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THE PRESHEAF MODEL FOR SET THEORY

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The whole discussion is just the putting together of two-plus-two from known facts in topos theory, but it is a useful exercise for me to get various things straight.

§1. The Construction. Let $\mathbb{C}$ be a fixed small category, called the <u>site</u>. It has <u>domains</u> (objects, types) A, B, C and <u>maps</u> $f: B \to A$, $g: C \to B$, etc. <u>Composition</u> $f \circ g: C \to A$ is written in the indicated order. The <u>identity</u> map on a domain A is written as 1_A. The usual axioms are satisfied about composition and identities. That $\mathbb{C}$ is small means the number of domains in $\mathbb{C}$ is limited and, for domains A, B, the collection $\{f \mid f: B \to A\}$ is always a set.

In making the model, we will often have need of a notation for functions (sets of ordered pairs). Thus:

$$(x_i)_{i \in I} = \{(i, x_i) \mid i \in I\},$$

where an ordered pair has $(a,b) = \{\{a\},\{a,b\}\}$. Note $(a,b) \neq \emptyset$. Therefore, if we also use the notation:

$$\langle y_j \rangle_{j \in J} = \{\emptyset\} \cup (y_j)_{j \in J},$$

then always $(x_i)_{i \in I} \neq \langle y_j \rangle_{j \in J}$. That is to say, we have functions (vectors, systems, families) in two "colours".

DEFINITION 1.1. Let A be a domain of $\mathbb{C}$. An <u>individual</u> (at stage A) is a system

$$a = (a_f)_{f: B \to A}$$

of arbitrary things indexed by $\{f \mid f: B \to A\}$. <u>Restriction</u> along a map $f: B \to A$ of $\mathbb{C}$ is given by:

$$a \upharpoonright f = (a_{f \circ g})_{g: C \to B}.$$

If we let I_A be the <u>class</u> of all individuals at stage A, then $a \upharpoonright f \in I_B$.

The notion of a set-valued pre-sheaf is assumed known; it is a functor $\mathcal{F}$ from $\mathbb{C}^{op}$ into Sets. If A is in $\mathbb{C}$, then $\mathcal{F}(A)$ is a set; and if $f: B \to A$ in $\mathbb{C}$, then $\mathcal{F}(f): \mathcal{F}(A) \to \mathcal{F}(B)$ is a function. We can define for $a \in \mathcal{F}(A)$ the family:

$$\bar{a} = (\mathcal{F}(f)(a))_{f:B\to A}\,.$$

Then $\bar{a}\upharpoonright f = \overline{\mathcal{F}(f)(a)}$. In this way the presheaf $\mathcal{F}$ is adequately represented by individuals; that is

$$\{\bar{a} \mid a \in \mathcal{F}(A)\} \subseteq I_A\,,$$

and $\bar{a} = \bar{b}$ iff $a = b$, because $(\bar{a})_{1_A} = a$.

DEFINITION 1.2. A set (at stage A) is a system

$$s = \langle s_f \rangle_{f:B\to A}$$

where for all $f: B \to A$, we have $s_f \subseteq I_B \cup V_B$, and whenever $g: C \to B$ and $b \in s_f$, then $b\upharpoonright g \in s_{f\circ g}$. Here V_B is the class of all sets at stage B. Restriction is defined by

$$s\upharpoonright f = \langle s_{f\circ g} \rangle_{g:B\to A}$$

and it easily follows that $s\upharpoonright f \in V_B$.

The definition above may seem circular because in defining V_A we use V_B. The apparent circularity is eliminated, however, if we argue by rank; that is, since $\mathbb{C}$ is small, a system has a limited rank, so we can consider the elements $b \in s_f$ as having lower rank. This means

that the model is "built from below", and there is no circularity. (This argument could be made more explicit by introducing ordinals as ranks.)

The formulae of set theory are compounded from atomic formulae $x = y$, $x \in y$, $x \in V$ in the usual way with logical symbols $\wedge, \vee, \neg, \to, \forall, \exists$. (Here $x \in V$ should be read as a one-place predicate "x is a set".) In defining "forcing" (truth in the model), we use the method of Joyal.

DEFINITION 1.3. For an assignment a of values in $I_A \cup V_A$ to variables we define $A \Vdash \Phi[a]$, read "A forces Φ at a" by these clauses:

(i) $A \Vdash x = y\,[a]$ iff $a(x) = a(y)$

(ii) $A \Vdash x \in y\,[a]$ iff $a(x) \in a(y)_{1_A}$ and $a(y) \in V_A$

(iii) $A \Vdash x \in V\,[a]$ iff $a(x) \in V_A$

(iv) $A \Vdash [\Phi \wedge \Psi][a]$ iff $A \Vdash \Phi[a]$ and $A \Vdash \Psi[a]$

(v) $A \Vdash [\Phi \vee \Psi][a]$ iff $A \Vdash \Phi[a]$ or $A \Vdash \Psi[a]$

(vi) $A \Vdash \neg\Phi[a]$ iff for no $f: B \to A$ we have $B \Vdash \Phi[a{\restriction}f]$

(vii) $A \Vdash [\Phi \to \Psi][a]$ iff whenever $f: B \to A$ and $B \Vdash \Phi[a{\restriction}f]$ then $B \Vdash \Psi[a{\restriction}f]$

(viii) $A \Vdash \forall x.\Phi[a]$ iff whenever $f: B \to A$ and $b \in I_B \cup V_B$, then $B \Vdash \Phi[a{\restriction}f(b/x)]$

(ix) $A \Vdash \exists x.\Phi[a]$ iff for some $b \in I_A \cup V_A$ $A \Vdash \Phi[a(b/x)]$

Here $a{\restriction}f$ is the assignment $(a{\restriction}f)(x) = a(x){\restriction}f$ and $a(b/x)$ is the assignment like a except $a(b/x)(x) = b$.

We should note that in the definition of forcing the quantifiers are <u>unbounded</u>. Thus, we must regard $A \Vdash \Phi[a]$ as a "translation" in the following sense: For each Φ separately the rules (i)-(ix) allow us to write out the definition of the corresponding relation between A and a. If we tried to Gödel number the formulae, we would have something very like the truth definition — which cannot

be formalized in set theory. But there is no bother here, since we work on formulae "one at a time".

§2. The Verification of the Axioms. Set theory in intuitionistic logic has — in basic outline — settled down, and I use the same axioms as from Fourman's paper, which have also been used elsewhere. It is intuitionistic logic we have to employ, since the law of the excluded middle is not generally forced — even for some very simple categories $\mathbb{C}$.

DEFINITION 2.1. The <u>axioms and axiom schemata</u> of IZF (intuitionistic Zermelo-Fraenkel Set theory) are:

(Decidability)

$$\forall x\, [x \in V \vee \neg x \in V]$$

(Membership)

$$\forall x,y\, [x \in y \to y \in V]$$

(Extensionality)

$$\forall x,y \in V\, [\forall z\, [z \in x \leftrightarrow z \in y] \to x = y]$$

(Pairing)

$$\forall x, y\ \exists z\, [x \in z \wedge y \in z]$$

(Separation)

$$\forall x\ \exists y \in V\ \forall z\, [z \in y \leftrightarrow z \in x \wedge \Phi(z)]$$

(Union)

$$\forall x\ \exists y\ \forall z\, [\exists w \in x.\, z \in w \rightarrow z \in y]$$

(Powerset)

$$\forall x\ \exists y\ \forall z\, [\forall w \in z.\, w \in x \rightarrow z \in y]$$

(Infinity)

$$\exists x\, [\exists y.\, y \in x \wedge \forall y \in x\ \exists z \in x,\ y \in z]$$

(Induction)

$$\forall x\, [\forall y \in x.\, \Phi(y) \rightarrow \Phi(x)] \rightarrow \forall x.\, \Phi(x)$$

(Collection)

$$\forall x\, [\forall y \in x.\ \exists z.\, \Phi(y,z) \rightarrow \exists w\ \forall y \in x\ \exists z \in w.\ \Phi(y,z)]$$

Where we use the notation $\Phi(z)$ in an informal sense to indicate principal free variables and substitution. (In (Separation), for instance, y must not be free in $\Phi(z)$.)

THEOREM 2.2. <u>All axioms of IZF are forced as well as all the axioms and rules of intuitionistic logic.</u>

The verification of the logic part is standard; the set-theory part will take a little time. The forcing clause 1.3 (i) looks rather stark; but even if we are assuming classical logic in our background logic, the <u>decidabi</u> of $=$ does not follow. Indeed

$$A \Vdash a = b \vee \neg a = b \text{ iff } a = b \text{ or for no } f: B \to A \text{ does } a1f = b1f.$$

The point here is that $a \neq b$ may result if $a_{1_A} \neq a_{1_A}$, but $a1f = b1f$ is perfectly possible when $f \neq 1_A$. Neither does it follow that equality is <u>stable</u>, for

$$A \Vdash \neg\neg a = b \text{ iff for all } f: B \to A \text{ there is } g: C \to B \text{ with } a1fog = b1fog$$

Taking $\mathbb{C}$ to be a partially ordered set/ahe is sufficient to see this does not mean the same as $A \Vdash a = b$. (Note we replace the assignments with constants when there is no

On the other hand we were at pains to make I_A disjoint from V_A, so the verification of (Decidability) is clear. (Membership) is also trivial in view of the extra clause of 1.3 (ii) – which is only sensible.

Before verifying the other axioms, it is useful to note the functionality of forcing:

LEMMA 2.3. <u>If $A \Vdash \Phi[a]$ and $f: B \to A$, then $B \Vdash \Phi[a1f]$.</u>

This can be proved in the usual way directly from 1.3. We then use this fact to note that to show an implication is forced at <u>all</u> stages it is enough to show that whenever the hypothesis is forced, the conclusion is forced.

Thus suppose $a, b \in V_A$ and $A \Vdash \forall z\, [z \in a \leftrightarrow z \in b]$. Then for all $f: B \to A$ and for all $c \in I_B \cup V_B$ we have $B \Vdash c \in a1f \leftrightarrow c \in b1f$. In particular $c \in (a1f)_{1_A} = a_f$ iff $c \in b_f$. But then $a_f = b_f$ for all $f: B \to A$, so $a = b$ follows. Indeed $A \Vdash a = b$. So (Extensionality) is verified

For (Pairing), let $a, b \in I_A \cup V_A$ and define

$$c = \langle \{a1f, b1f\} \rangle_{f: B \to A}$$

We find $A \Vdash a \in c \wedge b \in c$ because $c \in V_A$.

For (Separation), let $a \in I_A \cup V_A$ be given. Define

$$b = \langle \{c \in I_B \cup V_B \mid B \Vdash c \in a1f \wedge \Phi1f(c)\} \rangle_{f: B \to A}$$

where $\Phi1f$ has all parameters (from Stage A) restricted by f (to stage B). The indicated set <u>is</u> a set, because if $a \in I_A$ it is empty, but if $a \in V_A$ it is a subset of a_f. The reason that $b \in V_A$ is that by Lemma 2.3 we prove that if $g: C \to B$ and $c \in b_f$, then $c1g \in b_{f \circ g}$; indeed this is nothing more than a restatement of the lemma. The remainder of the verification is obvious.

For (Union), suppose $a \in V_A$ and define

$$b = \langle \{c \in I_B \cup V_B \mid B \Vdash \exists w \in a1f.\, c \in w\} \rangle_{f: B \to A}$$

This is a family of sets because

$$b_f = \bigcup \{ d_{1_B} \mid d \in a_f \text{ and } d \in V_B \}.$$

We have, of course, constructed b to satisfy the axiom.

For (Powerset), suppose $a \in V_A$ and define

$$b = \langle \{ c \in V_B \mid c_g \subseteq a_{f \circ g} \text{ for all } g : C \to B \} \rangle_{f : B \to A}$$

It is easy to check that $b \in V_A$ and satisfies the axiom.

We skip (Infinity) for the moment, as its truth will follow directly from the remarks of the next section.

To verify (Induction), we remark first that in order to check the <u>axiom</u> it is sufficient to check the following <u>rule</u>:

$$\frac{\forall x [\forall y \in x . \Psi(y) \to \Psi(x)]}{\forall x . \Psi(x)}$$

The reason is that the axiom follows from the rule by application to the formula defined this way:

$$\Psi(x) \leftrightarrow \forall \vec{z} [\forall x [\forall y \in x . \Phi(y) \to \Phi(x)] \to \Phi(x)] .$$

(Note that in this choice of $\Psi(x)$, the variable x is bound in the premiss but left free in the conclusion; while the quantifier $\forall \vec{z}$ is introduced to quantify over all the other parameters of $\Phi(x)$.) Now, in proving the rule in the interpretation, we assume that every A forces the hypothesis and show every A forces the conclusion. To this end, consider the pairs (f, b) where $f : B \to A$ is any map in $\mathbb{C}$ and $b \in I_B \cup V_B$.

Define between such pairs the relation $(g, c) \lessdot (f, b)$ which holds provided $g : C \to B$ and $c \in b_g$ and $b \in V_B$. We note that the rank of c must be strictly less than the rank of b, and so $\lessdot$ is a well-founded relation. Also for fixed (f, b) the collection of pairs $(g, c) \lessdot (f, b)$ must form a set because $\mathbb{C}$ is a small category; therefore by using (Induction) in the metalanguage, we know we can apply induction arguments to the relation $\lessdot$.

Now consider the property of pairs (f, b) where $B \Vdash \Psi(b)$. Since A forces the hypothesis to the rule, if all $(g, c) \lessdot (f, b)$ have this property then (f, b) has the property. Thus, by induction, all pairs (f, b) have the property. This is the same thing as saying that $A \Vdash \forall x . \Psi(x)$, as was to be proved.

Finally, we must consider (Collection). Suppose $a \in V_A$ and suppose $A \Vdash \forall y \in a \, \exists z . \Phi(y, z)$. This means:

$$\forall f : B \to A \; \forall b \in a_f \; \exists c \in I_B \cup V_B . \; B \Vdash \Phi{\restriction}f(b, c)$$

As the pairs (f, b) with $b \in a_f$ form a <u>set</u>, we can envoke the collection principle itself to secure a set S where:

$$\forall f : B \to A \; \forall b \in a_f \; \exists c \in S \cap (I_B \cup V_B) . \; B \Vdash \Phi{\restriction}f(b, c) .$$

We have, then, only to introduce

$$d = \langle s \cap (I_B \cup V_B) \rangle_{f:B\to A}$$

as an element $d \in V_A$ to find, as required:

$$A \Vdash \forall y \in a\, \exists z \in d\,.\, \Phi(y,z)\,.$$

Except for (Infinity) this completes the verification of the axioms.

§3. Global Elements. In the construction of the model, the elements of $I_A \cup V_A$ only "exist" at stage A — and, by restriction, at later stages when $f: B \to A$ is produced. It should be clear, however, that there are many sets and individuals that have a fuller existence. The way they exist can be defined as follows.

DEFINITION 3.1. A <u>global individual</u> is an assignment $a_A \in I_A$ for each A of $\mathbb{C}$ such that $a_A \restriction f = a_B$ whenever $f: B \to A$ in $\mathbb{C}$. <u>Global sets</u> are defined similarly.

LEMMA 3.2. <u>Any global element is completely determined by the family of objects a_{A1_A} for A of $\mathbb{C}$.</u>

<u>Proof</u>. The argument for sets and individuals is the same, so consider the latter. We have:

$$a_A = (a_{Af})_{f:B\to A} \quad \text{and}$$

$$a_A \restriction f = (a_{Af\circ g})_{g:C\to B}$$

But, since the element is global, $a_{Bg} = a_{Af\circ g}$ for all $g: C \to B$. Taking $g = 1_B$ we find

$$a_A = (a_{B1_B})_{f:B\to A}\,,$$

which establishes what we want.

As a consequence of this easy lemma, we see that to specify a global <u>individual</u> we only need to give an <u>arbitrary</u> family of objects a_A $(= a_{A1_A})$ over the A of $\mathbb{C}$. (That is, we may as well simplify the notation.) To specify a global <u>set</u>, what we need is a subset $S_A \subseteq I_A \cup V_A$ where restriction along $f: B \to A$ always maps S_A into S_B. Then $\langle S_B \rangle_{f:B\to A} \in V_A$ as required. This perhaps seems familiar? We can make the connection with known constructs quite precise with the help of a definition.

DEFINITION 3.3. A <u>global map</u> is a family of relations $r_A \subseteq (I_A \cup V_A) \times (I_A \cup V_A)$ where restriction along $f: B \to A$ always maps the pairs in r_A into pairs in r_B; further each r_A should be single valued. For global sets s and t we also say r <u>maps</u> s <u>into</u> t ($r: s \to t$) provided the domain of r_A is s_A and the range is contained in t_A.

PROPOSITION 3.4. <u>The category of global sets and global maps is equivalent to the category of set-valued pre-sheaves on $\mathbb{C}^{op}$ and natural transformations.</u>

<u>Proof</u>. We had already remarked after Definition 1.1 how every functor $\mathcal{F}: \mathbb{C}^{op} \to \mathrm{Sets}$ creats individuals (<u>not</u> global individuals note!), and we now recognize the sets

$$\bar{\mathcal{F}}_A = \{\bar{a} \mid a \in \mathcal{F}_A\}$$

as being a family making a global set $\bar{\mathcal{F}}$ corresponding to the functor $\mathcal{F}$. Conversely, every global set taken together with the induced restriction maps is a pre-sheaf. The correspondence is only an equivalence because $\bar{\mathcal{F}}_A \subseteq I_A$ while a little more broadly $s_A \subseteq I_A \cup V_A$, but the difference does not matter once we agree to look at these systems only up to one-one correspondences under global maps.

With this picture in mind it can be safely left to the reader to verify that the notion of a global map translates <u>word-for-word</u> into the definition of natural trans-formation between functors.

Where would global maps come from? Suppose s and t are global sets and $\Phi(x,y)$ is a formula with two free variables and which has as parameters <u>only global elements</u>. Suppose then that every A forces $\forall x \in s\, \exists! y \in t.\, \Phi(x,y)$. We then define

$$r_A = \{(a,b) \in s_A \times t_A \mid A \Vdash \Phi(a,b)\}.$$

We easily check that $r: s \to t$. Conversely, given any global map, we can introduce a global set that represents it; and so with a suitable formula we would be able to state the mapping relationship in formal set-theoretical terms. In other words the global maps correspond exactly to the functional relationships <u>definable</u> from global parameters. We can look at this remark as a justification of the notion of natural transformation:

with a suitable reinterpretation set-valued functors can be viewed as sets and the logical notion of function comes out exactly as that of natural transformation; the idea is forced on us by the logic of the situation. (By constructing models within models, this discussion could be carried over to functors between any two (small) categories and their natural transformations.)

A very special case of global elements are the constant elements. Suppose in our metatheory we think of an individual $a \in I$. Then the constant individual in the model $\check{a}_A = (a)_{f:B\to A}$ is obviously global. We extend this to sets $s \in V$. By assumption any set is a subset $s \subseteq I \cup V$, so naturally

$$\check{s}_A = \langle \{ \check{x}_B \mid x \in s \} \rangle_{f:B\to A}$$

and we see that this is a global set where

$$A \Vdash \check{a}_A \in \check{s}_A \quad \text{iff} \quad a \in s$$

This principle can be extended to all formulae, provided we introduce one-place predicates $x \in \check{I}$ and $y \in \check{V}$ with the obvious meanings. Then if Φ is any formula, we take $\check{\Phi}$ to be the result of relativizing

all quantifiers to $\check{I} \cup \check{V}$. We easily prove:

PROPOSITION 3.5. For any formula Φ with only constant parameters, we have $A \Vdash \check{\Phi}$ iff Φ is true.

The point of this result is that inside the model $\check{I} \cup \check{V}$ gives an exact picture of the external universe $I \cup V$; of course there is no reason to believe that $\check{I}$ and $\check{V}$ are definable properties; we had to impose them to be able to formulate $\check{\Phi}$. Nevertheless, many facts not mentioning $\check{I}$ and $\check{V}$ can be deduced with the aid of 3.5. For example, consider $\check{\omega}_A$ the image of the set of integers $\omega = \{0, 1, 2, \dots\}$. Because every integer (ordinal) is regarded as the set of all smaller numbers, it easily follows that $\check{\omega}_A$ validates the axiom of infinity in the model.

§4. Representing the Site. Not only can the external universe be mapped into the model, but so can the category $\mathbb{C}$

DEFINITION 4.1. For each map $f: B \to A$ in $\mathbb{C}$ we write:

$$\mathring{f} = (f \circ g)_{g: C \to B}.$$

For each domain A we also write

$$\hat{A}_B = \{\mathring{f} \mid f: B \to A\}.$$

It is clear that $\mathring{f} \in I_B$, but it is not a global individual. On the other hand $\hat{A}_B \in V_B$ is a global set; it is called the representable functor in category theory. We remark again that whenever it is convenient we drop the distinction between $\hat{A}_B$ and $\hat{A}_{B 1_B}$. Also in speaking of these global sets we often just say $\hat{A}$, $\hat{B}$, etc. when the stage is not important. We can also write $B \Vdash a \in \hat{A}$ to mean that $a = \mathring{f}$ for some $f: B \to A$, dropping the subscripts in the formula because they are determined in any case. We remark also that the correspondences $f \mapsto \mathring{f}$ and $A \mapsto \hat{A}$ are one-one.

Translating the well-known Yoneda Lemma into the present language (in a way that is in fact quite common in topos theory) we can state:

PROPOSITION 4.2. The category of global sets $\hat{A}$ and global maps results from a full and faithful embedding of the given category $\mathbb{C}$; moreover for any global set S there is a one-one correspondence between global maps from $\hat{A}$ into S and elements of S_A.

Proof. Suppose $h: A \to B$ in $\mathbb{C}$. If $f: C \to A$, then $h \circ f: C \to B$, so the relation

$$r_C \subseteq \hat{A}_C \times \hat{B}_C \quad \text{where}$$

$$r_C = \{(\mathring{f}, (h \circ f)^{\circ}) \mid f: C \to A\},$$

which we ought to call $\hat{h}$, is a global mapping $\hat{h}: \hat{A} \to \hat{B}$. Now $h \mapsto \hat{h}$ is one-one and $\widehat{h \circ k} = \hat{h} \circ \hat{k}$; so $\hat{}$ is a functor from $\mathbb{C}$ into the global set category, and the one-oneness means it is "faithfull".

Let now $r: \hat{A} \to \hat{B}$ be any other global map. Define h as the map $h: A \to B$ where

$(\dot{1}_A, \dot{h}) \in r_A$. It follows at once from restriction calculations that $(\dot{f}, (h\circ f)^{\cdot}) \in r_C$ for all $f: C \to A$; therefore $r = \hat{h}$.

In the same way one can show any global $r: \hat{A} \to S$ determines an element in S_A corresponding to the identity function and any such matching leads to a global map.

The result just established gives us an interesting reinterpretation of the site $\mathbb{C}$. To start with, $\mathbb{C}$ was just any abstract category. By passing to the model, we can regard $\mathbb{C}$ now as a "concrete" category where the $\hat{A}$ and $\hat{B}$ are "sets" and the maps $f: A \to B$ are "ordinary functions" $\hat{f}: \hat{A} \to \hat{B}$. Thus, if A and B are isomorphic in $\mathbb{C}$, then $\hat{A}$ and $\hat{B}$ are in a one-one correspondence in the model. These carry overs of abstract mapping properties may not seem particularly interesting, but they become much more interesting when $\mathbb{C}$ can be given more categorical structure.

DEFINITION. 4.3. The category $\mathbb{C}$ is said to be a <u>cartesian category</u> if it is provided with structure satisfying the following rules:

$$0_A : A \to 1 \qquad\qquad \frac{a : A \to 1}{a = 0_A}$$

$$\begin{array}{l} p_{AB} : A \times B \to A \\ q_{AB} : A \times B \to B \end{array} \qquad\qquad \frac{f : C \to A \quad g : C \to B}{\begin{array}{c} p_{AB} \circ \langle f, g\rangle = f \\ q_{AB} \circ \langle f, g\rangle = g \end{array}}$$

$$\frac{f : C \to A \quad g : C \to B}{\langle f, g\rangle : C \to A \times B} \qquad\qquad \frac{h : C \to A \times B}{\langle p_{AB} \circ h, q_{AB} \circ h\rangle = h}$$

In other words, we state abstractly in $\mathbb{C}$ that 1 is the one-element domain (terminator) which has one and only one map into it from any domain. In the model 1 will go over to $\hat{1}$ which proves to be a constant set — and obviously one which satisfies the formal statement that it has a unique element.

The construct $A \times B$ is a functor on the category, though we have said just a little less than that in the definition.

(What is missing is the definition of $h \times k$ for an arbitrary pair of maps.) When we translate the information into the model from the definition, we find that $\widehat{A \times B}$ as a global set is in a one-one correspondence with the set-theoretical cartesian product $\hat{A} \times \hat{B}$ in the model; moreover the maps p_{AB} and q_{AB} give just the projection maps, and the construct $\langle f, g \rangle$ is nothing other than the argument-wise pairing of functions.

It follows that if we give, with the aid of these notions, some mapping properties in $\mathbb{C}$, they will go over to their normal set-theoretical translations. For instance, a "group" in a cartesian category is in the first place a map $\mu : A \times A \to A$, for <u>multiplication</u>, together with a map $\iota : A \to A$ for <u>inversion</u>, and then lots of diagrams should commute to express the usual (equational) axioms of group theory. Surprise! $\hat{A}$ in the model is a group in the usual sense; that is, μ and ι become "actual" mappings (binary and unary operations) which provide

$\hat{A}$ with a group structure satisfying all the standard (formal) axioms.

In my paper in the Curry Festschrift I applied this kind of method for cartesian <u>closed</u> categories.

DEFINITION 4.4. The category $\mathbb{C}$ is said to be <u>cartesian closed</u> if in addition to the cartesian structure of 4.3 the category is provided with internal function spaces satisfying the following rules:

$$\varepsilon_{BC} : (B \to C) \times B \to C$$

$$\frac{h : A \times B \to C}{\Lambda_{AB} h : A \to (B \to C)}$$

$$\frac{h : A \times B \to C}{\varepsilon_{BC} \circ \langle \Lambda_{AB} h \circ p_{AB}, q_{AB} \rangle = h}$$

$$\frac{k : A \to (B \to C)}{\Lambda_{AB} (\varepsilon_{BC} \circ \langle k \circ p_{AB}, q_{AB} \rangle) = k}$$

PROPOSITION 4.5. <u>In the representation of the category $\mathbb{C}$ in the model, the construct $(B \to C)$ becomes a functor isomorphic to the set-theoretical function space functor.</u>