On Completing Ordered Fields

by DANA SCOTT

Every ordered field, not only the field of rationals, can be completed—in a suitable sense. This no doubt follows from the fact that an ordered field is a *uniform space*, but we shall show here that the completion process has a simple algebraic interpretation. The first step is to define what we mean by a *complete* field in a way applicable to arbitrary cardinalities, for the usual Dedekind notion of completeness for fields implies isomorphism with the reals. The definition that is appropriate is as follows:

Definition. A given ordered field is called *complete* if it has no proper extension to an ordered field in which the given field is order-dense.

If K and L are ordered fields and $K \subseteq L$, then K is *dense* in L (order-dense) if between any two distinct elements of L there lies an element of K. We say K is *cofinal* in L if every element of L is exceeded by an element of K. If K is cofinal in L, it is easy to see that it is also *coinitial* in the sense that for $\epsilon \in L$, $\epsilon > 0$, there always exists $\delta \in K$ with $0 < \delta < \epsilon$. Note, however, that K can be cofinal in L without being dense. [Give a non-Archimedean ordering to $K = Q((t))$, the field of formal power series. Let $L = \bar{K}$ be the real-closure of K, and ask yourself what lies between $\sqrt{t}$ and $2\sqrt{t}$.] Note also that every ordered field is cofinal in its real-closure, because easy estimates on the roots of a polynomial are always rationally computable from the coefficients.

We shall establish two results that seem to answer the obvious questions about fields complete in the sense of our chosen definition.

***Theorem* 1.** *Given any ordered field K, there is a complete ordered field $\hat{K}$ in which K is dense. Any other complete ordered field in which K is dense is isomorphic to $\hat{K}$ by a unique isomorphism that is the identity on K.*

It follows that if K is dense in L, then K is dense in $\hat{L}$. Thus $\hat{K}$ and $\hat{L}$ are isomorphic. This makes L isomorphic to a subfield of $\hat{K}$. This isomorphism is the identity on K and is unique—because density implies that each element of L is determined by the cut it makes in the ordered set K. Thus in many senses $\hat{K}$ is a maximal ordered field in which K is dense. Note that if K is the field of rationals, then clearly $\hat{K}$ is isomorphic to the reals.

***Theorem* 2.** *If K is dense in L, then passing to the real-closures we have $\bar{K}$ dense in $\bar{L}$.*

From Theorem 2 it follows that if K is real-closed, then so is $\hat{K}$. Or better, $\hat{K}$ is real-closed if and only if K is dense in its real-closure $\bar{K}$. Because if $\hat{K}$ is real-closed, then $K \subseteq \bar{K} \subseteq \hat{K}$, which means K is dense in $\bar{K}$. On the other hand, if K is dense in $\bar{K}$, then since K is dense in $\hat{K}$, the theorem implies $\bar{K}$ is dense in $\bar{\hat{K}}$. Therefore, $\hat{K}$ is dense in $\bar{\hat{K}}$, and so $\hat{K} = \bar{\hat{K}}$, which shows that the field is real-closed. [We note that the example of $Q((t))$ mentioned above shows that not every ordered field is dense in its real-closure.]

Proof of Theorem 1. Given an ordered field K we first extend K to an ordered field M in which *every* cut in K is filled. In other words, if sets $A,B \subseteq K$ are such that $A < B$ (that is, every element of A is less than every element of B), then there is at least one element $x \in M$ with $A < x < B$. The existence of such an M is very easy to establish. One can use the compactness theorem of first-order logic, or use the ultrapowers, or use the enlargements employed for nonstandard models of analysis, or simply adjoin indeterminates one after another until all the cuts are filled. Of course, M is a miserable field with many too many elements. The next step is to form a more reasonable quotient field of subring of M.

Let F be the subring of all *K-finite* elements of M; that is, F consists of those $x \in M$ for which there exists a $y \in K$ with $|x| < y$. We let I be the ideal of *K-infinitesimal* elements of F; that is, I consists of those $x \in F$ for which $|x| < y$ for *all* $y \in K$ where $y > 0$. Now I is a maximal ideal of F and we let $K' = F/I$. It is easy to see that K' is an ordered field and that K is isomorphic to a subfield by the obvious injection. Let us simply make $K \subseteq K'$. Now K' is less miserable than M for K is *cofinal* in K'. Furthermore, for every cut $A < B$ in K, there will exist $x \in K'$ with $A \leq x \leq B$, when A and B are *nonempty*. But generally K' is still too large to be $\hat{K}$; we shall find our completion as a subfield of K'.

To this end let K^+ be the set of strictly positive elements of K, and let $\hat{K}$ be the set of $x \in K'$ such that for every $a \in K^+$ there is an element of K between x and $x + a$. Note that $\hat{K}$ contains not only K but also all subfields of K' in which K is dense. Furthermore, K is dense in $\hat{K}$, because if x, $y \in \hat{K}$ and $x < y$, then we can choose $a \in K^+$ with $a \leq y - x$ (K is coinitial in K'). Next take $b \in K$ with $x < b < x + a$, which puts b between x and y. If we can only show that $\hat{K}$ is a subfield of K', then $\hat{K}$ will be *the* maximal subfield of K' in which K is dense, which is a more pleasant characterization of $\hat{K}$.

Well, let us show first that $\hat{K}$ is closed under *addition*. Suppose $x, y \in \hat{K}$ and $a \in K^+$. Choose b, $c \in K$ so that $x < b < x + \frac{1}{2}a$ and $y < c < y + \frac{1}{2}a$. Then $x + y < b + c < (x + y) + a$, and we see why $x + y \in \hat{K}$.

To show that $\hat{K}$ is closed under *minus*, let $x \in \hat{K}$ and $a \in K^+$. By the above, since $-a \in K \subseteq \hat{K}$, we know $x - a \in \hat{K}$. Thus for some $b \in K$ we have $x - a < b < (x - a) + a = x$. Hence $-x < -b < -x + a$, and we see why $-x \in \hat{K}$.

To show that $\hat{K}$ is closed under *product*, it is enough to consider $x, y \in \hat{K}$ with x, $y > 0$. First pick $b \in K^+$ with $b \leq a/2y$. Next pick $c \in K^+$ with $c \leq a/2(x + b)$. This is all arranged to make $by + cx + bc \leq a$. Now take d, $e \in K$ with $x < d < x + b$ and $y < e < y + c$. We find

$$xy < d \cdot e < x \cdot y + by + cx + bc \leq x \cdot y + a,$$

and we see why $x \cdot y \in \hat{K}$.

Finally, to show that $\hat{K}$ is closed under *inverse*, let $x \in \hat{K}$, $x > 0$, and let $a \in K^+$. Choose $b \in K^+$ so that $b \leq ax^2/(1 + ax)$, and pick $c \in K$ with $x - b < c < x$. Since this proof was found working backward, we are not surprised that

$$0 < \frac{x}{1 + ax} = x - \frac{ax^2}{1 + ax} \leq x - b < c < x,$$

whence $x^{-1} < c^{-1} < x^{-1} + a$, which shows us why $x^{-1} \in \hat{K}$.

Let us now establish the completeness of $\hat{K}$. Suppose $\hat{K} \subseteq L$ and $\hat{K}$ is dense in L. Then K is dense in L also. Let $x \in L$ and let

$$A = \{y \in K : y \leq x\},$$
$$B = \{y \in K : x < y\}$$

be the cut in K determined by x. Now in K' there is an element x' such that $A \leq x' \leq B$. We note that $x' \in \hat{K}$ because if $a \in K^+$, then for some $b \in K$ we have $x < b < x + a$. This means $b \in B$ while $b - a \in A$. Therefore, $x' \leq b \leq x' + a$. Now either the strict inequalities hold, or if not, $x' \in K$ and we can modify b to make the inequalities strict.

Hence $x' \in \hat{K}$. But then $x' \in L$ also, and in view of the density of K in L, it follows, from the fact that the two elements determine the same cut, that $x = x'$. This puts $x \in \hat{K}$; therefore, $L = \hat{K}$.

It is now time to reveal the nature of $\hat{K}$. The elements of $\hat{K}$ are in a one-to-one correspondence with the cuts in K that are *never* invariant under a nonzero translation by an element of K. We could have indeed constructed $\hat{K}$ this way. But then the proof that $\hat{K}$ is a field is quite tiresome. We have side-stepped this issue by starting with M (better, K') in which the axioms for a field are already satisfied. Then $\hat{K}$ can be a subfield. The necessary algebra is not avoided, however, because we still had to prove that $\hat{K}$ was closed under the field operations.

There is still one step missing, unfortunately: the uniqueness of $\hat{K}$. For this it seems best to use the cuts. Suppose L is another complete ordered field in which K is dense; then we can map each element $x \in L$ uniquely and one to one to an element $f(x) \in \hat{K}$ so that

$$\{y \in K: x < y\} = \{y \in K: f(x) < y\}.$$

This mapping is the identity on K. One can then proceed directly to show that this mapping preserves order, addition, minus, and products of positive elements. This is enough to conclude that L is mapped onto a subfield. But since L is complete, it must be mapped *onto* K.

Proof of Theorem 2. Suppose K is dense in L. We wish to show that $\bar{K}$ is dense in $\bar{L}$. (Obviously we are assuming that $\bar{K} \subseteq \bar{L}$, which is reasonable because $K \subseteq \bar{L}$ and $\bar{L}$ is real-closed.) Note that since L is cofinal in $\bar{L}$ and K is dense in L, then K is also cofinal in $\bar{L}$. To establish the denseness of $\bar{K}$ in $\bar{L}$ it is enough to prove:

(*) for each $x \in \bar{L}$ and $\eta \in K^+$, there exists $y \in \bar{K}$ with $|x - y| < \eta$

For if $u, v \in \bar{L}$ and $u < V$, then let $x = \frac{1}{2}(u + v)$ and choose $\eta \in K^+$ with $\eta \leq \frac{1}{2}(v - u)$. Then whenever $|x - y| < \eta$, obviously y must lie between u and v. So, by (*), some element of $\bar{K}$ lies between u and v.

Our proof of (*) will be based essentially on the principle that the roots of a polynomial depend continuously on the coefficients. Roughly, we take a polynomial which x satisfies with coefficients in L, then modify the coefficients ever so slightly to lie in K in such a manner that the new polynomial has a root within η of x which, of course lies in $\bar{K}$. It is possible to make this argument precise as it stands, but the following proof is slightly more elementary.

Let $f(x) = 0$ where

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n$$

with the $a_i \in L$. Now we can assume that $f(t)$ has no multiple roots and that it is *monotonic* in a neighborhood of x, say for t where $|x - t| \leq \eta' \leq \eta$. Thus $f(x - \eta')$ and $f(x + \eta')$ have opposite signs. Choose $\epsilon \in K^+$ so that $\epsilon < \min(|f(x - \eta')|, |f(x + \eta')|)$. Next choose $k \in K^+$ so large that $|x \pm \eta'|^i \leq k$ for $i = 0, \cdots, n - 1$. We can now pick $b_i \in K$, $i = 1, \cdots, n$, so that $|a_i - b_i| \leq \epsilon/nk$ for each i. Let

$$g(t) = t^n + b_1 t^{n-1} + \cdots + b_{n-1}t + b_n,$$

and note that

$$|f(x \pm \eta') - g(x \pm \eta')| \leq \sum_{i=1}^{n} |a_i - b_i|\, |x \pm \eta'|^{n-i} \leq \sum_{i=1}^{n} \left(\frac{\epsilon}{nk}\right) \cdot k = \epsilon.$$

Hence $g(x - \eta')$ has the same sign as $f(x - \eta')$, and $g(x + \eta')$ has the same sign as $f(x + \eta')$. Therefore, $g(t)$ changes sign between $x - \eta'$ and $x + \eta'$. Hence $g(y) = 0$ for some $y \in \bar{K}$ with $|x - y| < \eta' \leq \eta$, and the proof is complete.

Historical remarks. The results presented in this paper were obtained in the summer of 1961 while the author was a Miller Research Fellow at the University of California at Berkeley. The author profited at that time very much from conversations with A. Robinson and G. Kreisel. The paper was never published before because no application was apparent, and, in any case, the results are rather elementary. However, since several people have asked about them, the author is glad to have this opportunity to present them.

After reading a draft of the paper, E. Zakon kindly pointed out that similar results for ordered Abelian groups were formed by L. W. Cohen and Casper Goffman [*Trans. Am. Math. Soc. 67* (1949), 310–319]. Indeed, the idea of using the special cuts goes back to R. Baer [*J. Reine Angew. Math. 160* (1929), 208–226], who calls them Dedekindean cuts. We have in effect reproved some of the Cohen–Goffman results (sec. 2 of their paper) and extended the method to fields. They do not, for some reason, stress the fact that the original structure is order-dense in its completion and, further, their notion of completeness has to do with convergent (transfinite) sequences, but the notion proves to be equivalent to ours. They do, however, prove the interesting fact that the groups (and hence the real-closed fields) constructed by H. Hahn's method of formal power series are complete. Although the algebra is very simple, it is not at once obvious that the cuts they use for the additive structure are also appropriate for the full field structure. That, in a (rather small) nut shell, is the contribution of the present paper, for whatever it is worth.