. The category of sheaves over a topological space X is equivalent to the category of étale spaces over X with as morphisms $\pi_A : E_A \to X$ to $\pi_B : E_B \to X$, continuous maps $f : E_A \to E_B$ commuting with π.

Proof. Every such continuous map gives a sheaf map acting on sections by composition. Its inverse image $f^{-1} : \Pi_B \to \Pi_A$ is given by $[f^{-1}(P)](a) = P(f(a))$. Given a sheaf map $f : A \to B$ we can define an $\wedge\bigvee$- map $f^{-1} : \Pi_B \to \Pi_A$ in the same way and since sheaf maps preserve extents, this f^{-1} commutes with the maps π^{-1} and hence by duality arises from a continuous map commuting with π. But now every sheaf is the sheaf of sections of *some* étale space (4.21). $\square$

We continue our discussion by considering some particular categories of sheaves. Classically what we are about to look at is trivial. However, in §9 we shall apply these intuitionistic results to give information of interest to the purely classical mathematician.

5.11. THEOREM. If $\{\tau\}$ is a basis for Ω then $Sh(\Omega)$ is (equivalent to) a full subcategory of *Ens*, the category of sets.

Proof. Since global sections generate any Ω-sheaf, any morphism is determined by its action on global sections, and Γ , the global section functor, is faithful. Any map $f : \Gamma A \to \Gamma B$ on sections is extensional: since $[\![a = b]\!] = \top$ implies $a = b$ which leads to $f(a) = f(b)$ and hence $[\![f(a) = f(b)]\!] = \top$, we see that $[\![a = b]\!] \subseteq [\![f(a) = f(b)]\!]$. Thus Γ is full. $\square$

In particular, if $\Omega \cong P(\mathbb{1})$ then $Sh(\Omega) \cong \mathcal{E}ns$. In the case where Ω arises as a quotient of $P(\mathbb{1})$, we can characterize $Sh(\Omega)$ as a full subcategory of $\mathcal{E}ns$. Firstly we need some notation. Let j be a topology on $P(\mathbb{1})$ as in 2.21 . We shall say a property ϕ is *locally true* iff $0 \in j\{0 \mid \phi\}$ and may write $j\phi$ for " ϕ is locally true". We catalogue some evident properties for later use; three basic properties follow from the definition of topology:

(i) $\phi \to j\phi$;

(ii) $jj\phi \to j\phi$;

(iii) $(\phi \to \psi) \to (j\phi \to j\psi)$.

From these, the following facts follow formally:

$j(\phi \wedge \psi) \leftrightarrow j\phi \wedge j\psi$;

$j(\phi \to \psi) \to (j\phi \to j\psi)$;

$\exists x . j\phi \to j \exists x . \phi$;

$j \forall x \phi \to \forall x j\phi$.

We have stated these principles formally to save space. However, we shall use them informally in the proof of our next theorem.

5.12. THEOREM. A set A arises as the set of global sections of some Ω-sheaf, where $\Omega = P(\mathbb{1})/j$, iff

(*) for any $X \subseteq A$ we have $jE!x . x \in X \to E!x . j\, x \in X$.

In words: if locally there is a unique $x \in X$ then there is a unique x locally in X .

Proof. Suppose $A = \Gamma X$, where X is an Ω-sheaf, then for $x , y \in A$ we have $x = y$ iff $[\![x = y]\!] = \top$ iff $0 \in [\![x = y]\!]$; and, since $[\![x = y]\!]$ is a fixed point of j , this happens iff $0 \in j[\![x = y]\!]$ iff $j(0 \in [\![x = y]\!])$ iff $j\, x = y$. Thus,

(1) $\forall x , y \in A \; (jx = y \to x = y)$.

Certainly (1) follows from (*) (let $X = \{ x \mid x = y \}$). Thus it suffices to show that, for A satisfying (1) , condition (*) is equivalent to each inhabited singleton of A_Ω being realised uniquely by an element of A . So let A satisfy (1) . Now any subset $X \subseteq A$ gives a predicate on A_Ω :

$[\![x \in X]\!] = j\{0 \mid x \in X\}$ (or $[\![x \in X]\!] = \top$ iff $j\, x \in X$),

and every predicate arises in this way. For such a predicate to be a singleton we must have for $x, y \in X$ that

$$j\, x \in X \text{ and } j\, y \in X \rightarrow x = y \quad .$$

For this singleton to be inhabited

$$\bigvee_{x \in A} [\![x \in X]\!] = \top \quad (\text{or equivalently} \quad j\, \exists\, x \in A \,.\, x \in X)\,.$$

In the presence of (1) these are equivalent to $j E I x\ x \in X$.

Now given such a singleton, it is uniquely realised by an element x of A

iff $[\![x = y]\!] = [\![y \in X]\!]$ for $y \in A$

iff $[\![x = y]\!] = \top$ iff $j\, y \in X$ for $y \in A$

iff $x = y$ iff $j\, y \in X$ for $y \in A$;

(that is, iff x is the unique x such that $j\, x \in X$). □

We shall interpret this theorem later in an elementary topos.

We shall now, finally, see how Ω-sets, structured by operations and relations, may be used to interpret first-order logic. We begin by considering a language without descriptions but with an existence predicate as discussed in 0.1 and equality (see 0.2). We allow many sorts and infinitary disjunctions into our languages. This will allow us to say more about our models than we could otherwise. Given a first-order language L, an Ω-<u>structure</u> A suitable for L is a collection of Ω-sets, one for each sort of L, equipped with operations and relations of the appropriate types, to correspond with those of L. In general, we abuse notation by using the same symbol for a sort, operation or relation and its interpretation.

5.13. DEFINITION. Let A be an Ω-structure suitable for L. The language $L(A)$ is obtained by adding a constant of sort A for each element of $|A|$ (again we use the same symbol for an element and its name). For each sentence ϕ and closed term τ of $L(A)$ we define a value $[\![\phi]\!] \in \Omega$ and $[\![\tau]\!] \in |A|$ (where τ is of sort A).

SENTENCES

$$[\![R(\tau_0, \ldots, \tau_{n-1})]\!] = R([\![\tau_0]\!], \ldots, [\![\tau_{n-1}]\!])$$

$$[\![\tau = \sigma]\!] = [\![\, [\![\tau]\!] = [\![\sigma]\!]\,]\!]$$

$$[\![E\tau]\!] = E[\![\tau]\!]$$

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!] \qquad (\ \vee\ ,\ \rightarrow\ ,\ \neg\ \ \text{similarly})$$

$$[\![\forall x \in A \,.\, \phi(x)]\!] = \bigwedge \{ [\![Ea \to \phi(a)]\!] \mid a \in |A| \}$$

$$[\![\exists x \in A \,.\, \phi(x)]\!] = \bigvee \{ [\![Ea \wedge \phi(a)]\!] \mid a \in |A| \}$$

$$[\![\bigvee_{i \in I} \phi_i]\!] = \bigvee \{ [\![\phi_i]\!] \mid i \in I \}$$

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$$[\![a]\!] = a \quad \text{for } a \in |A|$$

$$[\![F(\tau_0,\ldots,\tau_{n-1})]\!] = F([\![\tau_0]\!],\ldots,[\![\tau_{n-1}]\!]) \ .$$

We say a formula $\phi(x_0,\ldots,x_{n-1})$ of L is <u>valid</u> in A (equivalently, A <u>satisfies</u> ϕ, or, in symbols $A \models \phi$) iff for every $a_0 \in |A_0|,\ldots,a_{n-1} \in |A_{n-1}|$ (of the appropriate sorts) we have $[\![\phi(a_0,\ldots,a_{n-1})]\!] = \top$. In particular, if ϕ is a sentence, $A \models \phi$ iff $[\![\phi]\!] = \top$.

5.14. THEOREM (Soundness). Let A be an Ω-structure suitable for L . The axioms of first-order logic are valid in A . Validity is preserved by the rules of first-order logic. In particular, if ϕ is provable ($\vdash \phi$) then ϕ is valid in every interpretation ($\models \phi$). Furthermore, $A \models \bigvee_{i \in I} \phi_i \to \psi$ iff $A \models \phi_i \to \psi$ for each $i \in I$.

Proof. It is well known that Heyting algebras soundly model propositional intuitionistic logic (Dummett [6]). It is also immediate from our definitions that the quantifier rules (see Ch. 0 , 1.1, 1.2, 1.4) and the rules of equality (0, 2.1, 2.2) are valid, as is our last claim. □

5.15. EXAMPLES.

(i) *The Pierce representation* (see [37] and [41]). Returning to example (iii) of 4.19 , we view a ring A as an $I(B)$-set using the cHa of ideals of the Boolean algebra B of central idempotents of A with the equality given by $[\![a = b]\!] = \{ e \in B \mid ea = eb \}$ for $a, b \in A$. In 5.2 we saw that the ring operations are strict total operations on this $I(B)$-set $k(A)$. Now, trivially, *viewed as an* $I(B)$-*structure*, $k(A)$ is a ring. So, what have we gained? We have some new properties. For example:

$$\models \ \forall x \, [x = x^2 \to x = 0 \vee x = 1] \ .$$

To check this out, suppose $ea = ea^2$ then $ea \in B$ and also $e - ea \in B$. Now $(e - ea)a = 0$ and $ea \cdot a = ea \cdot 1$, so $(e - ea) \in [\![a = 0]\!]$ and $ea \in [\![a = 1]\!]$. Thus $e \in [\![a = 0 \vee a = 1]\!]$.

We can write down other pleasant properties we might like our $I(B)$-ring to have. We list three, each one stronger (more demanding) than the one before:

(a) $\models \forall x\, [x = 0 \vee \quad x = 0]$

(b) $\models \forall x\, [x = 0 \vee \forall y\, [x . y = 0 \to y = 0]]$

(c) $\models \forall x\, [x = 0 \vee \exists y\, [x . y = 1]]$.

These, we do not get for nothing. We leave the reader to check that they are equivalent to the following requirements on the original ring A . In order:

(a)' $\forall x\, \exists e\, \forall e'\, [x \cdot e = x \wedge [x \cdot e' = 0 \to e \cdot e' = 0]]$

(b)' $\forall x\, \exists e\, \forall y\, [x \cdot e = x \wedge [x \cdot y = 0 \to y = 0]]$

(c)' $\forall x\, \exists y\, [x \cdot y \cdot x = x]$.

(A ring satisfying this last condition is said to be *regular*.)

(ii) For this example let $\Omega = \mathcal{O}(X)$ be the opens of a topological space. The sheaf R_X equipped with the operations + , × introduced in 5.2 is a ring. Furthermore, these ring operations are strictly extensional with respect to the apartness relation $\neq$ that is $\models a + b \neq a' + b \to a \neq a'$ (and similarly for multiplication). The apartness on this sheaf is a tight apartness in the sense of 0.4 ; in particular, $\models \forall x, y\, [\neg x \neq y \to x = y]$. In general of course $\not\models \forall x, y\, [x = y \vee \neg x = y]$. To be definite, consider the case where our underlying space is the real line $X = \mathbb{R}$, then, for $s \in \mathbb{R}$, define $a : \mathbb{R} \to \mathbb{R}$ by $a(t) = \max(0, t-a)$. We have $[\![a = 0]\!] = \{ t \mid t < s \}$ and $[\![\neg a = 0]\!] = \{ t \mid t > s \}$ thus, $s \notin [\![a = 0 \vee \neg a = 0]\!]$. In fact $\models \neg \forall x\, (x = 0 \vee \neg x = 0)$.

(iii) Now take $\Omega = \mathcal{O}(\mathbb{C})$ opens of the complex plane and consider the sheaf of holomorphic functions on H . If two holomorphic functions $f, g : U \to \mathbb{C}$ agree on some inhabited open set $V \subseteq U$ then they are equal intuitionistically just as classically (see Rousseau [43] for a constructive treatment of complex analysis). Thus $\models \forall x, y\, (x = y \vee \neg x = y)$. However, the theory of equality in this sheaf is not entirely classical. We have a relation $[\![a \neq b]\!] = \{ t \mid a(t) \neq b(t) \}$, which is a tight apartness relation, and it is easy to check that $\models \neg \forall x\, (x = 0 \vee x \neq 0)$.

(iv) For another algebraic example let A be a commutative ring with 1 and define the $\sqrt{A}$-set with

$$[\![a = b]\!] = \{ t \mid t^n a = t^n b \text{ for some } n > 0 \}$$

as in 4.19 . The operations inherited from A make this set into a ring. We now define a $\sqrt{A}$-valued subset by

$$[\![a \in M]\!] = \{ t \mid t^n = ar \text{ for some } n > 0 \text{ and } r \in A \} .$$

Note that if $t^n a = t^n b$ and $t^m = ar$, then $t^{n+m} = b(rt^n)$; so this subset is =-extensional. Since it is defined by a strict relation it is $\equiv$-extensional. Here are some of its properties :

(a) $\models\ 1\in M \wedge \neg\, 0\in M$

(b) $\models\ ab\in M \to a\in M$

(c) $\models\ a+b\in M \to a\in M \vee b\in M$

(d) $\models\ a\in M \wedge b\in M \to ab\in M$

(e) $\models\ \neg\, a\in M \leftrightarrow \bigvee_{n\in\mathbb{N}} a^n = 0$.

Classically, the first three properties describe the complement of an ideal and the fourth says that the ideal in question is prime. Intuitionistically there is no straightforward duality between ideals and their complements. In this case the dual notion seems more useful. We say M is a prime coideal or coprime. Putting $a\neq b$ iff $(a-b)\in M$ gives an apartness on M (this follows formally from the properties given), and our last property shows just how close this comes to being a tight apartness. Now to see that the properties hold. Obviously $[\![1\in M]\!] = A = \top$ in $\sqrt{A}$, and $[\![0\in M]\!]$ is just the nilradical $\bot\in\sqrt{A}$. Suppose $t^n = abr$; then $t^n = a(br)$, so the second property holds trivially. For the third property: if $t^n = (a+b)r$, we must express some power of t as $s_1 + s_2$, where $s_1^{n_1} = ar_1$ and $s_2^{n_2} = br_2$. This is already done: $t^n = ar + br$. If $t^n = ar$ and $t^m = bs$, then $t^{n+m} = ab(rs)$, so the fourth property holds. Now write $\sqrt{(a)}$ for the radical of $a\in A$. We have $t\in[\![\neg\, a\in M]\!]$ iff $\sqrt{(t)} \le \sqrt{(a)}$ iff $\sqrt{(t)}\wedge\sqrt{(a)} = \sqrt{(0)}$, but this happens if and only if at is nilpotent (in which case $t^n a^n = (at)^n = 0$); so $t\in[\![a^n = 0]\!]$ for some $n>0$. Also, of course, if $t^m a^n = 0$, then at is nilpotent.

We shall in general want to use interpretations in sheaves or complete Ω-sets in order to interpret descriptions. Often, as we have seen, it is simplest to arrive at a sheaf via a presentation of some generating Ω-set. Normally it is not straightforward to give an explicit description of the sections of the resulting sheaf. We therefore need to see how to use a generating set of sections to interpret the logic.

5.16. LEMMA. Let $B\subseteq A$ be a generating sub-Ω-set of A then

$$[\![\forall x\in A\,.\,\phi(x)]\!] = [\![\forall x\in B\,.\,\phi(x)]\!] \quad ;$$

$$[\![\exists x\in A\,.\,\phi(x)]\!] = [\![\exists x\in B\,.\,\phi(x)]\!] \quad .$$

Proof. Certainly $\bigwedge_{a\in|A|} [\![Ea \to \phi(a)]\!] \le \bigwedge_{b\in|B|} [\![Eb \to \phi(b)]\!]$. Now $\bigwedge_{b\in|B|} [\![Eb \to \phi(b)]\!] \wedge [\![a=d]\!] \le [\![\phi(a)]\!]$ for each $d\in|B|$ and $a\in|A|$. So

$$\bigwedge_{b\in|B|} [\![Eb \to \phi(b)]\!] \wedge \bigwedge_{d\in|B|} [\![a=d]\!] = [\![\forall x\in B\,.\,\phi(x)]\!] \wedge Ea \le [\![\phi(a)]\!] .$$

Thus $[\![\forall x\in B\,.\,\phi(x)]\!] = [\![\forall x\in A\,.\,\phi(x)]\!]$. The argument for $\exists$ is similar, but easier. □

5.17. DEFINITION. We say an Ω-structure A is <u>complete</u> iff each sort of L is interpreted by a complete Ω-set or sheaf.

We have seen that any Ω-set can be completed 4.17 and that any operations or relations can be extended to this completion (5.3). Thus, given any Ω-structure A, we can define its completion $\hat{A}$ in which we interpret each sort by the completion $\hat{A}$ of its interpretation A in A and each operation and relation by the extension (defined in 5.3) of the corresponding operation or relation in A.

5.18. THEOREM. For any closed term τ or sentence ϕ of $L(A)$ we have
$[\![\tau]\!]_{\hat{A}} = [\![\tau]\!]_A$ and $[\![\phi]\!]_{\hat{A}} = [\![\phi]\!]_A$.

Proof. This is by induction on the structure of τ and ϕ. Each step in the induction is trivial, with the exception of the quantifiers. These are dealt with by Lemma 5.16. $\square$

This theorem will allow us to use interpretations in sheaves while using only a generating set of sections to do our calculations. A sheaf or complete Ω-set has all the elements it should have. Note that for any formula $\phi(x)$ with one free variable, the map $a \mapsto [\![Ea \wedge \forall x\,(\phi(x) \leftrightarrow x = a)]\!]$ is a singleton:

$$[\![Eb \wedge \forall x\,(\phi(x) \leftrightarrow x = b)]\!] \leq [\![Eb \wedge \phi(b)]\!]$$

$$[\![\forall x\,(\phi(x) \leftrightarrow x = a)]\!] \wedge [\![Eb \wedge \phi(b)]\!] \leq [\![a = b]\!] \quad .$$

This allows us to interpret descriptions in any complete structure A.

5.19. DEFINITION. Let A be a complete Ω-structure for L. We extend the definition of valuation (5.13) to include descriptions (as described in 0.6) by defining $[\![Ix.\phi(x)]\!]$ to be the unique $b \in |A|$ such that for all $a \in |A|$ we have

$$[\![a = b]\!] = [\![Ea \wedge \forall x\,(\phi(x) \leftrightarrow x = a)]\!] \ .$$

5.20. THEOREM. The axioms and rules of first-order logic with descriptions are valid in any interpretation in a complete Ω-structure.

Proof. We need only check the axiom for descriptions (0.6.1). By definition, for $a \in |A|$ we have

$$[\![a = Ix.\phi(x)]\!] = [\![Ea \wedge \forall x\,(\phi(x) \leftrightarrow x = a)]\!] \quad .$$

So, since $=$ is strict,

$$Ea \rightarrow [a = Ix\,\phi(x) \leftrightarrow \forall x\,(\phi(x) \leftrightarrow x = a)] \quad ,$$

and, as this holds for any a, we have

$$\forall\, y\, [y = \mathrm{I}x\, \phi(x) \leftrightarrow \forall\, x\, (\phi(x) \leftrightarrow x = y)] \quad . \qquad \Box$$

5.21. EXAMPLES. We now show how internal and external properties of a sheaf may be related (see [9] and [39]).

(i) *Terminal objects.* A sheaf A is a terminal object in $Sh(\Omega)$ iff $\models \mathrm{EI}x \in A\,.\, x = x$. Suppose $\models \mathrm{EI}x \in A\,.\, x = x$ then $[\![\mathrm{I}x \in A\,.\, x = x]\!]$ is a global section $*$ of A such that, for any section $a \in |A|$, we have $Ea = [\![a = *]\!]$; that is, $a = * \restriction Ea$. hus A is certainly terminal: take $b \in |B|$ to $* \restriction Eb$. Conversely, if A is terminal, we certainly have at least one global section $*$ (map any sheaf with a global section to A). Now the identity $A \to A$ must be the same map as $a \to * \restriction Ea$; so $a = * \restriction Ea$ and $* = [\![\mathrm{I}x \quad x = x]\!]$.

(ii) *Finite products.* The cartesian product of two presheaves was dealt with briefly in 4.8 . We take another look. Let A_i be a presheaf for $i < n$. View each A_i as a functor and define $(\mathsf{X}_{i<n} A_i)(p) = \mathsf{X}_{i<n} (A_i(p))$ for $p \in \Omega$ with the obvious restrictions. With this definition it is easy to see that if each A_i is separated so is $\mathsf{X}_{i<n} A_i$ and that if each A_i is a sheaf, so is $\mathsf{X}_{i<n} A_i$. The maps giving tuples and projections are natural so we have strict total operations $\langle \ldots \rangle$ and π_i . It is also obvious that this construction gives a product either in the category of presheaves, or in $Sh(\Omega)$. Furthermore

(1) $\models \forall\, y \in \mathsf{X} A_i\,.\, y = \langle \pi_0 y, \ldots, \pi_{n-1} y \rangle$

(2) $\models \forall\, x_0 \in A_0, \ldots, x_{n-1} \in A_{n-1}\,.\, x_i = \pi_i \langle x_0, \ldots, x_{n-1} \rangle$.

It is easy to verify these, but we do not go through the details as their validity follows from the general result in 6.4 . In fact a categorical product in $Sh(\Omega)$ is completely characterized by these logical properties: Given $f_i : C \to A_i$ we can *define* the required morphism $\langle f_0, \ldots, f_{n-1} \rangle(c) = \langle f_0(c), \ldots, f_{n-1}(c) \rangle = [\![\mathrm{I}y. \bigwedge \pi_i(y) = f_i(c)]\!]$, thanks to the logical properties. In the other direction we can argue that since all products are isomorphic, they must share the same logical properties.

(iii) *Equalizers.* A diagram

$$C \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

is an equalizer in $Sh(\Omega)$ iff $\models \forall\, a \in A\, [f(a) = g(a) \to \mathrm{EI}c \in C\,.\, e(c) = a]$ *and* $\models \forall\, c \in C\, [f(e(c)) = g(e(c))]$. The latter condition tells us that our diagram commutes, and the former shows us how to construct the unique factorization of any e' with $f \circ e' = g \circ e'$; so the conditions imply that we have an equalizer. Conversely, it is again easiest to consider some particular equalizer of f and g and appeal to the uniqueness up to isomorphism. Let C be the subsheaf of A

with $|C| = \{ a \in |A| \mid f(a) = g(a) \}$ obviously we have an equalizer since we have taken the equalizer in *Sets* of f and g viewed as maps on sections. It is straightforward to check that in this case our formulae are valid.

We shall later (6.10) use the fact that the categorical properties of finite limits are simply characterized in the internal logic.

(iv) *Functions.* We can use descriptions to define (partial) functions. Let X be a topological space, R_X the sheaf of germs of continuous real functions on X structured as a ring as in 5.15. If we write $x^{-1} \equiv Iy\ x \times y = 1$ then for $a : U \to \mathbb{R}$ a section of R_X we have a section a^{-1} with $Ea^{-1} = \{ t \in U \mid a(t) \neq 0 \}$ and $a^{-1}(t) = 1/a(t)$ for $t \in Ea^{-1}$. This structure makes R_X almost a field

$$\models \ \forall x\,[x \neq 0 \leftrightarrow Ex^{-1}] \ . \qquad \Box$$

We state but do not prove a completeness theorem for sheaf models. A standard proof would give a completeness theorem for Ω-sets and the theory of the last two sections shows how these may be completed. For a fuller discussion of this point see Scott [46] §5.3.

5.22. THEOREM. If ϕ_i for $i \in I$ is a collection of sentences and ψ any formula, then $\{ \phi_i \mid i \in I \} \vdash \psi$ iff, for every sheaf model A, if $A \models \phi_i$ for all $i \in I$, then $A \models \psi$.

6. CHANGE OF BASE.

Thus far, we have considered sheaves over a fixed (but arbitrary) cHa. Here we consider a morphism $f^* : \Omega \to \Omega'$ of cHa and the interplay it induces between models over Ω and models over Ω'. This interplay introduces an aspect not found in classical model theory, a thoroughgoing pursuit of which leads inevitably to sites (see Makkai and Reyes [34]). For technical reasons (which will be explained) we restrict ourselves to *strict* operations and relations for the whole of this section. We start with a general definition. We are primarily interested in two special cases which will enable us to relate the properties of a sheaf to the properties of its stalks and sections.

6.1. DEFINITION. Let A be an Ω-set and $g : \Omega \to \Omega'$ a map between cHa preserving finite $\wedge$. We define an Ω'-set $g(A)$ by $|g(A)| = |A|$ and $[\![a = b]\!]_{g(A)} = g[\![a = b]\!]_A$. Any strict operation on A becomes a strict operation on $g(A)$ since, for strict operations, $\equiv$-extensionality follows from $=$-extensionality. (This was our reason for restricting our attention to strict operations.) Similarly any strict relation on A gives a strict relation on $g(A)$ by $[\![R(a_0,\dots,a_{n-1})]\!]_{g(A)} =$

$g[\![R(a_0,\ldots,a_{n-1})]\!]$. Thus to any structure A over Ω we associate a structure $g(A)$ over Ω' . In general, $g(A)$ is not complete, by abuse we also call its completion $g(A)$. Furthermore, since any strict total operation on A gives a strict total operation on $g(A)$, we have a functor $g : Sh(\Omega) \to Sh(\Omega')$. □

6.2. STALKS AND SECTIONS. (i) If X is a topological space any point $t \in X$ gives an $\wedge\bigvee$- map $t : \mathcal{O}(X) \to P(1\!\!1)$. For any $\mathcal{O}(X)$-structure A , we call the $P(1\!\!1)$-structure $t(A)$ the *stalk* of A at t . We also write A_t for $t(A)$ in this special case as this is the customary notation .

(ii) For any cHa Ω and $p \in \Omega$ we define an $\wedge$-map $p : \Omega \to P(1\!\!1)$ by $p(q) = \{ 0 \mid p \le q \}$. For an Ω-structure A , we call $p(A)$ the *structure of sections* over p. n this case it is customary to write $A(p)$ for $p(A)$.

(iii) A $P(1\!\!1)$-set A is generated by its global sections and operations and relations on A correspond exactly to ordinary operations and relations on this set of global sections. Thus $P(1\!\!1)$-structures correspond to structures in the usual sense. The global sections of A_t can be identified with the elements of the fibre over t of the étale space representing A (cf. 4.22), since $0 \in t[\![a=b]\!]$ iff $t \in [\![a=b]\!]$ iff $a(t) = b(t)$. The global sections of $p(A)$ may be identified with $A(p)$ (cf. 4.8 (v)) because $0 \in p[\![a=b]\!]$ iff $p \le [\![a=b]\!]$. □

We return to the general situation of 6.1 to see what can be said about the properties of $g(A)$. We say a formula is <u>basic</u> if it involves only $\top$, $\wedge$, $=$ and the primitive (strict) operations and relations. A <u>Horn formula</u> is one of the form $\forall \bar{x} \, (\phi(\bar{x}) \to \psi(\bar{x}))$ where ϕ and ψ are basic. A <u>Horn theory</u> is one all of whose axioms are (provably equivalent to) Horn sentences.

6.3. THEOREM. If ϕ is a basic sentence of $L(A)$, then $[\![\phi]\!]_{g(A)} = g[\![\phi]\!]_A$. Thus if θ is a Horn sentence and $A \models \theta$, then $g(A) \models \theta$; and if A is a model for some Horn theory, so is $g(A)$.

Proof. This is really all immediate: the first part holds for basic atomic sentences by definition, and for the rest because g preserves $\wedge$. The second and third parts follow as g is order preserving. □

In particular, we can see from 6.3 that if A is a model for a Horn theory T , so is the structure of sections $A(p)$ for each $p \in \Omega$. In fact, this is a sufficient condition for A to be a model for T .

6.4. THEOREM. Let A be an Ω-structure for L then for θ a Horn sentence (say $\forall \bar{x} \, (\phi(\bar{x}) \to \psi(\bar{x}))$), $A \models \theta$ iff for each $p \in \Omega$ the structure $A(p)$ of sections over p satisfies θ .

Proof. For each sequence $\bar{a}$ of elements of the appropriate sorts let $p \le \bigwedge_i Ea_i$. So the a_i may be regarded as naming elements of $A(p)$, and we have both:

$A(p) \models \phi(\bar{a})$ iff $p \le [\![\phi(\bar{a})]\!]$, and

$A(p) \models \psi(\bar{a})$ iff $p \le [\![\psi(\bar{a})]\!]$.

So if each $A(p)$ satisfies θ, we see that $A \models \forall \bar{x} \, (\phi(\bar{x}) \to \psi(\bar{x}))$. □

A special case deserves mention. Suppose A is a sheaf and on each set of sections $A(p)$ we have some algebraic structure compatible with restrictions. (So that A is a functor from Ω^{op} into some category of algebras.) This gives us a structure of strict total functions on A, and passing to structures of sections we regain the original structure on each $A(p)$. Thus, if each $A(p)$ is a model for some Horn theory, so is A. In particular, if the $A(p)$ all belong to some equational variety (say all are groups), so does A.

The examples we gave in 6.2 are rather special. More generally, we can consider an $\wedge,\bigvee$- map $f^* : \Omega \to \Omega'$; this is certainly an $\wedge$ map. Furthermore, as explained in 2.4, it determines an $\bigwedge$- map $f_* : \Omega' \to \Omega$ by the relationship $f^*p \le p'$ iff $p \le f_*p'$. Thus, we have a pair of $\wedge$-maps to which we now apply Definition 6.1.

6.4. INVERSE AND DIRECT IMAGES. The terminology here is, at first sight, rather confusing: Think of the $\wedge,\bigvee$- map $f^* : \Omega \to \Omega'$ as a "geometric map" going in the opposite direction. We call $f^*(A)$ the <u>inverse image</u> of A. In the special case where $f^* : \Omega \to \Omega/J$ is a quotient map, we write A/J for f_*A. Still thinking geometrically, we call $f_*(A)$ the <u>direct image</u> of A. Direct images are actually somewhat tricky to deal with. It will be useful to have an alternative description of them in terms of presheaves. For this purpose, let us view presheaves again as functors as in 4.8 (v). Since f^* is order preserving, it may be regarded as a functor. Given a presheaf $A : \Omega'^{op} \to \mathit{Sets}$, we can compose with f^* (or strictly speaking, its opposite) to obtain $f_*A : \Omega^{op} \to \Omega'^{op} \to \mathit{Sets}$, with $[f_*(A)](p) = A(f^*(p))$, for $p \in \Omega$, and the restriction maps inherited from A. Now if $p = \bigvee p_i$ in Ω, then $f^*p = \bigvee f^*p_i$ in Ω'; so that if A is separated, so is f_*A. The map $f_*A(p) \to \times_{i\in I} f_*A(p_i)$ is exactly the map $A(f^*p) \to \times_{i\in I} A(f^*p_i)$. Furthermore, since f^* preserves $\wedge$, we see that $\varprojlim_{[p_i]} f_*A(p_i) = \varprojlim_{[f^*p_i]} A(f^*p_i)$; and so, if A is a sheaf, so is B. Now let A be a separated presheaf over Ω', and for $f^* : \Omega \to \Omega'$ construct B, a presheaf over Ω, as above. We claim that f_*A can be embedded as a generating sub-Ω-set in B. For each $a \in |f_*A|$ with $Ea = f_*q'$ we have a section $a \in B(f_*q')$ corresponding to the restriction of b to $f^*f_*q' \le q'$ in A. Now such sections generate B: for any $p \in \Omega$ the restriction map $B(f_*f^*p) \to B(p)$ is an isomorphism because $f^*f_*f^*p = f^*p$. Thus it suffices to show that for these generating sections, $[\![a = b]\!]_B = [\![a = b]\!]_{f_*A} =$

$f_*[\![a=b]\!]_A$. Now, for $p \le Ea \wedge Eb \in \Omega$, we have $p \le f_*[\![a=b]\!]_A$ iff $f^*p \le [\![a=b]\!]_A$ iff $a \restriction f^*p = b \restriction f^*p$ (since A is separated) iff $a \restriction p = b \restriction p$ in B . □

Because f^* preserves $\bigvee$, the inverse image preserves much more than the basic sentences mentioned in 6.3 . Before seeing what is preserved, we look again at an example.

6.5. EXAMPLE. For any cHa Ω we have an $\wedge,\bigvee$- map $P(\mathbb{1}) \to \Omega$. As usual, we identify $P(\mathbb{1})$-structures with ordinary structures. Here, inverse image gives just the structure on constant sheaves described in 5.2 (ii) . Direct image corresponds to taking global sections as in 6.2 (ii) . □

6.6. GEOMETRIC FORMULAE. We say a formula is <u>positive</u> if it involves only $\top$, $\bot$, $\wedge$, $\bigvee$, $\exists$, $=$ and the primitive operations and relations. A <u>geometric formula</u> is one of the form $\forall \bar{x} \, (\phi \to \psi)$ where ϕ and ψ are positive. A theory T is geometric if each of its axioms is (provably equivalent to) a geometric sentence. □

Now let $f^* : \Omega \to \Omega'$ be an $\wedge,\bigvee$- map.

6.7. THEOREM. Let A be an Ω-structure for L . For ϕ a positive sentence of $L(A)$ we have

$$[\![\phi]\!]_{f^*A} = f^*[\![\phi]\!]_A \ .$$

Proof. An easy induction on the structure of ϕ . □

6.8. COROLLARY. For θ a geometric sentence, if $A \models \theta$, then $f^*A \models \theta$. Hence, if T is a geometric theory and A is a model of T, then so is f^*A . Furthermore, if f^* is an embedding, then $A \models \theta$ iff $f^*A \models \theta$. □

If we have enough points, then we can test the validity of a geometric sentence by testing it at each point.

6.9. THEOREM. Let A be an $O(X)$-structure for L, then $A \models \forall \bar{x} \, (\phi(\bar{x}) \to \psi(\bar{x}))$, a geometric sentence, iff for each $t \in X$ we have $A_t \models \forall \bar{x} \, (\phi(\bar{x}) \to \psi(\bar{x}))$.

Proof. For $t \in X$ and for each sequence a of elements of the appropriate sorts we have $t \in [\![\phi(a)]\!]$ iff $A_t \models \phi(a)$, and $t \in [\![\phi(a)]\!]$ iff $A_t \models \phi(a)$. Thus, if for each t we have $A_t \models \forall \bar{x} \, (\phi(\bar{x}) \to \psi(\bar{x}))$, then $[\![\phi(a)]\!] \subseteq [\![\psi(a)]\!]$; and we are done. □

So much for the logical properties of f^* . It also has interesting categorical properties which we deduce in the main from 6.8 .

6.10. THEOREM. The inverse image functor f^* is a left exact left adjoint to f_*.

Proof. To see that f_* is left exact, observe that the formulae characterizing finite limits, given in 5.15, are geometric. We must now convince ourselves that for A an Ω-set and B an Ω'-set, we have a natural isomorphism $[f^*A, B] \cong [A, f_*B]$. Using the fact that morphisms to a separated presheaf are uniquely determined by their action on a generating set, we see that a morphism from f^*A to B is just a map ϕ assigning to each section of A a section of B subject to the conditions: $E\phi(a) = f^*Ea$ and $f^*[\![a = a']\!] \leq [\![\phi(a) = \phi(a')]\!]$. But, if B is a sheaf, then a morphism from A to f_*B is just a map ϕ assigning to each section a of A a section of B over f^*Ea such that $[\![a = a']\!] \leq f_*[\![\phi(a) = \phi(a')]\!]$. By the adjointness relating f^* and f_* these are equivalent. □

Two special cases of this adjointness are worth mentioning. The $\wedge\bigvee$-map $P(\mathbb{1}) \to \Omega$ gives us an adjoint pair with left exact left adjoint $\mathit{Sets} \to Sh(\Omega)$ "taking constant sheaves" and right adjoint $\Gamma : Sh(\Omega) \to \mathit{Sets}$ "global sections". In §9 we shall see how the general situation may be retrieved from this special case by the technique of relativization. Another case is given by the $\wedge\bigvee$-map $K(\Omega) \to \Omega$ of 2.9. We have already remarked that $Sh(K(\Omega))$ is just the category of presheaves over Ω. Here the left adjoint functor may be viewed as "sheafification" and the right adjoint as the forgetful functor. Since we can easily form quotients of a cHa giving an $\wedge\bigvee$-map $\Omega \to \Omega/J$, Theorem 6.9 gives us a method of constructing models with particular geometric properties. This technique of *geometric forcing* is best described by means of an example. Let A be a commutative ring with 1. We set off now on a road which will lead to the spectrum of A.

Let Ω be the cHa of *multiplicative subsets* of A. That is

$$\Omega = \{ M \subseteq A \mid \text{for } a \in A \text{ we have } aM \subseteq M \} .$$

We equip the constant sheaf A_Ω with a non-standard relation of *separation* by defining $[\![a \neq b]\!] = (a - b)A$ for $a, b \in A$. We shall also make use of the ring structure on A_Ω. Now the following formulae are valid:

(i) $\models 0 \neq 1$

(ii) $\models \forall x, y, z \, (x + z \neq y + z \leftrightarrow x \neq y)$

(iii) $\models \forall x, y, z \, (x.z \neq y.z \to x \neq y)$.

Suppose for a moment that we have an $\wedge\bigvee$-map $f : \Omega \to \Omega'$. The inverse image structure $f(A)$ will again be a ring and have the properties (i) - (iii) above, since these are all geometric properties. We now ask ourselves when $f(A)$ will also have the following geometric properties:

(iv) $\forall\, x\, (\neg\, x \neq x)$

(v) $\forall\, x\, , y\, , z\, (x \neq y \to x = 0 \vee y \neq 0)$

(vi) $\forall\, x\, , y\, (x \neq 0 \wedge y \neq 0 \to x \cdot y \neq 0)$?

In answer, Theorem 6.7, gives us a condition on f for each sentence:

(iv)' $f(0) \leq f\bot$

(v)' $f[\![a \neq b]\!] \leq f[\![a \neq 0]\!] \vee f[\![b \neq 0]\!]$ for $a\, , b \in A$

(vi)' $f[\![a \neq 0]\!] \wedge f[\![b \neq 0]\!] \leq f[\![a \cdot b \neq 0]\!]$ for $a\, , b \in A$.

There is a universal solution to the problem of finding such an f . We simply form the $\wedge,\bigvee$- quotient of Ω by the least J map such that

(iv)" $\{0\} \leq J\bot$

(v)" $[\![a \neq b]\!] \leq J([\![a = 0]\!] \cup [\![b \neq 0]\!])$

(vi)" $[\![a \neq 0]\!] \wedge [\![b \neq 0]\!] \leq J[\![a \cdot b \neq 0]\!]$.

Then $f(A)$ will satisfy (iv) - (vi) iff f factors through $\Omega \to \Omega/J$. We now describe the J which forces our three properties more concretely.

6.11. LEMMA. The fixed points of J are the radical ideals of A .

Proof. We saw in 2.15 that the radical ideals of A form a cHa;thus,it suffices to show that (iv)" - (vi)" hold iff every fixed point of J is a radical ideal. Our first condition, (iv)" , will hold iff every fixed point contains $\{0\}$. Our second condition is that $(a-b)A \subseteq J(aA \cap bA)$: this will hold iff whenever a and b belong to a fixed point M then $(a-b) \in M$. Finally $aA \cap bA \subseteq JabA$ iff, given $ab \in M$, a fixed point, and $x\, , y \in A$ such that $ax = by$, we have $ax \in M$. But this happens iff whenever $c^2 \in M$ then $c \in M$. (In one direction, let $c = ax = by$; in the other, $x = y = 1$.) □

We see that A/J is a $\sqrt{A}$-ring equipped with a multiplicatively closed subset $S = \{ x \in A \mid x \neq 0 \}$. Of course we cannot strictly talk of subsets in this way until §7 ; however, we can formally form the corresponding ring of fractions and embed a in it $a \mapsto a/1$. The well known condition $a/1 = b/1$ iff $\exists\, c \in S\ (ac = bc)$ tells us that in the ring of fractions we shall have

$$[\![a/1 = b/1]\!] = [\![\exists\, c \neq 0\ (ac = bc)]\!] = \{ c \mid \exists\, n\ \ ac^n = bc^n \} .$$

This is reminiscent of 4.19 (iv) . We shall call our ring of fractions Spec(A) the <u>spectrum</u> of A . Now to the definition:

$$|\mathrm{Spec}(A)| = \{ ab^{-1} \mid a\, , b \in A \} .$$

Here we regard ab^{-1} as a formal symbol - strictly speaking this is just a set of ordered pairs-but the present notation is suggestive.

$$[\![ab^{-1} = cd^{-1}]\!] \;=\; [\![b \neq 0 \wedge d \neq 0 \wedge \exists\, e \neq 0 \;\; e(ad - bc) = 0]\!] \;.$$

Strictly speaking this is just a presentation of Spec(A) which is the completion of this $\surd A$-set. In the manner of 4.19 we can identify at least some of the sections of Spec(A) . We write $\surd(a) = \{ x \in A \mid \exists\, n \,.\, x^n \in aA \}$. for the radical of $a \in A$. These ideals form a basis for $\surd A$.

6.12. THEOREM. $\mathrm{Spec}(A)(\surd a) = A[a^{-1}]$.

Proof (see EGA [17]). Suppose s is a singleton and $Es = \bigvee \{ s(ab^{-1}) \mid a, b \in A \}$, then we can write $a^k = \Sigma_{i<n} u_i$, where $u_i \in s(ab^{-1})$. Hence we can write $u_i^m = r_i b_i$ and $(u_i u_j)^m (a_i b_j - a_j b_i) = 0$ for some large enough m . Write v_j for u_j^m . Taking a high enough power and suitable coefficients, we can now write $a^k = \Sigma\, z_i v_i b_i$. Let $b = \Sigma\, z_i v_i a_i$, then $bv_j b_j = a^k v_j b_j$; so $v_j \in [\![ba^{-k} = a_j b_j^{-1}]\!]$. Thus, every section of Spec(A) over $\surd(a)$ is of the form ba^{-k} . To tie up what remains, consider $b_1 a^{-k}$ and $b_2 a^{-k}$ (there is no loss in assuming a common denominator). If $a \in [\![b_1 a^{-k} = b_2 a^{-k}]\!]$, then for some N we have $a^N (b_1 - b_2) = 0$. So $b_1 a^{-k} = b_2 a^{-k}$ in $A[a^{-1}]$. □

In particular, we have shown that global sections of Spec(A) correspond exactly to the elements of A . The Ω-set of 4.19 (iv) is embedded in Spec(A) .

6.13. Spec(A) AS A LOCAL RING. If we define an operation $x^{-1} \equiv \mathrm{I}y.\, x \cdot y = 1$ on Spec(A), then for $ab^{-k} \in \mathrm{Spec}(A)(\surd b)$ we have $[\![E(ab^{-k})^{-1}]\!] = \surd(ab)$; because $\vdash E(ab^{-k})^{-1} \leftrightarrow Eb^{-1} \wedge Ea^{-1}$, and for $a \in A$ there is an inverse for a in $A[b^{-1}]$ just in case $\surd b \subseteq \surd a$. Thus, it is easy to check that Spec(A) is a local ring:

$$\models \;\; \forall\, x, y\, [E(x+y)^{-1} \rightarrow Ex^{-1} \vee Ey^{-1}] \;,$$

and in addition

$$\models \;\; \forall\, x\, [\neg\, Ex^{-1} \leftrightarrow \bigvee_{n \in \mathbb{N}} x^n = 0] \;.$$

(Compare 5.15 (vii) .)

We now see how the theory of Chapter 1 together with that of this section may be used to prove a conservative extension result.

6.14. LEMMA. If ϕ and ψ are first-order sentences, and for any structure $\mathcal{A}$ satisfying ϕ we have $\mathcal{A} \models \psi$, then for any structure $\mathcal{B}$ we have $\mathcal{B} \models \phi \rightarrow \psi$.

Proof. Let $\mathcal{B}$ be an Ω-structure and $p = [\![\phi]\!] \in \Omega$. The open quotient $f^* : \Omega \rightarrow \Omega_p = \{ q \mid q \leq p \}$ preserves all the logical structure $\bigwedge$, $\bigvee$, $\rightarrow$, $\top$, $\bot$. Therefore for any sentence θ we have by induction on structure:

$$[\![\theta]\!]_{f^*B} = f^*[\![\theta]\!]_B = [\![\theta]\!]_B \wedge p \ .$$

Thus $f^*B \models \phi$; whence by hypothesis $f^*B \models \psi$ and $B \models \phi \to \psi$. □

6.15. THEOREM. If ψ and ϕ_i, for $i < n$, are geometric sentences, and classically $\bigwedge\!\!\!\bigwedge_{i<n} \phi_i \vdash \psi$, then in any sheaf model we have

$$\models \bigwedge\!\!\!\bigwedge_{i<n} \phi_i \to \psi \ .$$

Proof. The $\wedge,\bigvee$- map $b^* : \Omega \to J(\Omega) \to J(\Omega)/\neg\neg$ is an embedding of Ω into a complete Boolean algebra. Let A be an Ω-structure such that $A \models \bigwedge\!\!\!\bigwedge_{i<n} \phi_i$, then $b^*A \models \bigwedge\!\!\!\bigwedge_{i<n} \phi_i$, because the ϕ_i are geometric. Moreover, $b^*A \models \psi$, since it is Boolean. Now because b^* is an embedding $A \models \psi$. Appealing to Lemma 6.14 we are done. □

The completeness theorem 5.22 gives as an immediate corollary the fact that if a geometric consequence is deducible *classically* from geometric axioms, then it is deducible *intuitionistically* from these axioms. Of course this result is also obtainable proof-theoretically [34].

CHAPTER III. HIGHER-ORDER STRUCTURES

In Chapter II we saw how a sheaf may be considered as a Heyting-valued set and used such variable sets to interpret first-order logic. Here we extend these interpretations to higher-order logic. In mathematics, we not only deal with objects of various types - integers, rationals, reals and so on - but also construct new types, for example pairs, sequences or sets of integers, from old. In Chapter 0 it was shown how such type constructions can be formalized once we have the basic type-forming operations: product and power types. In Section 4 , we described the corresponding basic constructions on sheaves (4.8 and 4.14). Having products of sheaves and power sheaves, we can extend our interpretations to higher-order logic. In Section 7 we do this and see how various constructions discussed formally in 0.7 look concretely when applied to sheaves. The next section is used to give some examples of higher-order structures. We give well-known representations for the models of Baire space and the Dedekind reals, which lead us to a general representation for internal topological spaces (announced in [10]). Our final section is an attempt to show that concentrating on sheaves over a cHa does not prevent us from saying something about more general topoi. Because we have taken the trouble to work intuitionistically, we can interpret out treatment in any topos. We obtain some basic results of the theory of sheaves for a topology on a topos in this way.

7. TYPES AND HIGHER-ORDER LOGIC

We begin by comparing the logical properties of product and power sheaves with the axioms discussed in 0.7 . The logical properties of products discussed in 5.15 corresoond precisely to those axiomatized in 0.7 .

7.1. THE POWER SHEAF (bis). Let A be a sheaf, the power sheaf $P(A)$ has as sections. pairs $\langle P,p\rangle$, where P is a strict extensional predicate on A, and $P(a) \leq p$ for all $a \in |A|$. For $a \in |A|$ and $\langle P,p\rangle \in |P(A)|$ define

$$[\![a \in \langle P,p\rangle]\!] = P(a) \quad .$$

This gives a strict <u>membership</u> <u>relation</u>. The global sections $\langle P,\top\rangle$ (for which we write simply P) generate $P(A)$. The formula

$$[\![P = Q]\!] = \bigwedge \{ P(a) \leftrightarrow Q(a) \mid a \in |A| \}$$

can be rewritten

$$[\![P = Q]\!] \; = \; \bigwedge \{ [\![Ea \to (a \in P \leftrightarrow a \in Q)]\!] \mid a \in |A| \} \quad ,$$

because P and Q are strict. Thus

$$\models \; \forall P, Q \in P(A) \; [P = Q \leftrightarrow \forall x \in A \; [x \in P \leftrightarrow x \in Q]] \quad .$$

Also, for any formula $\phi(x)$ with one free variable, $a \mapsto P(a) = [\![Ea \wedge \phi(a)]\!]$ gives a strict predicate on A and, hence, a global section of $P(A)$. For $a \in |A|$ we have $Ea \leq [\![a \in P \leftrightarrow \phi(a)]\!]$ and so

$$\models \; \exists P \in P(A) \; \forall x \in A \; [x \in P \leftrightarrow \phi(x)] \quad .$$

Thus, we have the two properties we need for power types. (In §9 we shall see that the power types just defined have the categorical properties necessary to make $Sh(\Omega)$ a topos. Our higher-order logic may be interpreted in any topos. What we are discussing concretely is a particular case of this abstract theory.) Since we use *all* strict, extensional predicates when constructing $P(A)$, these are the <u>standard</u> <u>Ω-valued</u> <u>models</u> <u>of</u> <u>higher-order</u> <u>logic</u>. We could (but do not) consider non-standard models, formed using only some of the predicates on A .

7.2. THE INTERPRETATION OF HIGHER-ORDER LOGIC. An <u>interpretation</u> of higher-order logic is given by assigning to each sort A a sheaf $[\![A]\!]$ in such a way that $[\![A]\!]$ commutes with powers and products. Thereafter interpreting higher-order logic is just like interpreting first-order logic. The syntactic operations of tuples $\langle \cdots \rangle$ and projections π_i and the membership relation $\in$ are interpreted by the corresponding operations and relation on sheaves (5.15 and 7.1). We add constants of sort A for the sections of $[\![A]\!]$ and define the <u>valuations</u> of terms and sentences, <u>validity</u> and <u>satisfaction</u> according to 5.13 and 5.19 just as in the first-order case.

7.3. THEOREM (Soundness). The axioms and rules of higher-order logic are valid for standard interpretations in sheaves.

Proof. The first-order axioms and rules were checked in Chapter III . They have not changed. The axioms concerning products are just the logical properties remarked in II.2.17 . Finally, (comp) follows formally from two principles we have noted (so-called "comprehension" and "extensionality") in 7.1 . □

Suppose we have an interpretation. We may identify the global sections of $[\![P(A)]\!]$ with the subsheaves of $[\![A]\!]$. If $\{ x \mid \phi(x) \}$ is a *closed* type, where $\# x = A$, we have a subsheaf $[\![\{ x \mid \phi(x) \}]\!]$ of $[\![A]\!]$. The relativization introduced in 0.7.7 corresponds to quantification over this subsheaf. Thus Metatheorem 0.7.8 is not so surprising. Extending an established abuse, we use the same notation henceforth for a type $\{ x \mid \phi(x) \}$ and the sheaf which interprets it.

Now, as promised, we give concrete descriptions of some of the types described formally in 0.7 .

7.4. QUOTIENTS. Suppose A is an Ω-set and R a symmetric, transitive relation on A . In 0.7.10 the type A/R is defined formally. Following through this formal description would lead us to A/R as a subsheaf of $P(A)$. However, we can describe immediately an Ω-set (which generates a sheaf isomorphic to) A/R : we take $|A/R| = |A|$ and $[\![a =_{A/R} b]\!] = [\![aRb]\!]$. Of course, in general this Ω-set will not be separated and, even if we took a quotient of $|A|$ to make it so, it would still not normally be complete. However, this Ω-set approach does provide a simple *presentation* of A/R .

7.5. FUNCTION SPACES. Here again we could follow through the formal description of function spaces (0.7.11); however, another presentation is more manageable. Let A and B be sheaves over Ω . Recall that a morphism from A to B is a map acting on sections which commutes with extents and restrictions. The sheaf B^A has these morphisms as its global sections and, as sections over $p \in \Omega$, the morphisms from $A \upharpoonleft p$ to $B \upharpoonleft p$. Restrictions are easy: for $f : A \upharpoonleft p \to B \upharpoonleft p$ we define $f \upharpoonleft q : A \upharpoonleft p \wedge q \to B \upharpoonleft p \wedge q$ by restricting f , as a map, to the appropriate collection of sections. With this definition it is easy to see that B^A is a sheaf; in fact, for this to be so, it is sufficient that B be a sheaf – A can be a presheaf. We have an obvious evaluation map $ev : A \times B^A \to B$ which, given a section (a,f) of $A \times B^A$ with $Ea = Ef$, gives us $ev(a,f) = f(a)$, a section of B with the same extent. Perhaps the easiest way to see that the sheaf B^A just defined is isomorphic to the one we would arrive at by following through the formal definition is to observe that this evaluation has the appropriate universal property. For the record:

7.6. LEMMA. $Sh(\Omega)$ is cartesian-closed. □

So far we have discussed certain type *constructions*. Now we turn to look at some particular basic types: $\emptyset$, $\mathbb{1}$, Ω , $\mathbb{N}$. In the next section we shall look at others: $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}$, our reason for deferring these is that they seem more properly mathematical, though the dividing line is hard to draw.

7.7. $\emptyset$, $\mathbb{1}$ AND SO ON ... . $\emptyset$ is the empty sheaf; it has one section $*$ with $E* = \bot$. As an Ω-set it can be presented as Ω-set with the empty set as its underlying set. As a sheaf, $\mathbb{1}$ has one global section, $0 = 0 \upharpoonleft \top$ and all its restrictions $0 \upharpoonleft p$ for $p \in \Omega$. As an Ω-set it can be presented as the simple Ω-set $\mathbb{1}_\Omega$. Categorically, $\emptyset$ is initial and $\mathbb{1}$ terminal in $Sh(\Omega)$. The various finite sets $\mathfrak{2}$, $\mathfrak{3}$ and so on are represented by simple sheaves 2_Ω , 3_Ω etc. Here a presentation is easily given, but an explicit description of all the sections would be messy.

7.8. $\mathbb{N}$. In order to verify Peano's Postulates (0.7.14), the sort $\mathbb{N}$ must be interpreted by (a sheaf isomorphic to) the simple sheaf $\mathbb{N}_\Omega$. It is easily checked that $\mathbb{N}_\Omega$ and its power sheaf form a model for Peano's axioms. The usual proof that these axioms are categorical is constructive and gives a *unique* isomorphism between any two models. Externally, this gives an isomorphism of sheaves. A categorical characterization of $\mathbb{N}$ is also possible ([29]).

7.9. THEOREM. $\mathbb{N}_\Omega$ is a natural number object in $Sh(\Omega)$. □

We described $\underset{\sim}{\Omega}$ as an Ω-set in Chapter II (1.4). Now we give a description of the corresponding sheaf.

7.10. $\underset{\sim}{\Omega}$ AND $\tilde{A}$. Let A be a sheaf; we describe the sheaf $\tilde{A}$ of singletons of A directly:

$$|\tilde{A}| \;=\; \{ (a,p) \mid a \in |A| \,,\ p \in \Omega \text{ and } Ea \leq p \} \,,$$

where $E(a,p) = p$ and $(a,p)\restriction q = (a\restriction q, p \wedge q)$. All we do is to confer upon each section of A an existence. Since $\underset{\sim}{\Omega} = \tilde{\mathbb{1}}$ we can describe it as follows:

$$|\underset{\sim}{\Omega}| \;=\; \{ (p,q) \mid p\,,q \in \Omega \text{ and } p \leq q \}$$

with $E(p,q) = q$ and $(p,q)\restriction r = (p \wedge r, q \wedge r)$. □

7.11. FINITE SETS AND FINITE SEQUENCES. We shall later need to consider the collections $P_{\mathrm{fin}}(A)$ of finitely indexed subsets of A and $A^* = \dot{\bigcup}_{n\in\mathbb{N}} A^n$ of finite sequences of elements of A . If A is a sheaf, then $P_{\mathrm{fin}}(A)$ may be presented as an Ω-set as follows:

$$P_{\mathrm{fin}}(A) \;=\; \{ F \subseteq |A| \mid F \text{ is finite and, for } a\,,b \in F,\ Ea = Eb \}$$

with $[\![F = G]\!] = \bigwedge\!\!\!\bigwedge_{a\in F} \bigvee\!\!\!\bigvee_{b\in G} [\![a = b]\!] \wedge \bigwedge\!\!\!\bigwedge_{b\in G} \bigvee\!\!\!\bigvee_{a\in F} [\![a = b]\!]$. We shall not attempt to describe explicitly all the sections of $P_{\mathrm{fin}}(A)$. A description of A^* is similar:

$$|A^*| \;=\; \{ \langle a_0,\ldots,a_{n-1}\rangle \mid n \in \mathbb{N} \,,\ a_i \in |A| \text{ and } Ea_i = Ea_j \}$$

with $[\![\langle a_0,\ldots,a_{n-1}\rangle = \langle b_0,\ldots,b_{n-1}\rangle]\!] = \bigwedge\!\!\!\bigwedge \{ [\![a_i = b_i]\!] \mid i < n \}$. Of course, $[\![\langle a_0,\ldots,a_{n-1}\rangle = \langle b_0,\ldots,b_{m-1}\rangle]\!] = \bot$ if $m \neq n$ (a decidable question). When A is a simple sheaf, these constructions become particularly simple. In this case, both A^* and $P_{\mathrm{fin}}(A)$ are simple sheaves. Thus, for example $\mathbb{N}$, $\mathbb{N}^*$ and $P_{\mathrm{fin}}(\mathbb{N})$ are all represented by simple sheaves: we say these structures are *absolute*. Essentially nothing is added by passing from sets to Ω-sets as far as these structures are concerned. We shall see that with $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}$ the situation is quite different: new sections arise which are not even locally equal to "constant sections" arising from the "real world". □

This concludes our general discussion of higher-order models apart from one remark: If A is a sheaf, any set of sections $B \subseteq |A|$ generates a subsheaf of A, and hence a global section of $P(A)$. We write B again for this subsheaf. This is not pernicious since $[\![b \in B]\!] = \bigvee \{ [\![b = a]\!] \mid a \in B \}$.

8. THE REALS AND TOPOLOGICAL SPACES.

Our purpose here is to give some examples of higher-order structures. It turns out that in modelling natural higher-order structures, we arrive at sheaves with their own mathematical interest. For much of this section, we restrict ourselves to sheaves over spatial cHa. These are easy to visualize and also have their own particular properties.

8.1. BAIRE SPACE. Our first example of a higher-order structure is the Baire space $\mathbb{N}^{\mathbb{N}}$. The first question is, what sheaf represents Baire space in our models? In discussing this, we restrict ourselves to spatial cHa, since the general case ([11] , [12]) involves Bar Induction or formal spaces, which we do not wish to discuss here. Our representation involves the usual Baire space endowed with the product topology (or finite information topology) having as subbasic opens the sets

$$V_{n,m} = \{ \alpha \in \mathbb{N}^{\mathbb{N}} \mid \alpha(n) = m \} .$$

This space is easily visualized: it is homeomorphic to the irrational numbers (those apart from every rational) topologized as a subspace of the reals. Our second question will be to represent this topology on the internal Baire space.

8.2. THEOREM. Let T be a topological space. $\mathbb{N}^{\mathbb{N}}$ is represented by $C(T,\mathbb{N}^{\mathbb{N}})$ the sheaf of (germs of) continuous maps from T to $\mathbb{N}^{\mathbb{N}}$.

Proof. Let a be a section of $\mathbb{N}^{\mathbb{N}}$. For $t \in Ea$ and $n \in \mathbb{N}$, define $a_t(n)$ to be the unique $m \in \mathbb{N}$ such that $t \in [\![a(n) = m]\!]$. This m certainly exists and so $a_t \in \mathbb{N}^{\mathbb{N}}$. As an element of Baire space, a_t depends continuously on $t \in Ea$ since $\{ t \mid a_t \in V_{n,m} \} = \{ t \mid a_t(n) = m \} = [\![a(n) = m]\!]$ is open. Thus, a is represented by a continuous function $a : Ea \to \mathbb{N}^{\mathbb{N}}$ and, as $[\![a(n) = m]\!] = \{ t \mid a_t(n) = m \}$, we have

$$[\![a = b]\!] = [\![\forall n . a(n) = b(n)]\!] = int \{ t \mid a_t = b_t \} .$$

Of course, it is obvious that every continuous map comes from some section of $\mathbb{N}^{\mathbb{N}}$. □

As always, we can regard this sheaf of germs of continuous maps as the sheaf of sections of the projection $T \times \mathbb{N}^{\mathbb{N}} \to T$. Doing this will enable us to represent not only the elements but also the topology of $\mathbb{N}^{\mathbb{N}}$.

8.3. TOPOLOGY ON $\mathbb{N}^{\mathbb{N}}$. Working internally we endow $\mathbb{N}^{\mathbb{N}}$ with the product topology: basic opens are given by finite information. For F a finite subset of $\mathbb{N}\times\mathbb{N}$ we have a basic open

$$V_F = \{ a \mid a(n) = m \text{ for } (n,m) \in F \}.$$

If a is a section of $T\times\mathbb{N}^{\mathbb{N}} \to T$, we have

$$[\![a \in V_F]\!] = a^{-1}(T\times V_F) = \{ t \mid a_t(n) = m \text{ for } (n,m)\in F \}.$$

A global internal open of $\mathbb{N}^{\mathbb{N}}$ is the union of an internal family $\mathcal{U}$ of basic opens. $\mathcal{U}$ is determined by the values $[\![V_F \in \mathcal{U}]\!]$ where F is a finite subset of $\mathbb{N}\times\mathbb{N}$ since $P_{fin}(\mathbb{N}\times\mathbb{N})$ is a simple sheaf. For a a section of $\mathbb{N}^{\mathbb{N}}$ we have

$$\begin{aligned}
[\![a \in \bigcup\mathcal{U}]\!] &= [\![\exists\, V_F \in \mathcal{U}\,.\, a \in V_F]\!] \\
&= \bigvee [\![a \in V_F \wedge V_F \in \mathcal{U}]\!] \\
&= \bigvee (a^{-1}(T\times V_F) \wedge [\![V_F \in \mathcal{U}]\!]) \\
&= \bigvee a^{-1}([\![V_F \in \mathcal{U}]\!] \times V_F) \\
&= a^{-1} \bigvee ([\![V_F \in \mathcal{U}]\!] \times V_F)
\end{aligned}$$

(where the sups are taken over all finite subsets F of $\mathbb{N}\times\mathbb{N}$). Global internal opens of $\mathbb{N}^{\mathbb{N}}$ are thus represented by opens of $T\times\mathbb{N}^{\mathbb{N}}$. Any open U of $T\times\mathbb{N}^{\mathbb{N}}$ gives an internal open U^* as follows:

$$[\![a \in U^*]\!] = a^{-1}(U);$$

and if $\models U^* = V^*$, then $U = V$, since $(t,\alpha)\in U$ iff $t \in [\![\underset{\sim}{\alpha} \in U^*]\!]$ (where $\underset{\sim}{\alpha}$ is the constant section). The same argument applied to $\mathbb{N}^{\mathbb{N}} \upharpoonright W$ will give us the following theorem:

8.4. THEOREM. $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$ is represented as the sheaf whose sections over $W \in \mathcal{O}(T)$ are the opens of $W\times\mathbb{N}^{\mathbb{N}}$. □

We shall soon see that this representation is not at all special: Essentially all internal topological spaces are represented in a similar way by external ones. Before going on to this general case, we consider another specific example, the Dedekind reals. These are constructed from $\mathbb{N}$ via the rationals. We leave it to the reader to prove that the construction of the rationals (as a partial quotient $\mathbb{Q}$ of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$) is absolute:

8.5. LEMMA. $\mathbb{Q}$ is represented by the simple sheaf $\mathbb{Q}_\Omega$. □

8.6. THE DEDEKIND REALS. These are constructed as a subtype of $P(\mathbb{Q})\times P(\mathbb{Q})$ as described briefly in 3.3. More formally, we consider those pairs

$\langle U,L\rangle \in P(Q)\times P(Q)$ satisfying the following conditions:

(1) $\exists\, p,q\in Q\ (p\in U \wedge q\in L)$

(2) $\forall\, p\ \neg(p\in U\wedge p\in L)$

(3) $\forall\, p\ (p\in L \leftrightarrow \exists\, q\in L\,.\, q>p)$

(4) $\forall\, p\ (p\in U \leftrightarrow \exists\, q\in U\,.\, q<p)$

(5) $\forall\, p,q\ (p>q \rightarrow p\in U\vee q\in L)$.

8.7. LEMMA. Let $r = \langle U,L\rangle$ be a section of R over $W\in \mathcal{O}(T)$. For $t\in W$ define

$$U_t = \{ q \mid t\in [\![q\in U]\!] \}$$

$$L_t = \{ q \mid t\in [\![q\in L]\!] \}$$

then $r_t = \langle U_t,L_t\rangle$ is a real and $\lambda t\cdot r_t$ defines a continuous map (which by abuse we call) $r : W\to \mathbb{R}$.

Proof. The axioms (2) for a Dedekind real are geometric, so r_t being the inverse image of r along the stalk map is a Dedekind real. For $q\in Q$ we have $t\in [\![q\in U]\!]$ iff $q>r_t$, and dually for L . Given $t\in W$ and $\varepsilon>0$, choose rationals q,q' within ε of r_t such that $q<r_t<q'$. Now, for any s in the open set $[\![q\in L\wedge q'\in U]\!]$ (a neighbourhood of t), we have $q<r_s<q'$ and, hence, $|r_s-r_t|<\varepsilon$.

8.8. THEOREM. R is represented by the sheaf $\underset{\sim}{C}(T,\mathbb{R})$ of germs of continuous real functions on T or equivalently as the sheaf of sections of the projection $\pi : T\times\mathbb{R}\to T$.

Proof. If $r : W\to\mathbb{R}$ is continuous, then setting $[\![q\in U]\!] = \{ t \mid r(t)<q \}$ and $[\![q\in L]\!] = \{ t \mid r(t)>q \}$ gives a pair of sections of $P(Q)$ whose inverse image at each stalk is a real $r(t) = \langle U_t,L_t\rangle$. As the axioms (2) are geometric, we have an internal real. We have a correspondence between sections of R over W and continuous functions $W\to\mathbb{R}$ which is obviously natural (i.e., restrictions are given by restriction). The last equivalence is trivial. $\square$

8.9. TOPOLOGY ON R . Working internally, we endow R with the usual topology having rational open intervals as a base. Considering a section of R over W as a section over W of the projection

$$\pi : T\times\mathbb{R}\to T ,$$

we have for $p,q\in Q$:

$$[\![a\in (p,q)]\!] = a^{-1}[T\times(p,q)] .$$

A global internal open of R is the union of an internal family $\mathcal{U}$ of rational opens. $\mathcal{U}$ is determined (as Q is a simple sheaf) by the values $[\![(p,q) \in \mathcal{U}]\!]$, where $p, q \in Q$. For a a section of R we have

$$[\![a \in \bigcup \mathcal{U}]\!] = a^{-1} \bigvee ([\![(p,q) \in \mathcal{U}]\!] \times (p,q))$$

(the sup being over all rational open intervals). The argument is just like the one we used for $\mathbb{N}^{\mathbb{N}}$ and leads along similar lines to, in essence, the same result:

8.10. THEOREM. $\mathcal{O}(R)$ is represented as the sheaf whose sections over $W \in \mathcal{O}(T)$ are the opens of $W \times \mathbb{R}$. □

As promised, we now generalize the representations just given to provide a representation for arbitrary internal topological spaces.

We recall that a topology on A is just a set of opens $\mathcal{O}(A) \subseteq P(A)$ closed under finite $\cap$ and arbitrary $\bigcup$. We give an example which we shall show to be quite general.

3.11. EXAMPLE. Let $\pi : Y \to T$ be a continuous map between topological spaces. We define a topology on π_T, the sheaf of sections of π, by letting $\mathcal{O}(\pi_T)$ be the subsheaf of $P(\pi_T)$ generated by the global sections U^* given by

$$[\![a \in U^*]\!] = a^{-1}(U) \quad \text{for } U \in \mathcal{O}(Y) .$$

We leave the reader to check that this gives a topology. We call this internal space the <u>space of sections</u> of π. □

We make use of the duality between the category of sober spaces and that of cHa with enough points to give a representation of internal sober spaces. For the moment, we consider an arbitrary cHa H in $Sh(\Omega)$. That is, H is an Ω-set or sheaf equipped with a relation $\leq$ and total operations $\top$ (unary), $\wedge$ (binary) and $\bigvee : P(H) \to H$ satisfying the axioms of 1.1. We spend some time getting a representation for such internal cHa.

8.12. LEMMA. (i) Every section of H is the restriction of some global section, in fact, $\models U \equiv \bigvee\{U\} \restriction EU$.

(ii) The lattice $\Gamma(H)$ of global sections of H is a cHa (where $U \leq V$ iff $[\![U \leq V]\!] = \top$), whose meets are just those calculated in H and whose joins are given by associating to a family $\{ U_i \mid i \in I \} \subseteq \Gamma(H)$ the corresponding global section of $P(H)$ and taking its join.

(iii) The map $\pi^{-1} = \lambda p \cdot \bigvee\{ \top \mid p \} : \Omega \to \Gamma(H)$ is an $\wedge,\bigvee$- map.

Proof. (i) $\{U\}$ is a shorthand for $\{ x \in H \mid x = U \}$. So we have

$$\models x \in H \, (x \leq \bigvee\{U\} \leftrightarrow \exists\, y \, (y = U \wedge x \leq y)) , \text{ and also}$$

$$\models \ EU \to \bigvee\{U\} \upharpoonleft EU \ = \ \bigvee\{U\} \quad .$$

Thus $\models \ EU \to U = \bigvee\{U\} \upharpoonleft EU$.

So $\models \ U \equiv \bigvee\{U\} \upharpoonleft EU$.

(ii) By abstract nonsense (6.4), $\Gamma(H)$ is a lattice with the finitary operations inherited from H . Now, given $\{U_i \mid i \in I\} \subseteq \Gamma(H)$ and $W \in \Gamma(H)$, we have

$$W \geq [\![\bigvee\{U_i \mid i \in I\}]\!] \quad \text{iff} \quad \models W \geq \bigvee\{U_i \mid i \in I\}$$

$$\text{iff, for each } U \in |H| , \quad \models U \in \{U_i \mid i \in I\} \to W \geq U$$

$$\text{iff, for each } U \in |H| , \quad \bigvee\{[\![U = U_i]\!]\} \leq [\![W \geq U]\!]$$

$$\text{iff, for each } i \in I , \quad \models W \geq U_i \quad .$$

So $[\![\bigvee\{U_i \mid i \in I\}]\!] \in \Gamma(H)$ is a sup for $\{U_i \mid i \in I\}$. Now for $V \in \Gamma(H)$ we have

$$\bigvee\{[\![U = U_i \wedge V]\!] \mid i \in I\} \ = \ \bigvee\{(\bigvee\{[\![W = U_i]\!] \mid i \in I\} \wedge [\![U = W \wedge V]\!])\}$$

so $[\![\{W \wedge V \mid W \in \{U_i \mid i \in I\}\}]\!] \ = \ [\![\{U_i \wedge V \mid i \in I\}]\!]$

and distributivity of $\Gamma(H)$ follows from that of H .

(iii) By internal distributivity $\bigvee\{\top \mid p \wedge q\} = \bigvee\{\top \mid p\} \wedge \bigvee\{\top \mid q\}$, so π^{-1} preserves $\wedge$ (the empty meet is a trivial case). Furthermore,

$$U \geq \bigvee\{\top \mid q\} \quad \text{iff} \quad \models q \to U = \top \quad .$$

Given $\{p_i \mid i \in I\} \subseteq \Omega$ we have a string of equivalences:

$$\text{for all } i \in I \quad U \geq \bigvee\{\top \mid p_i\} \qquad \text{iff}$$

$$\text{for all } i \in I \quad \models p_i \to U = \top \qquad \text{iff}$$

$$\models \bigvee p_i \to U = \top \qquad \text{iff}$$

$$U \geq \bigvee\{\top \mid \bigvee p_i\} \quad . \qquad \square$$

We have now seen that any internal cHa H gives rise to an $\wedge,\bigvee$-map $\pi_H^{-1} : \Omega \to \Gamma(H)$. We now show that every such $\wedge,\bigvee$-map arises from an (essentially unique) internal cHa, and that internal $\wedge,\bigvee$-maps correspond exactly to external $\wedge,\bigvee$-maps making the triangle

$$\begin{tikzcd} \Gamma(H) \arrow[rr] & & \Gamma(K) \\ & \Omega \arrow[ul, "\pi_H^{-1}"] \arrow[ur, "\pi_K^{-1}"'] & \end{tikzcd}$$

commute. An internal $\wedge\bigvee$-map is a strict total function commuting with $\wedge$ and $\bigvee$. Such a map restricts, obviously, to an $\wedge\bigvee$-map on global sections and, since $\top$ and $\bigvee$ (in terms of which π^{-1} is defined) are preserved, this map makes the triangle commute. In fact, we have a functor Γ from the category of internal cHa and (global) internal $\wedge,\bigvee$- maps to the category of cHa under Ω :

$$\Gamma : [cHa(\Omega)] \to \Omega/\mathrm{cHa} \quad .$$

8.13. THEOREM. Γ is an equivalence of categories.

Proof. We must show two things. Firstly that Γ is full and faithful, secondly that every cHa under Ω is (isomorphic to one) of the form $\pi_H^{-1} : \Omega \to \Gamma(H)$ for some internal cHa H . This will take some time.

Starting from an $\wedge,\bigvee$- map $\phi : \Gamma(H) \to \Gamma(K)$ making the triangle commute, we see that ϕ could arise from at most one sheaf map $\hat{\phi} : H \to K$, since every section is the restriction of a global section. Thus, Γ is faithful. We now show that defining (as is forced upon us)

$$\hat{\phi}(U) = \phi(\bigvee\{U\}) \upharpoonleft EU$$

gives a well defined internal $\wedge,\bigvee$- map extending ϕ . $\hat{\phi}$ obviously preserves extents. Now for global U , V we have

$$U \upharpoonleft p \;=\; V \upharpoonleft p \;\text{ iff }\; U \wedge \pi^{-1}p \;=\; V \wedge \pi^{-1}p$$

(both are equivalent to $\models \;\forall W\, (p \wedge W \leq U \leftrightarrow p \wedge W \leq V)$). Since ϕ preserves π^{-1} and $\wedge$, we see that $\hat{\phi}$ is compatible with restrictions and hence a sheaf map.

As $\hat{\phi}$ preserves finite infs of global sections and global sections generate H we see that $\hat{\phi}$ preserves $\wedge$ internally. The argument for $\bigvee$ is slightly more complicated. Global sections generate $\Gamma(H)$. So it suffices to show, for $\mathcal{U}$ a global section of $P(H)$, that

$$\hat{\phi}\bigvee\mathcal{U} \;=\; \bigvee\{\,\hat{\phi}(U) \mid U \in \mathcal{U}\,\} \quad ,$$

which is indeed the case, since we can translate everything in terms of global sections using

$$|\{\,\hat{\phi}(U) \mid U \in \mathcal{U}\,\}| \;=\; \{\,\hat{\phi}(U) \mid U \in |\mathcal{U}|\,\}$$

$$[\![\bigvee\mathcal{U}]\!] \;=\; \bigvee\{\,[\![\bigvee\{U\}]\!] \mid U \in |\mathcal{U}|\,\} \quad .$$

Thus $\hat{\phi}$ is an $\wedge,\bigvee$- map internally. It remains, given an $\wedge,\bigvee$- map $\theta : \Omega \to \Omega'$, to construct the appropriate internal cHa H . We construct H as an Ω-set : Take $|H| = \Omega'$ as underlying set and

$$[\![U \leq V]\!] = \bigvee\{\,p \in \Omega \mid U \wedge \theta(p) \leq V \wedge \theta(p)\,\} \;,$$

for $U, V \in \Omega'$ (and of course $[\![U = V]\!] = [\![U \leq V]\!] \wedge [\![V \leq U]\!]$). Infs in H are, on global sections, just those in Ω' . For $\mathcal{U}$ an Ω-valued subset of H , we define

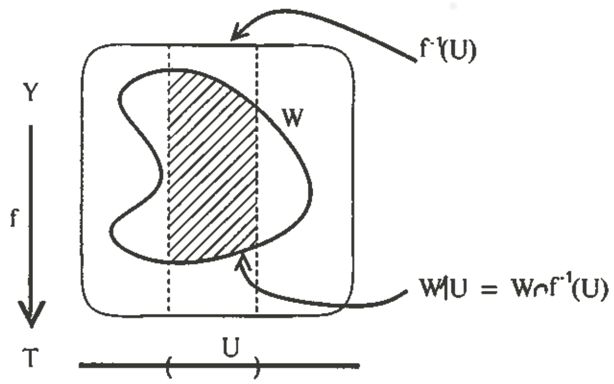
$$\bigvee\mathcal{U} \;=\; \bigvee\{\,U \wedge \theta([\![U \in \mathcal{U}]\!]) \mid U \in \Omega'\,\} \quad .$$

We leave to the reader the tedium of verifying that H is now a cHa internally and that the embedding $\Omega' \to \Gamma(H)$ is an isomorphism (under Ω). □

We look at some examples of internal cHa before returning to our proper study of topological spaces.

8.14. EXAMPLES. (i) The archetypical internal cHa is $\mathcal{P}(\mathbb{1})$ it is represented by the identity $\Omega \to \Omega$.

(ii) Our paradigm of an $\wedge,\bigvee$- map is the inverse image of a continuous map between spaces $f : Y \to T$ gives $f^{-1} : \mathcal{O}(T) \to \mathcal{O}(Y)$.



Here the cHa we get internally is easy to visualize. The global sections $\Gamma(H) = \mathcal{O}(Y)$, and for $V, W \in \Gamma(H)$ we have

$$[\![V \doteq W]\!] = \bigvee \{ U \in \mathcal{O}(T) \mid W \cap f^{-1}(U) = V \cap f^{-1}(U) \} .$$

The reader should recognize this cHa as the algebra of opens of the internal space of sections of f .

(iii) In Chapter 0 an example was promised to show the consistency of $\neg \forall p \in \Omega\, [p \vee \neg p]$. We can now give this. Take $\Omega = \mathcal{O}(\mathbb{R})$, the opens of the real line. $\mathcal{P}(\mathbb{1})$ is represented by the identity map (as in example (i)) and has as global sections $\mathcal{O}(\mathbb{R})$. Now for $r \in \mathbb{R}$ let $p = \{ t \in \mathbb{R} \mid t \neq r \} \in \mathcal{O}(\mathbb{R})$. We have $r \notin [\![p \vee \neg p]\!] = p$. As r was arbitrary, $[\![\forall p \in \Omega\, [p \vee \neg p]]\!] = \bot$.

In Chapter 0 various principles were disproved by showing they imply classical logic. Our present example shows that all such principles have counter-examples in the model of sheaves over $\mathcal{O}(\mathbb{R})$.

(iv) As an example (due to Grayson) of the pathology which is possible intuitionistically, we exhibit a model with a T_2 space which is not sober. Here we take $\Omega = \mathcal{O}(Q)$ the opens of the rationals. The internal space of rationals with the usual topology will give our example. The rationals are always given by the simple sheaf Q_Ω. Every section is locally constant. To represent the topology of Q is a matter of adapting the arguments leading to 5.4 and 5.10 . We see that $\mathcal{O}(Q)$ is represented by the $\wedge\bigvee$- map $\mathcal{O}(Q) \to \mathcal{O}(Q\times Q)$ dual to the projection $\pi_1 : Q \times Q \to Q$; thus, internal $\wedge\bigvee$- maps $\mathcal{O}(Q) \to P(1)$ correspond to commuting triangles

$$\begin{tikzcd} \mathcal{O}(Q\times Q) \arrow[r] & \mathcal{O}(Q) \\ \mathcal{O}(Q) \arrow[u] \arrow[ur] & \end{tikzcd}$$

or dually, to sections of the projection map. Of these, only the "locally constant" ones correspond to sections of Q . So Q is not sober. In any reasonable sense, Q is Hausdorff. □

We return to our representation of internal cHa to obtain a similar result for internal topological space. For this we specialize to the case where Ω is spatial, say $\Omega = \mathcal{O}(T)$.

8.16. LEMMA. Let H be a cHa with enough points in $Sh(T)$, then $\Gamma(H)$ has enough points.

Proof. We are given that

$$\models \ \forall\, V, W \in H \ (\ \bigvee\!\wedge\!\bigvee \ \phi : H \to \Omega \ \ \phi(V) = \phi(W) \to V = W) \ .$$

(Where we write Ω for the internal $P(\mathbb{1})$ to avoid confusion with the external $P(\mathbb{1})$ which we also need here.) Suppose now that $V, W \in \Gamma(H)$, and that for every $\wedge\bigvee$- map $\theta : \Gamma(H) \to P(\mathbb{1})$ we have $\theta(V) = \theta(W)$. We must show that $V = W$. For this it suffices to show that $\models \ \forall \wedge\bigvee \ \phi : H \to \Omega \ \ \phi(V) = \phi(W)$. Now let ϕ be an internal $\wedge\bigvee$- map and $t \in E\phi \in \mathcal{O}(T)$. The map ψ taking V to $\{ 0 \mid t \in [\![0 \in \phi(V)]\!] \}$ is an $\wedge\bigvee$-map $\Gamma(H) \to P(1)$ (being the composite of $\Gamma(\phi)$ and the characteristic map of t). Thus $\psi(V) = \psi(W)$; in other words

$$t \in [\![0 \in \phi(V)]\!] \ \text{ iff } \ t \in [\![0 \in \phi(W)]\!] \ .$$

So $E\phi \le [\![\phi(V) = \phi(W)]\!]$, and we are done. □

Assuming T is sober we now have a diagram of functors

$$\begin{tikzcd} & cHa(\mathcal{O}(T)) \arrow[r, phantom, "\cong"] & \mathcal{O}(T)/\mathrm{cHa} \\ & Pts(\mathcal{O}(T)) \arrow[u, hook] \arrow[r, hook] & \mathcal{O}(T)/\mathrm{Pts} \arrow[u, hook] \\ Top(\mathcal{O}(T)) \arrow[r] & Sob(\mathcal{O}(T)) \arrow[u, leftrightarrow] \arrow[r, hook] & \mathrm{Sob}/T \arrow[u, leftrightarrow] \end{tikzcd}$$

where the vertical arrows come from the abstract nonsense of 3.7 , and the top equivalence 8.13 drops down to a full faithful embedding by 8.16 . [$\cong$ denotes an equivalence, $\rtimes\!\ltimes$ duality, $\hookrightarrow$ full faithful embedding.]

8.17. REPRESENTATION OF INTERNAL SOBER SPACES. We have a full faithful functor $\mathit{Sob}(\mathcal{O}(T)) \to \mathrm{Sob}/T$ taking each internal sober space A to its representation which we write as $\rho_A : E_A \to T$. The special case where A is a discrete space internally gives the representation of sheaves by étale spaces 4.22 . □

8.18. THE SPACE OF SECTIONS OF A CONTINUOUS MAP (bis). We return to Example 8.11 . Given a commuting triangle of continuous maps

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "g"'] & & Z \arrow[dl, "h"] \\ & T & \end{tikzcd}$$

we have two spaces of sections g_T and h_T . Since global opens generate the internal topologies, it is easy to see that the sheaf map $f : g_T \to h_T$, acting on sections by composing with f, is continuous in the internal topology. Also, the map taking $U \in \mathcal{O}(Y)$ to $U^* \in \Gamma(\mathcal{O}(g_T))$ is an $\wedge\bigvee$- map so, if Y is sober, every internal $\wedge\bigvee$- map $\mathcal{O}(g_T) \to \mathcal{P}(\mathbb{1})$ gives externally $\mathcal{O}(Y) \to \Gamma(\mathcal{O}(g_T)) \to \mathcal{O}(T)$ (all under $\mathcal{O}(T)$) and, by duality, a section of g . Thus, if Y is sober, then g_T is sober internally, and we have a functor $\Gamma : \mathrm{Sob}/T \to \mathrm{Sob}(\ (T))$.

8.19. DEFINITION. We say $\pi : Y \to T$ is a <u>bundle</u> iff points through which a local section passes separate opens of Y .

Evidently, the representation of any internal space is a bundle. We leave the reader to prove the following theorem:

8.20. THEOREM. Γ is right adjoint to the representation and is an equivalence on the category of sober bundles. □

8.21. DEFINITION. We say $\pi : Y \to T$ is <u>full</u> iff there is a local section of through every point of Y .

Unfortunately not every sober bundle is full. However, any bundle has an obvious full subbundle having the same opens (take those points through which there is a local section). If $\pi_A : E_A \to T$ represents an internal space A, we call the associated full bundle $\pi_A : F_A \to T$ the *full representation* of A . External properties of the full representation are closely tied to internal propert-properties of A . We mention an example.

8.22. THEOREM. A is strongly T_2 internally; that is, $\models$ A is T_0 and $\models \forall y \in U \; \forall x \, [x \in U \vee \exists \, V, W \, (V \cap W = \emptyset \wedge x \in V \wedge y \in W)$ (where U, V, W range over $\mathcal{O}(A)$ and x, y over A) iff $\rho_A : F_A \to T$ is separated; (classically) distinct points in a fibre have disjoint neighbourhoods in the total space; intuitionistically, fibres are T_0 and

$$\forall y \in U \; \forall x \, [\pi_A(x) = \pi_A(y) \to x \in U \vee \exists \, V, W \, (V \cap W = \emptyset \wedge x \in V \wedge y \in W)]$$

(where U, V, W range over $\mathcal{O}(F_A)$ and x, y over F_A).

Proof. Constructively, trivial. Classically, show that classically the constructive and classical forms of separated are equivalent. □

We now return to spaces of the form X_T, sections of the projection $\pi : T \times X \to T$. Any projection is a full bundle. Examples are rife, in particular the internal spaces $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$, $2^{\mathbb{N}}$, $\mathbb{I}$ and $\mathcal{R}$ are represented as spaces of sections of projections.

8.23. THEOREM. Continuous maps $A \to X_T$ are represented by external continuous maps $E_A \to X$.

Proof. This is a corollary of our general representation, since commuting triangles

$$\begin{tikzcd} E_A \arrow[rr] \arrow[dr, "\pi_A"'] & & T \times X \arrow[dl, "\pi"] \\ & T & \end{tikzcd}$$

are just continuous maps $E_A \to X$. □

8.24. EXAMPLE. The special case of a continuous map $\mathcal{R} \to \mathcal{R}$ is well known. Such a map is represented by a continuous map $f : T \times \mathbb{R} \to \mathbb{R}$; the corresponding map on sections is given by $[\hat{f}(a)](t) = f(t, a(t))$. We shall see that in spatial topoi every continuous real function is uniformly continuous on closed intervals. Scott has shown that in some spatial topoi every function is continuous. □

8.25. LEMMA. Let $\mathcal{B}$ be a basis for the topology on X. The global opens $\hat{U}$, represented by $T \times U$ for $U \in \mathcal{B}$, generate (in the sheaf sense) a basis for the topology on X_T.

Proof. Just as for (8.9). □

8.26. THEOREM. If X is compact, then X_T is compact internally.

Proof. Let $\mathcal{B}$ be the internal basis generated by the opens $\hat{U}$ for $U \in \mathcal{O}(X)$. Suppose $\models \mathcal{U} \subseteq \mathcal{B}$ and $t \in [\![\forall x \in X_T \; \exists U \in \mathcal{U} \;\; x \in U]\!]$, then for each section x of X_T

$$t \in \bigvee \{ [\![x \in \hat{U} \wedge \hat{U} \in \mathcal{U}]\!] \mid U \in \mathcal{O}(X) \} \quad .$$

Specializing to constant sections, we obtain that, for each $x \in X$,

$$x \in \bigcup \{ U \mid t \in [\![\hat{U} \in \mathcal{U}]\!] \} \qquad \text{(i.e., this set covers } X \text{)} \quad .$$

Take a finite subcover $\{ U_i \mid i \in n \}$. Then

$$t \in [\![\bigwedge_i \hat{U}_i \in \mathcal{U} \wedge \forall x \in X_T \bigvee_i x \in U_i]\!] \quad . \qquad \square$$

This proof is entirely constructive. It shows us that if the unit interval is compact, then in any spatial topos the unit interval is again compact. From this it follows that if the unit interval is compact, then in any spatial topos a function continuous on a closed interval is uniformly continuous and its range has a least upper bound.

9. TOPOI AND INTERNALIZATION

The interpretation of higher-order logic in $Sh(\Omega)$ generalizes to any elementary topos (Fourman [9]). To every topos $\mathbb{E}$ there is associated a language $L(\mathbb{E})$ together with an interpretation in $\mathbb{E}$ for which the axioms and rules of higher-order logic are sound. Every object of $\mathbb{E}$ gives a sort of $L(\mathbb{E})$. To every closed term τ of sort $P(A)$ there corresponds a subobject $\{ x \in A \mid x \in \tau \}$ of A . Morphisms $f : A \to B$ in $\mathbb{E}$ correspond to closed terms of sort $P(A \times B)$ such that

$$\mathbb{E} \models \; f : A \to B$$

(" $f : A \to B$ " abbreviates " $Ef \wedge \forall a \in A \;\; E!b \in B. \langle a,b \rangle \in f$ "). Furthermore,

$$f \circ g = h \;\text{ iff }\; \mathbb{E} \models \; f \circ g = h \quad .$$

Throughout these notes we have taken care to work constructively. Statements appearing to talk about the class of all sheaves may be thought of as shorthand for schematic statements involving only a finite number of types. Our results are formalizable in intuitionistic type theory and are valid in any interpretation of that theory. Here we interpret some facts about sheaves on a cHa in an elementary topos. This process is called relativization – we interpret our results relative to the *base topos* $\mathbb{E}$ in place of *Ens* , the category of sets. Although we recall some basic definitions from topos theory, the reader is advised to consult the standard references ([13] , [23] , [51]) for more details. Unexplained categorical terminology can be found (explained) in MacLane [32] .

9.1. TOPOI. A topos is a cartesian closed category with finite limits and a subobject classifier.

We shall explain this definition in more detail shortly. Since it is often easier to see that a given category has these properties than to describe explicitly the corresponding interpretation of higher-order logic, this definition (due to Lawvere and Tierney [31]) has led to many interesting new models for intuitionistic logic. The interpretations we have been considering are a case in point.

9.2. PROPOSITION. The category $Sh(\Omega)$ is a topos.

Proof. Finite products and exponents have already been described explicitly: $Sh(\Omega)$ is cartesian closed. To provide finite limits it suffices to describe equalizers. Given $f, g : A \to B$, morphisms of sheaves, we can describe their equalizer as the subsheaf of A whose sections are those $a \in |A|$ such that $f(a) = g(a)$. Since this is essentially just the equalizer of f and g regarded as maps (on the sections), the universal property is easy to verify. We must next explain briefly what a subobject classifier is. Given a category C with finite limits we have a functor $P : C^{op} \to \mathcal{Sets}$ taking an object to its set of subobjects and a morphism to the corresponding inverse image function. A subobject classifier is an object Ω of C together with an isomorphism $P(A) \cong \mathrm{Hom}[A,\Omega]$ natural in A . We say each subobject of A is *classified* by the corresponding morphism $A \to \Omega$. The morphism classifying $\mathbb{1} \to \mathbb{1}$ is called *true* : $\mathbb{1} \to \Omega$ (where $\mathbb{1}$ is the terminal object of C). In our case, subobjects may be identified with subsheaves. Our subobject classifier is just $\underset{\sim}{\Omega}$. To a subsheaf B of A, we associate the morphism $A \to \underset{\sim}{\Omega}$ taking $a \in |A|$ to $([\![a \in B]\!], Ea) \in |\underset{\sim}{\Omega}|$. Naturality is immediate, since $[\![a \in f^{-1}B]\!] = [\![f(a) \in B]\!]$ defines inverse images in $Sh(\Omega)$. □

9.3. MORPHISMS OF TOPOI. A geometric morphism $f : \mathbb{E} \to \mathbb{F}$ of topoi is an adjoint pair $f^* \dashv f_*$ of functors

$$\mathbb{E} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbb{F}$$

where f^* is left exact (preserves finite limits). We call f^* the inverse image and f_* the direct image part of f .

9.4. EXAMPLES. (i) Any $\wedge\bigvee$- map $f^* : \Omega' \to \Omega$ induces a geometric morphism $f : Sh(\Omega) \to Sh(\Omega')$ by the results of §6 . The subobjects of $\mathbb{1}$ in $Sh(\Omega')$ may be identified with Ω' . Every object of $Sh(\Omega')$ is a quotient of a coproduct of subobjects of $\mathbb{1}$; since subobjects of $\mathbb{1}$ generate. The inverse image part of f

is the unique left exact continuous (preserving all colimits) functor extending f^* . (Whence the abuse of notation.)

(ii) Since there is a unique $\wedge\bigvee$- map $P(\mathbb{1}) \to \Omega$, for any cHa Ω , there is a geometric morphism

$$Sh(\Omega) \to \mathcal{E}ns$$

(In §5 we saw that $Sh(P(\mathbb{1})) \cong \mathcal{E}ns$.) □

We shall see that relativizing the second of these examples to $Sh(\Omega')$ gives us the first. Since our logic does not allow talk of classes, we need a definition.

9.5. DEFINITION. Let $\mathbb{E}$ be a topos and $H = \langle H,\leq\rangle$ a cHa in $\mathbb{E}$; that is, $\mathbb{E} \models$ "H is a cHa". The category $Sh_{\mathbb{E}}(H)$ has as objects structures $A = \langle |A|,E,1\rangle$ in $\mathbb{E}$ such that $\mathbb{E} \models$ " A is an H-sheaf " , and as morphisms, morphisms $f : |A| \to |B|$ of $\mathbb{E}$ such that $\mathbb{E} \models f : A \to B$. □

9.6. THEOREM. $Sh_{\mathbb{E}}(H)$ is a topos with a geometric morphism $\Gamma : Sh_{\mathbb{E}}(H) \to \mathbb{E}$ given by $\Gamma^*(A) = A_H$ and $\Gamma_*(A) = \{ a \in |A| \mid Ea = \top \}$ in $\mathbb{E}$.

Proof. This is almost as immediate as it seems. We interpret 9.2 in $\mathbb{E}$. The explicit constructions we have given provide finite limits, exponents and power types. We need to check however that they indeed perform these roles in $Sh_{\mathbb{E}}(H)$. Consider products: the statement that $A \leftarrow A \times B \to B$ is a product is formalized as a schema. For each C we have $\mathbb{E} \models \forall f : C \to A \ \forall g : C \to B \ (\langle f,g\rangle\circ\pi_1 = f \wedge \langle f,g\rangle\circ\pi_2 = g) \wedge \forall h : C \to A\times B \ \ \langle h\circ\pi_1,h\circ\pi_2\rangle = h$. Since the pairing and projection maps are given *explicitly* we can deduce that $A \leftarrow A\times B \to B$ is indeed a product in $Sh_{\mathbb{E}}(H)$. We apologize if this appears obvious: it seems worthwhile to spell out in some detail how it is that we can deduce *facts* about $Sh_{\mathbb{E}}(H)$ from the intuitionistic *provability* of the corresponding facts about $Sh(\Omega)$. In any case, we shall not go into such detail again: the argument for equalizers, exponents and the subobject classifier is analogous, and the geometric morphism which exists internally by 9.10 gives us the geometric morphism we want in the same way. □

We now look at this construction in the special case where $\mathbb{E} = Sh(\Omega)$. An internal cHa H is then represented by an $\wedge\bigvee$- map $\pi^{-1} : \Omega \to \Gamma(H)$ as in 8.8 . We give a similar representation for internal sheaves on H .

9.7. LEMMA. If $Sh(\Omega) \models$ " $A = \langle A, 1,E\rangle$ is an H-sheaf ", then $\Gamma(A)$ is a sheaf over $\Gamma(H)$.

Proof. Evidently $\Gamma(A)$ is a presheaf over $\Gamma(H)$, since the axioms are equational. By an argument analogous to that showing $\Gamma(H)$ is complete, we see it is a sheaf:

Every family $X \subseteq \Gamma(A)$ gives an internal family $X^* = \{ a \in A \mid \bigvee_{x \in X} a = x \}$. If X is compatible externally, then X^* is compatible internally because X generates X^* ; and certainly for $x, y \in X$ we have $[\![x \restriction Ey = y \restriction Ex]\!] = \top$, so $\models \forall a, b \in X^* \ a \restriction Eb = b \restriction Ea$. The internal join gives a global section $[\![\bigvee X^*]\!]$ which is a join for X in $\Gamma(A)$. □

9.8. LEMMA. Every sheaf A over $\Gamma(H)$ arises by taking global sections of some sheaf A over H in $Sh(\Omega)$ determined up to a canonical isomorphism.

Proof. We define $A = \langle A, \restriction, E \rangle$ a sheaf in $Sh(\Omega)$ with $\Gamma(A) = A$.

Let A be the Ω-sheaf with all sections of $A \restriction \pi^{-1}(U)$ as sections over U ; that is, $\Gamma(A,U) = \{ (a,U) \mid a \in |A| \text{ and } Ea \leq \pi^{-1}(U) \}$, and restrictions are given by $(a,U) \restriction W = (a \restriction \pi^{-1}W, W \cap U)$. It is obvious that this gives an Ω-presheaf. We claim that the join of a compatible family (a_i, U_i) of sections of A is given by $(\bigvee a_i, \bigvee U_i)$. If (a,U) and (b,V) are compatible, then $a \restriction \pi^{-1}V = b \restriction \pi^{-1}V$; so a and b are compatible in A, and this "join" is well defined. Furthermore, $a \restriction \pi^{-1}V = b \restriction \pi^{-1}U$ implies that $Eb \wedge \pi^{-1}(U) \leq Ea$. From this and $\bigvee a_i \restriction Ea_i = a_i$ it follows that $\bigvee a_i \restriction \pi^{-1}U_i = a_i$, which makes $(\bigvee a_i, \bigvee U_i)$ the required join.

Since $|A| \cong \Gamma(A)$ generates A , the structure making A into a $\Gamma(H)$-sheaf extends to make A an H-presheaf, A , the necessary compatibility with restrictions being immediate: $a \restriction \pi^{-1}U = b \restriction \pi^{-1}U \Rightarrow a \restriction V \restriction \pi^{-1}U = b \restriction V \restriction \pi^{-1}U$ and $Ea \cap \pi^{-1}U = Eb \cap \pi^{-1}U$. To show that A is an H-sheaf, let $X \subseteq A$ be a sub-Ω-sheaf of A such that

$$\forall\, x, y \in X .\ x \restriction Ey = y \restriction Ex \ .$$

Then $X^* = \{ a \in |A| \mid \text{for some } U \in \Omega \ (a,U) \in |X| \}$ is a compatible family in A . The join of X in A is given by $(\bigvee X^*, \top)$; to see it works argue just as we did to see A is complete. Since any internal compatible family is a restriction of a global compatible family, we are done (alternatively, localize the above argument over Ω). □

We are now in the position to spell out the appearance of $Sh_{\mathbb{E}}(H)$ when $\mathbb{E} \cong Sh(\Omega)$.

9.9. THEOREM.

$$\begin{array}{ccc} Sh_{\mathbb{E}}(H) & \cong & Sh(\Gamma(H)) \\ \Gamma_{\mathbb{E}} \big\downarrow & & \big\downarrow \pi \\ \mathbb{E} & \cong & Sh(\Omega) \end{array} \ .$$

That is to say, we have an equivalence of categories commuting with the two "structure maps" $\Gamma_{\mathbb{E}}$ and π .

Proof. Fortunately most of the work has been done. Now we fill in the arrows.

Any morphism $f : A \to B$ in $Sh_{\mathbb{E}}(H)$ restricts to a sheaf map on global sections. This gives a functor $Sh_{\mathbb{E}}(H) \to Sh(\Gamma(H))$. Any sheaf map $\phi : \Gamma(A) \to \Gamma(B)$ satisfies the compatibility condition

$$a \restriction \pi^{-1}U = b \restriction \pi^{-1}U \qquad (\phi a) \restriction \pi^{-1}U = (\phi b) \restriction \pi^{-1}U$$

and so extends to a sheaf map $A \to B$. Since global sections generate A, this extension is unique. This shows that our functor is full and faithful. By Lemma 9.8 it is an equivalence.

Take A a sheaf over Ω and let $\underset{\sim}{A}$ be the corresponding simple sheaf over H in $Sh_{\mathbb{E}}(H)$. Each section a of A gives a global section a^* of $\underset{\sim}{A}$ with $[\![a^* = b^*]\!] = \pi^{-1}[\![a = b]\!]$: these sections generate $\Gamma(\underset{\sim}{A})$ as a $\Gamma(H)$-sheaf. Since this is just a presentation of $\pi^{-1}(A)$, we see that the diagram commutes. $\square$

One case of the theory we have discussed in this section has received much attention. Let $\mathbb{E}$ be a topos and $j : \Omega \to \Omega$ a topology on $\mathbb{E}$ (see Johnstone [23] , p.76). Working internally, $\Omega \cong P(\mathbb{1})$ and we have a J-map $j : \Omega \to \Omega$ giving a quotient cHa $\Omega \to \Omega/j$. We discussed sheaves over quotients of $P(\mathbb{1})$ in II.2 . We introduce some more terminology from topos theory.

9.10. DEFINITION. A monomorphism $A' \rightarrowtail A$ in $\mathbb{E}$, classified by $\phi : A \to \Omega$, is j-<u>dense</u> iff $j \circ \phi : A' \to \Omega$ factors through *true* . An object B of $\mathbb{E}$ is a j-<u>sheaf</u> iff, whenever $A' \rightarrowtail A$ is j-dense, each $f : A' \to B$ has a unique extension $g : A \to B$ making the diagram

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[dr, "f"'] & A \arrow[d, dotted, "g"] \\ & B \end{tikzcd}$$

commute. We write $Sh_j(\mathbb{E})$ for the full subcategory of $\mathbb{E}$ whose objects are the j-sheaves.

Any endomorphism of Ω can be used to interpret a propositional operator. Thus j gives us a "modal" operator $[\![j\phi]\!]_\Delta = j \circ [\![\phi]\!]_\Delta$ (in the notation of [9]). Following Lawvere, we read $j\phi$ as "locally ϕ " .

9.11. LEMMA. (i) $A' \rightarrowtail A$ is j-dense iff $\models \forall x \in A \; j(x \in A')$.

(ii) B is a j-sheaf iff

(*) $\models \forall X \in P(B)\, [jEIx \in B.(x \in X) \to EIx \in B.j(x \in X)]$.

Proof. The first of these is immediate. As for the second, in one direction

($\Leftarrow$), we work internally: Suppose $\forall X \in P(B)\ jEIx \in B\ (x \in X) \to EIx \in B\ j(x \in X)$ and we are given $f : A' \to B$ with $A' \hookrightarrow A$ j-dense. Define $g(a) = Ix \in B.j(x = f(a))$. Since $j(a \in A')$ and $(a \in A') \to EIx \in B.(x = f(a))$ we have $jEIx \in B\ (x = f(a))$. But now, by our hypothesis, $EIx \in B.j(x = f(a))$ so $g(a)$ is well defined. It is easy to check that g extends f uniquely. Thus, B is a j-sheaf.

Conversely ($\Rightarrow$), let B be a j-sheaf. We write $A' \to A$ for the inclusion

$$\{ X \in P(B) \mid EIx\,.\, x \in X \} \hookrightarrow \{ X \in P(B) \mid jEIx\,.\, x \in X \}, \qquad (1)$$

which is j-dense by the first part of the lemma. But we have an obvious map $f : \{ X \in P(B) \mid EIx\ \ x \in X \} \to B$ defined by $f(X) = Ix\,.\, x \in X$. Let $\hat{f}$ be its unique extension along (1). Certainly, $\forall X \in A\,.\, j(\hat{f}(X) \in X)$, so

$$jEIx.x \in X \to \exists x \in B\,.\, j(x \in X) \qquad (2) \quad .$$

We must show this x is unique. To this end, consider

$$W = \{ \langle X,y,z \rangle \in P(B) \times B \times B \mid j(EIw\,.\, w \in X) \wedge j(y \in X) \wedge j(z \in X) \} \quad .$$

The subset $W' = \{ \langle X,y,z \rangle \in W \mid y = z \}$ is j-dense, and the two morphisms $W \to B$ induced by the projections agree on W'. Thus, by the sheaf condition they are equal; so $\forall \langle X,y,z \rangle \in W\ \ y = z$. Continuing this with (2) we are done. □

9.12. THEOREM. $Sh_{\mathbb{E}}(\Omega/j) \cong Sh_j(\mathbb{E})$.

Proof. Internalizing theorems 5.11 and 5.12 shows us that $Sh_{\mathbb{E}}(\Omega/j)$ is equivalent to the full subcategory of $\mathbb{E}$ consisting of those objects satisfying condition (*) of 9.11 . But, by that Lemma , this is just the category $Sh_j(\mathbb{E})$. □

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