Satz von Taylor für $f : \mathbb{R} \to \mathbb{R}$

sei $a \in \mathbb{R}$ fest gewählt, dann gilt

$$f(x) = T_m(x,a) + R_m(x,a)$$

Taylor-Polynom

$$T_m(x,a) = \sum_{j=0}^{m} \frac{(x-a)^j}{j!} f^{(j)}(a)$$

Restglied

$$R_m(x,a) = \int_a^x \frac{(x-t)^m}{m!} f^{(m+1)}(t)dt = \frac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(\xi)$$

Beweis: Induktion und partielle Integration:

$$\int_a^x \frac{(x-t)^m}{m!} f^{(m+1)}(t)dt = f^{(m+1)}(a)\frac{(x-a)^{m+1}}{(m+1)!} + \int_a^x \frac{(x-t)^{m+1}}{(m+1)!} f^{(m+2)}(t)dt$$

Satz von Taylor für $f : \mathbb{R}^d \to \mathbb{R}$

substituiere $u(\theta) = a + \theta h$, dann gilt $a = u(0)$, $x = a + h = u(1)$

def. $\tilde{f}(\theta) := f(u(\theta))$, $\quad \tilde{f} : \mathbb{R} \to \mathbb{R}$, $\quad$ Kettenregel

$$\begin{aligned} T_m(x,a) =& f(a) + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(a) h_i + \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \frac{h_i h_j}{2!} \\ &+ \ldots + \sum_{i_1,\ldots,i_m=1}^{d} \frac{\partial^m f}{\partial x_{i_1} \ldots \partial x_{i_m}}(a) \frac{h_{i_1} \cdots h_{i_m}}{m!} \end{aligned}$$

$$\begin{aligned} R_m(x,a) =& \sum_{i_1,\ldots,i_{m+1}=1}^{d} \int_0^1 \frac{(1-\theta)^m}{m!} \frac{\partial^{m+1} f}{\partial x_{i_1} \ldots \partial x_{i_{m+1}}}(a+\theta h) h_{i_1} \cdots h_{i_{m+1}} d\theta \\ =& \sum_{i_1,\ldots,i_{m+1}=1}^{d} \frac{\partial^{m+1} f}{\partial x_{i_1} \ldots \partial x_{i_{m+1}}}(a+\zeta h) \frac{h_{i_1} \cdots h_{i_{m+1}}}{(m+1)!} \\ & \zeta = \zeta(x,a) \in (0,1) \end{aligned}$$

Satz von Taylor für $f : \mathbb{R}^d \to \mathbb{R}^q$

sei $a \in \mathbb{R}$ fest gewählt, $x = a + h$, $f = (f_1, \ldots, f_q)^T$, dann gilt mit dem Satz von Taylor angewandt auf $f_k : \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned} T_{m,k}(x,a) =& f_k(a) + \sum_{i=1}^{d} \frac{\partial f_k}{\partial x_i}(a) h_i + \sum_{i,j=1}^{d} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(a) \frac{h_i h_j}{2!} \\ &+ \ldots + \sum_{i_1,\ldots,i_m=1}^{d} \frac{\partial^m f_k}{\partial x_{i_1} \ldots \partial x_{i_m}}(a) \frac{h_{i_1} \cdots h_{i_m}}{m!} \\ R_{m,k}(x,a) =& \sum_{i_1,\ldots,i_{m+1}=1}^{d} \frac{\partial^{m+1} f_k}{\partial x_{i_1} \ldots \partial x_{i_{m+1}}}(a+\zeta_k h) \frac{h_{i_1} \cdots h_{i_{m+1}}}{(m+1)!} \\ & \zeta_k = \zeta_k(x,a) \in (0,1) \end{aligned}$$

Satz von Taylor für $f : \mathbb{R}^d \to \mathbb{R}^q$

Kurzschreibweise

$$\begin{aligned} T_m(x,a) =& f(a) + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(a) h_i + \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \frac{h_i h_j}{2!} \\ &+ \ldots + \sum_{i_1,\ldots,i_m=1}^{d} \frac{\partial^m f}{\partial x_{i_1} \ldots \partial x_{i_m}}(a) \frac{h_{i_1} \cdots h_{i_m}}{m!} \\ =& f(a) + f'(a)h + \frac{1}{2!} f''(a)[h,h] + \ldots + \frac{1}{m!} f^{(m)}(a)[h,\ldots,h] \end{aligned}$$

$$\begin{aligned} R_m(x,a) =& \sum_{i_1,\ldots,i_{m+1}=1}^{d} \int_0^1 \frac{(1-\theta)^m}{m!} \frac{\partial^{m+1} f}{\partial x_{i_1} \ldots \partial x_{i_{m+1}}}(a+\theta h) h_{i_1} \cdots h_{i_{m+1}} d\theta \\ =& \int_0^1 \frac{(1-\theta)^m}{m!} f^{(m+1)}(a+\theta h)[h,\ldots,h] d\theta \end{aligned}$$